UNIVERSITAT DE BARCELONA

ADVANCED MATHEMATICS
MASTER'S FINAL PROJECT

Bridgeland Stability conditions on Surfaces

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June 28, 2023

Contents

Abstract

The main goal of this Master Thesis is to study the stability manifold on complex projective surfaces. In this case, the stability manifold is the complex manifold that parametrizes Bridgeland stability conditions on the derived category of coherent sheaves on the surface.

First of all, we present the classical study of stable sheaves on curves as the elementary model of Bridgeland stability to provide some intuition. Then we explain some basic definitions and results on triangulated categories and we construct the derived category of an abelian category.

Next, we introduce the concept of Bridgeland stability conditions and prove their existence on surfaces. The key result to prove the existence is the Bogomolov-Gieseker inequality.

The last part of this memoir explains the method that Feyzbakhsh, Li and Liu have developed to improve the Bogomolov-Gieseker inequality to enlarge the known region of the stability manifold for some specific surfaces. We have explored the possibility to apply this method to other surfaces.

Introduction

The Deligne-Mumford compactification of the moduli spaces of curves of a given genus gave rise to a big change in the way that classification or moduli problems where treated. In their original paper [DM69], Deligne and Mumford achieved the compactification on the moduli space of curves of a given genus g as the moduli space of stable curves of a given genus with marked points. A key point of this result is that this compactification had a structure of algebraic stack which allowed to treat it as an algebraic object. They achieved this "well-behaved" moduli space by imposing that the curves considered in this classification should be *stable*, ruling out all *unstable* curves from its construction. After imposing this condition, the moduli space was constructed via Geometric Invariant Theory.

Later on, the concept of stability was extended to other algebraic and geometric obtject. Stable sheaves were introduced by Mumford, and built upon by Gieseker, Maruyama and many others. The construction of moduli spaces of coherent sheaves on a fixed variety is a powerful method to construct new varieties with rich geometric properties. In particular, it has been the way to construct new hyperkähler manifolds.

In the mid 2000's, Bridgeland has extended the concept of stability to the derived category of coherent sheaves, and more generally to any triangulated category. To understand the significance of this extension is convenient to recall that the category of coherent sheaves preserves all the geometric information of a variety, while its derived category is a much subtle invariant as Mukai showed by proving that the derived category of an abelian variety is equivalent to that of its dual (although they are not in general isomorphic). The study of the derived category has a lot of interest in many areas such as representation theory, motivic theory and of course algebraic geometry among others.

Besides dealing with a very interesting object as the derived category is, stability for objects in the derived category is also very useful to study the classical moduli spaces of sheaves. Indeed, Bridgeland stability is much more flexible in the sense that the space of stability conditions forms a complex manifold which means that we can deform the stability condition in a controlled way. Moreover, although Geometric Invariant Theory is no longer available, it is still possible to obtain moduli spaces of stable objects in the derived category. For years the study of stability was reduced into finding a nice stability condition to produce a good moduli space by tuning a small number of "discrete" parameters. The perspective of Bridgeland is much more general, since we can vary the parameters continuously and control how the moduli space changes in the process.

However, Bridgeland stability conditions have a drawback. It is very difficult to prove that they exist on higher dimensional varieties. The existence of Bridgeland stability conditions is known for curves, surfaces, and some threefolds, and some very specific higher dimensional varieties (for example, the recent construction on very general abelian varieties by C. Li, E. Macrì, P. Stellari, and X. Zhao). Moreover, even if they are known to exist, studying the stability manifold is very hard and we only have a good description in few particular cases (e.g. curves, abelian surfaces, K3 surfaces of Picard rank one or varieties with finite Albanese morphism).

In this work, we start introducing the stability conditions on curves, where classical and Bridgeland stability basically coincide. Then, after introducing the derived categories and some formalism on triangulated categories, we prove that stability conditions exist for smooth projective surfaces. In the process of proving on dimension 2, we will identify the main difficulty: the (generalized) Bogomolov-Gieseker type inequalities. One of those inequalities allows to prove the support property of the stability condition (and the previous ones) and is the major struggle to prove existence. In this setting, it is possible to prove this equality and define the (α, β)-plane of stability conditions, which is a slice of the space of stability conditions $\mathrm{Stab}(X)$. Even in this simplified setting we won't have a precise description of the limits in the (α, β)-plane and we don't know whether it is possible to enlarge it.

There are conjectures about the hypothetical inequalities that may work in higher dimensional varieties but in general it remains an open problem. It has been seen that for threefolds, in some cases like specific complete intersections which are Calabi-Yau, this Bogomolov-type inequality can be reduced to the problem of improving a Bogomolov type inequality on a surface inside the threefold. This has been the main motivation to improve the known Boogomolov inequalities on surfaces. The last part of this Master thesis explains the main known method developed by Feyzbakhsh, Li and Liu to get better inequalities in the case of surfaces and we have explored if it may be applied to other surfaces. The main ingredient of this approach is to restrict further to curves and use generalized Clifford inequalities.

