Probing dark matter halos by gravitational waves

Author: Juan Fernández Esteban
*Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.**

Advisor: Oleg Bulashenko
(Dated: June 15, 2023)

Abstract: Gravitational waves open a new possibility in the detection of small dark matter halos. In this study we examine the gravitational lensing properties of the Navarro-Frenk-White (NFW) density profile, which is the most used model to describe cold dark matter halos. We also analyze the influence of the NFW halos on gravitational waves and, lastly, we study the effects of this interaction on the detection of the already mentioned waves. By exploring this effects, our objective is to understand better the interaction of gravitational waves and dark matter, two subjects shrouded in mistery.

I. INTRODUCTION

Gravitational waves (GWs) were first predicted in 1918 by Albert Einstein [1], without much hope to be detected. Nonetheless, they were first detected in 2015 by the *Laser Interferometry Gravitational-waves Observatory* (LIGO) [2]. Since then, a new astronomical observation window has been opened. So far, only (black holes and neutron stars) binary mergers have been detected [3], but there is hope to detect new phenomena, not only at the source, but also during its trajectory. In particular, we want to focus on gravitational lensing of gravitational waves.

We already know that light's path is curved with spacetime itself. This is called gravitational lensing [4, 5]. Gravitational waves follow the same geodesics as electromagnetic waves, so gravitational lensing should also work on them, and the detection of this phenomenon is really expected shortly.

On the other hand, dark matter is also a hot topic. While we do not know yet what it is, there are several candidates, as seen in [6]. We also have some candidates for its density distribution, but there is one that has been universally adopted since it adjusts accurately to the observations—the *Navarro-Frenk-White* (NFW) model. It is derived from N-body simulations to deal with cold dark matter [7]. It is expected for NFW profile to adjust also to small halos, but it is not as established as for big halos, since there is not as much evidence.

We may combine these concepts to study the lensing properties of NFW halos on gravitational waves. Despite NFW's wide mass range, we will focus on small to medium sized halos ($\sim 10^6 M_\odot$), which could in fact be potentially detected by currently existing or future detectors. Furthermore, GW lensing makes for a really good probe of this halos, since halos this size are not expected to have a nucleus of barionic matter, which means a really pure NFW lensing [8].

First, in section II we will discuss the general properties of the NFW model, and in section III we will illustrate the gravitational lensing theory. Then in section IV we will combine both to develop the NFW lens model, and in section V we will finally get to the main point, the study of NFW-lensed gravitational waves.

II. NFW DARK MATTER MODEL

The density profile provided by Navarro-Frenk-White is as follows [7]:

$$\rho(r) = \frac{\rho_s}{\frac{r}{r_0}\left(1+\frac{r}{r_0}\right)^2}, \tag{1}$$

where ρ_s and , the scale radius, r_0 are adjustable parameters, distinct for every halo. They are, therefore, obtained from observational data. The total mass of the halo diverges, so it is usually considered to a certain radius, the virial radius, $r_{vir} = c\, r_0$, where c is the concentration parameter. So the mass until this radius is called the viral mass, and it is enough to determine ρ_s and r_0.

III. GRAVITATIONAL LENSING THEORY

Let us suppose we have an extended lens, with a mass distribution $\rho(\vec{r})$. We will first state the deflection angle for a point mass lens, and then we will be integrating it for every mass differential. With perfectly reasonable approximations which always apply to the lensing framework, General Relativity predicts the deflection angle [9]:

$$\tilde{\alpha} = \frac{4GM}{c^2\, \xi} \tag{2}$$

with ξ the impact parameter and M the mass of the lens. As we said, for an extended lens we want to integrate over each mass differential to get the total deflection angle.

