Calculation of the equation of state of neutron star matter

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Symmetric nuclear matter, asymmetric nuclear matter and β-stable nuclear matter are discussed and modeled through a Lagrangian density of relativistic mean field (RMF) theory using a σ meson to describe the attractive interaction between nucleons, a ω meson to describe the repulsive interaction and a ρ meson to describe the asymmetry effects with the NL3 parametrization [1].

I. INTRODUCTION

Understanding the internal structure and properties of neutron stars has presented a challenge for the scientific community since their discovery. Over the last decades, great progress has been made [2].

In this project, the main focus will be studying the equation of state (*energy per particle* and *pressure*) of three different nuclear matter compositions by means of a relativistic mean field (RMF) formalism that correctly predicts known nuclear characteristics like the saturation point. Although there are non-relativistic models, observations of finite nuclei indicate that there must be some relativistic effects that are relevant [3], and also at the densities found in the core of neutron stars the nucleons and other components present a relativistic behaviour. Using the Dirac equation instead of the Schrödinger equation provides a more natural and consistent way of proceeding.

We know that the main interaction between nucleons, which are one of the main components of neutron stars or nuclei, is the strong interaction. The theory that explains this kind of interaction, *quantum chromodynamics*, is very complex and works with degrees of freedom that we cannot really observe, gluons and quarks. Can we be successful ignoring this underlying structure? Choosing hadrons (baryons and mesons) as our main degrees of freedom is a natural step considering our objectives and it is also a great step in making the theory much more accessible. This model is often referred to as *quantum hadrodynamics* [4].

We want to successfully reproduce the known results for the *energy per particle* (~ -16 MeV) and *density* (~ 0.15 fm^{-3}) at saturation using the NL3 values for the free parameters [1]. The project will be organised as follows. First, we will introduce the RMF formalism. Second, we will discuss the symmetric nuclear matter, where the density of protons and neutrons is equal. Then we will move onto asymmetric nuclear matter, where the density of protons and neutrons is different, and we will study how the proportion of each nucleon affects our magnitudes of interest. Lastly, we will introduce electrons to be able to get closer to the description of matter inside a neutron star and study the β-stable nuclear matter.

II. FORMALISM

As it has been briefly discussed before, our degrees of freedom will be hadrons (baryons and mesons). For the baryons we will be working with neutrons and protons, incorporated through a baryon field ψ. For the mesons we will be working with a scalar-isoscalar σ meson, to describe the attractive interaction between nucleons and represented by a scalar field ϕ. A vector-isoscalar ω meson, to describe short range-repulsion and introduced through a repulsive four-vector field V_μ. Finally, a vector-isovector ρ meson to take into account the asymmetry effects between nucleons, represented by the $\mathbf{R}_\mu$ field, which is a four-vector in Minkowski space and a three-vector in isospin space. Later on we will introduce electrons.

We can now build our Lagrangian density [1,2,6]:

$$\begin{aligned}\mathcal{L} \;=\; & \overline{\psi}[i\gamma_\mu\partial^\mu - g_V\gamma^0 V_0 - (M - g_S\phi_0)]\psi \\ & + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_S^2\phi_0^2) - \frac{1}{3}g_2\phi_0^3 - \frac{1}{4}g_3\phi_0^4 \\ & - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_V^2 V_\mu V^\mu \\ & - \frac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}^{\mu\nu} + \frac{1}{2}m_\rho^2\mathbf{R}_\mu\cdot\mathbf{R}^\mu - \frac{1}{2}g_\rho\overline{\psi}\gamma_\mu\tau\cdot\mathbf{R}^\mu\psi,\end{aligned} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and $\mathbf{G}_{\mu\nu} = \partial_\mu\mathbf{R}_\nu - \partial_\nu\mathbf{R}_\mu$ are the field tensors, M is the baryon mass (939 MeV) and non-linear terms on ϕ_0 have been included. The mass of the σ meson, m_S, and the couplings g_S, g_V, g_ρ, g_2 and g_3 are the free parameters tuned to fit the experimental values of the saturation point. Solving the equations derived directly from this Lagrangian density, quantizing every baryon and meson field, is extremely complicated.

