BLACK HOLE SHADOWS: SCHWARZSCHILD & REISSNER-NORDSTRÖM (anti-)dSITTER

Narcís Neher Mestre
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Catalonia, Spain.

Advisor: J. Russo

Abstract: In this work we discuss a precise derivation of shadows for spherically symmetric black holes. A brief overview of the historical Synge solution is also provided, along with a detailed explanation of the modern methodology that can be used with more complex metrics. The methodology will then be applied to some of the most significant black hole solutions. (Schwarzschild & Reissner-Nordström dSitter).

I. INTRODUCTION

Since the discovery of General Relativity, Black Holes (BH) have garnered considerable attention within the scientific community. These objects that are massive enough to curve space-time to the point where not even light can escape its gravitational pull, were initially regarded as purely abstract mathematical solutions, devoid of physical manifestation. Over time, however, the concept of their actual existence grew in credibility, culminating in the seminal work by Penrose, wherein he established, through his renowned paper, the inescapability of Black Holes within the framework of General Relativity [1].

The scientific community now generally accepts the existence of black holes, and research into them has made substantial progress as a result of advancements in measurement equipment (such as LIGO and the Event Horizon Telescope). Moreover, there has been a lot of expectation in the media as a result of recent publishing of the first image of a black hole [2]. This event has also shown that black holes may be studied not only through indirect methods, like gravitational waves or Doppler effect, but also through more direct methods like shadows, which can be useful to quantify the magnitudes from the black hole and even from space-time itself.

II. SYNGE'S SOLUTION

In 1965, Synge [3] published the first in-depth explanation of what is now known as a "Black Hole Shadow". Since they are the same entity, even though he spoke about *gravitationally intense stars*, his derivation is entirely valid. His starting point was the standard Schwarzschild in dimensionless spherical coordinates (i.e. $\rho \equiv r/2m$ and $\tau \equiv t/2m$), which is written as

$$ds^2 = 4m^2[(1-\rho^{-1})^{-1}d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) - (1-\rho^{-1})dt^2].$$

The spherical symmetry of the solution permits the establishment of a constant value for ϕ without sacrificing generality. By taking the second derivative with respect to an affine parameter for both sides of the metric equation and applying the condition that

$$\dot{s}^2 \equiv \left(\frac{ds}{d\lambda}\right)^2 = 0\,,$$

for a null geodesic, light rays trajectories must fulfill

$$(1-\rho^{-1})^{-1}\dot{\rho}^2 + \rho^2\dot{\theta}^2 - (1-\rho^{-1})\dot{t}^2 = 0\,. \tag{1}$$

Now, two constants of motion may be defined as $\rho^2\dot{\theta} = \alpha$ and $(1-\rho^{-1})\dot{t} = \alpha\beta$, associated with angular momentum and energy conservation. We find

$$\left(\frac{d\rho}{d\theta}\right)^2 = \rho^4 F(\beta,\rho) \tag{2}$$

$$F(\beta,\rho) = \beta^2 - (\rho-1)/\rho^3\,. \tag{3}$$

From (3) the forbidden region for the photons, where the condition $\beta^2 < (\rho-1)/\rho^3$ holds, can be obtained. Additionally, the LHS from (2) can be expressed in terms of the infinitesimal angle formed by the light ray trajectory and the line ($\theta = cte, \phi = cte$) that passes through the center of the black hole as follows (see **FIG.**1)

$$\cot\psi = \sqrt{\frac{g_{\rho\rho}}{g_{\theta\theta}}}\frac{d\rho}{d\theta}\,, \tag{4}$$

where $g_{\rho\rho}$ and $g_{\theta\theta}$ are the metric components that can be taken from (1).

With some trigonometric manipulation, (2) now reads as

$$\sin^2\psi = \frac{\rho-1}{\rho^3\beta^2}\,, \tag{5}$$

which holds at any point of the trajectory. The condition in β yields that $\beta^2_{max} = 4/27$ when $\rho = 3/2$.

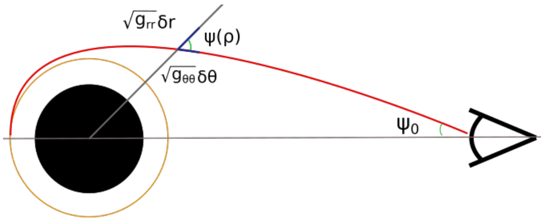

FIG. 1: In this illustration, the limiting light ray path is represented, as well as the definition of Synge's ψ angle. Light rays possessing a smaller angle would be absorbed by the Event Horizon, and the ones with greater angle would be deflected to infinity (which is the illuminated region the observer would see).

