ADVANCED MATHEMATICS
MASTER'S FINAL PROJECT

Equivariant cohomology and free $(\mathbb{Z}/2)^n$-complexes

Author:
Jordi Garriga i Puig

Supervisor:
Ignasi Mundet i Riera

Facultat de Matemàtiques i Informàtica

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Abstract

The field of transformation groups studies continuous actions of groups on topological spaces, in particular on CW-complexes. One of the fundamental questions that arises in this context is to determine those finite groups that can act effectively on a given topological space. A large amount of results are known about this issue, but it is not completely answered yet. Even in the case of abelian groups actions or elementary groups actions the question is highly nontrivial.

This project is devoted to a remarkable result regarding the description of those finite abelian groups that act freely on a CW-complex. The result states that if X is a finite CW-complex and $(\mathbb{Z}/p)^n$ acts freely on X, with p prime, then the sum of the lengths of the homology groups of X with coefficients in $\mathbb{Z}/p$ is bounded below by $n+1$. Our study has been restricted to the case $p = 2$, that was proved by Carlsson in 1983, with a modern approach based on cohomological methods.

Introduction

Along this work we will denote the p-tori by

$$\mathbb{Z}_p := \mathbb{Z}/(p).$$

for each $p \in \mathbb{Z}$ prime.

The theory of transformation group in as extensive field in mathematics that studies topological spaces endowed with a continuous action of a group. In this context there is a natural interest in necessary conditions for a topological space to admit a free action. This question has obtained different answers while the study of transformation groups has been developed.

This field was first introduced in the 1930's and the 1940's by P.A. Smith study of finite groups actions. His work stated several remarkable results on the fixed points set X^G of the group action. Regarding our question on free spaces, an important result due to Smith on free actions of p-tori is the following (see [Smi44]).

Theorem 0.1 (Smith)**.** *Let $G = (\mathbb{Z}_p)^n$ with p prime. Assume that G acts freely on the m-dimensional sphere $X = S^m$. Then $n = 1$.*

Substantial progress was achieved after the development of a modern approach based on cohomological methods, which was introduced in A. Borel seminar in Princeton University (see [Bor+60]). In particular, an important tool was described, the Borel construction.

The use of these reformulation implied an improvement and better understatement of P.A. Smith original results. Specifically, the study of free p-tori actions on finite CW-complexes lead to the following result, that was proved by Carlsson ([Car83]) for $p = 2$ and by Baumgartner ([Bau93]) for p odd.

Theorem 0.2 (Carlsson - Baumgartner)**.** *Let $G = (\mathbb{Z}_p)^n$ with p prime and let X be a finite CW-complex. Assume that G acts freely and cellularly on X. Then*

$$\sum_{n=-\infty}^{\infty} \lambda(H_n(X;\mathbb{Z}_2)) \geq n+1.$$

In the above statement one considers that for X a G-space, the G-action extends to the cohomology $H^\bullet(X;k)$, with k a fields, hence $H^\bullet(X;k)$ is a $k[G]$-modules. If J denotes the augmentation ideal of $k[G]$, for M a $k[G]$-module the length $\lambda(M)$ is defined as

$$\lambda(M) = \max\{\lambda : J^{\lambda-1}M \neq 0\}.$$

The aim of this project is to give a full description of the above result for the case $p = 2$ following [Car83]. For that we have structured the project in two sections and two additional appendices.

In the first chapter 1 we give an introduction to the basic concepts in the theory of transformation groups and we prove some basic results. In this sense, we start in the first section 1.1 by defining topological groups, G-spaces and G-CW-complexes, which are the main objects of study of this field. In the second section 1.2 we establish the definition of an equivariant cohomology theory over a category of G-spaces, and a comparison theorem of cohomology theories is proved. In the following section 1.3 we describe one of the most important tools in the study of transformation groups, which is the Borel construction, and we give both a topological and an algebraic approach to these method, which we prove to be equivalent. Finally, with a view towards studying $(\mathbb{Z}_2)^n$-complexes, in the last section 1.4 we give an explicit description of the algebraic version of the Borel construction for the case $G = (\mathbb{Z}_2)^n$.

The second chapter 2 is devoted to explain the proof of Theorem 2.23 following the procedure used by Carlsson in [Car83], which uses a purely algebraic approach. In the first section 2.1 we introduce some results on differential graded modules which lead up to the proof of Theorem 2.17, that is essential for the final prove. In the following section 2.2 we define the β functor, which latter on is observed to be analogous to the algebraic version of the Borel construction, and we state several results on its homology to finally prove Theorem 2.22, from which it is directly deduced the desired result. Finally, in 2.3 we give a geometric interpretation of the statements in the previous section considered over a finite CW-complex and we prove Theorem 2.23.

At the end of this project we have included two additional appendix chapters. In the first appendix A we prove some results on homotopy theory that are needed along the work, mainly the mapping cone construction and its main properties. In the second appendix B we give an introduction to principal G-bundles, which are essential in the definition of the Borel construction. After giving a proper description of these objects, we proceed in section B.1 by proving an important equivalence between equivariant maps and sections of bundles. Finally, in section B.2 we define universal principal G-bundles and classifying spaces, we prove one of the main theorems on principal G-bundles, and we conclude with the computation of some examples of universal principal G-bundles, which are used in the discussion of the Borel construction.

Chapter 1

Equivariant cohomology of G-CW-complexes and the Borel construction

In this chapter we give an introduction to the theory of transformation groups and we describe the category of G-spaces and G-CW-complexes. Moreover, we describe the Borel construction, which is an important tool in the study of transformation groups that allow us to define an equivariant cohomology theory $H^*_G(X)$ of a G-space X. In the last section we discuss the case of interest $G = (\mathbb{Z}_2)^n$. The chapter has been based on [AP93] and [Bre72].

1.1 Topological groups, G-spaces and G-CW-complexes

We start by introducing the object of study in the theory of transformation groups. In the most general setup we will consider an action of a topological group G on a topological space X.

By a **topological group** G we mean a Hausdorff topological space together with a continuous map $G \times G \to G$ which defines on G a group structure in such a way that the inverse map $x \mapsto x^{-1}$ is also continuous. The identity element of G is denoted by e.

Definition 1.1. *Let G a topological group. A **G-space** is a Hausdorff topological space X with a left action of G, that is a map $G \times X \to X$ which satisfies that*

a) $(gg')x = g(g'x)$ for all $x \in X$ and $g, g' \in G$;

b) $ex = x$ for all $x \in X$, where e denotes the identity element in G.

The action of G on X is ***free*** *if for each $x \in X$ the only element $g \in G$ that leaves x fixed, $gx = x$, is the identity $g = e$.*

Definition 1.2. *Let X, Y be G-spaces. A* ***G-equivariant map****, or a* ***G-map****, is a continuous map $f : X \to Y$ which commutes with the group action, that is*

$$f(gx) = gf(x) \quad \text{for all } g \in G \text{ and } x \in X.$$

Moreover, we say that two G-maps $h_0, h_1 : X \to Y$ are ***G-homotopic*** *if there exists a G-map*

$$H : X \times I \to Y$$

where $I = [0,1]$ with a trivial G-action, and such that $H_{|X\times\{0\}} = h_0$ and $H_{|X\times\{1\}} = h_1$.

From the above definitions it follows that for a fixed topological group G the set of G-spaces and G-equivariant maps define the category of G-spaces G**Top**.

In a similar way as one proceeds in non-equivariant topology it is convenient to restrict our study to the case of G-spaces that are G-homotopic to G-CW-complexes. The proper definition of G-CW-complexes will be the following step.

Definition 1.3. *An* ***n-dimensional G-cell of type*** *G/K is a G-space*

$$G/K \times D^n$$

with the G-action given by

$$\begin{array}{ccc} G \times \left(G/K \times D^n\right) & \to & G/K \times D^n \\ (g, ([g'], x)) & \mapsto & ([gg'], x), \end{array}$$

where D^n denotes the n-dimensional ball and $K \subset G$ is a subgroup. We say that $G/K \times S^{n-1}$ is the ***G-boundary*** *of the G-cell $G/K \times D^n$.*

In the construction of G-CW-complexes we say that the G-space X is obtained by attaching a disjoint union of n-dimensional G-cells to a G-space Y

$$\bigsqcup_{i\in I} \left(G/K_i \times D^n\right) \quad \text{with } K_i \subset G \text{ subgroup for each } i \in I,$$

along the G-maps

$$\phi_i : G/K_i \times S^{n-1} \to Y$$

if

$$X = Y \sqcup \left(\sqcup_{i\in I} G/K_i \times D^n\right) / \sim,$$

where for all $([g]_i, s) \in G/K_i \times S^{n-1}$

$$\phi_i([g]_i, s) \sim ([g]_i, s).$$

Notice that this procedure is completely analog to non-equivariant attaching maps in CW-complexes. Then we define G-CW-complexes as the following.

Definition 1.4. *A G-**CW-complex** is the colimit in the category of G-space G**Top** of a sequence of G-equivariant inclusions*

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots,$$

with X^0 a disjoint union of 0-dimensional G-cells and X^n obtained by attaching a disjoint union of n-dimensional G-cells to X^{n-1}.

We say that a G-CW-complex is **finite** if it is built from finitely many G-cells and that it is **finite-dimensional** if all its G-cells are of bounded dimension. For X a G-CW-complex we call X/G the **orbit space**, which naturally inherits a structure of CW-complex. Notice that the non-equivariant cells of X/G correspond to the orbit spaces of the G-cells of X and the attaching maps in X/G are induced by the corresponding maps in X modulo the G-action.

Since the construction of G-CW-complexes follows the same structure used in the definition of usual CW-complex, it can be proved that the standard topological and homological properties of CW-complexes have they analogous counterpart on the equivariant setting, which justifies the interest in focusing our study to G-spaces that are G-homotopic to G-CW-complexes.

Let us consider some examples of G-CW-complexes.

Example 1.5. Let $G = \mathbb{Z}_2 = \{\pm 1\}$ and consider the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ with the scalar multiplication G-action

$$\begin{array}{ccc} G \times S^{n-1} & \to & S^{n-1} \\ (g, x) & \mapsto & gx. \end{array}$$

Notice S^{n-1} is a free G-space with the following decomposition as a G-CW-complex

$$\begin{array}{l} X^0 = G \cong S^0 \\ X^m = X^{m-1} \cup_{\phi_m} (G \times D^m) \text{ for each } 1 \leq m < n \end{array}$$

where

$$\begin{array}{cccc} \phi_m : & G \times S^{m-1} & \to & X^{m-1} = S^{m-1} \\ & (g, x) & \mapsto & gx. \end{array}$$

The orbit space X/G corresponds to the real projective space $\mathbb{R}P^{n-1}$ with the standard cellular decomposition induced by the G-CW-complex structure of S^{n-1}.

Example 1.6. Let $G = S^1$ the unit sphere and consider $S^{2n-1} = \{x \in \mathbb{C}^n : \|x\| = 1\}$ with the G-action given in this case by complex multiplication

$$\begin{array}{ccc} G \times S^{2n-1} & \to & S^{2n-1} \\ (g, x) & \mapsto & gx. \end{array}$$

Again S^{n-1} is a free G-space with an analogous G-CW-complex decomposition

$$\begin{aligned} &X^0 = G \cong S^1 \\ &X^1 = X^0 \\ &X^{2m} = X^{2m-1} \cup_{\varphi_{2m}} (G \times D^{2m}) \text{ for each } 1 \leq m < n \end{aligned}$$

where

$$\begin{array}{rccc} \varphi_{2m}: & G \times S^{2m-1} & \to & X^{2m-1} = X^{2m-2} = S^{2m-1} \\ & (g,x) & \mapsto & gx. \end{array}$$

Now the orbit space X/G is the complex projective space $\mathbb{C}P^{n-1}$ again with the standard cellular decomposition induced by the G-CW-complex structure of S^{2n-1}.

In the following sections we will thoroughly use the universal free G-space EG, that is the total space of the universal principal G-bundle

$$EG \to BG$$

and which can be endowed with a G-CW-complex structure. See Appendix B for an introduction to principal G-bundles.

1.2 Equivariant cohomology theories of G-CW-complexes

The contemporary approach in the study of transformation groups emphasizes the importance of cohomological methods. In this section our objective is to provide a general description of the fundamental properties that a cohomological theory needs to satisfy within the equivariant context.

Several equivariant cohomology theories may be defined over distinct categories of G-spaces and the axioms that have to be fulfilled may depend on the context. Our focus will be on equivariant cohomology theories defined over the category of finite G-CW-complexes. We demand the following properties.

Definition 1.7. *An* ***equivariant cohomology theory*** $h_G^\bullet = \{h_G^n\}_{n\in\mathbb{Z}}$ ***on finite*** G***-CW-complex*** *is a family of contravariant functors from the category of finite G-CW-complex to the category of R-modules over R, with R a commutative ring, which fulfils*

A1) *$h_G^\bullet$ is G-homotopy invariant, that is that $h_G^\bullet(f_0) = h_G^\bullet(f_1)$ if $f_0, f_1 : X \to Y$ are G-homotopic;*

A2) *if X_1, X_2 are G-CW-complexes and $X_0 \subset X_2$ is a subcomplex, then if X is obtained by attaching X_2 to X_1 along the G-map $\phi : X_0 \to X_1$ there exists a long exact Mayer-Vietoris sequence given by*

$$\cdots \to h_G^q(X_0) \to h_G^{q+1}(X) \to h_G^{q+1}(X_1) \times h_G^{q+1}(X_2) \to h_G^{q+1}(X_0) \to \cdots .$$

An analog definition for an **equivariant homology theory** naturally arises. However, our interest lies in using a cohomology theory because its multiplicative structure will be essential in the study.

To conclude this section we will prove a comparison theorem between equivalent cohomology theories, which is a result that also has an analogous counterpart in the non-equivariant setting.

Theorem 1.8. *Let $\tau : h_G^\bullet \to k_G^\bullet$ be a natural transformation between equivariant cohomology theories in the sense of Definition 1.7. If $\tau(G/K)$ is an isomorphism for each $K \subset G$ subgroup, then for each finite G-CW-complex τ is an isomorphism .*

Proof. Let $K \subset G$ subgroup and consider the G-CW-complex

$$X = G/K \times S^{n-1} = X_1 \cup_\phi X_2$$

where $X_1 = G/K \times D^0$, $X_2 = G/K \times D^{n-1}$ and

$$\phi : X_0 = G/K \times S^{n-2} \to X_1$$

is given by collapsing S^{n-2} to one point. We proceed by induction on n to prove that $\tau(G/K \times S^{n-1})$ is an isomorphism. If $n = 1$ we have that $X_0 = \emptyset$, $X_1 = G/K \times D^0 \simeq G/K$ and $X_2 = G/K \times D^0 \simeq G/K$, hence by property A2 we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & h_G^q(X) \arrow[r] \arrow[d, "\tau(X)"] & h_G^q(X_1) \times h_G^q(X_2) \arrow[r] \arrow[d, "\tau(X_1)\times\tau(X_2)"] & 0 \\
0 \arrow[r] & k_G^q(X) \arrow[r] & k_G^q(X_1) \times k_G^q(X_2) \arrow[r] & 0
\end{tikzcd}$$

and by property A1 we obtain $\tau(X_1)$ and $\tau(X_2)$ are isomorphisms, hence $\tau(X)$ is also an isomorphism. For $n > 1$ by property A2 we also have a commutative diagram

$$\begin{tikzcd}
h_G^{q-1}(X_1) \times h_G^{q-1}(X_2) \arrow[r] \arrow[d, "\tau(X_1)\times\tau(X_2)"] & h_G^{q-1}(X_0) \arrow[r] \arrow[d, "\tau(X_0)"] & h_G^q(X) \arrow[r] \arrow[d, "\tau(X)"] & h_G^q(X_1) \times h_G^q(X_2) \arrow[r] \arrow[d, "\tau(X_1)\times\tau(X_2)"] & h_G^q(X_0) \arrow[d, "\tau(X_0)"] \\
h_G^{q-1}(X_1) \times h_G^{q-1}(X_2) \arrow[r] & k_G^{q-1}(X_0) \arrow[r] & k_G^q(X) \arrow[r] & k_G^q(X_1) \times k_G^q(X_2) \arrow[r] & k_G^q(X_0)
\end{tikzcd}$$

Then, by inductive hypothesis $\tau(X_0)$ is an isomorphism and by the same argument as before the same happens for $\tau(X_1)$ and $\tau(X_2)$, thus by the Five Lemma we obtain that $\tau(X)$ is an isomorphism, which concludes the inductive step.
To show that $\tau(X)$ is an isomorphism for each finite G-CW-complex we use induction on the skeleton of X. We can assume that the 0-skeleton X^0, that is a disjoint finite union of 0-dimensional G-cells, is constructed by attaching to an initial 0-cell the other 0-cells one after each other along empty maps. Then the result for the base case follows similarly

as the base case of the previous inductive process. Notice that this can be done because we have a finite number of 0-cells. For $n > 0$ we have that the n-skeleton X^n is obtained by attaching a disjoint finite union of n-dimensional G-cells to the $(n-1)$-skeleton X^{n-1}. Again we can split this process in a finite number of steps. At each step we construct

$$Y = Y_1 \cup_\phi Y_2$$

where Y_1 is assumed to satisfy that $\tau(Y_1)$ is an isomorphism, Y_2 is an n-dimensional G-cell $Y_2 = G/K \times D^n$ and

$$\phi : Y_0 = G/K \times S^{n-1} \to Y_1.$$

From property A2 we have a commutative diagram

$$\begin{tikzcd}
h_G^{q-1}(Y_1) \times h_G^{q-1}(X_2) \arrow[r] \arrow[d, "\tau(Y_1)\times\tau(Y_2)"] & h_G^{q-1}(Y_0) \arrow[r] \arrow[d, "\tau(Y_0)"] & h_G^q(Y) \arrow[r] \arrow[d, "\tau(Y)"] & h_G^q(Y_1) \times h_G^q(Y_2) \arrow[r] \arrow[d, "\tau(Y_1)\times\tau(Y_2)"] & h_G^q(Y_0) \arrow[d, "\tau(Y_0)"] \\
h_G^{q-1}(Y_1) \times h_G^{q-1}(X_2) \arrow[r] & k_G^{q-1}(Y_0) \arrow[r] & k_G^q(Y) \arrow[r] & k_G^q(Y_1) \times k_G^q(Y_2) \arrow[r] & k_G^q(Y_0)
\end{tikzcd}$$

Therefore, by property A1 we have that $\tau(Y_2)$ is an isomorphism and by the case proved before the same happens for $\tau(Y_0)$, hence by the Five Lemma we obtain $\tau(Y)$ is an isomorphism, which concludes the proof. $\square$

In the following section we are going to introduce an important case of an equivariant cohomology theory that is obtained using the Borel construction.

1.3 The Borel construction

In 1960 Armand Borel introduced a method to study the cohomology of G-spaces concerning the information associated to the G-action, the Borel construction. This has become a basic tool in the theory of transformation groups. Let us describe this construction.

We assume that G is a compact Lie group. Let X be a G-space and consider EG the total space of the universal principal G-bundle $EG \to BG$. For an introduction to universal principal G-bundles see Appendix B. The topological product $EG \times X$ can be endowed with a G-action given by

$$\begin{array}{ccc} G \times (EG \times X) & \to & EG \times X \\ (g, (p, x)) & \mapsto & (pg^{-1}, gx). \end{array}$$

Then we define the **Borel construction** on X as

$$X_G := EG \times_G X = EG \times X / \sim,$$

where $(p, x) \sim (pg^{-1}, gx)$ for each $g \in G$. Notice that X_G is the orbit space of the G-action on the product $EG \times X$.

In regard to Appendix B.1 we have that X_G is the fiber bundle associated to the universal principal G-bundle (EG, π) by X, hence X_G corresponds to the total space of the bundle

$$\pi_{EG} : X_G \to BG$$

with fiber X.

The Borel construction is significantly relevant because it provides the framework for defining an equivariant cohomology theory in the sense of Definition 1.7.

