Cosmic censorship in Lemaitre-Tolman-Bondi spacetimes: Conformal diagrams of locally and globally naked singularities

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Abstract: Naked singularities are highly curved regions of spacetime where classical general relativity fails, and whose effects are observable from infinity. The cosmic censorship conjecture states that such singularities must hide behind a black hole event horizon. We consider cosmic censorship in the context of a dust dominated universe, with a spherically symmetric primordial overdensity. This admits an exact description in terms of the Lemaitre-Tolman-Bondi (LTB) solution, in which both black holes and naked singularities can occur. We numerically construct causal diagrams for both cases, identifying the apparent horizons, event horizons and cauchy horizons when applicable.

I. INTRODUCTION

According to general relativity (GR), a massive enough star will undergo gravitational collapse once it has exhausted all its nuclear fuel. When a trapped surface occurs during this process, it implies the presence of some form of spacetime singularity [1].

According to the cosmic censorship (CC) conjecture, this spacetime singularity will be necessarily "hidden" behind a black hole (BH) horizon. Primordial black holes (PBH) may have originated in the early universe, and they are a possible candidate for dark matter if they formed during the radiation dominated era [2–4]. Here we are interested in PBH for different reasons, as a testing ground for CC, and for simplicity, we consider their formation in the matter dominated era.

The CC conjecture still stands as an unproved hypothesis. Furthermore, several scenarios have been found where gravitational collapse results in the formation of a naked singularity (NS). being the LTB solution one of the better known ones [5, 6]. The consequences of NS as well as their fate are still not totally clear. The semiclassical approximation is not valid in the vicinity of a NS, where curvature is Planckian. Their further evolution should be studied by including quantum gravity effects [6, 7].

In this work we will review CC and the formation of NS. For this purpose, we describe the formation of both a PBH and a NS in terms of the LTB solution for spherically symmetric dust collapse, considering an overdense region of the universe surrounded by a spatially flat homogeneous Friedmann-Lemaitre-Robertson-Walker (FLRW) space. We will numerically construct the causal diagram of both models, studying some of its physical properties and identifying the formation of horizons during the gravitational collapse. We will see that some parts of the universe crunch into singularities a finite time after the Big Bang, while in others the worldlines of dust particles continue forever into the future.

II. COSMIC CENSORSHIP

A *trapped surface* is defined as a closed, spacelike, two-surface from which null geodesics converge for both ingoing and outgoing congruences [8]. The boundary of a trapped surface is an *apparent horizon* (AH). In the present context, singularity theorems state that the existence of a trapped surface implies the existence of a spacetime singularity. Furthermore, Penrose's weak cosmic censorship (WCC) conjecture states that any spacetime singularity cannot be observable from infinity, and must then be hidden behind an *event horizon* (EH) [1]. In this sense, the WCC conjecture denies the existence of "globally naked" singularities for dynamical evolution of generic initial conditions.

In the strong cosmic censorship (SCC) conjecture, also "locally naked" singularities would not be admitted, where the observer is not necessarily at infinity [5]. In order to understand the SCC conjecture we shall introduce the *Cauchy horizon* (CH), the boundary of deterministic evolution. The CH separates the regions where the equations of GR can predict the future from those in which it cannot [9]. So as to preserve the determinism in Einstein's equations, regions beyond the CH should have no influence for any distant observer. The reasoning behind SCC is that Einstein's equations are unstable under perturbations when approaching the CH. An observer approaching the CH would see an exponential blue-shift effect of perturbations associated to a divergent energy flux, which would imply the existence of a singularity [9]. This property, proven in Reissner-Nordström BHs [10], is assumed to extend to the general case.

Nevertheless, the validity of CC is still under discussion. Recent articles related to the stability of the CH present several examples which would contradict CC [9, 11, 12], as well as refinements which would reinforce the conjecture [13]. A part from these cases, beyond the scope of this work, there are also many known models where gravitational collapse from physically reasonable initial data leads to the formation of a NS [14].

Among these models, the LTB solution is the one best understood. Aside from the conditions under which NS arise in this model, we are also interested in the implications of their existence in relation to CC. Classical physics ceases to be valid about one Planck time before the formation of the singularity as curvature becomes Planckian and quantum gravity effects must be taken into account [7, 15]. In this context, it is interesting to introduce the concept of *effective* naked singularities, understood as sufficiently high-curvature regions with observable effects for distant observers [7, 16]. May these regions of spacetime exist, we would be able to observe fully quantum gravitational effects.

