UNIVERSITAT DE BARCELONA

ADVANCED MATHEMATICS
MASTER'S FINAL PROJECT

On the Gromov-Hausdorff distance between compact spaces

Author:
Rafael Martínez Vergara

Supervisor:
Carles Casacuberta Vergés

Facultat de Matemàtiques i Informàtica

June 28, 2023

Contents

Summary

This work provides an introduction to the Gromov-Hausdorff distance, discussing its original definition and its relationship with correspondences between spaces. We prove that the Gromov-Hausdorff distance serves as a metric for the set of isometry classes of compact metric spaces. The primary objectives of this study are to establish the existence of a pseudo-metric on the disjoint union $X \sqcup Y$ that achieves the Gromov-Hausdorff distance between compact spaces X and Y, and to establish bounds for the Gromov-Hausdorff distance between spheres of different dimensions.

2020 Mathematics Subject Classification: 51F30, 53C23, 55N31, 55P91.

1. Introduction and background

When exploring Topological Data Analysis, the concept of Gromov-Hausdorff distance arises as a means of quantifying dissimilarity between data sets. Within this context, the Gromov-Hausdorff distance is primarily studied in relation to finite sets, as it is specifically designed for analyzing such datasets. However, an intriguing question arises when we shift our focus to infinite sets: What does it happen when we extend our consideration to compact metric spaces?

The Gromov-Hausdorff distance between metric spaces X and Y, denoted by $d_{GH}(X,Y)$, quantifies the extent to which X and Y fail to be isometric. The Gromov-Hausdorff distance is used in many areas of geometry [BBI01, CC97, Col96, Pet06]. In applications to shape and data comparison/classification, one aims to estimate either the Gromov-Hausdorff distance between spaces [Mém07, MS04, MS05] or the Gromov-Wasserstein distance [AMJ18], which is one of its optimal transport induced variants.

However, both distances are hard to compute, both analytically and algorithmically [AFN$^+$18, Mém12]. Despite the interest in this type of distances, the exact values of Gromov-Hausdorff distance between standard compact spaces such as spheres are known in only a small number of cases.

In Section 2, following [BBI01] and [KO99], we introduce the definition of Gromov-Haudorff distance between pseudo-metric spaces using isometric embeddings and the Hausdorff distance.

Definition 1.1. Let A, B be pseudo-metric spaces. The *Gromov-Hausdorff distance* between A and B, denoted by $d_{GH}(A,B)$, is the infimum of all $\varepsilon \geq 0$ so that there is a pseudo-metric space M and isometric embeddings $i_A\colon A \to M$ and $i_B\colon B \to M$ such that $d_M(i_A(A), i_B(B)) \leq \varepsilon$, where d_M denotes Hausdorff distance in M.

Then we prove that we can actually restrict ourselves to pseudo-metrics on the disjoint union of A and B.

We introduce correspondences between sets and the concept of distortion of a correspondence in order to prove that the Gromov-Hausdorff distance can be computed using them.

Theorem 1.2. For any two pseudo-metric spaces X and Y,

$$d_{GH}(X,Y) = \frac{1}{2}\inf_{C}\{\text{dis(C)}\},$$

where the infimum is taken over all correspondences C between X and Y.

We prove that the Gromov-Hausdorff distance is nonnegative, symmetric, and it satisfies the triangle inequality. Moreover, we have that $d_{GH}(X,Y) = 0$ if and only if X and Y are isometric.

Theorem 1.3. The set of isometry classes of compact metric spaces endowed with the Gromov-Hausdorff distance is a metric space.

We introduce the concept of length spaces from [BBI01], which are spaces where one can define a class of admissible paths. A particular case of a length space is a Riemmanian manifold endowed with the geodesic distance. We can study the Gromov-Hausdorff distance between spaces with the following result.

Proposition 1.4. Every compact length space can be obtained as a Gromov-Hausdorff limit of finite graphs.

This theorem, which is proved in Section 2, gives a way for computing an approximation of the Gromov-Hausdorff distance between compact length spaces.

In Section 3, following [ŠTZ92, ŠTZ93], we study the structure of the metric space of metrics on a given set. We focus on the case where the given space is a complete and compact metric space. Then, following [IIT16], we study the set of closed relations and the subset of closed correspondences, which turns out to be a compact set. We prove that the distortion function is a continuous function. Hence we obtain the following result.

Theorem 1.5. For any two compact metric spaces X and Y there exists a correspondence R such that $d_{GH}(X,Y) = \frac{1}{2}\text{dis}(R)$.

This allows us to infer that there exists a pseudo-metric ρ on the disjoint union $X \sqcup Y$ associated with the correspondence R where the infimum is actually achieved. Proving this fact was the main motivation of the present work.

In Section 4, following [KO99] and [Kal95], we introduce a generalization of the Gromov-Hausdorff distance to Banach spaces. This is useful to work with l^p spaces, which, for $p > 2$, are Banach spaces but not Hilbert spaces. We prove that this generalization is only useful when comparing real Banach spaces, because there are real-isomorphic Banach spaces which are not complex-isomorphic.

In Section 5, following [LMS21] and [ABC$^+$22], we focus on the case of estimating Gromov-Hausdorff distances between spheres of different dimensions. We relate Gromov-Hausdorff distance, Borsuk-Ulam theorems, and Vietoris-Rips complexes as follows. Estimating the Gromov-Haudorff distance $d_{GH}(X,Y)$ for metric spaces X and Y involves bounding the distortion of a function $f\colon X \to Y$, which measures the extent to which f fails to preserve distances; the more functions between X and Y distort the metrics, the larger $d_{GH}(X,Y)$ must be. When X and Y are spheres, [ABC$^+$22] show that it suffices to consider odd functions. We transform an odd function $f\colon \mathbb{S}^k \to \mathbb{S}^n$ into a continuous odd map between Vietoris-Rips complexes. Then we obstruct the existence of such maps with the $\mathbb{Z}/2$ equivariant topology of Vietoris-Rips complexes, measured via the following quantity.

Definition 1.6. For $k \geq n$, we define

$$c_{n,k} = \inf\{r \geq 0 \mid \text{ there exists an odd map } \mathbb{S}^k \to VR(\mathbb{S}^n; r)\}.$$

Due to a theorem of Hausmann [H$^+$95], there is a homotopy equivalence $VR(\mathbb{S}^n; r) \simeq \mathbb{S}^n$ for sufficiently small r, and moreover there is an odd map $f\colon VR(\mathbb{S}^n; r) \to \mathbb{S}^n$. The Borsuk-Ulam theorem then implies that no odd map $\mathbb{S}^k \to VR(\mathbb{S}^n; r)$ exists for such r unless $k \leq n$. In particular, $c_{n,n} = 0$. Therefore, the quantity $c_{n,k}$ represents the amount by which $\mathbb{S}^n$ needs to be "thickened" until it admits an odd map from $\mathbb{S}^k$.

The main results in Section 5 are bounds for the Gromov-Hausdorff distance between spheres.

Theorem 1.7. For all $k \geq n$, the following inequalities hold:

$$2 \cdot d_{GH}(\mathbb{S}^n, \mathbb{S}^k) \geq \inf\{\operatorname{dis}(f) \mid f\colon \mathbb{S}^k \to \mathbb{S}^n \text{ is odd}\} \geq c_{n,k}.$$

Theorem 1.8. For every $n \geq 1$, we have that $d_{GH}(\mathbb{S}^n, \mathbb{S}^{n+1}) \leq \pi/3$.

2. Gromov-Hausdorff distance between compact metric spaces

In this section we focus on the study of the Gromov-Hausdorff distance on compact metric spaces following [KO99].

We recall the notion of Gromov-Hausdorff distance between metric spaces. It will be convenient to define it, more generally, for pseudo-metric spaces.

Definition 2.1. A *pseudo-metric* on a set M is a map $d\colon M \times M \to [0, \infty)$ which is symmetric, satisfies the triangle law and the condition $d(x, x) = 0$ for all $x \in M$. We say that (M, d) is a *pseudo-metric space*.

Note that a pseudo-metric d does not necessarily satisfy the condition that $d(x, y) = 0 \implies x = y$. If we have two metric spaces $(X, d_X), (Y, d_Y)$ and a relation $R \subset X \times Y$, we define the *distortion* of R by

$$\operatorname{dis}(R) = \sup_{(x,y),(x',y') \in R} |d_X(x, x') - d_Y(y, y')|.$$

This notion can be applied to arbitrary maps g between metric spaces X and Y. Thus we denote the distortion of g by $\operatorname{dis}(g) := \operatorname{dis}(R_g)$ where R_g denotes the fact that a map g is also a relation. In this case we have

$$\operatorname{dis}(g) = \sup_{x,x' \in X} |d_X(x, x') - d_Y(g(x), g(x'))|.$$

Given two maps $g\colon X \to Y$ and $h\colon Y \to X$ between metric spaces, the *codistortion* of g and h is defined as

$$\operatorname{codis}(g, h) = \sup_{x \in X, y \in Y} |d_X(x, h(y)) - d_Y(g(x), y)|.$$

The codistortion allows one to bound the extent to which the maps g and h fail to be inverses of each other. Indeed, if $\operatorname{codis}(g, h) < \varepsilon$, then one has $d_X(x, h(g(x))) < \varepsilon$ for all x and $d_Y(g(h(y)), y) < \varepsilon$ for all y.

We need to introduce the Hausdorff distance between subsets of a metric space. Let (M, d) be a metric space, and let $S \subset M$ be a subset. We define the distance of a point $x \in M$ to the set S by

$$d(x, S) = \inf_{s \in S} d(x, s).$$

We denote by $U_r(S)$ the r-neighborhood of the set S, i.e., the set of points $x \in M$ such that $d(x, S) < r$.

Definition 2.2. Let (M, d) be a metric space and $A, B \subset M$. We define the *Hausdorff distance* between A and B by

$$d_H(A, B) = \inf\{r > 0 : A \subset U_r(B) \text{ and } B \subset U_r(A)\}.$$

We can specify the metric d by using the notation $d_H(A, B, d)$. If it is clear which metric we are using, then we use the notation $d_M(A, B)$.

The next proposition from [BBI01] contains reformulations of the Hausdorff distance.

Proposition 2.3. Let (M, d) be a metric space. Suppose given $A, B \subset M$ and $r > 0$. Then

(1) $d_H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$.
(2) $d_H(A, B) \leq r$ if and only if $d(a, B) \leq r$ for all $a \in A$ and $d(b, A) \leq r$ for all $b \in B$.

Proof. Both statements follow directly from the definition. Details are given in [BBI01, 7.3.2]. □

Proposition 2.4. Let (M, d) be a metric space. Then

(1) d_H is a pseudo-metric on the set of all subsets of M.
(2) $d_H(A, \overline{A}) = 0$ for any $A \subset M$, where $\overline{A}$ is the closure of A.
(3) If $A, B \subset M$ are closed and $d_H(A, B) = 0$, then $A = B$.

Proof. 1. Since the Hausdorff distance is an infimum of positive numbers, we have that d_H is non-negative. From the definition we can swap A and B and the definition remains unchanged, so, d_H is symmetric. The triangle inequality follows from the fact that for A, B, C subsets of M, using Proposition 2.3, we have that

$$d(a, C) \leq d(a, b) + d(b, C) \leq d(a, b) + d_H(B, C),$$

for all $b \in B$ and $a \in A$. Taking the infimum over the elements in B we obtain

$$d(a, C) \leq d(a, B) + d_H(B, C) \leq d_H(A, B) + d_H(B, C).$$

Taking the supremum over all elements $a \in A$ we have that

$$\sup_{a \in A} d(a, C) \leq d_H(A, B) + d_H(B, C).$$

With the same arguments we can prove that

$$\sup_{c \in C} d(A, c) \leq d_H(A, B) + d_H(B, C).$$

Hence we have proved that

$$d_H(A, C) = \max \left\{ \sup_{a \in A} d(a, C), \sup_{c \in C} d(A, c) \right\} \leq d_H(A, B) + d_H(B, C).$$

2. Since $A \subset \overline{A}$ we have that $d(a, \overline{A}) = 0$ for all $a \in A$. Also by the definition of the closure of the set A we have that $d(a, A) = 0$ for all $a \in \overline{A}$. Hence $d(A, \overline{A}) = 0$.
3. Suppose that there exists $x \in A \setminus B$. Since B is closed, there exists an $r > 0$ such that $B_r(x)$ does not intersect with B. Then $x \notin U_r(B)$, and hence $d_H(A, B) \geq r > 0$. □

In particular, Proposition 2.4 proves that the set of closed subsets of M equipped with the Hausdorff distance is a metric space.

Definition 2.5. Let A, B be pseudo-metric spaces. The *Gromov-Hausdorff distance* between A and B, denoted by $d_{GH}(A, B)$, is the infimum of all $\varepsilon \geq 0$ susch that there is a pseudo-metric space M and there are isometric embeddings $i_A \colon A \to M$ and $i_B \colon B \to M$ such that $d_M(i_A(A), i_B(B)) \leq \varepsilon$.

We denote by $\mathcal{F}(A, B)$ the set of all pairs (ϕ, ψ) of maps $\phi \colon A \to B$ and $\psi \colon B \to A$. We define $G(\phi, \psi)$ for $(\phi, \psi) \in \mathcal{F}(A, B)$ as the union of the graphs of ϕ and ψ. We define $D(\phi, \psi)$ as the supremum of all quantities

$$\frac{1}{2} |d_A(a_1, a_2) - d_B(b_1, b_2)|$$

such that $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $(a_i, b_i) \in G(\phi, \psi)$ for $i = 1, 2$.

Theorem 2.6. Let A and B be bounded metric spaces. Then

$$d_{GH}(A, B) = \inf_{(\phi, \psi) \in \mathcal{F}(A, B)} D(\phi, \psi).$$

Proof. We first suppose that M is a pseudo-metric space and that A, B are isometrically embedded in M. We denote by d_A the metric induced by the isometric embedding in $i_A(A)$, and similarly for $i_B(B)$. Let ε be the Hausdorff distance between $i_A(A)$ and $i_B(B)$ in M. If $\sigma > \varepsilon$ then we can define $\phi\colon i_A(A) \to i_B(B)$ such that $d_M(a, \phi(a)) < \sigma$ and $\psi\colon i_B(B) \to i_A(A)$ such that $d_M(\psi(b), b) < \sigma$ for all $a \in i_A(A)$ and $b \in i_B(B)$. Hence we have that, for $(a_i, b_i) \in G(\phi, \psi)$,

$$|d_A(a_1, a_2) - d_B(b_1, b_2))| \leq d_M(a_1, \psi(b_3)) + d_M(b_1, \phi(a_3))) < 2\sigma$$

with $a_2 = \psi(b_3)$ and $b_2 = \phi(a_3)$. We can take $\sigma \to \varepsilon$, so we have that

$$\inf_{(\phi,\psi)\in\mathcal{F}(A,B)} D(\phi, \psi) \leq d_{GH}(A, B).$$

Now suppose given $(\phi, \psi) \in \mathcal{F}(A, B)$, and let $\sigma = D(\phi, \psi)$. We let $M := A \bigsqcup B$ and define a pseudo-metric d_M as follows. We let d_M coincide with d_A on $i_A(A)$ and with d_B on $i_B(B)$. If $a \in i_A(A)$ and $b \in i_B(B)$ then

$$d_M(a, b) := \inf_{(a',b')\in G(\phi,\psi)} (d_A(a, a') + d_B(b', b)) + \sigma.$$

We have to check that d_M is a pseudo-metric on M. The symmetry property follows from the definition and, since d_M restricted to A and B is a metric, it satisfies that $d_M(m, m) = 0$ for all $m \in M$. Hence there only remains to check the triangle law.

Let $a_1, a_2 \in i_A(A)$ and $b \in i_B(B)$. Suppose that $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in G(\phi, \psi)$. Then we have that

$$d_B(\beta_1, b) + d_B(\beta_2, b) \geq d_B(\beta_1, \beta_2) \geq d_A(\alpha_1, \alpha_2) - 2\sigma.$$

Hence we have that

$$d_A(a_1, \alpha_1) + d_B(\beta_1, b) + d_A(\alpha_2, a_2) + d_B(\beta_2, b) \geq d_A(a_1, a_2) - 2\sigma.$$

Thus we have that

$$d_A(a_1, a_2) \leq d_A(a_1, \alpha_1) + d_B(\beta_1, b) + \sigma + d_A(\alpha_2, a_2) + d_B(\beta_2, b) + \sigma.$$

Since $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in G(\phi, \psi)$ are arbitrary, we obtain that

$$\begin{aligned} d_A(a_1, a_2) &\leq \inf_{(\alpha_1,\beta_1)\in G(\phi,\psi)} (d_A(a_1, \alpha_1) + d_B(\beta_1, b)) + \sigma \\ &+ \inf_{(\alpha_2,\beta_2)\in G(\phi,\psi)} (d_A(\alpha_2, a_2) + d_B(\beta_2, b)) + \sigma \\ &= d_M(a_1, b) + d_M(b, a_2). \end{aligned}$$

So we have that $d_M(a_1, a_2) \leq d_M(a_1, b) + d_M(b, a_2)$. We prove that $d_M(a_1, b) \leq d_A(a_1, a_2) + d_M(a_2, b)$ by contradiction, so assume the opposite inequality. Then

$$\begin{aligned} d_M(a_1, b) &> d_A(a_1, a_2) + \inf_{(\alpha,\beta)\in G(\phi,\psi)} d_A(a_2, \alpha) + d_B(\beta, b) + \sigma \\ &\geq \inf_{(\alpha,\beta)\in G(\phi,\psi)} d_A(a_1, \alpha) + d_B(\beta, b) + \sigma = d_M(a_1, b), \end{aligned}$$

which is clearly a contradiction.