1. Stability on smooth curves

1.1. **Classification of vector bundles over $\mathbb{P}^1$.**

Along this work every variety of any dimension will be considered over a field k of characteristic zero. As the beginning point for the case of curves we first see how is the situation on a very simple smooth curve: the projective line. In this case we have a complete classification of vector bundles stated in the following theorem:

Theorem 1.1. *(Classification of vector bundles over $\mathbb{P}^1_k$) Let V be a vector bundle over $\mathbb{P}^1$ of rank r. Then there exist integers $k_i, r_i \geq 1$ such that*

$$V \cong \mathcal{O}(k_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}(k_q)^{\oplus r_q}$$

with $k_1 > \cdots > k_q$.

Before proceeding to the proof we need to state (without proof) three lemmas.

Lemma 1.2. *Let $\varphi : W \longrightarrow V$ an injective map of vector bundles over $\mathbb{P}^1_k$. Then there exists an extension $W \hookrightarrow W' \subset V$. such that V/W' is a vector bundle. Moreover the rank of W' and the generic rank of V/W' coincides with the rank of W and the generic rank of V/W.*

Lemma 1.3. *Let V be vector bundle over $\mathbb{P}^1_k$. Then we have an isomorphism of functors*

$$\mathrm{Ext}^i(V, -) \cong H^i(V^* \otimes -) : \mathrm{Coh}(\mathbb{P}^1_k) \longrightarrow Vect_k$$

for all $i \geq 0$.

Lemma 1.4. *Let F be an $\mathcal{O}_{\mathbb{P}^1_k}$-module. Then either F has torsion or is locally free. Moreover F fits in a short exact sequence:*

$$0 \longrightarrow F_T \longrightarrow F \longrightarrow F' \longrightarrow 0$$

where F_T is the torsion submodule and F' is a locally free sheaf.

Now we have everything ready for proving Theorem 1.1.

Proof. (Theorem 1.1) Since we want to see that V is a direct sum of Serre twisting sheaves and the tensor product is distributive respect direct sums the idea is twisting V until we reach the maximal point where we kill all nontrivial global sections (the degree of this twisted sheaf will be $-k_1$ as in the statement). From that point our goal is to isolate this larger component and work via induction on the rank (i.e. $V \cong \mathcal{O}(n_1)^{r_1} \oplus W$ with W vector bundle over $\mathbb{P}^1$). The last step is showing that this decomposition is in fact unique.

Let V be a nontrivial (otherwise $V \cong \mathcal{O}^n_{\mathbb{P}^1_k}$ and we are done) vector bundle over $\mathbb{P}^1_k$. *Step 1:* Finding k_1. If V has no global sections we can twist by $\mathcal{O}(1)$ until V becomes globally generated (this is possible because $\mathcal{O}(1)$ is ample). So we can find an m such that $V(m) := V \otimes \mathcal{O}(m)$ has global section but $V(m-1)$ does not. On the other hand if V has global sections then by Serre duality:

$$H^0(\mathbb{P}^1_k, V(-m)) \cong H^1(\mathbb{P}^1_k, V^*(m) \otimes \omega_{\mathbb{P}^1_k}) \cong H^1(\mathbb{P}^1_k, V^*(m-2))$$

By Serre's vanishing theorem there exists an integer m_0 such that $H^1(\mathbb{P}^1_k, V^*(m_0 - 2)) \neq 0$ but $H^1(\mathbb{P}^1_k, V^*(m_0 + l - 2)) = 0$ for every $l \geq 1$. This means that $V(-m_0)$ has global sections and is maximal respect to this property.

Step 2: Isolating $\mathcal{O}(k_1)^{n_1}$. We have seen that, independently of the existence of global sections of V, there is an integer m such that $V(-m)$ has nontrivial global sections but $V(-m-l)$ does not for every $l \geq 1$. Recall that $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = k$ so $H^0(\mathbb{P}^1_k, V(-m))$ is a k-vector space and then $H^0(\mathbb{P}^1_k, V(-m)) \cong k^n$ for some positive integer n. This implies that there is a basis of global sections $e_1, \ldots, e_n$ that generates $V(-m)$ locally. We pick a global section, for example, e_1. Now we define a map $\mathcal{O}_{\mathbb{P}^1_k} \longrightarrow V(-m)$ given by $1 \mapsto e_1$. This is clearly locally injective since $V(-m)$ is globally generated so it's also injective as a morphism of sheaves. Now, since we aim to decompose V in two direct summands, we wish that $\mathcal{O}$ was a sub-vector bundle. If this is the case we are in the right path. If not, since the map we've built is injective and $V(-m)$ and $\mathcal{O}_{\mathbb{P}^1_k}$ are vector bundles there is an extension $\mathcal{O}'$ of $\mathcal{O}_{\mathbb{P}^1_k}$ of rank 1 such that $\mathcal{O}'$ is a sub-vector bundle of $V(-m)$. Since $\mathcal{O}$ is a proper line bundle with global sections must be of the form $\mathcal{O}_{\mathbb{P}^1_k}(l)$ for some $l \geq 1$. But, after one negative twist, $\mathcal{O}(-1) \hookrightarrow V(-m-1)$. Recall that $V(-m-1)$ has no global sections and so does $\mathcal{O}(-1)$ but since $\mathcal{O}(-1) \cong \mathcal{O}_{\mathbb{P}^1_k}(l-1)$ with $l-1 \geq 0$ this contradicts the fact that $\mathcal{O}_{\mathbb{P}^1_k}(l-1)$ has global sections. Then $\mathcal{O}_{\mathbb{P}^1_k}$ is a sub-vector bundle of $V(-m)$. Therefore we have a short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow V(-m) \longrightarrow V(-m)/\mathcal{O}_{\mathbb{P}^1_k} \longrightarrow 0$$

