Fractals in the Propagation of Forest Fires and Stochastic Models

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Abstract: The goal of this study is to compare models and real data to better understand the role of stochastic dynamics in forest-fire propagation rendering fractal geometries. I analysed empirical data and developed stochastic models to simulate fire spread. The main results show that forest fires have fractal dimensions, shown as repetitive and self-similar patterns in their structure. This discovery suggests that stochastic models can be useful in understanding and predicting the dynamics of forest fire spread. Furthermore, the simulation using the bond percolation stochastic model has yielded more similar results to the empirical data compared to site percolation, although there is room for improvement. Discrepancies might be caused by not considering the existence of natural boundaries, changes in atmospheric conditions, or human intervention in the evolution of a forest fire.

I. INTRODUCTION

Forest fires pose a serious threat to natural ecosystems and human communities. Every year, thousands of hectares of forests are engulfed in flames, causing loss of biodiversity, habitat destruction, and material damage. Moreover, forest fires jeopardize the safety and well-being of people living near wooded areas. Therefore, the prevention and control of forest fires are essential for preserving biodiversity and protecting human communities.

Understanding the spread of these fires is crucial for the development of effective forest management policies and strategies. The study of fire dynamics has been a subject of research for decades, aiming to develop predictive methods and tools to prevent and mitigate the impacts of these natural disasters.

Forest fires exhibit fractal forms in their structure, influenced by the underlying geometry of vegetation and fire propagation dynamics. The fractal arrangement of trees, the branching and the repetitive patterns of flame fronts contribute to these fractal shapes.

Fractals are objects or phenomena that exhibit self-similarity and repetitive structures at different scales. Their application in understanding forest fires allows us to explore the roughness and complexity of fire spread and identify emerging patterns and behaviours.

On the other hand, stochastic models are mathematical tools that enable the simulation and analysis of random processes. In the context of forest fires, these models capture the inherent variability of fire propagation and generate scenarios that can aid in decision-making and planning preventive measures.

In this work, I aim to explore the presence and application of fractals in the propagation of forest fires and investigate the fractal properties of forest fires. This will be accomplished through the analysis of empirical data from the forest fire cartographic database of the *Department d'Acció Climàtica, Alimentació i Agenda Rural* [1], using statistical analysis techniques for fractal characterization.

I will also implement a simple stochastic model to simulate the propagation of forest fires. This simulation will employ the percolation method on a two-dimensional square lattice [2]. Through the simulation, I will assess the model's ability to reproduce patterns observed in empirical data and seek correlations between fractal properties and propagation dynamics.

The objective of this study is to compare mathematical models with real data to understand how stochastic dynamics influence forest fire propagation and the formation of fractal geometries.

II. METHODS

I employed a methodology combining fractal analysis and stochastic modelling to investigate the propagation of forest fires. The study focuses on two aspects: the application of fractals to understand fire propagation and the implementation of a stochastic model to simulate this propagation.

A. SIMULATION

I developed a stochastic model to simulate the propagation of forest fires using the percolation method as a foundation. There are more sophisticated models of fire simulation[3], but I work with percolation models as simplified versions that replicate fractal geometries.

This method consists of representing the forest landscape as a two-dimensional square grid of dimension LxL with periodic boundary conditions. Initially, each grid cell is assigned as an unexplored zone with a value of -1, indicating that no fire has been detected in that area.

The fire propagation starts in a specific zone of the grid, which is marked as a burned zone with a value of 1. From there, we impose an equal probability to propagate the fire to any neighbouring site. I call this probability the percolation threshold (p). If the fire spreads to a neighbouring zone, it becomes a burned zone with a value

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of 1. If the fire does not spread to the neighbouring zone, it becomes an explored but unburned zone, with a value of 0, who depending on the model, it can either burn or not.

Using this propagation method, starting from an initial site the fire propagates shell by shell across the grid until it no longer propagates. This process allows for simulating the dynamics of forest fire propagation and observing the properties of the clusters that form, such as their area and perimeter.