III. THE LTB SOLUTION

The LTB solution is given by a spherically symmetric metric in the form

$$ds^2 = -dt^2 + \frac{R'(t,r)^2}{1-k(r)}dr^2 + R(t,r)^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

together with the stress-energy tensor for dust particles

$$T_{ab} = \rho(t,r)\delta_a^t\delta_b^t. \quad (2)$$

In what follows, a prime denotes derivatives with respect to the radial coordinate, r, and a dot denotes derivatives with respect to the time coordinate, t. $R(t,r)$ is the areal radius, $k(r)$ is a free function, and we use natural units $G = c = 1$.

In this context, Einstein's equations are

$$\dot{R}^2 = 2mR^{-1} - k, \quad (3)$$

$$\dot{k} = 0, \quad (4)$$

$$\dot{m} = 0, \quad (5)$$

where $m(r)$ represents the mass function, which must satisfy the constraint

$$m' = 4\pi R^2 R' T_{tt}. \quad (6)$$

The later equations together with the metric determine the LTB general solution. For $k = 0$ we have the homogeneous Einstein-de Sitter solution:

$$R(t,r) = \left(\frac{9}{2}\right)^{1/3} m^{1/3} t^{2/3}. \quad (7)$$

For $k > 0$, by parametric integration we find:

$$t(\eta, r) = mk^{-3/2}(\eta + \pi + \sin\eta), \quad (8)$$

$$R(\eta, r) = 2mk^{-1}\cos^2(\eta/2), \quad (9)$$

where $-\pi \le \eta \le \pi$. Thus, $t(-\pi, r) = 0$ represents the Big Bang and $t_c(r) = t(\pi, r) = 2\pi m k^{-3/2}$ gives us the time of collapse, for both of which $R(-\pi, r) = R(\pi, r) = 0$. This solution is then determined by the free functions m and k. For the models later reviewed, we take k as follows:

$$k = \begin{cases} \frac{3}{4}\sin^2(r) & r < \pi \\ 0 & r \ge \pi, \end{cases} \quad (10)$$

while the mass function will be chosen below. If $\bar{\rho}$ is the mean density

$$\bar{\rho} = \frac{3m}{4\pi R^3}, \quad (11)$$

then one can see for $t \ll t_c$ and $k > 0$ that

$$\frac{\delta\rho}{\rho} = \frac{\bar{\rho} - \rho_\infty}{\rho} \approx \frac{3}{20}\left(\frac{6\pi t}{t_c}\right)^{2/3} > 0, \quad (12)$$

being ρ_∞ the density at infinity. In this sense, the region $r < \pi$ represents an initial overdensity of spacetime going smoothly to 0 at $r = \pi$ and then becoming spatially flat. In this spacetime, the overdense region expands after the Big Bang and then undergoes gravitational collapse, while the homogeneous FLRW space continues forever into the future.

A. Naked singularities in LTB spacetimes

Two types of singularities can develop during gravitational collapse of inhomogeneous dust distributions. The ones of our interest are the so-called shell focusing singularities, which are naked when they occur on the central worldline [17]. In order to understand when central shell-focusing singularities arise, we consider the time of formation of the AH, determined by $R = 2m$. From (8) and (9) we find

$$\eta^\pm_{ah}(r) = \pm 2\arccos\sqrt{k}, \quad (13)$$

$$t^\pm_{ah}(r) = mk^{-3/2}(\eta^\pm_{ah} + \pi + \sin\eta^\pm_{ah}). \quad (14)$$

The result for the $+$ sign represents the time at which a trapped surface is formed when approaching t_c, while the $-$ sign would give the analogous time approaching the Big Bang. In this context, the formation of a naked singularity in marginally bound collapse has been widely reviewed [5, 7, 18]. For this case it is found that $t_c > t^+_{ah}$ for any $r > 0$, and thus any region beyond $r = 0$ cannot be naked [5]. Nevertheless, null rays emanate from $(t_c(0), 0)$ under some conditions for the mass function. In particular, NS arise for $k = 0$ and a mass function

$$m(r) = M_3 r^3 + M_4 r^4 + M_5 r^5 + M_6 r^6 + ..., \quad (15)$$

for $M_4 = M_6 = 0$, $M_5 < 0$ and also for $M_4 = M_5 = 0$, $M_6 < -\left(26 + 15\sqrt{3}\right) M_3^{5/2}/2$ [5, 18]. Then, local visibility is determined only by the central expansions of the mass function, while global visibility requires of a functional form such as (15) in the whole range of r. The physical reason behind the exposure of the singularity would be related to shearing effects which, when strong enough near the central worldline, would delay the formation of the AH [17].