The next step is to adopt a mean field approximation. The way to proceed is quantize only the baryon field and substitute the meson fields by their expectation value, which are constants:

$$\begin{gathered}\phi \longrightarrow \langle\phi\rangle \equiv \phi_0 \quad , \quad V^\mu \longrightarrow \langle V^\mu\rangle \equiv V_0, \\ \mathbf{R}^\mu \longrightarrow \langle\, \mathbf{R}^\mu\rangle \equiv R_{0,3}.\end{gathered} \tag{2}$$

The 3 in the meson ρ subscript is for the third component of the isospin vector. We will be working with infinite uniform matter, so there is no spatial variation in any magnitude, including the spatial contribution to the expectation value of the mesons. Taking into account the RMF approximation and the vanishing of spatial dependence in nuclear matter the Lagrangian density will be [1,2,6]:

$$\begin{aligned}\mathcal{L} &= \overline{\psi}[i\gamma_\mu\partial^\mu - g_V\gamma^0 V_0 - (M - g_S\phi_0)]\psi \\ &-\frac{1}{2}m_S^2\phi_0^2 + \frac{1}{2}m_V^2 V_0^2 - \frac{1}{3}g_2\phi_0^3 - \frac{1}{4}g_3\phi_0^4 \\ &+\frac{1}{2}m_\rho^2 R_{0,3}^2 - \frac{1}{2}g_\rho\overline{\psi}\gamma_\mu\tau_3 R_{0,3}\psi.\end{aligned} \tag{3}$$

To obtain the *mean-field equations* obeyed by the nucleon and meson fields we have to derive the Euler-Lagrange equations from the Lagrangian density. Let us define the *baryon density* ρ and the *scalar density* ρ_S:

$$\rho = \langle\psi^\dagger\psi\rangle = \sum_k \varphi_k^\dagger\varphi_k \tag{4}$$

$$\rho_S = \langle\bar{\psi}\psi\rangle = \sum_k \varphi_k^\dagger\beta\varphi_k. \tag{5}$$

The nucleons are fermions, so they will occupy states till a *Fermi momentum* k_F. We are considering an infinite system so we can transform the sums to integrals:

$$\rho = \frac{g}{(2\pi)^3}4\pi\int_0^{k_F} k^2 dk = \frac{g}{2\cdot 3\pi^2}k_F^3 \tag{6}$$

$$\begin{aligned}\rho_S &= \frac{g}{(2\pi)^3}\int d^3k\frac{M^*}{\epsilon_k} \\ &= \frac{gM^*}{4\pi^2}\left[k_F\epsilon_K - M^{*2}\ln\left(\frac{k_F+\epsilon_F}{M^*}\right)\right],\end{aligned} \tag{7}$$

where g is the *degeneracy*, ϵ_k is

$$\epsilon_k = \sqrt{k^2 + M^{*2}}, \tag{8}$$

$\epsilon_F = \epsilon_k(k_F)$ and M^* is the effective mass of the nucleons, and it is defined as

$$M^* = M - g_S\phi_0. \tag{9}$$

The σ meson acts upon the nucleon by reducing its mass. With those definitions made, let us write the *mean-field equations*:

$$m_V^2 V_0 = g_V\rho \tag{10}$$

$$m_S^2\phi_0 = g_S\rho_S - g_2\phi_0^2 - g_3\phi_0^3 \tag{11}$$

$$m_\rho^2 R_{0.3} = \frac{1}{2}g_\rho(\rho_p - \rho_n) \tag{12}$$

$$(\gamma^\mu(i\partial_\mu - g_V V_\mu) - (M - g_S\phi_0))\psi = 0. \tag{13}$$

The first equation determines by itself the value of V_0 given a value for the *baryon density* ρ. The second equation, that determines ϕ_0, needs to be iterated because of the dependence $\rho_S(\phi_0)$. It will be numerically solved by Newton-Raphson. The third equation determines $R_{0,3}$ for a given value of both the *proton* and *neutron density.* It will only be relevant in cases with asymmetry.

The fourth equation is a Dirac equation for the nucleons, with minimal couplings from the mesons. Its solution is relevant to successfully compute the *scalar* (7) and *baryon* (6) *density.* In nuclear matter we can treat the "modified nucleons" as free particles, so we seek stationary-state solutions of the form

$$\psi = \varphi_k e^{i(\vec{k}\cdot\vec{r} - E_k t)}\frac{1}{(2\pi)^{\frac{3}{2}}}. \tag{14}$$

