Gravitational lensing on a binary mass lens: caustics and critical lines

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Abstract: LIGO and Virgo interferometers have been able to detect gravitational waves resulting of the energy loss experienced by two compact objects orbiting around their common center of mass. When encountering massive objects, these gravitational waves can experience gravitational lensing while propagating through the Universe. This could lead to distortion effects encoded inside the detected signal, which means that one would have to consider these effects in order to obtain correct results. In this work, we want to study the gravitational wave lensing by a binary system, setting the bases for a future research of this effect near caustics and its application in gravitational wave detection.

I. INTRODUCTION

Gravitational waves (GWs) are disturbances in spacetime caused by accelerated massive objects. Although they have been an object of study since Albert Einstein's General Theory of Relativity, interferometers have only recently been able to detect them, due to their extremely small effect. The first direct observation of GWs was made on September 14, 2015 [1]. Since then, 90 events have been detected and a new observing run has recently started.

What is important in our study is the fact that GWs can be affected by the geometry of spacetime. As the GW propagates through the Universe, the masses along its way change the curvature of spacetime, which results in the wave getting distorted. This phenomenon is called gravitational lensing [2]. Our interest lies in the case where the lenses are two point masses: the binary lens.

Although gravitational lensing by GWs is now getting more attention, most studies are dedicated to the gravitational lensing of light. This has become an active field in astronomy [3, 4]. For example, it allows us to calculate the Hubble constant using the time delay between multiple images of a quasar, once the lens mass has been obtained from the location and shape of the images. What makes gravitational lensing of light so interesting is its effects, such as magnification or multiple imaging. Observing these effects gives us information about the cosmological parameters and the large scale structure distribution in the Universe. It also allows us to measure the amount of matter in the Universe and its spatial distribution at different scales. Gravitational lensing of light is also used to search for compact dark objects and as a way to get some insight into the dark matter problem [3, 4]. We expect that gravitational lensing of GWs could be useful to detect non-emitting objects that curve the path of the waves, causing some of the effects mentioned above. There is also a possibility that the matched filtering technique used to retrieve the characteristic parameters of the objects involved in the merger, which does not take into account gravitational lensing, could be leading to slightly mistaken results.

In this work, we will first discuss the basic concepts concerning gravitational lensing in Sec. II. In Sec. III, we will talk about the magnification effect experienced by a gravitationally lensed source and the directions along which this magnification diverges: the caustics. Then, we will briefly introduce the lens equation of the binary lens model and the possible outcomes of its resolution in Sec. IV. In Sec. V, we will talk about the numerical method we have used to obtain the magnification of the source and the lens plane, and the steps we have followed to program it. In Sec. VI, we will explain how we have obtained the transmission factor and its use in future work. Finally, in Sec. VII, we will review the most important aspects of the work and we will discuss the future of this project.

II. GRAVITATIONAL LENSING

An important parameter in gravitational lensing is the Einstein angle. When the source, the lens and the observer are aligned within approximately the Einstein angle $\theta_E = R_E/D_L$, we expect the lensing effects to be significant. Here, the Einstein radius R_E is given by [2]

$$R_E = \sqrt{2R_S \frac{D_L D_{LS}}{D_S}},$$

where R_S is the Schwarzschild radius of the lens, D_L, D_{LS} and D_S are the distances between the observer and the lens, the lens and the source, and the observer and the source, respectively. The Einstein radius will be useful in order to work with dimensionless expressions.

The gravitational deflection of waves can be described by a mapping from the lens plane to the source plane. In the absence of a lens, an observer would see the source at a position $\vec{\beta}$ (in angular units). In contrast, when the lens deflects the wave, the observer sees the source at a position $\vec{\theta}$. Given that the source's image position originates in the lens plane, we will often refer to it as the image plane. The equation that gives us the relation between the source's true position and its observed position is called the *lens equation* [5]-[8]:

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}).$$

For simplicity, we will be using the dimensionless form of the above equation,

$$\vec{y} = \vec{x} - \vec{\alpha}(\vec{x}), \tag{1}$$

where $\vec{x} = \vec{\theta}/\theta_E$ and $\vec{y} = \vec{\beta}/\theta_E$.