Next, we will make further considerations for a more general case. If we recover the idea of effective NS, we should consider the physical singularity not to occur at t_c, but rather at $t_s(r) = t_c - 1$ (one Planck time before t_c, where spacetime curvature and density are already Planckian [7, 15]). With this assumption, we find from (14) that $t_{ah}^+ > t_s$ for a non-zero range of r, meaning that null geodesics would emerge not only from the central worldline.

When this consideration is taken into account, we will see that NS appear for new different choices of the parameters from (15). Unfortunately, there is no analytical treatment to determine what factors make the NS arise for the $k > 0$ case, or to understand whether it will be global or local. In general, a null ray which propagates from this region and later crosses t_{ah}^+ will fall again into the singularity, thus being locally naked. On the other hand, a globally NS will have formed if null rays reach the surface of the overdense region instead of crossing t_{ah}^+.

IV. CONFORMAL DIAGRAMS

In the following section, we introduce an algorithm to numerically construct conformal diagrams. In conformal diagrams, cosmological infinities are reduced to finite distances and light rays are represented by ingoing or outgoing lines of slope equal to one. This algorithm will then be used to represent the conformal diagrams of both a locally and a globally NS. For the purpose of this work, it will be convenient to use two different coordinatizations.

A. Coordinatization and compactification

1. Coordinates $U, V \to T, X$

Given an arbitrary point P in our (t, r) space, null coordinates U and V are defined such that

$$V = t_i + r_i, \tag{16}$$
$$U = t_i - r_i, \tag{17}$$

being t_i and r_i the initial values of t, r for either an ingoing (V) or an outgoing (U) light ray passing through P. The trajectories of these rays are obtained by numerically integrating the null geodesics differential equation for metric (1) together with expressions (8) and (9):

$$\frac{dt}{dr} = \frac{\pm R'}{\sqrt{1-k}}. \tag{18}$$

One can see that any ingoing ray will start propagating from the Big Bang, taking $V = r_i$, while outgoing rays propagate both from the Big Bang, $U = -r_i$ or from $r = 0$. Nevertheless, coordinates U, V have the particularity that for $r = 0$, $U = V$, so that we can associate to any outgoing null geodesic starting at $(t_i, 0)$ an auxiliary ingoing ray such that $U = V_{aux} = r_{i,aux}$. An example of this is shown in Fig. 1.

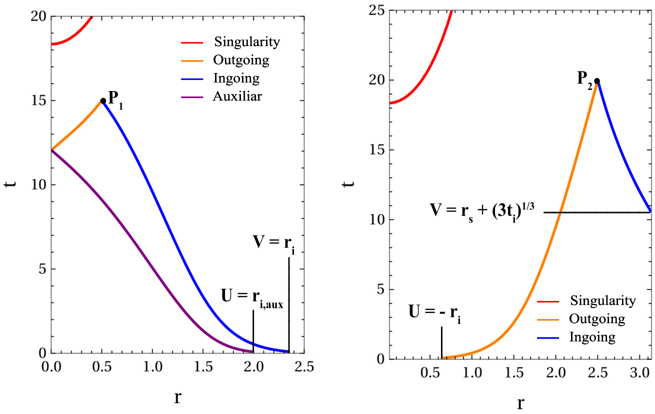

FIG. 1: Example of the assignment of U and V to points P_1, P_2. The red line represents the physical singularity. Blue and purple lines represent ingoing rays. Outgoing rays are represented in orange. In the left panel, an auxiliary ray is used to define U. In the right panel, r_s represents the value of r at the surface of the overdensity. Also, the expression given for V is obtained by propagating an ingoing ray starting at the Big Bang in the $k = 0$ region that crosses the overdensity surface at (t_i, r_s).