There are two types of propagation, depending on the type of percolation being followed:

- Bond percolation: The available neighbours are the four immediate neighbours (top, bottom, left, and right), if they are not burnt areas (value 1).
- Site percolation: The available neighbours are the four immediate neighbours (top, bottom, left, and right), if they are unexplored areas (value -1).

The difference between the two types of propagation is that, in the propagation of simulations following bond percolation, explored but unburnt areas (value 0) can be burnt and change to a value of 1. On the other hand, in site percolation, explored but unburnt areas (value 0) can no longer be modified.

Another significant difference between the two types of percolation is the critical percolation threshold. This is the critical probability of interest, where scale-free and self-similar behaviour appears as I will focus our study on this value and its vicinity. The critical percolation threshold for bond percolation is 0.5, while for site percolation it is 0.5927 [4].

For each type of propagation, I have implemented two different codes to determine the fractal dimension and the distribution of cluster areas.

One cluster script

The main objective is to simulate the spread of fire by initiating a forest fire at the centre of the network at the percolation threshold. The simulation stops once the fire extinguishes. Then, the process is repeated 10,000 times to obtain a statistically significant sample. For each simulation, the area and the perimeter are recorded. The area, A, is the number of grid squares that have been burned, represented by the value 1, and the perimeter, P, is the number of squares forming the boundary, represented by the value 0.

Once the simulations are completed, the focus shifts to analysing the relationship between the perimeter and the area of the generated clusters. This relationship is analysed as a bivariate distribution where some scatter is observed around a power-law model, where the exponent of this distribution is related to the fractal dimension of the clusters. The fractal dimension reflects the fractal structure of the clusters and provides information on how they are organized and distributed within space [5]. Specifically, I focus our study on the Hausdorff dimension, HD, to describe the relationship between the perimeter and area, and it is represented by the following equation:

$$P \sim A^{H_D/2} \quad \text{Ref. [6]} \tag{1}$$

Full lattice script

In this code, the goal is to determine the distribution of cluster sizes found in a lattice given a fixed percolation threshold. To do this, I start the first fire at point 1,1 of the grid, propagate it, and then navigate through the grid to find the next unexplored zone with a value of -1, where I start another fire. This process is repeated successively until there are no more unexplored zones in the grid with a value of -1.

For each simulation, I record the area of all the clusters formed, which is the number of grid points with a value of 1. I repeat this process 1000 times, and with the data of the cluster areas, I calculate the number of fires with the same area.

As the probability distribution function of areas, pdf(A) follows a power-law [7], I can estimate the exponent using maximum likelihood [8]. First, I define the distribution function of the power-law as:

$$p(x) = \frac{\tau - 1}{x_{min}} \left(\frac{x}{x_{min}} \right)^{-\tau} \quad for\ x \geq x_{min} \tag{2}$$

where the term $\tau - 1 / x_{min}$ is included to normalize the function.

The logarithm of the likelihood function becomes:

$$L(\tau) = \ln \prod_{i=1}^{n} \frac{\tau - 1}{x_{min}} \left(\frac{x}{x_{min}} \right)^{-\tau} \quad for\ x \geq x_{min} \tag{3}$$

Differentiating with respect to τ and setting it equal to zero, I find the maximum of the likelihood function. Rearranging, I obtain the following expression for the exponent.

$$\hat{\tau} = 1 + n \left[\sum_{i=1}^{n} \ln \frac{x_i}{x_{min}} \right]^{-1} \quad for\ x \geq x_{min} \tag{4}$$

where $\{x_i\}$ are the n points $x_i \geq x_{min}$ [9].

Then, the pdf(A) depends on the area in the following way:

$$pdf(A) \sim A^{-\tau} \tag{5}$$

I perform this simulation for probabilities that are lower but close to the critical percolation probability.