We can argue symmetrically with A, B interchanged and obtain that d_M is a pseudo-metric. With this definition of d_M it is clear that $d_M(a, \phi(a)) \leq \sigma$ and $d_M(\psi(b), b) \leq \sigma$. This shows that $d_{GH}(A, B) \leq \sigma = D(\phi, \psi)$, and we obtain that

$$d_{GH}(A, B) \leq \inf_{(\phi,\psi)\in\mathcal{F}(A,B)} D(\phi, \psi).$$

Since we have shown both inequalities, equality holds. $\square$

Definition 2.5 is the definition that N. J. Kalton and M. I. Ostrovskii used in [KO99]. The definition of the Gromov-Hausdorff distance between pseudo-metric spaces A and B deals with a huge class of pseudo-metric spaces. It is possible to reduce this class the to disjoint union of A and B. More precisely, the Gromov-Hausdorff distance between two metric spaces (A, d_A) and (B, d_B) is the infimum of $r > 0$ such that exists a pseudo-metric d on the disjoint union $A \sqcup B$ such that the restrictions of d to A and B coincide with d_A and d_B, and $d_H(A, B) < r$ in the space $(A \sqcup B, d)$.

To prove this, fix isometries $f\colon A \to A'$ and $g\colon B \to B'$, and define a distance between $a \in A$ and $b \in B$ by

$$d(a, b) = d_Z(f(a), g(b)).$$

This definition yields a pseudo-metric on $A \sqcup B$ for which $d_H(A, B) < r$. The quotient metric space $(A \sqcup B)/d$ is isometric to $A' \sqcup B' \subset Z$, where Z is a common pseudo-metric space and A', B' are isometric copies of A and B inside Z. To obtain a metric on $A \sqcup B$, define

$$d(a, b) = d_Z(f(a), g(b)) + \delta,$$

with δ an arbitrary positive constant. Then $d_H(A, B) < r + \delta$.

Hence we have obtained the following result.

Theorem 2.7. The Gromov-Hausdorff distance between pseudo-metric spaces A and B is the infimum of all $\varepsilon \geq 0$ so that there is a pseudo-metric on $A \sqcup B$ such that $d_{A\sqcup B}(A, B) \leq \varepsilon$.

Moreover, from [KO99], we obtain that

$$2d_{GH}(A, B) = \inf_{g,h}\{\mathrm{dis}(g), \mathrm{dis}(h), \mathrm{codis}(g, h)\},$$

where $g \colon X \to Y$ and $h \colon Y \to X$ are arbitrary maps. Anyway, in order to work with the Gromov-Hausdorff distance we will only use Theorem 2.6 for the disjoint union and Definition 2.5.

Proposition 2.8. Let X, Y, Z be metric spaces. Then

$$d_{GH}(X, Z) \leq d_{GH}(X, Y) + d_{GH}(Y, Z).$$

Proof. Let $d_{X\sqcup Y}$ and $d_{Y\sqcup Z}$ be metrics on $X \sqcup Y$ and $Y \sqcup Z$, respectively, that extend metrics of X, Y and Z. Define the distance between $x \in X$ and $z \in Z$ by

$$d_{X\sqcup Z}(x, z) = \inf_{y\in Y}\{d_{X\sqcup Y}(x, y) + d_{Y\sqcup Z}(y, z)\}.$$

We can check that $d_{X\sqcup Z}$ is a metric on $X \sqcup Z$. If $x_1, x_2 \in X$ and $z \in Z$ then we have that

$$\begin{aligned} d_{X\sqcup Z}(x_1, z) &= \inf_{y\in Y}\{d_{X\sqcup Y}(x_1, y) + d_{Y\sqcup Z}(y, z)\} \\ &\leq \inf_{y\in Y}\{d_{X\sqcup Y}(x_1, x_2) + d_{X\sqcup Y}(x_2, y) + d_{Y\sqcup Z}(y, z)\} \\ &= d_{X\sqcup Z}(x_1, x_2) + d_{X\sqcup Z}(x_2, z). \end{aligned}$$

If $x \in X$ and $z_1, z_2 \in Z$ then we have that

$$\begin{aligned} d_{X\sqcup Z}(x, z_1) &= \inf_{y\in Y}\{d_{X\sqcup Y}(x, y) + d_{Y\sqcup Z}(y, z_1)\} \\ &\leq \inf_{y\in Y}\{d_{X\sqcup Y}(x, y) + d_{Y\sqcup Z}(y, z_2) + d_{Y\sqcup Z}(z_2, z_1)\} \\ &= d_{X\sqcup Z}(x, z_2) + d_{X\sqcup Z}(z_2, z_1). \end{aligned}$$

This proves that the triangle inequality holds for $d_{X\sqcup Z}$. Since $d_{X\sqcup Z}$ extends the metrics in X and Z, it follows that $d_{X\sqcup Z}$ restricted to X and Z satisfies the conditions of metric. Hence $d_{X\sqcup Z}$ is a metric on $X \sqcup Z$. The definition of $d_{X\sqcup Z}$ yields that $d_H(X, Z) \leq d_H(X, Y) + d_H(Y, Z)$ where d_H refers to the metrics in the disjoint union of each pair. Thus taking the infimum over all metrics $d_{X\sqcup Y}$ and $d_{Y\sqcup Z}$ we obtain that

$$d_{GH}(X,Z) \le d_{GH}(X,Y) + d_{GH}(Y,Z). \qquad \square$$

We now introduce a particular case of relations in order to reduce further the space where the infimum is taken.

Definition 2.9. Let X and Y be sets. A *correspondence* between X and Y is a surjective multivalued function from X to Y. That is, a subset $C \subset X \times Y$ such that for all $x_0 \in X$ there is some $(x_0, y) \in C$ and for all $y_0 \in Y$ there is some $(x, y_0) \in C$.

Note that a correspondence is a particular case of a relation. If C is a correspondence, then C^{-1} is also a correspondence.

Theorem 2.10. For any two metric spaces X and Y,

$$d_{GH}(X,Y) = \frac{1}{2}\inf_C\{\text{dis(C)}\},$$

where the infimum is taken over all correspondences C between X and Y.

Proof. Let $d_{GH}(X,Y) < r$ for $r > 0$. We may assume that X and Y are subspaces of some metric space Z and $d_H(X,Y) < r$ in Z. Define

$$C := \{(x,y) : x \in X, y \in Y, d_Z(x,y) < r\}.$$

Since $d_H(X,Y) < r$, the set C is a correspondence between X and Y. If $(x,y),(x',y') \in C$ then

$$|d_Z(x,x') - d_Z(y,y')| \le d_Z(x,y) + d_Z(x',y') < 2r.$$

Hence we have that $\text{dis}(C) < 2r$.

Let $\text{dis}(C) = 2r$. It suffices to show that there is a pseudo-metric d on the disjoint union $X \sqcup Y$ such that d restricted to X is d_X, d restricted to Y is d_Y and $d_H(X,Y) \le r$ in $(X \sqcup Y, d)$. The idea is to set the distance from x to y equal to r whenever x and y correspond to each other, and take the minimal metric d generated by this condition. Formally, for $x \in X$ and $y \in Y$ we define

$$d(x,y) = \inf\{d_X(x,x') + d_Y(y',y) + r : (x',y') \in C\},$$

and the distances within X and Y are defined by d_X and d_Y. We need to check the triangle inequality. If $(x', y'), (x'', y'') \in C$ then

$$\operatorname{dis}(C) \geq |d_X(x', x'') - d_Y(y', y'')|.$$

Thus we have that, for all y,

$$d_Y(y, y') + d_Y(y, y'') \geq d_Y(y', y'') \geq d_X(x', x'') - \operatorname{dis}(C).$$

For $x_1, x_2 \in X$ and $y \in Y$, if $(x', y'), (x'', y'') \in C$ attain the respective infima, then

$$\begin{aligned} d(x_1, y) &= d_X(x_1, x') + d_Y(y', y) + r, \\ d(x_2, y) &= d_X(x_2, x'') + d_Y(y'', y) + r. \end{aligned}$$

Consequently,

$$\begin{aligned} d(x_1, y) + d(x_2, y) &= d_X(x_1, x') + d_Y(y', y) + d_X(x_2, x'') + d_Y(y'', y) + 2r \\ &\geq d_X(x_1, x') + d_X(x', x'') + d_X(x'', x_2) = d(x_1, x_2). \end{aligned}$$

Hence the triangle inequality follows by symmetry. For every $x \in X$ there is some $y \in Y$ such that $(x, y) \in C$. Hence $d(x, y) = r$. Thus $d(x, Y) = r$ and using symmetry we obtain that $d_H(X, Y) = r$. Therefore

$$d_{GH}(X, Y) \leq d_H(X, Y) = r.$$

So we have that the equality holds. $\square$

We now want techniques for handling Gromov-Hausdorff distances. While the previous results do not give an explicit expression for the distance, they provide another quantity which differs from the distance by no more than twice the distortion. Note that an estimate of this type is sufficient to study the topology determined by the Gromov-Hausdorff distance.

If (M, d) is a metric space and X is a subset of M, then the *packing radius* of X is half of the infimum of distances between different elements of X. If the packing radius is r, then open balls of radius r centered at the points of X will all be disjoint from each other, and each open ball centered at one of the points of X with radius $2r$ will be disjoint from the rest of X. The *covering radius* of X is the infimum of the numbers r such that every point of M is within distance r of at least one point in X, that is, it is the smallest radius such that closed balls of that radius centered at the points of X have all M as their union. An *ε-packing* is a set X of packing radius $\geq \varepsilon/2$; an *ε-covering* is a set X of covering radius $\leq \varepsilon$; and an *ε-net* is a set X that is both an ε-packing and ε-covering.

Definition 2.11. Let X and Y be metric spaces and $\varepsilon > 0$. A map $f\colon X \to Y$ is called an *ε-isometry* if $\operatorname{dis}(f) \leq \varepsilon$ and $f(X)$ is an ε-net in Y.

It is important to note that an ε-isometry does not need to be continuous.

Theorem 2.12. Let X and Y be metric spaces and $\varepsilon > 0$. Then

(1) If $d_{GH}(x, Y) < \epsilon$, then there exists a 2ε-isometry from X to Y.
(2) If there exists an ε-isometry from X to Y, then $d_{GH}(X, Y) < 2\varepsilon$.

Proof. 1. Let C be a correspondence between X and Y with $\operatorname{dis}(C) < 2\varepsilon$. For every $x \in X$, choose $f(x) \in Y$ such that $(x, f(x)) \in C$. This defines a map $f\colon X \to Y$. It is clear that $\operatorname{dis}(f) \leq \operatorname{dis}(C) < 2\varepsilon$. Let us show that $f(X)$ is a 2ε-net in Y. For $y \in Y$ consider $x \in X$ such that $(x, y) \in C$. Since both y and $f(x)$ are in correspondence with x, we have that

$$d(y, f(x)) \leq d(x, x) + \operatorname{dis}(C) < 2\varepsilon.$$

Hence we have that $d_H(y, f(X)) < 2\varepsilon$.

2. Let f be an ε-isometry. Define $C \subset X \times Y$ by

$$C = \{(x, y) \in X \times Y : d(y, f(x)) \leq \varepsilon\}.$$

Since $f(X)$ is an ε-net in Y, we have that C is a correspondence. If (x, y), $(x', y') \in C$ we have that

$$\begin{aligned}|d(y, y') - d(x, x')| &\leq |d(f(x), f(x')) - d(x, x')| + d(y, f(x)) + d(y', f(x')) \\ &\leq \operatorname{dis}(f) + 2\varepsilon \leq 3\varepsilon.\end{aligned}$$

Hence $\operatorname{dis}(C) \leq 3\varepsilon$, and by Theorem 2.10 we have that

$$d_{GH}(X, Y) \leq \frac{3}{2}\varepsilon < 2\varepsilon. \qquad \square$$

It is important that we do not require continuity of ε-isometries. Even if two spaces are very close with respect to the Gromov-Hausdorff distance, it can happen that there are no continuous maps with small distortion, for example spheres with small handles.

Let X be the standard two-dimensional sphere and X_n be the same sphere with a small handle attached to it. Let the diameter of the handle be less that $1/n$. As n grows, handles become smaller and smaller and the spaces X_n become more and more similar to X. One could say that handles vanish to a point and the sequence $\{X_n\}$ converges to X. However X_n is not homeomorphic to X, so a different notion of convergence is needed.

First we need two useful results, one for Lipschitz maps and the other one for distance-preserving maps.

Proposition 2.13. Let X be a metric space, and X' a dense subset of X. Let Y be a complete metric space and $f\colon X' \to Y$ a Lipschitz map. Then there exists a unique continuous map $\overline{f}\colon X \to Y$ such that $\overline{f}|_{X'} = f$. Moreover $\overline{f}$ is a Lipschitz map with the same Lipschitz constant as f.

Proof. Le C be a Lipschitz constant for f. For every $x \in X$ we define $\overline{f}(x) \in Y$ as follows. Choose a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in X'$ for all n, and $x_n \to x$ as $n \to \infty$. We can see that $\{f(x_n)\}$ is a Cauchy sequence. Indeed, we have that

$$d(f(x_i), f(x_j)) \leq Cd(x_i, x_j)$$

for all i, j. Hence $\{f(x_n)\}$ converges. We define $\overline{f}(x) = \lim_n f(x_n)$. So we have defined $\overline{f}\colon X \to Y$. If we have that

$$\begin{aligned} x &= \lim_n x_n, \\ x' &= \lim_n x'_n, \\ \overline{f}(x) &= \lim_n f(x_n), \\ \overline{f}(x') &= \lim_n f(x'_n), \end{aligned}$$

then we have that

$$\begin{aligned} d(\overline{f}(x), \overline{f}(x')) &= \lim_n d(f(x_n), f(x'_n)) \\ &\leq C \lim_n d(x_n, x'_n) = Cd(x, x'). \end{aligned}$$

Therefore f is Lipschitz with a Lipschitz constant $\leq C$.

If two continuous maps coincide on a dense set, then the two maps coincide everywhere. Hence the map $\overline{f}$ is unique. $\square$

Proposition 2.14. Let X be a compact metric space and $f\colon X \to X$ be a distance-preserving map. Then $f(X) = X$.

Proof. Suppose the contrary, i.e., let $p \in X \setminus f(X)$. Since $f(X)$ is compact and hence closed, there exists an $\varepsilon > 0$ such that $B_\varepsilon(p) \cap f(X) = \emptyset$. Let n be the maximal possible cardinality of an ε-separated set in X and $S \subset X$ be an ε-separated set of cardinality n, i.e., for any $x, y \in S$ we have that $d(x,y) \geq \varepsilon$. It can be proved that if there exists an $(\varepsilon/3)$-net of cardinality n, then an ε-separated set cannot contain more than n points. A maximal ε-separated set is an ε-net. ([BBI01, p.14]). Since f is a distance-preserving map, the set $f(S)$ is also ε-separated. On the other hand we have that

$$d(p, f(S)) \geq d(p, f(X)) \geq \varepsilon.$$

Therefore $f(S) \cup \{p\}$ is an ε-separated set of cardinality $n+1$. This is a contradiction with the choice of n. $\square$

Theorem 2.15. The Gromov-Hausdorff distance defines a metric on the space of isometry classes of compact metric spaces. In other words, the Gromov-Hausdorff distance is nonnegative, symmetric and satisfies the triangle inequality; moreover we have that $d_{GH}(X,Y) = 0$ if and only if X and Y are isometric.

Proof. In the statements above we have proved that the Gromov-Hausdorff distance is a pseudo-distance, so it only remains to prove that $d_{GH}(X,Y) = 0$ implies that X and Y are isometric. Let X and Y be two compact metric spaces such that $d_{GH}(X,Y) = 0$. From Theorem 2.12, there exists a sequence of maps $f_n\colon X \to Y$ such that $\operatorname{dis}(f_n) \to 0$. Since X is a compact metric space, we can fix a countable dense set $S \subset X$. Using the following Cantor diagonal procedure we obtain a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that for every $x \in S$ the sequence $\{f_{n_k}(x)\}$ is convergent in Y.