If we apply the cohomology functor we get a long exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \longrightarrow H^0(\mathbb{P}^1_k, V(-m)) \longrightarrow H^0(\mathbb{P}^1, V(-m)/\mathcal{O}_{\mathbb{P}^1_k}) \longrightarrow H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \longrightarrow \cdots$$

Since $\mathcal{O}_{\mathbb{P}^1_k}$ has no higher order cohomology $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = 0$ and this implies that

$$0 \longrightarrow H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \longrightarrow H^0(\mathbb{P}^1_k, V(-m)) \longrightarrow H^0(\mathbb{P}^1, V(-m)/\mathcal{O}_{\mathbb{P}^1_k}) \longrightarrow 0.$$

Now we have that $H^0(\mathbb{P}^1_k, V(-m)/\mathcal{O}_{\mathbb{P}^1_k}) \cong H^0(\mathbb{P}^1_k, V(-m))/H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \cong k^n/k \cong k^{n-1}$. Observe that, since $V(-m-l)/\mathcal{O}_{\mathbb{P}^1_k} \hookrightarrow V(-m-l)$ and $V(-m-l)$ does not have global sections for every $l \geq 1$, $V(-m-l)/\mathcal{O}_{\mathbb{P}^1_k}$ can't have global sections as well. So m is also maximal for the same property for V and for $V/\mathcal{O}_{\mathbb{P}^1_k}$. Now we have everything set to start with induction on n. If $n = 1$ then the second lemma gives us that $\mathrm{Ext}^1(V(-m)/\mathcal{O}_{\mathbb{P}^1_k}, \mathcal{O}_{\mathbb{P}^1_k}) = H^1(\mathbb{P}^1_k, (V(m)/\mathcal{O}_{\mathbb{P}^1_k})^*)$ and by Serre duality $H^1(\mathbb{P}^1_k, (V(m)/\mathcal{O}_{\mathbb{P}^1_k})^*) = H^0(\mathbb{P}^1_k, V(-m-2)/\mathcal{O}_{\mathbb{P}^1_k}) = 0$ by the previous observation. Then we have one unique Yoneda class for the exact sequence of vector bundles so $V(-m) \cong \mathcal{O}_{\mathbb{P}^1_k} \oplus (V(-m)/\mathcal{O}_{\mathbb{P}^1_k})$. Tensoring by $\mathcal{O}_{\mathbb{P}^1_k}(m)$ we get that $V \cong \mathcal{O}_{\mathbb{P}^1_k}(m) \oplus (V/\mathcal{O}_{\mathbb{P}^1_k})$. Since $n = 1$, $V(-m)/\mathcal{O}_{\mathbb{P}^1_k}$ does not have global sections. If $n > 1$ since $H^0(\mathbb{P}^1_k, V(-m)/\mathcal{O}_{\mathbb{P}^1_k}) \cong k^{n-1}$ by induction $V(-m)/\mathcal{O}_{\mathbb{P}^1_k} = \mathcal{O}^{n-1}_{\mathbb{P}^1_k} \oplus W(-m)$ where $W(-m)$ is a vector bundle with no global sections (and neither does $W(-m-l)$ for every $l \geq 0$). The same argument that we used for $V(-m)/\mathcal{O}_{\mathbb{P}^1_k}$ shows that $\mathrm{Ext}^1(W(-m), \mathcal{O}_{\mathbb{P}^1_k}) = 0$. Since Ext^1 is linear for direct sums

$$\mathrm{Ext}^1(V(-m)/\mathcal{O}_{\mathbb{P}^1_k}, \mathcal{O}_{\mathbb{P}^1_k}) \cong \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1_k}, \mathcal{O}_{\mathbb{P}^1_k})^{n-1} \oplus \mathrm{Ext}^1(W(-m), \mathcal{O}_{\mathbb{P}^1_k})$$

and by lemma 1.2. and Serre duality

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1_k}, \mathcal{O}_{\mathbb{P}^1_k}) \cong H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \cong H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2)) \cong 0.$$

It follows that $\mathrm{Ext}^1(V(-m)/\mathcal{O}_{\mathbb{P}^1_k}, \mathcal{O}_{\mathbb{P}^1_k}) = 0$ also. Then $V(-m) \cong \mathcal{O}_{\mathbb{P}^1_k} \oplus (\mathcal{O}^{n-1}_{\mathbb{P}^1_k} \oplus W(-m))$ and tensing by $\mathcal{O}_{\mathbb{P}^1_k}(m)$ gives $V \cong \mathcal{O}_{\mathbb{P}^1_k}(m)^n \oplus W$. We set $k_1 = m$ and $r_1 = n$. Repeating the same procedure for W gives a new decomposition and n_2 and r_2. Since the rank pf V is finite this process is indeed finite so it only takes a finite amount of steps to lead to the decomposition.

