EXACT SOLUTIONS IN SCALAR FIELD COSMOLOGY

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Abstract: The super-potential method is used to obtain exact solutions for scalar fields interacting with gravity. Exponential and hyperbolic potentials are studied and their cosmological behaviour is analysed. Universes exhibiting diverse expansion and contraction phenomena arise from this models and are characterised.

I. INTRODUCTION

Cosmology has successfully addressed some of the most challenging questions posed by nature, yet many key unknowns of our Universe remain unresolved [1, 2]. Recently, there has been an increasing interest in formulating models that employ scalar fields as actors within diverse contexts. The discovery of the Higgs Boson and the relevance of the inflation field have greatly influenced the popularity of such models; and other theoretical scalar particles, like the *axion* or the *dilaton*, have gained recognition. What is interesting to us is that working with scalar fields grants the opportunity to obtain exact solutions using the super-potential method.

The super-potential method is a standard approach to obtain exact solutions using Hamilton-Jacobi theory with scalar fields, as extensively described in [3–5]. This method allows us to investigate various potentials for a set of scalar fields ϕ^i interacting with gravity and analyse the cosmological behaviour when the potential leads to an exactly solvable problem. In brief, given a Lagrangian of the form

$$L = \frac{1}{2} G_{ij} \partial_t \phi^i \partial_t \phi^j - V(\phi^i), \tag{1}$$

if the potential fulfills

$$V = -\frac{1}{2} G^{ij} \frac{\partial W}{\partial \phi^i} \frac{\partial W}{\partial \phi^j} \tag{2}$$

the solutions of the first-order equations of motion also solve the second-order equations, and they satisfy

$$\partial_t \phi^i = \epsilon G^{ij} \frac{\partial W}{\partial \phi^j} \tag{3}$$

(with $\epsilon = \pm 1$). So, based on this, we can skillfully select a super-potential W that leads to first-order exactly solvable equations and a physically interesting potential. Our text will be guided by the work of J. G. Russo and P. K. Townsend in [6–9].

We will consider a flat ($k = 0$) Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology:

$$ds^2 = -e^{2\alpha\varphi} f^2 d\tau^2 + e^{2\beta\varphi} (dx_1^2 + ... + dx_{d-1}^2) \tag{4}$$

where

$$\alpha = (d-1)\beta \qquad \beta = \frac{1}{\sqrt{2(d-1)(d-2)}} \tag{5}$$

are normalisation constants in d dimensions, and

$$\varphi = \varphi(\tau) \qquad f = f(\tau) \qquad \phi = \phi(\tau) \tag{6}$$

in order to maintain isotropy. Here we shall study the $d = 4$ case. The function $f(\tau)$ allows us to make an alternative choice for the time coordinate; unless stated otherwise, our choice will be $f = e^{-\alpha\varphi}$, which corresponds to the usual cosmological time t. It should be noted that $\varphi(t)$ is a function that dictates the behaviour of the scale factor in the FLRW metric, expressed as $a(t) = e^{\beta\varphi}$; and it can be related to the Hubble parameter since $H(t) = \dot{a}(t)/a(t) = \beta\dot{\varphi}$.

A general expression for the Lagrangian of a set of gravity coupled scalar fields can be written using the proper choice of units $\kappa^2 = 8\pi G = 1/2$ [1] as

$$L = \frac{1}{2}\sqrt{-|g|}\left(2R - g^{\mu\nu} G_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j - 2V(\phi)\right), \tag{7}$$

which for the FLRW metric, reduces to the effective Lagrangian [6]

$$L_{\text{eff}} = \frac{1}{2f}\left(-\dot{\varphi}^2 + G_{ij}\dot{\phi}^i\dot{\phi}^j\right) - f e^{2\alpha\varphi} V(\phi). \tag{8}$$

A potential fulfilling (2) will be of the form

$$2e^{2\alpha\varphi} V(\phi) = \left(\partial_\varphi W\right)^2 - G^{ij} \partial_i W \partial_j W, \tag{9}$$

and if we choose a super-potential $W = e^{\alpha\varphi} F(\phi)$, we will be able to express the potential solely in terms of the scalar fields:

$$V = \frac{1}{2}\left(\alpha^2 F^2 - G^{ij} \partial_i F \partial_j F\right). \tag{10}$$