Now, since the deflection angle is small, it is reasonable to assume that light does not curve until it reaches the lens, so we can consider the thin lens approximation [4]. We want to have all the mass projected into a plane,

*Electronic address: juanfernandezesteban@gmail.com

whose name we will choose to be the lens plane, simply enough. This defines the surface mass density, as follows [4]:

$$\Sigma(\vec{\xi}) = \int_{\mathbb{R}} \rho(\vec{\xi}, z)dz, \tag{3}$$

where z points at the direction perpendicular to the lens plane, and $\vec{\xi}$ will be the coordinates on the lens plane. But first we want to introduce the necessary notation for

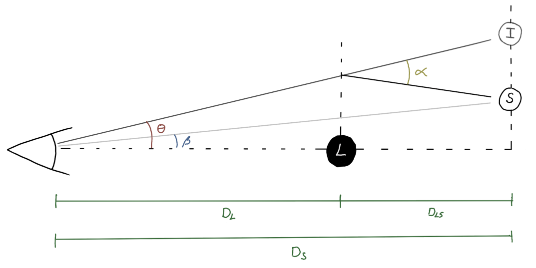

FIG. 1: Gravitational lensing diagram. I, S and L are the image, source and lens respectively. The eye represents the observer.

gravitational lensing. The angle from the center of the lens plane to the image is denoted by $\vec{\theta}$ (it is a vector, since we are working with a lens plane), and $\vec{\beta}$ is the angle to the source (the source will produce one or more images [5]). In fact they are the positions on the lens plane (source plane), divided by the distance, which makes for an angle in the paraxial approximation.

With some trigonometry, we can see that if we define D_l, D_s and D_{ls} as the distances to the lens, to the source and from the lens to the source respectively (can be seen in figure 1), we get

$$\vec{\beta} = \vec{\theta} - \frac{D_{ls}}{D_s}\vec{\hat{\alpha}}(\vec{\theta}) = \vec{\theta} - \vec{\alpha}(\vec{\theta}) \tag{4}$$

With the new scaled deflection angle $\vec{\alpha}$ [9]. This is called the lens equation, and reveals a comfortable relation for the source position and the image position. Now we need to compute our scaled deflection angle by combining Eqs. (2) and (3):

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{ls}}{D_s}\int_{\mathbb{R}^2} \frac{4G}{c^2}\frac{\Sigma(\vec{\xi})}{|D_l\vec{\theta} - \vec{\xi}|^2}(D_l\vec{\theta} - \vec{\xi})d\vec{\xi} \tag{5}$$

which can be reduced if we consider the dimensionless value $\kappa(\vec{\xi})$ named the convergence, and defined as follows:

$$\kappa(\vec{\xi}) = \frac{\Sigma(\vec{\xi})}{\Sigma_{cr}}, \quad \Sigma_{cr} = \frac{c^2}{4\pi G}\frac{1}{D_{\text{eff}}}, \quad D_{\text{eff}} = \frac{D_{ls}D_l}{D_s} \tag{6}$$

We defined the critical surface density Σ_{cr} and the effective lens distance D_{eff} as well. This way, the scaled deflection angle becomes:

$$\vec{\alpha}(\vec{\theta}) = \frac{1}{\pi}\int_{\mathbb{R}^2} \kappa(\vec{\xi})\frac{D_l\vec{\theta} - \vec{\xi}}{|D_l\vec{\theta} - \vec{\xi}|^2}d\vec{\xi} \tag{7}$$

Now, since $\vec{\nabla}\log|\vec{\theta}| = \vec{\theta}/|\vec{\theta}|^2$ we know that $\vec{\alpha}(\vec{\theta}) = \vec{\nabla}\,\psi(\vec{\theta})$ for some ψ named deflection potential, defined as:

$$\psi(\vec{\theta}) = \frac{1}{\pi}\int_{\mathbb{R}^2} \kappa(\vec{\xi})\log|D_l\vec{\theta} - \vec{\xi}|d\vec{\xi} \tag{8}$$

This is much more useful than it first seems, because it allows the definition of a new function such that the points on which its gradient vanishes is exactly the lens equation. This function is the Fermat potential or delay time [9]:

$$\tau(\vec{\beta}, \vec{\theta}) = \frac{1}{2}(\vec{\beta} - \vec{\theta})^2 - \psi(\vec{\theta}) \tag{9}$$

Now this is called delay time because it is actually the delay time of the wave, and can be split into a geometric part (first addend) and a gravitational part, the deflection potential [5].

