SPECIAL VALUES OF TRIPLE-PRODUCT p-ADIC L-FUNCTIONS AND NON-CRYSTALLINE DIAGONAL CLASSES

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Abstract. The main purpose of this note is to understand the arithmetic encoded in the special value of the p-adic L-function $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})$ associated to a triple of modular forms (f,g,h) of weights $(2,1,1)$, in the case where the classical L-function $L(f\otimes g\otimes h,s)$ –which typically has sign $+1$– does not vanish at its central critical point $s=1$. When f corresponds to an elliptic curve $E/\mathbb{Q}$ and the classical L-function vanishes, the Elliptic Stark Conjecture of Darmon–Lauder–Rotger predicts that $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ is either 0 (when the order of vanishing of the complex L-function is >2) or related to logarithms of global points on E and a certain Gross–Stark unit associated to g (when the order of vanishing is exactly 2). We complete the picture proposed by the Elliptic Stark Conjecture by providing a formula for the value $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ in the case where $L(f\otimes g\otimes h,1)\neq 0$.

Contents

Date: December 16, 2020.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 682152). The authors were also partially supported by projects MTM2015-66716-P and MTM2015-63829-P. The last author gratefully acknowledges Icrea for financial support through an Icrea Academia award.

1. Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$ and let $f \in S_2(N_f)$ be the newform attached to E. Let

$$g \in S_1(N_g, \chi)_L, \qquad h \in S_1(N_h, \bar{\chi})_L$$

be two cuspforms of weight one, inverse nebentype characters and with Fourier coefficients contained in a number field L. Let ρ_g and ρ_h be the Artin representations attached to g and h. The tensor product $\rho_g \otimes \rho_h$ is a self-dual Artin representation of dimension 4 of the form

$$\rho := \rho_g \otimes \rho_h : \mathrm{Gal}(H/\mathbb{Q}) \hookrightarrow \mathrm{Aut}(V_g \otimes V_h) \cong \mathrm{GL}_4(L),$$

where $H/\mathbb{Q}$ is a finite extension.

In this setting, the complex L-function $L(E \otimes \rho, s)$ attached to the (Tate module $V_p(E)$ of the) elliptic curve E twisted by the Artin representation ρ coincides with the Garrett–Rankin L-function $L(f \otimes g \otimes h, s)$ attached to the triple (f, g, h) of modular forms. By multiplying this L-function by an appropriate archimedean factor $L_\infty(f \otimes g \otimes h, s)$ one obtains an entire function $\Lambda(f \otimes g \otimes h, s)$ which satisfies a functional equation of the form

$$\Lambda(f \otimes g \otimes h, s) = \epsilon \cdot \Lambda(f \otimes g \otimes h, 2 - s), \tag{1.1}$$

where $\epsilon \in \{\pm 1\}$. Moreover, $L_\infty(f \otimes g \otimes h, s)$ does not have zeros nor poles at $s = 1$.

Denote N_g and N_h the level of g and h respectively. The sign can be written as a product of local factors $\epsilon = \prod_v \epsilon_v$ where v runs over the places of $\mathbb{Q}$, and $\epsilon_v = +1$ if v is a finite prime which does not divide $\mathrm{lcm}(N_f, N_g, N_h)$ or if $v = \infty$. We will work under the following assumption

Assumption 1.1. $\epsilon_v = +1$ for all v.

Assumption 1.1 holds most of the time: this is the case for instance if the greatest common divisor of the levels of f, g and h is 1.

Fix an odd prime number p such that

$$p \nmid N_f N_g N_h,$$

and denote by α_g, β_g the eigenvalues for the action of the Frobenius element at p acting on V_g. We use the analogous notation for h, and we assume

$$\alpha_g \neq \beta_g, \text{ and } \alpha_h \neq \beta_h.$$

Fix once and for all completions H_p, L_p of the number fields H, L at primes above p.

Choose an ordinary p-stabilisation of g, namely $g_\alpha(z) := g(z) - \beta_g g(pz)$ and define analogously h_α. Let

$$\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]], \qquad \mathbf{g} \in \Lambda_{\mathbf{g}}[[q]], \qquad \mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$$

be Hida families passing through the unique ordinary p-stabilisation of f and g_α and h_α respectively, where $\Lambda_{\mathbf{f}}, \Lambda_{\mathbf{g}}, \Lambda_{\mathbf{h}}$ are finite flat extensions of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[T]]$. Consider the Garret–Hida p-adic L-function

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$$

of [DR14] associated to the specific choice of test vectors $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ of [Hsi17, Chap. 3]. This p-adic L-function interpolates the square-roots of the central values of the classical L-function $L(\breve{f}_k \otimes \breve{g}_\ell \otimes \breve{h}_m, s)$ attached to the specializations of the Hida families at classical points of weights k, ℓ, m with $k, \ell, m \geq 2$ and $\ell \geq k + m$. Notice that the point $(2, 1, 1)$, which corresponds to our triple of modular forms (f, g, h), lies outside the region of classical interpolation for $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$. We are interested in studying the value

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$$

under the following assumption:

Assumption 1.2. $L(E\otimes\rho,1)\neq 0$ and $\mathrm{Sel}_p(E\otimes\rho)=0$.

Here $\mathrm{Sel}_p(E\otimes\rho)$ denotes the Bloch–Kato Selmer group attached to the representation

$$V:=V_p(E)\otimes V_g\otimes V_h.$$

Under Assumption 1.1, the sign ϵ of the functional equation (1.1) is $+1$, and thus the order of vanishing of $L(E\otimes\rho,s)$ at $s=1$ is even. One hence expects that $L(E\otimes\rho,1)$ is generically nonzero. If this L-value is nonzero, by [DR17] we know that the ρ-isotypical component $E(H)^\rho:=\mathrm{Hom}_{G_{\mathbb{Q}}}(V_g\otimes V_h,E(H)\otimes L)$ of the Mordell–Weil group $E(H)$ is trivial. By the Shafarefich–Tate conjecture one also expects the Selmer group $\mathrm{Sel}_p(E\otimes\rho)$ to be trivial, although this conjecture is widely open. It is also worth noting that the value $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ in the setting in which the complex L-function $L(E\otimes\rho,s)$ vanishes at $s=1$ has been analyzed in [DLR15], where the authors give a conjectural formula for this p-adic value as a 2×2-regulator of p-adic logarithms of global points.

Under our running assumption 1.2 one can not expect a similar formula for the above p-adic L-value, as no global points are naturally present in this scenario. The *main result of this paper* consists in an explicit formula for the value $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ which involves the algebraic part of the classical L-value $L(E\otimes\rho,1)$ and the *logarithm* of a canonical non-crystalline class along a certain *crystalline direction*.

In §2 we recall the basic definitions on Selmer groups and we give a precise description of the relaxed p-Selmer group $\mathrm{Sel}_{(p)}(E\otimes\rho)$ under Assumption 1.2. More precisely, the projection to the singular quotient gives an isomorphism

$$\partial_p:\mathrm{Sel}_{(p)}(E\otimes\rho)\xrightarrow{\cong}\mathrm{H}^1_s(\mathbb{Q}_p,V). \tag{1.2}$$

Let V_g^α,V_g^β, with basis v_g^α,v_g^β respectively, be the eigenspaces of V_g for the action of Frob_p with eigenvalues α_g,β_g, and use the analogous notation for V_h. The $G_{\mathbb{Q}_p}$-representation V decomposes as a direct sum as

$$V=V^{\alpha\alpha}\oplus V^{\alpha\beta}\oplus V^{\beta\alpha}\oplus V^{\beta\beta},$$

where $V^{\alpha\alpha}:=V_pE\otimes V_g^\alpha\otimes V_h^\alpha$ and similarly for the other pieces. It induces the decomposition

$$\mathrm{H}^1_s(\mathbb{Q}_p,V)=\mathrm{H}^1_s(\mathbb{Q}_p,V^{\alpha\alpha})\oplus\mathrm{H}^1_s(\mathbb{Q}_p,V^{\alpha\beta})\oplus\mathrm{H}^1_s(\mathbb{Q}_p,V^{\beta\alpha})\oplus\mathrm{H}^1_s(\mathbb{Q}_p,V^{\beta\beta}), \tag{1.3}$$

and the Bloch–Kato dual exponential gives isomorphisms

$$\exp^*_{\alpha\alpha}:\mathrm{H}^1_s(\mathbb{Q}_p,V^{\alpha\alpha})\xrightarrow{\cong}L_p$$

and similarly for the other pieces of the decomposition (1.3). Combining it with (1.2), we get a basis

$$\xi^{\alpha\alpha},\xi^{\alpha\beta},\xi^{\beta\alpha},\xi^{\beta\beta}$$

for $\mathrm{Sel}_{(p)}(E\otimes\rho)$ characterised by the fact that

$$\partial_p\xi^{\alpha\alpha}\in\mathrm{H}^1_s(\mathbb{Q}_p,V^{\alpha\alpha})\ \text{and}\ \exp^*_{\alpha\alpha}\partial_p\xi^{\alpha\alpha}=1,$$

and similarly for $\xi^{\alpha\beta},\xi^{\beta\alpha},\xi^{\beta\beta}$.

The $G_{\mathbb{Q}_p}$-cohomology of V and its submodule of crystalline classes $\mathrm{H}^1_f(\mathbb{Q}_p,V)\subseteq\mathrm{H}^1(\mathbb{Q}_p,V)$ also have decompositions analogous to (1.3). Moreover, if

$$\pi_{\alpha\beta}:\mathrm{H}^1(\mathbb{Q}_p,V)\longrightarrow\mathrm{H}^1(\mathbb{Q}_p,V^{\alpha\beta})$$

denotes the projection, then $\pi_{\alpha\beta}\xi^{\beta\beta}$ lies in $\mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta})$. Finally, we can write

$$\pi_{\alpha\beta}\xi^{\beta\beta} = R_{\beta\alpha} \otimes v_g^{\alpha} \otimes v_h^{\beta} \in (E(H_p) \otimes V_g^{\alpha} \otimes V_h^{\beta})^{G_{\mathbb{Q}_p}} \cong \mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta})$$

where $R_{\beta\alpha} \in E(H_p)$ is a local point on which Frob_p acts as multiplication by $\beta_g\alpha_h$.