The propagation of electromagnetic and GWs when the effect of lensing on polarization is negligible, can be described by a scalar wave equation [9]. This allows us to treat the GWs as scalars for the wave equation, instead of treating them as tensors. We will also consider the thin less approximation, which says that if the size of the lens is small compared to the cosmological distances travelled by the wave, we can neglect the extension of the lens along the line of sight in the calculation of the wave's deflection. This occurs in the so-called lens plane [2].

The gravitational lensing of light has been extensively studied [3]-[5], [7]-[11]. We can take advantage of the developed formalism to use it for the gravitational lensing of GWs. However, we must take into account the differences between electromagnetic (EM) waves and gravitational waves. In the case of EM waves, we can use the geometrical optics (GO) limit, since the wavelength is small in all observational situations. Instead, the GO approximation is not always valid for GWs due to their large wavelengths. Moreover, GWs propagate through matter with nearly no absorption, which means that we receive them only distorted by the gravitational lensing, while electromagnetic waves experience a large amount of absorption by galaxies and dust, so plenty of information is lost along the way. Because of that, the wave optic effects for GWs are more substantial than those for EM waves.

III. CAUSTICS

Caustics can be encountered on a daily basis. Whether it is in a glass of water or on the surface of a swimming pool (Fig. 1), everyone has seen these bright curves despite not knowing their actual name. In those cases, water is acting as a magnifying glass, creating these bright patterns because of the refraction and reflection taking place on the water waves. An analogous thing happens with gravitational waves. To have a better understanding of these curves, we will introduce some formulas.

FIG. 1: Caustics on a swimming pool.

The Jacobian of the mapping between the source and the lens plane is given by [5, 8]

$$J(\vec{x}) = \frac{\partial \vec{y}}{\partial \vec{x}}.$$

The source will experience magnification by the effect of gravitational lensing. This is due to the area distortion of the lens mapping, which is given by the inverse of the Jacobian. Then, for an image at $\vec{x}$, the magnification is [5]

$$\mu(\vec{x}) = \frac{1}{\det J(\vec{x})}.$$

For a more rigorous explanation, caustics are the directions in the source plane along which the amplification diverges, i.e. $\det J(\vec{x}) = 0$ [3]. When a source crosses these curves, additional images appear or disappear. The images of the caustics are called critical lines. Caustics are of great interest to us because when additional images appear, they do so on the critical line, attaining infinite magnification (in practice, the magnification does not diverge due to diffraction). This magnification implies a distortion of the wave in the frequency domain, which ends up affecting what is being detected.

For the binary lens, the expected behaviour is the following [8]. When the separation between the two lenses is bigger than $(m_1^{1/2} + m_2^{1/2})^{3/2}$, there are two separate extended caustics. For smaller distances, but bigger than $(m_1^{1/3} + m_2^{1/3})^{-3/4}$, there exists one caustic with six cusps, which are points where the caustics are not smooth. For smaller separation, there are three caustics: two are triangularly shaped, and the other one resembles an astroid with four cusps. The behaviour explained above will be confirmed by numerical calculations in Sec. V.

IV. BINARY LENS

It is convenient to define an Einstein radius for a mass equal to the sum of the two point masses, $M = M_1 + M_2$. We can write the reduced deflection angle as [5]-[8]:

$$\vec{\alpha}(\vec{\theta}) = m_1 \frac{\theta_E^2}{\left|\vec{\theta} - \vec{\theta_1}\right|^2}(\vec{\theta} - \vec{\theta_1}) + m_2 \frac{\theta_E^2}{\left|\vec{\theta} - \vec{\theta_2}\right|^2}(\vec{\theta} - \vec{\theta_2}).$$

We have set $m_1 = M_1/M$ and $m_2 = M_2/M$ in the above equation. The dimensionless lens equation (1) for a binary system then reads

$$\vec{y} = \vec{x} - \frac{m_1}{\left|\vec{x} - \vec{x_1}\right|^2}(\vec{x} - \vec{x_1}) + \frac{m_2}{\left|\vec{x} - \vec{x_2}\right|^2}(\vec{x} - \vec{x_2}). \tag{2}$$

In what follows, to speed up numerical calculations we use the lens equation in terms of complex numbers [3, 12]

$$z_S = z - \frac{m_1}{z^* - z_1^*} - \frac{m_2}{z^* - z_2^*} \tag{3}$$

where $z = x_1 + ix_2$ is the position on the lens plane, $z_S = y_1 + iy_2$ is the position on the source plane, and z^* denotes the complex conjugate of z.

