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HILBERT POINTS IN HARDY SPACES

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Dedicated, with admiration, to Nikolai Nikolski on the occasion of his 80th birthday

Abstract. A Hilbert point in $H^p(\mathbb{T}^d)$, for $d \geq 1$ and $1 \leq p \leq \infty$, is a nontrivial function φ in $H^p(\mathbb{T}^d)$ such that $\|\varphi\|_{H^p(\mathbb{T}^d)} \leq \|\varphi+f\|_{H^p(\mathbb{T}^d)}$ whenever f is in $H^p(\mathbb{T}^d)$ and orthogonal to φ in the usual L^2 sense. When $p \neq 2$, φ is a Hilbert point in $H^p(\mathbb{T})$ if and only if φ is a nonzero multiple of an inner function. An inner function on $\mathbb{T}^d$ is a Hilbert point in any of the spaces $H^p(\mathbb{T}^d)$, but there are other Hilbert points as well when $d \geq 2$. The case of 1-homogeneous polynomials is studied in depth and, as a byproduct, a new proof is given for the sharp Khinchin inequality for Steinhaus variables in the range $2 < p < \infty$. Briefly, the dynamics of a certain nonlinear projection operator is treated. This operator characterizes Hilbert points as its fixed points. An example is exhibited of a function φ that is a Hilbert point in $H^p(\mathbb{T}^3)$ for $p = 2, 4$, but not for any other p; this is verified rigorously for $p > 4$ but only numerically for $1 \leq p < 4$.

§1. Introduction

The prominence of inner functions (see for example [9] or [10]) arose from Beurling's landmark paper [3] on the shift operator on the Hardy space $H^2(\mathbb{T})$. One usually defines an inner function on the unit disk $\mathbb{D}$ as a bounded analytic function whose nontangential limits are unimodular at almost every point of the unit circle $\mathbb{T}$. In the spirit of Beurling's theorem, one could alternatively define inner functions as the norm 1 extremizers for point evaluation at the origin in invariant subspaces for the shift operator on $H^2(\mathbb{T})$, with an obvious modification should all functions in the space in question vanish at the origin.

The point of departure of this paper is another extremal property of inner functions that characterizes them in the one-variable case but leads to a wider and intrinsically interesting class of functions on the d-dimensional torus $\mathbb{T}^d$ for $d > 1$. The crucial definition is as follows. A nontrivial function φ in $H^p(\mathbb{T}^d)$ for $1 \leq p \leq \infty$ is said to be a *Hilbert point* in $H^p(\mathbb{T}^d)$ if

$$\|\varphi\|_{H^p(\mathbb{T}^d)} \leq \|\varphi + f\|_{H^p(\mathbb{T}^d)} \tag{1.1}$$

for every f in $H^p(\mathbb{T}^d)$ such that $\langle f, \varphi\rangle = 0$, where $\langle \cdot\,, \cdot\rangle$ is the usual inner product in $L^2(\mathbb{T}^d)$. Here no precaution is needed when $p \geq 2$; when $1 \leq p < 2$, we declare that $\langle f, \varphi\rangle = 0$ if f lies in the closure in $H^p(\mathbb{T}^d)$ of the space of polynomials g for which $\langle g, \varphi\rangle = 0$. We will see from (1.3) below that, *a posteriori*, this precaution is obsolete because a Hilbert point in $H^p(\mathbb{T}^d)$ automatically belongs to the dual space $(H^p(\mathbb{T}^d))^*$. All nontrivial functions in $H^2(\mathbb{T}^d)$ are clearly Hilbert points in $H^2(\mathbb{T}^d)$.

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Our usage of the term "Hilbert point" is intended to suggest that we are dealing with points in a Banach space around which the space locally "looks like" a Hilbert space. This point of view is perhaps most succinctly reinforced by the following interpretation in terms of Banach space geometry. Given a fixed function φ, we use the notation

$$B_p := \left\{ f \in H^p(\mathbb{T}^d) \,:\, \|f\|_{H^p(\mathbb{T}^d)} \leq \|\varphi\|_{H^p(\mathbb{T}^d)} \right\} \tag{1.2}$$

with the presumption that φ is in $H^p(\mathbb{T}^d)$. When $1 < p < \infty$, we will see that a function φ in $H^2(\mathbb{T}^d) \cap H^p(\mathbb{T}^d)$ is a Hilbert point in $H^p(\mathbb{T}^d)$ if and only if the supporting hyperplane T_2 to B_2 that contains φ coincides with the supporting hyperplane T_p to B_p that contains φ, in the sense that $T_2 \cap H^p(\mathbb{T}^d) = T_p \cap H^2(\mathbb{T}^d)$.

When $1 \leq p < \infty$, we will investigate Hilbert points in $H^p(\mathbb{T}^d)$ using duality techniques. A consequence of the Hahn–Banach theorem is the following description. A nontrivial function φ is a Hilbert point in $H^p(\mathbb{T}^d)$ for $1 \leq p < \infty$ if and only if there is a constant $\lambda > 0$ such that

$$P\left(|\varphi|^{p-2}\varphi\right) = \lambda\varphi, \tag{1.3}$$

where P denotes the *Riesz projection* from $L^2(\mathbb{T}^d)$ to $H^2(\mathbb{T}^d)$. The case of $p = \infty$ is less amenable to duality arguments; we will see that it often requires separate arguments.

Recall that I in $H^p(\mathbb{T}^d)$ is said to be an *inner function* if $|I(z)| = 1$ for almost every z in $\mathbb{T}^d$. From (1.3) it is evident that if $\varphi = CI$ for a constant $C \neq 0$, then φ is a Hilbert point in $H^p(\mathbb{T}^d)$ for every $p < \infty$. We obtain the same conclusion for the endpoint $p = \infty$ by taking the limit in (1.1). Our first main result, alluded to above, asserts that there are no other Hilbert points in $H^p(\mathbb{T})$ when $p \neq 2$.

Theorem 1.1. *Fix $1 \leq p \leq \infty$, $p \neq 2$. A nontrivial function φ is a Hilbert point in $H^p(\mathbb{T})$ if and only if φ is a nonzero multiple of an inner function.*

The situation becomes rather more complicated when $d \geq 2$. In what follows, we will mainly restrict our attention to one of the simplest nontrivial subspaces of $H^p(\mathbb{T}^d)$, namely that of 1-homogeneous polynomials. This means that we will be dealing with functions of the form

$$\varphi(z) = \sum_{j=1}^{d} c_j z_j. \tag{1.4}$$

Theorem 1.3 below reveals that there are Hilbert points in this subspace with function theoretic properties that effectively contrast those of inner functions.

Our study of 1-homogeneous polynomials as Hilbert points is chiefly based on (1.3) and the following remarkable formula.

Theorem 1.2. *Fix $1 \leq p < \infty$ and suppose that $\varphi(z) = \sum_{j=1}^{d} c_j z_j$. Then*

$$\left(P|\varphi|^{p-2}\varphi\right)(z) = \frac{p}{2}\sum_{j=1}^{d} c_j z_j \int_0^1 \int_{\mathbb{T}^d} |\varphi_j(\zeta, r)|^{p-2}\, dm_d(\zeta)\, 2r dr$$

where $\varphi_j(z, r) := \varphi(z_1, \ldots, rz_j, \ldots, z_d)$ for $j = 1, 2, \ldots, d$.

The integrals on the right-hand side of the formula in Theorem 1.2 only depend on the modulus of the coefficients of the 1-homogeneous polynomial (1.4). By symmetry, we can therefore easily obtain the following result from (1.3) and Theorem 1.2 for $p < \infty$, and then for $p = \infty$ using (1.1).

Theorem 1.3. *If the nonzero coefficients of $\varphi(z) = \sum_{j=1}^{d} c_j z_j$ all have the same modulus, then φ is a Hilbert point in $H^p(\mathbb{T}^d)$ for every $1 \leq p \leq \infty$.*

Functions of the form $\varphi(z) = \sum_{j=1}^d c_j z_j$ whose coefficients all have the same positive modulus, maximize the ratio $\|\varphi\|_{H^\infty(\mathbb{T}^d)}/\|\varphi\|_{H^2(\mathbb{T}^d)}$ among all 1-homogeneous polynomials, in stark contrast to what inner functions do. We are interested in whether there is any other possible choice of coefficients in (1.4) that yields Hilbert points for some $p \neq 2$. We will obtain the following partial converse to Theorem 1.3.

Theorem 1.4. *Suppose that $2 < p \leq \infty$. If $\varphi(z) = \sum_{j=1}^d c_j z_j$ is a Hilbert point in $H^p(\mathbb{T}^d)$, then the nonzero coefficients of φ all have the same modulus.*

We conjecture that Theorem 1.4 is true also for $1 \leq p < 2$. To obtain some evidence supporting this conjecture, we consider the following dynamical system. Let φ_0 be any 1-homogeneous polynomial with $\|\varphi_0\|_{H^2(\mathbb{T}^d)} = 1$. Based on (1.3) and Theorem 1.2, we iteratively define

$$\varphi_{n+1} := \frac{P\left(|\varphi_n|^{p-2}\varphi_n\right)}{\|P\left(|\varphi_n|^{p-2}\varphi_n\right)\|_{H^2(\mathbb{T}^d)}}. \tag{1.5}$$

In the range $2 < p < \infty$, we can completely describe the behavior of this dynamical system. It turns out that given any $\varphi_0(z) = \sum_{j=1}^d c_j z_j$, the iterates (1.5) will converge to a Hilbert point $\varphi(z) = \sum_{j=1}^d \widetilde{c}_j z_j$ with $\widetilde{c}_j \neq 0$ if and only if $c_j \neq 0$.

Based on an analysis of the simple case where $d = 2$ and numerical experiments in the case where $d = 3$, we observe that when $1 \leq p < 2$, the iterates will also converge to a Hilbert point, but now to a 1-homogeneous inner function, i.e., to a unimodular multiple of z_j for some j. If this convergence could be established for $d \geq 3$, we would have a proof of Theorem 1.4 also in the range $1 \leq p < 2$.