Then, a compactification and a rotation is applied:

$$T = \arctan U + \arctan V, \tag{19}$$
$$R = -\arctan U + \arctan V. \tag{20}$$

2. Coordinates $U', V' \to T', X'$

As will be seen later, it can be convenient to define a new coordinatization. Let be r_f, t_f the final values of t, r for an ingoing (V') or an outgoing (U') ray, where

$$V' = \left[\frac{t_f}{t_s(r_f)}\right]^{1/\gamma} + \frac{r_f}{r_s}, \tag{21}$$
$$U' = \left[\frac{t_f}{t_s(r_f)}\right]^{1/\gamma} - \frac{r_f}{r_s}. \tag{22}$$

In a similar way to what we saw for U, V, we find in Fig. 2 that any ingoing light ray propagating inside the overdensity must end either at $r = 0$ or at the singularity. Also, any outgoing ray will end at the surface r_s (in what follows, we take $r_s = \pi$) or at the singularity. In addition, the factor γ is used to visually enhance the range of values of V' where $t_f \ll t_s(r_f)$. Otherwise, V' would be very small in this region and would give no information about its causal structure. Since V' and U' are already compactified, they just need to be rotated:

$$T' = U' + V', \tag{23}$$
$$X' = -U' + V'. \tag{24}$$

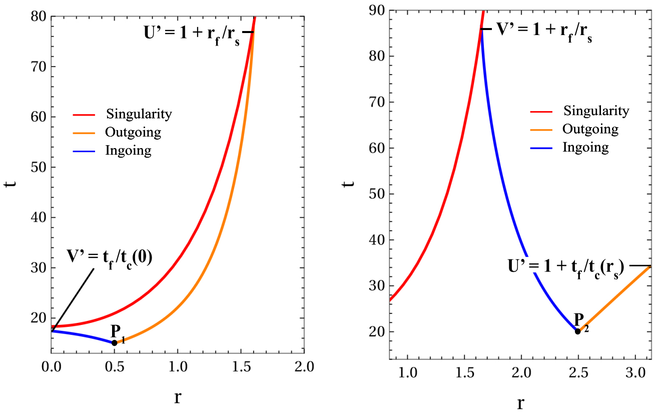

FIG. 2: Example of the assignment of U' and V' to points P_1, P_2. The red line represents the physical singularity. Blue lines are ingoing light rays. Outgoing light rays are shown in orange. For this example, $\gamma = 1$.

B. Conformal diagram of a locally naked singularity in a PBH

When considering k given in (10) and a mass function $m = 2r^3$, with a total mass of the overdense region $M \approx 62$, we find a locally naked singularity arise. Its diagram using null coordinates T, X is shown in Fig. 3.

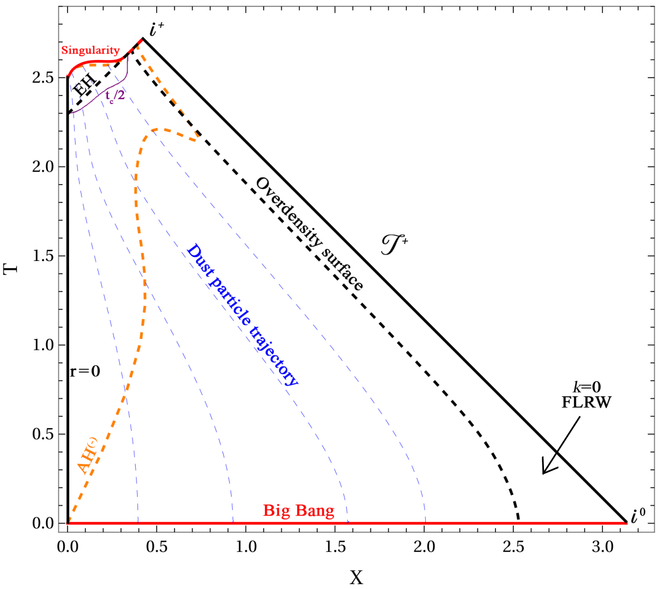

FIG. 3: Conformal diagram of a locally naked singularity in a PBH hole with mass function $m = 2r^3$ for null coordinates T, X. Plotted in the diagram are the singularity and the Big Bang (solid red), the symmetry time $t_c/2$ (thin purple), surfaces $r = 0$ and $r = \pi$ (solid and dashed black), minus (t_{ah}^-) and plus (t_{ah}^+) AH (dashed orange), EH (dashed black), and dust particle geodesics (dashed blue).