This will serve as a significant threshold that must be overcome by light rays originating from $\rho_0 > 3/2$. Taking the observer coordinates as the initial parameters together with these conditions, Synge equation is recovered

$$\sin^2 \chi = \frac{27}{4} \frac{\rho_0 - 1}{\rho_0^3} . \tag{6}$$

Although Synge applied this equation for escaping light rays on the surface of the "*gravitationally intense star*", which will escape if their angle fulfills $\psi_0 < \chi$, it can also be applied for computing the angular radius from the Black Hole Shadow seen by a stationary observer at ρ_0. Similarly, rays sent by the observer should fulfill $\psi_0 > \chi$ in order to be deflected and not captured, thus the shadow will have an angular radius of χ.

III. SPHERICALLY SYMMETRIC SOLUTIONS

A more efficient and comprehensive approach to tackle Synge's problem lies in resolving the generalized versions of spherically symmetric geometries and employing the Lagrange formalism to calculate the constants of motion. With this method, the first step is identifying the metric of the problem, which in this case, is

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + D(r)(d\theta^2 + \sin^2\theta d\phi^2) , \tag{7}$$

where $A(r)$,$B(r)$ and $D(r)$ are all positive definite. To obtain the conserved quantities, it is imperative to derive the Lagrangian associated with the system. In G.R, for the vacuum case without Electromagnetic interaction, this Lagrangian can be written as

$$\mathcal{L} := \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu .$$

Then, Euler-Lagrange equations impose that if a coordinate does not appear explicitly in the Lagrangian (i.e. $\partial_{x^\mu}\mathcal{L} = 0$) then it has an associated constant of motion which can be computed as

$$\mathcal{K}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} .$$

In the spherically symmetric Lagrangian there are two Killing fields, ∂_ϕ and ∂_t, which will give two constants of motion. In addition, the condition $\mathcal{L} = 0$ for photons is also taken into account.

From the general metric we can get these constants in terms of $A(r)$, $B(r)$ and $D(r)$:

$$\mathcal{K}_t \equiv -E = -A(r)\dot{t}$$
$$\mathcal{K}_\phi \equiv L_z = D(r)\dot{\phi} ,$$

which are usually redefined into a new constant b named *impact parameter* that is defined as

$$b := \frac{L_z}{E} = \frac{D(r)}{A(r)} \frac{\dot{\phi}}{\dot{t}} = \frac{D(r)}{A(r)} \frac{d\phi}{dt} .$$

Inserting these constants into the Lagrangian, the trajectory equation arises in its differential form

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{D(r)}{B(r)} \left(\frac{D(r)}{A(r)} \frac{1}{b^2} - 1\right) . \tag{8}$$

Although this equation can be integrated analytically, as in [4], which is the usual way to find shadows from more complex black holes, Synge's condition can also be found in terms of the metrics components. At this point, it is useful to compute the turning points from these trajectories, which is the same as finding forbidden regions from Synge equation (2). These points can be found with the condition

$$\left.\frac{dr}{d\phi}\right|_R = 0 ,$$

which implies the solution

$$b^2 = \frac{D(R)}{A(R)} . \tag{9}$$

Other solutions such as $D(r) = 0$ or $B(r) \to \infty$ are ruled out because they imply that the metric is singular outside the event horizon. Although it is also feasible to calculate these objects' shadows, it is not the primary aim of this work.

Now, it is useful to define the function $h(x)$ as

$$h^2(x) \equiv \frac{D(x)}{A(x)} \to b = h(R) . \tag{10}$$

Using (4) together with (8), the angular radius from the shadow will be

$$\cot^2 \psi = \frac{h(r)}{h(R)} - 1 ,$$

which holds at any point of the trajectory for the same reasons that (5) does. With some trigonometry, Synge's condition is recovered and it can be expressed in two forms

$$\sin^2\psi = \left(\frac{h(r_{ph})}{h(r_0)}\right)^2 \tag{11}$$

$$\sin^2\psi = \frac{b_{cr}^2 A(r_0)}{D(r_0)}\,, \tag{12}$$

where $b_{cr} \equiv h(r_{ph})$ and r_{ph} is the radial coordinate of the circular orbits of photons around the black hole (i.e. the photospheres) which can be found analogously as in Synge derivation. Firstly, the term that could lead to the RHS to be negative must be identified from (8). In this case, for spherically symmetric geometries this would be