Lastly we want to compute what we will call the transmission factor (also called amplification factor sometimes), which represents the ratio between lensed and unlensed amplitudes [10–12]. In simple terms, this is computed by calculating the amplification of each point in the lens plane (using the Fermat principle) and accounting for the delay time of each point in the lens plane to deal with interference.

For our case we will study the frame of geometrical optics, because it provides an easy solution for large frequencies as opposed to wave optics, which accounts for diffraction, but is much more complex and therefore requires much larger computation times.

If we define the Jacobian of the lens equation as $\det(\partial\vec{\beta}/\partial\vec{\theta})$, we can define the magnification of an image as $\mu(\vec{\theta}) = \det(\partial\vec{\beta}/\partial\vec{\theta})^{-1}$. Now, for our framework the transmission factor will be [12]:

$$F_{GO} = \sum_{Images}\sqrt{|\mu(\vec{\theta})|}\,e^{i(2\pi\nu\tau(\vec{\beta},\vec{\theta}) - n(\vec{\theta})\pi/2)} \tag{10}$$

where $n(\vec{\theta}) = 0, 1, 2$ for a minimum, saddle point and maximum of τ respectively, and is called the Morse index, and ν is the dimensionless frequency [12]. As expected from geometrical optics we are just dealing with images, assuming that the rest of the transmission factor cancels itself due to destructive interference, and we look only after their magnification and their path difference, to account for interference. This approximation is valid when the ratio between the Schwarzschild diameter of the lens and the wavelength is large. We want to define the phase function as [12]:

$$\Phi(\vec{\theta}) = \arg\left[e^{i\,2\pi\nu\tau(\vec{\beta},\vec{\theta})}\right], \quad -\pi < \Phi(\vec{\theta}) \le \pi \tag{11}$$

which will be an interesting visualization of what happens on the lens plane. It is, simply enough, the phase of a wave that arrives from the source on different parts of the lens plane.

IV. NFW LENS MODEL

We can, at last, get to the main point. As we exposed, our first objective is to find the lens equation of the NFW lens. For this we will have to figure the deflection angle, and for this we will first need to compute the convergence. It involves a really hard integral, but when solved becomes [13, 14]:

$$\kappa(x) = 2\kappa_s \frac{1-\mathcal{F}(x)}{x^2-1} \tag{12}$$

where $x = \xi/r_0$ (scaled radial position on the lens plane), $\kappa_s = \rho_s r_0/\Sigma_{cr}$ and the function $\mathcal{F}$ which is the core of the NFW lens model:

$$\mathcal{F}(x) = \begin{cases} \dfrac{\operatorname{arctanh}\sqrt{1-x^2}}{\sqrt{1-x^2}} & \text{for } x<1\,, \\ 1 & \text{for } x=1\,, \\ \dfrac{\arctan\sqrt{x^2-1}}{\sqrt{x^2-1}} & \text{for } x>1\,. \end{cases} \tag{13}$$

This function, even though it seems complex enough, is perfectly continuous and differentiable, and monotone. With some more calculations, we can get the lens equation [13]:

$$\vec{y} = \left[1 - \frac{4\kappa_s}{x^2}\left(\log\frac{x}{2} + \mathcal{F}(x)\right)\right]\vec{x} \tag{14}$$

We have $\vec{x} = \vec{\xi}/r_0 = \vec{\theta}D_l/r_0$, $\vec{y} = \vec{\beta}D_l/r_0$. The spherical symmetry of the density profile is obviously transferred to the lens equation in the form of a radial symmetry. Now this has one or three images (only at exactly the critical value y_{cr} it has 2), as can be seen in figure 2.