B. REAL FOREST FIRES

For the analysis of empirical data from the forest fire cartographic database of the *Department d'Acció Climàtica, Alimentació i Agenda Rural*, I have a text file per fire from 2007 to 2022, about 1200 fires, which simulates the shape of the fire using a matrix. In this matrix, the burned areas are represented with the value 1, the border areas with the value 0, and the unburned areas with the value -1. Each value in the matrix represents a one-hectare square in reality. This means that the border areas correspond to squares that are not completely burned, i.e., they have a combination of burned and unburned areas within the same square.

Additionally, I also have a file with the number of squares that form the areas and perimeters for each fire.

Firstly, I analyse the relationship between the perimeter and the area plus the perimeter because, as mentioned before, part of the fire area is in the border zones.

This relationship is modelled using a power law, where the exponent of this distribution is related to the Hausdorff dimension, as mentioned earlier, through the following equation:

$$P \sim (A+P)^{H_D/2} \tag{6}$$

Then, by calculating the distributions of perimeter and area plus perimeter using the data, I determine the exponents of the power laws through maximum likelihood as given in (4) for the distributions:

$$pdf(P) \sim P^{-\tau_P} \tag{7}$$

$$pdf(A+P) \sim (A+P)^{-\tau_{A+P}} \tag{8}$$

Using the equations (6), (7) and (8) and S=A+P, I can calculate the H_D using these exponents as follows:

$$pdf(S)dS = pdf(P(A+P))\frac{\partial P}{\partial S}dS \tag{9}$$

$$S^{\tau_S}dS = \left(S^{H_D/2}\right)^{-\tau_P}\frac{H_D}{2}S^{\frac{H_D}{2}-1} \tag{10}$$

$$H_D = 2\frac{\tau_{A+P}-1}{\tau_P-1} \tag{11}$$

Finally, I select the 3 largest wildfires during these 16 years: the Agramunt wildfire of 2009, the La Jonquera wildfire of 2012, and the Ribera d'Ebre wildfire of 2019 to reduce the size of the wildfire matrix, which has dimensions Lx·Ly, by a factor of 4. I reduce Lx by a factor of 2 and Ly by a factor of 2, meaning I go from having 4 squares in the matrix to having only 1, and so on until the matrix can no longer be reduced.

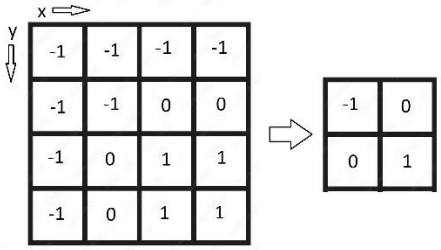

FIG:1. Example of reduction following the indicated method.

Before initiating a reduction, if Lx, Ly, or both dimensions are odd numbers, I add a row, a column, or both a row and a column of -1 to the respective matrix. To perform the reduction, I sum the values of the 4 squares. If the sum is -4, the new square has a value of -1. If the sum is 4, the new square has a value of 1. In any other case, the new square has a value of 0. For each reduction of the matrix, I record the reduction number, the number of squares with a value of 1, and the number of squares with a value of 0, that is, the area and perimeter of the reduced matrix. With this data, I analyse the evolution of the Minkowski-Bouligand dimension [5], D_{MB}. Which is an estimation of the fractal dimension using the area of the squares forming the boundary and the scale of the squares, according to the following equation:

$$D_{MB} = \lim_{r\to 0}\frac{\log P(r)}{\log 1/r} + 2 \tag{12}$$

where r is the scale of the squares, and P(r) is the perimeter of the boundary defined as $P(r) = N * r^2$ where N is the number of squares forming the boundary. Then, applying logarithms, the DMB is as follows:

$$D_{MB} = \lim_{r\to 0}\frac{\log N}{\log 1/r} \tag{13}$$

III. RESULTS AND DISCUSSION

A. RESULTS OF THE BOND PERCOLATION

Firstly, I define A as the number of grid squares forming the cluster area and P as the number of grid squares forming the cluster perimeter.