We can finally state the main result of the paper.

Theorem (cf Theorem 3.2)**.** *Under Assumptions 2.1 and 1.2,*

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1) = \frac{A \cdot \mathcal{E}}{\pi\langle f, f\rangle} \times \frac{\log_p(R_{\beta\alpha})}{\mathcal{L}_{g_\alpha}} \times \sqrt{L(E \otimes \rho, 1)}, \tag{1.4}$$

where $A \in \mathbb{Q}^\times$ is an explicit number, $\mathcal{E} \in L_p$ is a product of Euler factors, $\langle f, f\rangle$ denotes the Petersson norm of f, $\mathcal{L}_{g_\alpha} \in H_p$ is an element on which Frob_p acts as multiplication by $\frac{\beta_g}{\alpha_g}$ and which only depends on g_α, and $\log_p : E(H_p) \to H_p$ denotes the p-adic logarithm.

We refer to Theorem 3.2 for a more precise statement of the result and of the objects appearing in (1.4). In particular, the element $\mathcal{L}_{g_\alpha}$ is expected to be related to a so-called Gross–Stark unit attached to g_α, as conjectured in [DR16, Conjecture 2.1].

Under the additional assumption that g is not the theta series of a Hecke character of a real quadratic field in which p splits, the value $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ can be recast in a more explicit way in terms of *p-adic iterated integrals*, as explained in the introduction of [DLR15]. The numerical computations we offer in §5 are obtained by calculating such integrals, where a key input are Lauder's algorithms [Lau14] for the computation of overconvergent projections.

As an application of the main result, in Section 4 we explore the situation where g and h are theta series of the same imaginary quadratic field in which p splits. The following theorem is stated as Theorem 4.1 in the text.

Theorem (cf Theorem 4.1)**.** *Let K be an imaginary quadratic field in which p is split, and let ψ_g (resp. ψ_h) be a finite order Hecke character of K of conductor $\mathfrak{c}_g$ (resp. of conductor $\mathfrak{c}_h$). Denote by g and h the theta series attached to ψ_g and ψ_h, respectively. Suppose that $\gcd(N_f, \mathfrak{c}_g, \mathfrak{c}_h) = 1$ and that the Nebentype characters of g and h are inverses to each other. If $L(E, \rho_g \otimes \rho_h, 1) \neq 0$ then $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1) = 0$.*

2. The Selmer group of $f \otimes g \otimes h$

We begin this section by collecting some standard facts on Selmer groups of p-adic Galois representations that we will use. Then we introduce the Galois representation attached to the triple of modular forms f, g, and h of weights 2, 1, 1, and we study the corresponding Selmer groups. In particular, the structure of the relaxed Selmer group will be key in proving the main theorem of Section 3.

2.1. **Selmer groups.** Let V be a $\mathbb{Q}_p[G_{\mathbb{Q}}]$-module and let B_{cris} be Fontaine's p-adic crystalline period ring. For each prime number ℓ, denote

$$\mathrm{H}^1_f(\mathbb{Q}_\ell, V) := \begin{cases} \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, V) := \mathrm{H}^1(\mathbb{Q}_\ell^{\mathrm{ur}}/\mathbb{Q}_\ell, V^{I_\ell}) & \ell \neq p \\ \ker\left(\mathrm{H}^1(\mathbb{Q}_p, V) \to \mathrm{H}^1(\mathbb{Q}_p, V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})\right) & \ell = p, \end{cases} \tag{2.1}$$

and

$$\mathrm{H}^1_s(\mathbb{Q}_\ell, V) := \mathrm{H}^1(\mathbb{Q}_\ell, V) / \mathrm{H}^1_f(\mathbb{Q}_\ell, V).$$

The Bloch–Kato Selmer group of V is

$$\mathrm{Sel}_p(\mathbb{Q}, V) := \{x \in \mathrm{H}^1(\mathbb{Q}, V) \mid \mathrm{res}_\ell(x) \in \mathrm{H}^1_f(\mathbb{Q}_\ell, V) \text{ for all } \ell\},$$

where $\mathrm{res}_\ell \colon H^1(\mathbb{Q}, V) \to H^1(\mathbb{Q}_\ell, V)$ denotes the restriction map in Galois cohomology.

For each prime ℓ, we denote by ∂_ℓ the composition

$$\partial_\ell : \mathrm{H}^1(\mathbb{Q}, V) \xrightarrow{\mathrm{res}_\ell} \mathrm{H}^1(\mathbb{Q}_\ell, V) \longrightarrow \mathrm{H}^1_s(\mathbb{Q}_\ell, V),$$

where the second map is the natural quotient map.

The relaxed Selmer group is defined as

$$\mathrm{Sel}_{(p)}(\mathbb{Q}, V) := \{x \in \mathrm{H}^1(\mathbb{Q}, V) \mid \mathrm{res}_\ell(x) \in \mathrm{H}^1_f(\mathbb{Q}_\ell, V) \ \text{ for all } \ \ell \neq p\} \supseteq \mathrm{Sel}_p(\mathbb{Q}, V).$$

Let $V^* := \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1))$ be the Kummer dual of V. One can define a Selmer group $\mathrm{Sel}_{p,*}(\mathbb{Q}, V^*)$ for V^* which is dual to (2.1) with respect to the local Tate pairings

$$\langle \ , \ \rangle_\ell : \mathrm{H}^1(\mathbb{Q}_\ell, V) \times \mathrm{H}^1(\mathbb{Q}_\ell, V^*) \longrightarrow \mathbb{Q}_p. \tag{2.2}$$

For each ℓ, define $\mathrm{H}^1_{f,*}(\mathbb{Q}_\ell, V^*)$ to be the orthogonal complement of $\mathrm{H}^1_f(\mathbb{Q}_\ell, V)$ with respect to (2.2); the Selmer group attached to V^* is then

$$\mathrm{Sel}_{p,*}(\mathbb{Q}, V^*) := \{x \in \mathrm{H}^1(\mathbb{Q}, V^*) \mid \mathrm{res}_\ell(x) \in \mathrm{H}^1_{f,*}(\mathbb{Q}_\ell, V^*) \ \text{ for all } \ell\}.$$

Finally, the strict Selmer group of V^* is the subspace of $\mathrm{Sel}_{p,*}(\mathbb{Q}, V^*)$ defined as

$$\mathrm{Sel}_{[p],*}(\mathbb{Q}, V^*) := \{x \in \mathrm{H}^1(\mathbb{Q}, V^*) \mid \mathrm{res}_\ell(x) \in \mathrm{H}^1_{f,*}(\mathbb{Q}_\ell, V^*) \text{ for all } \ell \text{ and } \mathrm{res}_p(x) = 0\}.$$

By Poitou–Tate duality (see, for example, [MR04, Theorem 2.3.4]) there is an exact sequence

$$0 \to \mathrm{Sel}_p(\mathbb{Q}, V) \to \mathrm{Sel}_{(p)}(\mathbb{Q}, V) \to \mathrm{H}^1_s(\mathbb{Q}_p, V) \to \mathrm{Sel}_{p,*}(\mathbb{Q}, V^*)^\vee \to \mathrm{Sel}_{[p],*}(\mathbb{Q}, V^*)^\vee, \tag{2.3}$$

where $^\vee$ stands for the $\mathbb{Q}_p$-dual.

2.2. **Representations attached to modular forms.** In this section we review the main features of the representations, both p-adic and Λ-adic, attached to modular forms in the lines of [DR16, §2], which the reader can consult for more details.

Let $f \in S_2(N_f)$ be a weight two normalized eigenform of level N_f, trivial nebentype character and rational Fourier coefficients $a_n(f)$. Denote by E the elliptic curve over $\mathbb{Q}$ of conductor N_f associated to f by the Eichler–Shimura construction.

Let also

$$g \in S_1(N_g, \chi) \ \text{ and } \ h \in S_1(N_h, \bar{\chi})$$

be two normalized newforms of weight one, levels N_g and N_h, and nebentype characters χ and $\bar{\chi}$ respectively. Denote by K_g and K_h their fields of Fourier coefficients, and put $L := K_g \cdot K_h$ the compositum of these fields.

From now on, we fix a rational prime p, and we assume the following hypothesis.

Assumption 2.1. The prime p does not divide $N_f N_g N_h$.

Since we will be interested in putting f in a Hida family, we assume moreover that f is ordinary at p; that is to say, that $p \nmid a_p(f)$.

We denote the 2-dimensional p-adic representations attached to f by V_f. Since f corresponds to the curve E, the representation V_f is given by the rational Tate module $V_p(E) = T_p(E) \otimes \mathbb{Q}_p$. Denote by α_f, β_f the roots of the Hecke polynomial $X^2 - a_p(f)X + p$. Since f is ordinary at p, one of these roots, say α_f, is a p-adic unit; also, the restriction of V_f to a decomposition group $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$ admits a filtration of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules

$$0 \to V_f^+ \longrightarrow V_f \longrightarrow V_f^- \to 0$$

with the following properties:

(1) $\dim_{\mathbb{Q}_p} V_f^+ = \dim_{\mathbb{Q}_p} V_f^- = 1$;
(2) the group $G_{\mathbb{Q}_p}$ acts on the quotient V_f^- via ψ_f, where $\psi_f : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the unramified character that maps an arithmetic Frobenius Frob_p to α_f.
(3) the group $G_{\mathbb{Q}_p}$ acts on V_f^+ via the character $\chi_{\mathrm{cycl}}\psi_f^{-1}$ (here χ_{cycl} is the p-adic cyclotomic character).

There are Artin representations associated to g and h. Without loss of generality we can assume that they are defined over L, and that they factor through the same finite extension H of $\mathbb{Q}$. That is to say, they are of the form

$$\rho_g\colon \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{Aut}(V_g^\circ) \cong \mathrm{GL}_2(L), \quad \rho_h\colon \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{Aut}(V_h^\circ) \cong \mathrm{GL}_2(L)$$

for certain 2-dimensional L-vector spaces V_g° and V_h°.

