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THE ESSENTIAL COALITIONS INDEX IN GAMES WITH RESTRICTED COOPERATION

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Abstract:
We propose a new power index, which we call the essential coalitions index. Within the field of power indices, the new measure extends the Deegan-Packel power index to situations with restricted cooperation. In general, the class of games we study are not simple; with this in mind, we will introduce the essential coalitions as an analogue to the minimal winning coalitions of a simple game, since they generalize some relevant properties. We will first define the new index in terms of three reasonable assumptions, with a similar flavor to those used for the Deegan-Packel index; then, we will formally characterize the index. Finally, through numeric examples, we compare the essential coalitions index to the probabilistic Deegan-Packel index. We see that, in the latter's domain, the two indices only differ by a constant factor. Moreover, the new index is fit to analyze power in the formation of stable coalitions to run a government or a company board.

JEL Codes: C71, D71, D72.

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1 Introduction

Cooperative games have long been used to measure the power agents wield in a range of real-life settings. In Shapley and Shubik (1954), its authors use their namesake index to assess the influence a member of a committee has on the final decisions of the group. In the framework of the Shapley-Shubik index, these decisions are binary in nature: to pass a motion or not, in what is formally called a *simple* game. In these situations, the index measures the power of the agents by their decisiveness; more specifically, agents derive power from enabling a coalition to pass a motion.

Another approach to measuring power is that of the Deegan-Packel index. In this case, the main idea is that only a certain class of coalitions will actually form in practice. Namely, it is argued in Deegan and Packel (1979) that "only minimal winning coalitions emerge victorious". In the same committee framework we established before, a *minimal winning coalition* is one that is able to pass a motion, but for which this would cease to be the case should any of its members abandon it. In Deegan and Packel's approach, membership in minimal winning coalitions is the source of power for agents.

Regardless of the power measure, ever since the introduction of the Shapley-Shubik index, the literature has mostly focused on axiomatic characterizations of these indices. The first of many characterizations of the Shapley value, of which the Shapley-Shubik index is the restriction to simple games, appears in Shapley (1953), where the concept was introduced in the first place; similarly, the Deegan-Packel index is characterized in Deegan and Packel (1979). Moreover, as pointed out in the latter, the two characterizations differ only in one property, in spite of the apparent difference in their assessment of power.

There is, however, an underlying assumption that hinders the applicability of these indices. Namely, in these initial frameworks it was assumed that agents cooperate with each other without constraint. In such a setting, all coalitions are deemed equally likely to form. Indeed, in particular, Deegan and Packel assume that each minimal winning coalition has the same probability to arise.

It is not difficult to convince oneself that this is unlikely to be the case in real-life situations. For instance, in a parliament, committees are bound to have members from different parties, and coalitions will arguably tend to form along ideological lines. This motivates the development of models to restrict cooperation.

The most well studied such models are based on combinatorial structures that capture the relations between the agents. In particular, graphs have been used both to convey feasible channels of communication (Myerson, 1977) and describe pairwise incompatibilities between agents (Carreras, 1991). Both the Shapley-Shubik (Bergantiños et al., 1993; Myerson, 1977) and the Deegan-Packel (Alonso-Meijide, 2002) indices have been characterized on these restricted cooperation settings.

Arguably though, a more comprehensive restriction model is one that assesses how likely each coalition is to remain united. This is the intent behind *cooperation indices* (introduced in Amer and Carreras (1995)), which map each coalition to a value in the unit interval. In this same reference, a restriction to cooperation in TU games via cooperation indices is defined, and the Shapley value on the resulting class of games is characterized.

Finally, a similar but distinct approach to that of cooperation indices are *probabilistic* indices. In these, a probability distribution is defined on the set of feasible coalitions. Probabilistic generalizations of the Shapley value and the Deegan-Packel index are discussed in Dubey (1976) and Deegan and Packel (1980), respectively.

In both cases, though, the probabilistic indices assume probability distributions that are "symmetric with respect to cardinality"; that is, the probability that a coalition forms only depends on its size. This hampers generality. Once again referring to the parliamentary framework, while forming a larger coalition requires more agreements between parties, affecting its probability of occurring, the identities of the parties involved are also likely to have an effect on the probability of formation.

This issue does not arise in the cooperation index approach, which has not been studied as much. In particular, even though the Shapley value in restricted games has been characterized, this is not the case for other power indices.

In what follows, our goal will be to fill this gap. Namely, we shall introduce and characterize a Deegan-Packel inspired index on simple games restricted by a cooperation index. We will then compare the newly introduced index to the probabilistic one. The following section introduces the proper mathematical models needed to describe the aforementioned situations.

In Section 3 we will introduce the restriction models of interest and their potential applications, as well as an analogue to minimal winning coalitions on simple games restricted by a cooperation index. We will use this to motivate the new power index we propose, for which we will provide a characterization in Section 4. Finally, in Section 5 we shall compare the two indices conceptually and via some numerical examples.

2 Preliminaries

A cooperative transferable utility game (henceforth, a TU game, for short) is a pair (N, v) where $N = \{1, \ldots, n\}$, $n \geqslant 1$, is its set of players, and $v : 2^N \to \mathbb{R}$ is its characteristic function. The elements in $2^N = \{S : S \subseteq N\}$ are called *coalitions*, and the only constraint on v is $v(\varnothing) = 0$. We denote the set of TU games with player set N by $TU(N)$.

A simple game is a TU game such that $v(S) \in \{0, 1\}$ $\forall S \subseteq N$, $v(N) = 1$ and

$v(S) \leqslant v(T)$ whenever $S \subseteq T$. We denote the set of simple games with player set N by $SI(N)$. We refer to the coalitions $S \subseteq N$ such that $v(S) = 1$ ($v(S) = 0$, respectively) as winning (losing) coalitions. Thus, the game is completely determined by its set of winning coalitions, $\mathcal{W}(N, v) = \{S \subseteq N : v(S) = 1\}$. If no two winning coalitions of a game are disjoint, we say the game is *proper*; we denote the set of proper simple games with player set N by $PS(N)$.

The minimal winning coalitions of a simple game are those winning coalitions that become losing should any subset of its members be removed from them. We denote this set by $\mathcal{W}^m(N, v) = \{S \in \mathcal{W}(N, v) : T \notin \mathcal{W}(N, v)\ \forall T \subsetneq S\}$.

A *value* defined on a class of games $\mathcal{C}(N) \subseteq TU(N)$ is a function $f : \mathcal{C}(N) \to \mathbb{R}^n$. The i-th component of $f(N, v)$ represents the value of player i according to f. An *index* is a value defined on a subclass of simple games.