As stated, the equations of motion that arise from (8) when Euler-Lagrange theory is used will generally be of second-order and difficult to solve exactly. As an example, the equations derived from (8) for a single field have been widely studied and can be written in terms of the scalar field and the Hubble parameter $H(t)$

$$\begin{cases} \ddot{\phi} + 3\dot{\phi}H + \frac{\partial V(\phi)}{\partial \phi} = 0 \\ H^2 = \frac{1}{6}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \end{cases} . \tag{11}$$

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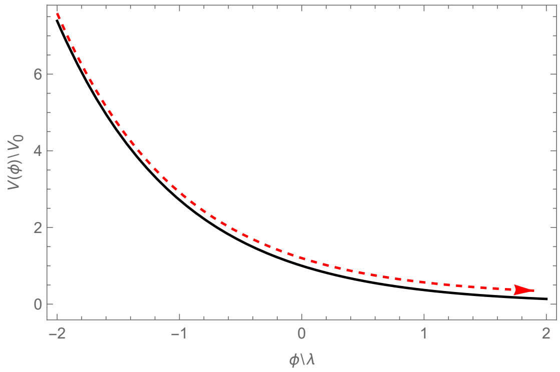

FIG. 1: Potential generated by function $F(\phi) = -c\exp(-\lambda\phi/2)$. It is shown for $0 < \lambda < \sqrt{3}$. The red-dashed line shows the evolution of the system with time.

The method grants us an opportunity to find simple solutions of the problem, since we know that if (10) is satisfied, then (3) will yield a solution of the second-order equations. The expression of this solutions in terms of $F(\phi)$ will come utmost useful:

$$\dot{\varphi} = -\alpha f e^{\alpha\varphi} F \qquad \dot{\phi}_i = f e^{\alpha\varphi} G^{ij}\partial_j F \tag{12}$$

II. SINGLE SCALAR FIELD

The most simple case we can analyse is a Lagrangian comprising a single scalar field contribution. This leads to two coupled differential equations governing the evolution of $\varphi(t)$ and $\phi(t)$. Of particular interest for analysis is the equation of state $p = \omega\rho$, where p represents the pressure and ρ the energy density. These variables are derived from the energy-momentum tensor $T_{\mu\nu}$ of a perfect fluid [1], and can be expressed as

$$\rho = \frac{\dot{\varphi}^2}{2} \qquad p = \frac{\dot{\varphi}^2}{2} - 2V(\phi), \tag{13}$$

therefore, ω can be written as

$$\omega = 1 - \frac{4V}{\alpha^2 F^2} = -1 + \frac{8}{3}\frac{(\partial_\phi F)^2}{F^2}. \tag{14}$$

A. EXPONENTIAL POTENTIAL

We can verify the efficacy of the super-potential method using it in a extensively studied scenario, such as the exponential potential. Choosing a function of the form $F(\phi) = -c\exp(-\lambda\phi/2)$ for the usual cosmological time coordinate ($f = e^{-\alpha\varphi}$), we obtain the potential

$$V(\phi) = \frac{c^2}{8}\left(3 - \lambda^2\right) e^{-\lambda\phi} \tag{15}$$

(shown in Figure 1). The parameters are chosen so $c > 0$ and $\lambda > 0$ yield an expanding solution for $t > 0$. The differential equations obtained from (12) are then

$$\dot{\phi} = \frac{c\lambda}{2} e^{-\lambda\phi/2} \qquad \dot{\varphi} = \alpha c e^{-\lambda\phi/2}, \tag{16}$$

which can be solved into

$$\phi(t) = \frac{2}{\lambda}\ln\left(\frac{\lambda^2 c}{4}t\right) \qquad a(t) = \left(\frac{\lambda c}{2}t\right)^{\frac{1}{\lambda^2}}. \tag{17}$$

We can verify these results with [7], and we learn this coincides with the late-time attractor solution. The comparison can be done using the parameter $V_0 = c^2\left(3 - \lambda^2\right)/8$ and $t_0^2 V_0 \lambda^2 = 2\left(3/\lambda^2 - 1\right)$. It is required that $\lambda < \sqrt{3}$ so the potential is positive.