Fix once and for all a prime $\mathfrak{p}$ of H and a prime $\mathfrak{P}$ of L above p. Denote the corresponding completions by $H_p := H_{\mathfrak{p}}$ and $L_p := L_{\mathfrak{P}}$. There are also p-adic Galois representations associated to g and h, that we will denote by V_g and V_h. There are non-canonical isomorphisms

$$j_g : V_g^\circ \otimes_L L_p \xrightarrow{\cong} V_g \quad \text{and} \quad j_h : V_h^\circ \otimes_L L_p \xrightarrow{\cong} V_h. \tag{2.4}$$

Since $p \nmid N_g N_h$ the representations V_g and V_h are unramified at p. We assume from now on that Frob_p acts on V_g and V_h with distinct eigenvalues. Let α_g, β_g be the eigenvalues for the action of Frob_p on V_g and let V_g^α, V_g^β be the corresponding eigenspaces. We will use the analogous notations α_h, β_h, V_h^α, and V_h^β for h.

Denote by g_α the p-stabilisation of g such that $U_p(g_\alpha) = \alpha_g g_\alpha$. The theory of Hida families ensures the existence of a Hida family $\mathbf{g}$ passing through g_α. This can be regarded as a power series $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$, where $\Lambda_{\mathbf{g}}$ is a finite flat extension of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[T]]$, with the property that, if we denote by $y_g\colon \Lambda_{\mathbf{g}} \to L_p$ the weight corresponding to g, then $y_g(\mathbf{g}) = g_\alpha$. There is a locally free $\Lambda_{\mathbf{g}}$-module $V_{\mathbf{g}}$ and a Λ-adic representation $\rho_{\mathbf{g}} : G_{\mathbb{Q}} \to \mathrm{GL}(V_{\mathbf{g}}) \cong \mathrm{GL}_2(\Lambda_{\mathbf{g}})$ that interpolates the p-adic representations associated to the specializations of $\mathbf{g}$. As a $G_{\mathbb{Q}_p}$-representation, $V_{\mathbf{g}}$ is equipped with a filtration of $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}_p}]$-modules

$$0 \to V_{\mathbf{g}}^+ \longrightarrow V_{\mathbf{g}} \longrightarrow V_{\mathbf{g}}^- \to 0, \tag{2.5}$$

where $V_{\mathbf{g}}^+$ and $V_{\mathbf{g}}^-$ are locally free of rank one and the action of $G_{\mathbb{Q}_p}$ on $V_{\mathbf{g}}^-$ is unramified, with Frob_p acting as multiplication by the p-th Fourier coefficient of $\mathbf{g}$. There is a perfect Galois equivariant pairing

$$\langle\ ,\ \rangle : V_{\mathbf{g}}^- \times V_{\mathbf{g}}^+ \longrightarrow \Lambda_{\mathbf{g}}(\det(\rho_{\mathbf{g}})). \tag{2.6}$$

For a crystalline $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-module W, denote $D(W) := (W \otimes B_{\mathrm{cris}})^{G_{\mathbb{Q}_p}}$. Recall that, if W is unramified, then

$$D(W) \cong (W \otimes \hat{\mathbb{Q}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}},$$

where $\hat{\mathbb{Q}}_p^{\mathrm{ur}}$ is the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. Denote by $\omega_{\mathbf{g}} \in D(V_{\mathbf{g}}^-)$ the canonical period associated to $\mathbf{g}$ constructed by Ohta [Oht95].

By specializing via y_g, we obtain the L_p-vector space $y_g(V_{\mathbf{g}}) := V_{\mathbf{g}} \otimes_{\Lambda_{\mathbf{g}}, y_g} L_p$, which can be identified with V_g. Using the functoriality of D and the identification $y_g(V_{\mathbf{g}}^+) = V_g^\beta,\ y_g(V_{\mathbf{g}}^-) = V_g^\alpha$ we obtain a pairing

$$\langle\ ,\ \rangle : D(V_g^\alpha) \times D(V_g^\beta) \longrightarrow D(L_p(\chi)) = (H_p \otimes L_p(\chi))^{G_{\mathbb{Q}_p}}. \tag{2.7}$$

Define

$$\omega_g := y_g(\omega_{\mathbf{g}}) \in D(V_g^\alpha)$$

and let $\eta_g \in D(V_g^\beta)$ be the element characterized by the equality

$$\langle \omega_g, \eta_g \rangle = \mathfrak{g}(\chi) \otimes 1 \in D(L_p(\chi)), \tag{2.8}$$

where $\mathfrak{g}(\chi)$ denotes the Gauss sum of χ viewed as an element of H_p. We define similarly $\omega_h \in D(V_h^\alpha)$ and $\eta_h \in D(V_h^\beta)$.

Using the isomorphisms (2.4) we can define an L structure on V_g by $V_g^L := j_g(V_g^\circ)$. Let v_g^α (resp. v_g^β) be an L-basis of $V_g^L \cap V_g^\alpha$ (resp. of V_g^β). Define

$$\Omega_g \in H_p^{1/\alpha_g},\ \ \Theta_g \in H_p^{1/\beta_g}$$

to be the elements such that

$$\Omega_g \otimes v_g^\alpha = \omega_g \in D(V_g^\alpha), \quad \Theta_g \otimes v_g^\beta = \eta_g \in D(V_g^\beta). \tag{2.9}$$

Let

$$V := V_f \otimes V_{gh}$$

be the p-adic representation given by the tensor product $V_f \otimes V_g \otimes V_h$. Since the product of the nebentype characters of f, g, and h is trivial we have that $V^* \cong V$. We next study the structure of several Selmer groups associated to V.

2.3. **Selmer groups of V.** Put $V_{gh}^\circ := V_g^\circ \otimes_L V_h^\circ$ and denote by ρ the representation afforded by this space:

$$\rho\colon\ \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{Aut}(V_{gh}^\circ).$$

Put $E(H)_L := E(H) \otimes_{\mathbb{Z}} L$ and denote by $E(H)^\rho$ the ρ-isotypical component of the Mordell–Weil group:

$$E(H)^\rho := \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}^\circ, E(H)_L).$$

Lemma 2.2. *There are isomorphisms*

$$\mathrm{H}^1(\mathbb{Q}, V) \cong (\mathrm{H}^1(H, V_f) \otimes V_{gh})^{\mathrm{Gal}(H/\mathbb{Q})} \cong \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}, \mathrm{H}^1(H, V_f)); \tag{2.10}$$

$$\mathrm{H}^1(\mathbb{Q}_p, V) \cong (\mathrm{H}^1(H_p, V_f) \otimes V_{gh})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \cong \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}, \mathrm{H}^1(H_p, V_f)). \tag{2.11}$$

Proof. We prove only (2.11), and (2.10) is proven similarly. By the inflation-restriction exact sequence we have the exact sequence

$$0 \to \mathrm{H}^1(\mathrm{Gal}(H_p/\mathbb{Q}_p), V^{G_{H_p}}) \to \mathrm{H}^1(\mathbb{Q}_p, V) \to \mathrm{H}^1(H_p, V)^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \to \mathrm{H}^2(\mathrm{Gal}(H_p/\mathbb{Q}_p), V^{G_{H_p}}).$$

Since $\mathrm{H}^1(\mathrm{Gal}(H_p/\mathbb{Q}_p), V^{G_{H_p}}) = \mathrm{H}^2(\mathrm{Gal}(H_p/\mathbb{Q}_p), V^{G_{H_p}}) = 0$, the restriction to G_{H_p} gives an isomorphism

$$\mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1(H_p, V)^{\mathrm{Gal}(H_p/\mathbb{Q}_p)}.$$

Composing it with the identifications

$$\mathrm{H}^1(H_p, V)^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} = \mathrm{H}^1(H_p, V_f \otimes V_{gh})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} = (\mathrm{H}^1(H_p, V_f) \otimes V_{gh})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)},$$

we get the first isomorphism of (2.11). Finally, the second isomorphism follows from the relation between Hom and tensor and from the selfduality $V_{gh}^{\vee} \cong V_{gh}$. $\square$

Let $E(H)_L^{V_{gh}} := \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}, E(H)_L)$. The Kummer homomorphism

$$E(H)_L \longrightarrow \mathrm{H}^1(H, V_f)$$

induces a homomorphism

$$\delta : E(H)_L^{V_{gh}} \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}, \mathrm{H}^1(H, V_f)) \cong \mathrm{H}^1(\mathbb{Q}, V),$$

which using (2.10) can be seen as a morphism

$$\delta : E(H)_L^{V_{gh}} \longrightarrow \mathrm{H}^1(\mathbb{Q}, V).$$

For $\triangle, \heartsuit \in \{\alpha, \beta\}$, denote

$$V_{gh}^{\triangle\heartsuit} := V_g^{\triangle} \otimes V_h^{\heartsuit} \quad \text{and} \quad V^{\triangle\heartsuit} := V_f \otimes V_g^{\triangle} \otimes V_h^{\heartsuit}.$$

Specializing (2.5) via y_g we obtain

$$0 \longrightarrow V_g^{\beta} \longrightarrow V_g \longrightarrow V_g^{\alpha} \longrightarrow 0.$$

For $\triangle \neq \heartsuit$ the pairing (2.6) and its analog for h induce perfect pairings

$$\langle\ ,\ \rangle : V_{gh}^{\triangle\triangle} \times V_{gh}^{\heartsuit\heartsuit} \longrightarrow L_p, \qquad \langle\ ,\ \rangle : V_g^{\triangle\heartsuit} \times V_g^{\heartsuit\triangle} \longrightarrow L_p. \tag{2.12}$$

The identifications

$$V_{gh}^{\triangle\triangle} \cong \mathrm{Hom}_{L_p[G_{\mathbb{Q}_p}]}(V_{gh}^{\heartsuit\heartsuit}, L_p) \quad \text{and} \quad V_{gh}^{\triangle\heartsuit} \cong \mathrm{Hom}_{L_p[G_{\mathbb{Q}_p}]}(V_{gh}^{\heartsuit\triangle}, L_p), \tag{2.13}$$

together with (2.11) give the following isomorphisms:

$$\mathrm{H}^1(\mathbb{Q}_p, V^{\triangle\triangle}) \cong (\mathrm{H}^1(H_p, V_f) \otimes V_{gh}^{\triangle\triangle})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \cong \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\heartsuit\heartsuit}, \mathrm{H}^1(H_p, V_f)); \tag{2.14}$$

$$\mathrm{H}^1(\mathbb{Q}_p, V^{\triangle\heartsuit}) \cong (\mathrm{H}^1(H_p, V_f) \otimes V_{gh}^{\triangle\heartsuit})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \cong \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\heartsuit\triangle}, \mathrm{H}^1(H_p, V_f)). \tag{2.15}$$