By replacing (14) in the Dirac equation (13), we obtain

$$\varphi_k = \sqrt{\frac{\epsilon_k + M^*}{2\epsilon_k}}\begin{pmatrix}1 \\ \frac{\vec{\sigma}\cdot\vec{k}}{\epsilon_k + M^*}\end{pmatrix} \tag{15}$$

$$E_K = \epsilon_k + g_V V_0. \tag{16}$$

The *energy density* is obtained by the usual means of QFT:

$$\varepsilon = \sum_\alpha \frac{\partial\mathcal{L}}{\partial(\partial\phi_\alpha/\partial t)}\frac{\partial\phi_\alpha}{\partial t} - \mathcal{L}. \tag{17}$$

We have now developed the ingredients needed to write the *pressure* p and the *chemical potential* μ:

$$p = \sum_i \rho_i\mu_i - \varepsilon \tag{18}$$

$$\mu_i = \frac{\partial\varepsilon}{\partial\rho_i}. \tag{19}$$

The index i is for protons, neutrons and electrons.

III. SYMMETRIC NUCLEAR MATTER (SNM)

Let us begin with the simpler case. In the symmetric case, we have

$$\rho = \rho_n + \rho_p \tag{20}$$

$$\rho_n = \rho_p = \frac{\rho}{2}. \tag{21}$$

We are considering baryons as a single degree of freedom so we will have spin-isospin *degeneracy* ($g = 4$). The RMF Lagrangian density for this case is

$$\begin{aligned}\mathcal{L}_{SNM} &= \bar{\psi}[i\gamma_\mu\partial^\mu - g_V\gamma^0 V_0 - (M - g_S\phi_0)]\psi \\ &- \frac{1}{2}m_S^2\phi_0^2 + \frac{1}{2}m_V^2 V_0^2 - \frac{1}{3}g_2\phi_0^3 - \frac{1}{4}g_3\phi_0^4.\end{aligned} \tag{22}$$

It is important to note that in the symmetric case, the ρ meson does contribute, so (12) will not be solved.

We can move on now onto the *energy density* ε_{SNM}:

$$\begin{aligned}\varepsilon_{SNM} &= \frac{1}{4\pi^2}\left[k_F\epsilon_F^3 + k_F^3\epsilon_F - M^{*4}\ln\left(\frac{k_F+\epsilon_F}{M^*}\right)\right] \\ &+g_V V_0\rho - \frac{1}{2}m_V^2 V_0^2 + \frac{1}{2}m_S^2\phi_0^2 + \frac{1}{3}g_2\phi_0^3 + \frac{1}{4}g_3\phi_0^4.\end{aligned} \tag{23}$$

Let us call the first term ε_0.

One of the requirements of our RMF theory is for the *energy per particle* ($\frac{\varepsilon}{\rho} - M$) to have a minimum of ~ -16 MeV at the saturation density of ~ 0.15 fm^{-3} [2]. We have obtained the behaviour of the *energy per particle* through numerical calculations using the NL3 values [1] of the parameters of the Lagrangian dennsity. Let us inspect it in FIG. 1:

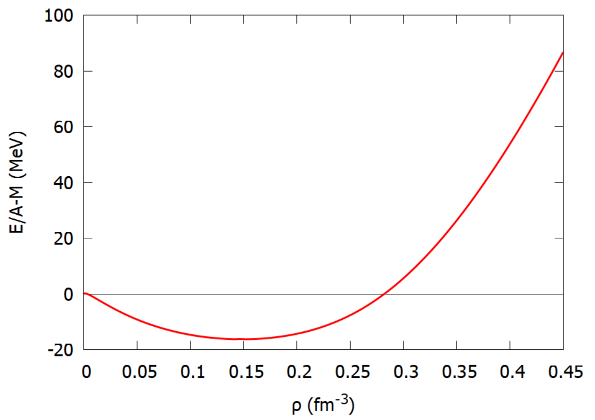

FIG. 1: Energy per particle as a function of the baryon density.

As we expected, there is a minimum around the said values. To calculate accurately this point, we need to search for the minimum of the *energy per particle*, $\left(\frac{\partial(\frac{\varepsilon}{\rho})}{\partial\rho}\right)_{\rho_0} = 0$. This derivative is equal to the *pressure* being 0. Developing this derivative, which is equal to (18), gives

$$p = \rho\left(\epsilon_F + g_V V_0\right) - \varepsilon_{SNM}. \quad (24)$$

Solving numerically the *mean-field equations* we can represent the *pressure* as a function of ρ in FIG. 2:

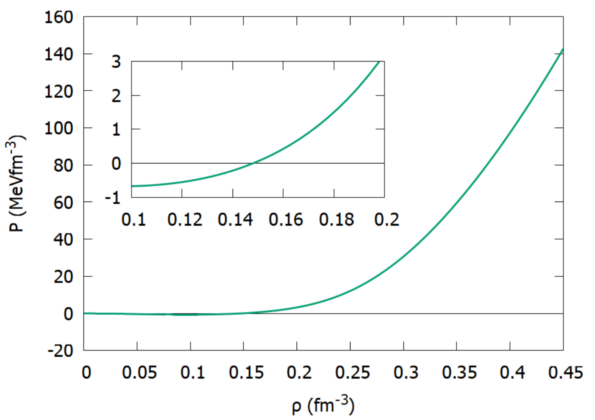

FIG. 2: Pressure as a function of the baryon density.

The pressure goes to 0 around the value of the density we expected. Both the *energy per particle* and *pressure* increase fast at high density. The behaviour at high density is important for the study of neutron stars and can be tuned with the free parameters of the Lagrangian density.

The exact value calculated for the saturation point is $\rho_0 = 0.148$ fm^{-3}, $E/A{-}M = -16.240$ MeV. For this density we have, $M^* = 558.806$, $M^*/M = 0.595$ MeV and $g_V V_0 = 307.935$ MeV. The symmetric matter is bound while the *energy per particle* is smaller than 0.

A notion of the behaviour at high density can be obtained through the *compression modulus* K. It defines the curvature of the *energy per particle* at saturation ρ_0:

$$K = 9\left[\rho^2 \frac{d^2}{d\rho^2}\left(\frac{\varepsilon_{SNM}}{\rho}\right)\right]_{\rho_0} = 272.527\,\text{MeV}. \quad (25)$$

IV. ASYMMETRIC NUCLEAR MATTER (ANM)

We no longer have the same density for protons and neutrons. We have now 2 different degrees of freedom for the baryons, and its *degeneracy* will be 2, only due to spin. Let us define the *asymmetry parameter*:

$$\delta = \frac{\rho_n - \rho_p}{\rho}, \quad \rho = \rho_n + \rho_p. \quad (26)$$

Writing the densities and *Fermi momenta* of the baryons in terms of the *asymmetry parameter* will allow us to study the behaviour of the *energy per particle* and *pressure* with the variation of the asymmetry of nuclear matter:

$$\rho_n = \rho\left(\frac{1+\delta}{2}\right) \quad (27)$$

$$\rho_p = \rho\left(\frac{1-\delta}{2}\right) \quad (28)$$

$$k_{F,n} = k_F\left(1+\delta\right)^{1/3} \quad (29)$$

$$k_{F,p} = k_F\left(1-\delta\right)^{1/3}. \quad (30)$$

The ρ meson is now important. We will have to add its contribution to the Lagrangian density, taking the form written in (3). Its field equation will determine its value for each asymmetry.

We need to add the free and interacting contributions of the ρ meson to the *energy density*:

$$\varepsilon_{ANM} = \varepsilon_{SNM} + \frac{1}{2} g_\rho R_{0,3}(\rho_p - \rho_n) - \frac{1}{2} m_\rho^2 R_{0,3}^2. \quad (31)$$

The term ε_{SNM} includes $\varepsilon_{0,p}$ and $\varepsilon_{0,n}$.

The expression of the *pressure* will also change due to the contribution of the ρ meson. Following the prescription given at (18), the expression for the *pressure* takes the following form:

$$p = \rho_n(\epsilon_{F,n} + g_V V_0 - \frac{1}{2} g_\rho R_{0,3}) + \rho_p(\epsilon_{F,p} + g_V V_0 + \frac{1}{2} g_\rho R_{0,3}) - \varepsilon_{ANM}. \quad (32)$$

In this case, we will not always have bound matter. Let us inspect the *energy per particle* for different δ in FIG. 3 and discuss its effect on the presence of bound matter.

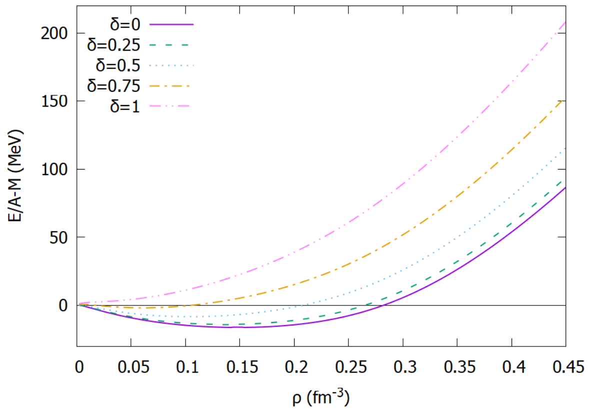

FIG. 3: Energy per particle as a function of the baryon density for different δ values, from symmetric nuclear matter ($\delta = 0$) to pure neutron nuclear matter ($\delta = 1$).