In particular, one can observe that $t_s < t_{ah}^+$ for any $r < 0.7$. Every light ray emanating from the singularity in this region then crosses the AH and thus falls again into the singularity, being locally naked. It is interesting to recall that in Fig. 3 all AH, EH, singularity and overdensity surface seem to merge together when approaching future infinity at the surface $r = \pi$. This is the result of joining two regions of spacetime, one in which a dust particle has a finite life-time (immediately before $r = \pi$), and another one with an infinite life-time (immediately after $r = \pi$). To see with more detail the region near future infinity, we can use the coordinates T', X', as shown in Fig. 4.

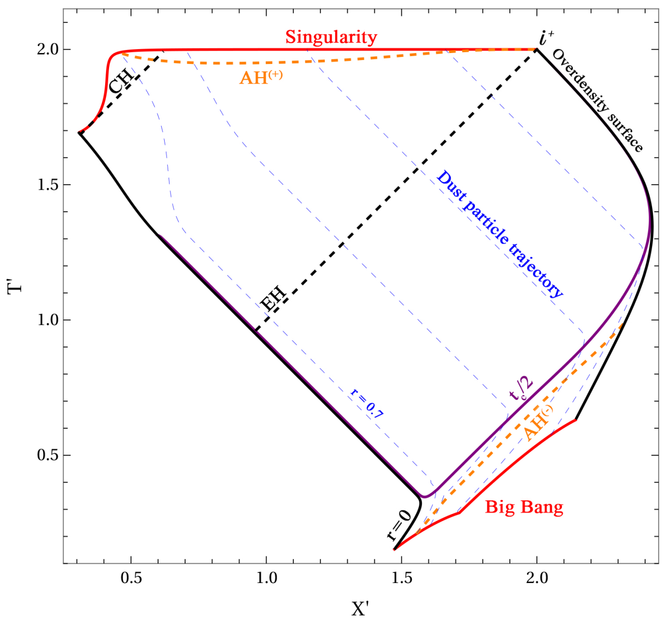

FIG. 4: Conformal diagram of a locally naked singularity in a PBH for null coordinates T', X', taking $\gamma = 14$. Plotted in the diagram are the singularity and the Big Bang (in solid red), $t_c/2$ (solid purple), surfaces $r = 0$ and $r = \pi$ (solid black), plus and minus AH (dashed orange), EH, CH (dashed black), and dust particle geodesics (dashed blue).

By contrast with Fig. 3, here the re-collapse phase ($t > t_c/2$) occupies the majority of the diagram. On the other hand, as an outgoing null geodesic propagating beyond $r > \pi$ will not necessarily cross the overdensity surface, we cannot study the outside region $r > \pi$ using coordinates U', V'.

C. Conformal diagram of a globally naked singularity

When we take $m = 2r^3 + 2.5r^4$, with a total mass $M \approx 305.5$, then $t_s < t_{ah}^+$ for $r < 0.59$ and a globally NS is formed. In this case, null geodesics starting at the NS for $r < 0.027$ reach the surface of the overdensity and propagate to future infinity. Those emanating from $0.027 < r < 0.59$ fall again into the singularity. As a consequence, for this mass function a non-zero range of r beyond the CH is exposed to distant observers and determinism is lost. Interestingly, neither the mass function taken here nor the one we used for the PBH fulfill the conditions under which NS arise in marginally bound collapse. We can say that both locally and globally NS arise in cosmological context, although we don't have an analytical treatment to determine under what conditions. The conformal diagram for this case is shown in Fig. 5.