Definition 1.9. *Let X be a G-space. We define the* ***equivariant cohomology of*** *X as*

$$H^\bullet(X_G) = H^\bullet(EG \times_G X).$$

We can notice that the equivariant cohomology $H^\bullet(X_G)$ has a structure of $H^\bullet(BG)$-module. Indeed, if $*$ denotes the singleton with a trivial G-action, we have that

$$H^\bullet(*_G) = H^\bullet(EG \times_G *) = H^\bullet(BG).$$

Therefore, the projection $X_G = EG \times_G X \to EG \times_G * = BG$ induces a map

$$H^\bullet(BG) \to H^\bullet(X_G)$$

that endows $H^\bullet(X_G)$ with a $H^*(BG)$-module structure.

Moreover, if G is a compact Lie group and acts freely on X we have a principal G-bundles given by the projection onto the orbit space

$$\pi : X \to X/G.$$

Therefore, by considering the associated bundle by EG we obtain

$$\pi_{EG} : EG \times_G X \to X/G$$

with fiber EG, that is contractible. It follows from the long exact sequence of homotopy groups (see [Hat02]) associated to the above fiber bundle that

$$H^\bullet(X_G) \cong H^\bullet(X/G).$$

This result will appear again in the context of the proof of the main theorem in Section 2.3.

With the above discussion we have defined the Borel construction and the equivariant cohomology of a G-space X merely using topological tools, since the equivariant cohomology of X has been defined as the usual topological cohomology of its Borel construction. However, there exists an equivalent procedure that allows us to describe an algebraic version of the Borel construction. The algebraic setting is particularly interesting because it provides the algebraic machinery needed to prove important results on the G-space X. We start with some definitions.

From now on we assume that G is a finite group.

Let EG the total space of the universal principal G-bundle, that is a free G-CW-complex and consider $\epsilon_\bullet(G) := W_\bullet(EG; k)$ the cellular chain complex with coefficients in a field k. Since $\epsilon_\bullet(G)$ inherits a G-action from the G-space structure of EG we may consider $\epsilon_\bullet(G)$ as a chain complex over the group ring $k[G]$.

Definition 1.10. *Let $C_\bullet$ (resp. $C^\bullet$) be a chain complex (resp. a cochain complex) over $k[G]$. We define*

$$\beta^G_\bullet(C_\bullet) := \epsilon_\bullet(G) \otimes_{k[G]} C_\bullet,$$

where $\beta^G_n(C_\bullet) = \bigoplus_i (\epsilon_i(G) \otimes_{k[G]} C_{n-i})$, and

$$\beta^\bullet_G(C^\bullet) = \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), C^\bullet),$$

where $\beta^n_G(C_\bullet) = \prod_i \mathrm{Hom}_{k[G]}(\epsilon_i(G), C^{n-i})$.

In the above definition we assume that $C_\bullet$ is a left $k[G]$-module and $C^\bullet$ is a right $k[G]$-module, and by defining

$$eg := g^{-1}e \quad \text{for } e \in EG, g \in G,$$

we convert the left G-action on EG to a right G-action that induces a right $k[G]$-module structure on $\epsilon_\bullet(G)$.

Notice that if $C^\bullet = \mathrm{Hom}_k(C_\bullet, k)$ by the tensor-hom adjunction we obtain that

$$\beta^\bullet_G(C^\bullet) = \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), \mathrm{Hom}_k(C_\bullet, k)) = \mathrm{Hom}_k(\epsilon_\bullet(G) \otimes_{k[G]} C_\bullet, k) = \mathrm{Hom}_k(\beta^G_\bullet(C_\bullet), k).$$

The above definitions allow us to define two homology and cohomology theories.

Definition 1.11. *Let $C_\bullet$ (resp. $C^\bullet$) be a chain complex (resp. a cochain complex) over $k[G]$. We define the **homology of G with coefficients in $C_\bullet$** as*

$$H^G_\bullet(C_\bullet) = H(\beta^G_\bullet(C_\bullet))$$

*and we define the **cohomology of G with coefficients in $C^\bullet$** as*

$$H^\bullet_G(C^\bullet) = H(\beta^\bullet_G(C^\bullet)).$$

It can be seen that if $C_\bullet$ is a $k[G]$-module concentrated at zero degree we obtain the usual homology and cohomology of the group G with coefficients in $C_\bullet$ and $C^\bullet$, respectively (see [Bro82]).

Let us assume that $C^\bullet$ is bounded below, which means that for some n_0 we have that $C^n = 0$ for each $n \leq n_0$. The following Proposition gives us an alternative description of $\beta^\bullet_G(C^\bullet)$. Let us denote

$$\epsilon^\bullet(G) = \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), k[G])$$

the dual $k[G]$-module of $\epsilon_\bullet(G)$. If $\epsilon_\bullet(G)$ is assumed to be a right $k[G]$-module, then $\epsilon^\bullet(G)$ can be considered with a structure of left $k[G]$-module given by

$$(g\phi)(e) = \phi(eg)$$

for each $g \in G$, $\phi \in \epsilon^\bullet(G)$ and $e \in \epsilon_\bullet(G)$.

Proposition 1.12. *Let $C^\bullet$ be a bounded below cochain complex over $k[G]$. We have a natural isomorphism of cochain complexes over k*

$$\begin{array}{rccc} \Phi: & C^\bullet \otimes_{k[G]} \epsilon^\bullet(G) & \to & \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), C^\bullet) \\ & c\otimes\phi & \mapsto & \Phi_{c\otimes\phi}: \quad \begin{array}{ccc} \epsilon_\bullet(G) & \to & C^\bullet \\ e & \mapsto & c\phi(e) \end{array} \end{array}$$

where $c \in C^\bullet$, $\phi \in \epsilon^\bullet(G)$ and $e \in \epsilon_\bullet(G)$.

Proof. We can see that the map is well-defined since it is compatible with the coboundaries. Indeed, for $c \in C^n$, $\phi \in \epsilon^i(G)$ and $e \in \epsilon_i(G)$ we have that

$$\begin{aligned} &\Phi(\partial(c\otimes\phi))(e) = \Phi(d_{C^\bullet}(c)\otimes\phi + (-1)^n c\otimes d_{\epsilon^\bullet(G)}(\phi))(e) = \\ &= \Phi_{d_{C^\bullet}(c)\otimes\phi}(e) + (-1)^n\Phi_{c\otimes d_{\epsilon^\bullet(G)}(\phi)}(e) = d_{C^\bullet}(c)\phi(e) + (-1)^n c d_{\epsilon^\bullet(G)}(\phi(e)) = \\ &= d_{C^\bullet}(c\phi(e)) - (-1)^{n+i}c\phi(d_{\epsilon^\bullet(G)}(e)) = d_{C^\bullet}\circ\Phi_{c\otimes\phi}(e) - (-1)^{n+i}\Phi_{c\otimes\phi}\circ d_{\epsilon^\bullet(G)}(e) = \\ &= \partial(\Phi(c\otimes\phi))(e). \end{aligned}$$

Moreover, we have that $\epsilon^\bullet(G)$ is a free $k[G]$-module, and $C^\bullet$ and $\epsilon_\bullet(G)$ are bounded below. Therefore, it is sufficient to prove the result with $\epsilon^\bullet(G)$ and $\epsilon_\bullet(G)$ being $k[G]$. Then we have

$$\begin{array}{rccccc} \Phi: & C^\bullet\otimes_{k[G]} k[G] & \cong & C^\bullet & \to & \mathrm{Hom}_{k[G]}(k[G], C^\bullet) \\ & c\otimes\phi & \leftrightarrow & c\phi & \mapsto & \Phi(c\otimes\phi): \quad \begin{array}{ccc} k[G] & \to & C^\bullet \\ e & \mapsto & c\phi(e) \end{array} \end{array}$$

which is clearly a natural isomorphism. $\square$

From the above definitions we conclude that we have two additive functors

$$\beta^G_\bullet : \partial\mathbf{g}k[G]\textbf{-Mod} \to \partial\mathbf{g}k\textbf{-Mod}$$
$$\beta^\bullet_G : \delta\mathbf{g}k[G]\textbf{-Mod} \to \delta\mathbf{g}k\textbf{-Mod}$$

from the category of chain complexes (resp. cochain complexes) over $k[G]$ to the category of chain complexes (resp. cochain complexes) over k. With the following Proposition we describe some properties of these functors.

Proposition 1.13. *The functor $\beta^G_\bullet$ satisfies the following properties:*

a) *$\beta^G_\bullet$ preserves homotopies, hence if $f_0, f_1 : C_\bullet \to C'_\bullet$ are homotopic chain maps over $k[G]$ then $\beta^G_\bullet(f_0)$ and $\beta^G_\bullet(f_1)$ are homotopic chain maps over k;*

b) *$\beta^G_\bullet$ is exact, hence if $C_\bullet \to C'_\bullet \to C''_\bullet$ is an exact sequence of chain complexes over $k[G]$ then*

$$\beta^G_\bullet(C_\bullet) \to \beta^G_\bullet(C'_\bullet) \to \beta^G_\bullet(C''_\bullet)$$

is an exact sequence of chain complexes over k;

c) *if $f : C_\bullet \to C'_\bullet$ is a map of chain complexes over $k[G]$ which is a chain equivalence over k, then $\beta^G_\bullet(f) : \beta^G_\bullet(C_\bullet) \to \beta^G_\bullet(C'_\bullet)$ is a chain equivalence of chain complexes over $k[G]$.*

Proof. To prove (a) consider $s : C_\bullet \to C'_\bullet$ a chain homotopy between f_0 and f_1 in $\partial\mathbf{g}k[G]$-**Mod**, hence s is a morphism of $k[G]$-modules such that

$$f_0 - f_1 = \partial s + s\partial.$$

Then we obtain that $id_{\epsilon_\bullet(G)} \otimes_{k[G]} s = \beta^G_\bullet(s)$ is the desired chain homotopy in $\partial\mathbf{g}k$-**Mod** since

$$\beta^G_\bullet(f_0) - \beta^G_\bullet(f_1) = \beta^G_\bullet(f_0 - f_1) = \beta^G_\bullet(\partial s + s\partial) = \partial\beta^G_\bullet(s) + \beta^G_\bullet(s)\partial.$$

Property (b) follows from the fact that $\epsilon_\bullet(G)$ is a free $k[G]$-module, hence $\epsilon_\bullet(G)$ is $k[G]$-flat and $\beta^G_\bullet = \epsilon_\bullet(G) \otimes_{k[G]} -$ is exact.
To prove (c) we will use the mapping cone. Consider $\mathrm{cone}(f)$ and notice that

$$\mathrm{cone}(\beta^G_\bullet(f)) = \beta^G_\bullet(\mathrm{cone}(f)),$$

hence by Proposition A.4 it is enough to see that if N is contractible, then $\beta^G_\bullet(N)$ is also contractible. For that consider a filtration of $\epsilon_\bullet(G)$ by the degree

$$F_n\epsilon_\bullet(G) = \bigoplus_{i=0}^{n} \epsilon_i(G)$$

which induces a filtration of $\epsilon_\bullet(G) \otimes_{k[G]} N$ given by

$$0 = F_{-1}(\epsilon_\bullet(G) \otimes_{k[G]} N) \subset \cdots \subset F_n(\epsilon_\bullet(G) \otimes_{k[G]} N) \subset F_{n+1}(\epsilon_\bullet(G) \otimes_{k[G]} N) \subset \cdots$$

such that

$$\varinjlim_n F_n(\epsilon_\bullet(G) \otimes_{k[G]} N) = \epsilon_\bullet(G) \otimes_{k[G]} N$$

and for each $n \geq 0$

$$\frac{F_{n+1}(\epsilon_\bullet(G) \otimes_{k[G]} N)}{F_n(\epsilon_\bullet(G) \otimes_{k[G]} N)} \cong \epsilon_n(G) \otimes_{k[G]} N,$$

which is contractible, because it is the direct sum of finitely many copies of N. Then, we obtain that $\epsilon_\bullet(G) \otimes_{k[G]} N$ is contractible (see Corollary B.1.18 in [AP93]), which concludes the proof. $\square$

Notice that the above Proposition is based on standard properties of the tensor product functor. Analogous statements for the functor $\beta_G^\bullet : \delta\mathbf{g}k[G]\text{-}\mathbf{Mod} \to \delta\mathbf{g}k\text{-}\mathbf{Mod}$ can be obtained using the corresponding properties for the Hom functor.

We also have that property (c) can be refined if $C_\bullet$ and $C'_\bullet$ are assumed to be free as k-modules. Indeed, since then the conditions of being acyclic and contractible are equivalent (see Proposition B.1.11 in [AP93]), then it suffices to assume that $H(f)$ is an isomorphism.

Once we have introduced the functors $\beta_\bullet^G$ and $\beta_G^\bullet$ we already have all the necessary information to describe the algebraic version of the Borel construction. Let X be a G-space. We can apply the above functors to the singular chain complex $S_\bullet(X;k)$ and to the singular cochain complex $S^\bullet(X;k)$, considered as $k[G]$-modules. We denote

$$\beta_\bullet^G(X;k) := \beta_\bullet^G(S_\bullet(X;k)) \qquad \beta_G^\bullet(X;k) := \beta_G^\bullet(S^\bullet(X;k))$$

and

$$H_\bullet^G(X;k) := H_\bullet(\beta_\bullet^G(X;k)) \qquad H_G^\bullet(X;k) := H^\bullet(\beta_G^\bullet(X;k)).$$

It can be seen that the above definitions correspond to equivariant homology and cohomology theories, respectively.

Theorem 1.14. *The functor $H_\bullet^G(-;k)$ (resp. $H_G^\bullet(-;k)$) is an equivariant homology theory (resp. equivariant cohomology theory) in the sense of Definition 1.7.*

Proof. See Theorem 1.2.6 in [AP93] for a proof of the equivariant cohomology case. □

It is important to notice that if X is a G-CW-complex we can equivalently use the cellular chain complex $W_\bullet(X;k)$ instead of the singular chain complex $S_\bullet(X;k)$ to define $H_\bullet^G(X;k)$ and $H_G^\bullet(X,k)$. We can state this result as a Proposition.

Proposition 1.15. *Let X be a G-CW-complex. We have natural isomorphisms*

$$H_\bullet(\beta_\bullet^G(S_\bullet(X;k))) \cong H_\bullet(\beta_\bullet^G(W_\bullet(X;k))) \quad \text{and} \quad H^\bullet(\beta_G^\bullet(S^\bullet(X;k))) \cong H^\bullet(\beta_G^\bullet(W^\bullet(X;k))).$$

Proof. Let

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots$$

be the structural skeletal filtration of X. We define the subcomplex

$$C_n(X;k) = \{c \in S_n(X;k) : \partial c \in S_{n-1}(X;k)\} \subset S_n(X;k),$$

where ∂ is the coboundary on the singular chain complex $S_\bullet(X;k)$. We have that $C_\bullet(X;k)$ is a $k[G]$-subcomplex of $S_\bullet(X;k)$ and we can consider a chain map over $k[G]$

$$p : C_\bullet(X;k) \to W_\bullet(X;k)$$

that sends $c \in C_n(X;k)$ to its class in $W_n(X;k) \cong H_n(X^n, X^{n-1};k)$. We want to prove that the maps

$$i : C_\bullet(X;k) \to S_\bullet(X;k) \qquad \text{and} \qquad p : W_\bullet(X;k) \to S_\bullet(X;k)$$

are homotopy equivalences over k. Therefore, since i and p are chain maps in $\partial\mathbf{g}k[G]$**-Mod** we obtain from Proposition 1.13(c) the desired result. Moreover, by the subsequent discussion it suffices to see that $H_\bullet(i)$ and $H_\bullet(p)$ are isomorphisms since $C_\bullet(X;k)$, $S_\bullet(X;k)$ and $W_\bullet(X;k)$ are free as k-modules.
We start proving that $H_\bullet(i)$ is an isomorphism. We have that $H_\bullet(i)$ is surjective because any homology class in $H_n(S_\bullet(X;k))$ is represented by a cycle in $S_n(X^n;k)$, thus also in $C_n(X;k)$. On the other hand, let us consider a cycle $c \in C_n(X;k)$ such that there exists another cycle $b \in S_{n+1}(X;k)$ with $\partial b = c$, that is that $H_\bullet(i)([c]) = 0$ in $H_n(S_\bullet(X;k))$. Then we can take $b' \in S_{n+1}(X^{n+1};k)$ such that $\partial b' = \partial b = c$ and by definition we obtain that $b' \in C_{n+1}(X;k)$, hence $[c] = 0$ in $H_n(C_\bullet(X;k))$ and $H_\bullet(i)$ is injective.
It remains to prove that $H_\bullet(p)$ is an isomorphism. Consider a cochain $c \in S_n(X^n,k)$ that represents a cycle in

$$W_n(X;k) = H_n(X^n, X^{n-1};k) = H_n\left(\frac{S_\bullet(X^n;k)}{S_\bullet(X^{n-1};k)}\right).$$

Then we have that $\partial c \in S_{n-1}(X^{n-2};k) + \partial(S_n(X^{n-1};k))$, so we can write $\partial c = a' + \partial b''$ with $a' \in S_{n-1}(X^{n-2};k)$ and $b'' \in S_n(X^{n-1};k)$. It is clear that $H_{n-1}(S_\bullet(X^{n-2};k)) = 0$ and we notice that $\partial a' = 0$. Therefore there exists $b' \in S_n(X^{n-2};k)$ such that $\partial b' = a'$. If we consider the element

$$c' = c - b' - b'' \in S_n(X^n,k)$$

we observe that it represents a cycle in $C_n(X^n;k)$ since $\partial c' = \partial c - a' - \partial b'' = 0$ and it is mapped by $H_\bullet(p)$ to the homology class represented by c, hence $H_\bullet(p)$ is surjective. On the other hand, let $c \in C_n(X^n;k)$ a cycle in $C_\bullet(X;k)$ such that $H_\bullet(p)([c]) = 0$, that is that there exist elements $b' \in S_{n+1}(X^{n+1};k)$ and $a'' \in S_n(X^{n-1};k)$ such that $\partial b' = c + a''$, and we also have that $\partial c = 0$, hence $\partial a'' = 0$. However, since $H_n(S_\bullet(X^{n-1};k)) = 0$ there exists $b'' \in S_{n+1}(X^{n-1};k)$ with $\partial b'' = a''$. Thus we obtain that $\partial(b' - b'') = c + a'' - a'' = c$ and since $b' - b'' \in C_{n+1}(X;k)$ we conclude that $[c] = 0$ in $H_n(C_\bullet(X;k))$, so $H_\bullet(p)$ is injective. This concludes the proof. □

Finally, we will give an explicit description of the multiplicative structure of this algebraic equivariant cohomology theory that we are considering. The cellular chain complex $\epsilon_\bullet(G)$ has a diagonal

$$\Delta : \epsilon_\bullet(G) \to \epsilon_\bullet(G) \otimes \epsilon_\bullet(G)$$

that is a map of chain complexes over $k[G]$. This diagonal map can be thought as being induced by the topological diagonal

$$EG \to EG \times EG$$

to $\epsilon_\bullet(G) = W_\bullet(EG;k)$ or as a lifting of the isomorphism

$$k \to k \otimes k$$

to the free resolutions $\epsilon_\bullet(G) \to k$ and $\epsilon_\bullet(G) \otimes \epsilon_\bullet(G) \to k \otimes k$, which is uniquely determined up to homotopy of chain complexes over $k[G]$.