Using a grid of dimension L=128 and considering the critical bond percolation probability, pc=0.5, I have executed the one cluster script, and the results obtained are as follows:

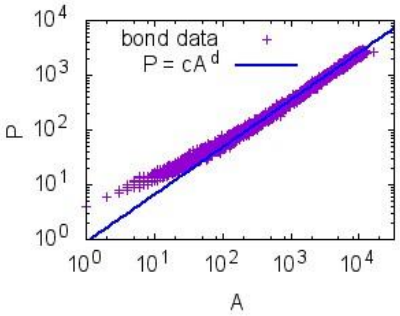

FIG. 1: Plot of P as a function of A from bond percolation, on a double logarithmic scale with base 10. The fitting was performed for values greater than 10^2.

The power-law regression equation is $P = 0.87(2)A^{0.867(3)}$. Comparing it with equation (1), I determine that the Hausdorff dimension for bond percolation is $H_D = 1.734(6)$.

Using a grid of dimension L=64, I have run the full lattice script for the range of probabilities between 0.46 and 0.49, with a step of 0.01. The results obtained are as follows:

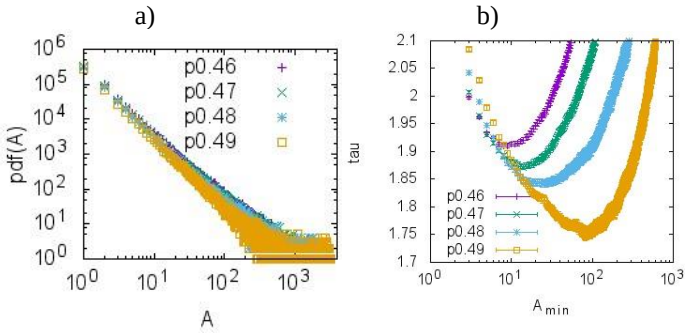

FIG. 2: a) Plot of the pdf(A) as a function of A from bond percolation, on a double logarithmic scale with base 10. b) Plot of the $\hat{\tau}$ as a function of A_{min} from bond percolation, on a logarithmic scale on the x-axis. I do this for different values of p to observe how τ evolves as it moves away from the critical percolation threshold.

Using the equation (5) and examining the data for p=0.49, I determine that $\tau = 1.75(1)$ for bond data.

B. RESULTS OF THE SITE PERCOLATION

Firstly, I define A as the number of grid squares forming the cluster area and P as the number of grid squares forming the cluster perimeter.

Using a grid of dimension L=256 and considering the critical site percolation probability, pc=0.5927, I have

executed the one cluster script, and the results obtained are as follows:

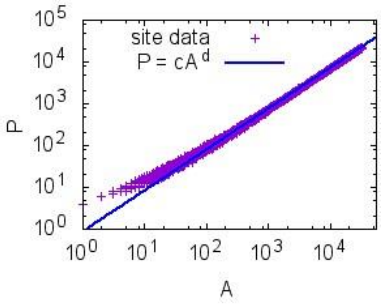

FIG. 3: Plot of P as a function of A from site percolation, on a double logarithmic scale with base 10.

The power-law regression equation is of the form $P = 0.843(3)A^{0.9797(4)}$. Comparing it with equation (1), I determine that the Hausdorff dimension for bond percolation is $H_D = 1.9594(8)$.