Let $S = \{x_i\}_{i=1}^\infty$. We first consider $\{f_n\}$. Since $\{f_n(x_1)\} \subset Y$ and Y is a compact metric space, there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that $\{f_{n_k}(x_1)\}$ is convergent in Y. For simplicity of notation we denote by $\{f_l\}$ the subsequence that applied to x_1 is convergent. Doing the same argument, since $\{f_l(x_2)\} \subset Y$ and Y is a compact metric space, there exists a subsequence $\{f_{l_k}\} \subset \{f_l\}$ such that $\{f_{l_k}(x_2)\}$ is convergent in Y. Since the sequence $\{f_l\}$ is a subsequence of $\{f_n\}$, we have that $\{f_l(x_1)\}$ converges in Y.

Assume we have a sequence $\{f_i\}$ that converges with $x_1, \ldots, x_m$. Since $\{f_i(x_{m+1})\} \subset Y$ and Y is a compact metric space, there exists a subsequence $\{f_{i_k}\} \subset \{f_i\}$ such that $\{f_{i_k}(x_{m+1})\}$ is convergent in Y. Hence $\{f_{i_k}(x_n)\}$ is convergent in Y for $n = 1, \ldots, m, m+1$. So by induction we can find a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that for every $x \in S$ the sequence $\{f_{n_k}(x)\}$ is convergent in Y.

For simplicity of notation we assume that $\{f_{n_k}\}$ is $\{f_n\}$. Then we can define a map $f \colon S \to Y$ as the limit of f_n, setting $f(x) = \lim_n f_n(x)$ for every $x \in S$. Since

$$|d(f_n(x), f_n(y)) - d(x,y)| \leq \operatorname{dis}(f_n) \to 0,$$

we have that $d(f(x), f(y)) = \lim_n d(f_n(x), f_n(y)) = d(x,y)$ for all $x, y \in S$. In other words, f is a distance-preserving map from S to Y. Hence using Proposition 2.13 we can extend f to all X.

In the same way we can obtain a map $g \colon Y \to X$ satisfying the same properties of f. Therefore we have that $f \circ g \colon Y \to Y$ is a distance-preserving map. Hence using proposition 2.14 the map $f \circ g$ is bijective. Then the map f is surjective and therefore is an isometry between X and Y. □

This last theorem allows us to consider compact metric spaces as points in the so-called *Gromov-Hausdorff space*, keeping in mind that isometric spaces represent the same "point". The topology of this space (determined by the Gromov-Hausdorff distance) is called the *Gromov-Hausdorff topology*.

In what follows, we consider converging sequences in the Gromov-Hausdorff space of compact metric spaces. By definition, a sequence $\{X_n\}_{n=1}^{\infty}$ of compact metric spaces converges to a compact metric space X if $d_{GH}(X_n, X) \to 0$ as $n \to \infty$. In this case we write $X_n \xrightarrow{GH} X$ and call X the *Gromov-Hausdorff limit* of $\{X_n\}_{n=1}^{\infty}$. Since d_{GH} is a metric, the limit is unique up to an isometry.

One of the reasons why we pay attention to the trivial case of finite spaces is that finite spaces form a dense set in the Gromov-Hausdorff space:

Example 2.16. Every compact metric space X is a limit of finite spaces. Indeed, take a sequence $\varepsilon_n \to 0$ of positive numbers and choose a finite ε_n-net S_n in X for every n. Then $S_n \xrightarrow{GH} X$, simply because

$$d_{GH}(X, S_n) \leq d_H(X, S_n) \leq \varepsilon_n.$$

Moreover, taking appropiate ε-nets one can essentially reduce convergence of arbitrary compact metric spaces to convergence of their finite subsets.

Definition 2.17. Let X and Y be two compact metric spaces, and $\varepsilon, \delta > 0$. We say that X and Y are (ε, δ)-approximations of each other if there exist finite collections of points $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$ in X and Y, respectively, such that:

(1) The set $\{x_i\}_{i=1}^N$ is an ε-net in X, and $\{y_i\}_{i=1}^N$ is an ε-net in Y.
(2) $|d_X(x_i, x_j) - d_Y(y_i, y_j)| < \delta$ for all $i, j \in \{1, \dots, N\}$.

An ε-approximation is an $(\varepsilon, \varepsilon)$-approximation.

Proposition 2.18. Let X and Y be compact metric spaces.

(1) If Y is an (ε, δ)-approximation of X, then $d_{GH}(X, Y) < 2\varepsilon + \delta$.
(2) If $d_{GH}(X, Y) < \varepsilon$, then Y is a 5ε-approximation of X.

Proof. (1) Let $X_0 = \{x_i\}_{i=1}^N$ and $Y_0 = \{y_i\}_{i=1}^N$ be as in Definition 2.17. The second condition in the definition means that the natural correspondence $\{(x_i, y_i) : 1 \leq i \leq N\}$ between X_0 and Y_0 has distortion less than δ. It follows that $d_{GH}(X_0, Y_0) < \delta/2$. Since X_0 and Y_0 are ε-nets in X and Y respectively, we have that $d_{GH}(X, X_0) \leq \varepsilon$ and $d_{GH}(Y, Y_0) \leq \varepsilon$. By the triangle inequality, we obtain that

$$\begin{aligned} d_{GH}(X, Y) &\leq d_{GH}(X, X_0) + d_{GH}(X_0, Y_0) + d_{GH}(Y_0, Y) \\ &< \varepsilon + \frac{\delta}{2} + \varepsilon < 2\varepsilon + \delta. \end{aligned}$$

(2) By Theorem 2.12, there is a 2ε-isometry $f \colon X \to Y$. Let $X_0 = \{x_i\}_{i=1}^N$ be an ε-net in X and $y_i = f(x_i)$. Then $|d(x_i, x_j) - d(y_i, y_j)| < 2\varepsilon < 5\varepsilon$ for all i, j. It remains to prove that $Y_0 = \{y_i\}_{i=1}^N$ is a 5ε-net in Y. Pick $y \in Y$. Since $f(X)$ is a 2ε-net in Y, there is an $x \in X$ such that $d(y, f(x)) \leq 2\varepsilon$. Since X_0 is an ε-net in X, there exists an $x_i \in X_0$ such that $d(x, x_i) \leq \varepsilon$. Then we have that

$$\begin{aligned} d(y, y_i) = d(y, f(x_i)) &\leq d(y, f(x)) + d(f(x), f(x_i)) \\ &\leq 2\varepsilon + d(x, x_i) + \operatorname{dis}(f) \leq 2\varepsilon + \varepsilon + 2\varepsilon \leq 5\varepsilon. \end{aligned}$$

Hence $d(y, Y_0) \leq d(y, y_i) \leq 5\varepsilon$. $\square$

The above proposition yields a criterion for convergence: $X_n \xrightarrow{GH} X$ if and only if, for any $\varepsilon > 0$, X_n is an ε-approximation of X for all large enough n. There is a more elegant formulation of this statement:

Theorem 2.19. Let X be a compact metric space and $\{X_n\}_n$ be a sequence of compact metric spaces. Then $X_n \xrightarrow{GH} X$ if and only if for every $\varepsilon > 0$ there exists a finite ε-net S in X and an ε-net S_n in each X_n such that $S_n \xrightarrow{GH} S$.

Moreover, the ε-net can be chosen so that, for all sufficiently large n, the set S_n has the same cardinality as S.

Proof. If such an ε-net exists, then X_n is an ε-approximation of X for all sufficiently large n. Then $X_n \xrightarrow{GH} X$ by the previous proposition. To prove the converse implication, take a finite $(\varepsilon/2)$-net S in X and construct corresponding nets S_n in X_n. Namely, pick a sequence of ε_n-approximations $f_n \colon X \to X_n$ where $\varepsilon_n \to 0$ and define $S_n = f_n(S)$. Then $S_n \xrightarrow{GH} S$ and, as in the previous proposition, S_n is an ε-net in X_n for all large enough n. $\square$

We need to introduce the concept of length space in order to obtain a quite useful result using finite graphs. We follow the definitions of [BBI01].

A *path* γ in a topological space X is a continuous map $\gamma \colon I \to X$ defined on an interval $I \subset \mathbb{R}$. By an *interval* we mean any connected subset of the real line; it may be open or closed, finite or infinite, and a single point is counted as an interval. Since a path is a map, one can speak about its image, restrictions, etc.

A *length structure* on a topological space X is a class A of *admissible* paths, which is a subset of all continuous paths in X, together with a map $L \colon A \to \mathbb{R}_+ \cup \infty$; the map L is called *path length*. The class A has to satisfy the following assumptions:

(1) The class A is closed under restrictions: if $\gamma \colon [a,b] \to X$ is an admissible path and $a \leq c \leq d \leq b$, then the restriction $\gamma|_{[c,d]}$ of γ is also admissible.
(2) A is closed under concatenations (products) of paths. Namely, if a path $\gamma \colon [a,b] \to X$ is such that its restrictions γ_1, γ_2 to $[a,c]$ and $[c,b]$ are both admissible paths, then so is γ.
(3) A is closed under linear reparametrizations: for an admissible path $\gamma \colon [a,b] \to X$ and a homeomorphism $\phi \colon [c,d] \to [a,b]$ of the form $\phi(t) = \alpha t + \beta$, the composition $\gamma \circ \phi$ is also an admissible path.

Every class of paths comes with its own class of reparametrizations. For example, consider the class of all continuous paths and the class of homeomorphisms, or the class of piecewise smooth paths and the class of diffeomorphisms. We only require that the class of reparametrizations includes all linear maps.

We require that L possesses the following properties:

(1) Length of paths is additive: $L(\gamma|_{[a,b]}) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]})$ for any $c \in [a, b]$.

(2) The length of a piece of a path continuously depends on the piece, i.e., for a path $\gamma\colon [a, b] \to X$ of finite lenght, denote by $L(\gamma, a, t)$ the length of the restriction of γ to the segment $[a, t]$. Then we require that $L(\gamma, a, \cdot)$ is a continuous function.

(3) The length is invariant under reparametrizations: $L(\gamma \circ \phi) = L(\gamma)$ for any homeomorphism ϕ such that γ and $\gamma \circ \phi$ are admissible.

(4) We require length structures to agree with the topology of X in the following sense: for a neighborhood U_x of a point x, the length of paths connecting x with points of the complement of U_x is separated from zero:

$$\inf\{L(\gamma) : \gamma(a) = x,\, \gamma(b) \in X \setminus U_x\} > 0.$$

Once we have a length structure, we are ready to define a metric associated with the structure. We will always assume that the topological space X carrying the length structure is a Hausdorff space. For two points $x, y \in X$ we set the associated distance $d(x, y)$ between them to be the infimum of lengths of admissible paths conecting these points:

$$d_L(x, y) = \inf\{L(\gamma) \mid \gamma\colon [a, b] \to X, \gamma \in A, \gamma(a) = x, \gamma(b) = y\}.$$

If it is clear from the context wich length structure L gives rise to d_L, we usually drop L in the notation d_L. Then (X, d_L) is a metric space.

Definition 2.20. A metric that can be obtained as the distance function associated to a length structure is called an *intrinsic* metric or *length* metric. A metric space whose metric is intrinsic is called a *length space*.

While any compact metric space can be obtained as a limit of finite spaces (see Example 2.16), these finite spaces do not carry length metrics. For length spaces, the role of finite spaces is played by the one-dimensional ones, i.e., graphs. Recall that a finite metric graph is a length space obtained by gluing together several spaces isometric to line segments in such a way that only endpoints may be shared between the segments. Equivalently, a finite metric graph is a finite topological graph equipped with a length metric.

Proposition 2.21. Every compact length space can be obtained as a Gromov-Hausdorff limit of finite graphs.

Proof. Let X be a length space. We want to obtain X as a Gromov-Hausdorff limit of finite graphs. Pick small positive numbers ε and δ (where δ is much smaller than ε), and choose a finite δ-net S in X. Then consider the following graph G: the set of vertices of G is S, two points $x, y \in S$ are connected by an edge if and only if $d(x,y) < \varepsilon$, and the length of this edge is equal to $d(x,y)$.

Let us show that G is an ε-approximation for X if δ is small enough, say, $\delta < \varepsilon^2/4\mathrm{diam}(X)$. We consider S both as a subset of X and a subset of G. Obviously S is an ε-net in both spaces, and $d_G(x,y) \geq d(x,y)$ for all x, y be a shortest path in X connecting x and y. Choose n points $x_1, \ldots, x_n$ where $n \leq 2L(\gamma)/\varepsilon$, dividing γ into intervals of lengths no greater than $\varepsilon/2$. For every $i = 1, \ldots, n$ there is a point $y_i \in S$ such that $d(x_i, y_i) \leq \delta$. In addition, set $x_0 = y_0 = x$ and $x_{n+1} = y_{n+1} = y$. Note that $d(y_i, y_{i+1}) \leq d(x_i, x_{i+1}) + 2\delta < \varepsilon$ for all $i = 0, \ldots, n$. In particular, y_i and y_{i+1} are connected by an edge in G provided that $\delta < \varepsilon/4$. Then

$$d_G(x,y) \leq \sum_{i=0}^{n} d(y_i, y_{i+1}) \leq \sum_{i=0}^{n} d(y_i, y_{i+1}) + 2\delta n = d(x,y) + 2\delta n.$$

Recall that $n \leq 2L(\gamma)/\varepsilon \leq 2\mathrm{diam}(X)/\varepsilon$; hence

$$d_G(x,y) \leq d(x,y) + \delta \frac{4\mathrm{diam}(X)}{\varepsilon} < d(x,y) + \varepsilon,$$

if $\delta < \varepsilon^2/4\mathrm{diam}(X)$.

Thus we have a finite graph which is an ε-approximation for X. Letting $\varepsilon \to 0$ yields a sequence of graphs converging to X. $\square$

The following result will be useful in the computation of Gromov-Hausdorff distances.

Proposition 2.22. Let X and Y be two compact metric spaces.

(1) Let $\{x_1, \ldots, x_n\} \subset X$ be an R-covering of the compact metric space X. Then $d_{GH}(X, \{x_1, \ldots, x_n\}) \leq R$.

(2) $\frac{1}{2}|\mathrm{diam}(X) - \mathrm{diam}(Y)| \leq d_{GH}(X,Y) \leq \frac{1}{2}\max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}$.

Proof. (1) This result can be found in [BBI01], and it is similar to the same result for ε-nets.

(2) The Gromov-Hausdorff distance between a one-point metric space $\{p\}$ and a bounded metric space X is

$$d_{GH}(\{p\}, X) = \frac{1}{2}\mathrm{diam}(X).$$

Hence using the triangle inequality we obtain that

$$d_{GH}(X,Y) \geq \frac{1}{2}|\mathrm{diam}(X) - \mathrm{diam}(Y)|.$$

We know that

$$d_{GH}(X,Y) = \frac{1}{2}\inf_{C}\{\mathrm{dis(C)}\},$$

where the infimum is taken over all correspondences C between X and Y. Therefore, we obtain that

$$d_{GH}(X,Y) \leq \frac{1}{2}\max\{\mathrm{diam}(X), \mathrm{diam}(Y)\}.$$ □

The study of length spaces is important because if we consider a Riemannian manifold M with the geodesic distance d then the Hopf-Rinow theorem states that M is geodesically complete if and only if (M,d) is a complete metric space. Recall that, by definition, M is geodesically complete if for every point $p \in M$ the exponential map $\exp_p$ is defined on the entire tangent space T_pM.

3. The metric space of metrics and the space of correspondences

In this section we follow [ŠTZ92], [ŠTZ93] and [IIT16].

Let X be a nonempty set. We denote by $\mathcal{M} = \mathcal{M}(X)$ the set of all metrics on X. We can define a finite metric d^* on $\mathcal{M}$ as follows: If $d, d' \in \mathcal{M}$ then

$$d^*(d,d') = \min\{1, \sup_{x,y\in X} |d(x,y) - d'(x,y)|\}.$$

Proposition 3.1. $(\mathcal{M}, d^*)$ is a metric space.

Proof. We have to show that d^* is a metric. We note that

$$d^*(d,d') = \min\{1, \|d - d'\|_\infty\}.$$

Therefore $d^*(d,d') = 0$ if and only if $\|d - d'\|_\infty = 0$, and this is equivalent to the statement that $d = d'$.

Symmetry follows from the equality $\|d - d'\|_\infty = \|d' - d\|_\infty$ and the triangle inequality is a consequence of the triangle inequality of the infinity norm. $\square$

We denote by t_α a discrete metric on X of value α, i.e., $t_\alpha(x,x) = 0$ and $t_\alpha(x,y) = \alpha$ for $x, y \in X$, $x \neq y$.

Proposition 3.2. The metric space $(\mathcal{M}, d^*)$ is not complete.