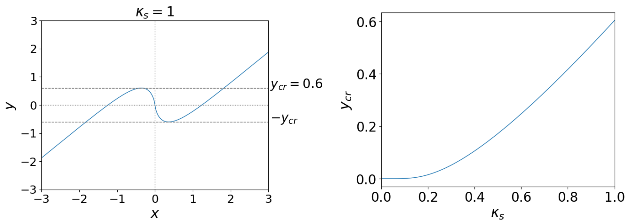

FIG. 2: Graphs of the lens equation: actual lens equation for $k_s = 1$ (left) and $y_{cr}(\kappa_s)$ (right)

The next objective is to get the deflection potential, and we may get it from the already shown Eq. (8). This yields [14]:

$$\psi_{\rm NFW}(x) = 2\kappa_s\left[\log^2\frac{x}{2} + (x^2-1)\mathcal{F}^2(x)\right] \tag{15}$$

With this, and the definition on Eq. (9), we get the delay time (normalised to x and y):

$$\tau(\vec{y},\vec{x}) = \frac{1}{2}(\vec{x}-\vec{y})^2 - \psi_{\rm NFW}(x) \tag{16}$$

And we can now compute the phase function, plotted in figure 3 with a density graph.

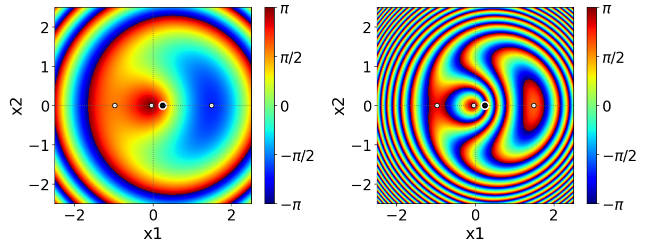

FIG. 3: Density plots for the phase function on the lens plane for frequency $\nu = 1$ (left) and $\nu = 5$ (right), both plots are done with $\kappa_s = 1$. The source $\vec{y} = (0.25, 0)$ and the images are also shown in both graphs.

Now to compute the transmission factor from Eq. (10), since we have the argument already, we first need the determinant [13]:

$$\begin{aligned} \det J(\boldsymbol{x}) = \left|\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right| &= \left[1 - \frac{4\,\kappa_{\rm s}}{x^2}\,\left(\log\frac{x}{2} + \mathcal{F}(x)\right]\right] \\ &\times \left[1 + \frac{4\,\kappa_{\rm s}}{x^2}\,\left(\log\frac{x}{2} + \mathcal{F}(x)\right) - 4\,\kappa_{\rm s}\,\frac{\mathcal{F}(x)-1}{1-x^2}\right] \end{aligned} \tag{17}$$

And now we can plot the transmission factor as seen in figure 4. We realize that when there is only one image, no interference can be appreciated, which makes sense for the geometrical optics approximation. Another downside of this approach is the divergence near the origin. Both artifacts would vanish if we used the wave optics approach.

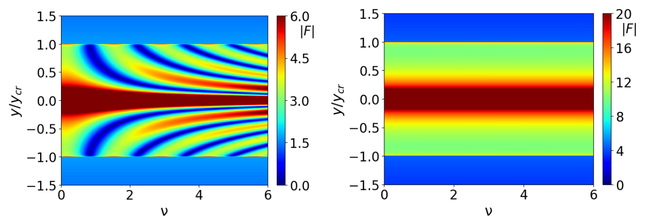

FIG. 4: These two density plots show the transmission factor in terms of y/y_{cr} and ν for $\kappa_s = 1, 0.2$, from left to right. Notice that the scales differ.

Nevertheless, we do not need a more advanced approach, since for small κ_s (small masses), y_{cr} is very small, and the interference zone is pretty much negligible, making for a transmission factor constant with ν. Also, as seen in figure 4, the critical zone also does tend to get more constant the smaller κ_s gets.