In Deegan and Packel (1979), the following assumptions on the behavior of players in a simple game are suggested:

- Only minimal winning coalitions will emerge victorious.
- Each minimal winning coalition has an equal probability of forming.
- Players in a minimal winning coalition divide the "spoils" equally.

From these, the Deegan-Packel power index, ρ, is derived, and defined for every simple game (N, v) and $i \in N$ by

$$\rho_i\,(N,\, v) = \frac{1}{|\mathcal{W}^m\,(N,\, v)|} \sum_{\substack{S \in \mathcal{W}^m(N,\, v) \\ i \in S}} \frac{1}{|S|}.$$

The index is characterized in the same paper that introduced it. Later on, in Deegan and Packel (1980), a probabilistic generalization of it is discussed and characterized.

To describe it, let $f : 2^N \to (0, 1)$ be a probability function, mapping each coalition to its probability of forming. It is assumed that if $|S| = |T|$, then $f(S) = f(T)$. Given such f and $(N, v) \in SI(N)$, define

$$P^f(S) = \begin{cases} \frac{f(S)}{\alpha^f(N,\, v)} \text{ if } S \in \mathcal{W}^m(N,\, v) \\ 0 \text{ otherwise,} \end{cases}$$

where $\alpha^f(N, v) = \sum_{S \in \mathcal{W}^m(N,\, v)} f(S)$. The probabilistic generalization of the Deegan-Packel index, ρ^f, is defined for every $(N, v) \in SI(N)$ and $i \in N$ by

$$\rho_i^f(N,\, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{v(S)}{|S|} P^f(S).$$

3 Games with cooperation indices

We begin this section by summarizing the introduction to cooperation indices in Amer and Carreras (1995). A cooperation index over player set N is a function $p : 2^N \to [0, 1]$ such that for every $i \in N$ we have $p(\{i\}) = 1$. We will denote the set of cooperation indices over N by $I(N)$.

For the purposes of this work, we will think of a cooperation index as conveying the *stability* of a coalition. Namely, given that S has formed, we will interpret $p(S)$ as the probability that S will remain united for a standardized period of time.

Condition $p(\{i\}) = 1 \ \forall i \in N$ reinforces this idea, since there is no coalition to break in these cases. Moreover, under this interpretation we can justify the absence of further structure on a cooperation index. The following example shows a reasonable scenario in which we can define a cooperation index that is non-monotonic with respect to inclusion.

Example 3.1. Consider the set of players $N = \{L, C, R\}$ representing a left-wing, a centrist, and a right-wing party, respectively. Arguably, the most stable non-trivial coalitions in 2^N are $S_1 = \{L, C\}$ and $S_2 = \{C, R\}$, since they only involve two ideologically close agents. Due to its increased size and diversity, it also seems logical for the grand coalition to have a lower cooperation index than both of these.

Finally, we can also support that $S_3 = \{L, R\}$ is less stable than the grand coalition. While $S_3 \subsetneq N$, and both coalitions contain the ideological extremes, one can argue that the presence of the centrist party in N increases communication between the other two.

All in all, it is not unsound to define $p \in I(N)$ so that $p(S_1) > p(N) > p(S_3)$.

Now, given $p \in I(N)$, consider the following equivalence relation $\sim$ in N. Two players $i, j \in N$ are related if and only if there exist coalitions $S_1, \ldots, S_k$ such that

- $p(S_r) > 0 \ \forall r \in \{1, \ldots, k\}$.
- $S_r \cap S_{r+1} \neq \varnothing \ \forall r \in \{1, \ldots, k-1\}$.
- $i \in S_1, j \in S_k$.

The equivalence classes under this relation are called *islands*. It can be shown that the connected components of an undirected graph $G = (N, E)$ coincide with the islands of any $p \in I(N)$ such that $p(S) > 0$ if $S \in E$ and $p(S) = 0$ whenever S is not connected.

Recall that, in Myerson's communication model (Myerson, 1977), players can only communicate within the connected components of a graph. The following result shows how, in the words of Amer and Carreras, "islands are the natural unities within which players can negotiate". Hence, the lemma will further validate the role of islands as a generalization of the connected components of a graph.

LEMMA 3.1. Let $n \geqslant 1$, $N = \{1, \ldots, n\}$, $p \in I(N)$. If $p(S) > 0$, then S must be fully contained within an island.

Proof. Since islands are the equivalence classes under $\sim$, it suffices to show that $i \sim j$ for any pair of players $i, j \in S$. This is seen to be true by letting $k = 1$ and $S_1 = S$ in the definition of $\sim$. □

In other words, $p(S) = 0$ whenever S contains players from different islands.

DEFINITION 3.1. A *game with a cooperation index* is a triple (N, v, p) where (N, v) is a TU game and $p \in I(N)$. We denote the set of games with a cooperation index with player set N by $GI(N)$.

Given $(N, v, p) \in GI(N)$, the *p-restriction* of (N, v) (as long as no confusion arises, the restricted game) is a new TU game (N, v_p) with

$$v_p(S) = \max_{\mathcal{P} \in \mathbf{P}^+(S,p)} \sum_{T \in \mathcal{P}} v(T)p(T)$$

for every $S \subseteq N$, where $\mathbf{P}^+(S, p)$ denotes the set of partitions of S into coalitions with positive index.

Note that the restriction of a simple game need not be a simple game itself. Nonetheless, the expression for the characteristic function of a restricted game is greatly reduced when the original game is a proper simple game. For the sake of notation, let $PI(N)$ denote the set of restricted proper simple games, that is, triples (N, v, p) such that (N, v) is a proper simple game, $p \in I(N)$ and (N, v_p) is not a null game.

LEMMA 3.2. Let $(N, v, p) \in PI(N)$. For every $S \subseteq N$, the restricted game satisfies

$$v_p(S) = \max_{T \subseteq S} v(T)p(T).$$

Proof. Note that, for any $S \subseteq N$ and partition $\mathcal{P} \in \mathbf{P}^+(S, p)$, there is at most one winning coalition $T \in \mathcal{P}$, since we are assuming that (N, v) is a proper game. Thus, as long as no winning $T \subseteq S$ with positive index exists, $v_p(S) = 0 = \max_{T \subseteq S} v(T)p(T)$.