This lens equation can be reduced to a fifth-degree complex polynomial equation [3]. Solving this equation will lead to finding the multiple images of a source at z_S. The three possible cases are:

- A source outside the caustics has three images, two inside the critical lines, and one outside.
- A source on the caustic implies two additional images on the critical line. This means that the images have infinite magnification.
- A source inside the caustics has five images.

The latter case is shown in Fig. 2.

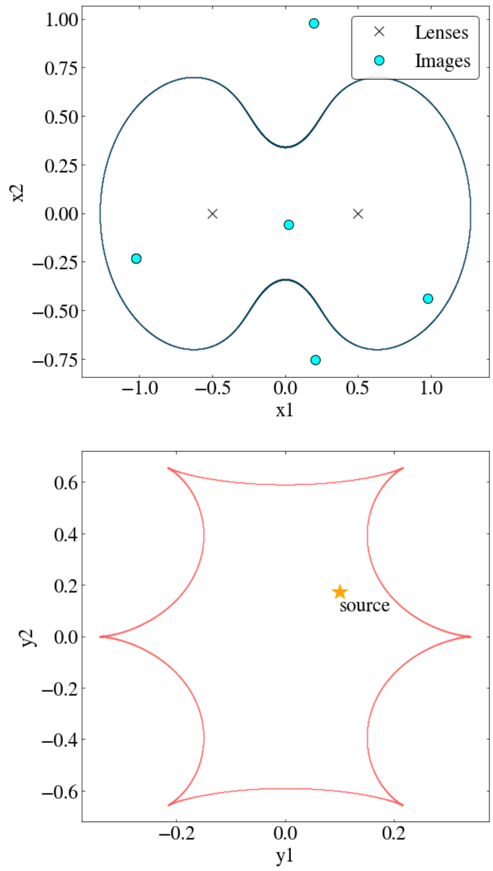

FIG. 2: Upper figure: representation of the critical lines on the image plane (continuous line), the two lenses (crosses) and the five images (blue circles). Bottom figure: plot of the caustics (red) and the source on the source plane. The lenses have a mass of $M = M_\odot$.

V. NUMERICAL METHOD

The method used to obtain the magnification maps of the source and the lens plane is called the inverse ray shooting method [7, 10, 11]. Using this method, one can map an area of the lens plane onto the source plane using the lens equation for arbitrary mass distribution. The rays sent from the lens plane are then collected in the source plane. In particular, we will obtain the magnification at a pixel in the source plane as the sum of rays that hit that pixel, up to a scaling factor. After that, we can obtain the magnification in the lens plane by tracing back the rays and assigning their pixel's magnification in the source plane to their corresponding pixel in the lens plane.

The code consists of the following steps. First of all, we create the image plane grid, a square-shape area divided into n_x^2 pixels. For this purpose, we define the image plane length (x_l) and the size of each pixel (x_s). The lens plane will then extend from $-x_l$ to x_l, in both of its directions. We then proceed to find the coordinates of the centre of each pixel.

It is also crucial to define the lenses that we are going to use. We must assign a mass to the first lens, define the mass ratio between the two lenses and choose the distance between them. For this study, we have chosen a binary system of two equal point mass lenses, and we have varied the distance between them to see its effects on the magnification, caustics and critical lines. Since the choice of the reference frame is arbitrary, we have chosen that the real axis passes through the two lenses, and have set the position of the second lens to $z_2 = -z_1$. That way, we must only define the distance between the two lenses instead of their respective positions in the image plane.