There is an interesting relationship between 1-homogeneous polynomials that are Hilbert points in $H^p(\mathbb{T}^d)$ and the sharp Khinchin inequality for Steinhaus variables. To see this, we begin by setting

$$a_p := \min\left(1,\, \Gamma\left(1+\frac{p}{2}\right)^{\frac{1}{p}}\right) \quad \text{and} \quad b_p := \max\left(1,\, \Gamma\left(1+\frac{p}{2}\right)^{\frac{1}{p}}\right)$$

for $1 \leq p < \infty$. Khinchin's inequality for Steinhaus variables can be formulated as the estimates

$$a_p\|\varphi\|_{H^2(\mathbb{T}^d)} \leq \|\varphi\|_{H^p(\mathbb{T}^d)} \leq b_p\|\varphi\|_{H^2(\mathbb{T}^d)} \tag{1.6}$$

for 1-homogeneous polynomials φ. The upper estimate when $1 \leq p < 2$ and the lower estimate when $2 < p \leq \infty$ are trivial consequences of Hölder's inequality. Otherwise, the constants a_p and b_p are optimal as $d \to \infty$. The case of $p = 1$ was established by Sawa [12], and the case of $1 < p < 2$ was proved by Kwapień and König [7]. The final case where $2 < p < \infty$ is independently due to Baernstein and Culverhouse [2] and to Kwapień and König [7]. The best constant in (1.6) is also known in the case of $0 < p < 1$ by a result of König [6].

The relationship between Hilbert points in $H^p(\mathbb{T}^d)$ and Khinchin's inequality is as follows.

Lemma 1.5. *Fix $d \geq 1$ and $1 \leq p < \infty$. Consider the functional defined on the unit sphere of $\mathbb{C}^d$ by $\mathscr{K}_p(c) := \|c_1 z_1 + \cdots + c_d z_d\|_{H^p(\mathbb{T}^d)}$. Then $c = (c_1, \ldots, c_d)$ is a critical point of $\mathscr{K}_p$ if and only if $\varphi(z) = c_1 z_1 + \cdots + c_d z_d$ is a Hilbert point in $H^p(\mathbb{T}^d)$.*

Combining Theorem 1.3, Theorem 1.4, and Lemma 1.5 with a computation, we obtain a new proof of the sharp Khinchin inequality for Steinhaus variables in the case where $2 < p < \infty$. In our proof, the heavy lifting is all done by Theorem 1.2.

In view of the results presented above, one might be tempted to conjecture that if φ is a Hilbert point in $H^p(\mathbb{T}^d)$ for *some* $p \neq 2$, then φ is a Hilbert point in $H^p(\mathbb{T}^d)$ for *all* $1 \leq p \leq \infty$. We will however show that this is not true. Specifically, we will consider

$$\varphi(z) = z_1^3 + z_2^3 + z_1 z_2 z_3. \tag{1.7}$$

Using (1.3) and an argument involving change of variables, we will see that φ is a Hilbert point in $H^p(\mathbb{T}^3)$ if and only if a certain Fourier coefficient of the function

$$|\zeta_1 + \zeta_2 + \zeta_3|^{p-2}(\zeta_1 + \zeta_2 + \zeta_3)$$

vanishes. This allows us to establish that (1.7) is a Hilbert point in $H^2(\mathbb{T}^3)$ and $H^4(\mathbb{T}^3)$, but not in $H^p(\mathbb{T}^3)$ for $4 < p \leq \infty$. Based on numerical evidence, we conjecture that φ is neither a Hilbert point in $H^p(\mathbb{T}^3)$ for $1 \leq p < 2$ nor for $2 < p < 4$.

To close this introduction, we give a brief overview of the contents of this paper. In §2, we reformulate our problem using duality techniques and establish Theorem 1.1. §3 is devoted to the proof of Theorem 1.2, Theorem 1.3, and Theorem 1.4. The dynamical system mentioned above is investigated in §4. In §5, we prove Lemma 1.5 and present a new proof of Khinchin's inequality for Steinhaus variables in the range $2 < p < \infty$. Finally, the function (1.7) is discussed in detail in §6.

§2. Duality reformulation and inner functions

Recall that every function f in $L^p(\mathbb{T}^d)$ can be represented by its Fourier series $f(z) \sim \sum_{\alpha \in \mathbb{Z}^d} \widehat{f}(\alpha)\, z^\alpha$, where $\widehat{f}(\alpha) := \int_{\mathbb{T}^d} f(z)\, \overline{z^\alpha}\, dm_d(z)$ and where m_d denotes the Haar measure of the d-dimensional torus $\mathbb{T}^d$. The Hardy space $H^p(\mathbb{T}^d)$ is the subspace of $L^p(\mathbb{T}^d)$ comprised of functions f such that $\widehat{f}(\alpha) = 0$ for every α in $\mathbb{Z}^d \setminus \mathbb{N}_0^d$, where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

We need a well-known consequence of the Hahn–Banach theorem concerning orthogonality in L^p spaces, which can be extracted from Shapiro's monograph [13].

Lemma 2.1. *Fix $1 \leq p < \infty$. If φ is a nontrivial function in $H^p(\mathbb{T}^d)$ and Y is a closed subspace of $L^p(\mathbb{T}^d)$, then the following are equivalent.*

(i) $\|\varphi\|_{L^p(\mathbb{T}^d)} \leq \|\varphi + f\|_{L^p(\mathbb{T}^d)}$ *for every f in Y.*
(ii) $\langle |\varphi|^{p-2}\varphi, f \rangle = 0$ *for every f in Y.*

Proof. If $1 < p < \infty$, this is a special case of [13, Theorem 4.2.1]. If φ is a nontrivial function in $H^1(\mathbb{T}^d)$, then $\log|\varphi|$ is in $L^1(\mathbb{T}^d)$ by [11, Theorem 3.3.5]. In particular,

$$m_d\left(\{z \in \mathbb{T}^d \,:\, \varphi(z) = 0\}\right) = 0.$$

This means that we obtain the assertion for $p = 1$ from [13, Theorem 4.2.2]. □

If φ is in $(H^p(\mathbb{T}^d))^*$, then the bounded linear functional generated by φ on $H^p(\mathbb{T}^d)$ can be represented as $L_\varphi(f) := \langle f, \varphi \rangle$, which implies that

$$\|\varphi\|_{(H^p(\mathbb{T}^d))^*} := \sup_{\substack{g \in H^p(\mathbb{T}^d) \\ g \not\equiv 0}} \frac{|\langle g, \varphi \rangle|}{\|g\|_{H^p(\mathbb{T}^d)}}. \tag{2.1}$$

We are now ready to reformulate the defining property of Hilbert points using duality. Note that the condition in Theorem 2.2(a) below coincides with (1.3) discussed in the introduction.

Theorem 2.2.

(a) *Fix $1 \leq p < \infty$. A nontrivial function φ in $H^p(\mathbb{T}^d)$ is a Hilbert point in $H^p(\mathbb{T}^d)$ if and only if there is some constant $\lambda > 0$ such that*

$$P\left(|\varphi|^{p-2}\varphi\right) = \lambda\varphi.$$

(b) *Fix $1 \le p \le \infty$. A nontrivial function φ in $H^p(\mathbb{T}^d) \cap H^2(\mathbb{T}^d)$ is a Hilbert point in $H^p(\mathbb{T}^d)$ if and only if*

$$\|\varphi\|_{H^p(\mathbb{T}^d)} \|\varphi\|_{(H^p(\mathbb{T}^d))^*} = \|\varphi\|^2_{H^2(\mathbb{T}^d)}.$$

Part (a) implies that a Hilbert point in $H^p(\mathbb{T}^d)$ also belongs to the dual space $(H^p(\mathbb{T}^d))^*$. This means that the requirement of part (b) that φ be in $H^2(\mathbb{T}^d)$ is automatically verified when φ is a Hilbert point in $H^p(\mathbb{T}^d)$. The assumption that φ is in $H^2(\mathbb{T}^d)$ is only needed to exclude from the statement the claim that if $1 \le p < 2$, then any φ in $H^p(\mathbb{T}^d) \setminus H^2(\mathbb{T}^d)$ is a Hilbert point in $H^p(\mathbb{T}^d)$.

The formula in part (b) further justifies our usage of the term "Hilbert point", because the identity $\|\varphi\|_{\mathscr{H}} \|\varphi\|_{\mathscr{H}^*} = \|\varphi\|^2_{\mathscr{H}}$ holds true for all vectors φ in a Hilbert space $\mathscr{H}$ by the Riesz representation theorem.

Remark. Theorem 1.3 can also be deduced from [4, Lemma 5] and Theorem 2.2(b), in addition to the proof based on Theorem 1.2 and Theorem 2.2(a) mentioned above.

Proof of Theorem 2.2(a). Throughout the proof, we will apply Lemma 2.1 with Y as the closure in $L^p(\mathbb{T}^d)$ of the set of analytic polynomials g satisfying $\langle g, \varphi \rangle = 0$.

We assume first that $P\big(|\varphi|^{p-2}\varphi\big) = \lambda\varphi$ for some $\lambda > 0$. By the implication (ii) $\implies$ (i) in Lemma 2.1, this means that

$$\|\varphi\|_{H^p(\mathbb{T}^d)} \le \|\varphi + f\|_{H^p(\mathbb{T}^d)}$$

for every f in Y. This shows that φ is a Hilbert point in $H^p(\mathbb{T}^d)$.

To prove the reverse implication, we begin with assuming that φ is a Hilbert point in $H^p(\mathbb{T}^d)$. By the implication (i) $\implies$ (ii) in Lemma 2.1, we have

$$\big\langle |\varphi|^{p-2}\varphi, f \big\rangle = 0$$

for every f in Y. But this means that the function $\psi := P\big(|\varphi|^{p-2}\varphi\big)$ also has the property that $\langle f, \psi \rangle = 0$ for all f in Y. Since φ is in $H^p(\mathbb{T}^d)$ and ψ is in $(H^p(\mathbb{T}^d))^*$, at least one of them belongs to $H^2(\mathbb{T}^d)$. If $1 \le p < 2$ so that ψ is in $H^2(\mathbb{T}^d)$, then every f in $H^p(\mathbb{T}^d)$ can be decomposed as

$$f = \frac{\langle f, \psi \rangle}{\|\psi\|^2_{H^2(\mathbb{T}^d)}} \psi + h,$$

where h belongs to Y. Since $H^q(\mathbb{T}^d)$ is contained in $H^p(\mathbb{T}^d)$ for $q = p/(p-1)$, this decomposition is in particular valid for every f in $H^q(\mathbb{T}^d)$. It follows that the action of the functional L_φ on $H^q(\mathbb{T}^d)$ can be computed explicitly:

$$L_\varphi(f) = \frac{\langle f, \psi \rangle}{\|\psi\|^2_{H^2(\mathbb{T}^d)}} \langle \psi, \varphi \rangle.$$

Since $\langle \psi, \varphi \rangle = \langle |\varphi|^{p-2}\varphi, \varphi \rangle = \|\varphi\|^p_{H^p(\mathbb{T}^d)}$, this means that φ must be a positive multiple of ψ. When $2 < p < \infty$, we may argue in the same way, with the roles of φ and ψ reversed. □

Proof of Theorem 2.2(b). If φ is a Hilbert point in $H^p(\mathbb{T}^d)$, then φ is also in the dual space $(H^p(\mathbb{T}^d))^*$. When $2 \le p \le \infty$, this is trivial. When $1 \le p < 2$, this follows from Theorem 2.2(a) and the fact that $|\varphi|^{p-2}\varphi$ is in $L^q(\mathbb{T}^d)$, where $q = p/(p-1)$. The inequality

$$\|\varphi\|_{H^p(\mathbb{T}^d)} \|\varphi\|_{(H^p(\mathbb{T}^d))^*} \ge \|\varphi\|^2_{H^2(\mathbb{T}^2)}$$

is true automatically by (2.1), so it suffices to prove that φ is a Hilbert point in $H^p(\mathbb{T}^d)$ if and only if the reverse inequality

$$\|\varphi\|_{H^p(\mathbb{T}^d)} \|\varphi\|_{(H^p(\mathbb{T}^d))^*} \le \|\varphi\|^2_{H^2(\mathbb{T}^2)} \tag{2.2}$$

is verified.