If, on the contrary, such T exists, then $v_p(S) = \max_{T \in \mathcal{W}(N,v) \cap 2^S} p(T) = \max_{T \subseteq S} v(T)p(T)$. □

In particular, if $(N, v, p) \in PI(N)$, then, for every $S \subseteq N$, $v_p(S)$ is either zero, or the value of the cooperation index on some coalition $T \subseteq S$. Intuitively, we observe that agents in a coalition S seek to organize themselves in the most stable winning subcoalition possible. All in all, Lemma 3.2 motivates the search for a set of coalitions whose cooperation indices capture all non-zero valued coalitions in the restricted game.

Definition 3.2. Given $(N, v, p) \in PI(N)$, we define the set of *essential coalitions* (of the restricted game) as

$$\mathcal{E}(N, v_p) = \{S \subseteq N : v_p(S) > v_p(T) \; \forall T \subsetneq S, \, S \neq \varnothing\}.$$

Below we introduce some relevant properties of the set $\mathcal{E}(N, v_p)$. In particular, Proposition 3.1 shows how its elements are indeed *essential* to define the restricted game.

Lemma 3.3. If $E \in \mathcal{E}(N, v_p)$, then $v_p(E) = p(E) > 0$.

Proof. Given an essential coalition E, it follows directly from the definition above that $v_p(E) > v_p(\varnothing) = 0$. On the other hand, by Lemma 3.2, for every $T \subsetneq E$,

$$\max_{R \subseteq E} v(R)p(R) = v_p(E) > v_p(T) = \max_{R' \subseteq T} v(R')p(R') \geqslant v(T)p(T).$$

Thus, $0 < v_p(E) = v(E)p(E)$, and so $v(E) > 0$, $p(E) > 0$ and $v_p(E) = p(E)$. □

Corollary 3.1. If E is an essential coalition, then it is fully contained in an island.

Proof. This follows from the previous result and Lemma 3.1. □

Proposition 3.1. Let $(N, v, p) \in PI(N)$, $\mathcal{C} \subseteq 2^N$. Suppose that for every S such that $v_p(S) > 0$ there is some $T \in \mathcal{C}$ satisfying $v_p(S) = v_p(T)$ and $T \subseteq S$; then, $\mathcal{E}(N, v_p) \subseteq \mathcal{C}$. Furthermore, the property holds for the set of essential coalitions.

Proof. The first part of the result is equivalent to no $\mathcal{C} \subsetneq \mathcal{E}(N, v_p)$ satisfying the property. To show this is true, let $\mathcal{C} \subseteq 2^N$ and suppose there exists $E \in \mathcal{E}(N, v_p)$ such that $E \notin \mathcal{C}$. By definition, $v_p(E) > v_p(T) \geqslant 0 \; \forall T \subsetneq E$, so the only $T \subseteq E$ for which it could be that $v_p(E) = v_p(T)$ is E itself. In particular, $\mathcal{C}$ does not satisfy the property of interest.

For the second part, let $S \subseteq N$ be such that $v_p(S) > 0$. By Lemma 3.2, we know that $v_p(S) = \max_{T \subseteq S} v(T)p(T)$. Let $R \subseteq S$ be the smallest coalition satisfying $v_p(S) = v(R)p(R)$.

We claim that $R \in \mathcal{E}(N, v_p)$. By construction, $v(R)p(R) > v(T)p(T) \; \forall T \subsetneq R$. In particular, $v(R)p(R) > \max_{T \subsetneq R} v(T)p(T)$, which implies $v_p(R) = v(R)p(R)$. Finally, since for every $T \subsetneq S$ we have $v_p(T) = v(R')p(R')$ for some $R' \subseteq T$, it also follows that $v_p(R) > v_p(R') \; \forall R' \subsetneq R$, which ends the proof. □

It is worth pointing out that Proposition 3.1 does not require that each non-zero valued coalition contains a unique essential coalition with the same value. Indeed, nothing prevents two essential coalitions from having the same cooperation index.

This is not unlike what occurs with minimal winning coalitions of a simple game, which all have the same value. On the other hand, any winning coalition contains *at least* one minimal winning coalition, and no smaller set of coalitions satisfies this property.

The connection between both sets is summarized in our definition of essential coalitions, which is evocative of the minimal winning coalitions satisfying

$$\mathcal{W}^m(N, v) = \{S \subseteq N : v(S) > v(T) \ \forall T \subsetneq S, S \neq \varnothing\}.$$

Now, recall Deegan and Packel's assumption that, in a simple game, only minimal winning coalitions "emerge victorious". They argue in Deegan and Packel (1979) that this is reasonable for players who maximize payoffs. This is the case in the fully cooperative setting, in which further agreements yield no increase in payoff.

However, for games with a cooperation index, reaching additional agreements may increase the stability of a coalition, thereby increasing its expected benefits. Furthermore, we have seen that if agents in a coalition organize themselves in the most stable winning subcoalition possible, the only coalitions that will form are the essential coalitions.

All in all, the motivation for a Deegan-Packel inspired power index on restricted proper simple games is established. The new index will be defined by the following assumptions on the behavior of the agents in these games:

1) Only essential coalitions will form.

2) Each of these coalitions have an equal probability of emerging.

3) Players in an essential coalition divide the benefits equally.

These assumptions define the *essential coalitions index*, $\mathfrak{e}$, which assigns each game $(N, v, p) \in PI(N)$ to an n-dimensional vector $\mathfrak{e}(N, v, p)$ whose i-th component is

$$\mathfrak{e}_i(N, v, p) = \frac{1}{|\mathcal{E}(N, v_p)|} \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ i \in E}} \frac{p(E)}{|E|}. \tag{1}$$

Note that the assumptions above are mere restatements of those defining the original Deegan-Packel power index, now in terms of essential coalitions. Since we have already discussed at length the first of these three assumptions, let us end this section with some observations on the other two.

Recall that the cooperation index is not meant to represent the probability that a coalition is formed, but that of it not breaking up before a standard period of time. Thus, lacking further information regarding their probability of formation, we argue in favor of still using a uniform probability distribution over the set of essential coalitions.

As for the utility distribution between agents, in the government formation framework we may think of agents in an essential coalition as taking turns in leading the group. Our assumption implies that each agent will lead the coalition for the same amount of time.

4 A characterization for the new index

In this section we will identify a set of more formal properties that uniquely define the essential coalitions index, $\mathfrak{e}$. The properties in our proposed characterization are very similar to those used in Amer and Carreras (1995) to characterize the Shapley value on restricted games.