We can now send a ray backwards from the centre of each of the pixels using the lens equation (3) to calculate the deflection. This will give us the position in the source plane at which the rays arrive. At this point, we can define the source plane length (y_l) and the size of each pixel (y_s) in order to maximize the area at which rays hit. The next step is to determine the pixels corresponding to each of the coordinates in the source plane at which rays hit. This is the most demanding part of the algorithm. We can create the source plane magnification array (of the same size as the source plane array) in which we will save the values of the magnification at each pixel. In order to calculate this value, for each time a ray hits a specific pixel, we will sum 1 to the square representing this pixel in the source plane magnification array. In the end, we will have an $n_y \times n_y$ (number of pixels in the source plane) matrix with the value of the magnification at each pixel in the source plane.

We must take into account the scaling factor associated with the method: $f_{sc} = x_l^2/n_y^2 \cdot x_s^2$. The final value of the magnification is obtained by multiplying the magnification array by this scaling factor. It is useful to visualize this magnification in a density plot.

It is in our interest to also determine the magnification in the lens plane. To do so, we trace back the rays and assign the value of the magnification in the pixel at which they hit the source plane to the pixel from which we have sent the rays. An example of our simulation can be seen in Fig. 3.

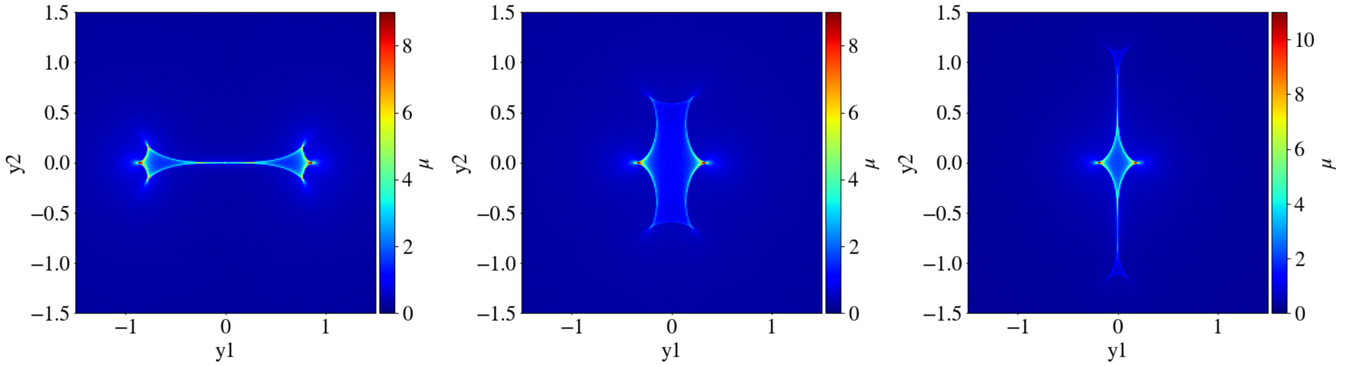

FIG. 3: Magnification of the source plane for a binary system with two masses of $M = 1M\odot$ and separation between the two masses of d=2, d=1 and d=0.7 (in units of the Einstein radius), from left to right. The magnification has been obtained by our code in Python. The caustics are in agreement with Ref. [8].

We can also see in Fig. 4 that using the suitable parameters, we can retrieve the expected result for a point mass lens: an Einstein ring.

VI. TRANSMISSION FACTOR

The transmission (or amplification) factor in the geometrical optics limit takes the form [13, 14]

$$F_{GO}(w, \vec{y}) = \sum_j |\mu_j|^{1/2} \exp\left[iwT_j - i\pi n_j\right], \tag{4}$$

where we sum over the j images and $n_j = 0,\ 1/2,\ 1$ when $\vec{x_j}$ is a minimum, saddle or maximum point of the time delay, respectively. We have used the dimensionless frequency w, defined by [14]

$$w = \frac{\omega}{c}\,(1 + z_L)\,\frac{D_L D_S}{D_{LS}}\,\theta_E^2, \tag{5}$$

where z_L is the redshift of the lens and ω is the GW frequency. The dimensionless time delay is given by

$$T(\vec{x}, \vec{y}) = \frac{1}{2}\,|\vec{x} - \vec{y}|^2 - \psi(\vec{x}) + \phi_m(\vec{y}), \tag{6}$$

where $\psi(\vec{x})$ is the dimensionless deflection potential and $\phi_m(\vec{y})$ is the arrival time for the ray with minimum delay. Accordingly, the minimum value of the time delay is zero. We can write the dimensionless deflection potential of a binary lens as [7]

$$\psi(\vec{x}) = m_1 \ln\left|\vec{x} - \frac{\vec{d}}{2}\right| + m_2 \ln\left|\vec{x} + \frac{\vec{d}}{2}\right|, \tag{7}$$

where d is the separation of the lenses in units of R_E.