It follows from [DR19, Lemma 4.1] that the submodule $H_f^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit})$ and the singular quotient $H_s^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit})$ can be written in terms of the filtration of V_f as follows:

$$\mathrm{H}_s^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit}) = \mathrm{H}^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\triangle\heartsuit}) \cong (V_{gh}^{\heartsuit\triangle} \otimes \mathrm{H}_s^1(H_p, V_f))^{\mathrm{Gal}(H_p/\mathbb{Q}_p)}; \tag{2.16}$$

$$\mathrm{H}_f^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit}) = \ker(\mathrm{H}^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit}) \to \mathrm{H}^1(I_p, V_f^- \otimes V_{gh}^{\triangle\heartsuit})) = \mathrm{H}^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}). \tag{2.17}$$

Lemma 2.3. *For $\triangle, \heartsuit \in \{\alpha, \beta\}$, $\triangle \neq \heartsuit$, there are isomorphisms*

$$\delta_p : E(H_p)_{L_p}^{V_{gh}} \longrightarrow \mathrm{H}_f^1(\mathbb{Q}_p, V);$$
$$\delta_p^{\triangle\heartsuit} : E(H_p)_{L_p}^{V_{gh}^{\triangle\heartsuit}} \longrightarrow \mathrm{H}_f^1(\mathbb{Q}_p, V^{\heartsuit\triangle});$$
$$\delta_p^{\triangle\triangle} : E(H_p)_{L_p}^{V_{gh}^{\triangle\triangle}} \longrightarrow \mathrm{H}_f^1(\mathbb{Q}_p, V^{\heartsuit\heartsuit}).$$

Proof. We prove the existence of the isomorphism $\delta^{\triangle\heartsuit}$, the others are similar. By Kummer theory, there is an injective morphism

$$E(H_p)_{L_p} \longrightarrow \mathrm{H}^1(H_p, V_f),$$

which is an isomorphism on its image $\mathrm{H}^1_f(H_p, V_f) \cong \mathrm{H}^1(H_p, V_f^+)$. It induces an homomorphism

$$\delta_p^{\triangle\heartsuit} : E(H_p)_{L_p}^{V_{gh}^{\triangle\heartsuit}} \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\triangle\heartsuit}, \mathrm{H}^1(H_p, V_f^+)).$$

Using the isomorphisms (2.13) we obtain

$$\mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\triangle\heartsuit}, \mathrm{H}^1(H_p, V_f^+)) \xrightarrow{\cong} (\mathrm{H}^1(H_p, V_f^+) \otimes V_{gh}^{\heartsuit\triangle})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)}.$$

Arguing as in the proof of Lemma 2.2, we get the isomorphisms

$$\begin{aligned}(\mathrm{H}^1(H_p, V_f^+) \otimes V_{gh}^{\heartsuit\triangle})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} &\cong \mathrm{H}^1(H_p, V_f^+ \otimes V_{gh}^{\heartsuit\triangle})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \\ &\cong \mathrm{H}^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\heartsuit\triangle}) \cong \mathrm{H}^1_f(\mathbb{Q}_p, V^{\heartsuit\triangle}).\end{aligned}$$

□

From now on we will make the following assumption on the Selmer group of V.

Assumption 2.4. $\mathrm{Sel}_p(\mathbb{Q}, V) = 0$.

Under this assumption one can identify the relaxed Selmer group with the singular quotient.

Lemma 2.5. *Under Assumptions 2.1 and 2.4 the natural map*

$$\partial_p : \mathrm{Sel}_{(p)}(\mathbb{Q}, V) \longrightarrow \mathrm{H}^1_s(\mathbb{Q}_p, V)$$

is an isomorphism. In particular, there is an isomorphism

$$\mathrm{Sel}_{(p)}(\mathbb{Q}, V) \cong \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\beta}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta}) \tag{2.18}$$

Proof. Since the representation V is self-dual there is an isomorphism $\mathrm{Sel}_{p,*}(V^*) \cong \mathrm{Sel}_p(V)$, (see, for example, [BK90] and [Bel, Theorem 2.1]). Then the lemma follows immediately from the exact sequence (2.3). □

In the next subsection we will describe the spaces in the right hand side of (2.18) in terms of dual exponential maps.

2.4. Bloch–Kato logarithms and exponentials.

The $G_{\mathbb{Q}_p}$-representations of the form $V_f^+ \otimes V_{gh}^{\triangle\heartsuit}$ are one dimensional and, therefore, given by characters. Indeed, $G_{\mathbb{Q}_p}$ acts on V_f^+ as $\chi_{\mathrm{cycl}}\psi_f^{-1}$, and it acts as ψ_g (resp. ψ_g^{-1}) on V_g^α (resp. V_g^β) and as ψ_h (resp. ψ_h^{-1}) on V_h^α (resp. V_h^β). Therefore we have that

$$\begin{aligned} V_f^+ \otimes V_{gh}^{\alpha\alpha} &= L_p(\chi_{\mathrm{cycl}}\psi_f^{-1}\psi_g\psi_h), \qquad & V_f^+ \otimes V_{gh}^{\alpha\beta} &= L_p(\chi_{\mathrm{cycl}}\psi_f^{-1}\psi_g\psi_h^{-1}), \\ V_f^+ \otimes V_{gh}^{\beta\alpha} &= L_p(\chi_{\mathrm{cycl}}\psi_f^{-1}\psi_g^{-1}\psi_h), \qquad & V_f^+ \otimes V_{gh}^{\beta\beta} &= L_p(\chi_{\mathrm{cycl}}\psi_f^{-1}\psi_g^{-1}\psi_h^{-1}). \end{aligned}$$

In particular $V_f^+ \otimes V_{gh}^{\triangle\heartsuit}$ is of the form $L_p(\psi\chi_{\mathrm{cycl}})$ for some nontrivial unramified character ψ. By (2.17) we have that $\mathrm{H}^1_f(\mathbb{Q}_p, V^{\triangle\heartsuit}) \cong \mathrm{H}^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit})$, and the Bloch–Kato logarithm gives an isomorphism (cf. [DR19, Example 1.6 (a)]):

$$\log_{\triangle\heartsuit} : \mathrm{H}^1_f(\mathbb{Q}_p, V^{\triangle\heartsuit}) \longrightarrow D(V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) = D(L_p(\psi\chi_{\mathrm{cycl}})). \tag{2.19}$$

For $(\triangle,\heartsuit)=(\alpha,\alpha)$, the pairings 2.7 and the analogous pairings for f and h give rise to a pairing

$$\langle\ ,\ \rangle : V_f^+\otimes V_g^\alpha\otimes V_h^\alpha\times V_f^-(-1)\otimes V_g^\beta\otimes V_h^\beta\longrightarrow L_p \tag{2.20}$$

which induces

$$\langle\ ,\ \rangle : D(V_f^+\otimes V_g^\alpha\otimes V_h^\alpha)\times D(V_f^-(-1)\otimes V_g^\beta\otimes V_h^\beta)\longrightarrow D(L_p)=L_p. \tag{2.21}$$

Denote by $\tilde{\omega}_f$ the differential form on $X_0(N_f)$ corresponding to f. It can be naturally viewed as an element of the de Rham cohomology group $H^1_{\mathrm{dR}}(X_0(N_f)/\mathbb{Q}_p)$. The comparison isomorphisms of p-adic Hodge theory provide a natural map

$$H^1_{\mathrm{dR}}(X_0(N_f)/\mathbb{Q}_p)(1)\longrightarrow D(V_f^-)$$

and therefore $\tilde{\omega}_f$ gives rise to an element $\omega_f\in D(V_f^-(-1))$. In (2.21), pairing with the class $\omega_f\otimes\eta_g\otimes\eta_h$ gives then an isomorphism

$$\langle\cdot,\omega_f\otimes\eta_g\otimes\eta_h\rangle : D(L_p(\psi\chi_{\mathrm{cycl}}))=D(V_f^+\otimes V_g^\alpha\otimes V_h^\alpha)\longrightarrow L_p. \tag{2.22}$$

There are similar pairings and isomorphisms for the remaining pairs $(\triangle,\heartsuit)$. We still denote

$$\log_{\triangle\heartsuit} : \mathrm{H}^1_f(\mathbb{Q}_p,V_f\otimes V_{gh}^{\triangle\heartsuit})\longrightarrow L_p \tag{2.23}$$

the map obtained by composing (2.19) with (2.22).

Remark 2.6. The logarithm maps of (2.23) are related to the usual p-adic logarithm on E as follows. The differential ω_f gives rise to an invariant differential on E, and we denote by

$$\log_{f,p}\colon E(H_p)\longrightarrow H_p$$

the corresponding formal group logarithm on E. The map $\log_{\alpha\beta}$ coincides with the inverse of the isomorphism of Lemma 2.3

$$E(H_p)_{L_p}^{V_{gh}^{\beta\alpha}}\cong (E(H_p)^{\beta_g\alpha_h}\otimes V_{gh}^{\alpha\beta})^{G_{\mathbb{Q}_p}}$$

composed with the maps

$$\begin{array}{ccccc} (E(H_p)^{\beta_g\alpha_h}\otimes V_{gh}^{\alpha\beta})^{G_{\mathbb{Q}_p}} & \longrightarrow & (H_p^{\beta_g\alpha_h}\otimes V_{gh}^{\alpha\beta})^{G_{\mathbb{Q}_p}}=D(V_{gh}^{\alpha\beta}) & \longrightarrow & L_p \\ x\otimes v_g^\alpha v_h^\beta & \longmapsto & \log_{f,p}(x)\otimes v_g^\alpha v_h^\beta & & \\ & & y & \longmapsto & \langle y,\eta_g\omega_h\rangle. \end{array}$$

Analogous equalities hold for the other maps $\log_{\triangle\heartsuit}$.