Definition 4.1. We say a value $\Psi : PI(N) \to \mathbb{R}^n$ satisfies:

- Average Essential Efficiency (AvEE) when for every $(N, v, p) \in PI(N)$ and every island I of p, if $I \notin \mathcal{W}(N, v)$ then $\sum_{i \in I} \Psi_i(N, v, p) = 0$, and if $I \in \mathcal{W}(N, v)$, then

$$\sum_{i \in I} \Psi_i(N, v, p) = \frac{1}{|\mathcal{E}(N, v_p)|} \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ E \subseteq I}} p(E).$$

- Essential Fairness (EF) when for every pair $(N, v, p_1), (N, v, p_2) \in PI(N)$ such that $p_1(S) = p_2(S)\ \forall S \neq R$ for some $R \subseteq N$, and for every $i, j \in R$,

$$\begin{aligned} &|\mathcal{E}(N, v_{p_1})| \left(\Psi_i(N, v, p_1) - \Psi_j(N, v, p_1)\right) \\ &\qquad = |\mathcal{E}(N, v_{p_2})| \left(\Psi_i(N, v, p_2) - \Psi_j(N, v, p_2)\right). \end{aligned}$$

- Essential Stability (ES) when, for every pair $(N, v, p_1), (N, v, p_2) \in PI(N)$ such that there is some $R \subseteq N$ for which $p_1(S) = p_2(S)\ \forall S \neq R$ and $p_2(R) = 0$, and for every $i \in R$,

$$|\mathcal{E}(N, v_{p_2})| \Psi_i(N, v, p_2) \leqslant |\mathcal{E}(N, v_{p_1})| \Psi_i(N, v, p_2).$$

- Essential Monotonicity (EM) when for every pair $(N, v, p_1), (N, v, p_2) \in PI(N)$,

$$|\mathcal{E}(N, v_{p_2})| \Psi_i(N, v, p_2) \leqslant |\mathcal{E}(N, v_{p_1})| \Psi_i(N, v, p_1)$$

  for every $i \in N$ for which $\mathcal{E}_i(N, v_{p_2}) \subseteq \mathcal{E}_i(N, v_{p_1})$ and $p_2(E) \leqslant p_1(E)$ for every $E \in \mathcal{E}_i(N, v_{p_2})$, where $\mathcal{E}_i(N, v_p)$ denotes the set of essential coalitions of (N, v_p) that contain player i.

Note that our notion of efficiency is well-defined due to Corollary 3.1, which established that any essential coalition is fully contained in an island. On the other hand, as long as the original game is simple and proper, at most one island can be a winning coalition. If no such island exists, then the restricted game is a null game. Any $(N,\, v,\, p)$ such that $(N,\, v_p)$ is null is excluded from $PI(N)$; therefore, the restricted games we shall consider have exactly one winning island.

All in all, (AvEE) states that the agents in the winning island divide a payoff of

$$\frac{1}{|\mathcal{E}\,(N,\, v_p)|} \sum_{E \in \mathcal{E}(N,\, v_p)} p(E).$$

This quantity has a natural interpretation tied to the notion of stability the cooperation index conveys. Namely, if each essential coalition E has an equal probability of forming a government and a probability of $p(E)$ of remaining united after a given amount of time, then the quantity above represents the expected time the governing coalition will stay united. In summary, those in the winning island divide among themselves the average time during which there will be a stable governing coalition.

As for the other properties, (EF) and (ES) are, respectively, normalizations of the fairness and stability properties (F) and (S) that appear in Amer and Carreras (1995). In Section 5 we will see that the essential coalitions index does not satisfy (ES); instead, it satisfies (EM).

It is also worth pointing out that both (EF) and (EM) resemble axioms that have already been used to characterize variations of the Deegan-Packel index. Indeed, essential monotonicity is reminiscent of the minimal monotonicity property used to characterize the unrestricted Deegan-Packel index in Alonso-Meijide et al. (2007). In Alonso-Meijide (2002), minimal fairness is used for that same index in games with cooperation restricted by a communication graph.

All in all, we argue that the properties we have introduced are reasonable. If the cooperation index of one coalition is changed, it is only *fair* that it produces the same difference in payoffs for every member of that coalition. Moreover, if the new cooperation index of that coalition is zero, it would be natural that the payoff allocated to its members to decrease. Similarly, the more essential coalitions an agent is a member of, and the more stable these are, the higher payoff we would expect to be allocated to them. Due to the relevance of essential coalitions in restricted games, it is justifiable to normalize by the number of such coalitions when defining these properties.

We will now show that (AvEE) and (EF) characterize $\mathfrak{e}$, which also satisfies (EM). To do so, we will first prove an auxiliary lemma regarding the essential coalitions of two similar restricted games. Then, we will show that the essential coalitions index satisfies the three properties at issue; finally, we will argue that no two different mappings can satisfy the first two properties.

LEMMA 4.1. Let $(N, v) \in PS(N)$, $p_1, p_2 \in I(N)$ and $R \subseteq N$ be such that, for every $S \neq R$, $p_1(S) = p_2(S)$. If $R \not\subseteq E$, then $E \in \mathcal{E}(N, v_{p_1})$ if and only if $E \in \mathcal{E}(N, v_{p_2})$.

Proof. By Lemma 3.2, if p_1 and p_2 may only differ at $R \not\subseteq E$, then for every $T \subseteq E$

$$v_{p_1}(T) = \max_{R' \subseteq E} v(R')p_1(R') = \max_{R' \subseteq T} v(R')p_2(R') = v_{p_2}(T).$$

In particular, $v_{p_1}(E) > v_{p_1}(T)$ for every $T \subsetneq E$ if and only if $v_{p_2}(E) > v_{p_2}(T)$ for every such T, and the result follows from Definition 3.2. □

PROPOSITION 4.1. The essential coalitions index satisfies average essential efficiency, essential fairness and essential monotonicity.

Proof. We start by the first property. By the monotonicity of v, if an island I of p is not winning in (N, v), then neither is any $T \subseteq I$. In particular, by Lemma 3.3, no essential coalitions are contained in such an island, and so $\mathfrak{e}_i(N, v, p) = 0 \ \forall i \in I$.