$$F(r;b) = \frac{D(r)}{A(r)}\frac{1}{b^2} - 1\,,$$

where $F(r;b)$ is formally the same as the one in Synge's derivation and only depends on the coordinate r and could depend on a set of parameters (i.e. $F(r;b) \to F(r;b_1,b_2...)$). At this point, in order to find the allowed regions for the r coordinate, the condition $F(r;b) > 0$ must be fulfilled. This condition can be visualized by plotting

$$\frac{1}{b^2} = \frac{A(r)}{D(r)}\,. \tag{13}$$

This curve will define the turning points of photons (i.e. it sets $d\phi/dr$ to 0). Photons above the curve (13) are permitted, whilst those below it are prohibited. What is interesting is that *photon orbits* can be found at the extreme points of this curve, and they will be stable when it is a minimum and unstable when it is a maximum. Applying this condition

$$\left.\frac{d(b^{-2})}{dr}\right|_{r_{ph}} = \frac{d}{dr}\left.\left(\frac{A(r)}{D(r)}\right)\right|_{r_{ph}} = 0\,, \tag{14}$$

enables very efficient computation of Shadows from spherically symmetric Black Holes.

Moreover, with this derivation it is interesting to note that the equation for shadows of spherically symmetric metrics is invariant under conformal transformations (i.e. $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f(r)g_{\mu\nu}$), being $f(r)$ an arbitrary function of the radial coordinate only. This transformation implies that every metric component from (7) will be multiplied by this general function and then, it is straightforward to show that $f(r)$ cancels out from (12) and (14), therefore neither r_{ph} nor the shadow will change.

A. SCHWARZSCHILD METRIC

In Schwarzschild solution, the metric's components are

$$A(r) = 1 - 1/r = B(r)^{-1}\,, \qquad D(r) = r^2\,.$$

r_{ph} can be computed using (14)

$$\frac{d}{dr}\left.\left(\frac{1-1/r}{r^2}\right)\right|_{r_{ph}} = 0 \;\to r_{ph} = 3/2\,.$$

Expression (6) can now be recovered using (12).

B. REISSNER-NORDSTRÖM METRIC

The Reissner-Nordström (RN) solution describes charged black holes with the following metric components

$$A(r) = 1 - 1/r + \beta^2/r^2 = B(r)^{-1}\,, \qquad D(r) = r^2$$
$$(2M)^2\beta^2 \equiv q_e^2 + q_m^2\,,$$

where M is the mass of the black hole (which is added to make β dimensionless), and (q_e, q_m) are the electric and the magnetic charge respectively. The quadratic dependence on β reflects the charge invariance of the metric. This is expected because negative and positive charge will affect equally the energy (i.e. the mass) of the black hole.

Using (14), r_{ph} can be computed as

$$\begin{gathered}\frac{d}{dr}\left.\left(\frac{1-1/r+\beta^2/r^2}{r^2}\right)\right|_{r_{ph}} = 0 \;\to \\ r_{ph\pm} = \frac{3}{4}\left(1 \pm \sqrt{1 - \frac{32}{9}\beta^2}\right)\,.\end{gathered} \tag{15}$$

In the RN BH, an inner and outer horizons arise and their location can be found by solving:

$$\begin{gathered}A(r_h) = 1 - 1/r_h + \beta^2/r_h^2 = 0 \to \\ r_{h\pm} = \frac{1}{2}\left(1 \pm \sqrt{1-4\beta^2}\right)\,.\end{gathered} \tag{16}$$

This equation gives an upper bound to the charge of the Black Hole which is $\beta^2 < 1/4$. It can be proved that $r_{ph+} > r_{h+} > r_{ph-} > r_{h-}$ for any physically acceptable value of β, which means that only the r_{ph+} photosphere coordinate will be relevant for the shadow.
Again, using (12), r_{ph} can be computed

$$\sin^2\psi_{RN} = \frac{(\rho_0^2 - \rho_0^2 + \beta^2)(3+\sqrt{9-32\beta^2})^4}{32\rho_0^4(3 - 8\beta^2 + \sqrt{9-32\beta^2})}\,. \tag{17}$$

Indeed, for $\beta = 0$ the Schwarzschild solution is recovered.