A similar discussion can be applied to the representations of the form $V_f^-\otimes V_{gh}^{\triangle\heartsuit}$. In this case we have the following isomorphisms of 1-dimensional representations:

$$\begin{aligned} V_f^-\otimes V_{gh}^{\alpha\alpha}&=L_p(\psi_f\psi_g\psi_h), & V_f^-\otimes V_{gh}^{\alpha\beta}&=L_p(\psi_f\psi_g\psi_h^{-1}\bar{\chi}),\\ V_f^-\otimes V_{gh}^{\beta\alpha}&=L_p(\psi_f\psi_g^{-1}\psi_h\chi), & V_f^-\otimes V_{gh}^{\beta\beta}&=L_p(\psi_f\psi_g^{-1}\psi_h^{-1}). \end{aligned}$$

Therefore, $V_f^-\otimes V_{gh}^{\triangle\heartsuit}$ is isomorphic to a representation of the form $L_p(\psi)$ for some unramified and nontrivial character ψ. By (2.16) there is an identification

$$\mathrm{H}^1_s(\mathbb{Q}_p,V^{\triangle\heartsuit})=\mathrm{H}^1(\mathbb{Q}_p,L_p(\psi)),$$

and by [DR17, Example 1.8 (b)] the dual exponential gives isomorphisms

$$\exp^*_{\triangle\heartsuit} : \mathrm{H}^1_s(\mathbb{Q}_p,V^{\triangle\heartsuit})\longrightarrow D(L_p(\psi))\cong L_p, \tag{2.24}$$

where the last isomorphism is induced by pairing with the appropriate class of $D(L_p(\psi^{-1})) = D(V_f^+(-1) \otimes V_g^{\heartsuit} \otimes V_h^{\triangle})$ similarly as in (2.22). Arguing as in Remark 2.6, let

$$\exp^*_{f,p} : \mathrm{H}^1_s(H_p, V_f) \longrightarrow H_p$$

denote the dual exponential on $\mathrm{H}^1_s(H_p, V_f)$. Then $\exp^*_{\beta\beta}$ can be identified with the composition

$$\begin{array}{ccccc} (\mathrm{H}^1_s(H_p, V_f)^{\alpha_g\alpha_h} \otimes V_{gh}^{\beta\beta})^{G_{\mathbb{Q}_p}} & \longrightarrow & (H_p^{\alpha_g\alpha_h} \otimes V_{gh}^{\beta\beta})^{G_{\mathbb{Q}_p}} = D(V_{gh}^{\beta\beta}) & \longrightarrow & L_p \\ x \otimes v_g^\beta v_h^\beta & \longmapsto & \exp^*_{f,p}(x) \otimes v_g^\beta v_h^\beta & & \\ & & y & \longmapsto & \langle y, \omega_g\omega_h \rangle, \end{array} \tag{2.25}$$

after taking into account the identification

$$\mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta}) \cong (\mathrm{H}^1_s(H_p, V_f)^{\alpha_g\alpha_h} \otimes V_{gh}^{\beta\beta})^{G_{\mathbb{Q}_p}}.$$

Analogous formulas hold for the dual exponentials $\exp_{\triangle\heartsuit}$ on the remaining components.

To sum up the discussion of this subsection, we conclude that the relaxed Selmer group of V admits a basis adapted to decomposition (2.18) with respect to the dual exponential maps.

Proposition 2.7. *Under Assumptions 2.4 and 2.1,* $\mathrm{Sel}_{(p)}(V)$ *has a basis*

$$\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\} \tag{2.26}$$

characterized by the fact that there exist elements $\Psi_{\beta\beta} \in \mathrm{H}^1_s(H_p, V_f)^{\beta_g\beta_h}$, $\Psi_{\beta\alpha} \in \mathrm{H}^1_s(H_p, V_f)^{\beta_g\alpha_h}$, $\Psi_{\alpha\beta} \in \mathrm{H}^1_s(H_p, V_f)^{\alpha_g\beta_h}$, $\Psi_{\alpha\alpha} \in \mathrm{H}^1_s(H_p, V_f)^{\alpha_g\alpha_h}$ *such that*

$$\partial_p\xi^{\alpha\alpha} = (\Psi_{\beta\beta} \otimes v_g^\alpha v_h^\alpha, 0, 0, 0),\ \ \partial_p\xi^{\alpha\beta} = (0, \Psi_{\beta\alpha} \otimes v_g^\alpha v_h^\beta, 0, 0)$$

$$\partial_p\xi^{\beta\alpha} = (0, 0, \Psi_{\alpha\beta} \otimes v_g^\beta v_h^\alpha, 0),\ \ \partial_p\xi^{\beta\beta} = (0, 0, 0, \Psi_{\alpha\alpha} \otimes v_g^\beta v_h^\beta)$$

and

$$\exp^*_{f,p}(\Psi_{\beta\beta}) = \exp^*_{f,p}(\Psi_{\beta\alpha}) = \exp^*_{f,p}(\Psi_{\alpha\beta}) = \exp^*_{f,p}(\Psi_{\alpha\alpha}) = 1.$$

Remark 2.8. Notice that the basis (2.26) depends on the choice of the L-basis v_g^α, v_g^β of V_g and the L-basis v_h^α, v_h^β of V_h. Then each element of the basis $\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\}$ depends on this choice up to multiplication by an element of $L^\times$.

3. Special value formula for the triple product p-adic $\mathcal{L}$-function in rank 0

We continue with the notation and assumptions of the previous section. In particular, $V := V_f \otimes V_g \otimes V_h$ is the tensor product of the p-adic representations attached to the newforms

$$f \in S_2(N_f)_{\mathbb{Q}}, \quad g \in M_1(N_g, \chi)_L, \quad h \in M_1(N_h, \bar{\chi})_L,$$

and we assume from now on that $\gcd(N_f, N_g, N_h)$ is square free. Recall that V_g° (resp. V_h°) stands for the Artin representation attached to g (resp. h) and ρ denotes the tensor product representation

$$\rho : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{GL}(V_g^\circ \otimes V_h^\circ) \cong \mathrm{GL}_4(L).$$

The complex L-function

$$L(E, \rho, s) = L(f \otimes g \otimes h, s)$$

has entire continuation and satisfies a functional equation relating the value at s with the value at $2 - s$. Let ϵ be the sign of this functional equation and denote $N := \mathrm{lcm}(N_f, N_g, N_h)$. Then ϵ is the product of local signs $\epsilon = \prod_v \epsilon_v$, where v runs over the places of $\mathbb{Q}$ dividing N or ∞. In this setting, $\epsilon_\infty = +1$. Assume also that $\epsilon_v = +1$ for all $v \mid N$. In particular, the global sign is $\epsilon = 1$ and the order of vanishing of $L(E, \rho, s)$ at the central point $s = 1$ is even.

Recall that p stands for a prime that does not divide N, and that $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ (resp. $\mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$) is a Hida family passing through the p-stabilization g_α (resp. h_α) such that $U_p g_\alpha = \alpha_g g_\alpha$ (resp. $U_p h_\alpha = \alpha_h h_\alpha$). Similarly, denote by $\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]]$ a Hida family passing through the p-stabilization f_α of f.

Denote by $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$ the triple product p-adic $\mathcal{L}$-function defined in [DR17], attached to the choice of Λ-adic test vector $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ of [Hsi17, Chap. 3]. The values $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m)$ of this p-adic $\mathcal{L}$-function at triples of integers (k, ℓ, m) with $\ell \geq k + m$ interpolate the square root of the algebraic part of

$$L(\breve{\mathbf{f}}_k \otimes \breve{\mathbf{g}}_\ell \otimes \breve{\mathbf{h}}_m, \frac{k + \ell + m - 2}{2}), \tag{3.1}$$

where $\breve{\mathbf{f}}_k, \breve{\mathbf{g}}_\ell, \breve{\mathbf{h}}_m$ denote the specializations of $\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}$ at weights k, ℓ, m.

There is an analogous triple product p-adic $\mathcal{L}$-function $\mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ that interpolates (3.1) but for the range of values (k, ℓ, m) with $k \geq \ell + m$. In particular, $\mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ is directly related to $L(E, \rho, 1)$.

The article [DLR15] studies the value $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ when $L(E, \rho, 1) = 0$. In particular, the Elliptic Stark Conjecture predicts that when $E(H)^\rho$ is 2-dimensional then $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ should encode the p-adic logarithms of global elements in $E(H)^\rho$.

In the present note, our running Assumption 2.4 is that $\mathrm{Sel}_p(\mathbb{Q}, V) = 0$. This implies that $E(H)^\rho = 0$ and, conjecturally, it also implies that $L(E, \rho, 1) \neq 0$.

The main result of this section is an explicit formula for $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ in this case, and this can be seen as completing the study of $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ initiated in [DLR15].

3.1. **Kato classes.** The main tool that we shall use are the generalized Kato classes

$$\kappa := \kappa(f, g_\alpha, h_\alpha) \in \mathrm{Sel}_{(p)}(\mathbb{Q}, V) \tag{3.2}$$

introduced in [DR17, §3]. While we refer to loc. cit. for the detailed construction of these classes, let us describe informally how they are defined. Although the very definition of (3.2) is not strictly necessary for our purposes below, we include it for the interest of the reader.

The class κ should be regarded as the limit as $\ell \to 1$ in weight space of a sequence of global cohomology classes $\kappa(f, \mathbf{g}_\ell, \mathbf{h}_\ell)$ indexed by weights $\ell \geq 2$. At $\ell = 2$ the class is constructed by means of the codimension 2 cycle Δ_2 in the cube $X_1(N)^3$ of the classical modular curve $X_1(N)$ given by the diagonal embedding $x \mapsto (x, x, x)$. This diagonal cycle is not trivial in cohomology, but it is possible to modify it slightly in order to make it null-homologous and $\kappa(f, \mathbf{g}_2, \mathbf{h}_2)$ is defined as the $(\breve{f}, \breve{\mathbf{g}}_2, \breve{\mathbf{h}}_2)$-isotypic component of the image of Δ_2 under the p-adic étale Abel-Jacobi map.