Proof. We consider the sequence $\{t_{1/k}\}_{k=1}^\infty$ of elements of $\mathcal{M}$. If $0 < m < n$ are natural numbers, then

$$d^*(t_{1/n}, t_{1/m}) = \min\{1, \|t_{1/n} - t_{1/m}\|_\infty\} = \left|\frac{1}{n} - \frac{1}{m}\right|.$$

Hence, the sequence $\{t_{1/k}\}_k$ converges. However, the limit is not a metric on X. Since uniform convergence implies pointwise convergence, it is enough to observe that for $x, y \in X$ with $x \neq y$ we have that

$$\lim_n t_{1/n}(x,y) = \lim_n \frac{1}{n} = 0,$$

and the zero function is not a metric on X. $\square$

We denote by $\aleph_0$ the cardinality of the set of positive integers $\mathbb{N}$.

In connection with this we mention some subspaces of $\mathcal{M}$ which are complete. Suppose $\alpha > 0$ and define

$$\mathcal{H}_\alpha = \{d \in \mathcal{M} : d(x,y) \geq \alpha \text{ for all } x, y \in X \text{ with } x \neq y\}.$$

Theorem 3.3. The subspace $\mathcal{H}_\alpha$ of $\mathcal{M}$ is a complete metric space.

Proof. We omit the details of this result, which can be found in [ŠTZ92]. □

We consider the following sets:

$$\mathcal{U} = \mathcal{U}(X) = \{d \in \mathcal{M} : (X,d) \text{ is a complete metric space}\},$$
$$\mathcal{K} = \mathcal{K}(X) = \{d \in \mathcal{M} : (X,d) \text{ is a compact metric space}\}.$$

Note that if X is finite then $\mathcal{U} = \mathcal{K}$.

Equivalence of metrics determines an equivalence relation $\sim$ on $\mathcal{M}$. We denote by $\mathcal{M}|_\sim$ the set of equivalence classes corresponding to $\sim$.

Let $\mathcal{O}_0$ be the class of $\mathcal{M}|_\sim$ whose elements d satisfy the following property: A sequence $x_k \in X$ converges with respect to d if and only if $\{x_k\}$ is almost stationary, i.e., $d(x_k, x) \to 0$ for some $x \in X$ implies that $x_k = x$ for all but at most finitely many k.

The class $\mathcal{O}_0$ contains all discrete metrics, but there are other metrics too. For instance, if $X = (0, \infty)$ and we define $\sigma(x,y) = \max\{x,y\}$ for $x \neq y$, and $\sigma(x,x) = 0$, then $\sigma \in \mathcal{O}_0$ but σ is not discrete.

Proposition 3.4. The set $\mathcal{K}$ is a union of classes from $\mathcal{M}|_\sim$.

Proof. If $d \in \mathcal{K}$ then (X,d) is compact and hence (X,d') is also compact for all $d' \sim d$. □

The analogous theorem for $\mathcal{U}$ does not hold in general, since completeness is not a topological property. Actually we have that

Theorem 3.5. (1) If X is finite then $\mathcal{U} = \mathcal{M} = \mathcal{O}_0$.
(2) If X is infinite, then $\mathcal{U} \cap \mathcal{O}_0 \neq \emptyset \neq \mathcal{O}_0 \cap (\mathcal{M} \setminus \mathcal{U})$.

Proof. (1) The case of a finite set X is clear, since all convergent sequences are almost stationary.

(2) Let X be an infinite set. Then there exists a one-to-one sequence $x_k \in X$, i.e, $x_i \neq x_j$ if $i \neq j$. Let $X' = X \setminus \{x_k\}$ and define a metric σ on X as follows: $\sigma(x,x) = 0$ for all $x \in X$, $\sigma(x,y) = 1$ if x and y are different and at least one of them belongs to X', while $\sigma(x_i, x_j) = \max\{1/i, 1/j\}$ for different $i, j \in \mathbb{N}$. We prove that $\sigma \in \mathcal{O}_0 \cap (\mathcal{M} \setminus \mathcal{U})$.

Suppose that $\{y_k\}$ is a sequence such that $\sigma(y_k, y) \to 0$ as $k \to \infty$. Then $\sigma(y_k, y) = 1$ if $y \in X'$ and $y_k \neq y$, and $\sigma(y_k, y) \geq 1/m$ if $y = x_m$ and $y_k \neq x_m$ for all k, m. Thus there exists a k_0 such that $y = y_k$ for every $k \geq k_0$, which implies that $\sigma \in \mathcal{O}_0$. Furthermore, the sequence $\{x_k\}$ is a Cauchy sequence but it does not converge with respect to σ because $\{x_k\}$ is not almost stationary. Consequently $\sigma \in \mathcal{M} \setminus \mathcal{U}$.

On the other hand, the discrete metric t_1 belongs to $\mathcal{O}_0 \cap \mathcal{U}$. □

We say that a subset A of a topological space X is *dense in itself* if A has no isolated points. Equivalently, A is dense in itself if every point of A is an accumulation point.

Lemma 3.6. Every class $\mathcal{O} \in \mathcal{M}|_\sim$ is a dense in itself subset of $\mathcal{M}$.

Proof. Let $\varepsilon > 0$. Let $\mathcal{O} \in \mathcal{M}|_\sim$, $d \in \mathcal{O}$. It suffices to consider the metrics $d_a = d + a\min\{1, d\}$ for $0 < a < \varepsilon$, and notice that $d_a \in B_\varepsilon(d) \cap \mathcal{O}$ for all $0 < a < \varepsilon$. □

Theorem 3.7. Each of the sets $\mathcal{K}, \mathcal{U}$ is a dense in itself subset of $(\mathcal{M}, d^*)$.

Proof. The union of an arbitrary system of dense in itself sets is dense in itself. Hence the statement for $\mathcal{K}$ follows from Lemma 3.6 and Proposition 3.4. In view of Theorem 3.5 we have to deal with $\mathcal{U}$ separately. Let $d \in \mathcal{U}$, $\varepsilon > 0$ and $0 < a < \varepsilon$. Define $d_a(x,y) = d(x,y) + a$ for different $x, y \in X$ and $d_a(x,x) = 0$ for all $x \in X$. Then $d_a \in \mathcal{U} \cap B_\varepsilon(d)$ for each $0 < a < \varepsilon$. □

The following result will be needed for the final statements of the section about the Gromov-Hausdorff distance between compact metric spaces.

Theorem 3.8. The set $\mathcal{K}$ is closed in $(\mathcal{U}, d^*)$.

Proof. If X is a finite set, then $\mathcal{K} = \mathcal{U}$. So we can suppose that $|X| \geq \aleph_0$. Let $d_n \in \mathcal{K}$ and $d \in \mathcal{U}$ and suppose that $d^*(d_n, d) \to 0$ as $n \to \infty$. Assume that $d \in \mathcal{U} \setminus \mathcal{K}$. Then (X, d) is not totally bounded. Hence for some $1 > \varepsilon_0 > 0$ the set X has a countable ε_0-discrete subset, i.e., there exists a sequence $x_n \in X$ such that

$$d(x_k, x_l) \geq \varepsilon_0$$

for all $k, l \in \mathbb{N}$ with $k \neq l$. Let $n \in \mathbb{N}$ be fixed. The metric space (X, d_n) is compact. So $\{x_k\}_k$ has a convergent subsequence $\{x_{k_j}\}_j$ in (X, d_n). Then

$$|d(x_{k_i}, x_{k_j}) - d_n(x_{k_i}, x_{k_j})| \geq \varepsilon_0 - d_n(x_{k_i}, x_{k_j})$$

for all $i, j \in \mathbb{N}$ with $i \neq j$. So we have that

$$\sup_{i,j \in \mathbb{N}} |d(x_{k_i}, x_{k_j}) - d_n(x_{k_i}, x_{k_j})| \geq \varepsilon_0$$

for all $i, j \in \mathbb{N}$ with $i \neq j$. This implies that $d^*(d_n, d) \geq \varepsilon_0 > 0$ for every n. This is a contradiction. $\square$

It is not true in general that $\mathcal{K}$ is closed in $(\mathcal{M}, d^*)$. To see this, let $X = \{x_1, \dots\}$ be a countable set. Define metrics $d_n, d \in \mathcal{M}$ as follows: $d(x, x) = 0$ for all $x \in X$ and $d(x_i, x_j) = \max\{1/i, 1/j\}$ for all $i, j \in \mathbb{N}$ different. For each n define $d_n(x, x) = 0$ for all $x \in X$ and for all $i, j \in \mathbb{N}$ different let $d_n(x_i, x_j) = d(x_i, x_j)$ if $\min\{i, j\} \neq n$ and $d_n(x_i, x_j) = \min\{1/i, 1/j\}$ if $\min\{i, j\} = n$.

In this case we have that $d \in \mathcal{M} \setminus \mathcal{U} \subset \mathcal{M} \setminus \mathcal{K}$. Further, $d_n \in \mathcal{K}$ for each $n \in \mathbb{N}$, since every sequence in (X, d_n) is either almost stationary or d_n-converges to x_n. On the other hand, as $n \to \infty$,

$$\begin{aligned} d^*(d, d_n) &= \sup_{\min\{i,j\}=n} |d(x_i, x_j) - d_n(x_i, x_j)| \\ &= \sup_{j>n} |d(x_i, x_j) - d_n(x_i, x_j)| = \sup_{j>n} \left| \frac{1}{n} - \frac{1}{j} \right| = \frac{1}{n} \to 0. \end{aligned}$$

Now we want to prove the existence of a pseudo-metric in the disjoint union that achieves the actual Gromov-Hausdorff distance. Before that, we need to consider the space of all correspondences between X and Y, denoted by $\mathcal{R}(X, Y)$, as a set inside the set of all nonempty relations $\mathcal{P}(X, Y) \subset X \times Y$ between X and Y. Notice that any $\sigma \in \mathcal{P}(X, Y)$ such that there exists an $R \in \mathcal{R}(X, Y)$, $\mathbb{R} \subset \sigma$, satisfies that $\sigma \in \mathcal{R}(X, Y)$. In particular, the closure $\overline{R}$ of a correspondence R is a correspondence itself.

Proposition 3.9. If $\sigma \in \mathcal{P}(X,Y)$, then $\text{dis}(\sigma) = \text{dis}(\overline{\sigma})$.

Proof. If $\overline{\sigma} \subset \sigma$, then $\text{dis}(\sigma) \leq \text{dis}(\overline{\sigma})$. Suppose that $\text{dis}(\sigma) < \text{dis}(\overline{\sigma})$. By definition, for each $\varepsilon > 0$ and any $(\bar{x},\bar{y}),(\bar{x}',\bar{y}') \in \overline{\sigma}$ there exist $(x,y),(x',y') \in \sigma$ such that

$$|d_X(\bar{x},\bar{x}') - d_X(x,x')| \leq \frac{\varepsilon}{3},$$
$$|d_Y(\bar{y},\bar{y}') - d_Y(y,y')| \leq \frac{\varepsilon}{3}.$$

Thus we have that

$$|d_X(\bar{x},\bar{x}') - d_Y(\bar{y},\bar{y}')| < |d_X(x,x') - d_Y(y,y')| + 2\frac{\varepsilon}{3} \leq \text{dis}(\sigma) + 2\frac{\varepsilon}{3}.$$

Hence, by taking the supremum, we conclude that $\text{dis}(\sigma) + 2(\varepsilon/3) > \text{dis}(\overline{\sigma})$. Since ε is arbitrary we have $\text{dis}(\sigma) \geq \text{dis}(\overline{\sigma})$, which is a contradiction. $\square$

We denote by $\mathcal{P}_c(X,Y)$ the set of all closed nonempty relations between X and Y. Similarly $\mathcal{R}_c(X,Y)$ stands for the set of all closed correspondences between X and Y.

Corollary 3.10. For any metric spaces X and Y we have

$$d_{GH}(X,Y) = \frac{1}{2}\inf\{\text{dis}(R) : R \in \mathcal{R}_c(X,Y)\}.$$

Now we can establish a link between correspondences from $\mathcal{R}(X,Y)$ and pseudo-metrics on $X \sqcup Y$.

Proposition 3.11. Let X and Y be metric spaces. Let $R \in \mathcal{R}(X,Y)$ be such that $\text{dis}(R) < \infty$. Let ρ_R be an extension of the metrics of X and Y defined by

$$\rho_R(x,y) = \inf\{d_X(x,x') + d_Y(y,y') + \frac{1}{2}\text{dis}(R) : (x',y') \in R\}.$$

Then ρ_R is a pseudo-metric on $X \sqcup Y$ and $d_H(X,Y,\rho_R) = \frac{1}{2}\text{dis}(R)$.

Proof. It is clear that ρ_R is symmetric. The triangle inequality follows from

$$d_X(x,x') + d_Y(y,y') \leq d_X(x,z) + d_X(z,x') + 2d_Y(y,y')$$

for $z \in X$ and similarly for $z \in Y$.

Recall that the Hausdorff distance is

$$d_H(X,Y) = \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(X,y)\right\},$$

and that

$$d(x,Y) = \inf_{y\in Y} d(x,y).$$

Hence we have that

$$\begin{aligned}\rho_R(x,Y) &= \inf_{y\in Y}\rho_R(x,y) = \inf_{y\in Y}\inf\{d_X(x,x') + d_Y(y,y') + \frac{1}{2}\mathrm{dis}(R) : (x',y') \in R\} \\ &= \inf\{d_X(x,x') + \frac{1}{2}\mathrm{dis}(R) : (x',y') \in R\} = \frac{1}{2}\mathrm{dis}(R).\end{aligned}$$

We can do the same for $\rho_R(y,X)$. Therefore we obtain that, with ρ_R,

$$d_H(X,Y) = \max\left\{\frac{1}{2}\mathrm{dis}(R), \frac{1}{2}\mathrm{dis}(R)\right\} = \frac{1}{2}\mathrm{dis}(R). \qquad \square$$

We denote by $H(X)$ the family of all nonempty subsets of a metric space X. Recall that d_H is a metric on $H(X)$. In [BBI01, Theorem 7.3.8] the following result is proved:

Theorem 3.12 (Blaschke)**.** Let X be a metric space. The metric spaces $H(X)$ and X are both compact or non-compact simultaneously.

We consider the metric space $X \times Y$ with the distance function

$$d_{X\times Y}((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}.$$

If X and Y are compact metric spaces, then $X \times Y$ is a compact metric space. We also have that $\mathcal{P}_c(X,Y) = H(X\times Y)$ and using Theorem 3.12 we obtain that $\mathcal{P}_c(X,Y)$ is a compact metric space.

Proposition 3.13. Let X and Y be compact metric spaces. The set $\mathcal{R}_c(X,Y)$ is closed in $\mathcal{P}_c(X,Y)$, and consequently, $\mathcal{R}_c(X,Y)$ is a compact metric space.

Proof. It is sufficient to prove that for each $\sigma \in \mathcal{P}_c(X,Y) \setminus \mathcal{R}_c(X,Y)$ there exists a neighborhood U which does not intersect with $\mathcal{R}_c(X,Y)$. Since $\sigma \notin \mathcal{R}_c(X,Y)$, either $\pi_X(\sigma) \neq X$ or $\pi_Y(\sigma) \neq Y$, where π_X and π_Y are the canonical projections. Suppose that the first condition holds, i.e., there exists $x \in X \setminus \pi_X(\sigma)$. Since σ is a closed subset of the compact space $X \times Y$, it is compact itself, and therefore $\pi_X(\sigma)$ is compact in X. Hence there exists an open ball $U_\varepsilon(x)$ such that $U_\varepsilon(x) \cap \pi_X(\sigma) = \emptyset$. We can take $U_\varepsilon(x) \times Y$ as U. □

Now we define a function

$$
\begin{aligned}
f\colon (X \times Y) \times (X \times Y) &\to \mathbb{R} \\
(x, y, x', y') &\mapsto |d_X(x, x') - d_Y(y, y')|.
\end{aligned}
$$

It is clear that f is continuous. Notice that for each $\sigma \in \mathcal{P}(X,Y)$ we have

$$\mathrm{dis}(\sigma) = \sup\{f(x, y, x', y') : (x, y), (x', y') \in \sigma\} = \sup f|_{\sigma\times\sigma}.$$

Proposition 3.14. Let X and Y be compact metric spaces. Then the function $\mathrm{dis}\colon \mathcal{P}_c(X,Y) \to \mathbb{R}$ is continuous.