To analyse the evolution of this cosmology, we will calculate $H(t)$ and $\ddot{a}(t)/a(t)$:

$$H(t) = \frac{1}{\lambda^2 t} \qquad \frac{\ddot{a}(t)}{a(t)} = \frac{1}{\lambda^4 t^2}\left(1 - \lambda^2\right). \tag{18}$$

We notice that $\dot{a}(t)$ is positive definite, indicating a expanding universe; and $\ddot{a}(t)$ corresponds to an accelerated expansion for $\lambda < 1$. A similar analysis can be carried out by examining the behaviour of ω, where values of $\omega < -1/3$ correspond to a positive $\ddot{a}$ [1]. In this case, ω maintains a constant value $\omega = -1 + 2\lambda^2/3$, which meets the constraint $\lambda < 1$ for an accelerated expansion.

A way to verify our approach is to find *de Sitter* phases, where the potential acts as a cosmological constant and the Hubble parameter remains constant. For example, if we choose $\lambda = 0$, solving (16) again yields a constant potential $V_0 = 3c^2/8$ and a vanishing kinetic energy since $\dot{\phi} = 0$. Such a case provides a solution with $H = c/4$.

B. HYPERBOLIC POTENTIALS: *cosh*(x)

Having established the utility of the method, we will seek for $F(\phi)$ functions that have the potential to result in interesting cosmologies. We will consider the specific case where $F(\phi) = -b\cosh(g\phi)$, and we will choose $b > 0$ and $|g| < \alpha$. This choice is made so the solution exhibits expansion for $t > 0$. The potential is then

$$V(\phi) = \frac{b^2}{2}\left(\alpha^2\cosh^2(g\phi) - g^2\sinh^2(g\phi)\right). \tag{19}$$

Figure 2 illustrates that the potential exhibits a well-like shape that does not vanish at its minimum. Again, the differential equations system given by (12) is

$$\dot{\phi} = -bg\sinh(g\phi) \qquad \dot{\varphi} = \alpha b\cosh(g\phi) \tag{20}$$

and can be solved into

$$\begin{aligned} \phi(t) &= \frac{1}{g}\ln\left[\coth\left(\frac{bg^2 t}{2}\right)\right] \\ a(t) &= \left[2\sinh\left(bg^2 t\right)\right]^{\frac{1}{4g^2}}. \end{aligned} \tag{21}$$

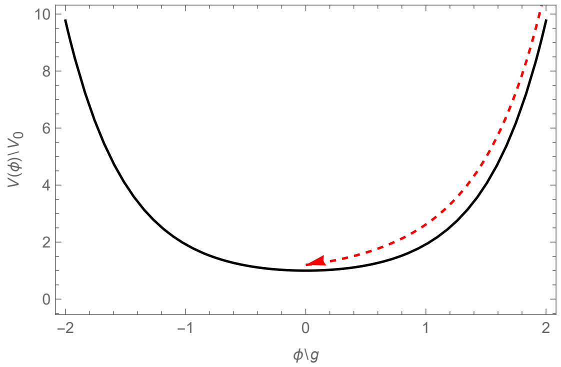

FIG. 2: Potential generated by function $F(\phi) = -b\cosh(g\phi)$. We assume $|g| < \alpha$ and $b > 0$. The red-dashed line shows the evolution of the system with time.

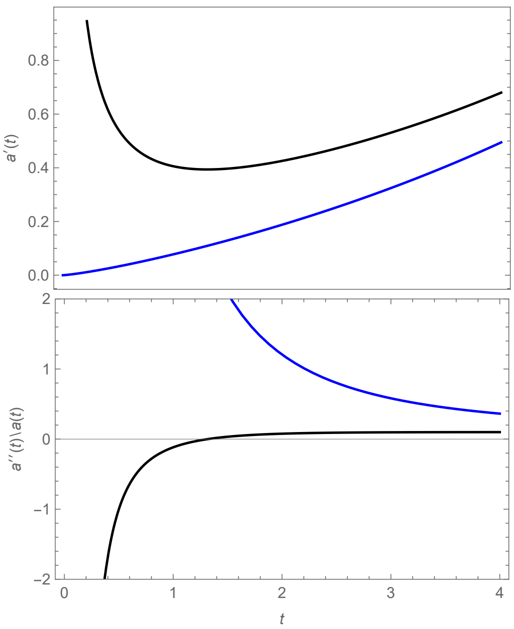

FIG. 3: $\dot{a}(t)$ (above) and $\ddot{a}(t)/a(t)$ (below) as a function of time. Solutions for $g^2 < 1/4$ (blue) and $g^2 > 1/4$ (black) are shown.