The fact that this decomposition is unique follows from the fact that $V(-n_1)$ has 0-th cohomology k^{r_1} and by the maximality of n_1 respect to the existence of global sections. □

we use this result in the following section to see which are some of the slope (semis)stable bundles over $\mathbb{P}^1_k$. Also this sets the path we should follow. Despite in a general projective smooth curve this fails we can modify the statement to make a work around. Our free sumands will be replaced by semistable sheaves, the direct sum decomposition will be replaced by a filtration by semistable sheaves and the chain on decreasing integers by a chain of decreasing slopes.

1.2. **Classical slope stability.**

Now, before introducing the classical slope stability we need a couple definitions. From now on until the next chapter C will denote a smooth projective curve over a field k.

Lemma 1.5. *If E is a coherent sheaf over C then there exist T_E a torsion sheaf and F_E a vector bundle such that the sequence*

$$0 \longrightarrow T_E \longrightarrow E \longrightarrow F_E \longrightarrow 0$$

is exact.

Proof. T_E can be obtained as the kernel of the canonical map $E \to E \otimes_{\mathcal{O}_X} K_C$ where K_C is the sheaf of total quotients ring. Since E and $E \otimes K_C$ have the same stalk at the generic point η and they are $\mathcal{O}_{C,\eta}$-vector spaces of finite dimension then $(T_E)_{C,\eta} = \ker(E \to E \otimes K_C)_{C,\eta} = 0$ which implies T_E is torsion. Then we set $F_E = E/T_E$ that locally is a linear subspace of $E(U) \otimes K_X(U)$ which is torsion-free. Since we're on a curve, the stalks of the structural sheaf at a point are PID's. This implies that the stalks of F_E are in fact free and then F_E is locally free. □

Definition 1.6. Given a line bundle L over C we define its degree $d(L)$ as the degree of the associated divisor (i.e. if $L = \mathcal{O}(D)$ and $D = \sum n_i P_i$ then $d(L) = \sum n_i$). If F is a vector bundle over C we define its degree $d(F)$ as the degree of the line bundle $\wedge^{\operatorname{rk} F} F$. If T is a torsion sheaf over C we define its degree $d(T)$ as $\displaystyle\sum_{p \in C} \operatorname{length} T_p$. Previous lemma also allows us to the define the degree of any coherent sheaf over C as $d(E) := d(T_E) + d(F_E)$.

Definition 1.7. Given $E \in \operatorname{Coh}(C)$ we define the slope of E as

$$\mu(E) := \frac{d(E)}{\operatorname{rk} E}$$

If $\operatorname{rk} E = 0$ we set $\mu(E) = +\infty$. Here $\operatorname{rk} E := \operatorname{rk} F_E$.

The slope is the key for talking about stability and is one of the elements we aim to generalize. Once we have the slope we can talk about stability:

Definition 1.8. A coherent sheaf E over C is called semistable if for any proper sub-sheaf $F \subset E$ then $\mu(F) \leq \mu(E)$. Moreover if the inequality is always strict we say that E is stable.

Example 1.9. Notice that in the case of $\mathbb{P}^1$ every coherent sheaf is free and the degree is given by the degree of the twisted sheaves of Serre. This means that for example the if V is a (semi)stable sheaf of rank 1 then V can't have torsion because the torsion would be supported in dimension 0 and therefore it would have infinite slope contradicting the stability of V. Thus V must be locally free and by the classification theorem free of the form $V = \mathcal{O}(a)$ for some a. Notice that then $\mu(V) = a$.

Another important property that one would like to have in a possible generalization is the existence of a filtration by semistable sheaves:

Theorem 1.10. *Let $E \in \mathrm{Coh}(C)$ non zero. Then there exists a unique filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that E_i/E_{i-1} is semistable and $\mu(E_1/E_0) > \cdots > \mu(E_{n-1}/E_n)$. This filtration is called the Harder-Narasimhan filtration and it can be proven that is in fact unique.

Will prove this in more generality in the case of stability functions over abelian categories.