Proof. Since $(X \times Y) \times (X \times Y)$ is compact, the function f is uniformly continuous. Thus for any $\sigma \in \mathcal{P}_c(X,Y)$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for the open ball $U = U_\delta^{X\times Y}(\sigma) \subset X \times Y$ of radius δ centered at σ we have that

$$\sup f|_{U\times U} \leq \sup f|_{\sigma\times\sigma} + \varepsilon.$$

We denote by V the open ball $U_\delta^{\mathcal{P}_c(X,Y)}(\sigma) \subset \mathcal{P}_c(X,Y)$ of radius δ centered at σ. Since for any $\sigma' \in V$ we have $\sigma' \subset U$, it follows that

$$\mathrm{dis}(\sigma') = \sup f|_{\sigma'\times\sigma'} \leq \sup f|_{U\times U} \leq \mathrm{dis}(\sigma) + \varepsilon.$$

Swapping σ and σ', we get that $|\mathrm{dis}(\sigma) - \mathrm{dis}(\sigma')| \leq \varepsilon$, and hence the function $\mathrm{dis}\colon \mathcal{P}_c(X,Y) \to \mathbb{R}$ is continuous. □

We say that a correspondence $R \in \mathcal{R}(X,Y)$ is *optimal* if $d_{GH}(X,Y) = \frac{1}{2}\mathrm{dis}(R)$. We denote by $\mathcal{R}_{\mathrm{opt}}(X,Y)$ the set of all optimal correspondences between X and Y.

Theorem 3.15. For any two compact metric spaces X and Y we have that $\mathcal{R}_{\mathrm{opt}}(X,Y) \neq \emptyset$.

Proof. By Proposition 3.14 the function dis: $\mathcal{R}_c(X, Y) \to \mathbb{R}$ is continuous, and by the Proposition 3.13 the space $\mathcal{R}_c(X, Y)$ is a compact metric space. Hence there exists $R \in \mathcal{R}_c(X, Y)$ such that $d_{GH}(X, Y) = \frac{1}{2}\mathrm{dis}(R)$, and we conclude that $R \in \mathcal{R}_{\mathrm{opt}}(X, Y)$. $\square$

From the construction that relates each correspondence with a specific pseudo-metric, we obtain the following result:

Corollary 3.16. For all compact metric spaces X and Y there exists a pseudo-metric ρ on $X \sqcup Y$ such that $d_{GH}(X, Y) = d_H(X, Y, \rho)$.

We cannot expect that the pseudo-metric ρ from Corollary 3.16 is actually a metric. For instance, let $X = Y = \mathbb{S}^1$. We know that $d_{GH}(X, Y) = 0$. If there exists a metric on $X \sqcup Y$ such that $0 = d_{GH}(X, Y) = d_H(X, Y, \rho)$, then there exist $x \in X$ and $y \in Y$ such that $x \neq y$ in $X \sqcup Y$ and $\rho(x, y) = 0$, which is contradictory with the fact that ρ is a metric.

4. Distances between Banach spaces

The standard notion of distance between two Banach spaces is the Banach-Mazur distance, which is defined by

$$d_{BM}(X,Y) = \log\inf\{\|T\|\|T^{-1}\| \mid T\colon X \to Y \text{ is an isomorphism}\}.$$

The Banach-Mazur distance is only finite when X and Y are isomorphic. We next introduce the Kadets distance and certain related notions of distance. The Kadets distance is closely related to the Gromov-Hausdorff distance; see [KO99]. We want to include Banach spaces in the range of the Gromov-Hausdorff distance, such as l_p spaces for $p > 2$, which are not Hilbert spaces.

We recall that if Z is a Banach space and X and Y are closed subspaces of Z then the *gap* or *opening* $\Lambda(X,Y)$ is defined as the Hausdorff distance between the closed unit balls B_X and B_Y of X and Y, i.e.,

$$\Lambda(X,Y) = \max\{\sup_{y\in B_Y} d(y,B_X), \sup_{x\in B_X} d(x,B_Y)\}.$$

For X and Y arbitrary Banach spaces, we define the *Kadets* distance as

$$d_K(X,Y) = \inf_{Z,\phi,\psi} \Lambda(\phi(X),\psi(Y)),$$

where the infimum is taken over all Banach spaces Z and all linear isometric embeddings $\phi\colon X \to Z$ and $\psi\colon Y \to Z$.

This distance was introduced by Kadets, who proved for example that

$$\lim_{p\to 2} d_K(l_p, l_2) = 0.$$

The Kadets distance is clearly related to the notion of Gromov-Hausdorff distance between metric spaces. It is natural to introduce the Gromov-Hausdorff distance $d_{GH}(X,Y)$ between two Banach spaces X and Y as the Gromov-Hausdorff distance between their closed unit balls. Formally it is defined as $d_{GH}(X,Y) = d_{GH}(B_X,B_Y)$. Suppose d is a metric on the union $B_X \cup B_Y$ which coincides with the respective norm-distances on B_X and B_Y. We can extend d to $X \cup Y$ by defining d to coincide with the norm-distance on each of X and Y and, for $x \in X$ and $y \in Y$,

$$d(x,y) = \inf_{u\in B_X, v\in B_Y} \{\|x-u\|_X + d(u,v) + \|y-v\|_Y\}.$$

Let us note that our definition applies to both real and complex Banach spaces, but there are complex Banach spaces which are real-isometric but not even complex-isometric. In view of this, the Gromov-Hausdorff distance is more natural for the category of real Banach spaces.

An elementary example of a two real-isomorphic Banach spaces which are not complex-isomorphic is given by Kalton in [Kal95]. Bourgain in [Bou86] gave an example with a probabilistic approach. Kalton gave an explicit construction of a Banach space X so that X is not isomorphic to its complex conjugate.

Given a Banach space X we define its complex conjugate $\hat{X}$ to be the space X equipped with the alternative scalar multiplication $\alpha \bigotimes x = \bar{\alpha} x$.

Let ω denote the space of all complex-valued sequences. We let e_n be the canonical basis vectors. Suppose $f\colon [0, \infty) \to \mathbb{C}$ is a Lipschitz map, with $f(0) = 0$. We define a map $\Omega\colon l_2 \to \omega$ by

$$\Omega_f(x)(n) = x(n) f\left(\log\left(\frac{\|x\|_2}{|x(n)|}\right)\right).$$

Here we interpret the right-hand side to be zero if $x(n) = 0$. We then define $Z_2(f)$ to be the space of pairs (x, y) in $l_2 \times \omega$ such that

$$\|(x, y)\|_f = \|x\|_2 + \|y - \Omega_f(x)\|_2 < \infty.$$

It follows that $Z_2(f)$ is a Banach space under a norm equivalent to the quasi-norm $\|\cdot\|_f$. This space was considered in [KP79], but only the real case was discussed. However, the switch to complex scalars and complex-valued f is not a problem.

If $s \in l_\infty$ with $\|s\|_\infty \leq 1$ then we have the estimate $\|\Omega_f(sx) - s\Omega_f(x)\|_2 \leq C_0\|x\|_2$ where C_0 depends on the Lipschitz constant of f, and this leads to the fact that there is a constant C such that $\|(sx, sy)\|_f \leq C\|(x, y)\|_f$. It is used frequently that if $\tau\colon \mathbb{N} \to \mathbb{N}$ is a permutation and $x_\tau(n) = x(\tau(n))$ then $\|(x_\tau, y_\tau)\|_f = \|(x, y)\|_f$.

We now specialise to the functions $f_\alpha(t) = t^{1+i\alpha}$ for $-\infty < \alpha < \infty$. We write $Z_2(\alpha)$ instead of $Z_2(f_\alpha)$ and Ω_α instead of Ω_{f_α}. The following statement can be found in [Kal95], it is important for the following discussion.

Proposition 4.1. The complex conjugate of $Z_2(\alpha)$ is isomorphic to $Z_2(-\alpha)$.

Theorem 4.2. If $Z_2(\alpha)$ is isomorphic to $Z_2(\beta)$ then $\alpha = \beta$.

Proof. We suppose that $\alpha \neq 0$ and that $Z_2(\alpha)$ is isomorphic to $Z_2(\beta)$. Let $a = 1 + i\alpha$ and $b = 1 + i\beta$.

We observe the following inequalities for $t > s \geq 0$:

$$|t^b - s^b| \geq t - s, \tag{1}$$

$$|t^b - s^b| \leq |b|(t - s), \tag{2}$$

$$|t^b - s^b - bs^{b-1}(t-s)| \leq \frac{1}{2}\frac{|b|^2}{|s|}(t-s)^2. \tag{3}$$

For $w \in l_2$ we define

$$\Omega'_\beta(w)(n) = w(n)\left(\log|w(n)^{-1}|\right)^b.$$

Note that

$$\|\Omega_\beta(w) - \Omega'_\beta(w)\|_2 \leq |b||\log\|w\|_2|\|w\|_2.$$

If $A \subset \mathbb{N}$ is finite then we let $\xi_A = \sum_{n \in A} e_n$.

We will suppose the existence of an operator $T\colon Z_2(\alpha) \to Z_2(\beta)$ such that $\|T\| < 1$ and $c > 0$, so that, for every n, $\|T(e_n, 0)\|_\beta, \|T(0, e_n)\|_\beta > c$. We say that T is *admissible* if it satisfies these properties for some $c > 0$. If we have an admissible operator, then we can find an admissible operator T and an increasing sequence of integers $\{p_n\}_n$ so that for suitable sequences $u, v, w, y \in \omega$ we have, setting $B_n = \{p_{n-1} + 1, \ldots, p_n\}$, $T(e_n, 0) = (u\xi_{B_n}, v\xi_{B_n})$ and $T(0, e_n) = (w\xi_{B_n}, y\xi_{B_n})$. Here $u\xi_{B_n} = \sum_{k \in B_n} u(k)e_k$.

We can show that we must have $\lim_n \|w\xi_{B_n}\|_2 = 0$. So it follows, by passing to a subsequence and rearranging that we can further suppose that $w = 0$. We then have $c \leq \|y\xi_{B_n}\|_2 \leq 1$ for all n and some $c > 0$. We can also show by contradiction that we cannot have $\inf_n \|u\xi_{B_n}\| = 0$.

So there exists an admissible T so that $\inf_n \|u\xi_{B_n}\| > 0$. Under these circumstances we can apply a diagonalization procedure and a subsequence argument to produce an operator $S\colon Z_2(\alpha) \to Z_2(\beta)$ with $\|S\| < 1$ and so that $S(e_n, 0) = (\lambda e_n, \mu e_n)$ and $S(0, e_n) = (0, \nu e_n)$ with $\lambda \neq 0$. Let $A = \{1, 2, \ldots, N\}$ and $\sigma = \frac{1}{2}\log(N)$. Then

$$\left\|S(N^{-1/2}\xi_A, \sigma^a N^{-1/2}\xi_A\right\|_2 < 1.$$

Hence,

$$\left|\nu\sigma^a - \lambda(\sigma + \log|\lambda|)^b + \mu\right| \leq 1.$$

As this holds for all N we must have $a = b$ or $\alpha = \beta$, as required. $\square$

With this theorem we have an immediate corollary which give us what we are looking for:

Corollary 4.3. The space $Z_2(\alpha)$ is not isomorphic to its complex conjugate when $\alpha \neq 0$.

This example shows that, in the case of Banach spaces, we need to restrict Gromov-Hausdorff distance to the real case.

5. Gromov-Hausdorff distances between spheres

In this section we follow the articles [LMS21] and [ABC$^+$22]. We consider the problem of estimating the Gromov-Hausdorff distance $d_{GH}(\mathbb{S}^n, \mathbb{S}^m)$ between spheres endowed with their geodesic distance.

We recall that the *diameter* of a bounded metric space (X, d_X) is the number

$$\operatorname{diam}(X) = \sup_{x,x' \in X} d_X(x, x').$$

For $m \in \mathbb{N} \cup \{\infty\}$, we view the m-dimensional sphere

$$\mathbb{S}^m = \{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \mid x_1^2 + \cdots + x_{m+1}^2 = 1\}$$

as a metric space by endowing it with the geodesic distance: For any two points $x, x' \in \mathbb{S}^m$ the geodesic distance between them is

$$d_{\mathbb{S}^m}(x, x') := \arccos(\langle x, x' \rangle) = 2 \arcsin\left(\frac{d_E(x, x')}{2}\right),$$

where d_E denotes the canonical Euclidean metric inherited from $\mathbb{R}^{m+1}$.

Note that for $m = 0$ this definition implies that $\mathbb{S}^0$ consists of two points at distance π, and that $\mathbb{S}^\infty$ is the unit sphere in l^2 with distance given by the expression above.

We recall that for any two bounded metric spaces X and Y one always has that

$$d_{GH}(X, Y) \leq \frac{1}{2} \max\{\operatorname{diam}(X), \operatorname{diam}(Y)\}.$$

This means that in the case of spheres with the geodesic metric we have the following bound:

$$d_{GH}(\mathbb{S}^m, \mathbb{S}^n) \leq \frac{\pi}{2} \text{ for all } 0 \leq m \leq n. \tag{4}$$

Here we state a theorem from Lyusternik and Schnirelmann, which can be found in [Bol06, p.118], that follows from the Borsuk-Ulam theorem. This theorem will be helpful for proving some bounds for the Gromov-Hausdorff distance in terms of correspondences.

Theorem 5.1 (Lyusternik- Schnirelmann)**.** Let $n \in \mathbb{N}$, and $\{U_1, \ldots, U_{n+1}\}$ be a closed cover of $\mathbb{S}^n$. Then there is $i_0 \in \{1, \ldots, n+1\}$ such that U_{i_0} contains two antipodal points.

Lemma 5.2. For any integer $m \geq 1$ and any finite metric space P with cardinality at most $m+1$ we have that $d_{GH}(P, \mathbb{S}^n) \geq \pi/2$.

Proof. Suppose $m \geq 1$ is given. Assume that R is an arbitrary correspondence between $\mathbb{S}^m$ and P. We claim that $\operatorname{dis}(R) \geq \pi$. For each $p \in P$ let $R(p) := \{z \in \mathbb{S}^m \mid (x,p) \in R\}$. Then $\{\overline{R(p)} \mid p \in P\}$ is a closed cover of $\mathbb{S}^m$. Since P has cardinality at most $m+1$, Theorem 5.1 yields that for some $p_0 \in P$ we have that $\operatorname{diam}(R(p_0) = \pi$. The claim follows since

$$\operatorname{dis}(R) \geq \max_{p \in P} \operatorname{diam}(R(p)). \qquad \square$$

Hence we have that for each integer $n \geq 1$, if P is a finite metric space with cardinality at most $n+1$ and $\operatorname{diam}(P) \leq \pi$ then $d_{GH}(P, \mathbb{S}^n) = \pi/2$.

Theorem 5.3 (Distance to $\mathbb{S}^0$)**.** For any integer $n \geq 1$,

$$d_{GH}(\mathbb{S}^0, \mathbb{S}^n) = \frac{\pi}{2}.$$

Proof. The proof follows from the fact that $\mathbb{S}^0$ is a finite metric space with cardinality 2 and that $\operatorname{diam}(\mathbb{S}^0) = \pi$. $\square$

We can use the same argument in the proof of Lemma 5.2 to obtain the following:

Corollary 5.4. Let R be any correspondence between a finite metric space P and $\mathbb{S}^\infty$. Then $\operatorname{dis}(R) \geq \pi$.

Proof. As in the proof of Lemma 5.2, the correspondence R induces a closed cover of $\mathbb{S}^\infty$. Thus, it induces a closed cover of any finite dimensional sphere $\mathbb{S}^{|P|-1} \subset \mathbb{S}^\infty$. Hence we can apply Theorem 5.1 to obtain the result. $\square$

Actually a small modification of the proof of Corollary 5.4 gives us the following stronger claim. We say that a metric space X is *totally bounded* if and only if for every real number $\varepsilon > 0$, there exists a finite collection of open balls of radius ε whose centers lie in X and whose union contains X. This is equivalent to the existence of a finite ε-net.

Theorem 5.5. Let X be any totally bounded metric space. Then

$$d_{GH}(X, \mathbb{S}^\infty) \geq \frac{\pi}{2}.$$

Proof. Fix $\varepsilon > 0$ and let P_ε be a finite ε-net for X. Then, by the triangle inequality and Corollary 5.4, we have that

$$d_{GH}(X, \mathbb{S}^\infty) \geq d_{GH}(\mathbb{S}^\infty, P_\varepsilon) - d_{GH}(X, P_\varepsilon) \geq \frac{\pi}{2} - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies the claim. □

The Borsuk-Ulam theorem implies that, for any positive integers $n > m$ and for any given continuous function $\phi\colon \mathbb{S}^n \to \mathbb{S}^m$, there exist two antipodal points in the higher dimensional sphere which are mapped to the same point in the lower dimensional sphere. This forces the distortion of any such continuous map to be π.

In contrast we prove that there always exists a surjective, antipode preserving and continuous map $\psi_{m,n}$ from $\mathbb{S}^m$ to $\mathbb{S}^n$.

Theorem 5.6. For all integers $0 < m < n < \infty$, there exists an antipode preserving continuous surjection $\psi_{m,n}\colon \mathbb{S}^m \to \mathbb{S}^n$, i.e., $\psi_{m,n}(-x) = -\psi_{m,n}(x)$ for every $x \in \mathbb{S}^m$.