Notice that $\phi(t)$ only exists for $t > 0$ and is positive definite. Computing the derivatives $\dot{a}(t)$ and $\ddot{a}(t)$ we obtain

$$\begin{aligned} \dot{a}(t) &= \frac{b}{4}\coth(bg^2t)\left[2\sinh\left(bg^2t\right)\right]^{\frac{1}{4g^2}} \\ \frac{\ddot{a}(t)}{a(t)} &= \frac{b^2}{16}\left[1 + (1-4g^2)\mathrm{csch}^2(bg^2t)\right]. \end{aligned} \tag{22}$$

Let us analyse these results and unmask the underlying physics. As stated, the solution exists for the normal flow of time $t > 0$ for any value of $|g| < \alpha$. As shown in Figure 3, $\dot{a}(t)$ is positive and indicates an expanding universe. Under time reversal change of variables $\tilde{t} \equiv -t$, $\phi(t)$ exists only for $t < 0$, when $a(t)$ is real and positive with the same constraint on g. Again, $\dot{a}(\tilde{t})$ is positive definite, so the same expanding solution is found. The acceleration parameter exhibits two distinct behaviours (see Figure 3) for values of g above and below $g^2 = 1/4$:

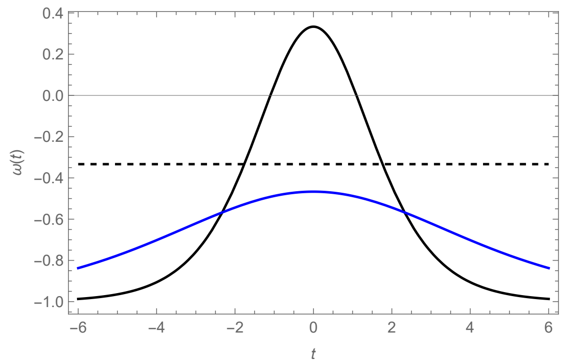

FIG. 4: $\omega(t)$ as a function of time. Solution for $g^2 < 1/4$ (blue) and $g^2 > 1/4$ (black) are shown. The dashed line represents the limit $\omega = -1/3$, above which acceleration is negative.

- For $g^2 \leq 1/4$ the acceleration is positive definite, resulting in a monotonically expanding universe. A slow early expansion is described. For $g^2 = 1/4$, the acceleration remains constant and positive.
- For $g^2 > 1/4$ the acceleration parameter exhibits an early deceleration phase with duration

$$\tau_{\mathrm{dec}} = \frac{\mathrm{arccosh}(2g)}{bg^2} \tag{23}$$

followed by an accelerated expansion for $t > \tau_{dec}$. In this case, $\dot{a}(t)$ presents a rapid expansion at early times and a minimum at $t = \tau_{dec}$.

Let us calculate the Hubble parameter and the equation of state for this specific potential:

$$\begin{aligned} H(t) &= \frac{b}{4}\coth(bg^2t) \\ \omega &= -1 + \frac{8}{3}g^2\,\mathrm{sech}^2(bg^2t). \end{aligned} \tag{24}$$

It is evident that the behaviour of $H(t)$ will be that of $\dot{a}(t)$. Given the expression of ω (see Figure 4) we can further verify some of the deductions made above:

- For $g^2 < 1/4$, $\omega < -1/3$ is fulfilled for all t and the acceleration is positive, as already stated.
- For $g^2 > 1/4$, the threshold $\omega < -1/3$ is surpassed and a period of deceleration is found. Equaling $\omega = -1/3$ in (24) and solving for t the same duration (23) is obtained.

In the limit $t \to \infty$, $\phi \to 0$, the system rolls down to the potential's minimum value $V_0 = 3b^2/8$ and stays there. The kinetic energy at the minimum of the well vanishes, therefore the potential acts as a cosmological constant and we encounter a *de Sitter* phase characterised by $\Lambda = V_0$. The Hubble parameter converges to a constant value $H = b/4$. For the case $g = 0$, not previously discussed, the potential is constant and a *de Sitter* universe emerges.

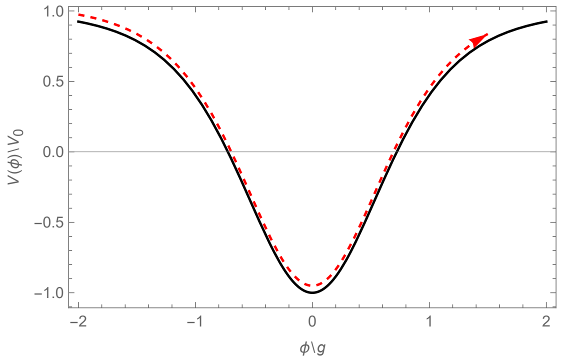

FIG. 5: Potential generated by $F(\phi) = b\,\tanh(g\phi)$. The red-dashed line shows the evolution of the system with time.