If, on the contrary, $I \in \mathcal{W}(N, v)$, then

$$|\mathcal{E}(N, v_p)| \sum_{i \in I} \mathfrak{e}_i(N, v, p) = \sum_{i \in I} \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ i \in E}} \frac{p(E)}{|E|} = \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ E \subseteq I}} \sum_{i \in E} \frac{p(E)}{|E|} = \sum_{\substack{E \in \mathcal{E}(N, v_p) \\ E \subseteq I}} p(E).$$

Moving on to essential monotonicity, let $(N, v, p_1), (N, v, p_2) \in PI(N)$ and $i \in N$ be such that $\mathcal{E}_i(N, v_{p_2}) \subseteq \mathcal{E}_i(N, v_{p_1})$ and $p_2(E) \leqslant p_1(E) \ \forall E \in \mathcal{E}_i(N, v_{p_2})$. Then,

$$\begin{aligned}
|\mathcal{E}(N, v_{p_2})| \, \mathfrak{e}_i(N, v, p_2) &= \sum_{\substack{E \in \mathcal{E}(N, v_{p_2}) \\ i \in E}} \frac{p_2(E)}{|E|} \\
&\leqslant \sum_{\substack{E \in \mathcal{E}(N, v_{p_2}) \\ i \in E}} \frac{p_1(E)}{|E|} \\
&\leqslant \sum_{\substack{E \in \mathcal{E}(N, v_{p_1}) \\ i \in E}} \frac{p_1(E)}{|E|} \\
&= |\mathcal{E}(N, v_{p_1})| \, \mathfrak{e}_i(N, v, p_1).
\end{aligned}$$

Finally, for (EF) note that for any pair $(N,\, v,\, p_1)\,,\,(N,\, v,\, p_2)\in PI(N)$ and $i,\, j\in N$,

$$\begin{aligned}
|\mathcal{E}\left(N,\, v_{p_1}\right)|\left(\mathfrak{e}_i\left(N,\, v,\, p_1\right)-\mathfrak{e}_j\left(N,\, v,\, p_1\right)\right| &= \sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_1}\right)\\ i\in E}}\frac{p_1(E)}{|E|}-\sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_1}\right)\\ j\in E}}\frac{p_1(E)}{|E|}\\
&= \sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_1}\right)\\ i,j\in E}}\frac{p_1(E)}{|E|}+\sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_1}\right)\\ i\in E, j\notin E}}\frac{p_1(E)}{|E|}\\
&\quad -\sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_1}\right)\\ i,j\in E}}\frac{p_1(E)}{|E|}-\sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_1}\right)\\ i\notin E, j\in E}}\frac{p_1(E)}{|E|}\\
&= \sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_1}\right)\\ i\in E, j\notin E}}\frac{p_1(E)}{|E|}-\sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_1}\right)\\ i\notin E, j\in E}}\frac{p_1(E)}{|E|}. \qquad (2)
\end{aligned}$$

Now, if p_1 and p_2 may only differ on one coalition R, then, by Lemma 4.1, for any $i,\, j\in R$ the latter expression does not change if the index is p_2 instead of p_1. Hence, for any such i and j,

$$\begin{aligned}
|\mathcal{E}\left(N,\, v_{p_1}\right)|\left(\mathfrak{e}_i\left(N,\, v,\, p_1\right)-\mathfrak{e}_j\left(N,\, v,\, p_1\right)\right| &= \sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_2}\right)\\ i\in E, j\notin E}}\frac{p_2(E)}{|E|}-\sum_{\substack{E\in\mathcal{E}\left(N,\, v_{p_2}\right)\\ i\notin E, j\in E}}\frac{p_2(E)}{|E|}\\
&= |\mathcal{E}\left(N,\, v_{p_2}\right)|\left(\mathfrak{e}_i\left(N,\, v,\, p_2\right)-\mathfrak{e}_j\left(N,\, v,\, p_2\right)\right|,
\end{aligned}$$

where the second equality summarizes a chain of equalities analogous to that in (2). All in all, it is shown that $\mathfrak{e}$ satisfies essential fairness, and the proof ends. □

THEOREM 4.1. There exists a unique map $\Psi:\, PI(N)\to\mathbb{R}^n$ satisfying (AvEE) and (EF).

Proof. Existence was shown in the previous proposition; thus, it only remains to prove that (AvEE) and (EF) uniquely define $\mathfrak{e}$. First of all, note that this is the case for $(N,\, v,\, p)\in PI(N)$ such that $p(S)=1$ if $|S|=1$ and $p(S)=0$ otherwise. In such case, the islands of p are the singletons. By our definition of $PI(N)$, exactly one of them is winning; let $\{i\}$ be this island. Then, $\{i\}$ is the only essential coalition of $(N,\, v_p)$ and so, by (AvEE), it must be the case that $\Psi_i\left(N,\, v,\, p\right)=1$, while $\Psi_j\left(N,\, v,\, p\right)=0$ if $j\neq i$.

Now, let $\Psi,\, \Phi:\ PI(N)\to\mathbb{R}^n$ satisfy (AvEE) and (EF), and let $p\in I(N)$ have the minimum amount of non-zero index coalitions so that there is some $(N,\, v,\, p)\in PI(N)$ such that $\Psi\left(N,\, v,\, p\right)\neq\Phi\left(N,\, v,\, p\right)$. By our previous discussion, we can find $R\subseteq N$ such that $|R|\geqslant 2$ and $p(R)>0$. Given such R, define a new index p' so that $p'(S)=p(S)$ if $R\neq S$ and $p'(R)=0$. By our choice of p, $\Psi\left(N,\, v,\, p'\right)=\Phi\left(N,\, v,\, p'\right)$.

On the other hand, by essential fairness, if $i,\, j \in R$,

$$\begin{aligned}|\mathcal{E}\left(N,\, v_p\right)|\left(\Phi_i\left(N,\, v,\, p\right) - \Phi_j\left(N,\, v,\, p\right)\right) &= |\mathcal{E}\left(N,\, v_{p'}\right)|\left(\Phi_i\left(N,\, v,\, p'\right) - \Phi_j\left(N,\, v,\, p'\right)\right)\\ &= |\mathcal{E}\left(N,\, v_{p'}\right)|\left(\Psi_i\left(N,\, v,\, p'\right) - \Psi_j\left(N,\, v,\, p'\right)\right)\\ &= |\mathcal{E}\left(N,\, v_p\right)|\left(\Psi_i\left(N,\, v,\, p\right) - \Psi_j\left(N,\, v,\, p\right)\right)\end{aligned}$$

and, rearranging terms, it follows that, for every pair $i,\, j \in R$,

$$\Phi_i\left(N,\, v,\, p\right) - \Psi_i\left(N,\, v,\, p\right) = \Phi_j\left(N,\, v,\, p\right) - \Psi_j\left(N,\, v,\, p\right).$$