In order to prove Theorem 5.6, we need to work a bit more. Spherical suspensions and space-filling curves are key technical tool for this purpose. The existence of space-filling curves is well known; see [Pea90].

Theorem 5.7 ([Pea90])**.** There exists a continuous and surjective map

$$H\colon [0,1] \to [0,1]^2.$$

By resorting to space-filling curves, we can prove the following proposition which will be crucial.

Proposition 5.8. There exists an antipode preserving continuous surjection

$$\psi_{1,2}\colon \mathbb{S}^1 \to \mathbb{S}^2.$$

Proof. We denote by $\mathrm{Conv}(v_1, \ldots, v_d)$ the convex hull of the vectors $v_1, \ldots, v_d$. Let e_i for $i = 1, 2, 3$ be the canonical orthonormal basis of $\mathbb{R}^3$. We define the 3-dimensional cross-polytope by

$$\hat{\mathbb{B}}^3 = \mathrm{Conv}(e_1, -e_1, e_2, -e_2, e_3, -e_3) \subset \mathbb{R}^3.$$

Then, its boundary $\partial\hat{\mathbb{B}}^3$, which consists of eight triangles

$$\mathrm{Conv}(e_1, e_2, e_3), \mathrm{Conv}(e_1, e_2, -e_3), \ldots, \mathrm{Conv}(-e_1, -e_2, -e_3),$$

is homeomorphic to $\mathbb{S}^2$.

Now, we divide $\mathbb{S}^1$ into eight closed circular arcs with length equal to $\pi/4$. We are able to build a continuous and surjective map

$$\phi_1\colon [0, \pi/4] \to \mathrm{Conv}(e_1, e_2, e_3) \text{ such that } \phi_1(0) = e_1, \phi_1(\pi/4) = e_2$$

as follows.

Since $\mathrm{Conv}(e_1, e_2, e_3)$ is homeomorphic to $[0, 1]^2$, by Theorem 5.7 there exists a continuous and surjective map $\phi_1'\colon [\pi/12, \pi/6] \to \mathrm{Conv}(e_1, e_2, e_3)$. Then we can extend its domain using linear interpolation between e_1 and $\phi'(\pi/12)$, and between e_2 and $\phi'(\pi/6)$ to give rise to ϕ_1. By using an analogous procedure, we can construct continuous and surjective maps

$$\phi_2\colon [\pi/4, \pi/2] \to \mathrm{Conv}(-e_1, e_2, e_3) \text{ such that } \phi_2(\pi/4) = e_2, \phi_2(\pi/2) = e_3,$$
$$\phi_3\colon [\pi/2, 3\pi/4] \to \mathrm{Conv}(e_1, -e_2, e_3) \text{ such that } \phi_3(\pi/2) = e_3, \phi_3(3\pi/4) = -e_2,$$
$$\phi_4\colon [3\pi/4, \pi] \to \mathrm{Conv}(-e_1, -e_2, e_3) \text{ such that } \phi_3(3\pi/4) = -e_2, \phi_3(\pi) = -e_1.$$

We construct the remaining ones, $\phi_5, \phi_6, \phi_7, \phi_8$ by suitably reflecting the ones already constructed:

$$\phi_5\colon [\pi, 5\pi/4] \to \mathrm{Conv}(-e_1, -e_2, -e_3) \text{ such that } \phi_5(x) = -\phi_1(-x),$$
$$\phi_6\colon [5\pi/4, 3\pi/2] \to \mathrm{Conv}(e_1, -e_2, -e_3) \text{ such that } \phi_6(x) = -\phi_2(-x),$$
$$\phi_7\colon [3\pi/2, 7\pi/4] \to \mathrm{Conv}(e_1, e_2, -e_3) \text{ such that } \phi_7(x) = -\phi_3(-x),$$
$$\phi_8\colon [7\pi/4, 2\pi] \to \mathrm{Conv}(-e_1, e_2, -e_3) \text{ such that } \phi_8(x) = -\phi_4(-x).$$

Finally, by gluing all eight maps ϕ_i, we build an antipode preserving continuous and surjective map $\hat{\psi}_{1,2}\colon \mathbb{S}^1 \to \partial\hat{\mathbb{B}}^3$. Using the canonical (closest point projection) homeomorphism between $\hat{\mathbb{B}}^3$ and $\mathbb{S}^2$, we obtain $\psi_{1,2}\colon \mathbb{S}^1 \to \mathbb{S}^2$.

From its construction the map $\psi_{1,2}$ is continuous, surjective, and antipode preserving. $\square$

Suppose given $m, n \in \mathbb{N}$ and a map $f\colon \mathbb{S}^m \to \mathbb{S}^n$. One can lift this map f to a map from $\mathbb{S}^{m+1}$ to $\mathbb{S}^{n+1}$ in the following way: A point from $\mathbb{S}^{m+1}$ can be expressed as $(p\sin\theta, \cos\theta)$ for some $p \in \mathbb{S}^m$ and $\theta \in [0, \pi]$. Then, the *spherical suspension of f* is the map

$$\begin{aligned} Sf\colon \mathbb{S}^{m+1} &\to \mathbb{S}^{n+1} \\ (p\sin\theta, \cos\theta) &\mapsto (f(p)\sin\theta, \cos\theta). \end{aligned}$$

Lemma 5.9. If the map $f\colon \mathbb{S}^m \to \mathbb{S}^n$ is continuous, surjective, and antipode preserving, then $Sf\colon \mathbb{S}^{m+1} \to \mathbb{S}^{n+1}$ is also continuous, surjective, and antipode preserving.

Proof. From the construction it is clear that Sf is continuous and surjective. Since f is antipode preserving, we know that $f(-p) = -f(p)$ for every $p \in \mathbb{S}^m$. Hence,

$$\begin{aligned} Sf(-p\sin\theta, -\cos\theta) &= Sf(-p\sin(\pi-\theta), \cos(\pi-\theta)) \\ &= (f(-p)\sin(\pi-\theta), \cos(\pi-\theta)) \\ &= (-f(p)\sin(\pi-\theta), \cos(\pi-\theta)) \\ &= -Sf(p\sin(\pi-\theta), \cos(\pi-\theta)) \end{aligned}$$

for any $p \in \mathbb{S}^m$ and $\theta \in [0\pi]$. Thus, Sf is also antipode preserving. $\square$

Using induction we can obtain the following corollary:

Corollary 5.10. For any integer $m > 0$, there exists a continuous, surjective, and antipode preserving map

$$\psi_{m,m+1}\colon \mathbb{S}^m \to \mathbb{S}^{m+1}.$$

Proof. We have the existence of $\psi_{1,2}$ from Proposition 5.8. For general m, it suffices to apply Lemma 5.9 in the induction hypothesis. $\square$

Proof of Theorem 5.6. By Corollary 5.10, there are continuous, surjective, and antipode preserving maps $\psi_{m,m+1}, \psi_{m+1,m+2}, \ldots, \psi_{n-1,n}$. Then the map

$$\begin{aligned}\psi_{m,n} \colon \mathbb{S}^m &\to \mathbb{S}^n \\ p &\mapsto \psi_{m,n}(p) := \psi_{n-1,n} \circ \cdots \circ \psi_{m,m+1}(p)\end{aligned}$$

is also continuous, surjective, and antipode preserving. This is due to the fact that composition map of two continuous, surjective, and antipode preserving maps is continuous, surjective, and antipode preserving. $\square$

Theorem 5.11. For all $0 < m < n < \infty$ we have that

$$d_{GH}(\mathbb{S}^m, \mathbb{S}^n) < \frac{\pi}{2}.$$

Proof. Let $\psi_{m,n} \colon \mathbb{S}^m \to \mathbb{S}^n$ be an antipode preserving continuous surjection. Recall that the graph of a surjective map can be seen as a correspondence, so let $R_{m,n} = \mathrm{graph}(\psi_{m,n})$. In order to prove the theorem, it is enough to prove that $\mathrm{dis}(R_{m,n}) = \mathrm{dis}(\psi_{m,n}) < \pi$.

Since $\psi_{m,n}$ is continuous and $\mathbb{S}^m$ is compact, the supremum in the definition of the distortion is a maximum:

$$\mathrm{dis}(\psi_{m,n}) = \max_{x,x' \in \mathbb{S}^m} |d_{\mathbb{S}^m}(x, x') - d_{\mathbb{S}^n}(\psi_{m,n}(x), \psi_{m,n}(x'))|.$$

Let $x_0, x_0' \in \mathbb{S}^m$ attain the maximum. We may assume that $x_0 \neq x_0'$, otherwise $d_{GH}(\mathbb{S}^m, \mathbb{S}^n) \leq \frac{1}{2}\mathrm{dis}(\psi_{m,n}) = 0$, which is a contradiction since $n \neq m$.

In this case suppose first that $-x_0 \neq x_0'$. We have that

$$0 < d_{\mathbb{S}^m}(x, x') < \pi \text{ and } 0 \leq d_{\mathbb{S}^n}(\psi_{m,n}(x), \psi_{m,n}(x')) \leq \pi.$$

Thus

$$|d_{\mathbb{S}^m}(x, x') - d_{\mathbb{S}^n}(\psi_{m,n}(x), \psi_{m,n}(x'))| < \pi.$$

Assume now that $-x_0 = x_0'$. In this case $d_{\mathbb{S}^m}(x, x') = d_{\mathbb{S}^n}(\psi_{m,n}(x), \psi_{m,n}(x')) = \pi$ since $\psi_{m,n}$ is antipode preserving. Thus, we also have that

$$0 = |d_{\mathbb{S}^m}(x, x') - d_{\mathbb{S}^n}(\psi_{m,n}(x), \psi_{m,n}(x'))| < \pi. \tag{5}$$

$\square$

Observe that the antipode preserving property of $\psi_{m,n}$ given in Theorem 5.6 is stronger than what we need in the proof of Theorem 5.11. Indeed, all one needs is that $\psi_{m,n}(x) \neq \psi_{m,n}(-x)$ for any $x \in \mathbb{S}^m$.

Now we focus on giving some non trivial lower bound for the Gromov-Hausdorff distance between spheres of different dimensions. We will make use of Vietoris-Rips complexes in order to transform an odd function f into a continuous odd map between Vietoris-Rips complexes of spheres.

We follow the definitions of [LMS21], but [MBZ03, Chapter 5] contains more background. We have the following concepts and properties:

- A $\mathbb{Z}/2$ *space* is a topological space X equipped with an *involution map*, denoted by $x \mapsto -x$, such that $-(-x) = x$ for all $x \in X$. We say that a $\mathbb{Z}/2$ space X is *free* if $-x \neq x$ for all $x \in X$.
- If X and Y are $\mathbb{Z}/2$ spaces, then a function $f\colon X \to Y$ is *odd* if $f(-x) = -f(x)$ for all $x \in X$.
- If X, Y, Z are $\mathbb{Z}/2$ spaces, and $f\colon Y \to Z$, $g\colon X \to Y$ are odd, then $f \circ g$ is odd.
- The sphere $\mathbb{S}^n$ is a $\mathbb{Z}/2$ space, since it inherits the involution map of $\mathbb{R}^{n+1}$.
- The *index* of a $\mathbb{Z}/2$ space X is defined to be

$$\mathrm{ind}(X) = \min\{k \geq 0 \mid \text{ there exists an odd map } X \to \mathbb{S}^k\}.$$

- The *coindex* of a $\mathbb{Z}/2$ space X is defined to be

$$\mathrm{coind}(X) = \max\{k \geq 0 \mid \text{ there exists an odd map } \mathbb{S}^k \to X\}.$$

- For all $n \geq 0$, we have $\mathrm{ind}(\mathbb{S}^n) = \mathrm{coind}(\mathbb{S}^n) = n$, by the Borsuk-Ulam theorem.
- For all $\mathbb{Z}/2$ spaces X, we have $\mathrm{ind}(X) \geq \mathrm{coind}(X)$, by the Borsuk-Ulam theorem.
- If there is an odd map $X \to Y$, then $\mathrm{ind}(X) \leq \mathrm{ind}(Y)$ and $\mathrm{coind}(X) \leq \mathrm{coind}(Y)$.
- If the $\mathbb{Z}/2$ space X is not free, then $\mathrm{ind}(X) = \mathrm{coind}(X) = \infty$ because we may construct an odd map $\mathbb{S}^k \to X$ for any $k \geq 0$ by taking the constant map to a fixed point of the $\mathbb{Z}/2$ action on X.
- A $\mathbb{Z}/2$ *metric space* is a $\mathbb{Z}/2$ space which is also a metric space, and that satisfies $d_X(x, x') = d_X(-x, -x')$ for all $x, x' \in X$.

- Let X, Y be $\mathbb{Z}/2$ spaces, and let $f_0, f_1\colon \to Y$ be odd maps. Then a $\mathbb{Z}/2$ *homotopy* from f_0 to f_1 is a homotopy $H\colon X \times [0,1] \to Y$ such that $H(\cdot, t)$ is odd for all $t \in [0,1]$. In this case we say that f_0 and f_1 are $\mathbb{Z}/2$ *homotopic*.
- Let X, Y be $\mathbb{Z}/2$ spaces. We say that X, Y are $\mathbb{Z}/2$ *homotopy equivalent*, denoted $X \sim_{\mathbb{Z}/2} Y$, if there exists odd maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ is $\mathbb{Z}/2$ homotopic to the identity map on Y and $g \circ f$ is $\mathbb{Z}/2$ homotopic to the identity map on X.

Note that if $X \sim_{\mathbb{Z}/2} Y$, then $\mathrm{ind}(X) = \mathrm{ind}(Y)$ and $\mathrm{coind}(X) = \mathrm{coind}(Y)$.

We say that a space X is *k-connected* if the homotopy groups $\pi_i(X)$ are trivial for $i \leq k$. An important property is the following:

(6) If a $\mathbb{Z}/2$ space X is $(k-1)$-connected, then $\mathrm{ind}(X) \geq \mathrm{coind}(X) \geq k$.

We identify a simplicial complex with its realization. For example, if $\{x_0, \ldots, x_m\}$ is a simplex in a simplicial complex, then we write $\sum_i \lambda_i x_i$ to refer to a point in the geometric realization of its simplicial complex, where the barycentric coordinates $\lambda_i \geq 0$ satisfy $\sum_i \lambda_i = 1$. A simplicial map between two simplicial complexes indeed induces a continuous function between their geometric realizations.

For a metric space X and $r \geq 0$, the *Vietoris-Rips simplicial complex* $VR(X;r)$ has vertex set X and a nonempty finite subset $\sigma \subset X$ is a simplex when $\mathrm{diam}(\sigma) \leq r$. The Vietoris-Rips complex is a clique complex, which means that for every nonempty finite $\sigma \subset X$, the simplex σ is in $VR(X;r)$ if and only if the edge $\{u, v\}$ is in $VR(X;r)$ for every pair $u, v \in \sigma$.

If X is a $\mathbb{Z}/2$ metric space and $r \geq 0$, we extend the involution on X to an involution on $VR(X;r)$ by defining

$$-\sum_i \lambda_i x_i := \sum_i \lambda_i(-x_i).$$

Note that if X is a free $\mathbb{Z}/2$ metric space then $VR(X;r)$ is a free $\mathbb{Z}/2$ space whenever $r < \inf_{x \in X} d_X(x, -x)$. In particular, $VR(\mathbb{S}^n; r)$ is a free $\mathbb{Z}/2$ space for $r < \pi$.

Definition 5.12. For $k \geq n$, we define

$$c_{n,k} = \inf\{r \geq 0 \mid \text{ there exists an odd map } \mathbb{S}^k \to VR(\mathbb{S}^n; r)\}.$$

We recall that a subset A of a metric space X is an *ε-covering* if for every point $x \in X$ there exists a point $a \in A$ such that $d_X(a, x) < \varepsilon$.

Lemma 5.13. For $X \subset \mathbb{S}^k$ a finite $(\varepsilon/2)$-covering with $X = -X$, there exists an odd map $\phi \colon \mathbb{S}^k \to VR(X;\varepsilon)$.

Proof. It suffices to consider $\varepsilon < \pi$, otherwise $VR(X;\varepsilon)$ is not a free $\mathbb{Z}/2$ space and $\mathrm{coind}(VR(X;\varepsilon)) = \infty$. Let $\{\rho_x\}_{x\in X}$ be a $\mathbb{Z}/2$ invariant partition of unity subordinated to the cover $\{B(x,\varepsilon/2)\}_{x\in X}$ of $\mathbb{S}^k$. That is,

- ρ_x is a nonnegative continuous real-valued function supported in the ball $B(x,\varepsilon/2)$.
- $\sum_{x\in X} \rho_x(y) = 1$ for all $y \in \mathbb{S}^k$.
- $\rho_{-x}(-y) = \rho_x(y)$ for all $x \in X$ and $y \in \mathbb{S}^k$.

To see that such a $\mathbb{Z}/2$ partition of unity exists, note that it can be obtained from a (standard) partition of unity on the quotient space $\mathbb{R}P^n$.