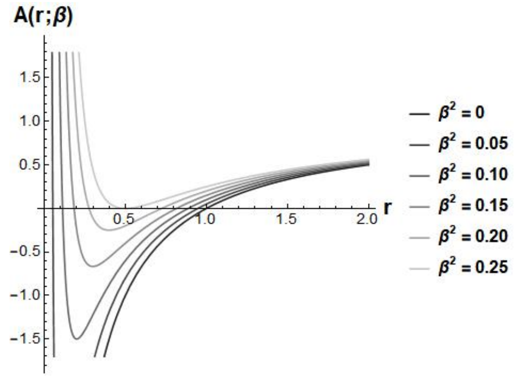

FIG. 2: This picture illustrates how the horizons coordinates (zeroes of the A(r) component) change with the charge β

C. REISSNER-NORDSTRÖM-KOTTLER (RNK)

In this section, charged BH in Universes with a non-vanishing cosmological constant are studied. In the former case, the components of the metric will be

$$A(r) = 1 - 1/r + \beta^2/r^2 - \Lambda r^2 = B(r)^{-1}\,, \qquad D(r) = r^2\,,$$

where $\Lambda \equiv \frac{(2M)^2}{3}\tilde{\Lambda}$ is defined as a dimensionless Cosmological constant. A brief discussion without taking into account the Black Hole charge can be found in [5].

In the above RNK solution, the number of horizons depends on the values of the pair of parameters $\{\beta,\Lambda\}$. Only three types of solutions will arise, which are presented in **FIG**3. It is useful to examine these three solutions in terms of the critical values of this parameters.

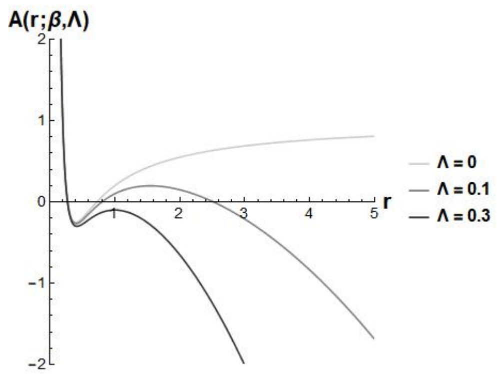

FIG. 3: In this figure $\beta^2 = 0.2$ and the different types of solutions one can obtain are displayed

While for $\Lambda > 0$ a cosmological horizon, r_c, is formed, for $\Lambda \leq 0$ there is no cosmological horizon. In addition, the solutions might have the inner and the outer horizons of the charged BH (RN solution), when the parameters are within certain critical values. These values correspond to the ones which cause the polynomial equation, $A(r)r^2 = 0$, to vanish. This leads to

$$\Lambda(256\beta^6\,\Lambda^2 + (128\beta^4 - 144\beta^2 + 27)\Lambda + 16\beta^2 - 4) = 0\,,$$

therefore,

$$\Lambda_c^{\pm} = \frac{-27 + 16\beta^2(9 - 8\beta^2) \pm \sqrt{(9 - 32\beta^2)^3}}{512\beta^6}\,. \tag{18}$$

BH's will then only exist in the interval

$$\Lambda_c^{-}(\beta^2) < \Lambda_{\rm OBS} < \Lambda_c^{+}(\beta^2)\,. \tag{19}$$

The different regions are illustrated in **FIG.**4.

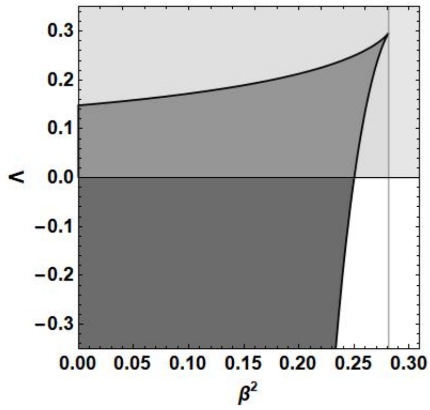

FIG. 4: The real and positive *zeroes* of $A(r)$ in a (Λ,β) diagram. White (no *zeroes*), light gray (one *zero*), dark gray (three *zeroes*) and darker gray (two *zeroes*).