Example 1.11. We continue in the case of $\mathbb{P}^1_k$, let $E \in \mathrm{Coh}(\mathbb{P}^1_k)$ of rank 2. If $T_E \neq 0$ the filtration will look $0 \subset T_E \subset \cdots \subset E$ and we can quotient everything by T_E and our conclusions will be the same but with one factor less in the filtration so there is no problem assuming E is locally free (and by the classification theorem free) of rank 2. This implies that $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$. The first factor has to be a semistable sheaf inside V. If $a > b$ we can take $V_1 = \mathcal{O}(a)$ (that clearly destabilizes V because $a > \frac{a+b}{2}$) and $V_2 = V$ because $\mathcal{O}(a) \oplus \mathcal{O}(b)/\mathcal{O}(a) \cong \mathcal{O}(b)$ and $\mu(V_1) = a > b = \mu(V_2/V_1)$. If $a = b$ then we want to see that V is already semistable. In this case, since V has no torsion the only possible destabilizing object has to be $\mathcal{O}(c) \subset V$. Since it destabilizes V then $c > a$. This gives a short exact sequence

$$0 \longrightarrow \mathcal{O}(c) \longrightarrow \mathcal{O}(a) \oplus \mathcal{O}(b) \longrightarrow Q \longrightarrow 0.$$

Twisting by $-c$ we have

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(a-c) \oplus \mathcal{O}(b-c) \longrightarrow Q(-c) \longrightarrow 0.$$

Applying $H^0(\mathbb{P}^1_k, -)$ we have that $H^0(\mathbb{P}^1_k, \mathcal{O}(a-c) \oplus \mathcal{O}(b-c)) = 0$ which implies $H^0(\mathbb{P}^1_k, \mathcal{O}) = 0$ that is a contradiction. This shows that V is semistable if $a = b$. From this example one may be convinced that this is the right way to go in order to generalize theorem 1.1.

1.3. **A way to generalize stability.** This more classical stability can be generalized to higher dimension as follows. The first key fact is that for a coherent sheaf E over C, $\mathrm{ch}_0(E) = \mathrm{rk}\, E$ and $\mathrm{ch}_1(E) = d(E)$. So one would think that the Chern character is a good point to start in the path of generalizing slope stability. Since we want a number the idea is basically to multiply our rank and our degree with a divisor so we end up in the top cohomology class which ends up being $\mathbb{Z}$ because X is a smooth projective surface and hence connected.

Definition 1.12. In the case of surfaces we take two divisors $\omega, B \in N^1(X) := NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (where $NS(X)$ is the Néron-Severi group of Cartier divisors up to numerical equivalence) with ω being ample. Then we set

$$\mu_{\omega,B}(E) := \frac{\omega \cdot \mathrm{ch}_1^B(E)}{\omega^2 \cdot \mathrm{ch}_0^B(E)} = \frac{\omega \cdot \mathrm{ch}_1(E)}{\omega^2 \cdot \mathrm{ch}_0(E)} - \frac{\omega \cdot B}{\omega^2}$$

where $\mathrm{ch}_i^B(E) := \mathrm{ch}_i(E)e^{-B}$ is the twisted Chern character.

Notice that this is an integer because all quantities are in the top class of the Chow ring an therefore we can take its degree. So this is in fact, as in the case of curves, a rational number. Stability now is not exactly defined in the same way as before but in a weaker form and we refer to this stability in particular as slope stability. In this case we say that $E \in \mathrm{Coh}(X)$ is slope semistable if for all $F \subset E$ we have that $\mu_{\omega,B}(F) \leq \mu_{\omega,B}(E/F)$ and that is stable if the inequality is always strict.

Notice that in the case of X a smooth curve (taking $\mu = \mathrm{ch}_1^B / \omega \cdot \mathrm{ch}_0^B$) this notion coincides for all divisors up to a positive constant so all slopes defined this way end up being equivalent to classical slope.

2. Stability over smooth surfaces

2.1. **Triangulated categories.**

First of all, before introducing Bridgeland stability and since we mainly work over the bounded derived category of coherent sheaves, we talk about it and its triangulated structure. This is a little survey on triangulated categories and the construction of the derived category of an abelian category. we only present the basic notions and results needed to prove further results on stability.

Definition 2.1. Let $\mathcal{T}$ be an additive category (i.e a category where there is a zero object, finite coproducts exist and all Hom carry a structure of abelian group composite-distributive). We say that $\mathcal{T}$ has a triangulated structure or simply that it is triangulated if it has a shift functor $[1] : \mathcal{T} \longrightarrow \mathcal{T}$ (i.e additive and auto-equivalent) and a class of distinguished triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

that satisfies the following axioms

T1 (a) For any $A \in \mathcal{T}$ the triangle $A \xrightarrow{id} A \longrightarrow 0 \longrightarrow A[1]$ is distinguished.
(b) Any triangle isomorphic to a distinguished one is also distinguished.
(c) Any morphism $f : A \to B$ completes to a distinguished triangle $A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$.

T2 $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is distinguished iff $B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1]$ is distinguished

T3 For any commutative diagram of distinguished triangles of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, dashed, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

there exists a morphism γ that completes it to a morphism of triangles

T4 (Octahedral Axiom) For any pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ there exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[r, "h"] \arrow[d, "g"] & C' \arrow[d] \arrow[dr] & \\
& C \arrow[r] \arrow[dr] & B' \arrow[r] \arrow[d] & A[1] \arrow[d, "{f[1]}"] \\
& & A' \arrow[r] \arrow[d] \arrow[dr] & B[1] \\
& & B[1] \arrow[r, "{h[1]}"] & C'[1]
\end{tikzcd}$$

where all triangles beginning at A, B and C' are distinguished.