In this figure we can see the extreme cases for δ and some intermediate ones.

The $\delta = 0$ case corresponds to the symmetric nuclear matter, which we have just discussed in section III.

The $\delta = 1$ case corresponds to pure neutron matter, which works as a first approximation to the interior of a neutron star. We can see that the bound matter is found in smaller intervals of density for increasing δ till it is no longer bound. The *energy per particle* also grows with δ for a fixed density. The same happens with the pressure. It increases with δ for a fixed density. We computed the behaviour of the *pressure* with the density and the asymmetry in FIG. 4.

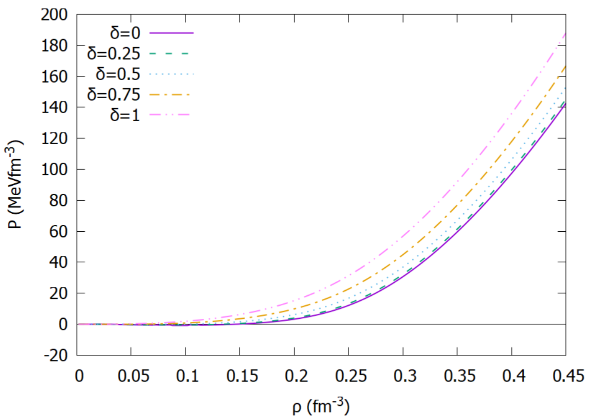

FIG. 4: Pressure as a function of the baryon density for different δ values, from symmetric nuclear matter ($\delta = 0$) to pure neutron nuclear matter ($\delta = 1$).

It is important to note that neutron matter needs something more apart from the strong interaction to stay bound, a gravitational pull. Neutron stars are stable because both nuclear forces and gravity are present.

V. β-STABLE NUCLEAR MATTER

Our last section will be devoted to the β equilibrium. Free neutrons are not stable. They decay to protons and electrons. In the densities found inside neutron stars, the inverse reaction can also take place:

$$n \longrightarrow p + e^- + \bar{\nu}_e \tag{33}$$
$$p \longrightarrow n + e^+ + \nu_e \tag{34}$$

We will impose two conditions for the β equilibrium: Charge neutrality:

$$\rho_p = \rho_e. \tag{35}$$

Chemical potentials satisfying:

$$\mu_n - \mu_p = \mu_e, \tag{36}$$

modeling the electrons as a relativistic Fermi sea and with

$$\mu_n = \epsilon_{F,n} + g_V V_0 - \frac{1}{2} g_\rho R_{0,3} \tag{37}$$
$$\mu_p = \epsilon_{F,p} + g_V V_0 + \frac{1}{2} g_\rho R_{0,3} \tag{38}$$
$$\mu_e = \sqrt{k_{F,e}^2 + m_e^2}. \tag{39}$$

For every baryon density, there will only be one possible value for the *asymmetry parameter* δ that satisfies these conditions. We can apply these conditions to our model and compute the correct value of δ for each density. The results are presented in FIG. 5:

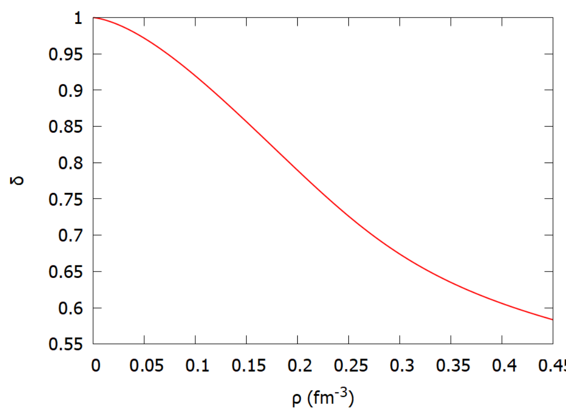

FIG. 5: Evolution of the *asymmetry parameter* δ with the baryon density to satisfy the β equilibrium.

We can see that the system tends to lower asymmetry with increasing density. The behaviour of $\delta(\rho)$ depends on the symmetry energy of the chosen parametrization [1].