Now, small subhalo masses ($\sim 10^3 M_\odot - 10^6 M_\odot$) lead through a power formula with exponent 0.18 max [8] or 0.11 min [18] to $\kappa_s \sim 10^{-3}$.

V. GW LENSING AND SNR

We will be using PyCBC [15] to generate the gravitational waveforms. PyCBC is a package for python that has data about gravitational wave detections, models and tools to manipulate waveforms. We will be working with $\kappa_s = 0.001$ from now on.

First we are generating the waveform, with the model "IMRPhenomPv2", with masses $36M_\odot$ and $29M_\odot$ (two black holes) at a distance of 1Gpc.

Then we want to apply the lensing, which is done through the transmission factor. We Fourier transform the strain signal $h(t)$, so we get it in the frequency domain $\tilde{h}(f)$, and then we just scalarly multiply it by the transmission factor $F(f, y)$ (after setting down the source position y), so the amplitude of each frequency gets multiplied by its corresponding factor. We have then the lensed wave in the frequency domain, $\tilde{h}_L = \tilde{h} \cdot F$, which we can Fourier transform again so we get it in the time domain, $h_L(t)$.

In our case, for small κ_s, the critical zone is negligible, so we get transmission factor independent of the frequency. And that means that we just get the waveform multiplied by a constant, which means we get just amplification, and the wave is not distorted.

To detect the GWs we want to distinguish the noise, coming from different non ideal conditions, from the actual signal. This is the main challenge faced in GW detection.

We need to introduce now one last concept, the signal to noise ratio. It is an integral over the signal and the noise around the signal, which gives us (not formally) how much signal there is hidden in the noise. The formal definition is [16, 17]:

$$SNR = \int_{f_l}^{f_h} \frac{|\tilde{h}(f)|^2}{S_n(f)} df \qquad (18)$$

where $S_n(f)$ is the noise power spectral density (noise can vary with frequency), $h(f)$ is the amplitude of the incoming signal in the frequency domain (its Fourier transform), and the minimum and maximum frequencies are determined by the frequency bandwidth of the detector. A detected GW usually has a SNR of about 10, but it can vary from 7 to over 20, for very extreme cases [3].

We will use for calculations the noise profiles of LIGO during Observing Runs O3 and O4. With $\kappa_s = 0.001$, $y_{cr} \sim 10^{-4}$, so establishing $y = 10^{-3}$ we are well outside the interference zone. With these parameters, for the O3 noise, SNRs are 24.4 and 24.7 for the unlensed and lensed waves respectively, and for O4, it goes higher (noise has been decreased), SNRs are 25.4 and 25.8.

As expected, we get a greater SNR with the lensed wave, but for small halos this is really small. It would require great sensitivity to distinguish the case of a wave coming from closer or with a bigger source from a lensed one. The possibility of distinguishing both cases cannot be discarded, however.

Lastly, it is also interesting to visualize the spectrograms of both the lensed and the unlensed waves. The lensed wave has a more diffuse appearance which is surprising.

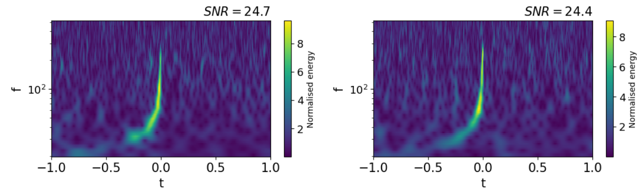

FIG. 5: Spectrograms of the lensed (left) and unlensed (right) gravitational waves. They both represent the energy of the waves in terms of the frequency and the time.

VI. CONCLUSIONS

We were able to create with Python a complete model of the NFW lens, so we could study much more than there is reflected on this project about the model. We changed parameters to study this lens, and we found out that NFW lens is a really complex model, with some vagueness in some aspects, so there is still a lot to learn.

We notice that strong lensing (having multiple images separated by times longer than the signal itself) is not too likely for NFW halos, and it gets more unlikely the smaller the halo gets. When multiple images are discarded, the main effect that persists is amplification.