Remark 2.2.

- We don't know if the octahedral axiom is independent from the previous three. Usually if $\mathcal{T}$ satisfies **T1** − **T3** is called pre-triangulated but it's not known if there are pre-triangulated categories which are not triangulated.
- The octahedral axiom can be presented in a different way: Denote by $C(f)$ (the cone of f) the object from $\mathcal{T}$ resulting from the completion of $A \xrightarrow{f} B$ into a distinguished triangle. Suppose we're given a pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$. Then the octahedral axiom is equivalent to the triangle

  $$C(f) \longrightarrow C(g \circ f) \longrightarrow C(g) \longrightarrow C(f)[1]$$

  being distinguished. If one sets $C' = C(f), B' = C(g \circ f), A' = C(g)$ this becomes clear.

The following consequences of the definition will be very useful.

Proposition 2.3. *Let $A \xrightarrow{f} B \xrightarrow{g} C \to A[1]$ a distinguished triangle in $\mathcal{T}$. Then for any A_0 the sequences*

$$\mathrm{Hom}(A_0, A) \longrightarrow \mathrm{Hom}(A_0, B) \longrightarrow \mathrm{Hom}(A_0, C)$$

and

$$\mathrm{Hom}(C, A_0) \longrightarrow \mathrm{Hom}(B, A_0) \longrightarrow \mathrm{Hom}(A, A_0)$$

are exact.

Proof. If $h : A_0 \to B$ such that $g \circ h = 0$. By **T1** and **T3** we have a morphism $m : A_0 \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
A_0 \arrow[r, "\mathrm{id}"] \arrow[d, dashed, "m"] & A_0 \arrow[r] \arrow[d, "h"] & 0 \arrow[r] \arrow[d] & A_0[1] \arrow[r, "{\mathrm{id}[1]}"] \arrow[d, dashed, "{m[1]}"] & A_0[1] \arrow[d, "{h[1]}"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & A[1] \arrow[r, "{f[1]}"] & B[1]
\end{tikzcd}$$

Therefore $f \circ m = h$ and we're done with $\ker \subset \mathrm{Im}$. The other inclusion follows from the fact that $g \circ f = 0$ which is also a consequence of **T1** and **T3**. Indeed, since we have the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & A[1] \end{tikzcd}$$

we have $g \circ f = 0$. The second exact sequence is proved in the same way but now the triangle $A_0 \to A_0 \to 0 \to A_0[1]$ is in the bottom row, A goes to 0 and we complete with a morphism $m' : C \to A_0$. ☐

Proposition 2.4. *If we have a morphism of triangles*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

where two of the three vertical arrows are isomorphisms so is the third.

Proof. By **T2** we can suppose without any kind of loss of generality that this isomorphisms are α and β. Applying $\mathrm{Hom}(C', -)$ we have

$$\begin{tikzcd} \mathrm{Hom}(C',A) \arrow[r] \arrow[d, "\alpha\circ-"] & \mathrm{Hom}(C',B) \arrow[r] \arrow[d, "\beta\circ-"] & \mathrm{Hom}(C',C) \arrow[r] \arrow[d, "\gamma\circ-"] & \mathrm{Hom}(C',A'[1]) \arrow[r] \arrow[d, "\alpha[1]\circ-"] & \mathrm{Hom}(C',B[1]) \arrow[d, "\beta[1]\circ-"] \\ \mathrm{Hom}(C',A') \arrow[r] & \mathrm{Hom}(C',B') \arrow[r] & \mathrm{Hom}(C',C') \arrow[r] & \mathrm{Hom}(C',A'[1]) \arrow[r] & \mathrm{Hom}(C',B'[1]) \end{tikzcd} .$$

By the five lemma $\gamma \circ -$ is an isomorphism and hence γ has a right inverse. Applying $\mathrm{Hom}(-, C)$ and the same argument we get a left inverse. Therefore γ is an isomorphism. ☐

Example 2.5. Our main example of triangulated category is the derived category of an abelian category. In particular when we take this abelian category to be the category of coherent sheaves over a curve or a surface.

Also it would be nice to recall the construction of the derived category. we explain the process via "localization". We use this term because the idea is the same as the classical localization of rings. First we begin with an abelian category $\mathcal{A}$. Now this gives new categories $\mathrm{Kom}(\mathcal{A})$ and $\mathrm{Kom}^b(\mathcal{A})$ which are the categories of complexes and bounded complexes respectively. The objects of this categories are (cohomological) complexes and bounded complexes respectively and the morphism are collections of morphisms in $\mathcal{A}$ that commute with the differential of the complex i.e a morphism in $\mathrm{Kom}^*(\mathcal{A}), f : (A^\bullet, d_A) \longrightarrow (B^\bullet, d_B)$ with $* \in \{\ , b\}$ is a collection $\{f^k : A^k \longrightarrow B^k\}$ of morphisms such that $f^{k+1} \circ d_A = d_B \circ f^k$. Notice that the category of bounded complexes is a full additive sub-category of $\mathrm{Kom}(A)$ (both are additive since $\mathcal{A}$ is so). Then, since we have complexes and additivity, we can talk about homotopy.