For higher weights $\ell > 2$, one defines in a similar way a null-homologous cycle Δ_ℓ in the product $X_1(N) \times \mathcal{E}_1^{\ell-1}(N) \times \mathcal{E}_1^{\ell-1}(N)$ where $\mathcal{E}_1^{\ell-1}(N)$ denotes the Kuga-Sato variety over $X_1(N)$ whose generic fiber over a point x is the $(\ell - 1)$-th self-product of the marked elliptic curve associated to x under the moduli interpretation.

The class $\kappa(f, \mathbf{g}_\ell, \mathbf{h}_\ell)$ is then again defined as the $(\breve{f}, \breve{\mathbf{g}}_\ell, \breve{\mathbf{h}}_\ell)$-isotypic component of the image of Δ_ℓ under the p-adic étale Abel-Jacobi map. In [DR17] it is shown that these classes can be packaged into a Λ-adic cohomology class $\kappa(f, \mathbf{g}, \mathbf{h})$ and then $\kappa = \kappa(f, g_\alpha, h_\alpha)$ is defined as the specialization of $\kappa(f, \mathbf{g}, \mathbf{h})$ at $\ell = 1$.

Next proposition pins down the relation between the generalized Kato class κ with the p-adic L-values $\mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ and $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$. To lighten the notation, let us denote

$$\mathcal{L}_p^f := \mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1), \quad \mathcal{L}_p^g := \mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1).$$

Let

$$\pi_{\alpha\beta} : \mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V^{\alpha\beta})$$

be the projection map induced by the natural map $V \to V^{\alpha\beta}$.

Proposition 3.1 (Darmon–Rotger)**.**

(1) The element $\partial_p\kappa$ lies in the image of the natural map

$$\mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta}) \longrightarrow \mathrm{H}^1_s(\mathbb{Q}_p, V)$$

and

$$\exp^*_{\beta\beta}(\partial_p\kappa) = \frac{2(1 - p\alpha_f\alpha_g^{-1}\alpha_h^{-1})}{\alpha_g\alpha_h(1 - \alpha_f^{-1}\alpha_g\alpha_h)(1 - \chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})} \times \mathcal{L}_p^f. \tag{3.3}$$

(2) The element $\pi_{\alpha\beta}\,\mathrm{res}_p\,\kappa \in \mathrm{H}^1(\mathbb{Q}_p, V^{\alpha\beta})$ belongs to $\mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta})$ and

$$\log_{\alpha\beta}(\pi_{\alpha\beta}\,\mathrm{res}_p\,\kappa) = 2(1 - \chi(p)p^{-1}\alpha_f a_p(g)^{-1}a_p(h))^{-1} \times \mathcal{L}_p^g. \tag{3.4}$$

Proof. The fact that $\partial_p\kappa$ is the image of an element in $\mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta})$ is [DR17, Proposition 2.8]. The equality (3.3) follows from Proposition 5.2 and Theorem 5.3 of [DR17]. By part (1) of the proposition $\pi^s_{\alpha\beta}\partial_p\kappa = 0$ in the singular quotient $\mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\beta})$. This means that $\pi_{\alpha\beta}\,\mathrm{res}_p\,\kappa$ belongs to $\mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta})$. Equality (3.4) follows from Proposition 5.1, Theorem 5.3 of [DR17]. □

3.2. **Main formula.** Using the class κ introduced above and the basis (2.26) of $\mathrm{Sel}_{(p)}(V)$, we can give a precise formula for $\mathcal{L}_p^g$ in the rank 0 setting. Define the local points $R_{\beta\alpha} \in E(H_p)^{\beta\alpha}$ by the equality

$$\pi_{\alpha\beta}\,\mathrm{res}_p\,\xi^{\beta\beta} = R_{\beta\alpha} \otimes v_g^\alpha v_h^\beta \in \mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta}) = (E(H_p)^{\beta\alpha} \otimes V_{gh}^{\alpha\beta})^{G_{\mathbb{Q}_p}}. \tag{3.5}$$

Theorem 3.2. *The class κ is a multiple of $\xi^{\beta\beta}$. More precisely,*

$$\kappa = \frac{\Theta_g\Theta_h 2(1 - p\alpha_f\alpha_g^{-1}\alpha_h^{-1})\mathcal{L}_p^f}{\alpha_g\alpha_h(1 - \alpha_f^{-1}\alpha_g\alpha_h)(1 - \chi^{-1}(p)\alpha_f^{-1}a_p(g)a_p(h)^{-1})} \cdot \xi^{\beta\beta}.$$

Moreover, if we define the quantities

$$\mathcal{L}_{g_\alpha} := \frac{\Omega_g}{\Theta_g}, \quad \mathcal{E} := \frac{(1 - \chi(p)p^{-1}\alpha_g^{-1}\alpha_h)(1 - p\alpha_f\alpha_g^{-1}\alpha_h^{-1})}{\alpha_g\alpha_h(1 - \alpha_f^{-1}\alpha_g\alpha_h)(1 - \chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})}$$

then we have that

$$\mathcal{L}_p^g = \mathcal{E} \times \frac{\log_p(R_{\beta\alpha})}{\mathcal{L}_{g_\alpha}} \times \mathcal{L}_p^f \mod L^\times.$$

Proof. By Proposition 3.1, κ is an element of $\mathrm{Sel}_{(p)}(\mathbb{Q}, V)$ such that

$$\exp^*(\partial_p\kappa) = (0, 0, 0, \frac{2(1 - p\alpha_f\alpha_g^{-1}\alpha_h^{-1})}{\alpha_g\alpha_h(1 - \alpha_f^{-1}\alpha_g\alpha_h)(1 - \chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})} \times \mathcal{L}_p^f). \tag{3.6}$$

Then κ is a multiple of the element $\xi^{\beta\beta}$; indeed

$$\kappa = \frac{\exp^*_{\beta\beta}(\partial_p\kappa)}{\exp^*_{\beta\beta}(\partial_p\xi^{\beta\beta})}\xi^{\beta\beta}.$$

Observe that (3.6) gives us the expression for the numerator. We now compute the denominator.

$$\begin{aligned}\exp^*_{\beta\beta}(\partial_p\xi^{\beta\beta}) &= \langle \exp^*_{f,p}(\Psi_{\alpha\alpha})\otimes v_g^\beta v_h^\beta, \omega_g\omega_h\rangle = \frac{\exp^*_{f,p}(\Psi_{\alpha\alpha})}{\Theta_g\Theta_h}\\ &= \frac{1}{\Theta_g\Theta_h}.\end{aligned}$$

Here we used the fact that $\eta_g\eta_h = \Theta_g\Theta_h v_g^\beta v_h^\beta$. So we get

$$\begin{aligned}\kappa &= \frac{\exp^*_{\beta\beta}(\partial_p\kappa)}{\exp^*_{\beta\beta}(\partial_p\xi^{\beta\beta})}\cdot\xi^{\beta\beta}\\ &= \frac{2(1-p\alpha_f\alpha_g^{-1}\alpha_h^{-1})\Theta_g\Theta_h}{\alpha_g\alpha_h(1-\alpha_f^{-1}\alpha_g\alpha_h)(1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})}\times\mathcal{L}_p^f\cdot\xi^{\beta\beta}\end{aligned}$$

By (3.4),

$$\begin{aligned}\mathcal{L}_p^g &= \frac{1}{2}(1-\chi(p)p^{-1}\alpha_g^{-1}\alpha_h)\log_{\alpha\beta}(\pi_{\alpha\beta}\operatorname{res}_p\kappa)\\ &= \frac{\Theta_g\Theta_h(1-\chi(p)p^{-1}\alpha_g^{-1}\alpha_h)(1-p\alpha_f\alpha_g^{-1}\alpha_h^{-1})}{\alpha_g\alpha_h(1-\alpha_f^{-1}\alpha_g\alpha_h)(1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})}\times\mathcal{L}_p^f\log_{\alpha\beta}(\pi_{\alpha\beta}\operatorname{res}_p\xi^{\beta\beta})\\ &= \mathcal{E}\Theta_g\Theta_h\mathcal{L}_p^f\log_{\alpha\beta}(\pi_{\alpha\beta}\operatorname{res}_p\xi^{\beta\beta}) = \mathcal{E}\Theta_g\Theta_h\mathcal{L}_p^f\langle\log_p(R_{\beta\alpha})\otimes v_g^\alpha v_h^\beta, \eta_g\omega_h\rangle\\ &= \mathcal{E}\mathcal{L}_p^f\frac{\Theta_g\Theta_h}{\Omega_g\Theta_h}\log_p(R_{\beta\alpha}) = \mathcal{E}\mathcal{L}_p^f\frac{\Theta_g}{\Omega_g}\log_p(R_{\beta\alpha})\\ &= \frac{\mathcal{E}\mathcal{L}_p^f}{\mathcal{L}_{g_\alpha}}\log_p(R_{\beta\alpha})\end{aligned}$$

since $\omega_g\eta_h = \Omega_g\Theta_h\otimes v_g^\alpha v_h^\beta$. □

We end this section by noting that $\mathcal{L}_{g_\alpha}$ is often expected to be related to the Gross–Stark Unit u_{g_α} attached to the modular form g_α as defined in [DLR15, §1]. More precisely, under the additional assumption that g is not the theta series of a Hecke character of a real quadratic field in which p splits, [DR16, Conjecture 2.1] predicts that

$$\mathcal{L}_{g_\alpha} \stackrel{?}{=} \log_p(u_{g_\alpha}) \mod L^\times. \tag{3.7}$$

Thus we obtain the following consequence of Theorem 3.2, under the aforementioned hypothesis:

Corollary 3.3. *Assuming the equality* (3.7), *if* $\mathrm{Sel}_p(\mathbb{Q}, V) = 0$ *then*

$$\mathcal{L}_p^g = \mathcal{E}\times\frac{\log_p(R_{\beta\alpha})}{\log_p(u_{g_\alpha})}\times\mathcal{L}_p^f.$$

4. The case of theta series of an imaginary quadratic field K where p splits

In this section we will consider a particular case where g and h are theta series of the same imaginary quadratic field in which p splits. We will see that in this setting the representation V decomposes in a way that forces $\mathcal{L}_p^g$ to vanish when the complex L-function does not vanish at the central critical point; that is, the special value of the p-adic $\mathcal{L}$-function vanishes in analytic rank 0.