We begin with the necessity of (2.2). To this end, we assume that φ is a Hilbert point in $H^p(\mathbb{T}^d)$. Since φ is in $(H^p(\mathbb{T}^d))^*$ and since

$$H^p(\mathbb{T}^d) \cap (H^p(\mathbb{T}^d))^* \subset H^2(\mathbb{T}^d),$$

we may decompose every g in $H^p(\mathbb{T}^d)$ as

$$g = \frac{\langle g, \varphi\rangle}{\|\varphi\|^2_{H^2(\mathbb{T}^d)}}\varphi + \left(g - \frac{\langle g, \varphi\rangle}{\|\varphi\|^2_{H^2(\mathbb{T}^d)}}\varphi\right). \tag{2.3}$$

If $g \not\equiv 0$, then we use the decomposition (2.3) and the assumption that φ is a Hilbert point in $H^p(\mathbb{T})$ to see that

$$\frac{|\langle g, \varphi\rangle|}{\|g\|_{H^p(\mathbb{T}^d)}} \leq \frac{\|\varphi\|^2_{H^2(\mathbb{T}^d)}}{\|\varphi\|_{H^p(\mathbb{T}^d)}}.$$

Since g is arbitrary, we get the desired inequality (2.2).

To prove the sufficiency of (2.2), we suppose next that φ satisfies (2.2). By (2.1) we then get

$$\frac{|\langle g, \varphi\rangle|}{\|g\|_{H^p(\mathbb{T}^d)}} \leq \frac{\|\varphi\|_{H^2(\mathbb{T}^d)}}{\|\varphi\|_{H^p(\mathbb{T}^d)}}$$

for every nontrivial g in $H^p(\mathbb{T}^d)$. In particular, choosing $g = \varphi + f$ with $\langle f, \varphi\rangle = 0$, we see that φ is a Hilbert point in $H^p(\mathbb{T}^d)$. □

Now, we formulate three corollaries to Theorem 2.2. The first simply makes explicit an immediate consequence of the fact that a Hilbert point φ in any of the spaces $H^p(\mathbb{T}^d)$ is in $H^2(\mathbb{T}^d) \cap (H^p(\mathbb{T}^d))^*$. Then the orthogonal projection

$$P_\varphi f := \frac{\langle f, \varphi\rangle}{\|\varphi\|_{H^2(\mathbb{T}^d)}}\varphi$$

is a well-defined operator on $H^p(\mathbb{T}^d)$, and we obtain the following from part (b).

Corollary 2.3. *A nontrivial function φ in $H^p(\mathbb{T}^d)$ is a Hilbert point in $H^p(\mathbb{T}^d)$ for $1 \leq p \leq \infty$ if and only if φ belongs to $H^2(\mathbb{T}^d) \cap (H^p(\mathbb{T}^d))^*$ and P_φ is a contraction on $H^p(\mathbb{T}^d)$.*

The Hahn–Banach theorem supplies a contractive projection onto every one-dimensional subspace of a Banach space. Corollary 2.3 can be reformulated as follows. If φ is in $H^2(\mathbb{T}^d) \cap (H^p(\mathbb{T}^d))^*$, then φ is a Hilbert point in $H^p(\mathbb{T}^d)$ if and only if the projection from $H^p(\mathbb{T}^d)$ to $\operatorname{span}(\{\varphi\})$ coincides with the orthogonal projection from $H^2(\mathbb{T}^d)$ to $\operatorname{span}(\{\varphi\})$. We refer to [8] and our recent paper [5] for studies of other contractive projections on Hardy spaces.

We next come back to our interpretation of Hilbert points in terms of Banach space geometry. We retain the notation from the introduction (see (1.2)) and stress that the supporting hyperplane T_p of B_p is well defined because $H^p(\mathbb{T}^d)$ is uniformly convex when $1 < p < \infty$.

Corollary 2.4. *Suppose that φ is in $H^2(\mathbb{T}^d) \cap H^p(\mathbb{T}^d)$ with $1 < p < \infty$. Then φ is a Hilbert point in $H^p(\mathbb{T}^d)$ if and only if $T_p \cap H^2(\mathbb{T}^d) = T_2 \cap H^p(\mathbb{T}^d)$.*

Proof. We begin with assuming that φ is a Hilbert point in $H^p(\mathbb{T}^d)$. Then the inclusion $T_2 \cap H^p(\mathbb{T}^d) \subset T_p \cap H^2(\mathbb{T}^d)$ is immediate. To prove the reverse inclusion, we note that if f is in T_p, then

$$\|\varphi + f\|_{H^p(\mathbb{T}^d)} \geq \|\varphi\|_{H^p(\mathbb{T}^d)}.$$

Thus Lemma 2.1 implies that $\langle f, |\varphi|^{p-2}\varphi\rangle = 0$ for every f in T_p. Therefore, since P is selfadjoint, we have

$$\big\langle f, P(|\varphi|^{p-2}\varphi)\big\rangle = 0.$$

Now invoking Theorem 2.2(a), we see that $\langle f, \varphi\rangle = 0$, whence f is in T_2.

We assume next that $T_p \cap H^2(\mathbb{T}^d) = T_2 \cap H^p(\mathbb{T}^d)$. If $\langle f, \varphi\rangle = 0$, then f will be in T_p which implies that $\|\varphi + f\|_{H^p(\mathbb{T}^d)} \geq \|\varphi\|_{H^p(\mathbb{T}^d)}$. This means by definition that φ is a Hilbert point in $H^p(\mathbb{T}^d)$. □

As mentioned in the introduction, the following result is a direct consequence of Theorem 2.2(a) for $p < \infty$ and by a limiting argument for $p = \infty$. It is also possible to deduce this result from Theorem 2.2(b), (2.1), and Hölder's inequality.

Corollary 2.5. *Fix $d \geq 1$ and suppose that $\varphi = CI$ for a constant $C \neq 0$ and an inner function I. Then φ is a Hilbert point in $H^p(\mathbb{T}^d)$ for every $1 \leq p \leq \infty$.*

Now, we turn to the proof of Theorem 1.1, which states that there are no other Hilbert points in $H^p(\mathbb{T})$ when $p \neq 2$.

Proof of Theorem 1.1. The sufficiency part is the case where $d = 1$ of Corollary 2.5, so it remains to settle the necessity part.

We begin with the case where $1 \leq p < \infty$, $p \neq 2$. We assume that φ is a Hilbert point in $H^p(\mathbb{T}^d)$ and use Theorem 2.2(a) to infer that

$$P\left(|\varphi|^{p-2}\varphi\right) = \lambda\varphi$$

for some $\lambda > 0$. We may assume without loss of generality that $\lambda = 1$ by rescaling φ if necessary. Suppose that f is an arbitrary analytic polynomial. Then since the Riesz projection is selfadjoint, we see that

$$\big\langle |\varphi|^2, f\big\rangle = \langle \varphi, \varphi f\rangle = \big\langle P(|\varphi|^{p-2}\varphi), \varphi f\big\rangle = \big\langle |\varphi|^{p-2}\varphi, \varphi f\big\rangle = \langle |\varphi|^p, f\rangle\,.$$

The same identity is valid also with f replaced by $\bar{f}$. Hence $|\varphi|^p = |\varphi|^2$ almost everywhere, so φ must be an inner function.

We assume next that φ is a Hilbert point in $H^\infty(\mathbb{T})$. We factor φ in the usual way as $\varphi = IE$, where I is an inner function and E is an outer function. For a real number η, consider the function $f_\eta = IE^{1+\eta}$, which is in $H^\infty(\mathbb{T})$ and satisfies $\|f_\eta\|_{H^\infty(\mathbb{T})} = \|\varphi\|_{H^\infty(\mathbb{T})}^{1+\eta}$. Since

$$\|\varphi\|_{(H^\infty(\mathbb{T}))^*} \geq \operatorname{Re}\frac{\langle f_\eta, \varphi\rangle}{\|f_\eta\|_{H^\infty(\mathbb{T})}}, \tag{2.4}$$

Theorem 2.2(b) shows that φ is a Hilbert point in $H^\infty(\mathbb{T})$ only if the quantity on the right in (2.4) is maximized for $\eta = 0$. Using that $|\varphi| = |E|$ almost everywhere, we find that

$$0 = \frac{d}{d\eta}\operatorname{Re}\frac{\langle f_\eta, \varphi\rangle}{\|f_\eta\|_{H^\infty(\mathbb{T})}}\bigg|_{\eta=0} = \frac{1}{\|E\|_{H^\infty}}\int_{\mathbb{T}} |E(z)|^2 \log\left(\frac{|E(z)|}{\|E\|_{H^\infty(\mathbb{T})}}\right) dm_1(z),$$

which is true if and only if φ is a constant multiple of an inner function. □

§3. 11-homogeneous polynomials

To prove Theorem 1.2, we require some basic facts. We first recall that the Riesz projection P can be expressed by using the Szegő kernel as

$$Pf(z) = \int_{\mathbb{T}^d} f(\zeta)\prod_{j=1}^{d}\frac{1}{1 - z_j\bar{\zeta_j}}\,dm_d(\zeta). \tag{3.1}$$

Next, a function f in $L^p(\mathbb{T}^d)$ is called 1-homogeneous if

$$f(e^{i\theta}z_1, e^{i\theta}z_2, \ldots, e^{i\theta}z_d) = e^{i\theta}f(z_1, z_2, \ldots, z_d).$$

It is clear that if φ is a 1-homogeneous polynomial, then $|\varphi|^{p-2}\varphi$ is a 1-homogeneous function. Hence $P(|\varphi|^{p-2}\varphi)$ is a 1-homogeneous polynomial whenever φ is a 1-homogeneous polynomial.