If $C^\bullet$ is a cochain complex over $k[G]$, then the above diagonal map induces a product

$$\beta_G^\bullet(C^\bullet) \otimes \beta_G^\bullet(k) = \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), C^\bullet) \otimes \mathrm{Hom}_{k[G]}(k, C^\bullet) \to \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), C^\bullet) = \beta_G^\bullet(C^\bullet)$$

given by $\phi \otimes \psi \mapsto \phi \cup \psi$, for $\phi \in \beta_G^\bullet(C^\bullet)$, $\psi \in \beta_G^\bullet(k)$ and $\phi \cup \psi$ given by

$$(\phi \cup \psi)(e) = \eta((\phi \otimes \psi)(\Delta e))$$

with $e \in \epsilon_\bullet(G)$, that is the composition

$$\epsilon_\bullet(G) \xrightarrow{\Delta} \epsilon_\bullet(G) \otimes \epsilon_\bullet(G) \xrightarrow{\phi \otimes \psi} C^\bullet \otimes k \xrightarrow{\eta} C^\bullet,$$

where $\eta : C^\bullet \otimes k \to C^\bullet$ is the k-module structure natural map. This product induces a map in cohomology

$$H_G^\bullet(C^\bullet) \otimes H_G^\bullet(k) \to H_G^\bullet(C^\bullet),$$

hence we have a $H_G^\bullet(k)$-module structure on $H_G^\bullet(C^\bullet)$. More precisely, this induces a right $H_G^\bullet(k)$-module structure, and we can define an analog left structure. Since Δ is commutative up to homotopy, the left and right structure coincide on cohomology level $H_G^\bullet(C^\bullet)$, and they are equivalent on cochain level only up to homotopy. Notice that if $C^\bullet = k$, we obtain the usual cup product on $H_G^\bullet(k)$.

Furthermore, if $C^\bullet$ has a product, that is a morphism

$$\mu : C^\bullet \otimes C^\bullet \to C^\bullet$$

in $\partial\mathbf{g}k[G]$**-Mod** such that $C^\bullet \otimes C^\bullet$ has a diagonal map, then with a similar argument we have an induced product

$$\beta_G^\bullet(C^\bullet) \otimes \beta_G^\bullet(C^\bullet) \to \beta_G^\bullet(C^\bullet)$$

given by $\phi_1 \otimes \phi_2 \mapsto \phi_1 \cup \phi_2$, for $\phi_1, \phi_2 \in \beta_G^\bullet(C^\bullet)$ and $\phi_1 \cup \phi_2$ given by

$$(\phi_1 \cup \phi_2)(e) = \mu((\phi_1 \otimes \phi_2)(\Delta e))$$

with $e \in \epsilon_\bullet(G)$, that is the composition

$$\epsilon_\bullet(G) \xrightarrow{\Delta} \epsilon_\bullet(G) \otimes \epsilon_\bullet(G) \xrightarrow{\phi_1 \otimes \phi_2} C^\bullet \otimes C^\bullet \xrightarrow{\mu} C^\bullet.$$

Finally, we can apply the above discussion to our case of interest. Let X be a G-space and take $C^\bullet = S^\bullet(X;k)$ the singular cochain complex, which has a product

$$\mu : S^\bullet(X;k) \otimes S^\bullet(X;k) \to S^\bullet(X;k)$$

induced by the diagonal map $X \to X \times X$. This induces a multiplicative structure on cohomology

$$H^\bullet(X;k) = H^\bullet(\beta_G^\bullet(S^\bullet(X;k))).$$

To conclude our introduction to the Borel construction we will see that both equivariant cohomology theories defined above indeed coincide.

Theorem 1.16. *Let G be a finite group. The homology (resp. cohomology) theories $H^G_\bullet(-;k)$ and $H_\bullet(EG\times_G -;k)$ (resp. $H^\bullet_G(-;k)$ and $H^\bullet(EG\times_G -;k)$) are naturally isomorphic over the category of G-spaces.*

Proof. We start proving the statement for the homology theories. Let X a G-space. Recall that

$$H^G_\bullet(X;k) = H_\bullet(\beta^G_\bullet(X;k)),$$

with $\beta^G_\bullet(X;k) = \epsilon_\bullet(G)\otimes_{k[G]} S_\bullet(X;k)$ and $\epsilon_\bullet(G) = W_\bullet(EG;k)$. We can observe that we can replace the cellular chain complex $W_\bullet(EG;k)$ by the singular chain complex $S_\bullet(X;k)$ since both are free resolutions of k over $k[G]$, hence are chain homotopic over $k[G]$. By Eilenberg-Zilberg Theorem we can consider two chain maps Φ and Ψ

$$S_\bullet(EG;k)\otimes S_\bullet(X;k) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} S_\bullet(EG\times X;k)$$

such that $\Phi\Psi$ and $\Psi\Phi$ are naturally chain equivalent to the corresponding identity maps. Moreover, since the G-action is given by a continuous action we have that naturality assures us that Φ and Ψ are compatible with the diagonal G-action, hence they induce chain homotopy equivalences over k given by

$$S_\bullet(EG;k)\otimes_{k[G]} S_\bullet(X;k) \underset{\overline{\Psi}}{\overset{\overline{\Phi}}{\rightleftarrows}} k\otimes_{k[G]} S_\bullet(EG\times X;k).$$

We also have that $\pi : EG\times X \to EG\times_G X$ is a covering map with deck transformation group G, that is that

$$G = \{f\in \mathrm{Aut}(EG\times X) : \pi f = \pi\} \subset \mathrm{Aut}(EG\times X).$$

Therefore we obtain that the map

$$\overline{\Theta} : k\otimes_{k[G]} S_\bullet(E\times X;k) \to S_\bullet(EG\times_G X;k)$$

is an isomorphism. Indeed, $\overline{\Theta}$ is injective because if two singular simplices $\sigma^1_q, \sigma^2_q : \Delta_q \to EG\times X$ coincide when they are composed with π, then there exists $g\in G$ such that $g\sigma^1_q = \sigma^2_q$, hence they represent the same class in $k\otimes_{k[G]} S_\bullet(E\times X;k)$. On the other hand, $\overline{\Theta}$ is surjective since any singular simplex $\sigma_q : \Delta_q \to EG\times_G X$ can be lifted to a simplex on $EG\times X$ because Δ_q is contractible. Thus, the composition $\overline{\Theta\Phi}$ induces the desired isomorphism on homology

$$H^G_\bullet(X;k) \cong H_\bullet(EG\times_G X;k).$$

With a similar argument we can obtain the isomorphism for cohomology, but we have to additionally check the compatibility with the ring structure. Recall that

$$H^\bullet_G(X;K) = H^\bullet(\beta^\bullet_G(X;k)),$$

with $\beta_G^\bullet(X;k) = \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), S^\bullet(X;k))$. As before we replace the cellular chain complex $W_\bullet(EG;K)$ by the singular chain complex $S_\bullet(EG;k)$, and we consider the diagonal on $S_\bullet(EG;k)$ given by the composition of the map induced by the topological diagonal $EG \to EG \times EG$ and an Eilenberg-Zilberg map

$$S_\bullet(EG;k) \to S_\bullet(EG \times EG;k) \to S_\bullet(EG;k) \otimes S_\bullet(EG;k).$$

We can observe that by the tensor-hom adjunction we have natural isomorphisms

$$\begin{aligned}&\beta_G^\bullet(S^\bullet(X;k)) = \mathrm{Hom}_{k[G]}(S_\bullet(EG;k), S^\bullet(X;k)) = \\ &= \mathrm{Hom}_{k[G]}(S_\bullet(EG;k), \mathrm{Hom}_k(S_\bullet(X;k),k)) \cong \mathrm{Hom}_k(S_\bullet(EG;k) \otimes_{k[G]} S_\bullet(X;k), k),\end{aligned}$$

hence we obtain the desired isomorphism in cohomology. Regarding the multiplicative structure we have that the product in $H_G^\bullet(X;k) = H^\bullet(\beta_G^\bullet(S^\bullet(X;k)))$ is induced by the composition

$$\begin{aligned}&S_\bullet(EG;k) \otimes S_\bullet(X;k) \xrightarrow{\Delta_{EG} \otimes \Delta_X} (S_\bullet(EG;k) \otimes S_\bullet(EG;k)) \otimes (S_\bullet(X;k) \otimes S_\bullet(X;k)) \xrightarrow{\tau} \\ &\xrightarrow{\tau} (S_\bullet(EG;k) \otimes S_\bullet(X;k)) \otimes (S_\bullet(EG;k) \otimes S_\bullet(X;k)) \xrightarrow{\phi_1 \otimes \phi_2} k \otimes k \cong k\end{aligned}$$

where $\phi_1, \phi_2 \in \mathrm{Hom}_k(S_\bullet(EG;k) \otimes_{k[G]} S_\bullet(X;k), k)$, Δ_{EG} and Δ_X are the corresponding diagonals and τ denotes the twist of the middle terms in the tensor product. On the other hand, the product in $H^\bullet(EG \times_G X;k)$ is induced by the diagonal on $S_\bullet(EG \times X;k)$

$$S_\bullet(EG \times X;k) \xrightarrow{\Delta_{EG \times X}} S_\bullet(EG \times X;k) \otimes S_\bullet(EG \times X;k) \xrightarrow{\psi_1 \otimes \psi_2} k \otimes k \cong k$$

for $\psi_1, \psi_2 \in S^\bullet(EG \times X;k)$. It is clear that the following diagram

$$\begin{tikzcd}
S_\bullet(EG;k) \otimes S_\bullet(X;k) \arrow[r, "\Phi"] \arrow[d, "\Delta_{EG} \otimes \Delta_X"] & S_\bullet(EG \times X;k) \arrow[dd, "\Delta_{EG \times X}"] \\
(S_\bullet(EG;k) \otimes S_\bullet(EG;k)) \otimes (S_\bullet(X;k) \otimes S_\bullet(X;k)) \arrow[d, "\tau"] & \\
(S_\bullet(EG;k) \otimes S_\bullet(X;k)) \otimes (S_\bullet(EG;k) \otimes S_\bullet(X;k)) \arrow[r, "\Phi \otimes \Phi"] & S_\bullet(EG \times X;k) \otimes S_\bullet(EG \times X;k)
\end{tikzcd}$$

is commutative up to homotopy. Therefore, it follows from the above discussion that we have a commutative diagram up to homotopy given by

$$\begin{tikzcd}
S_\bullet(EG;k) \otimes_{k[G]} S_\bullet(X;k) \arrow[r] \arrow[d, "\overline{\Theta\Phi}"] & (S_\bullet(EG;k) \otimes_{k[G]} S_\bullet(X;k)) \otimes (S_\bullet(EG;k) \otimes_{k[G]} S_\bullet(X;k)) \arrow[d, "(\overline{\Theta\Phi}) \otimes (\overline{\Theta\Phi})"] \\
S_\bullet(EG \times_G X;k) \arrow[r, "\Delta_{EG \times_G X}"] & S_\bullet(EG \times_G X;k) \otimes S_\bullet(EG \times_G X;k)
\end{tikzcd}$$

where the horizontal maps induce the products in cohomology. Finally, since the isomorphism $H_G^\bullet(X;k) \cong H^\bullet(EG \times_G X;k)$ is given by the dual complexes map of $\overline{\Theta\Phi}$ we deduce that this isomorphism is compatible with the multiplicative structure, as we wanted to obtain. $\square$

1.4 The Borel construction for 2-tori

In this last section we will provide an explicit description of the algebraic version of the Borel construction for the group $G = (\mathbb{Z}_2)^n$. This detailed exposition will be indispensable to understand the forthcoming chapter's proof of the main theorem.

In the first place we consider the case $n = 1$, that is $G = \mathbb{Z}_2$. From the discussion in Appendix B.2 we know that the total space of the universal principal G-bundle is given by

$$EG = \varinjlim_n S^n = S^\infty$$

Considering a natural decomposition of EG as a G-CW-complex we obtain that the cellular chain complex $\varepsilon_\bullet(G) = W_\bullet(EG)$ is given by a free chain complex over $\mathbb{Z}[G]$ with one generator in each non-negative dimension, hence we have that

$$\varepsilon_\bullet(G) = W_\bullet \otimes \mathbb{Z}[G],$$

where $W_\bullet$ is a free abelian group generated by $\{w_0, w_1, w_2, \ldots\}$, with $\deg(w_i) = i$ for each $i \geq 0$. The boundary on $\epsilon_\bullet(G)$ is given by

$$\partial w_n = \left\{ \begin{array}{ll} (1+g)w_{n-1} & \text{if } n \text{ is even,} \\ (1-g)w_{n-1} & \text{if } n \text{ is odd} \end{array} \right\},$$

where 1 denotes the unit element and g the generator of G. The following result gives us the homology of the classifying space BG.

Proposition 1.17. *Let k be a field of characteristic 2 and let $G = \mathbb{Z}_2$. For each $n \geq 0$ we have*

$$H_n(BG;\mathbb{Z}) \cong H_n(\epsilon_\bullet(G) \otimes_{k[G]} S_\bullet(*;k)) \cong \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}_2 & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise} \end{array} \right\}$$

and

$$H_n(BG;k) \cong H_n(\epsilon_\bullet(G) \otimes_{k[G]} k) \cong k.$$

Proof. The first isomorphisms follow from Theorem 1.16 since

$$H_n(BG;k) = H_n(EG \times_G *;k) \cong H_n^G(*;k) = H_n(\epsilon_\bullet(G) \otimes_{k[G]} S_\bullet(*;k)) = H_n(\epsilon_\bullet(G) \otimes_{k[G]} k).$$

where $*$ denotes the singleton with the trivial G-action. The same argument holds for $H_n(BG;\mathbb{Z})$. We can observe that the boundary on $\epsilon_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is given by

$$\overline{\partial}(w_n \otimes 1) = \left\{ \begin{array}{ll} ((1+g)w_{n-1}) \otimes 1 = w_{n-1} \otimes 2 & \text{if } n \text{ is even,} \\ ((1-g)w_{n-1}) \otimes 1 = 0 & \text{if } n \text{ is odd} \end{array} \right\},$$

hence the result follows immediately. Finally, since k is of characteristic 2 we deduce that the boundary is trivial on $\epsilon_\bullet(G) \otimes_{k[G]} k$, thus we also obtain the result for $H_n(BG;k)$. □

We can compute the cohomology of the classifying space BG by taking the dual complexes, or by using the Universal Coefficient Theorem. Nevertheless, to obtain the multiplicative structure on $H^\bullet(BG;k)$ we need to consider a diagonal map

$$\Delta : \epsilon_\bullet(G) \to \epsilon_\bullet(G) \otimes \epsilon_\bullet(G).$$

Let us consider the diagonal given by

$$\Delta(w_n) = \sum_{p+q=n} \Delta_{p,q},$$

where

$$\Delta_{p,q} = \left\{ \begin{array}{ll} w_p \otimes w_q & \text{if } p \text{ is even,} \\ w_p \otimes g w_q & \text{if } p \text{ is odd and } q \text{ is even,} \\ -w_p \otimes g w_q & \text{if } p \text{ and } q \text{ are odd} \end{array} \right\}.$$

Since the desired map is a lifting of the natural isomorphism $k \to k \otimes k$ to the free resolutions $k[G] \to k$ and $k[G] \otimes k[G] \to k \otimes k$, which is uniquely determined up to homotopy, it suffices to prove that the above definition of Δ is a well-defined morphism of chain complexes over $k[G]$, that is that commutes with the boundary. We can write

$$\Delta(w_n) = \left\{ \begin{array}{ll} \sum_{m=0}^{\frac{n-1}{2}} w_{2m} \otimes w_{n-2m} + w_{2m+1} \otimes g w_{n-2m-1} & \text{if } n \text{ is odd,} \\ w_n \otimes w_0 + \sum_{m=0}^{\frac{n}{2}-1} w_{2m} \otimes w_{n-2m} + w_{2m+1} \otimes w_{n-2m-1} & \text{if } n \text{ is even.} \end{array} \right\},$$

Then denote by $\partial_\otimes$ to the boundary on $\epsilon_\bullet(G) \otimes \epsilon_\bullet(G)$ and notice that for n odd we have that

$$\begin{aligned} \partial_\otimes(\Delta(w_n)) = & \sum_{m=0}^{\frac{n-1}{2}} (1+g) w_{2m-1} \otimes w_{n-2m} + w_{2m} \otimes (1-g) w_{n-2m-1} + \\ & + (1-g) w_{2m} \otimes g w_{n-2m-1} - w_{2m+1} \otimes (1+g) w_{n-2m-2} = \\ & = w_{n-1} \otimes (1-g) w_0 + (1-g) w_{n-1} \otimes g w_0 + \\ & + \sum_{m=0}^{\frac{n-1}{2}-1} (1-g) w_{2m} \otimes g w_{n-2m-1} + w_{2m} \otimes (1-g) w_{n-2m-1} + \\ & + (1+g) w_{2m+1} \otimes w_{n-2m-2} - w_{2m+1} \otimes (1+g) w_{n-2m-2} = \\ & = (1-g) w_{n-1} \otimes w_0 + (1-g) \sum_{m=0}^{\frac{n-1}{2}-1} w_{2m} \otimes w_{n-2m-1} - w_{2m+1} \otimes g w_{n-2m-2} = \\ & = (1-g)\Delta(w_{n-1}) = \Delta((1-g) w_{n-1}) = \Delta(\partial(w_n)). \end{aligned}$$

With a similar procedure we may obtain the same result for n even.

Proposition 1.18. *Let k be a field of characteristic 2 and let $G = \mathbb{Z}_2$. We have*

$$H^\bullet(BG;\mathbb{Z}) \cong H^\bullet(\mathrm{Hom}_{\mathbb{Z}[G]}(\epsilon_\bullet(G), \mathbb{Z})) \cong \frac{\mathbb{Z}[t]}{(2t)}, \quad \text{with } \deg(t) = 2$$

and

$$H^\bullet(BG;k) \cong H^\bullet(\mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), k)) \cong k[t], \quad \text{with } \deg(t) = 1.$$

Proof. The initial isomorphisms again follow from Theorem 1.16

$$\begin{aligned} H^\bullet(BG;k) &= H^\bullet(EG\times_G *;k) \cong H^\bullet_G(*;k) = \\ &= H^\bullet(\mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), S^\bullet(*;k))) = H^\bullet(\mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), S^\bullet(*;k))) \end{aligned}$$

where $*$ again denotes the singleton with the trivial G-action and the same can be argued for $H_n(BG;\mathbb{Z})$. Consider the element $\overline{w}^n \in \mathrm{Hom}_{\mathbb{Z}[G]}(\epsilon_\bullet(G),\mathbb{Z})$ defined over $W_\bullet$ by

$$\overline{w}^n(w_m)\left\{\begin{array}{ll} 1 & \text{if } n=m, \\ 0 & \text{if } n\neq m \end{array}\right\},$$

which extends to $\epsilon_\bullet(G)$ as a $\mathbb{Z}[G]$-module morphism. Since $\delta\overline{w}^n(w_{n+1}) = \overline{w}^n(\partial w_{n+1})$ we deduce that

$$\delta\overline{w}^n = \left\{\begin{array}{ll} (1-g)\overline{w}^{n+1} & \text{if } n \text{ is even,} \\ (1+g)\overline{w}^{n+1} & \text{if } n \text{ is odd} \end{array}\right\}.$$

For $\overline{w}^p$ and $\overline{w}^q$ we can also consider the cup product, that is given by the composition

$$\epsilon_\bullet(G) \xrightarrow{\Delta} \epsilon_\bullet(G)\otimes\epsilon_\bullet(G) \xrightarrow{\overline{w}^p\otimes\overline{w}^q} \mathbb{Z}\otimes\mathbb{Z}\to\mathbb{Z}$$

and we can observe that satisfies

$$\overline{w}^p\cup\overline{w}^q(w_m) = (\overline{w}^p\otimes\overline{w}^q)(\Delta w_m) = \left\{\begin{array}{ll} 1 & \text{if } p+q=m, \\ 0 & \text{if } p+q\neq m \end{array}\right\}.$$

Indeed, this result is immediate for p even, and for p odd and q even. If p and q are odd we have to take into account the sign convention to obtain that

$$(\overline{w}^p\otimes\overline{w}^q)(-w_p\otimes gw_q) = -(-1)^{|p||q|}\overline{w}^p(w_p)\otimes\overline{w}^q(gw_q) = 1.$$

Therefore, $\overline{w}^p\cup\overline{w}^q = \overline{w}^{p+q}$. With this discussion we can compute the cohomology rings. For n even the element $\overline{w}^n$ is a representative of the generator of $H^n(BG;\mathbb{Z})$ and since $2\overline{w}^n \sim 0$ we obtain that $H^n(BG;\mathbb{Z})$ is zero for n odd. Given the multiplicative structure induced by the cup product we obtain the desired result. On the other hand, since k is of characteristic two we have that the coboundary is trivial, and again considering the cup product we conclude the proof. □

Notice that we have observed that the cochain complex $\beta^\bullet_G(k) = \mathrm{Hom}_{k[G]}(\epsilon\bullet(G),k)$, together with the multiplicative structure induced by the cup product induced by the diagonal map defined above, is isomorphic to the polynomial algebra $k[t]$ with $\deg(t)=1$

$$\beta^\bullet_G(k)\cong k[t]$$

not only on cohomology, but also on cochain level given by $\overline{w}^n\mapsto t^n$. Moreover, we have seen that the coboundary on $k[t]$ is given by the derivation

$$\delta(t) = 2t^2,$$

which holds for any coefficient ring k, and if k is a field of characteristic 2 we have that this coboundary is trivial.