Note that this will remain true for any $R \subseteq N$ such that $p(R) > 0$. Therefore, it will still hold for any pair $i,\, j \in N$ such that $i \sim j$ (see page 4 for the definition of the relation $\sim$). To see this, given such a pair let $S_1,\, \ldots,\, S_r$ be a sequence of coalitions with positive index p such that $i \in S_1$, $j \in S_k$ and $S_r \cap S_{r+1} \neq \varnothing\ \forall r \in \{1,\, \ldots,\, k-1\}$. Thus, for each $r \in \{1,\, \ldots,\, k-1\}$ we can take $i_r \in S_r \cap S_{r+1}$. Now, by repeatedly applying our previous findings, we obtain

$$\begin{aligned}\Phi_i\left(N,\, v,\, p\right) - \Psi_i\left(N,\, v,\, p\right) &= \Phi_{i_1}\left(N,\, v,\, p\right) - \Psi_{i_1}\left(N,\, v,\, p\right)\\ &= \Phi_{i_2}\left(N,\, v,\, p\right) - \Psi_{i_2}\left(N,\, v,\, p\right)\\ &= \cdots = \Phi_{i_{k-1}}\left(N,\, v,\, p\right) - \Psi_{i_{k-1}}\left(N,\, v,\, p\right)\\ &= \Phi_j\left(N,\, v,\, p\right) - \Psi_j\left(N,\, v,\, p\right).\end{aligned}$$

To summarize, it is shown that $d(i) = \Phi_i\left(N,\, v,\, p\right) - \Psi_i\left(N,\, v,\, p\right)$ is constant for $i \in I$, where I is an island of p. Thus, for each island I we can define $\widetilde{d}_p(I) = d(i)$ with $i \in I$.

By (AvEE), for every island I of p, we also have $\sum\limits_{i\in I} \Phi_i\left(N,\, v,\, p\right) = \sum\limits_{i\in I} \Psi_i\left(N,\, v,\, p\right)$, so

$$0 = \sum_{i\in I}\left(\Phi_i\left(N,\, v,\, p\right) - \Psi_i\left(N,\, v,\, p\right)\right) = \sum_{i\in I} d(i) = |I|\widetilde{d}_p(I).$$

It follows that $\widetilde{d}_p(I) = 0$ for every island I, which contradicts our original assumptions, since it implies $\Phi\left(N,\, v,\, p\right) = \Psi\left(N,\, v,\, p\right)$. Thus, it must be the case that Φ and Ψ are equal everywhere, as desired. □

5 Examples and counterexamples

Now that we have characterized the new index, we will proceed to analyze its behavior on a series of numerical examples. The first of them will show that the essential coalitions index does not satisfy the essential stability property introduced in Definition 4.1.

Example 5.1. Consider a parliament where four parties seek to form a stable government. Assume these parties are numbered in decreasing order of the share of seats they control. Furthermore, suppose the simple game $(N,\, v)$ defined by

$$\mathcal{W}^m\,(N,\, v) = \{\, \{1,\, 2\},\ \{1,\, 3\},\ \{2,\, 3,\, 4\}\,\}$$

describes the struggle, with the winning coalitions representing those that could form a government. With this information we can see that the Deegan-Packel index of $(N,\, v)$ is $\rho\,(N,\, v) = \left(\frac{1}{3},\, \frac{5}{18},\, \frac{5}{18},\, \frac{1}{9}\right) \approx (0.33,\, 0.28,\, 0.28,\, 0.11)$.

Now, consider the diagram in Figure 1. Note that the cooperation index of losing coalitions has no effect on the restricted game. By inspection, it is readily seen that

$$\mathcal{E}\,(N,\, v_{p_1}) = \{\, \{1,\, 2\},\ \{1,\, 3\},\ \{1,\, 2,\, 4\},\ \{2,\, 3,\, 4\}\,\}$$

is the set of essential coalitions of the restricted game. Therefore, the essential coalitions index for this game is $\mathfrak{e}\,(N,\, v,\, p_1) = \left(\frac{0.7}{4},\, \frac{0.55}{4},\, \frac{0.65}{4},\, \frac{0.45}{4}\right) \approx (0.18,\, 0.14,\, 0.16,\, 0.11)$.

Observe that the essential coalitions index takes lower values than the unrestricted Deegan-Packel index. This comes as a consequence of the latter being efficient, i.e. adding up to one, while the former may add up to less than that. Moreover, while parties 2 and 3 were symmetric in the unrestricted game, the essential coalition index allocates a bigger

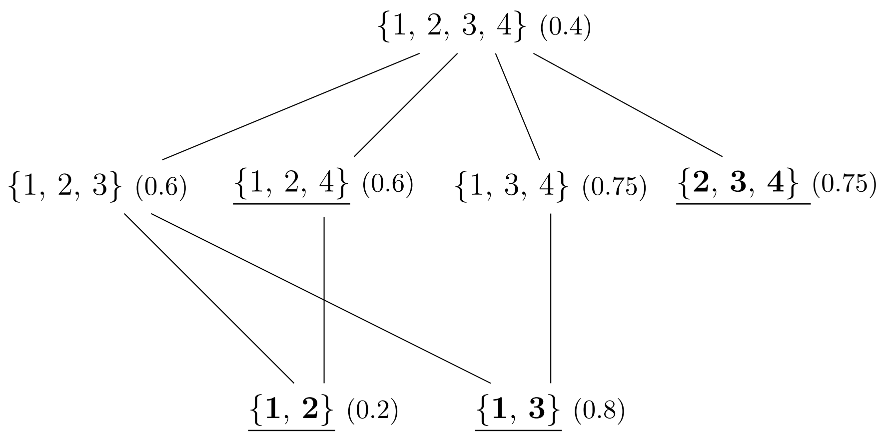

Figure 1: The winning coalitions of $(N,\, v)$ in Example 5.1, with the value of the cooperation index p_1 on each of them in parentheses. The minimal winning coalitions of $(N,\, v)$ are in **boldface**; the essential coalitions of $(N,\, v_{p_1})$ are underlined.

payoff to the latter. In particular, despite party 2 appearing in more essential coalitions (i.e. having more options to reach a stable pact), the difference does not compensate for the fact that an agreement only between parties 1 and 3 is significantly more stable than any of the options available to party 2.