Proof of Theorem 1.2. Our goal is to establish that if $\varphi(z) = \sum_{j=1}^d c_j z_j$, then

$$\left(P|\varphi|^{p-2}\varphi\right)(z) = \frac{p}{2}\sum_{j=1}^d c_j z_j \int_0^1 \int_{\mathbb{T}^d} |\varphi_j(\zeta, r)|^{p-2}\, dm_d(\zeta)\, 2r\, dr, \tag{3.2}$$

where $\varphi_j(z,r) := \varphi(z_1, \ldots, rz_j, \ldots, z_d)$ for $j = 1, 2, \ldots, d$. By the discussion above, we know that the left-hand side of (3.2) is a 1-homogeneous polynomial. Hence we do not need to consider any other Fourier coefficients when computing the Riesz projection.

Let us first demonstrate that, without loss of generality, we may assume that $c_j > 0$ for $j = 1, 2, \ldots, d$. Suppose that we have established (3.2) for $c_j > 0$. Given any φ, we define

$$\widetilde{\varphi}(z) = \sum_{j=1}^d e^{i\theta_j} c_j z_j$$

where $e^{i\theta_j}$ is chosen so that $\widetilde{c_j} := e^{i\theta_j} c_j > 0$. Using (3.1), a change of variables, and the rotation invariance of $\mathbb{T}^d$, we find

$$\left(P|\varphi|^{p-2}\varphi\right)(z) = \left(P|\widetilde{\varphi}|^{p-2}\widetilde{\varphi}\right)(e^{-i\theta_1}z_1, e^{-i\theta_2}z_2, \ldots, e^{-i\theta_d}z_d).$$

Using (3.2) for $\widetilde{\varphi}$ we obtain (3.2) for φ, because the integrals on the right-hand side of (3.2) are the same for φ and $\widetilde{\varphi}$, again by rotation invariance.

Considering the Fourier series of $|\varphi|^{p-2}$, we compute

$$(P|\varphi|^{p-2}\varphi)(z) = A\varphi(z) + \sum_{j=1}^d z_j \sum_{\substack{k=1 \\ k\neq j}}^d B_{j,k} c_k, \tag{3.3}$$

for $A = \int_{\mathbb{T}^d} |\varphi(\zeta)|^{p-2}\, dm_d(\zeta)$ and $B_{j,k} = \int_{\mathbb{T}^d} |\varphi(\zeta)|^{p-2}\, \zeta_j \overline{\zeta_k}\, dm_d(\zeta)$ with $j \neq k$. In what follows, let α in $\mathbb{N}_0^d$ denote a multi-index, and set $c = (c_1, c_2, \ldots, c_d)$. Suppose that $p - 2 = 2n$ for a nonnegative integer n. By the multinomial theorem,

$$(\varphi(z))^n = \sum_{|\alpha|=n} \binom{n}{\alpha} c^\alpha z^\alpha \quad \text{where} \quad \binom{n}{\alpha} = \frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_d!}. \tag{3.4}$$

We will use (3.4) to obtain expressions for A and $B_{j,k}$. It is clear that

$$A = \sum_{|\alpha|=n} \binom{n}{\alpha}^2 c^{2\alpha}.$$

Given some α with $\alpha_k \geq 1$, let β denote the multi-index obtained by subtracting 1 from the kth coordinate of α and adding 1 to the jth coordinate of α. Note that

$$\binom{n}{\beta} = \binom{n}{\alpha}\frac{\alpha_k}{\alpha_j + 1}.$$

By using (3.4) twice, we see that

$$B_{j,k} = \sum_{\substack{|\alpha|=n \\ \alpha_k \geq 1}} \binom{n}{\alpha}\binom{n}{\beta} c^\alpha c^\beta = \sum_{\substack{|\alpha|=n \\ \alpha_k \geq 1}} \binom{n}{\alpha}^2 \frac{\alpha_k}{\alpha_j + 1}\frac{c_j}{c_k} c^{2\alpha} = \frac{c_j}{c_k}\sum_{|\alpha|=n} \binom{n}{\alpha}^2 \frac{\alpha_k}{\alpha_j + 1} c^{2\alpha}.$$

Consequently,

$$Ac_j + \sum_{\substack{k=1\\k\neq j}}^{d} B_{j,k}c_k = c_j(n+1)\sum_{|\alpha|=n}\binom{n}{\alpha}^2\frac{c^{2\alpha}}{\alpha_j+1}. \tag{3.5}$$

A direct computation shows that

$$\sum_{|\alpha|=n}\binom{n}{\alpha}^2\frac{c^{2\alpha}}{\alpha_j+1} = \int_0^1\int_{\mathbb{T}^d}|\varphi_j(\zeta,r)|^{2n}\,dm_d(\zeta)2r\,dr$$

for $\varphi_j(z,r) = \varphi(z_1,\ldots,rz_j,\ldots,z_d)$ for $j=1,2,\ldots,d$. By (3.3) and (3.5) we have now established the formula

$$(P|\varphi|^{2n}\varphi)(z) = (n+1)\sum_{j=1}^{d}c_jz_j\int_0^1\int_{\mathbb{T}^d}|\varphi_j(\zeta,r)|^{2n}\,dm_d(\zeta)\,2r\,dr \tag{3.6}$$

for $n=0,1,2,\ldots$. We can extend (3.6) to the case of $n=p-2$ for $p\geq 1$ by polynomial approximation. Let D denote the differential operator

$$Df(x) := \frac{d}{dx}\big(xf(x)\big).$$

If $f(x)=x^n$ for $n=0,1,2,\ldots$, then (3.6) can be restated as

$$P\left(f\big(|\varphi|^2\big)\varphi\right)(z) = \sum_{j=1}^{d}c_jz_j\int_0^1\int_{\mathbb{T}^d}Df\big(|\varphi_j(\zeta,r)|^2\big)\,dm_d(\zeta)2r\,dr. \tag{3.7}$$

By the linearity of P and D, it is clear that (3.7) is fulfilled for any polynomial f. Suppose that f is continuously differentiable on $[0,C]$ for

$$C = (c_1+c_2+\cdots+c_d)^2.$$

Then (3.7) is true for f because both f and f' can be simultaneously uniformly approximated by polynomials on $[0,C]$. In particular, (3.7) holds true for

$$f(x) = (\delta+x)^{p/2-1}$$

for $\delta>0$. By Fubini's theorem, we may let $\delta\to 0^+$ when $p\geq 1$ and obtain (3.2) from (3.7). □

Remark. We may replace the above polynomial approximation argument by an appeal to analytic continuation in the variable p and the fact that the sequence $2n$ is not a Blaschke sequence in the right half-plane. The latter kind of argument is used in the proof of Theorem 6.2 below.

Note that we may interpret the integrals on the right-hand side of (3.2) as area integrals over the unit disk with respect to the variable $w=rz_j$. Let A denote the Lebesgue measure of $\mathbb{C}$, normalized so that $A(\mathbb{D})=1$. The following result is pertinent to our analysis of the right-hand side of (3.2).

Lemma 3.1. *Fix* $2<p<\infty$. *If* $a>b>0$, *then*

$$\int_{\mathbb{T}}\int_{\mathbb{D}}|aw+bz+c|^{p-2}\,dA(w)\,dm_1(z) < \int_{\mathbb{T}}\int_{\mathbb{D}}|az+bw+c|^{p-2}\,dA(w)\,dm_1(z)$$

for every complex number c.

Proof. We begin with interchanging the order of integration and using rotation invariance to obtain

$$\int_{\mathbb{T}}\int_{\mathbb{D}} |aw+bz+c|^{p-2}\,dA(w)\,dm_1(z) = \int_{\mathbb{D}}\int_{\mathbb{T}} \big||aw+b|z+c\big|^{p-2}\,dm_1(z)\,dA(w). \tag{3.8}$$

Notice that since $a>b$, the Möbius transformation

$$w \mapsto \frac{aw+b}{a+bw} = \frac{w+b/a}{1+(b/a)w}$$

maps the unit disk into itself. Hence

$$|aw+b| < |a+bw| \tag{3.9}$$

for every w in $\mathbb{D}$. The function $z\mapsto |z+c|^{p-2}$ is subharmonic for each fixed complex number c. We can therefore use (3.9) to conclude that

$$\int_{\mathbb{T}} \big||aw+b|z+c\big|^{p-2}\,dm_1(z) < \int_{\mathbb{T}} \big||a+bw|z+c\big|^{p-2}\,dm_1(z).$$

Integrating over $\mathbb{D}$ with respect to w and using (3.8) twice, we obtain the stated inequality. □

We are now ready to proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4 *for* $2<p<\infty$. We may assume without loss of generality that $c_j>0$ for $j=1,2,\ldots,d$. If $d=1$ there is nothing to prove, so we also assume that $d\geq 2$. We appeal to Theorem 2.2(a) and Theorem 1.2 to conclude that φ is a Hilbert point in $H^p(\mathbb{T}^d)$ if and only if

$$\frac{p}{2}\sum_{j=1}^d c_j z_j \int_0^1 \int_{\mathbb{T}^d} |\varphi_j(z,r)|^{p-2}\,dm_d(z) 2r\,dr = \lambda \sum_{j=1}^d c_j z_j$$

for some constant $\lambda>0$. Of course, this is equivalent to the claim that

$$I_1 = I_2 = \cdots = I_d, \tag{3.10}$$

where for $j=1,2,\ldots,d$ we define

$$I_j := \int_0^1 \int_{\mathbb{T}^d} |\varphi_j(z,r)|^{p-2}\,dm_d(z)\,2r\,dr.$$

We will use a contrapositive argument, so assume that $a=c_1$ and $b=c_2$ for some $a>b>0$. By rotations and changing the order of integration, we find

$$\begin{aligned} I_1 &= \int_{\mathbb{T}^{d-2}}\int_{\mathbb{T}}\int_{\mathbb{D}} \left|aw+bz+\sum_{j=2}^d c_j z_j\right|^{p-2} dA(w)\,dm_1(z)\,dm_{d-2}(z) \\ &< \int_{\mathbb{T}^{d-2}}\int_{\mathbb{T}}\int_{\mathbb{D}} \left|az+bw+\sum_{j=2}^d c_j z_j\right|^{p-2} dA(w)\,dm_1(z)\,dm_{d-2}(z) = I_2, \end{aligned}$$

where we have used Lemma 3.1 for each $c=\sum_{j=2}^d c_j z_j$. Now, it is clear from (3.10) that φ cannot be a Hilbert point in $H^p(\mathbb{T}^d)$. □

For the proof of Theorem 1.4 in the case of $p=\infty$, we require a well-known result, which will also be used in §5 for $p<\infty$. Let $H^p_1(\mathbb{T}^d)$ be the subspace of $H^p(\mathbb{T}^d)$ comprised of 1-homogeneous polynomials.