Notice that we can equip $\mathrm{Kom}(\mathcal{A})$ with a shift functor taking $A^\bullet[1]^k := A^{k+1}$ with differential $d[1]^k := -d^{k+1}$ and for morphisms just taking $f[1]^k := f^{k+1}$. As in the classical definition of topological homotopy, a morphism is homotopic to 0 if there is a collection of morphisms $\{s^k : A^k \longrightarrow B^{k-1}\}$ such that $f^k = d_B s^k + s^{k+1} d_A$ an two morphisms f, g are homotopic if their difference is homotopic to zero. From this relation we can define the homotopy category of $\mathcal{A}$, denoted $K(A)$, which objects are the ones from $\mathrm{Kom}(\mathcal{A})$ and their morphisms are classes of homotopy of those from $\mathrm{Kom}(\mathcal{A})$. Note that we still have $K^b(\mathcal{A})$ as an additive full subcategory of $K(\mathcal{A})$. One could ask *why?*. Why do we need homotopy? The answer is clear: $K(A)$ can be equipped with a natural structure of triangulated category with the shift functor of $\mathrm{Kom}(\mathcal{A})$ because, in opposition to $\mathrm{Kom}(\mathcal{A})$, we can now produce cones of morphisms that are well-behaved. If $f : A^\bullet \longrightarrow B^\bullet$ is a morphism in $\mathrm{Kom}(A)$ take the following complex:

$$(C(f)^\bullet)^k := A^{k+1} \oplus B^k$$

with differential

$$d^k_{C(f)} := \begin{pmatrix} d_A^{k+1} & 0 \\ f^{k+1} & d_B^k \end{pmatrix}$$

We have a natural injection $\iota(f) : B^\bullet \to C(f)^\bullet$ and a natural projection $\pi(f) : C(f)^\bullet \to A^\bullet[1]$ that behave well respect to the differential. In order to produce a triangulated structure we need the existence of a morphism between the triangles

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\iota(f)} C(f)^\bullet \longrightarrow A^\bullet[1]$$

and

$$C(\iota(f))^\bullet[-1] \xrightarrow{f} B^\bullet \xrightarrow{\iota(f)} C(f)^\bullet \longrightarrow C(\iota(f))^\bullet.$$

Propositions 2.3 and 2.4 force this morphism to be an isomorphism if it exists. This is where homotopy enters. We can produce such a morphism in $\mathrm{Kom}(\mathcal{A})$ but we will only be able to assure that is an isomorphism up to homotopy. Basically we take the morphism

$$\phi := \begin{pmatrix} -f[1] \\ \mathrm{id}_{A^\bullet[1]} \\ 0 \end{pmatrix} : A^\bullet[1] \longrightarrow C(\iota(f)) = B[1] \oplus C(f) = B[1] \oplus A[1] \oplus B$$

with inverse-homotopy map

$$\psi := (0, \mathrm{id}_{A^\bullet[1]}, 0) : C(\iota(f))^\bullet \longrightarrow A^\bullet[1].$$

Despite $\psi \circ \varphi = \mathrm{id}_{A^\bullet[1]}$ and $\phi \circ \psi \neq \mathrm{id}_{C(\iota(f))^\bullet}$ in general, as we said before, they are homotopic via

$$s = \begin{pmatrix} 0 & 0 & \mathrm{id}_{B^\bullet} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On the one hand we have

$$\mathrm{id}_{C(\iota(f))^\bullet} - \phi \circ \psi = \begin{pmatrix} \mathrm{id}_{B^\bullet[1]} & 0 & 0 \\ 0 & \mathrm{id}_{A^\bullet[1]} & 0 \\ 0 & 0 & \mathrm{id}_{B^\bullet} \end{pmatrix} - \begin{pmatrix} 0 & -f[1] & 0 \\ 0 & \mathrm{id}_{A^\bullet[1]} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{B^\bullet[1]} & f[1] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{B^\bullet} \end{pmatrix}.$$