Consider now a cooperation index p_2 such that $p_2(S) = p_1(S)$ for every $S \neq \{1, 3\}$, and $p_2(\{1, 3\}) = 0$. We may interpret this as the relationship between parties 1 and 3 having broken down; however, broader agreements are assumed to be robust enough to withstand their mutual animosity. The essential coalitions of (N, v_{p_2}) are

$$\mathcal{E}(N, v_{p_2}) = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.$$

Hence, the essential coalitions index is now $\mathfrak{e}(N, v, p_2) = \left(\frac{0.75}{5}, \frac{0.75}{5}, \frac{0.7}{5}, \frac{0.7}{5}\right)$. Note that parties 1 and 2 appear in the same number of essential coalitions, and with the same cooperation indices; the same occurs for parties 3 and 4. As a consequence, parties 1 and 2 have the same essential coalition index in this game, and so do parties 3 and 4.

On the other hand, essential stability has been violated. Indeed,

$$|\mathcal{E}(N, v_{p_2})|\, \mathfrak{e}_1(N, v, p_2) = 0.75 > 0.7 = |\mathcal{E}(N, v_{p_1})|\, \mathfrak{e}_1(N, v, p_1)$$

and

$$|\mathcal{E}(N, v_{p_2})|\, \mathfrak{e}_3(N, v, p_2) = 0.7 > 0.65 = |\mathcal{E}(N, v_{p_1})|\, \mathfrak{e}_3(N, v, p_1).$$

We do observe that, for parties 1 and 3, their essential coalitions index has decreased due to having $p_2(\{1, 3\}) = 0$: we have $\mathfrak{e}_1(N, v, p_2) = \frac{0.75}{5} = 0.15 < 0.175 = \mathfrak{e}_1(N, v, p_1)$ and $\mathfrak{e}_3(N, v, p_2) = 0.14 < 0.1625 = \mathfrak{e}_3(N, v, p_1)$, respectively. In light of this, we may suspect that such decline occurs in general, but the following example shows otherwise.

Example 5.2. Consider the six player simple game (N, v) with

$$\mathcal{W}^m(N, v) = \{\{1, 2\}, \{2, 3, 4, 5, 6\}\}$$

and the cooperation index $p_1(S) = 1\ \forall S \subseteq N$. Note that $\mathcal{E}(N, v_{p_1}) = \mathcal{W}^m(N, v)$; moreover, the essential coalitions index coincides with the Deegan-Packel index of the original game, $\mathfrak{e}(N, v, p_1) = \rho(N, v) = \left(\frac{1}{4}, \frac{7}{20}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$.

Consider now the cooperation index p_2 defined by $p_2(S) = p_1(S)\ \forall S \neq \{1, 2\}$ and $p_2(\{1, 2\}) = 0$. The essential coalitions of the restricted game (N, v_{p_2}) are

$$\mathcal{E}(N, v_{p_2}) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{2, 3, 4, 5, 6\}\}$$

Hence, the essential coalitions index is $\mathfrak{e}(N, v, p_2) = \left(\frac{4}{15}, \frac{23}{75}, \frac{8}{75}, \frac{8}{75}, \frac{8}{75}, \frac{8}{75}\right)$.

Even though the index for player 2 has decreased from $\mathfrak{e}_2(N,\,v,\,p_1)=\frac{7}{20}=0.35$ to $\mathfrak{e}_2(N,\,v,\,p_2)=\frac{23}{75}\approx 0.307$, player 1 has actually improved from $\mathfrak{e}_1(N,\,v,\,p_1)=\frac{1}{4}=0.25$ to $\mathfrak{e}_1(N,\,v,\,p_2)=\frac{4}{15}\approx 0.267$.

The previous example hints at the following property of essential coalitions. For any simple game $(N,\,v)$, if $p\in I(N)$ is monotonically non-increasing with respect to inclusion, then $E\in\mathcal{E}(N,\,v_p)$ if and only if $E\in\mathcal{W}^m(N,\,v)$ and $p(E)>0$. In other words, the essential coalitions of the restricted game are those minimal winning coalitions of the original game with positive cooperation index. This follows directly from Definition 3.2.

Moreover, if p is also symmetric with respect to cardinality, then the normalization of the essential coalitions index coincides with the probabilistic Deegan-Packel index. Indeed, given $(N,\,v,\,p)\in PI(N)$, if p is symmetric with respect to cardinality, then so is the probability function $f(S)=\frac{p(S)}{\sum\limits_{T\subseteq N}p(T)}$. Now, for each player $i\in N$, the probabilistic index satisfies

$$\begin{aligned}
\rho_i^f(N,\,v) &= \sum_{\substack{S\subseteq N\\ i\in S}}\frac{v(S)}{|S|}P^f(S)=\sum_{\substack{S\in\mathcal{W}^m(N,\,v)\\ i\in S}}\frac{v(S)}{|S|}\cdot\frac{f(S)}{\sum\limits_{W\in\mathcal{W}^m(N,\,v)}f(W)}\\
&=\left(\sum_{W\in\mathcal{W}^m(N,v)}f(W)\right)^{-1}\sum_{\substack{S\in\mathcal{W}^m(N,v)\\ i\in S}}\frac{f(S)}{|S|}\\
&=\left(\sum_{W\in\mathcal{W}^m(N,v)}\frac{p(W)}{\sum\limits_{T\subseteq N}p(T)}\right)^{-1}\sum_{\substack{S\in\mathcal{W}^m(N,v)\\ i\in S}}\frac{p(S)}{\sum\limits_{T\subseteq N}p(T)}\cdot\frac{1}{|S|}\\
&=\left(\sum_{W\in\mathcal{W}^m(N,v)}p(W)\right)^{-1}\sum_{\substack{S\in\mathcal{W}^m(N,v)\\ i\in S}}\frac{p(S)}{|S|}.
\end{aligned}$$

On the other hand, if $\mathcal{E}(N,\,v_p)=\mathcal{W}^m(N,\,v)$, then for every $i\in N$ we have

$$\mathfrak{e}_i(N,v,p)=|\mathcal{E}(N,v_p)|^{-1}\sum_{\substack{S\in\mathcal{E}(N,v_p)\\ i\in S}}\frac{p(S)}{|S|}=|\mathcal{W}^m(N,v)|^{-1}\sum_{\substack{S\in\mathcal{W}^m(N,v)\\ i\in S}}\frac{p(S)}{|S|}.$$

In particular, the normalization $\widetilde{\mathfrak{e}}$ defined by

$$\widetilde{\mathfrak{e}}_i(N,\,v,\,p)=\frac{\mathfrak{e}_i(N,\,v,\,p)}{\sum\limits_{j\in N}\mathfrak{e}_j(N,\,v,\,p)}$$

coincides with the probabilistic index in this case. In other words, $\widetilde{\mathfrak{e}}$ generalizes ρ^f, since the probabilistic index is only applied when f defines a probability function over

2^N and is symmetric with respect to cardinality. We have already established that a cooperation index is not as constrained.