Let K be an imaginary quadratic field of discriminant D_K. Let $\psi_g, \psi_h\colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ be two finite order Hecke characters of K of conductors $\mathfrak{c}_g, \mathfrak{c}_h$ and central characters $\varepsilon, \bar\varepsilon$ respectively. Here

$\varepsilon\colon \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ is a finite order character of and $\bar{\varepsilon}$ denotes is complex conjugate. Let g and h be the theta series attached to ψ_g and ψ_h. They are modular forms of weight one, and their levels and nebentype characters are given by

$$N_g := D_K \cdot \mathrm{N}_K(\mathfrak{c}_g), \quad N_h := D_K \cdot \mathrm{N}_K(\mathfrak{c}_h), \quad \chi := \chi_K \cdot \varepsilon, \quad \bar{\chi} = \chi_K \cdot \bar{\varepsilon},$$

where N_K stands for the norm on ideals of K and we regard ε and $\bar{\varepsilon}$ as Dirichlet characters via class field theory. That is to say,

$$g \in M_1(N_g, \chi), \quad \text{and} \quad h \in M_1(N_h, \bar{\chi}).$$

Let $f \in S_2(N_f)$ be a newform with rational coefficients and let E be the associated elliptic curve over $\mathbb{Q}$. We will particularize some of the results of the previous sections to this choice of forms f, g, and h, so we will use the same notations as before. In particular, ρ stands for the Artin representation afforded by $V_g \otimes V_h$ and p is a prime that does not divide $N_f \cdot N_g \cdot N_h$. In this section, we will make the following additional assumptions:

(1) $\gcd(N_f, \mathfrak{c}_g\mathfrak{c}_h) = 1$;
(2) p splits in K.

A finite order Hecke character ψ of K can be regarded, via class field theory, as a Galois character $\psi\colon G_K \to \mathbb{A}_K^\times$. Let σ_0 be any element in $G_{\mathbb{Q}} \setminus G_K$. We denote by ψ' the character defined by $\psi'(\sigma) := \psi(\sigma_0\sigma\sigma_0^{-1})$ (this does not depend on the particular choice of σ_0). Also, ψ gives rise to a 1-dimensional representation of G_K, and we let $V_\psi = \mathrm{Ind}_K^{\mathbb{Q}}(\psi)$ denote the induced representation; it is a 2-dimensional representation of $G_{\mathbb{Q}}$. Observe that, with this notation, $V_g = V_{\psi_g}$ and $V_h = V_{\psi_h}$.

There is a well-known decomposition of $V_g \otimes V_h$ as the direct sum of two representations:

$$V_g \otimes V_h = V_{\psi_1} \oplus V_{\psi_2}, \tag{4.1}$$

where the characters ψ_1 and ψ_2 are

$$\psi_1 := \psi_g\psi_h, \quad \text{and} \quad \psi_2 := \psi_g\psi_h'.$$

This induces a decomposition of the representation $V = V_f \otimes V_g \otimes V_h$ as a direct sum of two representations:

$$V = V_1 \oplus V_2, \tag{4.2}$$

where

$$V_1 := V_f \otimes V_{\psi_1}, \quad \text{and} \quad V_2 := V_f \otimes V_{\psi_2}.$$

This induces a factorization of complex L-functions

$$L(E, \rho, s) = L(E, \psi_1, s) \cdot L(E, \psi_2, s).$$

Under our assumption that $\gcd(N_f, \mathfrak{c}_g\mathfrak{c}_h) = 1$ the local signs of $L(E, \psi_1, s)$ and $L(E, \psi_2, s)$ are equal, so that the local signs of $L(E, \rho, s)$ are all equal to $+1$ and therefore the assumption on local signs of Section 3 is satisfied.

Theorem 4.1. *In the setting of this section, if $L(E, \rho, 1) \neq 0$ then $\mathcal{L}_p^g = 0$.*

Proof. If $L(E, \rho, 1) \neq 0$ then $L(E, \psi_i, 1) \neq 0$ for $i = 1, 2$. Note that ψ_1 and ψ_2 are ring class characters of the imaginary quadratic field K. Then, by results of Gross–Zagier and Kolyvagin

$$\mathrm{Sel}_p(\mathbb{Q}, V_i) = 0 \ \text{ for } i = 1, 2. \tag{4.3}$$

The decomposition (4.2) induces a decomposition of the Selmer groups

$$\mathrm{Sel}_p(\mathbb{Q}, V) = \mathrm{Sel}_p(\mathbb{Q}, V_1) \oplus \mathrm{Sel}_p(\mathbb{Q}, V_2), \tag{4.4}$$

and analogously for the relaxed and the strict Selmer groups of V. In particular, $\mathrm{Sel}_p(\mathbb{Q}, V) = 0$.

Since p splits in K we can write $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, and from our assumption that $p \nmid N_f \cdot N_g \cdot N_h$ we see that $p \nmid \mathfrak{c}_g \mathfrak{c}_h$. Without loss of generality we can suppose that

$$\psi_g(\mathfrak{p}) = \alpha_g, \quad \psi_g(\bar{\mathfrak{p}}) = \beta_g, \quad \psi_h(\mathfrak{p}) = \alpha_h, \quad \psi_h(\bar{\mathfrak{p}}) = \beta_h,$$

so that

$$V_1 = V^{\alpha\alpha} \oplus V^{\beta\beta} \quad \text{and} \quad V_2 = V^{\alpha\beta} \oplus V^{\beta\alpha}.$$

By (4.3), the same computations as in §2.1 show that there are isomorphisms

$$\mathrm{Sel}_{(p)}(\mathbb{Q}, V_1) \xrightarrow{\partial_p} \mathrm{H}^1_s(\mathbb{Q}_p, V_1) \xrightarrow{(\pi^s_{\alpha\alpha}, \pi^s_{\beta\beta})} \mathrm{H}^1_s(\mathbb{Q}_p, V_1^{\alpha\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V_1^{\beta\beta}),$$

where $\pi^s_{\alpha\alpha}$ denotes the natural map in the singular quotient induced by the projection $V \to V^{\alpha\alpha}$, and analogously for $\pi^s_{\beta\beta}$. Similarly, there are dual exponential maps

$$\exp^*_{\alpha\alpha} : \mathrm{H}^1_s(\mathbb{Q}_p, V_1^{\alpha\alpha}) = \mathrm{H}^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\alpha\alpha}) \longrightarrow L_p$$

and

$$\exp^*_{\beta\beta} : \mathrm{H}^1_s(\mathbb{Q}, V_1^{\beta\beta}) = \mathrm{H}^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\beta\beta}) \longrightarrow L_p$$

which are in fact isomorphisms.

Then $\mathrm{Sel}_{(p)}(\mathbb{Q}, V_1)$ has dimension 2 over $\mathbb{Q}_p$ with the canonical basis

$$\zeta^{\alpha\alpha}, \ \zeta^{\beta\beta},$$

where $\zeta^{\alpha\alpha}$ is characterized (up to scalars in $L^\times$) by the fact that

$$\exp^*_{\alpha\alpha}(\pi_{\alpha\alpha}\partial_p(\zeta^{\alpha\alpha})) = 1, \quad \text{and} \quad \exp^*_{\beta\beta}(\pi_{\beta\beta}\partial_p(\zeta^{\alpha\alpha})) = 0.$$

Similarly,

$$\exp^*_{\alpha\alpha}(\pi_{\alpha\alpha}\partial_p(\zeta^{\beta\beta})) = 0, \quad \text{and} \quad \exp^*_{\beta\beta}(\pi_{\beta\beta}\partial_p(\zeta^{\beta\beta})) = 1.$$

Analogously, $\mathrm{Sel}_{(p)}(\mathbb{Q}, V_2)$ has dimension 2 with basis $\zeta^{\alpha\beta}, \zeta^{\beta\alpha}$.

By Theorem 3.2, the value $\mathcal{L}_p^g$ is a multiple of $\log_{\alpha\beta}(\mathrm{res}_p\, \xi^{\beta\beta})$. On the other hand, using the decomposition

$$\mathrm{Sel}_{(p)}(\mathbb{Q}, V) = \mathrm{Sel}_{(p)}(\mathbb{Q}, V_1) \oplus \mathrm{Sel}_{(p)}(\mathbb{Q}, V_2),$$

the element $\xi^{\beta\beta} \in \mathrm{Sel}_{(p)}(\mathbb{Q}, V)$ corresponds to a multiple of $(0, \zeta^{\beta\beta})$, and this implies that

$$\pi_{\beta\alpha}\, \mathrm{res}_p\, \xi^{\beta\beta} = 0.$$

□

5. Numerical computations

In this section we present a few numerical examples illustrating the phenomena studied in this note. They have been computed with a Sage ([S+20]) implementation of Lauder's algorithms ([Lau14]), adapted to work in the current setting. The code is available at `github.com/mmasdeu/ellipticstarkconjecture`. The data for the weight-one modular forms can be found in Alan Lauder's website.[1]

The aim of this section is threefold: first of all, we illustrate and numerically verify the vanishing predicted by Theorem 4.1; we also provide various other examples not covered by Theorem 4.1 where $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ vanishes, and for which we suspect there should be a systematic explanation;

[1]See `http://people.maths.ox.ac.uk/lauder/weight1/`.

finally, we present numerical data where $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ does not vanish, therefore confirming that these quantities are certainly not always 0.

This raises the natural question about what is the arithmetic meaning encoded by the p-adic L-value $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$. When the analytic rank of the associated classical L-function is 2, the authors of [DLR15] proposed a conjectural interpretation of these p-adic iterated integrals, predicting that $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ should encode a 2×2-regulator given by the p-adic logarithms along suitable directions of global points on E, rational over the number field cut out by the tensor product $\rho_g\otimes\rho_h$ of the Artin representations attached to g and h.