Let $C^\bullet$ be a bounded below cochain complex over $k[G]$, that is an object in $\delta\mathbf{g}k[G]$-**Mod**. By Proposition 1.12 we have a natural isomorphism

$$\beta_G^\bullet(C^\bullet) \cong C^\bullet \otimes_{k[G]} \epsilon^\bullet(G)$$

with the usual coboundary considered in the tensor product of cochain complexes. We can observe that we have a natural isomorphism of $k[G]$-modules

$$\epsilon^\bullet(G) = \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), k[G]) \cong \mathrm{Hom}_{k[G]}(W_\bullet \otimes k[G], k[G]) \cong k[G] \otimes \mathrm{Hom}_k(W_\bullet, k) = k[G] \otimes W^\bullet,$$

where we denote $W^\bullet = \mathrm{Hom}_k(W_\bullet, k)$. Therefore, it induces an isomorphism of k-modules

$$\beta_G^\bullet(C^\bullet) \cong C^\bullet \otimes_{k[G]} \epsilon^\bullet(G) \cong C^\bullet \otimes_{k[G]} k[G] \otimes W^\bullet \cong C^\bullet \otimes W^\bullet.$$

Under these isomorphisms the coboundary, which we will denote by $\overline{\delta}$, is given by

$$\overline{\delta}(c \otimes w^n) = \left\{ \begin{array}{ll} \delta c \otimes w^n + (-1)^{|c|} c(1-g) \otimes w^{n+1} & \text{if } n \text{ is even,} \\ \delta c \otimes w^n + (-1)^{|c|} c(1+g) \otimes w^{n+1} & \text{if } n \text{ is odd} \end{array} \right\}.$$

where w^n is the dual element to w_n, δ is the coboundary in $C^\bullet$ and $|c|$ denotes the degree of the element $c \in C^\bullet$. The above formula follows from the fact that the coboundary of $\epsilon^\bullet(G) \cong k[G] \otimes W^\bullet$ is given by

$$\delta(1 \otimes w^n) = \left\{ \begin{array}{ll} (1-g) \otimes w^{n+1} & \text{if } n \text{ is even,} \\ (1+g) \otimes w^{n+1} & \text{if } n \text{ is odd} \end{array} \right\},$$

that the cochain complex $C^\bullet \otimes_{k[G]} \epsilon^\bullet(G)$ has the usual tensor product coboundary and that

$$cg \otimes (1 \otimes w) = c \otimes (g \otimes w)$$

in the isomorphism $C^\bullet \otimes_{k[G]} \epsilon^\bullet(G) \cong C^\bullet \otimes_{k[G]} (k[G] \otimes W^\bullet)$. To avoid confusions with the standard tensor product cochain complex, we denote $C^\bullet \overline{\otimes} W^\bullet = (C^\bullet \otimes W^\bullet, \overline{\delta})$.

Proposition 1.19. *Let $C^\bullet$ be a bounded below cochain complex over $k[G]$. Then $\beta_G^\bullet(C^\bullet) \cong C^\bullet \overline{\otimes} W^\bullet$ is naturally isomorphic to $C^\bullet \otimes k[t]$ as a right module over $\beta_G^\bullet(k) \cong k[t]$.*

Proof. We have to see that the structure of $\beta_G^\bullet(k)$-module of $\beta_G^\bullet(C^\bullet) \cong C^\bullet \overline{\otimes} W^\bullet$ is equivalent to $C^\bullet \otimes k[t]$. For that we take $c \otimes w^p \in \beta_G^\bullet(C^\bullet) \cong C^\bullet \overline{\otimes} W^\bullet$ and $\overline{w}^q \in \beta_G^\bullet(k)$ and we compute the product $(c \otimes w^p) \cup \overline{w}^q$ given by the composition

$$\epsilon_\bullet(G) \xrightarrow{\Delta} \epsilon_\bullet(G) \to \epsilon_\bullet(G) \xrightarrow{(c \otimes w^p) \otimes \overline{w}^q} C^\bullet \otimes k \xrightarrow{\eta} C^*,$$

where $\eta : C^\bullet \otimes k \to C^\bullet$ is the k-module structure map of $C^\bullet$. If follows from definition that

$$((c \otimes w^p) \cup \overline{w}^q)(w_m) = \eta((c \otimes w^p) \otimes \overline{w}^q)(\Delta w_m) = \left\{ \begin{array}{ll} c & \text{if } p+q = m, \\ 0 & \text{if } p+q \neq m \end{array} \right\},$$

hence we obtain that $(c \otimes w^p)\overline{w}^q = c \otimes w^{p+q}$ in $C^\bullet \overline{\otimes} W^\bullet$. Therefore we deduce that

$$\begin{array}{rccc} \Phi: & C^\bullet \overline{\otimes} W^\bullet & \to & C^\bullet \otimes k[t] \\ & c \otimes w^n & \mapsto & c \otimes t^n \end{array}$$

is the desired isomorphism of $k[t]$-modules. □

We also denote $C^\bullet \overline{\otimes} k[t]$ to the complex with underlying module $C^\bullet \otimes k[t]$ and coboundary induced by the isomorphism from $C^\bullet \overline{\otimes} W^\bullet$. Then $C^\bullet \overline{\otimes} k[t]$ is a differential graded module over $k[t]$. In particular, the product

$$\beta^\bullet_G(C^\bullet) \otimes \beta^\bullet_G(k) \to \beta^\bullet_G(C^\bullet)$$

is already associative on the cochain level, while the analogous left product in only associative up to homotopy.

Once we have properly discusses the base case we can extend the above results to the description of the algebraic Borel construction for the case $G = (\mathbb{Z}_2)^n$ with $n > 1$. The corresponding total space of the universal principal G-bundle $EG \to BG$ can be given by the n-fold product of the $\mathbb{Z}_2$-CW-complexes $E\mathbb{Z}_2$

$$EG = E\mathbb{Z}_2 \times \cdots \times E\mathbb{Z}_2.$$

Therefore, the cellular chain complex $\epsilon_\bullet(G) = W_\bullet(EG; k)$ corresponds to the n-fold tensor product of $\epsilon_\bullet(\mathbb{Z}_2) = W_\bullet(E\mathbb{Z}_2; k)$

$$\epsilon_\bullet(G) = \epsilon_\bullet(\mathbb{Z}_2) \otimes \cdots \otimes \epsilon_\bullet(\mathbb{Z}_2)$$

considered as a complex over $k[G] = k[\mathbb{Z}_2] \otimes \cdots \otimes k[\mathbb{Z}_2]$ with the componentwise action. Given these considerations we can easily extend the calculation for $G = \mathbb{Z}_2$.

Proposition 1.20. *Let k a field of characteristic 2 and let $G = (\mathbb{Z}_2)^n$. We have*

$$H^\bullet(BG; k) = H^\bullet_G(*; k) \cong k[t_1, \cdots, t_n]$$

with $\deg(t_i) = 1$ *for each* $1 \le i \le n$.

Proof. From the above discussion we deduce that

$$BG = EG/G = B\mathbb{Z}_2 \times \cdots \times B\mathbb{Z}_2.$$

Therefore, since by Proposition 1.18 we know that $H^\bullet(B\mathbb{Z}_2; k) \cong k[t]$, using Künneth Theorem we can obtain the desired result

$$\begin{aligned} H^\bullet(BG; k) = H^\bullet(B\mathbb{Z}_2 \times \cdots \times B\mathbb{Z}_2; k) \cong H^\bullet(B\mathbb{Z}_2; k) \otimes \cdots \otimes H^\bullet(B\mathbb{Z}_2; k) \cong \\ \cong k[t] \otimes \cdots \otimes k[t] \cong k[t_1, \cdots, t_n], \end{aligned}$$

where we have used that k is a field. □

Finally, we can extend the latter result in Proposition 1.19 to $G = (\mathbb{Z}_2)^n$.

Proposition 1.21. *Let $C^\bullet$ be a bounded below cochain complex over $k[G]$. Then $\beta^\bullet_G(C^\bullet)$ is naturally isomorphic to $C^\bullet \otimes k[t_1, \dots, t_n]$ as a right $k[t_1, \dots, t_n]$-module.*

Proof. By Proposition 1.12 we have an isomorphism of k-modules

$$\begin{aligned}
&\beta^\bullet_G(C^\bullet) \cong C^\bullet \otimes_{k[G]} \epsilon^\bullet(G) = C^\bullet \otimes_{k[G]} \mathrm{Hom}_{k[G]}(\epsilon_\bullet(G), k[G]) \cong \\
&\cong C^\bullet \otimes_{k[G]} \mathrm{Hom}_{k[G]}((W_\bullet \otimes k[\mathbb{Z}_2]) \otimes \cdots \otimes (W_\bullet \otimes k[\mathbb{Z}_2]), k[G]) \cong \\
&\cong C^\bullet \otimes_{k[G]} \mathrm{Hom}_{k[G]}((W_\bullet \otimes \cdots \otimes W_\bullet) \otimes k[G]), k[G]) \cong \\
&\cong C^\bullet \otimes_{k[G]} k[G] \otimes \mathrm{Hom}_k(W_\bullet \otimes \cdots \otimes W_\bullet, k) \cong \\
&C^\bullet \otimes \mathrm{Hom}_k(W_\bullet \otimes \cdots \otimes W_\bullet, k) \cong C^\bullet \otimes W^\bullet \otimes \cdots \otimes W^\bullet
\end{aligned}$$

and from Proposition 1.19 it follows the desired isomorphism. $\square$

As before we denote by $C^\bullet \overline{\otimes} k[t_1, \dots, t_n]$ to the complex $\beta^\bullet_G(C^\bullet)$ with the twisted differential of $C^\bullet \overline{\otimes} k[t_1, \dots, t_n]$.

And with this final result we bring our general introduction to the theory of transformation groups to a close.

Chapter 2

The homology of finite free $(\mathbb{Z}_2)^n$-Complexes

This chapter is dedicated to the proof of the main theorem based on Carlsson method [Car83], which is strongly based on commutative algebra. In the first place Carlsson devotes a section to the study of differential graded modules and proves two key results, that are Theorems 2.16 and 2.17. In a second part the author defines the functor β and proves a general version of the main theorem for any free bounded above chain complex $C_\bullet$ over $k[G]$, which is Theorem 2.22. Finally, this result is used to obtain the main theorem.

2.1 Differential graded modules

Let k a field of characteristic 2 and let $A = k[x_1, \dots, x_n]$ be the polynomial ring in n variables, which we grade by considering $\deg(x_i) = -1$ for all $1 \leq i \leq n$.

Along this chapter we will assume that any graded A-module M is **bounded above**, that is that $M_n = 0$ for some n sufficiently large, and **locally finite**, that is that $\dim_k(M_n)$ is finite for each $n \in \mathbb{Z}$.

To start with, we consider a standard result on free graded A-modules.

Proposition 2.1. *Let M, N be free graded A-modules and let $f : M \to N$ be a morphism of A-modules of degree 0 such that $f \otimes id : M \otimes_A k \to N \otimes_A k$ is an isomorphism. Then f is an isomorphism. Moreover, if $f \otimes id$ is injective, then f is injective onto a direct summand.*

Proof. Since M is free and bounded above we can assume that

$$M = Am_1 \oplus Am_2 \oplus Am_3 \oplus \cdots = \bigoplus_{i \in I} Am_i,$$

where we order the elements by degree, $\deg(m_i) \geq \deg(m_{i+1})$ for each $i \in I$. Moreover, since f is a morphism of A-modules of degree 0, the same happens for $f \otimes id$ and we have

that for each $i \in I$
$$(f \otimes id)(m_i \otimes 1) = \sum_{j=1}^{k_i} n_j^i \otimes k_j^i,$$
where $k_j^i \in k$ and $n_j^i \in N$ with $\deg(n_j^i) = \deg(m_i)$. Notice that the above sum is finite because N is assumed to be locally finite. Therefore, since $f \otimes id$ is an isomorphism we deduce that
$$N = An_1 \oplus An_2 \oplus An_3 \oplus \cdots = \bigoplus_{i \in I} An_i,$$
with $n_i = \sum_{j=1}^{k_i} k_j^i n_j^i$. Let us define a morphism of A-modules given by
$$\begin{array}{rccc} g: & N & \to & M \\ & n_i & \mapsto & m_i \end{array}$$
We can observe that the composition of f and g is given by
$$(g \circ f)(m_i) = m_i + \sum_{j=i+1}^{\infty} p_i^j m_j = id_M(m_i) + h(m_i)$$
with $p_i^j \in A$ and then we have that the inverse of f is given by $f^{-1} = Id + \sum_{n=1}^{\infty} (-1)^n h^n$, hence we deduce f is an isomorphism. If $f \otimes id$ is injective, the same argument considered over $f \otimes id : M \otimes k \to \operatorname{im}(f \otimes id)$ implies the desired result. □

Definition 2.2. *A **differential graded (DG)** A**-module** M is a graded A-module with a morphism of A-modules*
$$d : M \to M$$
*such that $\deg(d) = -1$ and $d^2 = 0$. We say that M is a **free** (resp. **finitely generated**) DG A-module if M is free (resp. finitely generated) as an A-module.*

Definition 2.3. *Let M be a DG A-module and let N be a graded A-module. We define $H_\bullet(M, N)$ as the homology of the DG A-module $M \otimes_A N$ with the differential $d \otimes id$.*

The homology $H_\bullet(M \otimes_A N)$ is defined as usual
$$H_\bullet(M \otimes_A N) := \frac{\ker(d_n \otimes id : M_n \otimes_A N \to M_{n-1} \otimes_A N)}{\operatorname{im}(d_{n+1} \otimes id : M_{n+1} \otimes_A N \to M_n \otimes_A N)}.$$

Let $I \subset A$ be the maximal ideal $I = (x_1, \ldots, x_n)$. If M is free DG A-module, we can consider the filtration given by the powers $I^k M$
$$\cdots \to I^{k+1}M \to I^k M \to I^{k-1}M \to \cdots \to IM \to M.$$
Moreover, notice that the morphism of A-modules $I^k \otimes_A M \to I^k M$ induces an isomorphism
$$\frac{I^k M}{I^{k+1} M} \cong \frac{I^k}{I^{k+1}} \otimes_A \frac{M}{IM}.$$

Since M is locally finite and bounded above, then the filtration $\{I^kM\}_k$ is finite in each dimension, that is that for all $n \in \mathbb{Z}$ there exists some $k \geq 0$ such that $(I^kM)_n = 0$. Finally, for each $k \geq 0$ we have that I^k/I^{k+1} is a direct sum of copies of the field k. We denote

$$\mathrm{gr}_I(A) = \bigoplus_{k \geq 0} \frac{I^k}{I^{k+1}}.$$

Lemma 2.4. *VLet M be a free DG A-module. There exists an spectral sequence with*

$$E^1_{-p,q} \cong \frac{I^p}{I^{p+1}} \otimes_A H_q(M,k) \quad \text{and} \quad d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

that converges to $H_\bullet(M)$. Moreover, $\{E^r_{p,q}, d^r\}_r$ is a spectral sequence of modules over $\mathrm{gr}_I(A)$ and

$$E^\infty_{0,q} \cong \mathrm{im}(H_p(M) \to H_p(M,k)).$$

Proof. We have a filtration of DG A-modules

$$\cdots \to I^{k+1}M \to I^kM \to I^{k-1}M \to \cdots \to IM \to M$$

that determines a spectral sequence $\{E^r_{p,q}, d^r\}_{r\geq 1}$ with d^r of bidegree $(-r, r-1)$ and such that

$$E^1_{p,q} \cong H_{p+q}\left(\frac{I^pM}{I^{p+1}M}\right) \cong H_{p+q}\left(\frac{I^k}{I^{k+1}} \otimes_A \frac{M}{IM}\right) \cong \frac{I^p}{I^{p+1}} \otimes_A H_q\left(\frac{M}{IM}\right) \cong$$
$$\cong \frac{I^p}{I^{p+1}} \otimes_A H_q\left(\frac{M}{IM}\right) \cong \frac{I^p}{I^{p+1}} \otimes_A H_q(M \otimes_A k) = \frac{I^p}{I^{p+1}} \otimes_A H_q(M,k).$$

where we have used that the differential is trivial on the quotient I^k/I^{k+1} and that

$$M/IM \cong M \otimes_A A/I \cong M \otimes_A k.$$

Since the filtration is finite in each dimension it is bounded, hence the spectral sequence converges to $H_\bullet(M)$. It is clear that $\{E^r_{p,q}, d^r\}_{r\geq 1}$ is a spectral sequence of $\mathrm{gr}_I(A)$-modules since d is a morphism of A-modules. Finally, we have that

$$E^r_{0,q} = \frac{\{x \in M_q : d(x) \in I^rM_{q-1}\}}{IM_q + d(M_{q+1})} \cong k \otimes_A \frac{\{x \in M_q : d(x) \in I^rM_{q-1}\}}{d(M_{q+1})}.$$

Therefore, any element in $E^\infty_{0,q} \subset E^1_{0,q}$ admits a representative cycle that has a lifting $\zeta \in M_q$ such that $d(\zeta) \in I^rM_{q-1}$ for all $r \geq 1$. Since the filtration is finite in each dimension, for r sufficiently large we have that $(I^rM)_{q-1} = 0$, hence the element belongs to $H_q(M)$. Notice that the spectral sequence in the statement is obtained by considering $p := -p$. □

Let us define the operator

$$\theta_i : H_\bullet(M,k) \to H_\bullet(M,k)$$

given by

$$\begin{array}{rccc} d^1: & E^1_{0,q} \cong H_q(M,k) & \to & E^1_{-1,q} \cong (I/I^2) \otimes_A H_q(M,k) \\ & 1 \otimes \zeta & \mapsto & \sum_{i=1}^n \overline{x_i} \otimes \theta_i(\zeta) \end{array}$$

where $\overline{x_i}$ denotes the image of x_i in I/I^2. Since $\{\overline{x_i}\}_i$ is a basis of I/I^2 over k, we have that the images $\theta_i(\zeta)$ are uniquely determined.

Proposition 2.5. *For each $1 \leq i, j \leq n$ we have that $\theta_i^2 = 0$ and $\theta_i\theta_j = \theta_j\theta_i$.*

Proof. Since $(d^1)^2 = 0$ and d^1 is a morphism of A-modules we obtain

$$\begin{aligned} 0 = (d^1)^2(1 \otimes \zeta) = d^1 \left(\sum_{i=1}^n \overline{x_i} \otimes \theta_i(\zeta) \right) = \sum_{i=1}^n \overline{x_i} d^1(1 \otimes \theta_i(\zeta)) = \\ = \sum_{i=1}^n \overline{x_i} \sum_{j=1}^n \overline{x_j} \otimes \theta_j\theta_i(\zeta) = \sum_{i=1}^n \sum_{j=1}^n \overline{x_i x_j} \otimes \theta_j\theta_i(\zeta). \end{aligned}$$

Therefore, as $\{\overline{x_i x_j}\}_{1 \leq i,j \leq n}$ is a basis of I^2/I^3 over k we obtain the desired result. □

Lemma 2.6. *Let N be the largest integer such that $H_N(M,k) \neq 0$. Let $\zeta \in H_N(M,k)$ with $\theta_i(\zeta) = 0$ for all $1 \leq i \leq n$. Then $\zeta \in \operatorname{im}(H_N(M) \to H_N(M,k))$.*

Proof. Since $\theta_i(\zeta) = 0$ for all $1 \leq i \leq n$ we obtain $d^1(\zeta) = 0$. Then for any $r > 1$ we have

$$d^r(\zeta) \in E^r_{-r,N+r-1} \subset E^1_{-r,N+r-1} \cong \frac{I^r}{I^{r+1}} \otimes_A H_{N+r-1}(M,k) = 0$$

because $N < N + r - 1$. Therefore, $d^r(\zeta) = 0$ for all $r \geq 1$, and by Lemma 2.4 we conclude

$$\zeta \in E^\infty_{0,N} \cong \operatorname{im}(H_N(M) \to H_N(M,k)).$$

□

We say that a map between DG A-modules is a **chain map** and the usual notions of **chain homotopy** and **chain equivalence** are considered.