Many power indices are normalized to one, including the Shapley-Shubik index and the Deegan-Packel index. This feature is desirable in order to make comparisons regarding how different power indices distribute payoffs in the same scenario. Even though we will use $\widetilde{\mathfrak{e}}$ to compare the essential coalitions index to other indices, we will argue that it is meaningful for the $\mathfrak{e}$ to not be efficient.

We will delve deeper into this in the next section. For now, in our final numerical example we show that the probabilistic index and the essential coalitions index may not coincide if the cooperation index is symmetric with respect to cardinality, but non-monotonic with respect to inclusion.

EXAMPLE 5.3. Let (N, v) be the five player simple game with

$$\mathcal{W}^m(N, v) = \{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4, 5\}\}$$

and consider the cooperation index $p \in I(N)$ defined as $p(S) = \frac{1}{|S|}$ as long as $|S| \neq 2$, and let $p(S) = \frac{1}{5}$ if $|S| = 2$. Then,

$$\mathcal{E}(N, v_{p_2}) = \mathcal{W}^m(N, v) \cup \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\},$$

and so the essential coalitions index evaluates to

$$\mathfrak{e}(N, v, p) \approx (0.0958, 0.0620, 0.0495, 0.0495, 0.0495).$$

By taking $f(S) = \frac{p(S)}{\sum_{T \subseteq N} p(T)}$ and computing the probabilistic Deegan-Packel index $\rho^f(N, v)$, we obtain

$$\rho^f(N, v) \approx (0.299, 0.112, 0.196, 0.196, 0.196).$$

For the sake of comparison, let us normalize the essential coalitions index,

$$\widetilde{\mathfrak{e}}(N, v, p) \approx (0.313, 0.202, 0.162, 0.162, 0.162).$$

Note that while $\widetilde{\mathfrak{e}}$ and ρ^f agree in that player 1 should be allocated the most payoff and players 3 through 5 are symmetric, they differ on whether the latter should get more or less than player 2. Namely, the (normalized) essential coalitions index allots more to player 2 than to players 3 through 5.

This is explained by this player appearing in more essential coalitions. These have a positive effect in the namesake power index, and compensate for the low cooperation index of $\{1, 2\}$. On the contrary, the probabilistic index only captures the negative consequences of player 2 being involved in said minimal winning coalition, compared to those with higher cooperation index in which players 3 through 5 participate.

6 Final remarks

Models to restrict cooperation are of great importance in game theory, as they provide a more realistic approach to analyze the results of the interactions between agents. In this work, we studied a broad restriction model in cooperation indices; more specifically, we focused on the study of power in so restricted games.

Our main contribution is the *essential coalitions index*, a new index fit to analyze several real-life situations. In particular, it is suitable to assess power in government formation. In such a process, the cohesion of the governing coalition is a relevant factor, and this is one feature the restriction model describes.

By design, the new index bears resemblance with the preexisting Deegan-Packel index and its probabilistic version. Indeed, our motivation for the essential coalitions index is the assessment that some of the assumptions that define these are not realistic. Namely, we argued that, in some occasions, having more agents involved in a coalition will make it more stable, and so a non-minimal winning coalition may arise; on the other hand, the probability that a coalition forms or breaks is likely to depend on more than its size.

The flexibility of the cooperation index as a restriction model aids in solving these issues; a cooperation index is merely a function from the set of coalitions to the unit interval with the only condition of mapping singletons to the value one. Also, crucially, it encompasses other models of restriction, including Myerson's communication model (Myerson, 1977), which has been more thoroughly studied (see Amer and Carreras (1995) for more models that can be expressed as specific cases of the cooperation index).

In spite of their apparently similar inspiration, there is a key difference between the probabilistic index and the essential coalitions index in regards to efficiency. In general, the new index is not efficient; instead, it satisfies the property of *average essential efficiency*, introduced in Section 4. This is a consequence of the conceptual differences between the restricted cooperation models from which they arise.

The probability function used for the probabilistic index measures the chances a given coalition has of forming, and once a coalition has formed, it is assumed unbreakable. In contrast, the cooperation index conveys the stability of a coalition once it has been formed. For instance, in the government formation framework the cooperation index of a coalition is the probability it has to remain united until the end of the term.

Thus, the "efficiency gap", that is, the quantity $1 - \sum_{i \in N} \mathfrak{e}_i(N, v, p)$ accounts for the possibility that the government collapses at some point, and a caretaker government is put in place. More specifically, this efficiency gap is interpreted as the expected time for which a caretaker government is running the country or institution at issue.

All of this is begging for the combination of the two approaches: a probability function assessing the odds a coalition has to emerge victorious, and a cooperation index indicating the stability of each coalition. We hope to explore this in the future.

Now, in spite of its desirable features, the new index is not without its questionable assumptions. In acknowledging them, we open the possibility to amend them in future works. With this in mind, we dedicate the closing paragraphs to mentioning two of them.

The first one regards the equal division of payoffs. Within the framework of government formation, at the end of Section 3 we interpreted this hypothesis as the members of the governing coalition taking turns in leading it. The assumption is that at the end of the term each member has been the leader for the same amount of time.

This may be a strong assumption. However, we point out that this issue already appeared in the definition of the original Deegan-Packel index. In other words, its authors did consider reasonable that members of a minimal winning coalition evenly divide their collective payoff. So, instead of disregarding the assumption altogether, we offer the alternative justification of each member having an equal influence on policy. While still debatable, this is closer to reality: even small partners will make demands in exchange for their support to the governing coalition.

The final issue we will address is how fit the essential coalitions are for the assumptions that define their namesake index. By definition, a coalition S is essential if and only if it obtains a higher payoff than any of its subcoalitions. But it may be the case that the worth of S is only slightly above that of a coalition $T \subsetneq S$. At the extreme, it could occur that, assuming equal division of payoffs, the agents in T have no incentive to enlarge the coalition; and yet, S would be deemed an essential coalition.

This is a rather contentious point. One could be in favor of using a family of coalitions that more accurately reflects the agents' rationality to define an analogous power index. There are possibly several alternatives in this direction.

All in all, it is worth noting that the essential coalitions were defined by extending to restricted games a property of minimal winning coalitions of a simple game. As such, in any case, we consider that the essential coalitions index is a reasonable extension to the Deegan-Packel index, although it is likely not the only one.

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