In this figure, the darker region is limited by the critical curves Λ_c^+ on the top and Λ_c^- on the right (represented as black lines) while the gray vertical line corresponds to $(\beta_{RNK}^{max})^2 = 9/32$ which is the maximum charge that a RNK BH can have. The function with two real positive roots (in **FIG.**3) corresponds to the darkest region (RN-like BH) and the one that has three roots to the dark gray region (RN with cosmological horizon). Finally, if it only has one root a BH can not be formed.

Because of the quadratic dependence on r of the Cosmological Constant term in $A(r)$, the expression of the radial coordinate of the photosphere will be the same as in Reissner-Nordström B.H. (15). It can also be shown that r_{ph-} will be always inside the event horizon, whereas r_{ph+} will be located between the event horizon and the cosmological horizon. The expression of the shadow for RNK B.H. can now be obtained using r_{ph+} instead of r_{ph-}:

$$\sin^2\psi_{RNK} = \frac{G(\beta^2)^4 P(\rho_0;\beta^2,\Lambda)}{-128\beta^4\Lambda - G(\beta^2)(-4+27\Lambda) + 8\beta^2(-4 + D(\beta^2)\Lambda)}\,, \tag{20}$$

where $G(\beta^2) \equiv 3 + \sqrt{9 - 32\beta^2}$, $D(\beta^2) \equiv 6(3 + G(\beta^2))$ and $P(\rho_0; \beta^2, \Lambda) \equiv \frac{\beta^2 + \rho_0(-1 + \rho_0 - \Lambda\rho_0^3)}{8\rho_0^4}$. Solutions for RN and Schwarzschild can be recovered by taking the limits $\Lambda \to 0$ and $\Lambda, \beta \to 0$ respectively.

It is interesting to see that, positive values of Λ allow charges of BH to be greater than $\beta^2_{RNK} > \frac{1}{4}$. For negative and small $|\Lambda|$ one recovers the R.N condition of $\beta^2 < 1/4$ and, for more negative values of Λ, the maximum value permitted for β^2 gets monotonically smaller.

D. SHADOW AT LARGE DISTANCES

From (11)

$$\sin^2\psi = \left(\frac{\beta^2}{r_0^4} - \frac{1}{r_0^3} + \frac{1}{r_0^2} - \Lambda\right) h^2(r_{ph}) \,. \tag{21}$$

If the limit for $r_0 >> 1$ (i.e. r>>m) without crossing the cosmological horizon is now taken, the expression obtained is

$$\sin^2\psi = \left(\frac{\beta^2}{r_0^4} - \Lambda\right) h^2(r_{ph}) = \left(\frac{1}{r_0^2} - \frac{\tilde{\Lambda}}{3}\right) h^2(\tilde{r}_{ph}) \,, \tag{22}$$

where in the second equality, the original coordinates (i.e. with dimensions) are restored. This expression agrees with equation (39) in [6].

It is useful to derive a formula for the angular size of the shadow as seen by a comoving observer. Using the aberration formula given in [6] (equations (23) & (34)) and taking into account the identification $f(r) \to B(r)$, it can be applied to equation (21) and compute the limit for a very distant observer

$$\lim_{r_0\to\infty} \sin^2\psi_{Comov} = \frac{1}{1 + 1/(\Lambda\, h^2(r_{ph}))} \,,$$

which is the same equation as (38) in [6].

IV. CONCLUSIONS AND FURTHER RESEARCH

This paper presents a comprehensive analysis of the Shadows cast by spherically symmetric Black Holes. Section {III} establishes a general approach for calculating Shadows in spherically symmetric geometries, which is subsequently applied to investigate the properties of the most general stationary and spherically symmetric Black Holes, namely the Reissner-Nordström (anti-)dSitter metric. The methodology employed in this study demonstrates the conformal invariance of the shadow associated with any spherically symmetric solution. In section {III B} upper and lower boundaries for the pair of parameters $\{\beta, \Lambda\}$ are presented and discussed. The determination of the maximum charge achievable by a Black Hole could lead, in theory, to the value of the Cosmological constant. However, due to the small predicted values of this constant in modern theories, its impact on the results is deemed negligible. Finally, the dependence of the Shadow in $\{\beta, \Lambda\}$ is also found analytically in (22). To our knowledge, the derivation of shadows for RNK metric remains unexplored within the existing body of literature.

Further research can be conducted to explore the properties inherent in the shadows of alternative type of Black Holes, such as axially symmetric or time-dependent mass configurations. By scrutinizing the shadow, one could derive intrinsic characteristics of the object.

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