Define the map $\phi \colon \mathbb{S}^k \to VR(X;\varepsilon)$ by $\phi(y) = \sum_{x\in X} \rho_x(y)x$. Note that any point x whose coefficient in $\phi(y)$ is positive must have $d_{\mathbb{S}^k}(x,y) < \varepsilon/2$ because ρ_x is supported in $B(x,\varepsilon/2)$. Therefore, $\mathrm{diam}\{x \in X \mid \rho_x(y) > 0\} < \varepsilon$, so $\phi(y)$ is a well defined point in $VR(X;\varepsilon)$. Since each ρ_x is continuous, so is ϕ. Since $-X = X$, we can compute that

$$\begin{aligned}\phi(-y) = \sum_{x\in X} \rho_x(-y)x = \sum_{x\in X} \rho_{-x}(y)(x) = \sum_{x\in -X} \rho_x(y)(-x)\\ = \sum_{x\in -X} \rho_x(y)(-x) = \sum_{x\in X} \rho_x(-y)(x) = \phi(-y).\end{aligned}$$

Thus ϕ is an odd map. ☐

Choosing a different partition of unity will produce a map that is homotopic to ϕ. Indeed, given two partitions of unity $\{\rho_x^1\}_{x\in X}$ and $\{\rho_x^2\}_{x\in X}$, a homotopy between the corresponding maps ϕ_1, ϕ_2 can be given by a straight line homotopy $H(\cdot,t) = t\phi_1 + (1-t)\phi_2$.

Lemma 5.14. A function $f \colon X \to Y$ between metric spaces induces a simplicial map $\hat{f} \colon VR(X;r) \to VR(Y;r + \mathrm{dis}(f))$ for any $r \geq 0$. If X, Y are $\mathbb{Z}/2$ metric spaces and f is odd, then $\hat{f}$ is also odd.

Proof. We define $\hat{f} \colon VR(X;r) \to VR(Y;r + \mathrm{dis}(f))$ by sending a vertex $x \in X$ to $f(x) \in Y$, and then extending linearly to simplices. In other words, $\hat{f}([x_0,\ldots,x_m]) = [f(x_0),\ldots,f(x_m)]$. If $\mathrm{diam}(\sigma) \leq r$, then by definition of distortion, $\mathrm{diam}(f(\sigma)) \leq r + \mathrm{dis}(f)$. Thus, $\hat{f}$ is well defined, simplicial, and continuous on the underlying geometric realizations.

If both X, Y are $\mathbb{Z}/2$ metric spaces and f is an odd function, then

$$\hat{f}\left(-\sum_i \lambda_i x_i\right) = \hat{f}\left(\sum_i \lambda_i(-x_i)\right) = \sum_i \lambda_i f(-x_i) = -\sum_i \lambda_i f(x_i). \qquad \square$$

This lemma shows how to turn a possibly discontinuous function into a continuous one.

For each integer $n \geq 1$, recall the natural isometric embedding of $\mathbb{S}^{n-1}$ to the equator $E(\mathbb{S}^n)$ of $\mathbb{S}^n$:

$$\begin{aligned} t_{n-1} \colon \mathbb{S}^{n-1} &\hookrightarrow \mathbb{S}^n \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_n, 0). \end{aligned}$$

Also, define the sets $A(\mathbb{S}^n) \subset \mathbb{S}^n$, which are usually called *helmets*:

Definition 5.15. Let us define

$$\begin{aligned} &A(\mathbb{S}^0) := \{1\}, \\ &A(\mathbb{S}^1) := \{(\cos\theta, \sin\theta) | \theta \in [0, \pi)\}. \end{aligned}$$

Moreover, for general $n \geq 1$, we define inductively

$$A(\mathbb{S}^n) := \mathbb{H}_{>0}(\mathbb{S}^n) \cup t_{n-1}(A(\mathbb{S}^{n-1})),$$

where $\mathbb{H}_{>0}(\mathbb{S}^n) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_{n+1} > 0\}$.

Observe that, for any $n \geq 0$,

$$A(\mathbb{S}^n) \cap (-A(\mathbb{S}^n)) = \emptyset \text{ and } A(\mathbb{S}^n) \cup (-A(\mathbb{S}^n)) = \mathbb{S}^n.$$

The following lemma is simple but critical.

Lemma 5.16. For any $m, n \geq 0$, let $C \subset \mathbb{S}^n$ be a nonempty set such that $C \cap (-C) = \emptyset$ and let $\phi \colon C \to \mathbb{S}^m$ be any map. Then, the extension $\phi^* \colon C \cup (-C) \to \mathbb{S}^m$ of ϕ defined by $\phi^*(x) = \phi(x)$ for all $x \in C$ and $\phi^*(-x) = -\phi(x)$ for all $x \in -C$ is an antipode preserving map and satisfies that $\mathrm{dis}(\phi^*) = \mathrm{dis}(\phi)$.

Proof. By definition ϕ^* is antipode preserving. Now for $x, x' \in C$ we have

$$\begin{aligned} |d_{\mathbb{S}^n}(x, -x') - d_{\mathbb{S}^m}(\phi^*(x), \phi^*(-x'))| &= |(\pi - d_{\mathbb{S}^n}(x, x')) - (\pi - d_{\mathbb{S}^m}(\phi(x), \phi(x')))| \\ &= |d_{\mathbb{S}^n}(x, x') - d_{\mathbb{S}^m}(\phi(x), \phi(x'))| \leq \mathrm{dis}(\phi) \end{aligned}$$

and

$$|d_{\mathbb{S}^n}(-x,-x') - d_{\mathbb{S}^m}(\phi^*(-x),\phi^*(-x'))| = |d_{\mathbb{S}^n}(x,x') - d_{\mathbb{S}^m}(\phi(x),\phi(x'))|$$
$$\leq \mathrm{dis}(\phi).$$

Hence we have that $\mathrm{dis}(\phi^*) = \mathrm{dis}(\phi)$. □

Theorem 5.17. For all $k \geq n$, the following inequalities hold:

$$2 \cdot d_{GH}(\mathbb{S}^n, \mathbb{S}^k) \geq \inf\{\mathrm{dis}(f) \mid f\colon \mathbb{S}^k \to \mathbb{S}^n \text{ is odd}\} \geq c_{n,k}.$$

Proof. Let $k \geq n$ and let $f\colon \mathbb{S}^k \to \mathbb{S}^n$ be an odd function. The last inequality follows from the claim that $\mathrm{dis}(f) \geq c_{n,k}$. Indeed, let $\varepsilon > 0$ and choose a finite $\mathbb{Z}/2$ invariant ε-covering $X \subset \mathbb{S}^k$. By Lemma 5.13 we get an odd map $\mathbb{S}^k \to VR(X;\varepsilon)$, and by Lemma 5.14 the restriction map $f_{|X}\colon X \to \mathbb{S}^n$ induces a continuous odd map $VR(X;\varepsilon) \to VR(X;\varepsilon + \mathrm{dis}(f))$. Their composition

$$\mathbb{S}^k \to VR(X;\varepsilon) \to VR(X;\varepsilon + \mathrm{dis}(f))$$

is continuous and odd, showing that $\varepsilon + \mathrm{dis}(f) \geq c_{n,k}$ for all $\varepsilon > 0$. Hence $\mathrm{dis}(f) \geq c_{n,k}$.

The first inequality is the "helmet trick" from [LMS21]. Lemma 5.16 states that any function $h\colon \mathbb{S}^k \to \mathbb{S}^n$ can be modified to obtain an odd function $f\colon \mathbb{S}^k \to \mathbb{S}^n$ with $\mathrm{dis}(f) \leq \mathrm{dis}(h)$. Therefore, since

$$2 \cdot d_{GH}(\mathbb{S}^n, \mathbb{S}^k) = \inf_{g,h} \max\{\mathrm{dis}(h), \mathrm{dis}(g), \mathrm{codis}(h,g)\}$$

where the infimum is taken over all maps $h\colon \mathbb{S}^k \to \mathbb{S}^n$ and $g\colon \mathbb{S}^n \to \mathbb{S}^k$,

$$2 \cdot d_{GH}(\mathbb{S}^n, \mathbb{S}^k) \geq \inf\{\mathrm{dis}(h) | h\colon \mathbb{S}^k \to \mathbb{S}^n\} \geq \inf\{\mathrm{dis}(f) | f\colon \mathbb{S}^k \to \mathbb{S}^n \text{ is odd }\}.$$

□

Now we show how strong is Theorem 5.17 by describing the known values of the constants $c_{n,k}$. These results depend on the topology of Vietoris-Rips complexes and thickenings of spheres. Indeed, the topology of $VR(\mathbb{S}^n;r)$ constrains how large the scale r must be in order for the complex to admit an odd map from the k-sphere.

We begin with some basic properties that follow from the definition of $c_{n,k}$. The inclusion $\mathbb{S}^k \hookrightarrow \mathbb{S}^{k'}$ implies that $c_{n,k} \leq c_{n,k'}$ for $k \leq k'$. Furthermore, the inclusion $VR(\mathbb{S}^{n'};r) \hookrightarrow VR(\mathbb{S}^n;r)$ implies that $c_{n,k} \leq c_{n',k'}$ for $k \leq k'$ and

$n' \leq n$. Since the diameter of $\mathbb{S}^n$ is π, the complex $VR(\mathbb{S}^n;\pi)$ is contractible and therefore $c_{n,k} \leq \pi$ for all $n \leq k$.

We have that $c_{n,n} = 0$ since $VR(\mathbb{S}^n;\varepsilon) \simeq_{\mathbb{Z}/2\mathbb{Z}} \mathbb{S}^n$ for all $\varepsilon > 0$ sufficiently small. Also we can see that $c_{0,k} = \pi$ for all $k > 0$.

Theorem 5.18. For all $l \geq 1$, we have $c_{1,2l+1} = c_{1,2l} = \frac{2\pi l}{2l+1}$.

In order to prove this theorem we have to introduce what Vietoris-Rips metric thickenings are. We replace the points of the simplices in this definition with the corresponding finitely supported probability measures. That is, we define the *Vietoris-Rips metric thickening* $VR^m(X;r)$ to be the subspace of the finitely supported probability measures $\mathcal{P}$ consisting of all measures μ such that $\mathrm{supp}(\mu)$ has diameter at most r. The topology of $VR^m(X;r)$ is the metric topology given by the 1-Wasserstein distance. For more details, see [Moy22].

Proof. These values are related to the homotopy types of the simplicial complexes $VR(\mathbb{S}^1;r)$ and of the metric thickenings $VR^m(\mathbb{S}^1;r)$. We can compute the homotopy types of these simplicial complexes $VR(\mathbb{S}^1;r) \simeq \mathbb{S}^{2l+1}$ for $\frac{2\pi l}{2l+1} \leq \frac{2\pi(l+1)}{2l+3}$. The homotopy types of these metric thickenings are determined in [Moy22] as $VR^m(\mathbb{S}^1;r) \simeq \mathbb{S}^{2l+1}$ for $\frac{2\pi l}{2l+1} \leq \frac{2\pi(l+1)}{2l+3}$.

For $l > \frac{2\pi l}{2l+1}$, $VR(\mathbb{S}^1;r)$ is $2l$-connected. Then from (6) we obtain an odd map $\mathbb{S}^{2l+1} \to VR(Sp^1;r)$. This shows that $c_{1,2l} \leq c_{1,2l+1} \leq \frac{2\pi l}{2l+1}$. On the other hand, as in Section 5.1 of [ABF20], we can produce an odd map $VR^m(\mathbb{S}^1;r) \to \mathbb{R}^{2l} \setminus \{0\} \simeq_{\mathbb{Z}/2\mathbb{Z}} \mathbb{S}^{2l-1}$ for $r < \frac{2\pi l}{2l+1}$. This shows that $c_{1,2l} \geq c_{1,2l+1} \geq \frac{2\pi l}{2l+1}$. Hence $c_{1,2l} = c_{1,2l+1} = \frac{2\pi l}{2l+1}$. □

Theorem 5.19. For all $n \geq 1$, we have $c_{n,n+2} = c_{n,n+1} = r_n := \arccos\left(\frac{-1}{n+1}\right)$.

Proof. These values follow from knowledge about the homotopy types of $VR(\mathbb{S}^n;r)$ and $VR^m(\mathbb{S}^n;r)$. From results in [AAF18, Proposition 5.3] and [LMO20, Corollary 7.1] we have that $VR^m(\mathbb{S}^n;r) \simeq \mathbb{S}^n$ and $VR(\mathbb{S}^n;r) \simeq \mathbb{S}^n$ for all $r < r_n$. Recall that the alternating group $A(n+2)$ can be seen as a subgroup of $SO(n)$ even though there is no canonical way to do this; see [AHP22, Section 5.1] for more details. Furthermore, [AAF18, Theorem 5.4] provides a homotopy equivalence $VR^m(\mathbb{S}^n;r_n) \simeq \mathbb{S}^n * \frac{SO(n+1)}{A(n+2)}$. Since S^n is

$(n-1)$-connected and $\frac{SO(n+1)}{A(n+2)}$ is 0-connected, their join is $(n+1)$-connected. This shows that $c_{n,n+1} \leq c_{n,n+2} \leq r_n$. On the other hand, [AAF18, Proposition 5.3] produces an odd map $VR^m(\mathbb{S}^n; r) \to \mathbb{R}^{n+1} \setminus \{0\} \simeq_{\mathbb{Z}/2\mathbb{Z}} \mathbb{S}^n$ for $r < r_n$, and the same construction provides an odd map $VR(\mathbb{S}^n; r) \to \mathbb{R}^{n+1} \setminus \{0\}$. Therefore, the Borsuk-Ulam theorem implies that there cannot exist odd maps $\mathbb{S}^{n+1} \to VR^m(\mathbb{S}^n; r)$ or $\mathbb{S}^{n+1} \to VR(\mathbb{S}^n; r)$ for $r < r_n$. This shows that $c_{n,n+1} \geq c_{n,n+2} \geq r_n$. Hence $c_{n,n+1} = c_{n,n+2} = r_n$. $\square$

Exact values of $c_{n,k}$ are not known for $n \geq 2$ and $k \geq n+3$, but we can provide some bounds on $c_{n,k}$ in terms of coverings of projective spaces. For a metric space X, we denote by $\mathrm{cov}_X(k)$ the infimum over all $\varepsilon > 0$ for which there exists a finite set $A \subset X$ of cardinality $|A| \leq k$ such that the balls of radius ε about A cover X, i.e., such that A is an ε-covering of X. Let $\mathbb{R}P^n$ be the projective space obtained as the quotient $\mathbb{S}^n/(x \sim -x)$, and equipped with the quotient metric. Explicitly,

$$d_{\mathbb{R}P^n}(\{x, -x\}, \{x', -x'\}) = \min\{d_{\mathbb{S}^n}(x, x'), d_{\mathbb{S}^n}(-x, -x')\},$$

so $\mathbb{R}P^n$ has diameter $\pi/2$.

Theorem 5.20. For all $k \geq n \geq 1$, we have that $c_{n,k} \geq \pi - 2\mathrm{cov}_{\mathbb{R}P^n}(k)$.

Proof. In [ABF21, Theorem 3] it is proved that if $\delta \geq \mathrm{cov}_{\mathbb{R}P^n}(k)$ then there is an odd map $VR^m(\mathbb{S}^n; \pi - 2\delta) \to \mathbb{S}^{k-1}$, and so $\mathrm{coind}(VR^m(\mathbb{S}^n; \pi - 2\delta)) \leq \mathrm{ind}(VR^m(\mathbb{S}^n; \pi - 2\delta)) \leq k-1$. Therefore, there is no odd map defined on $\mathbb{S}^k \to VR^m(\mathbb{S}^n; \pi - 2\delta)$ unless $\delta \leq \mathrm{cov}_{\mathbb{R}P^n}(k)$. Replacing $\pi - 2\delta$ by r we conclude that there is no odd map $\mathbb{S}^k \to VR^m(\mathbb{S}^n; r)$ unless $r \geq \pi - 2\mathrm{cov}_{\mathbb{R}P^n}(k)$. This is tight when $n = 1$ and k is odd. $\square$

For any $n \geq 1$, we have that

$$\lim_{k \to \infty} 2\mathrm{cov}_{\mathbb{R}P^n}(k) = 0.$$

Corollary 5.21. Let $n \geq 1$ be fixed. The distortion of an odd function $f\colon \mathbb{S}^k \to \mathbb{S}^n$ tends towards its maximum possible value π as k goes to infinity.