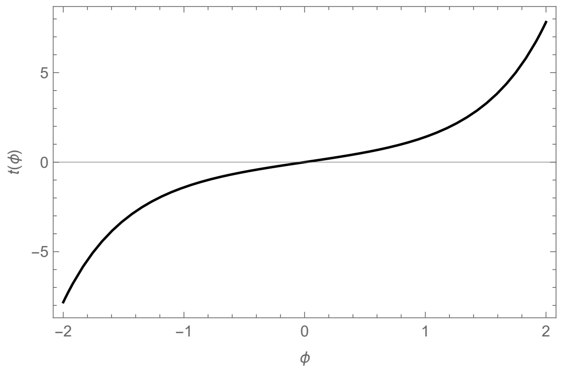

FIG. 6: t as a function of $\tilde{t} \equiv \phi$.

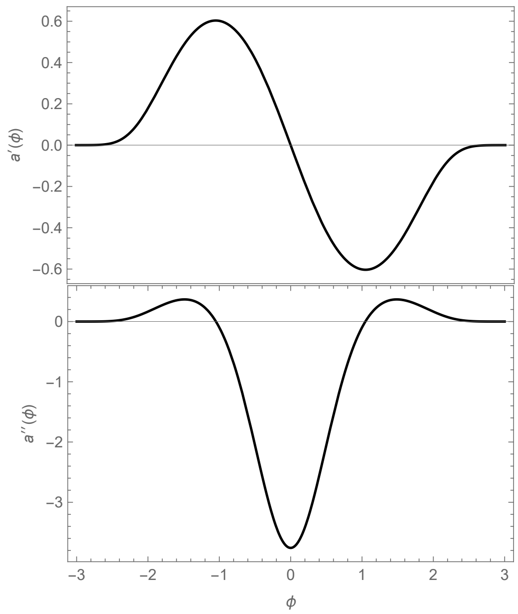

FIG. 7: $\dot{a}(\tilde{t})$ and $\ddot{a}(\tilde{t})$ as a function of $\tilde{t} \equiv \phi$. The representation is made using $g = 1$.

C. HYPERBOLIC POTENTIALS: *tanh*(x)

Consider the case where a function of the form $F(\phi) = b\,\tanh(g\phi)$ results in the potential

$$V(\phi) = \frac{b^2}{2}\left(\alpha^2 \tanh^2(g\phi) - g^2 \mathrm{sech}^4(g\phi)\right). \tag{25}$$

Note that the potential is bound from below but is not positive definite (see Figure 5). This may result in a model with negative energy phases, and it will be of interest to analyse its cosmological implications. The differential equations obtained from (12) are now

$$\dot{\phi} = bg\,\mathrm{sech}^2(g\phi) \qquad \dot{\varphi} = -\alpha b\,\tanh(g\phi), \tag{26}$$

and they are not so easily solved since the solution for $\phi(t)$ is not invertible:

$$2bgt = \frac{\sinh(2g\phi)}{2g} + \phi. \tag{27}$$

For the choice $g > 0$, $b > 0$, the dependence $t(\phi)$ is illustrated in Figure 6. $t(\phi)$ is a monotonic function and it exhibits the following limits:

$$\phi \to -\infty \Longrightarrow t \to -\infty \qquad \phi \to +\infty \Longrightarrow t \to +\infty$$
$$\phi = 0 \Longrightarrow t = 0;$$

therefore we can use $\tilde{t} \equiv \phi$ as a time variable (so the time variable is no longer the usual cosmological time, as for the other cases). Now $\frac{\partial\varphi}{\partial t} = \dot{\phi}\frac{\partial\varphi}{\partial\phi}$ results in a differential equation for $\varphi(\phi)$ that can be solved, yielding the scale factor solution

$$a(\phi) = exp\left[-\frac{\cosh(2g\phi)}{16g^2}\right]. \tag{28}$$

As usual, we will now proceed to calculate meaningful cosmological parameters and interpret their behaviour. With $\phi = \phi(t)$, we obtain

$$\begin{aligned} H(\phi) &= -\frac{b}{4}\tanh(g\phi) \\ \frac{\ddot{a}(\phi)}{a(\phi)} &= \frac{b^2}{16}\left(\tanh^2(g\phi) - 4g^2\mathrm{sech}^4(g\phi)\right) \\ \omega &= -1 + \frac{32}{3}g^2\,\mathrm{csch}^2(2g\phi). \end{aligned} \tag{29}$$