We can see from Fig. 5 that using our algorithm and imposing a separation between the two masses of $d = 0$, i.e. imposing a point mass lens, we retrieve the expected results for a point mass lens [15].

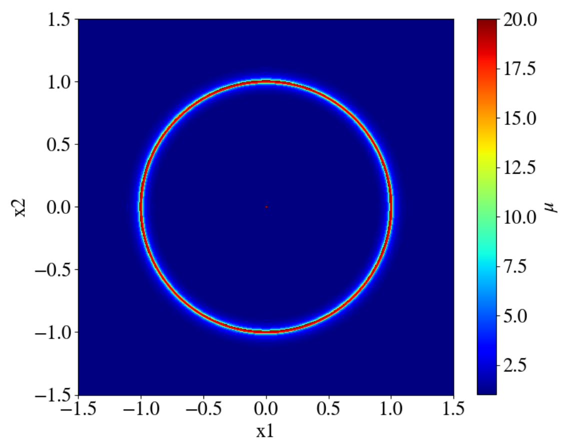

FIG. 4: Magnification on the lens plane for a point mass lens, using our algorithm with a distance between the two masses of zero (in units of the Einstein radius).

VII. CONCLUSIONS

We have studied the lensing properties of a binary mass system. We have created a program to calculate the magnification on the source and the lens plane, using the inverse ray shooting method (IRS). The program is able to calculate the magnification once we have specified one of the masses, the mass ratio and distance between the masses, and the size of the planes (specifying the number of pixels). It also plots the caustics and critical lines. The results agree with those in the bibliography. Moreover, we can reproduce the expected results for a point mass lens with the same algorithm.

Finally, we have created another program which calculates the transmission factor in the geometrical optics

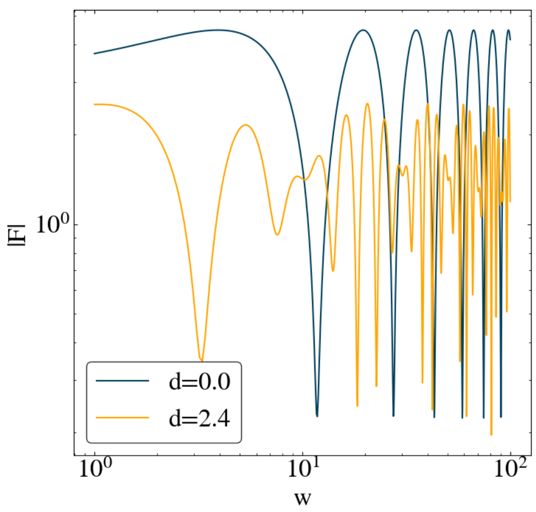

FIG. 5: Absolute value of the transmission factor for different frequencies, for a binary lens with separation d=2.4 and a point mass lens (binary lens with d=0) and the source position $y = \frac{1+\sqrt{3}i}{10}$.

limit. For that, we have needed to calculate the delay time, the dimensionless deflection potential, as well as knowing if each of the images of the source were a minimum, a maximum or a saddle point of the time delay. To validate the algorithm, we have studied the limit when the distance between the masses goes to zero, and we have obtained the expected results for the point mass lens, i.e., equally spaced frequency peaks for the transmission factor.

The main point is: wave effects are important when the wavelength of the gravitational wave is of the order of the Schwarschild radius. In this limit, neglecting them will induce an error in our results, since the GO approximation will be no longer valid.

The next step would be calculating the signal to noise ratio by Fourier transforming the product of an unlensed waveform and the transmission factor. Then, we could study what happens exactly near caustics and, since the geometrical optics limit stops being suitable in that case, to look for a better approximation in order to understand the real effect of the magnification there and use it as an astrophysical tool. I intend to continue with this research in my Master's thesis.

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