Using a grid of dimension L=128, I have run the full lattice script for the range of probabilities between 0.56 and 0.59, with a step of 0.01. The results obtained are as follows:

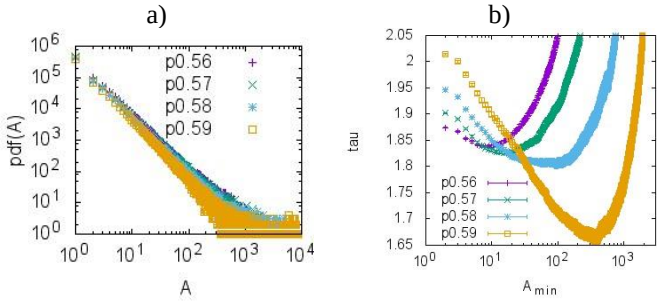

FIG. 4: a) Plot of the distribution of A as a function of A from site percolation, on a double logarithmic scale with base 10. b) Plot of the $\hat{\tau}$ as a function of A_{min} from site percolation, on a logarithmic scale on the x-axis. I do this for different values of p to observe how τ evolves as it moves away from the critical percolation threshold.

Using the equation (5) and examining the data for p=0.59, I determine that $\tau = 1.67(1)$ for site data.

C. RESULTS FROM THE REAL DATA

Firstly, I define A as the area of the cluster in hectares and P as the perimeter of the cluster in hectares.

From the fractal dimension study:

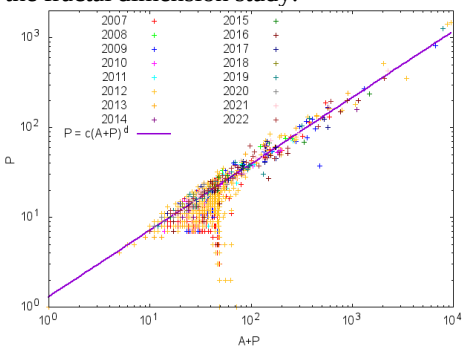

FIG. 5: Plot of P as a function of A+P on a double logarithmic scale with base 10 to observe its behaviour as a power law.

The power-law regression equation is $P = 1.3(2)(A+P)^{0.74(2)}$. Comparing it with equation (1), I determine that the Hausdorff dimension for bond percolation is $H_D = 1.48(4)$.

To determine de exponent of pdf (P) and of pdf(A+P) I obtained:

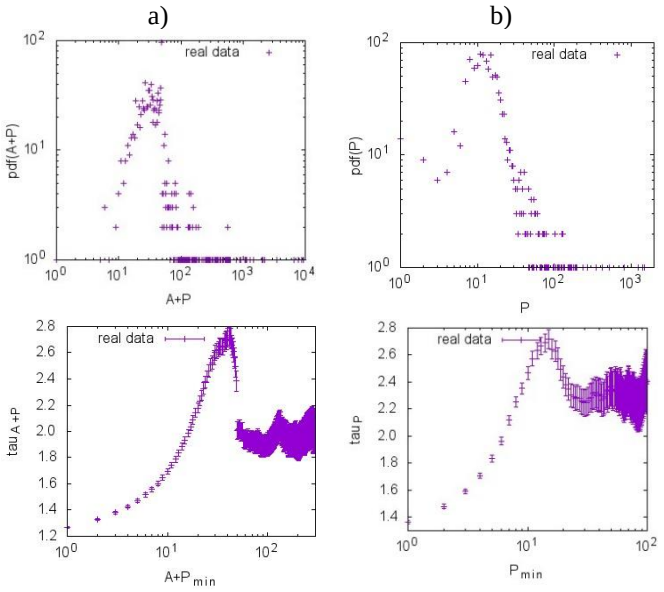

FIG. 6: a) Plot of pdf(A+P) as a function of A+P on a double logarithmic scale with base 10 and plot of τ_{A+P} as a function of $(A+P)_{min}$ on a logarithmic scale on x-axis. b) Plot of pdf(P) as function of P on a double logarithmic scale with base 10 and plot of τ_P as function of P_{min} on a logarithmic scale on x-axis.

Using the equations (8) and (7), I determine that:

$\tau_{A+P} = 1.9(1)$ and $\tau_P = 2.3(1)$.

Using equation (11) the $H_D = 1.4(2)$.