Now we give an upper bound on the Gromov-Hausdorff distance between $\mathbb{S}^n$ and $\mathbb{S}^{n+1}$ for all $n \geq 1$. For this we need to introduce several geometric objects. For all $n \geq 1$, we may inscribe a regular $(n+1)$-simplex in $\mathbb{S}^n$. Any pair of vertices of the inscribed simplex lie on the same geodesic distance apart, and this distance is exactly $r_n = \arccos(\frac{-1}{n+1})$. The facets of the inscribed simplex may be projected radially outward, obtaining $(n+2)$ sets that cover $\mathbb{S}^n$, and which are additionally closed, geodesically convex, and pairwise isometric. We call these radially projected facets *regular geodesic simplices* in $\mathbb{S}^n$. The diameters of these simplices are computed in [San46] and the value t_n is $\arccos(-\frac{n+1}{n+3})$ whenever n is odd and $\arccos(-\sqrt{\frac{n}{n+4}})$ whenever n is even.

This diameter is achieved between points at the centers of opposite faces which each contains half the vertices of the simplex and rounded appropriately whenever there is an odd number of vertices. Notice that $r_n \leq t_n$ for every n, and the equality holds only for $n = 1$. As $n \to \infty$ we have that $r_n \to \pi/2$ and $t_n \to \pi$. In particular, $t_n > 2\pi/3$ for all $n \geq 2$.

These quantities r_n, t_n played an essential role in the work of Lim, Mémoli, and Smith [ABC+22], who showed that $r_n \leq 2d_{GH}(\mathbb{S}^n, \mathbb{S}^{n+1}) \leq t_n$.

We first make an important observation regarding the distortion of relations which only pair together points that lie a bounded distance from one another.

Lemma 5.22. Let (X, d_X) be a metric space, and $Y \subset X$ be a subspace with induced metric d_Y. Let $R \subset X \times Y$ be any relation, and define

$$\varepsilon_R := \sup\{d_X(x,y) \mid (x,y) \in R\}.$$

Then the distortion of R is at most $2\varepsilon_R$.

Proof. Let (x,y) and (x',y') be in R. We want to bound the quantity $|d_X(x,x') - d_Y(y,y')|$. Applying the triangle inequality we obtain

$$d_X(x,x') \leq d_X(x,y) + d_X(y,y') + d_X(y',x') \leq 2\varepsilon_R + d_X(y,y').$$

We can also use the symmetry to see that $d_X(y,y') \leq 2\varepsilon_R + d_X(x,x')$. Hence $\mathrm{dis}(R) \leq 2\varepsilon_R$. $\square$

Recall that $\overline{H(\mathbb{S}^{n+1})}$ denotes the closed upper half hemisphere of $\mathbb{S}^{n+1}$, and let $N \in \overline{H(\mathbb{S}^{n+1})}$ denote the north pole. We consider the map $\tau \colon \overline{H(\mathbb{S}^{n+1})} \setminus$

$\{N\} \to \mathbb{S}^n$ which sends a point in the upper hemisphere to the unique nearest point on the equator.

We require two important facts for the statement that follows. The most crucial is that in order to bound $d_{GH}(\mathbb{S}^n, \mathbb{S}^{n+1})$ it suffices to bound the distortion of correspondences between the upper hemisphere $\overline{H(\mathbb{S}^{n+1})}$ and the equator of $\mathbb{S}^n$ (see Lemma 5.5 of [ABC+22]). Second, note that if $x \neq N$ and x' are points in $\overline{H(\mathbb{S}^{n+1})}$ and $d_{\mathbb{S}^{n+1}}(x, x') \geq \pi/2$, then $d_{\mathbb{S}^{n+1}}(\tau(x), x') \geq d_{\mathbb{S}^{n+1}}(x, x')$. Indeed, $d_{\mathbb{S}^{n+1}}(x, x') \geq \pi/2$ if and only if $\langle x, x' \rangle \leq 0$, and since both x and x' have nonnegative last coordinate we see that $\langle \tau(x), x' \rangle \leq \langle x, x' \rangle$, which implies that $d_{\mathbb{S}^{n+1}}(\tau(x), x') \geq d_{\mathbb{S}^{n+1}}(x, x')$.

Theorem 5.23. For every $n \geq 1$, we have that $d_{GH}(\mathbb{S}^n, \mathbb{S}^{n+1}) \leq \pi/3$.

Proof. We first construct a correspondence between $\mathbb{S}^n$ and $\mathbb{S}^{n+1}$, and then we bound its distortion.

Let $P = \{p_1, \ldots, p_{n+2}\}$ be the vertices of an inscribed regular $(n+1)$- simplex in $\mathbb{S}^n$. For each $i \in \{1, \ldots, n+2\}$, let F_i be the geodesic convex hull of $P \setminus \{p_i\}$. So, for each i the set F_i is a regular geodesic simplex in $\mathbb{S}^n$, and its barycenter is $-p_i$. We define

$$\begin{aligned} E :=& \{p \in \overline{H(\mathbb{S}^{n+1})} \mid d_{\mathbb{S}^{n+1}}(p, N) > \pi/3\}, \\ C_i :=& \{p \in \overline{H(\mathbb{S}^{n+1})} \mid p \neq N, \tau(p) \in F_i, \text{ and } d_{\mathbb{S}^{n+1}}(p, N) \leq \pi/3\} \cup \{N\}. \end{aligned}$$

We define a correspondence R between $\overline{H(\mathbb{S}^{n+1})}$ and $\mathbb{S}^n$ as follows:

$$R := \{(p, \tau(p)) \mid p \in E\} \sqcup \{(p, -p_i) \mid p \in C_i \text{ for some } i\}.$$

This is a correspondence since E and C_i cover $\overline{H(\mathbb{S}^{n+1})}$, and since $(p, p) \in R$ for every $p \in \mathbb{S}^n$.

Now we argue that the distortion of R is at most $2\pi/3$. Let $(x, y), (x', y') \in R$. We consider the following cases:

(1) Both x and x' lie in E. By Lemma 5.22, the relation between E and $\mathbb{S}^n$ consist of pairs $(x, \tau(x))$ has distortion at most $\pi/3$. Here we have $y = \tau(x)$ and $y' = \tau(x')$, so $|d_{\mathbb{S}^{n+1}}(x, x') - d_{\mathbb{S}^n}(y, y')|$ is at most $\pi/3$.

(2) Neither x and x' lie in E. Then we must have $d_{\mathbb{S}^{n+1}}(x, N) \leq \pi/3$ and $d_{\mathbb{S}^{n+1}}(x', N) \leq \pi/3$. Hence $d_{\mathbb{S}^{n+1}}(x, x') \leq 2\pi/3$. Moreover both y and y' lie in P so $d_{\mathbb{S}^n}(y, y') \leq r_n \leq 2\pi/3$. Thus we have that $|d_{\mathbb{S}^{n+1}}(x, x') - d_{\mathbb{S}^n}(y, y')|$ is at most $2\pi/3$.

(3) $x \in E$, $x' \notin E$, and $d_{\mathbb{S}^{n+1}}(x,x') \leq \pi/2$. It is sufficient to show that $d_{\mathbb{S}^n}(y,y') - d_{\mathbb{S}^{n+1}}(x,x') \leq 2\pi/3$. Since $y = \tau(x)$, we have that $d_{\mathbb{S}^{n+1}}(x,y) \leq \pi/6$. Moreover, for some $i \in \{1,\dots,n+2\}$ we have that $x' \in C_i$ and $y' = -p_i$. Every point in C_i has nonnegative inner product with $-p_i$, so $d_{\mathbb{S}^{n+1}}(x',y') \leq \pi/2$. Applying the triangle inequality twice, we obtain

$$\begin{aligned} d_{\mathbb{S}^n}(y,y') &\leq d_{\mathbb{S}^{n+1}}(y,x) + d_{\mathbb{S}^{n+1}}(x,x') + d_{\mathbb{S}^{n+1}}(x',y') \\ &\leq \frac{\pi}{6} + d_{\mathbb{S}^{n+1}}(x,x') + \frac{\pi}{2}. \end{aligned}$$

Hence $|d_{\mathbb{S}^n}(y,y') - d_{\mathbb{S}^{n+1}}(x,x')| \leq 2\pi/3$.

(4) $x \in E$, $x' \notin E$, and $d_{\mathbb{S}^{n+1}}(x,x') \leq \pi/2$. As in the previous case it is sufficient to show that $d_{\mathbb{S}^n}(y,y') - d_{\mathbb{S}^{n+1}}(x,x') \leq 2\pi/3$. We have that $y = \tau(x)$, and since $d_{\mathbb{S}^{n+1}}(x,x') > \pi/2$ we infer that

$$d_{\mathbb{S}^{n+1}}(x,x') \leq d_{\mathbb{S}^{n+1}}(y,x') \leq d_{\mathbb{S}^{n+1}}(y,y') + d_{\mathbb{S}^{n+1}}(x',y').$$

Consequently, $d_{\mathbb{S}^{n+1}}(x,x') - d_{\mathbb{S}^n}(y,y') \leq d_{\mathbb{S}^{n+1}}(x',y')$. By the previous case we have that $d_{\mathbb{S}^{n+1}}(x',y') \leq \pi/2$. □

In view of Theorems 5.17, 5.18 and 5.23, we have shown that in particular the Gromov-Hausdorff distance between $\mathbb{S}^1$ and $\mathbb{S}^2$ is equal to $\pi/3 \simeq 1.04719755120$.

6. Discussion

In this work we have seen how the Gromov-Hausdorff distance between pseudo-metric spaces X and Y is defined in terms of embeddings into a common pseudo-metric space containing isometric copies of X and Y, following [BBI01] and [KO99]. Then we have proved that this definition is equivalent to considering only pseudo-metrics on the disjoint union $X \sqcup Y$ as explained in [BBI01]. This restricts greatly the collection of spaces that we need to consider and allows us to understand that the Gromov-Hausdorff distance cannot necessarily be achieved inside any common metric space. The simplest example is to consider $X = Y$ and suppose that the Gromov-Hausdorff distance is attained within $X \sqcup X$. Since the Gromov-Hausdorff distance between X and X itself is 0, there would exist different points in $X \sqcup X$ for which the distance is 0. This is not allowed for a metric. Hence the remaining possibility is that there exists a pseudo-metric on $X \sqcup X$ for which the Gromov-Hausdorff distance is achieved.

We related the original definition of the Gromov-Hausdorff distance between pseudo-metric spaces X and Y with the distortion of correspondences between X and Y. Specifically, the Gromov-Hausdorff distance is half the infimum of the distortion of all correspondences between the given spaces X and Y, using [BBI01] as a reference. This relation allowed us to study the Gromov-Hausdorff distance by considering the distortion as a continuous function on the space of relations between X and Y. If we consider X and Y to be compact metric spaces, then the distortion function is a continuous function and the set of closed correspondences between X and Y is a compact set; therefore there exists a correspondence for which the infimum is achieved.

Following [KO99], we have introduced a generalization of the Gromov-Hausdorff distance for Banach spaces. We proved that this generalization is only useful when comparing real Banach spaces, since there exist real-isometric Banach spaces that are not complex-isometric.

We have studied an estimate of bounds for the Gromov-Hausdorff distance between spheres of different dimensions. In this case, bounding the Gromov-Hausdorff distance involves bounding the distortion of functions between spheres of different dimensions. Using ideas from [LMS21] we have proved that, in the case of spheres, it suffices to consider odd functions only. Therefore, using Vietoris-Rips complexes and the Borsuk-Ulam theorem we can establish a lower bound for the Gromov-Hausdorff distance between spheres. We can also provide an upper bound for the Gromov-Hausdorff distance between spheres of consecutive dimensions. Therefore, in particular, we could prove that the Gromov-Hausdorff distance between $\mathbb{S}^1$ and $\mathbb{S}^2$ is $d_{GH}(\mathbb{S}^1, \mathbb{S}^2) = \pi/3$, as first shown in [LMS21].

References

[AAF18] Michał Adamaszek, Henry Adams, and Florian Frick. Metric reconstruction via optimal transport. *SIAM Journal on Applied Algebra and Geometry*, 2(4):597–619, 2018.

[ABC$^+$22] Henry Adams, Johnathan Bush, Nate Clause, Florian Frick, Mario Gómez, Michael Harrison, R Amzi Jeffs, Evgeniya Lagoda, Sunhyuk Lim, Facundo Mémoli, et al. Gromov-Hausdorff distances, Borsuk-Ulam theorems, and Vietoris-Rips complexes. *arXiv preprint arXiv:2301.00246*, 2022.

[ABF20] Henry Adams, Johnathan Bush, and Florian Frick. Metric thickenings, Borsuk–Ulam theorems, and orbitopes. *Mathematika*, 66(1):79–102, 2020.

[ABF21] Henry Adams, Johnathan Bush, and Florian Frick. The topology of projective codes and the distribution of zeros of odd maps. *arXiv preprint arXiv:2106.14677*, 2021.

[AFN$^+$18] Pankaj K. Agarwal, Kyle Fox, Abhinandan Nath, Anastasios Sidiropoulos, and Yusu Wang. Computing the Gromov-Hausdorff distance for metric trees. *ACM Transactions on Algorithms (TALG)*, 14(2):1–20, 2018.

[AHP22] Henry Adams, Mark Heim, and Chris Peterson. Metric thickenings and group actions. *Journal of Topology and Analysis*, 14(03):587–613, 2022.

[AMJ18] David Alvarez-Melis and Tommi S Jaakkola. Gromov-Wasserstein alignment of word embedding spaces. *arXiv preprint arXiv:1809.00013*, 2018.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. A course in metric geometry. Graduate Studies in Mathematics vol. 33., Providence, RI: American Mathematical Society. 2001.

[Bol06] Béla Bollobás. *The art of mathematics: Coffee time in Memphis*. Cambridge University Press, 2006.

[Bou86] J Bourgain. Real isomorphic complex Banach spaces need not be complex isomorphic. *Proceedings of the American Mathematical Society*, 96(2):221–226, 1986.

[CC97] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. i. *Journal of Differential Geometry*, 46(3):406–480, 1997.

[Col96] Tobias H. Colding. Large manifolds with positive Ricci curvature. *Inventiones mathematicae*, 124(1):193–214, 1996.

[H$^+$95] Jean-Claude Hausmann et al. On the Vietoris-Rips complexes and a cohomology theory for metric spaces. *Annals of Mathematics Studies*, 138:175–188, 1995.

[IIT16] Alexander Ivanov, Stavros Iliadis, and Alexey Tuzhilin. Realizations of Gromov-Hausdorff distance. *arXiv preprint arXiv:1603.08850*, 2016.

[Kal95] Nigel J. Kalton. An elementary example of a Banach space not isomorphic to its complex conjugate. *Canadian Mathematical Bulletin*, 38(2):218–222, 1995.

[KO99] Nigel J. Kalton and Mikhail I. Ostrovskii. Distances between Banach spaces. 1999.

[KP79] N.J. Kalton and N.T. Peck. Twisted sums of sequence spaces and the three space problem. *Transactions of the American Mathematical Society*, 255:1–30, 1979.

[LMO20] Sunhyuk Lim, Facundo Mémoli, and Osman Berat Okutan. Vietoris-Rips persistent homology, injective metric spaces, and the filling radius. *arXiv preprint arXiv:2001.07588*, 2020.

[LMS21] Sunhyuk Lim, Facundo Mémoli, and Zane Smith. The Gromov-Hausdorff distance between spheres. *arXiv preprint arXiv:2105.00611*, 2021.

[MBZ03] Jiří Matoušek, Anders Björner, and Günter M Ziegler. *Using the Borsuk-Ulam theorem: Lectures on topological methods in combinatorics and geometry*, volume 2003. Springer, 2003.

[Mém07] Facundo Mémoli. On the use of Gromov-Hausdorff distances for shape comparison. 2007.

[Mém12] Facundo Mémoli. Some properties of Gromov–Hausdorff distances. *Discrete & Computational Geometry*, 48:416–440, 2012.

[Moy22] Michael Moy. Vietoris-Rips metric thickenings of the circle. *arXiv preprint arXiv:2206.03539*, 2022.

[MS04] Facundo Mémoli and Guillermo Sapiro. Comparing point clouds. In *Proceedings of the 2004 Eurographics/ACM SIGGRAPH Symposium on Geometry Processing*, pages 32–40, 2004.

[MS05] Facundo Mémoli and Guillermo Sapiro. A theoretical and computational framework for isometry invariant recognition of point cloud data. *Foundations of Computational Mathematics*, 5:313–347, 2005.

[Pea90] Giuseppe Peano. *Sur une courbe qui remplit toute une aire plane*. Springer, 1990.

[Pet06] Peter Petersen. *Riemannian Geometry*, volume 171. Springer, 2006.

[San46] Luis A. Santaló. Convex regions on the n-dimensional spherical surface. *Annals of Mathematics*, pages 448–459, 1946.

[ŠTZ92] T. Šalát, J. Tóth, and L. Zsilinszky. Metric space of metrics defined on a given set. *Real Analysis Exchange*, 18(1):225–231, 1992.

[ŠTZ93] T. Šalat, J. Toth, and L. Zsilinszky. On the structure of the space of metrics defined on a given set. 1993.