In the setting of this note, where the analytic rank is 0, one can not expect global points on E appearing in the picture, because according to the Birch and Swinnerton-Dyer conjecture the eigenspace of the Mordell–Weil group of E cut out by $\rho_g\otimes\rho_h$ should be trivial (and this is indeed the case in many instances, as proved in [DR17]). The analogous motivic class that one does expect to show up in our scenario is a global cohomology class with values in $V_p(E)\otimes\rho_g\otimes\rho_h$ that should fail to be crystalline at p, and $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ should be interpreted as some sort of p-adic invariant attached to such class. Our main Theorem 3.2 claims precisely a statement along these lines: there exists a specific global cohomology class (namely, the generalized Kato class constructed in [DR17]) which fails to be crystalline along the direction in $V_g\otimes V_h$ on which Frob_p acts with eigenvalue $\beta_g\beta_h$ (and therefore is not crystalline at p), *but* remains crystalline along a different direction, namely the line in $V_g\otimes V_h$ on which Frob_p acts with eigenvalue $\beta_g\alpha_h$. It thus makes sense to compute the Bloch-Kato logarithm of this class along the latter direction, and $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ essentially encodes the output of that operation, together with other global invariants. This is how the numerical data below should be understood.

In some instances one can go further and understand better these p-adic iterated integrals by relating them to well-known constructions in the literature. Namely, in [GR20] the first and last authors focus on the case where E has multiplicative reduction at p, while g and h are theta series associated to characters of the same imaginary quadratic field K, in which p is assumed to remain inert. Under these hypotheses, they prove a formula relating $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ to the Kolyvagin classes constructed by Bertolini and Darmon in [BD97] by means of the tower of Heegner points of conductor p^r with $r\geq 1$.

5.1. **Dihedral case.**

(a) We computed $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})(2,1,1)$ with $\mathbf{f}$ the Hida family passing through the modular form f_E of weight 2 attached to an elliptic curve $E/\mathbb{Q}$ of conductor N_f and $\mathbf{g}$ attached to the weight-one modular form $g=\theta(1_K)$ for some imaginary quadratic field K. The modular form g belongs then to $M_1(N_g,\chi_K)_{\mathbb{Q}}$. For each of the entries in the table we give the Cremona label for the elliptic curve E_f, its conductor N_f, the field K, the level N_g of g, the level N such that $p^\alpha N=\mathrm{lcm}(N_f,N_g)$ with $\alpha\geq 0$ and $p\nmid N$. In all of these cases, we obtained $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})=0$ up to the working precision of p^{10}. Due to computational restrictions, only in the ramified case we have been able to compute examples where p divides the conductor of the elliptic curve.

Note that all the elliptic curves arising in Table 1 below have rank 0 over K, and thus the zeros obtained in this table are accounted for by Theorem 4.1.

In tables 2 and 3 below, we see instances of zeros which we expect are explained by the sign of the action of the level N Atkin-Lehner operator although we have not verified this in detail.

In what follows we illustrate with examples the fact that the quantity $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})$ is not always zero.

E_f	K	N_g	p	N	$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})$
11a	$\mathbb{Q}(\sqrt{-5})$	20	7	220	0
11a	$\mathbb{Q}(\sqrt{-11})$	11	5	11	0
19a	$\mathbb{Q}(\sqrt{-19})$	19	5	19	0
19a	$\mathbb{Q}(\sqrt{-19})$	19	7	19	0
39a	$\mathbb{Q}(\sqrt{-39})$	39	5	39	0
51a	$\mathbb{Q}(\sqrt{-51})$	51	5	51	0
55a	$\mathbb{Q}(\sqrt{-55})$	55	7	55	0
187a	$\mathbb{Q}(\sqrt{-187})$	187	7	187	0

TABLE 1. Cases with p split in K.

E_f	K	N_g	p	N	$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})$
11a	$\mathbb{Q}(\sqrt{-3})$	3	5	33	0
11a	$\mathbb{Q}(\sqrt{-11})$	11	7	11	0
15a	$\mathbb{Q}(\sqrt{-15})$	15	7	15	0
39a	$\mathbb{Q}(\sqrt{-39})$	39	7	39	0
51a	$\mathbb{Q}(\sqrt{-51})$	51	7	51	0
67a	$\mathbb{Q}(\sqrt{-67})$	67	5	67	0
67a	$\mathbb{Q}(\sqrt{-67})$	67	7	67	0
187a	$\mathbb{Q}(\sqrt{-187})$	187	5	187	0

TABLE 2. Cases with p inert in K.

E_f	K	N_g	p	N	$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})$
15a	$\mathbb{Q}(\sqrt{-15})$	15	5	3	0
35a	$\mathbb{Q}(\sqrt{-35})$	35	5	7	0
35a	$\mathbb{Q}(\sqrt{-35})$	35	7	5	0
55a	$\mathbb{Q}(\sqrt{-55})$	55	5	11	0

TABLE 3. Cases with p ramified in K.

(b) In this example we fix f to be attached to the elliptic curve $E_f\colon y^2 = x^3 + x^2 - 15x + 18$, of conductor $N_f = 120$. The weight-one form g we consider has level $N_g = 120$ also, and has q-expansion

$$\begin{aligned} g(q) &= q + iq^2 + iq^3 - q^4 - iq^5 - q^6 - iq^8 - q^9 + q^{10} - iq^{12} + q^{15} + q^{16} - iq^{18} \\ &\quad + iq^{20} + q^{24} - q^{25} - iq^{27} + iq^{30} - 2q^{31} + iq^{32} + O(q^{34}), \end{aligned}$$

where $i^2=-1$. It is the theta series attached to the Dirichlet character ϵ modulo 120 defined by

$$\epsilon(97)=-1, \quad \epsilon(31)=1, \quad \epsilon(41)=-1, \quad \epsilon(61)=-1.$$

The field cut out by ϵ is $K=\mathbb{Q}(\sqrt{-6})$, and we take $p=5$ which is split in both $L=\mathbb{Q}(\sqrt{-1})$ and K. Note that p divides N_f and N_g. We compute to precision 10 the quantity

$$\mathcal{L}_5^g(\mathbf{f},\mathbf{g},\mathbf{g})(2,1,1)=4\cdot 5+3\cdot 5^2+4\cdot 5^3+3\cdot 5^5+4\cdot 5^6+3\cdot 5^7+5^8+2\cdot 5^9+O(5^{10}).$$

With the same setting, we take $p=13$ (now p is split in L but inert in K). We obtain

$$\mathcal{L}_{13}^g(\mathbf{f},\mathbf{g},\mathbf{g})(2,1,1)=7+3\cdot 13+10\cdot 13^2+13^4+11\cdot 13^5+13^6+6\cdot 13^7+4\cdot 13^8+5\cdot 13^9+O(13^{10})$$

(c) Let E_f be the elliptic curve $y^2+y=x^3+x^2+42x-131$ with label `175c1`. It has conductor $N_f=175$ and rank 0. Let $g=h$ be the theta series of the character ϵ_1 of $K=\mathbb{Q}(\alpha)$ with α satisfying $\alpha^2-\alpha+2=0$, of discriminant $D_K=-7$ and conductor $5\mathcal{O}_K$ (which is inert, of norm 25), satisfying

$$\epsilon_1(127)=-1, \quad \epsilon_1(101)=-1.$$

The modular form g has q-expansion

$$g(q)=q+iq^2-iq^7+iq^8-q^9-q^{11}+q^{14}-q^{16}-iq^{18}-iq^{22}-iq^{23}+q^{29}+O(q^{30}),$$

where again $i^2=1$. For $p=13$ (which is inert in K and split in L), we obtain

$$\mathcal{L}_{13}^g(\mathbf{f},\mathbf{g},\mathbf{g})(2,1,1)=1+3\cdot 13+2\cdot 13^2+13^3+12\cdot 13^4+9\cdot 13^5+3\cdot 13^8+5\cdot 13^9+O(13^{10}).$$

(d) Finally, consider the elliptic curve E_f of conductor 175 from the previous example, and for $g=h$ consider the theta series of another character ϵ_2 of $K=\mathbb{Q}(\alpha)$, $\alpha^2-\alpha+2=0$, of discriminant $D_K=-7$ and conductor $5\mathcal{O}_K$ (inert, of norm 25), now taking the values

$$\epsilon_2(127)=1, \quad \epsilon_2(101)=-1.$$

This yields a modular form g with q-expansion

$$g(q)=q+q^2-q^7-q^8+q^9-q^{11}-q^{14}-q^{16}+q^{18}-q^{22}+q^{23}-q^{29}+O(q^{30}).$$

We numerically obtain for $p=13$ that

$$\mathcal{L}_{13}^g(\mathbf{f},\mathbf{g},\mathbf{g})(2,1,1)=0.$$

Again, we do not have a way to prove that $\mathcal{L}_{13}^g(\mathbf{f},\mathbf{g},\mathbf{g})(2,1,1)$ is actually zero.

5.2. **Exotic image case.** In the non-CM setting, we have been able to compute the following example. Consider $E_f\colon y^2=x^3-17x-27$, which has conductor $N_f=124$. Let g be the modular form of level $N_g=124$ and projective image A_4, defined as the theta series of the character ϵ of conductor 124 having values

$$\epsilon(65)=\alpha^2-1, \quad \epsilon(63)=-1,$$

where α satisfies $\alpha^4-\alpha^2+1=0$. The modular form g has q-expansion

$$\begin{aligned} g(q)=\ & q-\alpha^3q^2+\left(-\alpha^3+\alpha\right)q^3-q^4+\left(\alpha^2-1\right)q^5-\alpha^2q^6+\left(\alpha^3-\alpha\right)q^7+\alpha^3q^8+\alpha q^{10}-\alpha q^{11} \\ & +\left(\alpha^3-\alpha\right)q^{12}+\left(-\alpha^2+1\right)q^{13}+\alpha^2q^{14}+\alpha^3q^{15}+q^{16}-\alpha^2q^{17}+\left(-\alpha^3+\alpha\right)q^{19}+\left(-\alpha^2+1\right)q^{20} \\ & +\left(\alpha^2-1\right)q^{21}+\left(\alpha^2-1\right)q^{22}+\alpha^2q^{24}-\alpha q^{26}+\alpha^3q^{27}+O(q^{28}). \end{aligned}$$

We let $h=g^*$ its complex conjugate, and compute with $p=13$, obtaining

$$\mathcal{L}_{13}^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)=1+5\cdot 13+5\cdot 13^2+4\cdot 13^3+6\cdot 13^4+6\cdot 13^5+6\cdot 13^6+13^7+3\cdot 13^8+9\cdot 13^9+9\cdot 13^{10}+O(13^{11}).$$

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