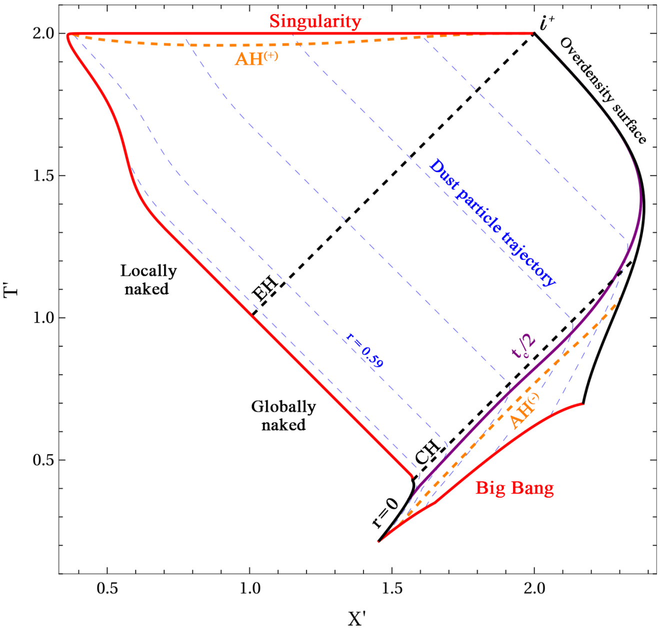

FIG. 5: Conformal diagram of a globally naked singularity with mass function $m = 2r^3 + 2.5r^4$ using coordinates T', X', with $\gamma = 16$. Plotted in the diagram are the singularity and the Big Bang (solid red), $t_c/2$ (purple), surfaces $r = 0$ and $r = \pi$ (solid black), plus and minus AH (dashed orange), EH, CH (dashed black), and dust particle geodesics (dashed blue).

V. CONCLUSIONS

We have reviewed the status of the cosmic censorship conjecture in the context of the LTB solution. One finds that naked singularities can arise for physically reasonable initial data. This scenario, widely analysed for marginally bound collapse, has been studied here in a cosmological context. Introducing the idea of effective naked singularities, we see that locally and globally naked singularities appear. These regions of extremely high curvature would presumably be governed by quantum effects.

We have described two different algorithms to numerically build conformal diagrams. We have seen that using different coordinatizations we can study in more detail different regions of our spacetime, depending on what we are most interested in.

In particular, we have analysed the conformal diagrams of both locally and globally naked singularities in figures 3, 4 and 5 . These examples violate strong and weak cosmic censorship, respectively. For the second case, spacetime regions beyond the Cauchy horizon are exposed to future infinity. This implies a breakdown of determinism in classical physics, and suggests the idea that new physics beyond general relativity may be exposed to distant observers. It would be interesting to clarify the regime of parameters where global or local naked singularities are formed in the context of gravitationally bound collapse. This issue is left for further research.

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[1] S. W. Hawking and R. Penrose: Proc. Roy. Soc., A (in press) (1969).
[2] A.M. Green and B.J. Kavanagh 2021 J. Phys. G: Nucl. Part. Phys. 48 043001. (2021).
[3] S. Hawking. *Gravitationally Collapsed Objects of Very Low Mass.*, Monthly Notices of the Royal Astronomical Society, 152(1):75–78, 1971.
[4] S.Clesse, J. García-Bellido. Phys. Rev. D **92**, 023524 (2015).
[5] R. P. A. C. Newman, Class. Quantum Grav. 3, 527 (1986)
[6] T.P. Singh. *Gravitational Collapse, Black Holes and Naked Singularities.* J.Astrophys. Astron. 20,221. (1999).
[7] T. Harada, H. Iguchi, K. Nakao, T. P. Singh, T. Tanaka, C. Vaz. Phys. Rev. D **64**, 041501(R) (2001).
[8] R. Penrose, Phys. Rev. Lett. **14**, 57. (1965)
[9] Kyriakos Destounis, Rodrigo D.B. Fontana, and Filipe C. Mena Phys. Rev. D **102**, 104037. (2020).
[10] Chandrasekhar, Subrahmanyan, and J. B. Hartle. *Crossing the Cauchy Horizon of a Reissner-Nordstrom Black-Hole.* Proceedings of the Royal Society of London. Series A, Mathematical and physical sciences 384, no. 1787 (1982).
[11] R. Luna, M. Zilhão,V. Cardoso, J.L. Costa, J. Natário. Phys. Rev. D **99**, 064014 (2019).
[12] O. J. C. Dias, H. S. Reall, and J. E. Santos (2018), arXiv:1808.02895.
[13] M. Dafermos and Y. Shlapentokh-Rothman 2018 Class. Quantum Grav. **35** 195010.
[14] P. S. Joshi and I. H. Dwivedi, Phys. Rev. D **47**, 5357 (1993).
[15] H. Iguchi and T. Harada 2001 Class. Quantum Grav. 18 3681.
[16] T. Harada, Pramana 63, 741–754 (2004)
[17] Pankaj S. Joshi, Naresh Dadhich, and Roy Maartens Phys. Rev. D **65**, 101501(R) (2002).
[18] T. Harada, H. Iguchi, K. Nakao . Prog.Theor.Phys. **107** (2002) 449-524