Lemma 2.7. *Let M be a free DG A-module such that $H_\bullet(M,k) = 0$. Then M is contractible.*

Proof. Since $H_\bullet(M,k) = 0$ we can assume $M \otimes_A k$ has a basis over k given by $\{e_\alpha, f_\alpha\}_{\alpha \in A}$ for some indexing set A. Let us consider the free graded A-module F generated by $\{e'_\alpha, f'_\alpha\}_{\alpha \in A}$, where $\deg(e'_\alpha) = \deg(e_\alpha)$ and $\deg(f'_\alpha) = \deg(f_\alpha)$ with the differential d' given by $d'(e'_\alpha) = f'_\alpha$ and $d'(f'_\alpha) = 0$ for each $\alpha \in A$. It is clear that F is a free DG A-module. Let us define a morphism of A-modules

$$\begin{array}{rccc} \phi: & F & \to & M \\ & e'_\alpha & \mapsto & \overline{e_\alpha} \\ & f'_\alpha & \mapsto & d(\overline{e_\alpha}) \end{array}$$

where $\overline{e_\alpha}$ is a lifting of $e_\alpha \in M \otimes_A k$ to $\overline{e_\alpha} \in M$. We have an induced isomorphism

$$\begin{array}{rccl} \phi \otimes id: & F \otimes_A k & \to & M \otimes_A k \\ & e'_\alpha \otimes 1 & \mapsto & \phi(e'_\alpha) \otimes 1 = e_\alpha \\ & f'_\alpha \otimes 1 & \mapsto & \phi(f'_\alpha) \otimes 1 = d(\overline{e'_\alpha}) \otimes 1 = f_\alpha, \end{array}$$

that is indeed a chain map. Since M and F are free graded A-modules, by Proposition 2.1 we have ϕ is also an isomorphism. Finally, we have that F is chain equivalent to 0. Consider $\alpha : F \to 0$, $\beta : 0 \to F$ and notice $\beta \circ \alpha = id_0$ and $\phi : F \to F$ given by $\phi(e'_\alpha) = 0$ and $\phi(f'_\alpha) = e'_\alpha$ is a chain equivalence between id_F and $\alpha \circ \beta$ because

$$(\phi \circ d + d \circ \phi)(e'_\alpha) = g(f'_\alpha) = e'_\alpha = (id_F - \alpha \circ \beta)(e'_\alpha),$$

$$(\phi \circ d + d \circ \phi)(f'_\alpha) = d(e'_\alpha) = f'_\alpha = (id_F - \alpha \circ \beta)(f'_\alpha).$$

Therefore, M is contractible. □

Corollary 2.8. *Let M, N be free DG A-modules and let $f : M \to N$ be a chain map. If $H_\bullet(f, k)$ is an isomorphism, then f is a chain equivalence.*

Proof. Since $f \otimes k$ is a quasi-isomorphism, by Corollary A.3 we have that $\text{cone}(f \otimes k)$ is acyclic. We notice that

$$\text{cone}(f \otimes k) = (M \otimes_A k)[-1] \oplus (N \otimes_A k) \cong (M[-1] \oplus N) \otimes_A k = \text{cone}(f) \otimes_A k,$$

and

$$D_{\text{cone}(f\otimes k)} = \begin{pmatrix} -d_M \otimes id & 0 \\ -f \otimes id & d_N \otimes id \end{pmatrix} = \begin{pmatrix} -d_M & 0 \\ -f & d_N \end{pmatrix} \otimes id = D_{\text{cone}(f)} \otimes id.$$

Therefore, $\text{cone}(f) \otimes_A k$ is acyclic and by Lemma 2.7 $\text{cone}(f)$ is contractible. Finally, by Proposition A.4 we conclude that f is a chain equivalence. □

Definition 2.9. *Let M be a DG A-module. A* ***composition series*** *for M is a sequence*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_q = M$$

of DG A-modules such that for each $1 \leq j \leq q$ the quotient M_j/M_{j-1} is a free DG A-module with zero differential. We say M is ***solvable*** *if there exists a free finitely generated DG A-module $\overline{M}$ that admits a composition series and a chain equivalence*

$$f : \overline{M} \to M,$$

We define the ***length*** *$l(M)$ of M as the length of the shortest composition series that admits any free finitely generated DG A-module $\overline{M}$ which is chain equivalent to M.*

Lemma 2.10. *Let M_1, M_2, M_3 be free finitely generated DG A-modules such that the differential on M_1 is trivial and assume we have a short exact sequence*

$$0 \to M_1 \xrightarrow{q} M_2 \xrightarrow{p} M_3 \to 0$$

If $l(M_3) = l$, then there exists a free finitely generated DG A-module $\overline{M_2}$ admitting a composition series of length $l+1$ and a chain equivalence between M_2 and $\overline{M_2}$.

Proof. Since $l(M_3) = l$ we know there exist a free finitely generated DG A-module $\overline{M_3}$ that admits a composition series of length l and a chain equivalence $f : \overline{M_3} \to M_3$. Let us consider the pullback

$$\overline{M_2} = \{(m, \overline{m}) \in M_2 \oplus \overline{M_3} : p(m) = f(m)\}$$

and the natural map

$$\begin{array}{rccc} \phi : & \overline{M_2} & \to & M_2 \\ & (m, \overline{m}) & \mapsto & m. \end{array}$$

We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "q'"] \arrow[d, equal] & \overline{M_2} \arrow[r, "p'"] \arrow[d, "\phi"] & \overline{M_3} \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r, "q"] & M_2 \arrow[r, "p"] & M_3 \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact and the top row maps are given by $q'(m') = (q(m'), 0)$ and $p'(\overline{m}, m) = \overline{m}$. Then, we have a morphism between the induced long exact sequences in homology, so for each $k \in \mathbb{Z}$ we have a commutative diagram

$$\begin{tikzcd} H_{k+1}(\overline{M_3}) \arrow[r] \arrow[d, "H_{k+1}(f)"] & H_k(M_1) \arrow[r] \arrow[d, equal] & H_k(\overline{M_2}) \arrow[r] \arrow[d, "H_k(\phi)"] & H_k(\overline{M_3}) \arrow[r] \arrow[d, "H_k(f)"] & H_{k-1}(M_1) \arrow[d, equal] \\ H_{k+1}(M_3) \arrow[r] & H_k(M_1) \arrow[r] & H_k(M_2) \arrow[r] & H_k(M_3) \arrow[r] & H_{k-1}(M_1) \end{tikzcd}$$

Since by hypothesis $H_{k+1}(f)$ and $H_k(f)$ are isomorphisms, by the Five Lemma we obtain that $H_k(\phi)$ is also an isomorphism, hence ϕ is a chain equivalence by Corollary 2.8. Moreover, we have that $\overline{M_2}$ is free and finitely generated. Finally, the exact sequence

$$0 \to M_1 \to \overline{M_2} \to \overline{M_3} \to 0$$

splits because $\overline{M_3}$ is free, hence projective. Therefore, we have that $\overline{M_2} \cong M_1 \oplus \overline{M_3}$ and we have that

$$\overline{M_2} / \overline{M_3} \cong M_1$$

is a free DG A-module with zero differential. Thus,

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = \overline{M_3} \subseteq \overline{M_2}$$

is a composition series of length $l+1$ for $\overline{M_2}$, which concludes the proof. $\square$

Proposition 2.11. *Let M be a free finitely generated DG A-module. Then M is solvable.*

Proof. We proceed by induction on the rank of M, which is defined as

$$\operatorname{rank}(M) = \dim_k(M \otimes_A k).$$

If $\operatorname{rank}(M) = 1$, then $M \cong A$. Since A is an integral domain, for any $x \in A$ we have

$$d(x)d(x) = d(xd(x)) = xd^2(x) = 0,$$

hence $d = 0$ in M and $0 \subseteq M$ is a composition series for M, thus M is solvable.
Assume that the result is satisfied for all M' with $\operatorname{rank}(M') < \operatorname{rank}(M)$. If $H_\bullet(M,k) = 0$, by Lemma 2.7 we have M is solvable. If not, take N to be the largest integer such that $H_N(M,k) \neq 0$. By Proposition 2.5 we have that $H_\bullet(M,k)$ is an $\Lambda(\theta_1, \cdots, \theta_n)$-module, where $\Lambda(\theta_1, \cdots, \theta_n)$ is the exterior algebra generated by the operators θ_i. Consider the ideal $J = (\theta_1, \cdots, \theta_n) \subset \Lambda(\theta_1, \cdots, \theta_n)$ and let l be the largest integer such that

$$J^l H_N(M,k) \neq 0.$$

Notice that this is well-defined because $J^{n+1}H_N(M,k) = 0$. For $\zeta \in J^l H_N(M,k)$ we have that $j\zeta = 0$ for any $j \in J$, and in particular $\theta_i(\zeta) = 0$ for all $1 \leq i \leq n$. Then, by Lemma 2.6 we have that

$$\zeta \in \operatorname{im}(H_N(M) \to H_N(M,k)),$$

so there exists $\overline{\zeta} \in M_N$ with $d\overline{\zeta} = 0$ that projects to a cycle that represents ζ in $M \otimes_A k$. Since $\overline{\zeta}$ projects to a nonzero element in $M \otimes_A k$, by Proposition 2.1 it generated a free summand of M. Consider Ae_N a free graded A-module generated by an element of degree N and with trivial differential. We can consider the map

$$\begin{array}{rccc} i: & Ae_N & \to & M \\ & e_N & \mapsto & \overline{\zeta} \end{array}$$

that is trivially a chain map and induces an inclusion into a direct summand of M. Hence, we have that $\operatorname{rank}(\operatorname{coker}(i)) = \operatorname{rank}(M) - 1$. Therefore, by the inductive hypothesis and Lemma 2.10 considered over the short exact sequence

$$0 \to Ae_N \xrightarrow{i} M \to \operatorname{coker}(i) \to 0$$

we obtain that M is solvable, which concludes the proof. $\square$

Definition 2.12. *Let M be a DG A-module. We say that M is **totally finite** if $H_\bullet(M)$ is a a k-vector space of finite dimension.*

Let M_i be a graded $k[x_i]$-module for each $1 \leq i \leq n$. Then we have that

$$M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n$$

is a graded module over $k[x_1] \otimes_k \cdots \otimes_k k[x_n] \cong k[x_1, \dots, x_n] = A$ by considering the componentwise action. For each $1 \leq i \leq n$ let us consider the $k[x_i]$-modules

$$A_i = k[x_i] \quad B_i = k[x_i, x_i^{-1}] \quad C_i = \frac{B_i}{A_i}.$$

Then, for $0 \leq j \leq n$ we define the A-modules

$$\begin{aligned} P_j &= A_1 \otimes_k A_2 \otimes_k \cdots \otimes_A A_j \otimes_k C_{j+1} \otimes_k \cdots \otimes_k C_n \\ Q_j &= A_1 \otimes_k A_2 \otimes_k \cdots \otimes_A A_j \otimes_k B_{j+1} \otimes_k C_{j+2} \otimes_k \cdots \otimes_k C_n \end{aligned}$$

Notice that for each $1 \leq i \leq n$ we have a short exact sequence of $k[x_i]$-modules

$$0 \to A_i \to B_i \to C_i \to 0.$$

Then, by successively applying the functors $- \otimes_k A_k$ for $1 \leq k \leq i-1$ and $- \otimes_j C_{k'}$ for $i+1 \leq k' \leq n$, which are exact, we conclude that for each $0 \leq j \leq n-1$ there is a short exact sequence of A-modules

$$0 \to P_{j+1} \to Q_j \to P_j \to 0.$$

Lemma 2.13. *Let M be a free finitely generated totally finite DG A-module. Then $H_\bullet(M, Q_j) = 0$ for all $0 \leq j \leq n-1$.*

Proof. Let S_{j+1} be the multiplicative system of power of x_{j+1}. It follows from the definitions that $Q_j = (P_{j+1})_{S_{j+1}}$. Moreover, since localization commutes with tensor products and it is exact, which implies that it preserves homology, we deduce that for each $0 \leq j \leq n-1$

$$\begin{aligned} &H_\bullet(M, P_{j+1})_{S_{j+1}} = H_\bullet(M \otimes_A P_{j+1})_{S_{j+1}} \cong H_\bullet((M \otimes_A P_{j+1})_{S_{j+1}}) \cong \\ &\cong H_\bullet(M \otimes_A (P_{j+1})_{S_{j+1}}) \cong H_\bullet(M \otimes_A Q_j) = H_\bullet(M, Q_j). \end{aligned}$$

We proceed by induction downward on the index j. If $j = n-1$ with the same argument as before we have

$$H_\bullet(M, Q_j) \cong H_\bullet(M, A)_{S_n} \cong H_\bullet(M, A) \otimes_A A[x_n^{-1}].$$

Since $H_\bullet(M, A) \cong H_\bullet(M)$ is a finite dimensional k-vector space, it follows that some power of x_n needs to annihilate $H_\bullet(M)$, hence $H_\bullet(M)_{S_n} = 0$ and we are done.

Assume that the result holds for $j > m$, with $0 \leq m < n-1$. For each $1 \leq k \leq n-1-j$ we have a short exact sequence of A-modules

$$0 \to P_{j+k+1} \to Q_{j+k} \to P_{j+k} \to 0.$$

We apply the covariant functor $M \otimes_A -$, which is exact, and we consider the connecting morphism of the associated long exact sequence on homology

$$\partial_{j+k} : H_\bullet(M, P_{j+k}) \to H_\bullet(M, P_{j+k+1}).$$

By inductive hypothesis we obtain that ∂_j is an isomorphism for all $k \geq 1$, because $H_\bullet(M, Q_{j+k}) = 0$. Therefore, the composition

$$\partial_{n-1} \circ \cdots \circ \partial_{j+2} \circ \partial_{j+1} : H_\bullet(M, P_{j+1}) \to H_\bullet(M, P_n)$$

is also an isomorphism, hence

$$H_\bullet(M, P_{j+1}) \cong H_\bullet(M, P_n) \cong H_\bullet(M, A) \cong H_\bullet(M),$$

and by the same argument for the base case we obtain that

$$H_\bullet(M, Q_j) \cong H_\bullet(M, P_{j+1})_{S_{j+1}} \cong H_\bullet(M)_{S_{j+1}} = 0,$$

which concludes the proof for the inductive step. □

Corollary 2.14. *Let M be a free finitely generated totally finite DG A-module. Then the connecting morphisms*

$$\partial_j : H_\bullet(M, P_j) \to H_\bullet(M, P_{j+1})$$

of the long exact sequence in homology induced by the short exact sequence

$$0 \to P_{j+1} \to Q_j \to P_j \to 0$$

are isomorphisms.

Lemma 2.15. *Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of graded A-modules and let*

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

be a short exact sequence of free DG A-modules such that M_3 has trivial differential. Then the connecting morphism δ of the long exact sequence in homology induced by the sequence of graded A-modules satisfy that

$$\operatorname{im}(\delta : H_\bullet(M_2, N_3) \to H_\bullet(M_2, N_1)) \subset \operatorname{im}(H_\bullet(M_1, N_1) \to H_\bullet(M_2, N_1)).$$

Proof. Let $\zeta \in H_\bullet(M_2, N_3)$ and $z = \sum_{i \in I} m_i \otimes n_i \in M_2 \otimes_A N_3$ a representative cycle, hence $\sum_{i \in I} d(m_i) \otimes n_i = 0$. We want to compute $\delta\zeta$. For that, we consider a lifting $\overline{n_i} \in N_2$ of each n_i with $i \in I$ and we compute

$$(d \otimes id)\left(\sum_{i \in I} m_i \otimes \overline{n_i}\right) = \sum_{i \in I} d(m_i) \otimes \overline{n_i} \in M_2 \otimes_A N_1.$$

On the other hand, since the differential on M_3 is trivial we have

$$d(m_i) \in \ker(M_2 \to M_3) = M_1,$$

where we identify M_1 with $\mathrm{im}(M_1 \to M_2)$, which is a direct summand of M_2 since the A-modules are free. From the above statements one concludes that

$$\sum_{i \in I} d(m_i) \otimes \overline{n_i} \in (M_2 \otimes_A N_1) \cap (M_1 \otimes N_2),$$

but again since the A-modules are free we have that

$$(M_2 \otimes_A N_1) \cap (M_1 \otimes N_2) = M_1 \otimes_A N_1.$$

Therefore, we deduce that $\sum_{i \in I} d(m_i) \otimes \overline{n_i}$ is a cycle contained in $M_1 \otimes_A N_1 \subset M_2 \otimes N_1$, which is what we wanted to prove. $\square$

Theorem 2.16. *Let M be a free finitely generated totally finite DG A-module with $H_\bullet(M) \neq 0$. Then $l(M) \geq n+1$.*

Proof. We can assume that M admits a composition series of length $l = l(M)$. If that is not the case, we can replace M by a chain equivalent free finitely generated DG A-module admitting such composition series. Let

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M$$

be this composition series. In the first place we will prove that for each $0 \leq j \leq l$ the morphism

$$H_\bullet(i_j, P_j) : H_\bullet(M_{l-j}, P_j) \to H_\bullet(M, P_j)$$

is surjective, where

$$i_j : M_{l-j} \to M$$

is the inclusion associated to the composition series. We proceed by induction on j. If $j = 0$ the result is immediate. For $j > 0$ assume that i_j is surjective and consider the commutative diagram

$$\begin{array}{ccc} H_\bullet(M_{l-j}, P_j) & \xrightarrow{H_\bullet(i_j, P_j)} & H_\bullet(M, P_j) \\ \big\downarrow{\scriptstyle \partial} & & \big\downarrow{\scriptstyle \partial} \\ H_\bullet(M_{l-j}, P_{j+1}) & \xrightarrow[H_\bullet(i_j, P_{j+1})]{} & H_\bullet(M, P_{j+1}) \end{array}$$

where the vertical maps are induced by the connecting homomorphism associated to the short exact sequence

$$0 \to P_{j+1} \to Q_j \to P_j \to 0$$

as it was done in Lemma 2.13. By Corollary 2.14 the right vertical map is an isomorphism. Let $\zeta \in H_\bullet(M, P_{j+1})$ and take $\zeta' \in H_\bullet(M, P_j)$ such that $\partial\zeta' = \zeta$. By inductive hypothesis we have that there exists some $\zeta'' \in H_\bullet(M_{l-j}, P_j)$ such that $\zeta' = H_\bullet(i_j, P_j)(\zeta'')$. Then, by the commutative diagram above we obtain

$$H_\bullet(i_j, P_{j+1})(\partial\zeta'') = \zeta.$$

If we apply Lemma 2.15 to the short exact sequences of A-modules

$$0 \to P_{j+1} \to Q_j \to P_j \to 0 \qquad\qquad 0 \to M_{l-j-1} \to M_{l-j} \to \frac{M_{l-j}}{M_{l-j-1}} \to 0$$

we obtain that

$$\operatorname{im}(\delta : H_\bullet(M_{l-j}, P_j) \to H_\bullet(M_{l-j}, P_{j+1})) \subset \operatorname{im}(H_\bullet(M_{l-j-1}, P_{j+1}) \to H_\bullet(M_{l-j}, P_{j+1})),$$

hence there exists some $\zeta''' \in H_\bullet(M_{l-j-1}, P_{j+1})$ such that its image in $H_\bullet(M_{l-j}, P_{j+1})$ is $\partial(\zeta'')$. Therefore, $H_\bullet(i_{j+1}, P_{j+1})(\zeta''') = \zeta$, so $H_\bullet(i_{j+1}, P_{j+1})$ is surjective as we wanted to see. Finally, assume that $l \leq n$. Then the map

$$H_\bullet(i_l, P_l) : H_\bullet(M_0, P_l) \to H_\bullet(M, P_l)$$

is surjective. However, $M_0 = 0$ and by Corollary 2.14 we have that

$$H_\bullet(M, P_l) \cong H_\bullet(M, P_n) \cong H_\bullet(M, A) \cong H_\bullet(M),$$

so $H_\bullet(M) = 0$, which is a contradiction. Hence, $l \geq n + 1$. □

Recall that for each $1 \leq i \leq n$ we have an operator θ_i on $H_\bullet(M, k)$ and by Proposition 2.5 we have that if M is a free DG A-module, then $H_\bullet(M, k)$ is an $\Lambda(\theta_1, \cdots, \theta_n)$-module. We denote $E = \Lambda(\theta_1, \cdots, \theta_n)$.