Lemma 3.2. *The orthogonal projection* $P_1\colon H^2(\mathbb{T}^d)\to H^2_1(\mathbb{T}^d)$ *extends to a contraction on* $H^p(\mathbb{T}^d)$ *for every* $1\leq p\leq\infty$.

Proof. The claim follows from the formula

$$P_1 f(z) = \int_{\mathbb{T}} f(z_1\zeta, z_2\zeta, \ldots, z_d\zeta)\, \bar{\zeta}\, dm_1(\zeta)$$

and Minkowski's integral inequality. □

Proof of Theorem 1.4 *for* $p = \infty$. As in the case of $2 < p < \infty$, we assume without loss of generality that $c_j > 0$ for $j = 1, 2, \ldots, d$. By Lemma 3.2, we know that the projection from $H^\infty(\mathbb{T}^d)$ to $H^\infty_1(\mathbb{T}^d)$ is contractive, which implies that

$$\|\varphi\|_{(H^\infty(\mathbb{T}^d))^*} = \|\varphi\|_{(H^\infty_1(\mathbb{T}^d))^*}.$$

Observing that

$$\|\varphi\|_{H^\infty(\mathbb{T}^d)} = \sum_{j=1}^d |c_j| = \sum_{j=1}^d c_j$$

because $c_j > 0$ by our assumption, we clearly have $\|\varphi\|_{(H^\infty_1(\mathbb{T}^d))^*} = \max_{1\leq j\leq d} c_j$. Hence from Theorem 2.2(b) we deduce that φ is a Hilbert point in $H^\infty(\mathbb{T}^d)$ if and only if

$$\Big(\max_{1\leq j\leq d} c_j\Big) \sum_{j=1}^d c_j = \sum_{j=1}^d c_j^2.$$

By the assumption that $c_j > 0$, we see that φ is a Hilbert point in $H^\infty(\mathbb{T}^d)$ if and only if $c_1 = c_2 = \cdots = c_d$. □

The conclusion of Theorem 1.4 also holds true if $d = 2$ and $1 \leq p < 2$. To see this, it is sufficient to establish the following result. It replaces Lemma 3.1 in the proof of Theorem 1.4, but the inequality goes in the reverse direction.

Lemma 3.3. *Fix* $1 \leq p < 2$. *If* $\varphi(z) = az_1 + bz_2$ *for* $a > b > 0$, *then*

$$\int_0^1 \int_{\mathbb{T}^2} |\varphi_1(z,r)|^{p-2}\, dm_d(z)\, 2rdr > \int_0^1 \int_{\mathbb{T}^2} |\varphi_2(z,r)|^{p-2}\, dm_d(z)\, 2r\, dr.$$

Proof. As in the proof of Lemma 3.1, we see that $|aw + b| < |a + bw|$ for every $w \in \mathbb{D}$. The statement now follows from the fact that $p - 2 < 0$. □

Based on Theorem 1.3, Theorem 1.4, and Lemma 3.3, we offer now the following.

Conjecture 3.1. Suppose that $1 \leq p < 2$. If $\varphi(z) = \sum_{j=1}^d c_j z_j$ is a Hilbert point in $H^p(\mathbb{T}^d)$, then the nonzero coefficients of φ all have the same modulus.

The conjecture is open for $d \geq 3$. In the next section we will obtain some evidence in support of Conjecture 3.1.

§4. Dynamics of the nonlinear projection operators

It may be easier to understand the action of the nonlinear projection operator $\varphi \mapsto P(|\varphi|^{p-2}\varphi)$ if we normalize it in the following way:

$$\mathscr{P}_p(\varphi) := \frac{P(|\varphi|^{p-2}\varphi)}{\|P(|\varphi|^{p-2}\varphi)\|_{H^2(\mathbb{T}^d)}}. \tag{4.1}$$

Then $\mathscr{P}_p$ maps the unit sphere of $H^2_1(\mathbb{T}^d)$ into itself by Theorem 1.2 and φ is a fixed point of $\mathscr{P}_p$ if and only if it is a Hilbert point in $H^p(\mathbb{T}^d)$ by Theorem 2.2 (a).

Consider a 1-homogeneous polynomial

$$\varphi_0(z) = \sum_{j=1}^d c_j z_j,$$

normalized such that $\|\varphi_0\|_{H^2(\mathbb{T}^d)} = 1$. We define inductively $\varphi_{n+1} := \mathscr{P}_p(\varphi_n)$ for every nonnegative integer n. By Theorem 2.2(a), we know that $\varphi_{n+1} = \varphi_n$ if and only if φ_n is a Hilbert point in $H^p(\mathbb{T}^d)$. We let $c_j^{(n)}$ denote the coefficient of z_j in φ_n (so that $c_j = c_j^{(0)}$). What can we say about the behavior of these coefficients as $n \to \infty$? We begin with two obvious conclusions, which follow at once from Theorem 1.2.

(i) If $c_j = 0$, then $c_j^{(n)} = 0$ for every $n \geq 0$.

(ii) If $c_j \neq 0$, then $\arg(c_j^{(n)}) = \arg(c_j)$ for every $n \geq 0$.

For simplicity, we shall in what follows assume that $c_j > 0$ for every $j = 1, 2, \ldots, d$. Next, let us compare two coefficients.

(iii) If $c_j = c_k$, then $c_j^{(n)} = c_k^{(n)}$ for every $n \geq 0$. This follows at once from Theorem 1.2 and symmetry.

To see what happens when $c_j \neq c_k$, we establish a result that complements Lemma 3.1 by giving an inequality in the opposite direction. While the proof of Lemma 3.1 relied crucially on an argument involving subharmonicity, the next result follows from a purely geometric consideration.

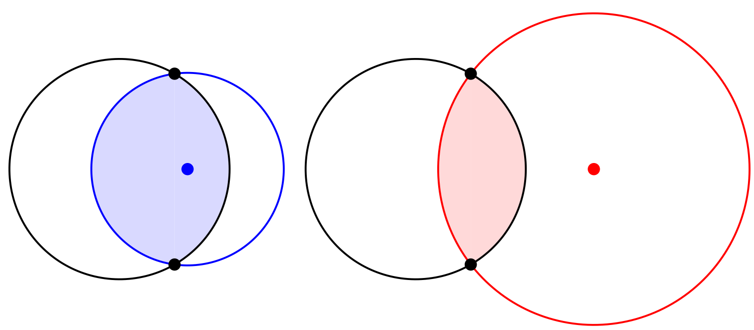

FIGURE 4.1. The areas $A_1(r)$ and $A_2(r)$ in the proof of Lemma 4.1, for $x = \frac{1+\sqrt{5}}{2}$ and $\frac{1}{x} = \frac{\sqrt{5}-1}{2}$. The black circle is the unit circle. We have chosen $r = \sqrt{2}$ or, equivalently, $\theta = \frac{\pi}{3}$.

Lemma 4.1. *Fix $1 \leq p < \infty$. If $a > b > 0$, then*

$$a \int_{\mathbb{T}} \int_{\mathbb{D}} |aw + bz + c|^{p-2} \, dA(w) \, dm_1(z) > b \int_{\mathbb{T}} \int_{\mathbb{D}} |az + bw + c|^{p-2} \, dA(w) \, dm_1(z)$$

for every complex number c.

Proof. Replacing c by c/b, we assume without loss of generality that $a = x > 1$ and $b = 1$. Set $\Phi(r, z, c) = |rz + c|^{p-2}$. As in the proof of Lemma 3.1, we write

$$\int_{\mathbb{T}} \int_{\mathbb{D}} |xz + w + c|^{p-2} \, dA(w) \, dm_1(z) = \int_{\mathbb{T}} \int_{\mathbb{D}} \Phi(|1 + xw|, z, c) \, dA(w) \, dm_1(z),$$
$$\int_{\mathbb{T}} \int_{\mathbb{D}} |z + xw + c|^{p-2} \, dA(w) \, dm_1(z) = \int_{\mathbb{T}} \int_{\mathbb{D}} \Phi(|x + w|, z, c) \, dA(w) \, dm_1(z).$$

For a fixed z on $\mathbb{T}$ and a complex number c, we consider

$$I_1(x) := \int_{\mathbb{D}} \Phi(|1 + xw|, z, c) \, dA(w) \quad \text{and} \quad I_2(x) := \int_{\mathbb{D}} \Phi(|x + w|, z, c) \, dA(w).$$

For $0 \leq r \leq x + 1$, we define

$$A_1(r) := A\big(\{w \in \mathbb{D} : |1 + xw| \leq r\}\big) \quad \text{and} \quad A_2(r) := A\big(\{w \in \mathbb{D} : |x + w| \leq r\}\big).$$

By symmetry, we note that $A_1(r)$ is equal to the area of the intersection of the disk $\mathbb{D}(1/x, r/x)$ and the unit disk $\mathbb{D}$ and, similarly, that $A_2(r)$ is equal to the area of the intersection of the disks $\mathbb{D}(x,r)$ and $\mathbb{D}$. See Figure 4.1. Since $A_2(r) = 0$ for $r < x-1$, we rewrite the integrals as

$$I_1(x) = \int_0^{x+1} \Phi(r,z,c) A_1'(r)\,dr \qquad \text{and} \qquad I_2(x) = \int_{x-1}^{x+1} \Phi(r,z,c) A_2'(r)\,dr,$$

by polar coordinates and a change of variables. Since Φ is nonnegative, we are done if we can prove that $A_2'(r) \leq x A_1'(r)$ for $x-1 \leq r \leq 1$. We restrict r to this interval henceforth.