The *energy density* will only have a new contribution from the electrons, $\varepsilon_{0,e}$. The *pressure* will have a new contribution of the form $\rho_e \mu_e - \varepsilon_{0,e}$. Let us see how it differs from the nucleon pressure (without the electron contribution) and from fixed δ. Once again through numerical calculations, we present the results in FIG. 6:

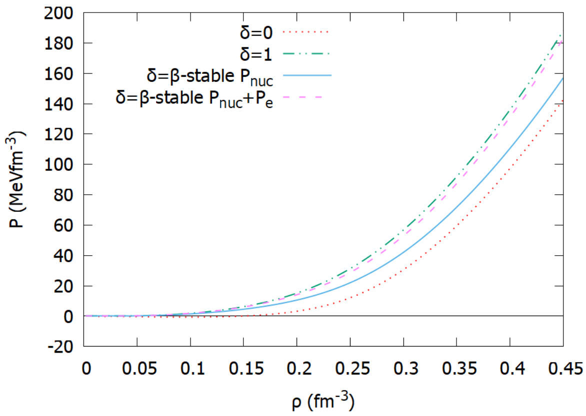

FIG. 6: Pressure for the β-stable matter with and without the electron contribution compared to symmetric nuclear matter ($\delta = 0$) and pure neutron matter ($\delta = 1$).

We see that the nucleon pressure is between the symmetric and neutron matter pressures and consistent with FIG. 4 and FIG. 5. The electron contribution to the *pressure* is large, increasing the pressure to almost the same as the pure neutron matter at high density. With some other parametrization, the nucleon-plus-electron *pressure* can even surpass the pure neutron matter pressure. This important contribution can be better understood looking at the evolution of the chemical potentials. We have calculated their evolution with the density, as can be seen in FIG. 7:

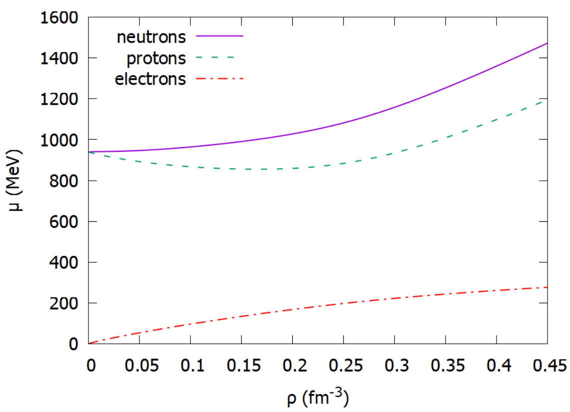

FIG. 7: Evolution of the chemical potentials with the density.

Electrons end up with a chemical potential larger than 200 MeV at $\rho = 0.45$ fm^{-3}, more than 400 times their rest mass. Their contribution to the pressure is proportional to its chemical potential, explaining why it is relevant.

VI. CONCLUSIONS

The RMF theory proves to be able to reproduce some central known values for symmetric nuclear matter. Symmetric nuclear matter ($\delta = 0$) is self-bound and has a saturation or equilibrium point at $\rho = 0.148$ fm^{-3}. Neutron matter ($\delta = 1$) is unbound at all densities. The inclusion of electrons has a significant effect on the *pressure*.

To get closer to a real neutron star one could include other leptons such as muons to the β equilibrium. The inclusion of hyperons would be also a good generalization of the project. Also, one could apply other parametrizations like the one presented at [6]. Finally, a natural continuation of the project would be to apply the calculated *equation of state* to study the mass-radius relation of neutron stars [2].

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[1] G.A. Lalazissis, J. König and P. Ring, *New parametrization for the Lagrangian density of relativistic mean field theory*, Phys. Rev. C **55**, 540 (1997)
[2] N.K. Glendenning, *Compact Stars: Nuclear Physics, Particle Physics, and General Relativity*, Springer, 2000
[3] B.D. Serot and J. D. Walecka, Advances in Nuclear Physics **16**, 1 (1986)
[4] J.D. Walecka, *Theoretical Nuclear and Subnuclear Physics*, Oxford University Press, 2004
[5] B.D. Serot, *Building Atomic Nuclei with the Dirac Equation*, International Journal of Modern Physics A19, 107 (2004)
[6] B.G. Todd-Rutel and J. Piekarewicz, *Neutron-Rich Nuclei and Neutron Stars: A New Accurately Calibrated Interaction for the study of Neutron-Rich Matter*, Phys. Rev. Lett. **95**, 122501 (2005)