Let us consider the ideal $J = (\theta_1, \ldots, \theta n)$. For F an E-module we define the **length** $\lambda(E)$ as the largest λ such that $J^{\lambda-1}F \neq 0$

$$\lambda(F) = \max\{\lambda : J^{\lambda-1}F \neq 0\}.$$

For M a free finitely generated DG A-module we define the **homological length** $\mathcal{L}(M)$ as

$$\mathcal{L}(M) = \sum_{n=-\infty}^{\infty} \lambda(H_n(M, k)).$$

Notice that the sum is finite because M is finitely generated.

Theorem 2.17. *Let M be a free finitely generated DG A-module. Then $\mathcal{L}(M) \geq l(M)$.*

Proof. We want to see that there exists a free finitely generated DG A-module admitting a composition series of length $\mathcal{L}(M)$ which is chain equivalent to M. We proceed by induction on the homological length $\mathcal{L}(M)$.

Assume that $\mathcal{L}(M) = 1$. Then there exists $N \in \mathbb{Z}$ such that $H_N(M,k)$ is the only non-vanishing homology group, hence $H_n(M,k) = 0$ for all $n \neq N$. Since J annihilates $H_N(M,k)$ it follows from Lemma 2.6 that the map

$$H_N(M) \to H_N(M,k)$$

is surjective. We can consider a basis of $H_N(M,k)$ given by $\zeta_1, \dots, \zeta_s$ and take cycles in M $\overline{\zeta_1}, \dots, \overline{\zeta_s}$ that are projected to representative cycles of $\zeta_1, \dots, \zeta_s$. Construct a $A_s = \oplus_{i=1}^s A\rho_i$ generated by $\rho_1, \dots, \rho_s$ with $\deg(\rho_i) = N$ for all $1 \leq i \leq s$ and with trivial differential. We can consider the map

$$\begin{array}{rccc} f: & A_s & \to & M \\ & \rho_i & \mapsto & \overline{\zeta_i} \end{array}$$

which is trivially a chain map and induces an isomorphism

$$H_\bullet(f,k) : H_\bullet(A_s,k) \to H_\bullet(M,k).$$

Therefore, by Corollary 2.8 we obtain that f is a chain equivalence. It is clear that A_s admits a composition series of length 1, hence $l(M) = 1$ and we conclude the proof for the base case.

For the inductive step assume that the result hold for $\mathcal{L}(M) \leq j$ and let M be a free finitely generated DG A-module with $l(M) = j+1$. Let N be the largest integer such that $H_N(M,k) \neq 0$ and take the length $\lambda = \lambda(H_N(M,k))$. Notice that

$$J(J^{\lambda-1}H_N(M,k)) = J^\lambda H_N(M,k) = 0,$$

hence by Lemma 2.6 we obtain that $J^{\lambda-1}H_N(M,k) \subset \operatorname{im}(H_N(M) \to H_N(M,k))$. Similarly as before, choose a basis of $J^{\lambda-1}H_N(M,k)$ given by $\zeta_1, \dots, \zeta_s$ and take cycles in M $\overline{\zeta_1}, \dots, \overline{\zeta_s}$ that are projected to representative cycles of $\zeta_1, \dots, \zeta_s$. Let $A_s = \oplus_{i=1}^s A\rho_i$ be as before and consider the map

$$\begin{array}{rccc} f: & A_s & \to & M \\ & \rho_i & \mapsto & \overline{\zeta_i}, \end{array}$$

which is again a chain map. After applying the functor $- \otimes_A k$ we obtain the map

$$\begin{array}{rccc} f \otimes id: & A_s \otimes_A k & \to & M \otimes_A k \\ & \rho_i & \mapsto & \zeta_i', \end{array}$$

where ζ_i' is a cycle in $M \otimes_A k$ representing ζ_i, hence $f \otimes id$ is injective. Therefore, by Proposition 2.1 we deduce that f is an inclusion to a direct summand of M, which allows us to consider the free DG A-module

$$\overline{M} = \frac{M}{f(A_s)},$$

that is also finitely generated. From the short exact sequence of A-modules

$$0 \to A_s \xrightarrow{f} M \to \overline{M} \to 0$$

and after applying the functor $- \otimes_A k$ we obtain a long exact sequence given by

$$\begin{aligned} \cdots \to & H_{N+1}(\overline{M}, k) \to H_N(f(A_s), k) \to H_N(M, k) \to H_N(\overline{M}, k) \to H_{N-1}(f(A_s), k) \to \cdots \\ & \cdots \to H_n(f(A_s), k) \to H_n(M, k) \to H_n(\overline{M}, k) \to H_{n-1}(f(A_s), k) \to \cdots \end{aligned}$$

because the A-modules are free. Moreover, since $H_n(\overline{M}, k) = H_{N+1}(M, k) = 0$ for $n > N$ and $H_n(f(A_s), k) = 0$ for $n \neq N$ we deduce that

$$H_n(\overline{M}, k) \cong \left\{ \begin{array}{ll} H_n(M, k) & \text{if } j < N, \\ H_N(M, k)/(J^{\lambda-1} H_N(M, k)) & \text{if } j = N \end{array} \right\}.$$

Therefore, $\mathcal{L}(\overline{M}) = \mathcal{L}(M) - 1$. From the inductive hypothesis it follows that $\overline{M}$ is chain equivalent to a free finitely generated DG A-module $\overline{\overline{M}}$ that admits a composition series of length $\mathcal{L}(M) - 1$. Finally, by Lemma 2.10 applied to the above short exact sequence we obtain that there exists a free finitely generated DG A-module chain equivalent to M with a composition series of length $\mathcal{L}(M)$, which concludes the proof. $\square$

Remark 2.18. In the original version of Carlsson's paper [Car83] Theorem 2.16 states that $l(M) \geq n$. Nevertheless, we have observed that the argument can be extended to $l(M) \geq n + 1$ without any additional difficulty. We want to emphasize this fact because this extended version is essential to obtain the main theorem in the most general terms (see Theorem 1.4.14 in [AP93]), and even though all the bibliography attributes the proof of the main theorem to Carlsson, inexplicably the author does not state the result with the standard lower bound.

2.2 The β functor

Let $G = (\mathbb{Z}_2)^n$ and consider the group ring $E = k[G]$. Notice that as a k-algebra we have an isomorphism between E and the exterior algebra $\Lambda(y_1, \ldots, y_n)$ given by

$$\begin{array}{rcl} \Lambda(y_1, \ldots, y_n) & \to & k[G] \\ y_i & \mapsto & 1 + T_i, \end{array}$$

where $T_1, \dots, T_n$ is a system of generators of G. We assume E is concentrated at zero degree.

We say that a **cochain complex over** E is a bounded above and locally finite free graded E-module $C_\bullet$ such that $\deg(\delta) = -1$, where δ denotes the differential δ. For $C_\bullet$ a cochain complex over E we define a DG A-module $\beta C_\bullet$ with underlying module

$$A \otimes_k C_\bullet.$$

and differential given by

$$d = id \otimes \delta + \sum_{i=1}^{n} x_i \otimes y_i,$$

where x_i and y_i denote the multiplication by x_i and y_i, for each $1 \leq i \leq n$. Notice that β defines a functor

$$\beta : E\text{-}\mathbf{CoCh} \to k\text{-}\mathbf{CoCh}$$

from the category of cochain complexes over E to the category of cochain complexes over k. With the following result we will compare the structures of the E-module $H_\bullet(C_\bullet)$ and $H_\bullet(\beta C_\bullet, k)$ as a $\Lambda(\theta_1, \dots, \theta_n)$-module.

Proposition 2.19. *Let $C_\bullet$ be a cochain complex over E. There is a natural isomorphism*

$$b : H_\bullet(C_\bullet) \to H_\bullet(\beta C_\bullet, k)$$

compatible with the multiplicative structures, that is that the following diagram commutes

$$\begin{array}{ccc} H_\bullet(C_\bullet) & \xrightarrow{b} & H_\bullet(\beta C_\bullet, k) \\ \big\downarrow{\scriptstyle y_i} & & \big\downarrow{\scriptstyle \theta_i} \\ H_\bullet(C_\bullet) & \xrightarrow{b} & H_\bullet(\beta C_\bullet, k) \end{array}$$

Proof. Let us consider the chain map

$$\begin{array}{rccl} \overline{b} : & C_\bullet & \to & k \otimes_A \beta C_\bullet \cong k \otimes_A (A \otimes_k C_\bullet) \\ & c & \mapsto & 1 \otimes 1 \otimes c, \end{array}$$

that is well-defined because

$$\begin{aligned} ((1 \otimes d)(\overline{b}(c)) = (1 \otimes d)(1 \otimes 1 \otimes c) = 1 \otimes 1 \otimes \delta(c) + \sum_{i=1}^{n} 1 \otimes x_i \otimes y_i c = \\ = 1 \otimes 1 \otimes \delta(c) = \overline{b}(\delta(c)), \end{aligned}$$

where we have used that $1 \otimes x_i \otimes y_i c = \overline{x_i} \otimes 1 \otimes y_i c = 0$ for each $1 \leq i \leq n$. It follows from definition that $\overline{b}$ is an isomorphism, hence $b = H_\bullet(\overline{b})$ is also an isomorphism. To prove that the diagram commutes consider z a cycle in $C_\bullet$, hence satisfies $\delta(z) = 0$. Notice that a lifting of $\overline{b}(z) = 1 \otimes 1 \otimes z$ from $k \otimes_A \beta C_\bullet$ to $A \otimes_A \beta C_\bullet$ may be given by $\zeta = 1 \otimes 1 \otimes z$. If $\overline{z}$

is the homology class represented by the cycle z, we know that to compute $\theta_i(b(\overline{z}))$ we can use $d^1(b(\overline{z})) = \overline{d(\overline{b}(z))}$. We have

$$d(\zeta) = 1 \otimes 1 \otimes \delta(z) + \sum_{i=1}^{n} 1 \otimes x_i \otimes y_i z = \sum_{i=1}^{n} 1 \otimes x_i \otimes y_i z = \sum_{i=1}^{n} x_i \otimes 1 \otimes y_i z,$$

hence reducing $d(\zeta)$ modulo I^2 we conclude that

$$d^1(b(\overline{z})) = \sum_{i=1}^{n} \overline{x_i} \otimes y_i \overline{z}.$$

Therefore, recalling the definition of the operators θ_i for $1 \leq i \leq n$ we obtain the commutativity of the diagram. □

Let M be an E-module. We define

$$M^E = \{m \in M : y_i m = 0 \text{ for all } 1 \leq i \leq n\},$$

which also determines a functor

$$-^E : E\text{-}\mathbf{CoCh} \to k\text{-}\mathbf{CoCh}$$

from the category of cochain complexes over E to the category of cochain complexes over k.

Proposition 2.20. *Let $C_\bullet$ be a cochain complex over E. There is a natural isomorphism of k-vector spaces*

$$s : H_\bullet(C_\bullet^E) \to H_\bullet(\beta C_\bullet).$$

Proof. We consider the map

$$\begin{array}{cccc} \overline{s}: & C_\bullet^E & \to & \beta C_\bullet \\ & z & \mapsto & 1 \otimes z. \end{array}$$

Notice that for any $z \in C_\bullet^E$ we have that

$$d\overline{s}(z) = d(1 \otimes z) = 1 \otimes \delta(z) + \sum_{i=1}^{n} x_i \otimes y_i z = 1 \otimes \delta(z) = \overline{s}(\delta(z))$$

because $y_i z$ for all $1 \leq i \leq n$, hence $\overline{s}$ is a chain map.
We can observe that the cochain complex $C_\bullet$ can be filtered with the cochain subcomplexes

$$F_n C_\bullet = \bigoplus_{j \geq n} C_j$$

and this filtration satisfies that the successive quotients $F_n C_\bullet / F_{n+1} C_\bullet$ are finitely generated free E-modules with trivial differential. We also have that the filtration $\{F_n C_\bullet\}_n$ is finite in each dimension, that is that for all $k \in \mathbb{Z}$ we have that $(F_nC_\bullet)_k = 0$ for k sufficiently large, because $C_\bullet$ is bounded above and locally finite. Moreover, we also have that $\beta C_\bullet$ is bounded above. Notice that if we prove the statement for the cochain complexes $F_nC_\bullet$ for each $n \in \mathbb{Z}$ we are done. Furthermore, we claim that it suffices to prove the statement for the quotients $F_nC_\bullet/F_{n+1}C_\bullet$ for each $n \in \mathbb{Z}$. Indeed, assume that the result holds for the quotients and proceed by induction downwards on the filtration degree n. Since $C_\bullet$ is bounded above, for n_0 sufficiently large we have that $C_n = 0$ for all $n \geq n_0$, hence $F_nC_\bullet = 0$ for all $n \geq n_0$ and the proof for base case follows immediately. For the inductive step consider the short exact sequence of E-modules

$$0 \to F_{n+1}C_\bullet \to F_nC_\bullet \to \frac{F_nC_\bullet}{F_{n+1}C_\bullet} \to 0.$$

Since A is k-flat we obtain that the functor β is exact. On the other hand, we have that $F_nC_\bullet/F_{n+1}C_\bullet$ is a free E-module, in particular a projective E-module. Therefore the surjective projection $F_nC_\bullet \to F_nC_\bullet/F_{n+1}C_\bullet$ has a section which allows us to assure that $-^E$ preserves the exactness of the above short exact sequence. Then after applying these two functors to the aforementioned sequence and by considering the chain map $\bar{s}$ we obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (F_{n+1}C_\bullet)^E \arrow[r] \arrow[d, "\bar{s}"] & (F_nC_\bullet)^E \arrow[r] \arrow[d, "\bar{s}"] & \left(\frac{F_nC_\bullet}{F_{n+1}C_\bullet}\right)^E \arrow[r] \arrow[d, "\bar{s}"] & 0 \\
0 \arrow[r] & \beta(F_{n+1}C_\bullet) \arrow[r] & \beta(F_nC_\bullet) \arrow[r] & \beta\left(\frac{F_nC_\bullet}{F_{n+1}C_\bullet}\right) \arrow[r] & 0
\end{tikzcd}$$

Therefore, we have a map between the long exact sequences associated to the horizontal short exact sequences of the above diagram given by

$$\begin{tikzcd}
H_{k+1}\left(\left(\frac{F_nC_\bullet}{F_{n+1}C_\bullet}\right)^E\right) \arrow[r] \arrow[d, "s"] & H_k(E_{n+1}) \arrow[r] \arrow[d, "s"] & H_k(E_n) \arrow[r] \arrow[d, "s"] & H_k\left(\left(\frac{F_nC_\bullet}{F_{n+1}C_\bullet}\right)^E\right) \arrow[r] \arrow[d, "s"] & H_{k-1}(E_{n+1}) \arrow[d, "s"] \\
H_{k+1}\left(\beta\left(\frac{F_nC_\bullet}{F_{n+1}C_\bullet}\right)\right) \arrow[r] & H_k(\beta_{n+1}) \arrow[r] & H_k(\beta_n) \arrow[r] & H_k\left(\beta\left(\frac{F_nC_\bullet}{F_{n+1}C_\bullet}\right)\right) \arrow[r] & H_{k-1}(\beta_{n+1})
\end{tikzcd}$$

where we denote $\beta_n = \beta(F_nC_\bullet)$ and $E_n = (F_nC_\bullet)^E$ to simplify the notation. By the inductive hypothesis and the assumption on the quotients $F_nC_\bullet/F_{n+1}C_\bullet$ we obtain that the first and the last two vertical maps are isomorphisms, and by the Five Lemma we conclude the proof for the inductive step of the claim.
Finally, since each of the quotients $F_nC_\bullet/F_{n+1}C_\bullet$ is a direct sum of free E-modules modules

with trivial differentials, it is enough to consider the cochain complex $D_\bullet$ given by

$$D_n = \left\{ \begin{array}{ll} E & \text{if } n = 0, \\ 0 & \text{if } n \neq 0 \end{array} \right\}$$

as the functors β and $-^E$ commute with direct sums. Now, on the one hand with a carefully observation of $\beta D_\bullet$ we can notice that it corresponds to the Koszul resolution of k over A. Therefore, since the Koszul resolution is acyclic (see Proposition 4.2.2 in [Ser75]) we conclude

$$H_n(\beta D_\bullet) = \left\{ \begin{array}{ll} k & \text{if } n = 0, \\ 0 & \text{if } n \neq 0 \end{array} \right\}.$$

On the other hand, it is clear that we also have that

$$H_n(D_\bullet^E) = \left\{ \begin{array}{ll} k & \text{if } n = 0, \\ 0 & \text{if } n \neq 0 \end{array} \right\}.$$

Hence, since the map s matches the generators of $H_\bullet(D_\bullet^E)$ and $H_\bullet(\beta D_\bullet)$ we finish the proof. $\square$

Corollary 2.21. *Let $C_\bullet$ be a cochain complex over E finitely generated as an E-module. Then $\beta C_\bullet$ is totally finite.*

Proof. We have that C^E is a finite-dimensional k-vector space because $C_\bullet$ is finitely generated, hence the same happens to $H_\bullet(C^E)$. Therefore, it follows from Proposition 2.20 that $H_\bullet(\beta C_\bullet)$ is also a finite-dimensional k-vector space, which means that $\beta C_\bullet$ is totally finite. $\square$

Let us consider the ideal $J = (y_1, \ldots, y_n) \subset E$. Recall that if M is a E-module, we define the length $\lambda(M)$ as the largest integer λ such that $J^{\lambda-1}M \neq 0$.

Theorem 2.22. *Let $C_\bullet$ be a cochain complex over E finitely generated as an E-module. Then*

$$\sum_{n=-\infty}^{\infty} \lambda(H_n(C_\bullet)) \geq n.$$

Proof. It follows from Proposition 2.19 that

$$\sum_{n=-\infty}^{\infty} \lambda(H_n(C_\bullet)) = \sum_{n=-\infty}^{\infty} \lambda(H_n(\beta C_\bullet, k)) = \mathcal{L}(\beta C_\bullet).$$

By Corollary 2.21 we have that $\beta C_\bullet$ is totally finite, and it is clear that $\beta C_\bullet$ is also locally finite and bounded above. Therefore, by Theorem 2.16 we deduce that $\beta C_\bullet$ admits a composition series of length n, that is $l(\beta C_\bullet) \geq n$. Finally, from Theorem 2.17 we conclude that $\mathcal{L}(\beta C_\bullet) \geq l(\beta C_\bullet) \geq n$, which finishes the proof. $\square$

2.3 The proof of the main theorem

We have already introduced all the necessity algebraic results needed to properly prove the main theorem. Before concluding this chapter we will provide a geometric interpretation to the these statements in order to obtain a more illustrative characterisation of the main theorem.