The unit circle intersects the circles $|1+wx| = r$ and $|x+w| = r$ at the same two points $e^{\pm i\theta}$ for some $0 < \theta < \pi$. We find it convenient to view θ as a function of r. Let ℓ_1 and ℓ_2 denote the arc length of the part of the circles intersecting the unit disk, respectively. Then

$$\ell_1(r) = 2\frac{r}{x} \arctan\Big(\frac{\sin\theta}{\cos\theta + 1/x}\Big) \quad \text{and} \quad \ell_2(r) = 2r\arctan\Big(\frac{\sin\theta}{\cos\theta + x}\Big),$$

where arctan takes values in $[0,\pi]$. Inspecting Figure 4.1 again, we see that

$$A_1(r) = \pi\Big(\frac{x-1}{x}\Big)^2 + \int_{\frac{x-1}{x}}^{\frac{r}{x}} \ell_1(xs)\,ds \quad \text{and} \quad A_2(r) = \int_{x-1}^{r} \ell_2(s)\,ds,$$

when $x-1 < r < x+1$, from which we obtain

$$A_1'(r) = \frac{2r}{x^2}\arctan\frac{\sin\theta}{\cos\theta + 1/x} \quad \text{and} \quad A_2'(r) = 2r\arctan\frac{\sin\theta}{\cos\theta + x}.$$

Now, we fix $x-1 < r < x+1$, or equivalently $0 < \theta < \pi$. To establish the desired estimate $A_2'(r) \leq xA_1'(r)$, it suffices to check that $F_\theta(x) > 0$ for $x > 1$, where

$$F_\theta(x) = \arctan\Big(\frac{\sin\theta}{\cos\theta + 1/x}\Big) - x\arctan\Big(\frac{\sin\theta}{\cos\theta + x}\Big).$$

Since $F_\theta(1) = 0$, we compute

$$F_\theta'(x) = \frac{(x+1)\sin\theta}{1+2x\cos\theta + x^2} - \arctan\Big(\frac{\sin\theta}{\cos\theta + x}\Big) \geq \frac{(x+1)\sin\theta}{1+2x\cos\theta+x^2} - \frac{\sin\theta}{\cos\theta + x},$$

using the estimate $\arctan(y) \leq y$ for $y \geq 0$. Since

$$\frac{(x+1)\sin\theta}{1+2x\cos\theta+x^2} - \frac{\sin\theta}{\cos\theta+x} = \frac{\sin\theta(1-\cos\theta)(x-1)}{(1+2x\cos\theta+x^2)(\cos\theta+x)}$$

we conclude that $F_\theta'(x) > 0$, which completes the proof. □

We may now make the following additional assertion.

(iv) If $c_j > c_k$, then $c_j^{(n)} > c_k^{(n)}$ for every $n \geq 0$. This is a consequence of Theorem 1.2 and Lemma 4.1.

Combining assertions (i)—(iv) with Theorem 1.2 and Lemma 3.1, we obtain the following result.

Theorem 4.2. *Fix $2 < p < \infty$. Suppose that $\varphi_0(z) = \sum_{j=1}^d c_j z_j$ is an arbitrary point in the unit sphere of $H_1^2(\mathbb{T}^d)$ and that $c_j \neq 0$ for $j = 1,2,\ldots,d$. Then*

$$\lim_{n\to\infty} (\mathscr{P}_p^n \varphi_0)(z) = \frac{1}{\sqrt{d}} \sum_{j=1}^d \frac{c_j}{|c_j|} z_j.$$

Proof. We may assume without loss of generality that $c_1 \geq c_j > 0$ for $j = 2, 3, \ldots, d$. By (iv), this ordering will persist under iterations by $\mathscr{P}_p$ so that we will have $c_1^{(n)} \geq c_j^{(n)}$ for all n and $j = 2, 3, \ldots, d$. In particular, this implies that $c_1^{(n)} \geq d^{-1/2}$ by the normalization. The crux of the proof will be to show that $n \mapsto c_1^{(n)}$ is a strictly monotone decreasing sequence whenever $c_1^{(0)} > d^{-1/2}$.

We begin with showing how to conclude once we know that $n \mapsto c_1^{(n)}$ is strictly monotone decreasing. If

$$c_1^{(\infty)} := \lim_{n \to \infty} c_1^{(n)} = \frac{1}{\sqrt{d}},$$

then we are done because the ordering of the coefficients persists under iterations. We will next rule out the possibility that $c_1^{(\infty)} > d^{-1/2}$. If this were the case, we could by compactness find a subsequence n_k and coefficients $c_j^{(\infty)}$ such that

$$c_j^{(\infty)} = \lim_{k \to \infty} c_j^{(n_k)}$$

for $j = 1, 2, \ldots, d$. Clearly, the ordering persists in the limit so that $c_1^{(\infty)} \geq c_j^{(\infty)}$ for every $j = 2, 3, \ldots, d$. If we now start iterating from

$$\varphi_\infty(z) := \sum_{j=1}^{d} c_j^{(\infty)} z_j,$$

then the largest coefficient of the iterates will again be a strictly monotone decreasing sequence. However, this would violate the fact that the coefficients of $\mathscr{P}_p(\varphi)$ for φ in the unit sphere of $H_1^2(\mathbb{T}^d)$ depend continuously on the coefficients of φ.

It remains to show that $n \mapsto c_1^{(n)}$ is strictly monotone decreasing when $c_1^{(0)} > d^{-1/2}$. Then there exists a j_0 such that $c_{j_0}^{(0)} < d^{-1/2} < c_1^{(0)}$. By (iv) and induction on n, we have then $c_{j_0}^{(n)} < c_1^{(n)}$ for all nonnegative integers n. Now invoking Lemma 3.1 and taking Theorem 1.2 into account, we see that the ratios $c_1^{(n)}/c_j^{(n)}$ are monotone nonincreasing for $j = 2, 3, \ldots, d$. This allows us to draw the desired conclusion because we have seen that at least one of these sequences of ratios is strictly monotone decreasing. □

Remark. Theorem 1.3 and Theorem 1.4 can be obtained as direct corollaries to Theorem 4.2 through Theorem 2.2(a).

Suppose that we start iterating from $\varphi_0(z) = az_1 + bz_2$ for $|a|^2 + |b|^2 = 1$ when $1 \leq p < 2$. Replacing Lemma 3.1 by Lemma 3.3, by similar considerations as above we obtain the following conclusions.

- If $|a| = |b| = 1/\sqrt{2}$, then $\varphi_\infty(z) = \varphi_0(z)$.
- If $|a| > |b|$, then $\varphi_\infty(z) = \frac{a}{|a|} z_1$.
- If $|a| < |b|$, then $\varphi_\infty(z) = \frac{b}{|b|} z_2$.

The key difference between the cases where $1 \leq p < 2$ and $2 < p < \infty$ is that if $a > b > 0$, then the sequence $a^{(n)}$ is strictly monotone increasing in the former and monotone decreasing in the latter.

Consider now φ_0 in the unit sphere of $H_1^2(\mathbb{T}^d)$ and apply the nonlinear projection operator (4.1) for $d \geq 3$ and $1 \leq p < \infty$. Repeating the arguments of the first part of the proof of Theorem 4.2, we see that to extend Theorem 1.2 to the range $1 \leq p < 2$, it would suffice to show that in this case, the largest coefficient of the iterates is strictly monotone increasing. We have performed some numerical experiments when $d = 3$ and $1 \leq p < 2$, picking many random polynomials from $H_1^2(\mathbb{T}^3)$ as the initial point and applying the iteration. A representative example (with $p = 1$) can be found in Table 4.1.

n	$a^{(n)}$	$b^{(n)}$	$c^{(n)}$
0	0.7256	0.6766	0.1251
1	0.7577	0.6346	0.1520
2	0.8259	0.5413	0.1576
3	0.9191	0.3762	0.1175
4	0.9742	0.2152	0.0686
5	0.9931	0.1120	0.0359
6	0.9982	0.0566	0.0182
7	0.9996	0.0284	0.0091
8	0.9999	0.0142	0.0046

TABLE 4.1. Iterations of (4.1) with $p = 1$ starting with $\varphi_0(z) = a^{(0)}z_1 + b^{(0)}z_2 + c^{(0)}z_3$, computed numerically within the precision of 10^{-5}.

Table 4.1 reveals another difference between $1 \leq p < 2$ and $2 < p < \infty$, because the ratio $n \mapsto a^{(n)}/c^{(n)}$ is not monotone. In this example, the ratio decreases in the first two iterations and then increases thereafter. This indicates that the case of $1 \leq p < 2$ is subtler, because it is not sufficient to consider the pairwise interaction of coefficients under the iterations.

Question 4.1. Suppose that $\varphi_0(z) = \sum_{j=1}^{d} c_j z_j$ is in the unit sphere of $H^2_1(\mathbb{T}^d)$ and that $c_1 > c_j \geq 0$ for every $j = 2, 3, \ldots, d$. Is it true that $\lim_{n\to\infty}(\mathscr{P}_p^n \varphi_0)(z) = z_1$ whenever $1 \leq p < 2$?

From the above discussion, it follows that a positive answer to Question 4.1 would lead to a proof of Conjecture 3.1.

It is natural to ask how the dynamics of $\mathscr{P}_p$ may be in a more general situation. Notice however that $\mathscr{P}_p$ is not well defined on the unit sphere of $H^2(\mathbb{T}^d)$ when $p > 2$, so that it is not clear how to proceed in full generality. One could imagine modifiying the definition of $\mathscr{P}_p$ or restricting again to some submanifold of the unit sphere of $H^2(\mathbb{T}^d)$ that is preserved by $\mathscr{P}_p$, such as that consisting of m-homogeneous polynomials. It would be interesting to know in which generality the facts observed above may be true, namely that inner functions are attracting fixed points for $1 \leq p < 2$ and repelling fixed points for $2 < p < \infty$.

§5. KHINCHIN'S INEQUALITY FOR STEINHAUS VARIABLES

Now we are going to see how Theorem 1.4 (and Theorem 1.2) can be applied to give a proof of the sharp Khinchin inequality (1.6) in the range $2 < p < \infty$. Recall that a Steinhaus random variable by definition is uniformly distributed on $\mathbb{T}$ with respect to the Lebesgue arc length measure. Hence, if $(z_j)_{j\geq 1}^d$ is a sequence of independent Steinhaus variables and $(c_j)_{j\geq 1}^d$ are complex numbers, then

$$\mathbb{E}\bigg|\sum_{j=1}^{d} c_j z_j\bigg|^p = \int_{\mathbb{T}^d} |c_1 z_1 + \cdots + z_d z_d|^p \, dm_d(z) = \|\varphi\|^p_{H^p(\mathbb{T}^d)}.$$

The novelty of our proof of Khinchin's inequality is that we avoid using bisubharmonic functions as was done in [2]. It may be observed, however, that subharmonicity plays an essential role, namely in the proof of Lemma 3.1.

Our proof begins with Lemma 1.5, where we consider critical points of the functional

$$\mathscr{K}_p(c) := \|c_1 z_1 + \cdots + c_d z_d\|_{H^p(\mathbb{T}^d)} \tag{5.1}$$

defined for $c = (c_1, \ldots, c_d)$ on the unit sphere of $\mathbb{C}^d$. Recall that $H^p_1(\mathbb{T}^d)$ is the d-dimensional subspace of $H^p(\mathbb{T}^d)$ comprised of 1-homogeneous polynomials.