Firstly, the Hubble parameter yields an expanding universe for $-\infty < \tilde{t} < 0$ that stops at $\tilde{t} = 0$ and transitions into a contracting phase for $0 < \tilde{t} < \infty$. On the other hand, the acceleration parameter displays four distinct periods, which can be easily distinguished in Figure 7:

- An initial period of accelerated expansion for $-\infty < \tilde{t} < -\phi_0$
- A period of deceleration that stops the expansion, for $-\phi_0 < \tilde{t} < 0$
- Followed by a period of accelerating contraction for $0 < \tilde{t} < \phi_0$
- And a final period of decelerating contraction for $\phi_0 < \tilde{t} < \infty$

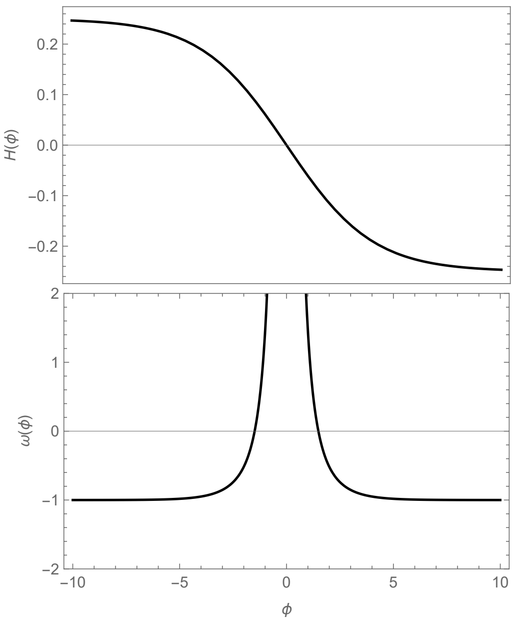

FIG. 8: $H(\tilde{t})$ and $\omega(\tilde{t})$ as a function of $\tilde{t} \equiv \phi$. The representation is made using $g = 1/4$. Notice that ω diverges at $\phi = 0$.

where

$$\phi_0 = \frac{1}{2g}\text{arcsinh}(4g). \tag{30}$$

Through the analysis of the equation of state we can identify periods of acceleration for values of ϕ that meet $\omega < -1/3$, resulting in the same discussion made above and the result (30). However, it is important to keep in mind that the equation of state is typically bounded to $|\omega| \leq 1$ [2, 10], and yet the expression of ω in (29) is not bounded from above (see Figure 8). This behaviour is anticipated in (14) for potentials that are not positive definite. The values of ϕ that violate this restriction are

$$|\phi| < \frac{1}{2g}\text{arcsinh}\left(\frac{4g}{\sqrt{3}}\right) \tag{31}$$

which, as expected, is the period at which the potential is negative.

The expression of $\dot{\phi}$ in (26) is positive definite. Hence one would expect the system rolling down the potential from $-\infty$ and up again to ∞. Its expression also shows that $\dot{\phi}$ vanishes at $\pm\infty$, and since the potential is constant and not negligible at this limits, we find *de Sitter* phases characterised by $\Lambda = 3b^2/8$ and $H = \mp b/4$.

III. CONCLUSIONS

We have successfully used the super-potentials method, initially verifying our ability to utilise it applying it to the exponential potential case, which resulted in the expected outcome. In the case of the potential derived from a $cosh(g\phi)$ function, we have successfully characterised an expanding solution exhibiting two distinct behaviours: a monotonically expanding universe, and a universe with an initial phase of deceleration.

For the potential (25), we encounter a universe with an initial expansion and a following contraction. The expansion is accelerated up to a turning point at $\tilde{t} = -\phi_0$, where it starts to slow down and eventually comes to a stop. Similarly, the contraction undergoes initial acceleration, and at $\tilde{t} = \phi_0$ it starts to decelerate. This universe goes through a phase of negative potential energy density, which is theorized in some Early Universe models.

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