Let $G = (\mathbb{Z}_2)^n$ and $k = \mathbb{Z}_2$. Let X be a free finite G-CW-complex and consider $C_\bullet = W^\bullet(X;k)$ the cellular cochain complex negatively graded, where we also assume that the differential δ is of degree -1.

We can observe that $C_\bullet$ is indeed a free graded E-module, since the G-action is free. Moreover, $C_\bullet$ is locally finite, because X is finite, and bounded above. Then we can consider the construction $\beta C_\bullet$ that we defined in the previous section 2.2. From the discussion in section 1.4 we can notice that $\beta C_\bullet$ corresponds to the DG A-module $C^\bullet \otimes k[t_1, \dots, t_n]$ negatively graded with the twisted differential, and by Proposition 1.21 we obtain that the construction $\beta C_\bullet$ is naturally isomorphic to $\beta^\bullet_G(X;k)$, again negatively graded. Therefore we can conclude that by defining the construction $\beta C_\bullet$ we are obtaining an algebraic model whose homology corresponds to the equivariant cohomology of the G-CW-complex X

$$H_\bullet(\beta C_\bullet) \cong H^\bullet_G(X;k).$$

We can understand 2.19 as the fact that by considering the tensor product with the field k on cochain level we recover the usual cohomology of the G-CW-complex X

$$H_\bullet(\beta C_\bullet, k) \cong H_\bullet(C_\bullet) = H^\bullet(X;k).$$

Furthermore, we can observe that Proposition 2.20 is an equivalent result to the fact that if X is a free G-space, we have that

$$H^\bullet_G(X;k) \cong H^\bullet(X/G;k).$$

Indeed, we already know that the equivariant cohomology of X can be obtained by the homology of the construction $\beta C_\bullet$. On the other hand, we can prove that $C^E_\bullet = W^\bullet(X/G;k)$. Let $c \in C^E_\bullet$ be a cochain and notice that for each $1 \leq i \leq$ we have

$$y_i c = 0 = (1 + T_i)c \iff T_i c = c,$$

that corresponds to say $c \in W^\bullet(X/G;k)$. Therefore we obtain

$$H_\bullet(\beta C_\bullet) \cong H^\bullet_G(X) \cong H^\bullet(X/G;k) \cong H_\bullet(C^E_\bullet)$$

which is equivalent to Proposition 2.20.

Finally, by Theorem 2.22 applied to $C_\bullet = W^\bullet(X;k)$ we can prove the main theorem of this project.

Theorem 2.23. *Let X be a finite CW-complex such that $(\mathbb{Z}_2)^n$ acts freely and cellularly on X. Then*

$$\sum_{n=-\infty}^{\infty} \lambda(H_n(X;\mathbb{Z}_2)) \geq n.$$

Proof. Let $C_\bullet = W^\bullet(X;k)$ be cellular cochain complex negatively graded, which by the previous discussion is a cochain complex over E and it is finitely generated as an E-module. If follows from 2.22 that

$$\sum_{n=-\infty}^{\infty} \lambda(H^n(X;k)) = \sum_{n=-\infty}^{\infty} \lambda(H_n(C_\bullet)) \geq n.$$

Finally, by the Universal Coefficients Theorem we deduce that $H^n(X;k) \cong \mathrm{Hom}_k(H_n(X;k),k)$ as $k[G]$-modules. Then it is immediate to check that $\lambda(H_n(X;k)) = \lambda(\mathrm{Hom}_k(H_n(X;k),k))$ for each $n \in \mathbb{Z}$, which is reasonable because they are dual representations. And we conclude the proof. □

Appendix A

Basic notions on homotopy theory

A.1 The mapping cone

Throughout this section we assume that all the objects are considered in an abelian category. We have based the result on [Wei94].

Definition A.1. *Let $A_\bullet, B_\bullet$ be chain complexes and let $f : A_\bullet \to B_\bullet$ be a morphism of chain complexes. The **mapping cone** of f is the chain complex given by*

$$\operatorname{cone}(f) = A_\bullet[-1] \oplus B_\bullet$$

with the differential

$$D(a,b) = (-d_A(a), d_B(b) - f(a)).$$

It is standard to see that $\operatorname{cone}(f)$ is a well-defined chain complex.

Lemma A.2. *Let $f : A_\bullet \to B_\bullet$ be a morphism of chain complexes. There is an exact sequence of chain complexes*

$$0 \to B_\bullet \xrightarrow{i} \operatorname{cone}(f) \xrightarrow{j} A_\bullet[-1] \to 0,$$

where $i(c) = (0, c)$ and $j(a, c) = -a$. Moreover, the connecting morphism in the associated long exact sequence is $H_\bullet(f)$.

Proof. It is straightforward to check the exactness of the sequence of chain complexes. To compute the connecting morphism in the associated long exact sequence consider $a \in A_n[-1] = A_{n-1}$ with $d_A(a) = 0$, take the element $(a, 0) \in \operatorname{cone}(f)_n$ and notice that

$$D(a,0) = (-d_A(a), f(a)) = (0, f(a)),$$

so the element of B_{n-1} having $(0, f(a))$ as image is $f(a)$, hence the connecting morphism corresponds to $H_{n-1}(f)$. $\square$

Corollary A.3. *Let $f : A_\bullet \to B_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if* $\mathrm{cone}(f)$ *is acyclic.*

Proof. From the long exact sequence associated to the short exact sequence in Lemma A.2

$$\cdots \to H_{n+1}(\mathrm{cone}(f)) \to H_{n+1}(A_\bullet[-1]) = H_n(A_\bullet) \xrightarrow{H_n(f)} H_n(B_\bullet) \to H_n(\mathrm{cone}(f)) \to \cdots$$

the result follows immediately. □

Proposition A.4. *Let $f : A_\bullet \to B_\bullet$ be a morphism of chain complexes. If* $\mathrm{cone}(f)$ *is contractible, then f is a chain equivalence.*

Proof. Since $\mathrm{cone}(f)$ is contractible, it is chain equivalent to the zero module, so there exists a map $\phi : \mathrm{cone}(f) \to \mathrm{cone}(f)$ such that $\phi D + D\phi = id_{\mathrm{cone}(f)}$. Assume that ϕ is given by the matrix

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From the equality $\phi D + D\phi = id_{\mathrm{cone}(f)}$ we obtain that

$$\begin{pmatrix} -ad_A - bf - d_A a & bd_B - d_A b \\ -cd_A - df - fa + d_B c & dd_B - fb + d_B d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then since $bd_B - d_A b = 0$ we deduce that $-b : B_\bullet \to A_\bullet$ is a chain map. And it follows from

$$\begin{aligned} -ad_A - bf - d_A a = 1 &\implies 1 - (-b)f = (-a)d_A + d_A(-a) \\ dd_B - fb + d_B d = 1 &\implies 1 - f(-b) = dd_B + d_B d \end{aligned}$$

that $-b$ is indeed the chain homotopy inverse of f that satisfies $(-b)f \simeq_{-a} id_A$ and $f(-b) \simeq_d id_B$, hence f is a chain equivalence. □

Appendix B

Principal G-bundles

In this appendix we give an introduction to principal G-bundles. We have followed [Nab18] and [Mit01a]. We start by defining this object.

Definition B.1. *Let G be a topological group. Let E and B be right G-spaces such that the action of G on B is trivial and consider a G-map*

$$\pi : E \to B$$

We say that (E, π) is a ***principal G-bundle over B*** *if there exists an open cover $\{U_i\}_{i \in I}$ of B such that for each $i \in I$ there exists a G-map $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ which is an homeomorphism and makes the following diagram commutative*

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[rr, "\phi_i"] \arrow[dr, "\pi"'] & & U_i \times G \arrow[dl, "p_1"] \\
& U_i &
\end{tikzcd}$$

where p_1 denotes the projection onto the first component. The space E is called the ***total space*** *of the principal G-bundle (E, π).*

Notice that since G acts trivially on B we have that π factors through the orbit space E/G. Moreover, we also have that $U_i \times G$ has a right G-action given by

$$\begin{array}{ccc} (U_i \times G) \times G & \to & U_i \times G \\ ((x, h), g) & \mapsto & (x, hg), \end{array}$$

hence we obtain that G acts freely on E. Therefore π factors through E/G with an homeomorphism $\overline{\pi} : E/G \to B$

$$\begin{tikzcd}
E \arrow[rr, "\pi"] \arrow[dr] & & B \\
& E/B \arrow[ur, "\overline{\pi}"'] &
\end{tikzcd}$$

In conclusion, we can understand a principal G-bundle over B as a locally trivial free G-space with orbit space B.

Definition B.2. *Let (E, π) be a principal G-bundles over B and let (E', π') be principal G-bundles over B'. A **morphism of principal G-bundles** is a G-map $\sigma : E \to E'$.*

Notice that for $\sigma : E \to E'$ a morphism of principal G-bundles we have an associated map $\overline{\sigma} : B \to B'$ which makes the following diagram commutative

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{\overline{\sigma}} & B' \end{array}$$

defined by $\overline{\sigma}(b) = \pi'(\sigma(e))$, where $\pi(e) = b$. We have that $\overline{\sigma}$ is well-defined because if $e_1, e_2 \in \pi^{-1}(b)$ we have that $e_1 = e_2 g$ for some $g \in G$ and then

$$\pi'(\sigma(e_1)) = \pi'(\sigma(e_2 g)) = \pi'(\sigma(e_2)g) = \pi'(\sigma(e_2)).$$

We also have that $\overline{\sigma}$ is continuous because for each $b \in B$ there exists some open set $U_i \subset B$ such that $b \in U_i$ and there exists a local section $s : U_i \to \pi^{-1}(U_i) \cong U_i \times G$. Then we have that

$$\overline{\sigma}_{|U_i} = \pi' \circ \sigma \circ s,$$

hence $\overline{\sigma}$ is continuous on U_i, and since $B = \bigcup_{i \in I} U_i$ we deduce $\overline{\sigma}$ is continuous on B.

With the following result we can observe that morphisms of principal G-bundles are very rigid structures.

Proposition B.3. *Let $\sigma : E \to E'$ be a morphism of principal G-bundles such that $\overline{\sigma} : B \to B'$ is the identity. Then σ is an isomorphism.*

Proof. To see ϕ is injective consider $e_1, e_2 \in E$ such that $\sigma(e_1) = \sigma(e_2)$. From the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{Id} & B \end{array}$$

it follows that $\pi' \circ \sigma = \pi$, hence we obtain $\pi(e_1) = \pi(e_2)$, which implies that $e_1 = e_2 g$ for some $g \in G$. Therefore we have that

$$\sigma(e_1) = \sigma(e_2 g) = \sigma(e_2)g = \sigma(e_2)$$

and since the action of G on E' is free we deduce that $g = e$, thus $e_1 = e_2$. To prove subjectivity let $e_2 \in E'$ and consider $b = \pi'(e_2)$. We can take $e \in \pi^{-1}(b)$ and again from the above commutative diagram it follows that

$$\pi'(\sigma(e)) = \pi(e) = b = \pi'(e_2),$$

thus $\sigma(e) = e_2 g$ for some $g \in G$. Then if we define $e_1 = eg^{-1}$ we can observe that

$$\sigma(e_1) = \sigma(eg^{-1}) = \sigma(e)g^{-1} = e_2 g g^{-1} = e_2,$$

thus ϕ is surjective. It remains to see that ϕ^{-1} is continuous. For $b \in B$ we choose $U_i \subset B$ with $b \in U_i$ and such that the bundles are trivial over U_i, hence there exist G-equivariant homeomorphisms

$$\phi : U_i \times G \to \pi^{-1}(U_i) \qquad \phi' : U_i \times G \to (\pi')^{-1}(U_i).$$

Since σ is surjective we have that $\phi(\pi^{-1}(U_i)) = (\pi')^{-1}(U_i)$, and we have a well-defined morphism of principal G-bundles

$$\begin{tikzcd} U_i \times G \arrow[r] \arrow[d] & U_i \times G \arrow[d] \\ U_i \arrow[r, "Id"] & U_i \end{tikzcd}$$

that is necessarily of the form

$$\phi_2^{-1} \circ \phi \circ \phi_1(x, g) = (\phi_2^{-1} \circ \phi \circ \phi_1(x, e))g = (x, \tau(x, e))g = (x, \tau(x, e)g),$$

with $\tau : U \to G$ a continuous map. We can observe that the inverse map is given by

$$\phi_1^{-1} \circ \phi^{-1} \circ \phi_2(x, g) = (x, \tau(x, e)^{-1} g),$$

which is also continuous, and then we obtain that ϕ is continuous on $\phi_2^{-1}(U)$. Finally, since $B = \bigcup_{i \in I} U_i$ we are done. □

B.1 Equivariant maps and sections of bundles

If (E, π) is a principal G-bundle over B and X is a left G-space, we can define the **associated fiber bundle to** (E, π) **by** X as

$$\begin{array}{rccc} \pi_X : & E \times_G X & \to & B \\ & [p, x] & \mapsto & \pi(p) \end{array}$$

where $E \times_G X$ corresponds to the space $E \times X$ modulo the equivalence relation $(p, x) \sim (pg, g^{-1}x)$ for each $g \in G$. In this section we will see that we have a bijective correspondence between sections of the above fiber bundle and G-equivariant maps $E \to X$.

Theorem B.4. *Let (E, π) be a principal G-bundle over B, let X be a G-space and consider $E \times_G X$ the associated bundle. For any open set $U \subset B$ we have a bijective correspondence*

$$\Gamma(U, E \times_G X) \longleftrightarrow \mathrm{Hom}(\pi^{-1}(U), X)^G,$$

where $\Gamma(U, E \times_G X)$ denotes the set of sections from U to $E \times_G X$ and $\mathrm{Hom}(\pi^{-1}(U), X)^G$ is the set of G-equivariant maps $\pi^{-1}(U) \to X$.

Proof. Let $U \subset B$ be an open subset. Consider $\phi : \pi^{-1}(U) \to E \times_G X$ a G-map and define the section

$$\begin{array}{rccc} s_\phi : & U & \to & E \times_G X \\ & b & \mapsto & [p, \phi(p)] \end{array}$$

The section s_ϕ is well-defined because for $p, q \in \pi^{-1}(b)$ we have that $p = qg$ for some $g \in G$ and then

$$[p, \phi(p)] = [qg, \phi(qg)] = [qg, g^{-1}\phi(q)] = [q, \phi(q)].$$

Moreover, it is clear that $\pi_X \circ s_\phi = Id_U$. It remains to prove that s_ϕ is continuous. Take an open set $V_i = U_i \cap U \subset U$ such that the bundle is trivial over V_i and consider a local section $s_{V_i} : V_i \to E$. For each $v \in V_i$ we have that

$$s_\phi(b) = [s_{V_i}(b), \phi \circ s_{V_i}(b)],$$

hence we have that $s_{\phi|V_i}$ is continuous. Finally, since $\bigcup_{i \in I} U_i \cap U = U$ we obtain that s_ϕ is continuous on U. Therefore we conclude that we have defined a map

$$\mathrm{Hom}(\pi^{-1}(U), X)^G \to \Gamma(U, E \times_G X).$$

To show that this map is bijective we will construct its inverse. Consider a section $s \in \Gamma(U, E \times_G X)$ and define the map given by

$$\begin{array}{rccc} \phi_s : & \pi^{-1}(U) & \to & E \times_G X \\ & p & \mapsto & f \end{array}$$

where $s(\pi(p)) = [p, f]$. Notice that this map is well-defined because if $s(\pi(p)) = [p, f_1] = [p, f_2]$, then we have that $(p, f_1) = (pg, g^{-1}f_2)$ for some $g \in G$, and since the action is free we deduce $g = e$, hence $f_1 = f_2$. Observe then that $s(\pi(p)) = [p, \phi_s(p)]$. To see that ϕ_s is G-equivariant take $g \in G$ and notice that for any $p \in \pi^{-1}(U)$ we have that

$$[p, \phi_s(p)] = s(\pi(p)) = s(\pi(sg)) = [pg, \phi_s(pg)],$$

where we have used that $\pi(pg) = \pi(g)$. Thus there exists $h \in G$ such that $(pg, \phi_s(pg)) = (ph, h^{-1}\phi(p))$. We have that $pg = ph$ and again since the action is free we deduce $g = h$, hence we obtain $\phi_s(pg) = g^{-1}\phi(p)$ as we desired. Finally, to prove that ϕ_s is continuous we consider an open subset $V_i = U_i \cup U \subset U$ such that the bundle is trivial over V_i, hence we have an G-equivariant homeomorphism

$$\phi : V_i \times G \to \pi^{-1}(V_i).$$

This induce a local trivialization on the associated bundle given by

$$\begin{array}{rccc} \phi_X : & V_i \times X & \to & (E \times_G X)_{|V_i} \\ & (b, x) & \mapsto & [\phi(b, e), x] \end{array}$$

If we consider $\phi_X^{-1}(q) = (\pi_X(q), \tau_X(q))$, where $\tau_X : (E \times_G X)_{|V_i} \to X$ is a G-equivariant map, we obtain that

$$(b, x) = \phi_X^{-1}([\phi(b, e), x]) = (\pi_X \circ \phi(b, e), \tau_X([\phi(b, e), x])) = (b, \tau_X([\phi(b, e), x])),$$

so we have that $x = \tau_X([\phi(b, e), x])$ for all $x \in X$. Moreover, we also have that

$$\tau_X \circ s(b) = \tau_X \circ s \circ \pi_1(b, e) = \tau_X \circ s \circ \pi \circ \phi(b, e) = \tau_X([\phi(b, e), \phi_s(\phi(b, e))]) = \phi_s \circ \phi(b, e).$$

Therefore $\phi_s \circ \phi$ is continuous, so ϕ_s is continuous on $\phi^{-1}(V_i)$. Finally, since $\bigcup_{i \in I} U_i \cap U = U$ we conclude that ϕ_s is continuous on $\pi^{-1}(U)$, as we desired. Thus we conclude that we have constructed a map

$$\Gamma(U, E \times_G X) \to \mathrm{Hom}(\pi^{-1}(U), X)^G$$

and it is immediate to check it is an inverse to the previous map, which finishes the proof. □

If particular, for $U = X$ we obtain the bijective correspondence

$$\Gamma(B, E \times_G X) \longleftrightarrow \mathrm{Hom}(E, X)^G.$$

As a consequence of the previous result we can state the following Corollary.

Corollary B.5. *Let (E, π) be a principal G-bundles over B and let (E', π') be principal G-bundles over B'. There is a bijective correspondence*

$$\Gamma(B, E \times_G E') \longleftrightarrow \mathrm{Hom}_G(E, E'),$$

where $\mathrm{Hom}_G(E, E')$ denotes the set of morphisms of principal G-bundles.

Proof. By Definition B.2 we have that $\mathrm{Hom}_G(E, E')$ corresponds to $\mathrm{Hom}(E, E')^G$, hence the result follows from Theorem B.4. □

B.2 Universal principal G-bundles and classifying spaces

Let (E,π) be a principal G-bundle and let $f : B' \to B$ be a continuous map, with B' another trivial G-space. We can consider the **pullback bundle** $E' = f^*E = B' \times_B E$

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ B' & \xrightarrow[f]{} & B \end{array}$$

and we can observe that $(f^*(E),\pi)$ has a natural structure of principal G-bundle over B'. It is clear that $\pi : f^\bullet E \to B'$ is a G-map. Moreover, if $\{U_i\}_{i\in I}$ is a open cover of B with $\{\phi_i : \pi^{-1}(U_i) \to U_i \times G\}_{i\in I}$ local trivializations, then we consider $\{f^{-1}(U_i)\}_{i\in I}$ as an open cover of B' and it follows from the definitions that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(f^{-1}(U_i)) \arrow[rr, "\phi_i'"] \arrow[dr, "\pi"'] & & f^{-1}(U_i) \times G \arrow[dl] \\ & f^{-1}(U_i) & \end{tikzcd}$$

with $\phi_i = \pi_2 \circ \phi$, where π_2 denotes the projection onto the second component.

With the following result we can observe that pullback bundles preserve homotopies.