Proof of Lemma 1.5. Fix $2 < p < \infty$. For c in the unit sphere of $\mathbb{C}^d$, let φ denote the associated 1-homogeneous polynomial. By the Lagrange multiplier theorem, any critical point $(c_1, \ldots, c_d)$ of the functional (5.1) satisfies

$$\nabla \|\varphi\|_{H^p(\mathbb{T}^d)} = \lambda \nabla \|\varphi\|_{H^2(\mathbb{T}^d)}$$

for some constant λ. This means that the complex tangent space to the closed ball in $H^p_1(\mathbb{T}^d)$ centered at the origin with radius $\|\varphi\|_{H^p(\mathbb{T}^d)}$ at the point φ is the same as the complex tangent space to the closed unit ball in $H^2_1(\mathbb{T}^d)$ at the point φ. But this condition means that for any 1-homogeneous polynomial f such that $\langle f, \varphi\rangle = 0$ we have

$$\|\varphi + f\|_{H^p(\mathbb{T}^d)} \geq \|\varphi\|_{H^p(\mathbb{T}^d)}. \tag{5.2}$$

By Lemma 3.2, we see that (5.2) holds true for all 1-homogeneous polynomials f satisfying $\langle f, \varphi\rangle = 0$ if and only if φ is a Hilbert point in $H^p(\mathbb{T}^d)$. □

Remark. The above proof is a finite-dimensional version of the argument used to establish Corollary 2.4, where we saw that $T_p \cap H^2(\mathbb{T}^d) = T_2 \cap H^p(\mathbb{T}^d)$ at a Hilbert point in $H^p(\mathbb{T}^d)$.

By Lemma 1.5 and Theorem 1.4, we know that to get the optimal upper and lower bounds in Khinchin's inequality when $2 < p < \infty$, we only need to investigate the 1-homogeneous polynomials for which all the nonzero coefficients have the same modulus. Hence we require the following result.

Lemma 5.1. *If* $2 < p < \infty$, *then*

$$d \mapsto \left\| \frac{1}{\sqrt{d}} \sum_{j=1}^d z_j \right\|_{H^p(\mathbb{T}^d)}$$

is strictly monotone increasing for $d \geq 1$.

Proof. Fix $2 < p < \infty$. For $0 \leq t \leq 1$, we consider the function

$$\varphi_t(z) := \left(\frac{1-t}{d} + \frac{t}{d+1}\right)^{1/2} (z_1 + \cdots + z_d) + \left(\frac{t}{d+1}\right)^{1/2} z_{d+1}$$

and define $\Phi(t) := \|\varphi_t\|^p_{H^p(\mathbb{T}^d)}$. We need to prove that $\Phi(0) < \Phi(1)$, which we will do by showing that $\Phi'(t) > 0$ for $0 < t < 1$. Writing $|\varphi_t|^p = (\varphi_t \overline{\varphi_t})^{p/2}$ and using the chain rule and the product rule, we find that

$$\Phi'(t) = \frac{p}{2} \int_{\mathbb{T}^{d+1}} |\varphi_t(z)|^{p-2} \varphi_t(z) \cdot 2 \frac{d}{dt} \varphi_t(\overline{z})\, dm_{d+1}(z).$$

To proceed, we first note that we may replace $|\varphi_t|^{p-2}\varphi_t$ by $P(|\varphi_t|^{p-2}\varphi_t)$ by orthogonality. Next we compute

$$2\frac{d}{dt}\varphi_t(\overline{z}) = -\frac{1}{d(d+1)} \left(\frac{1-t}{d} + \frac{t}{d+1}\right)^{-1/2} (\overline{z_1} + \cdots + \overline{z_d}) + \left(\frac{1}{(d+1)t}\right)^{1/2} \overline{z_{d+1}}.$$

By Theorem 1.2, we see that

$$\Phi'(t) = \frac{p^2}{4(d+1)} \Bigg(- \int_0^1 \int_{\mathbb{T}^{d+1}} |(\varphi_t)_1(\zeta, r)|^{p-2} 2r\, dr\, dm_{d+1}(\zeta) + \int_0^1 \int_{\mathbb{T}^{d+1}} |(\varphi_t)_{d+1}(\zeta, r)|^{p-2} 2r\, dr\, dm_{d+1}(\zeta) \Bigg).$$

Since $p > 2$ and $\frac{1-t}{d} + \frac{t}{d+1} > \frac{t}{d+1}$ for $0 < t < 1$, we obtain $\Phi'(t) > 0$ by using Lemma 3.1 as in the proof of Theorem 1.4. □

Theorem 5.2 (Khinchin's inequality [2, 7]). *Fix $2 < p < \infty$. We have*

$$\Bigg\| \sum_{j=1}^{d} c_j z_j \Bigg\|_{H^p(\mathbb{T}^d)} \leq \Gamma\left(1 + \frac{p}{2}\right)^{\frac{1}{p}} \left(\sum_{j=1}^{d} |c_j|^2\right)^{\frac{1}{2}}$$

for all complex numbers $c_1, \ldots, c_d$. The constant $\Gamma\big(1 + \frac{p}{2}\big)^{\frac{1}{p}}$ is optimal.

Proof. By Lemma 1.5, Theorem 1.4, and Lemma 5.1, we deduce that the asserted inequality holds true with optimal constant equal to

$$\lim_{d \to \infty} \Bigg\| \frac{1}{\sqrt{d}} \sum_{j=1}^{d} z_j \Bigg\|_{H^p(\mathbb{T}^d)} = \Gamma\left(1 + \frac{p}{2}\right)^{\frac{1}{p}}.$$

The limit can be evaluated by the central limit theorem, because the independent complex-valued random variables $(z_j)_{j \geq 1}$ have mean 0 and variance 1. □

Remark. The proof of Khinchin's inequality for $1 \leq p < 2$ in [7] requires rather technical estimates. This indicates why Conjecture 3.1 could be more difficult to establish compared to Theorem 1.4, because a positive answer to the former would simplify the proof of Khinchin's inequalty for $1 \leq p < 2$ substantially.

§6. A Hilbert point in $H^4(\mathbb{T}^3)$H4(T3)

We have so far devoted our attention to two classes of Hilbert points. If $\varphi = CI$ for a constant $C \neq 0$ and an inner function I, then φ is a Hilbert point for every $1 \leq p \leq \infty$ by Corollary 2.5. If φ is a 1-homogeneous polynomial, then from Theorem 1.3 and Theorem 1.4 it follows that if φ is a Hilbert point in $H^p(\mathbb{T}^d)$ for *some* $2 < p \leq \infty$, then it is a Hilbert point in $H^p(\mathbb{T}^d)$ for every $1 \leq p \leq \infty$. Conjecture 3.1 implies that the same statement should be true if $2 < p \leq \infty$ is replaced by $1 \leq p < 2$.

The purpose of the present section is to demonstrate that in general, when $d \geq 2$, the Hilbert points depend on p. We begin with the following result, which is inspired by [5, Example 3.4].

Theorem 6.1. *The function*

$$\varphi(z) = c_1 z_1^3 + c_2 z_2^3 + c_3 z_3^3 + c_4 z_1 z_2 z_3$$

is a Hilbert point in $H^4(\mathbb{T}^3)$ if and only if the nonzero coefficients of φ all have the same modulus.

Proof. We will use Theorem 2.2(a), and begin by expanding

$$\begin{aligned}
|\varphi(z)|^2 = \|\varphi\|_{H^2(\mathbb{T}^3)}^2 &+ c_1 z_1^3 \left(\overline{c_2 z_2^3} + \overline{c_3 z_3^3} + \overline{c_4 z_1 z_2 z_3}\right) \\
&+ c_2 z_2^3 \left(\overline{c_1 z_1^3} + \overline{c_3 z_3^3} + \overline{c_4 z_1 z_2 z_3}\right) \\
&+ c_3 z_3^3 \left(\overline{c_1 z_1^3} + \overline{c_2 z_2^3} + \overline{c_4 z_1 z_2 z_3}\right) \\
&+ c_1 c_2 c_3 z_1 z_2 z_3 \left(\overline{c_1 z_1^3} + \overline{c_2 z_2^3} + \overline{c_3 z_3^3}\right).
\end{aligned}$$

Hence, we find that

$$P\left(|\varphi|^2 \varphi\right) = \sum_{j=1}^{3} c_j z_j \left(2\|\varphi\|_{H^2(\mathbb{T}^3)}^2 - |c_j|^2\right) + c_4 z_1 z_2 z_3 \left(2\|\varphi\|_{H^2(\mathbb{T}^3)}^2 - |c_4|^2\right),$$

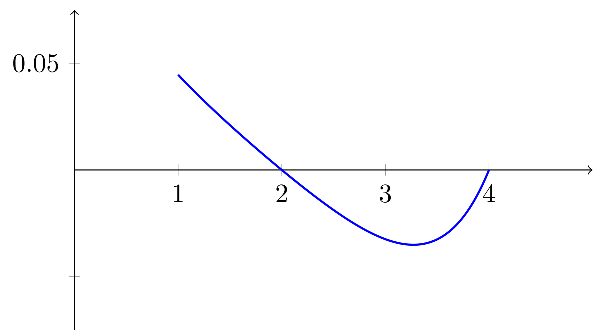

FIGURE 6.1. The Fourier coefficient (6.1) for $1 \leq p \leq 4$.

from which we easily deduce that the solutions of the equation $P(|\varphi|^2\varphi) = \lambda\varphi$ have the stated form. □

If $c_4 = 0$, then the conclusion of Theorem 6.1 can be obtained directly from Theorem 1.3 and Theorem 1.4 by substituting $\zeta_1 := z_1^3$, $\zeta_2 := z_2^3$, and $\zeta_3 := z_3^3$. However, the same argument shows that φ is a Hilbert point also in $H^p(\mathbb{T}^d)$ for every $1 \leq p \leq \infty$. We now turn to the main result of this section, where we see that putting $c_3 = 0$ leads to a completely different situation.

Theorem 6.2. *The function $\varphi(z) = z_1^3 + z_2^3 + z_1 z_2 z_3$ is a Hilbert point in $H^2(\mathbb{T}^3)$ and $H^4(\mathbb{T}^3)$, but not in $H^p(\mathbb{T}^3)$ for any $4 < p \leq \infty$.*

Numerical evidence (see Figure 6.1) suggests that this φ is a Hilbert point in $H^p(\mathbb{T}^d)$ only when $p = 2, 4$. Unfortunately, we are only able to verify analytically that there is possibly a finite number of p in $[1, 4) \setminus \{2\}$ for which φ is a Hilbert point in $H^p(\mathbb{T}^d)$.