This makes small halos ($\sim 10^6 M_\odot$) hard to detect, since we have small amplifications which can be easily confused with a closer source. If we get strange mass combinations, or multimessenger signals, they can, however be detected through gravitational waves analysis. Our plan is to continue with this research in the Master thesis.

Acknowledgments

I have nothing but good words to say about my advisor Dr. Oleg Bulashenko, who not only helped me with this work, but also to dig further into the topic. He has made all this much easier, providing a critical point of view when I programmed something and helping when I did not know where to continue. I would like to thank my family as well, because they have been supporting me since my passion for physics started long ago, and through all my degree they have been a main pilar, and the TFG is no exception. My friends have been also a huge support for me and I thank all of them, in part also because they all looked at the figures that were generated for this work, and they gave me their feedback.

I would also love to thank all the rest of people that made it possible for me to get here, and that I will not explicitly mention, but I do have them on my mind.

[1] A. Einstein, *Über Gravitationswelle*, Sitzungsber. K. Preuss. Akad. Wiss. **1**: 154 (1918).

[2] B.P. Abbott et al. (LIGO Scientific and Virgo Collaborations), "Observation of Gravitational Waves from a Binary Black Hole Merger", Phys. Rev. Lett. **116**: 061102 (2016).

[3] LIGO Scientific Collaboration detections, `https://www.ligo.org/detections/`.

[4] P. Schneider, J. Ehlers, and E. Falco, *Gravitational Lenses* (Springer, New York, 1992).

[5] S. Mollerach and E. Roulet, *Gravitational Lensing and Microlensing*, (World Scientific Publishing, Singapore, 2002).

[6] G. Bertone, T.M.P. Tait, "A new era in the search for dark matter". Nature **562**: 51 (2018).

[7] J.F. Navarro, C.S. Frenk and S.D.M. White, "The structure of cold dark matter halos", Astrophys. J. **462**: 563 (1996).

[8] H.G. Choi, Ch. Park, and S. Jung, "Small-scale shear: Peeling off diffuse subhalos with gravitational waves", Phys. Rev. D **104**: 063001 (2021).

[9] P. Schneider, Ch.S. Kochanek and J. Wambsganss, *Gravitational Lensing: Strong, Weak and Micro*, (Springer, New York, 2006).

[10] M. Born and E. Wolf, *Principles of optics*, 7th ed., (Cambridge University Press, Cambridge, U.K., 1999).

[11] R. Takahashi and T. Nakamura, "Wave effects in the gravitational lensing of gravitational waves from chirping binaries", Astrophys. J. **595**: 1039 (2003).

[12] O. Bulashenko and H. Ubach, "Lensing of gravitational waves: universal signatures in the beating pattern", Journal of Cosmology and Astroparticle Physics **2022**: 022 (2022).

[13] M. Karamazov, L. Timko and D. Heyrovský, "Gravitational Lensing by a Massive Object in a Dark Matter Halo. I. Critical Curves and Caustics", Astrophys. J. **922**: 72 (2021).

[14] R. Takahashi, "Quasi-geometrical optics approximation in gravitational lensing", Astron. Astrophys. **423**: 787 (2004).

[15] PyCBC Development Team (2023), `https://doi.org/10.5281/zenodo.7547919`.

[16] C. Cutler and E.E. Flanagan, "Gravitational waves from merging compact binaries: How accurately can one extract the binary's parameters from the inspiral waveform?", Phys. Rev. D **49**: 2658 (1994).

[17] E.E. Flanagan and S.A. Hughes, "Measuring gravitational waves from binary black hole coalescences. I. Signal to noise for inspiral, merger, and ringdown", Phys. Rev. D **57**: 4535 (1998).

[18] X. Guo and Y. Lu, "Probing the nature of dark matter via gravitational waves lensed by small dark matter halos", Phys. Rev. D **106**: 023018 (2022).