Proposition B.6. *Let X be a topological space, let (E,π) be a principal G-bundle over X and let B be a CW-complex. If $f,g : B \to X$ are homotopic maps, then f^*E and g^*E are isomorphic as principal G-bundles.*

Proof. Consider $H : B \times I \to X$ a homotopy between f and g and let H^*E be the pullback bundle

$$\begin{array}{ccc} F^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ B \times I & \xrightarrow[F]{} & X \end{array}$$

Then the result is deduced from the following Lemma B.7. □

Lemma B.7. *Let $\pi : Q \to B \times I$ be a principal G-bundle and consider Q_0 its restriction to $B \times \{0\}$. Then Q is isomorphic to $Q_0 \times I$. In particular, Q_0 is isomorphic to Q_1.*

Proof. To say that Q_0 is the restriction of Q to $B \times \{0\}$ means that Q_0 is the pullback bundle

$$\begin{tikzcd} Q_0 \arrow[r] \arrow[d] & Q \arrow[d, "\pi"] \\ B \times \{0\} \arrow[r, "i_0"'] & B \times I \end{tikzcd}$$

where i_0 denotes the natural inclusion. By Proposition B.3 it suffices to find an equivariant map $Q \to Q_0 \times I$ of principal G-bundles over $(B \times \{0\}) \times I$. Moreover, by Theorem B.4 this is equivalent to find a section of the associated bundle

$$Q \times_G (Q_0 \times I) \to B \times I.$$

However, again by Theorem B.4 we can assure that we have a section $s_0 : B \times \{0\} \to Q \times_G (Q_0 \times I)$ associated to the equivariant map $Q \to Q_0 \times I$ that makes the following diagram commutative

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "\pi"'] & Q_0 \times I \arrow[r] \arrow[d] & Q \arrow[d, "\pi"] \\ B \times \{0\} \arrow[r, "Id"'] & B \times \{0\} \arrow[r, "i_0"'] & B \times I \end{tikzcd}$$

Finally, by a general property of Serre fibrations (see Corollary 5.3 in [Mit01b]) we can extend s_0 to a global section over $B \times I$ as we desired. Therefore we obtain that $Q_0 \times I$ is isomorphic to $Q_1 \times I$, which implies the last statement. ☐

If we denote by $[X, B]$ the set of homotopic classes of maps $X \to B$, from Proposition B.6 if follows that we have a well-defined map given by

$$\begin{array}{ccc} [X,B] & \to & \mathcal{P}_G(X) \\ f & \mapsto & f^*E, \end{array}$$

where $\mathcal{P}_G(X)$ denotes the set of isomorphism classes of morphisms of principal G-bundles over X. From this observation it arises the natural question of which conditions are needed so as the above map is a bijection, from which arises the definition of universal principal G-bundles. Recall that we say that a topological space X is **weakly contractible** if $\pi_i(X) = 0$ for each $i > 0$.

Definition B.8. *A principal G-bundle $\pi : EG \to BG$ is **universal** if the total space EG is weakly contractible.*

We will prove that when $\pi : EG \to BG$ is a universal principal G-bundle and X is a CW-complex, then the map $[X, B] \to \mathcal{P}_G(X)$ is a bijection. We start with a Lemma.

Lemma B.9. *Let (B, A) a CW-complex pair and let $\pi : E \to B$ be a fiber bundle with fiber F. Assume that $\pi_k(F) = 0$ whenever $B \setminus A$ contains a $(k+1)$-dimensional cell. Then any section $s \in \Gamma(A, E)$ can be extended to a global section $\bar{s} \in \Gamma(B, E)$. In particular, the fiber bundle $\pi : E \to B$ admits a global section if $A = \emptyset$ and F is weakly contractible.*

Proof. We will proceed by induction on the skeleton of the CW-complex pair (B, A), so we will see that any section $s_{|A^k} : A^k \to E$ can be extended to a global section $s_k : B^k \to E$. If $k = 0$ the result is immediate because any extension of $s_{|A^0}$ is continuous because B^0 is a discrete space. Assume that $s_{|A^k} : A^k \to E$ extends to a global section $s_k : B^k \to E$. We denote by e^k_α an arbitrary k-dimensional cell in B and we define

$$C_1^{k+1} = \{\alpha : e_\alpha^{k+1} \subset A\} \qquad C_2^{k+1} = \{\alpha : e_\alpha^{k+1} \subset B \setminus A\}.$$

We distinguish the following two cases.
a) If $C_2^{k+1} = \emptyset$, then there are no $(k+1)$-dimensional cells in $B \setminus A$. We define $\overline{s_{k+1}} : (\bigcup_\alpha D_\alpha^{k+1}) \sqcup B^k \to E$ by

$$\overline{s_{k+1}} = \left\{ \begin{array}{ll} s_k & \text{on } B, \\ s \circ \phi_\alpha^{k+1} & \text{on } D_\alpha^{k+1} \end{array} \right\},$$

where $\phi_\alpha^{k+1} : D_\alpha^{k+1} \to B$ is the characteristic map of the cell e_α^{k+1}, that satisfies $\phi_\alpha^{k+1}(D_\alpha^{k+1}) \subset A$. It is clear that $\overline{s_{k+1}}$ is continuous so it only remains to prove that $\overline{s_{k+1}}$ induces a continuous map on B^{k+1}. We consider the attaching maps $\varphi_\alpha^{k+1} : \partial D_\alpha^{k+1} \to B^k$ given by ${\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}}$. Since for each $x \in \partial D_\alpha^{k+1}$ we clearly have that

$$\overline{s_{k+1}}(x) = \overline{s_{k+1}}(\varphi_\alpha^{k+1})$$

we obtain a continuous map $s_{k+1} : B^{k+1} \to E$. It is immediate to check that $\pi \circ s_{k+1} = Id_{B^{k+1}}$ and $s_{k+1|A^{k+1}} = s_{|A^{k+1}}$, so we conclude that $s_{k+1} : B^{k+1} \to E$ is a section that extends $s_{k+1|A^{k+1}}$.
b) If $C_2^{k+1} \neq \emptyset$, let e_α^{k+1} be a $(k+1)$-dimensional cell in $B \setminus A$ and let $\phi_\alpha^{k+1} : D_\alpha^{k+1} \to B$ be its characteristic map. We have that $\phi_\alpha^{k+1}(\partial D_\alpha^{k+1}) \subset B^k$, so the composition map

$$\bar{s}_\alpha^k = s_k \circ \phi_\alpha^{k+1} : \partial D_\alpha^{k+1} \to E$$

is well-defined and one may notice that $\pi \circ \bar{s}_\alpha^k = {\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}}$. Therefore, $\bar{s}_\alpha^k$ defines a section of the pullback bundle $({\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}})^*(E)$ over ∂D_α^{k+1}

$$\begin{tikzcd}
({\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}})^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\
\partial D_\alpha^{k+1} \arrow[r, "{\phi_\alpha^{k+1}}_{|\partial D_\alpha^{k+1}}"'] \arrow[ur, dashed, "\bar{s}_\alpha^k"] & B
\end{tikzcd}$$

Since D_α^{k+1} is contractible, it follows from Proposition B.6 that the pullback bundle $(\phi_\alpha^{k+1})^*(E)$ over D_α^{k+1}

$$\begin{tikzcd}
(\phi_\alpha^{k+1})^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\
D_\alpha^{k+1} \arrow[r, "\phi_\alpha^{k+1}"'] & B
\end{tikzcd}$$

is trivial, that is that $(\phi_\alpha^{k+1})^*(E) \cong D_\alpha^{k+1} \times F$. Therefore, we also have that the bundle $(\phi_\alpha^{k+1}{}_{|\partial D_\alpha^{k+1}})^*(E)$ is also trivial, as we can observe in the following diagram

$$\begin{tikzcd}
\partial D_\alpha^{k+1} \times F \arrow[r] \arrow[d] & D_\alpha^{k+1} \times F \arrow[r] \arrow[d] & E \arrow[d] \\
\partial D_\alpha^{k+1} \arrow[r, hook, "i"'] \arrow[ur, dashed, "\overline{s}_\alpha^k"] & D_\alpha^{k+1} \arrow[r, "\phi_\alpha^{k+1}"'] & B
\end{tikzcd}$$

We can consider $\overline{s}_\alpha^k$ as a map $\overline{s}_\alpha^k = (x, \overline{\tau}_\alpha^k(x))$, where $\overline{\tau}_\alpha^k : \partial D_\alpha^{k+1} \to F$. By hypothesis we have that $\pi_k(F) = 0$, so we can extend $\overline{\tau}_\alpha^k$ to a continuous map $\tau_\alpha^{k+1} : D_\alpha^{k+1} \to F$. Therefore we can extend $\overline{s}_\alpha^k$ to a continuous section $s_\alpha^{k+1} : D_\alpha^{k+1} \to (\phi_\alpha^{k+1})^*(E)$. Finally, we define $\overline{s_{k+1}} : (\bigcup_\alpha D_\alpha^{k+1}) \sqcup B^k \to E$ by

$$\overline{s_{k+1}} = \left\{ \begin{array}{ll} s_k & \text{on } B, \\ s_\alpha^{k+1} & \text{on } D_\alpha^{k+1} \text{ if } \alpha \in C_2^{k+1}, \\ s \circ \phi_\alpha^{k+1} & \text{on } D_\alpha^{k+1} \text{ if } \alpha \in C_1^{k+1} \end{array} \right\}.$$

It is clear that $\overline{s_{k+1}}$ is continuous and as before we can check that for attaching maps $\varphi_\alpha^{k+1} : \partial D_\alpha^{k+1} \to B^k$ given by $\phi_\alpha^{k+1}{}_{|\partial D_\alpha^{k+1}}$ we have that $\overline{s_{k+1}}(x) = \overline{s_{k+1}}(\varphi_\alpha^{k+1})$ for all $x \in \partial D_\alpha^{k+1}$. Therefore $\overline{s_{k+1}}$ induces a continuous map $s_{k+1} : B^{k+1} \to E$ and as before it is immediate to see that $\pi \circ s_{k+1} = Id_{B^{k+1}}$ and $s_{k+1|A^{k+1}} = s_{|A^{k+1}}$, hence s_{k+1} is the desired section. ☐

Finally, we will prove the main theorem of this section.

Theorem B.10. *Let $\pi : EG \to BG$ be a universal principal G-bundle and let X be a CW-complex. Then we have a bijective correspondence*

$$\begin{array}{ccc} [X, BG] & \to & \mathcal{P}_G(X) \\ f & \mapsto & f^*(EG). \end{array}$$

Proof. To prove subjectivity we consider a principal G-bundle (E, π) over X and we want to prove that there exists a map $f : X \to BG$ such that $f^*(EG)$ is isomorphic to E. By Proposition B.3 this it suffices to find a morphism of G-bundles $\sigma : P \to f^*(EG)$ such that $\overline{\sigma} : X \to X$ is the identity

$$\begin{tikzcd}
E \arrow[r, "\sigma"] \arrow[d, "\pi"'] & f^*(EG) \arrow[d] \\
X \arrow[r, "Id"] & X
\end{tikzcd}$$

Moreover, since pullback objects are unique up to isomorphism the above is again equivalent to obtaining a G-equivariant map $\phi : P \to EG$ which makes the following diagram commutative

$$\begin{array}{ccc} E & \xrightarrow{\phi} & EG \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

By Corollary B.5 we know that this is equivalent to find a section of the associated bundle $E \times_G EG \to X$, which can be obtained using Theorem B.9 since the fiber EG is weakly contractible. Therefore the map is surjective. On the other hand, let us consider $f_0, f_1 : X \to BG$ two maps which satisfy that the pullback bundles are isomorphic $f_0^*(EG) \cong f_1^*(EG)$. We want to see that f_0 and f_1 are homotopic maps. We have an induced morphism of principal G-bundles$f_0 : X \to BG$

$$\begin{array}{ccc} f_0^*(EG) & \xrightarrow{\phi_0} & EG \\ {\scriptstyle\pi_0}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{f_0} & BG \end{array}$$

Moreover, the isomorphism of principal G-bundles $f_0^*(EG) \cong f_1^*(EG)$ is equivalent to the existence of a morphism of principal G-bundles $\phi_1 : f_0^*(EG) \to EG$ that makes the following diagram commutative

$$\begin{array}{ccc} f_0^*(EG) & \xrightarrow{\phi_1} & EG \\ {\scriptstyle\pi_0}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{f_1} & BG \end{array}$$

From the above two commutative diagrams and by Proposition B.4 we obtain two sections $s_0, s_1 : X \to f_0^*(EG) \times_G EG$ defined by

$$s_0(\pi_0(x)) = [x, \phi_0(x)] \qquad s_1(\pi_0(x)) = [x, \phi_1(x)]$$

for $x \in X$. We denote $P = f_0^*(EG) \times I$ and $X_t = X \times \{t\}$. If we see s_i as a section in $\Gamma(X_i, P \times_G EG)$ for $i = 0, 1$, we can consider $s_0 \sqcup s_1 \in \Gamma(X_0 \sqcup X_1, P \times_G EG)$ given by $(s_0 \sqcup s_1)_{|X_i} = s_i$ for $i = 0, 1$. Since $(X \times I, X_0 \sqcup X_1)$ is a CW-complex pair and EG is weakly contractible, it follows from Lemma B.9 that there exists a section $s \in \Gamma(X \times I, P \times_G EG)$ that extends $s_0 \sqcup s_1$. Let $\phi : P \to EG$ be the associated morphism, which satisfies that $s(\phi(z), t) = [(z, t), \phi(z, t)]$, and which induces a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi} & EG \\ {\scriptstyle(\pi_0, Id)}\downarrow & & \downarrow{\scriptstyle\pi} \\ X \times I & \xrightarrow{F} & BG \end{array}$$

If follows from the definition that $s(\pi_0(z), 0) = s_0(\pi_0(z))$ and $s(\pi_1(z), 1) = s_1(\pi_0(z))$, we deduce that

$$\phi(z, 0) = \phi_0(z) \qquad \phi(z, 1) = \phi_1(z).$$

Therefore, for all $x \in X$ we have that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$, that provides us with the desired homotopy between f_0 and f_1. This concludes the proof. $\square$

We say that the space BG is the **classifying space** of the group G in view of the fact given X a CW-complex the space BG classifies isomorphism classes of principal G-bundles over X. Moreover, we have uniqueness of universal principal G-bundles up to G-homotopy equivalence. To prove this we would first need to show that the classifying space BG admits a structure of CW-complex. For a proof see [Nab18]

Lemma B.11. *The classifying space BG admits a structure of CW-complex*

Theorem B.12. *Let $EG \to BG$ and $E'G \to B'G$ be two universal principal G-bundles. Then there exists an homotopy equivalence $B'G \to BG$ and a G-equivariant homotopy equivalence $E'G \to EG$ which makes the following diagram commutative*

$$\begin{array}{ccc} E'G & \longrightarrow & EG \\ \downarrow & & \downarrow \\ B'G & \longrightarrow & BG \end{array}$$

Proof. We consider maps $f : B'G \to BG$ and $g : BG \to B'G$ such that $E'G \cong f^*(EG)$ and $EG \cong g^*(E'G)$. Then the composition map $f \circ g : BG \to BG$ satisfies that

$$(f \circ g)^*(EG) \cong g^*(f^*(EG)) \cong g^*(E'G) \cong EG.$$

Therefore, by Lemma B.11 we have a bijective correspondence

$$[BG, BG] \leftrightarrow \mathcal{P}_G(BG)$$

and we obtain that the map $f \circ g$ is homotopic to the identity Id_{BG}. With a similar argument we obtain that $g \circ f$ is homotopic to the identity $Id_{B'G}$. Therefore we conclude that $f : B'G \to BG$ is a homotopy equivalence, as we wanted to see. $\square$

It can also be proved that the total space of the universal principal G-bundle has a structure of a G-CW-complex in the sense of Definition 1.4. For an explicit construction see [AP93](1.1.2). To finish this section we will consider two examples of universal principal G-bundles.

One of the most simple examples arises from $G = \mathbb{Z}$. We can consider the usual covering map of the unit sphere

$$\mathbb{R} \to S^1,$$

which is clearly a universal principal $\mathbb{Z}$-bundle, since $\mathbb{R}$ is clearly contractible.

In the discussion of the Borel construction for $G = (\mathbb{Z}_2)^n$ in Sections 1.3 and 1.4 we strongly use the case $G = \mathbb{Z}_2$. For this discussion we need to define the infinite unit sphere by

$$S^\infty = \varinjlim_n S^n.$$

Lemma B.13. *The infinite unit sphere S^∞ is contractible.*

Proof. To prove that S^∞ is contractible we will construct an homotopy between the identity on S^∞ and the constant map

$$f : S^\infty \to S^\infty$$

given by $f(x_1, x_2, x_3, \dots) = (1, 0, 0, \dots)$. Consider the linear transformation map $g : S^\infty \to S^\infty$ given by a shift of coordinates $g(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$, which is continuous because $\|g(x)\| = \|x\|$. Then we can define an homotopy between g and the identity given by

$$\begin{array}{rccc} H: & S^\infty \times I & \to & S^\infty \\ & (x,t) & \mapsto & \frac{tg(x)+(1-t)x}{\|tg(x)+(1-t)x\|} \end{array}$$

Since $f(x)$ and x are linearly independent for all $x \in S^\infty$ we have that H is well-defined, and it is clearly continuous. Moreover we have that $H(x,0) = x$ and $H(x,1) = g(x)$, thus g is homotopic to the identity. On the other hand we can consider another homotopy between g and f given by

$$\begin{array}{rccc} H: & S^\infty \times I & \to & S^\infty \\ & (x,t) & \mapsto & \frac{tf(x)+(1-t)g(x)}{\|tf(x)+(1-t)g(x)\|} \end{array}$$

which follows similarly. $\square$

We can consider a structure of G-space on S^∞ given by

$$\begin{array}{ccc} \mathbb{Z}_2 \times S^\infty & \to & S^\infty \\ (g,x) & \mapsto & gx \end{array}$$

which is clearly continuous because the restriction of the action on S^n is continuous. In particular, this action is free. Therefore, the orbit map on S^∞ induces a principal $\mathbb{Z}_2$-bundle $S^\infty \to S^\infty/\mathbb{Z}_2$ with contractible total space, hence we obtain a universal principal $\mathbb{Z}_2$-bundle

$$E\mathbb{Z}_2 = S^\infty \to S^\infty/\mathbb{Z}_2 = B\mathbb{Z}_2.$$

With this computation we finish our this introduction to universal principal G-bundles.

Bibliography

[AP93] Christopher Allday and Volker Puppe. *Cohomological Methods in Transformation Groups*. Cambridge University Press, 1993.

[Bau93] Christoph Baumgartner. “On the cohomology of free p-torus actions”. In: *Arch. Math. (Basel)* 61.2 (1993), pp. 137–149.

[Bor+60] Armand Borel et al. *Seminar on Transformation Groups. (AM-46)*. Princeton University Press, 1960. (Visited on 06/15/2023).

[Bre72] Glen E. Bredon. *Introduction to Compact Transformation Groups*. Academic Press, Inc., 1972.

[Bro82] Kenneth S. Brown. *Cohomology of Groups*. Springer-Verlag, 1982.

[Car83] G. Carlsson. “On the Homology of Finite Free $(\mathbb{Z}/2)^n$-Complexes”. In: *Inventiones mathematicae* 74 (1983), pp. 139–147.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[Mit01a] Stephen A. Mitchell. *Notes on Principal Bundles and Classifying Spaces*. 2001.

[Mit01b] Stephen A. Mitchell. *Notes on Serre fibrations*. 2001.

[Nab18] Mehdi Nabil. *Universal Principal Bundles and Classifying Spaces*. 2018.

[Ser75] Jean-Pierre Serre. *Algèbre Locale, Multiplicités*. Springer Berlin, Heidelberg, 1975.

[Smi44] P. A. Smith. “Permutable Periodic Transformations”. In: *Proceedings of the National Academy of Sciences of the United States of America* 30.5 (1944), pp. 105–108.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.