Proof. We begin with the case of $p = \infty$, where we need to establish the estimate

$$\left\|z_1^3 + z_2^3 + z_1 z_2 z_3 - \varepsilon z_3^3\right\|_{H^\infty(\mathbb{T}^3)} < 3$$

to see that φ is not a Hilbert point in $H^\infty(\mathbb{T}^3)$ by (1.1). To this end set $\zeta_1 := \bar{z_1} z_2$ and $\zeta_2 := \bar{z_1}^2 z_2 z_3$ so that our task is to show that

$$\left\|1 + \zeta_1 + \zeta_2 - \varepsilon\bar{\zeta_1}\zeta_2^3\right\|_{H^\infty(\mathbb{T}^3)} < 3.$$

Now if $|1 + \zeta_1| \leq 2 - 2\varepsilon$ or $|1 + \zeta_2| \leq 2 - 2\varepsilon$, then trivially

$$\left|1 + \zeta_1 + \zeta_2 - \varepsilon\bar{\zeta_1}\zeta_2^3\right| \leq 3 - \varepsilon.$$

It therefore suffices to consider ζ_1 and ζ_2 such that

$$|1 + \zeta_1| > 2 - 2\varepsilon \quad \text{or} \quad |1 + \zeta_2| > 2 - 2\varepsilon.$$

In this case, we may finish the proof by an easy computation that is essentially identical to that given in the proof of [5, Lemma 2.5].

We assume from now on that $p < \infty$. It is clear that we may rewrite the Fourier series of φ as a Fourier series in the variables

$$\zeta_1 := z_1^3, \quad \zeta_2 := z_2^3, \quad \zeta_3 := z_1 z_2 z_3.$$

Such a rewriting reveals, by symmetry, that the Fourier coefficients of $P(|\varphi|^{p-2}\varphi)$ with respect to the three monomials z_1^3, z_2^3, $z_1 z_2 z_3$ are identical. Since the function φ is 3-homogeneous, there can be at most one additional term in the Fourier series of $P(|\varphi|^{p-2}\varphi)$, namely a multiple of z_3^3. We deduce from this that φ is a Hilbert point in $H^p(\mathbb{T}^3)$ if and only if $\Phi(p) = 0$, where

$$\Phi(p) := \int_{\mathbb{T}^3} |\varphi(z)|^{p-2}\varphi(z)\,\overline{z_3^3}\,dm_3(z). \tag{6.1}$$

Using the notation $\psi(\zeta) := \zeta_1 + \zeta_2 + \zeta_3$, by the change of variables introduced above we see that

$$\Phi(p) = \int_{\mathbb{T}^3} |\psi(\zeta)|^{p-2}\psi(\zeta)\,\zeta_1\zeta_2\overline{\zeta_3^3}\,dm_3(\zeta). \tag{6.2}$$

Assume that $p - 2 = 2n$, where n is a nonnegative integer and expand

$$\big(\psi(\zeta)\big)^{n+1} = \sum_{|\alpha|=n+1} \binom{n+1}{\alpha}\zeta^\alpha \qquad \text{and} \qquad \big(\overline{\psi(\zeta)}\big)^n = \sum_{|\beta|=n} \binom{n}{\beta}\overline{\zeta^\beta}.$$

We only get a contribution to (6.2) for $\alpha = \beta + (-1,-1,3)$, when

$$\binom{n+1}{\alpha}\binom{n}{\beta} = (n+1)\binom{n}{\beta}^2 \frac{\beta_1\beta_2}{(\beta_3+1)(\beta_3+2)(\beta_3+3)}.$$

This shows that

$$\Phi(2(n+1)) = (n+1)\sum_{|\beta|=n}\binom{n}{\beta}^2 \frac{\beta_1\beta_2}{(\beta_3+1)(\beta_3+2)(\beta_3+3)}. \tag{6.3}$$

Note that the numerator $\beta_1\beta_2$ ensures that $\Phi(2) = \Phi(4) = 0$, so φ is a Hilbert point in $H^2(\mathbb{T}^3)$ and $H^4(\mathbb{T}^3)$. It is also clear that $\Phi(2(n+1)) > 0$ for every integer $n \geq 2$. We need an analytic expression for (6.3). This can be established directly by using Bergman norms when $n \geq 2$. If we write $\psi_n(w) = (w_1 + w_2 + w_3)^n$, then

$$\begin{aligned}\Phi(2(n+1)) &= \frac{(n+1)}{3!}\int_{\mathbb{D}^3}\left|\frac{\partial^2}{\partial w_1\partial w_2}\psi_n(w)\right|^2 3(1-|w_3|^2)^2\,dA_3(w)\\ &= \frac{(n+1)n^2(n-1)^2}{3!}\int_{\mathbb{D}^3}|\psi_{n-2}(w)|^2\,3(1-|w_3|^2)^2\,dA_3(w),\end{aligned}$$

where $dA_3(w) := dA(w_1)dA(w_2)dA(w_3)$. Returning to (6.1), we have established the identity

$$\Phi(p) = \binom{p/2}{3}\frac{(p-2)(p-4)}{4}\int_{\mathbb{D}^3}|w_1+w_2+w_3|^{p-6}\,3(1-|w_3|^2)^2\,dA_3(w) \tag{6.4}$$

when $p > 4$ is a positive even integer. Since the sequence of positive integers violates the Blaschke condition in the right half-plane and since Φ grows at most exponentially as $p \to \infty$, by analytic continuation it follows that (6.4) is valid also for any noninteger $p > 4$. This breaks down at $p = 4$ because integrability fails. The expression on the right-hand side of (6.4) is positive for $p > 4$, so we see that the Fourier coefficient (6.1) does not vanish. Hence φ is not a Hilbert point in $H^p(\mathbb{T}^3)$ for $p > 4$. □

Remark. One could offer a rigorous computer assisted proof that $\Phi(p) \neq 0$ also when $1 \leq p < 4$, $p \neq 2$, by estimating the integral in (6.2) using interval arithmetic in the intervals $[1,2)$ and $(2,4)$ and analyzing separately the behavior near $p = 2$ and $p = 4$.

We believe that the Fourier coefficients of

$$|\zeta_1+\zeta_2+\zeta_3|^{p-2}(\zeta_1+\zeta_2+\zeta_3) = \sum_{|\alpha|=1} c_p(\alpha)\zeta^\alpha \tag{6.5}$$

may be of some independent interest. In the proof of Theorem 6.2 we investigated the Fourier coefficient corresponding to $\alpha = (-1,-1,3)$.

Our interest in (6.5) stems from the fact that when $d = 2$, we have easy access to all the corresponding Fourier coefficients. By a computation in [4, Section 3], it follows that

$$|\zeta_1 + \zeta_2|^{p-2}(\zeta_1 + \zeta_2) = \sum_{|\alpha|=1} \frac{\Gamma(p)}{\Gamma(p/2+\alpha_1)\Gamma(p/2+\alpha_2)}\zeta^\alpha. \tag{6.6}$$

Let us sketch a different proof of (6.6) in the spirit of Theorem 6.2. We consider first $p = 2n$ for a positive integer n and write

$$|\zeta_1 + \zeta_2|^{2n-2}(\zeta_1 + \zeta_2) = \overline{(\zeta_1\zeta_2)^n}(\zeta_1 + \zeta_2)^{2n-1} = \sum_{j=0}^{2n-1} \binom{2n-1}{j} z_1^{n-1-j} z_2^{j-n}, \tag{6.7}$$

to establish (6.6) when p is an even integer. By analytic continuation as in the proof of Theorem 6.2, we obtain (6.8) for $1 \leq p < \infty$. This proof shows that if $p = 2n$, then the nonzero Fourier coefficients in (6.6) are precisely the entries in the row $2n-1$ of Pascal's triangle. Since $|\zeta_1 + \zeta_2|^2 = 2 + \zeta_1\overline{\zeta_2} + \overline{\zeta_1}\zeta_2$, we see that

$$c_{p+2}(\alpha_1, \alpha_2) = 2c_p(\alpha_1, \alpha_2) + c_p(\alpha_1 - 1, \alpha_2 + 1) + c_p(\alpha_1 + 1, \alpha_2 - 1), \tag{6.8}$$

where $\alpha_1 + \alpha_2 = 1$. If $p = 2n$, then the recursion (6.8) corresponds to the three applications of Pascal's formula needed to go from the row $2n - 1$ to the row $2n + 1$.

Returning to (6.5), we similarly expand $|\zeta_1 + \zeta_2 + \zeta_3|^2$ to get the recursion

$$\begin{aligned} c_{p+2}(\alpha_1, \alpha_2, \alpha_3) = 3c_p(\alpha_1, \alpha_2, \alpha_3) &+ c_p(\alpha_1 + 1, \alpha_2 - 1, \alpha_3) + c_p(\alpha_1 + 1, \alpha_2, \alpha_3 - 1) \\ &+ c_p(\alpha_1 - 1, \alpha_2 + 1, \alpha_3) + c_p(\alpha_1, \alpha_2 + 1, \alpha_3 - 1) \\ &+ c_p(\alpha_1 - 1, \alpha_2, \alpha_3 + 1) + c_p(\alpha_1, \alpha_2 - 1, \alpha_3 + 1) \end{aligned}$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Specializing to the case where $p = 2n$ for positive integers n as above, we observe that the nonzero Fourier coefficients in (6.5) correspond precisely to the entries in the slice $2n - 1$ of Pascal's hexagonal pyramid. We refer to the On-Line Encyclopedia of Integer Sequences [1] and note that our numbering of the slices differs by 1. The numbers in Pascal's hexagonal pyramid do not have a known closed form similar to the binomials appearing in (6.7), so it is not clear how to proceed to get a formula for general $1 \leq p < \infty$.

Further examples can be generated starting with any of the sets found in [5, Section 3]. It is clear that for every $n > 1$ we could construct functions that are Hilbert points in $H^p(\mathbb{T}^4)$ for $p = 2, 4, \ldots, 2n$ and "most likely" for no other p in the range $1 \leq p \leq \infty$. We use quotation marks here to indicate that verifying rigorously the last assertion would be difficult if not impossible.

Beyond inner functions, we have so far only seen polynomial Hilbert points in $H^p(\mathbb{T}^d)$ for $p \neq 2$, which reflects that our understanding of the general situation is very limited. We do not know, for instance, whether there exists an unbounded Hilbert point in $H^p(\mathbb{T}^d)$ for some $p \neq 2$. It remains also to be seen whether Hilbert points in $H^p(\mathbb{T}^d)$ may have an operator theoretic role to play when $d \geq 2$, as they do with such distinction when $d = 1$, in view of Beurling's theorem.

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