The new value, $H_D = 1.4(2)$, is compatible with the previously calculated H_D value using FIG. 5, $H_D = 1.48(4)$, with a relative error of 5%.

The evolution of P and D_{MB} with the reductions taking the smallest value of P and the respective reduction as a reference from the three largest forest fires are:

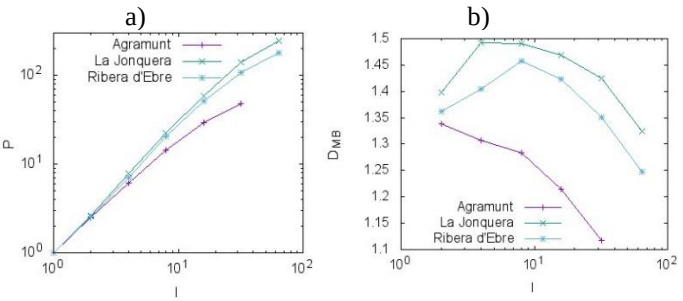

FIG. 7: a) Plot of P as a function of l, on a double logarithmic scale with base 10. b) Plot of D_{MB} as function of l on a logarithmic scale on x-axis.

Where l is the scale of the reference value grid divided by the scale of the corresponding value grid. I can observe how the D_{MB} converges to a value close to 1.5 the grid scale is close to 10 hectares, which is approximately the H_D. Then D_{MB} tends

towards 1 when the grid scale is close to 1 hectare, the D_{MB} of the polygon, because the data is given in polygons, and therefore, there is a scale at which the perimeter becomes a line.

D. DISCUSSION

Using the values obtained from the simulations and the real data, I assess whether the simulations are a good method for predicting real forest fires.

TABLE I: Obtained statistics.

	Real data	**Bond data**	**Site data**
HD	1.48(2)	1.734(6)	1.9594(8)
τ	1.9(1)	1.75(1)	1.67(1)

As we can see, the HD values from the simulations are higher than those from the real data, with the value of bond percolation being the closest with a relative error of 17% compared to the HD value of the real data.

The tau values from the simulations, in this case, are lower than those from the real data, with the value of bond percolation being the closest with a relative error of 8% compared to the tau value of the real data.

On their own, these simple models are not a good method for predicting the fractal shapes of forest fires, as they depend on external agents such as forest distribution, weather conditions, natural barriers or human intervention, which these models do not implement. These factors influence the fire grade distribution to a minor order of magnitude.

IV. CONCLUSIONS

In summary, for the real data, we obtain $H_D = 1.48(4)$ and $\tau = 1.9(1)$.

For the simulations using bond percolation, we obtain $H_D = 1.728(6)$ and $\tau = 1.75(1)$, and for the simulations using site percolation, we obtain $H_D = 1.9594(8)$ and $\tau = 1.67(1)$.

Ultimately, bond percolation is better than site percolation for predicting forest fires, although the simulation should be improved to achieve lower relative errors. One reason why it is better is that in the mesh, a region defined with a value of 0, meaning that it has been explored, and the fire has not spread there, it is possible for the fire to spread again using bond percolation. However, when using site percolation, a region of the mesh defined with a value of 0 becomes immutable.

One way to improve the simulations would be to introduce boundaries within the mesh, such as rivers, roads, or cliffs, which act as natural firebreaks. Other way would be applying changes in atmospheric conditions or considering the human intervention in the evolution of a forest fire.

On the other hand, we observe that in the real data, we can calculate the H_D in two different ways. Regarding the D_{MB}, which theoretically is larger than the H_D, we see that it closely approximates the value of the H_D in average values.

To carry out this work, I have applied knowledge from Computational Physics, Statistical Physics, and Collective Phenomena and Phase Transitions. These fields of knowledge have allowed me to analyse and simulate the spread of forest fires, investigate the fractal properties of fire boundaries, and perform statistical analyses for result characterization. This combination of disciplines has been essential in addressing the objectives of the study and achieving the obtained results.

Acknowledgments

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