On the other hand

$$s[1] \circ d_{C(\iota(f))^\bullet} = \begin{pmatrix} 0 & 0 & \mathrm{id}_{B^\bullet[1]} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} d_{B^\bullet[1]} & 0 & 0 \\ 0 & d_{A^\bullet}[1] & 0 \\ \mathrm{id}_{B^\bullet}[1] & f[1] & d_{B^\bullet} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{B^\bullet}[1] & f[1] & d_{B^\bullet} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$d_{C(\iota(f))^\bullet}[-1]\circ s = \begin{pmatrix} d_{B^\bullet} & 0 & 0 \\ 0 & d_{A^\bullet} & 0 \\ -\mathrm{id}_{B^\bullet} & -f & d_{B^\bullet[-1]} \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & \mathrm{id}_{B^\bullet} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & d_{B^\bullet} \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_{B^\bullet} \end{pmatrix}$$

so

$$s[1]\circ d_{C(\iota(f))^\bullet} - d_{C(\iota(f))^\bullet}[-1]\circ s = \begin{pmatrix} \mathrm{id}_{B^\bullet}[1] & f[1] & d_{B^\bullet} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & d_{B^\bullet} \\ 0 & 0 & 0 \\ 0 & 0 & -\mathrm{id}_{B^\bullet} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathrm{id}_{B^\bullet[1]} & f[1] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{B^\bullet} \end{pmatrix} = \mathrm{id}_{C(\iota(f))^\bullet} - \phi\circ\psi$$

With this in mind we have now a very clear class of distinguished triangles for $K(\mathcal{A})$: those that are isomorphic to the cone of a morphism $f : A^\bullet \longrightarrow B^\bullet$. This class along with the shift functor give a triangulated structure on $K(\mathcal{A})$.

Now the second big step in constructing the derived category is localizing. Since we want to keep track of the cohomology of the complex, we want to somehow relate objects that share the same cohomology in a suitable way. This is the main goal of the derived category. First of all, since $\mathcal{A}$ is abelian, we have a cohomology functor $H^\bullet : K(\mathcal{A}) \longrightarrow K(\mathcal{A})$ that sends an object $(A^\bullet, d_A)$ to $H^\bullet(A^\bullet, d_A) := \ker(d_A)/\operatorname{Im}(-d_A[-1])$. Defining the map $H^\bullet(f)$ is quite more delicate since we need the Freyd-Mitchell embedding theorem that says that any small abelian subcategory can be fully embedded into the category of modules over some ring in a way that kernels and cokernels are preserved. This allows to define this morphism as the usual way if we were working over modules and then pulling it back to $K(\mathcal{A})$. The usual way basically sends a morphism $f : A \to B$ to the morphism $H(f) : H^\bullet(A^\bullet) \to H^\bullet(B^\bullet)$ defined by $H(f)^k([a]) := [f^k(a)]$. This is well defined since $a \in \operatorname{Im}(d_{A^\bullet}^{k-1})$ implies that $a = d_{A^\bullet}^{k-1}(b)$. Then $f^k(a) = f^k(d_{A^\bullet}^{k-1}(b)) = d_{B^\bullet}^{k-1}(f^{k-1}(b)) \in \operatorname{Im}(d_{B^\bullet}^{k-1})$. This functor is called the canonical cohomology functor. If $H^\bullet(A^\bullet) = 0$ we say that $A^\bullet$ is acyclic and if $H^\bullet(f)$ is an isomorphism, then we say that f is a quasi-isomorphism. This two concepts are deeply related as the following proposition states:

Proposition 2.6. *A morphism $f : A^\bullet \longrightarrow B^\bullet$ is a quasi isomorphism if and only if $C(f)$ is acyclic.*

Proof. Recall that by definition of $C(f)^\bullet$ we have an exact sequence

$$0 \longrightarrow B^\bullet \xrightarrow{\iota(f)} C(f)^\bullet \xrightarrow{\pi(f)} A^\bullet[1] \longrightarrow 0.$$

As in the module case, the snake lemma gives a long exact sequence on cohomology

$$\cdots \to H^{-1}(A^\bullet[1]) \xrightarrow{H^0(f)} H^0(B^\bullet) \xrightarrow{H^0(\iota(f))} H^0(C(f)^\bullet) \xrightarrow{H^0(\pi(f))} H^0(A^\bullet[1]) \xrightarrow{H^1(f)}$$

$$\xrightarrow{H^1(f)} H^1(B^\bullet) \xrightarrow{H^1(\iota(f))} H^1(C(f)^\bullet) \xrightarrow{H^1(\pi(f))} H^1(A^\bullet[1]) \longrightarrow \cdots$$

Therefore $H^\bullet(C(f)^\bullet) = 0$ iff $H^\bullet(f)$ is an isomorphism. ∎

So our goal now is to invert all quasi-isomorphisms. For this purpose it's more convenient to change our morphisms in a way that we still have a copy of all morphisms and then invert those that we want (in our case the quasi-isomorphisms). First we need to check that our class of candidates to isomorphisms behaves well, i.e forms a multiplicative system in the following sense:

Definition 2.7. A family of morphisms $\mathcal{S}$ in a category $\mathcal{C}$ is a multiplicative system if:

S1. For all $X \in \mathcal{C},\ id_X \in \mathcal{S}$.
S2. If $f, g \in \mathcal{S}$ and $g \circ f$ exists, then it belongs to $\mathcal{S}$.
S3. i For any pair of morphisms $X \xrightarrow{f} Y, Z \xrightarrow{g} Y$ there exist morphisms $W \xrightarrow{h} X, W \xrightarrow{i} Z$ such that $f \circ h = i \circ g$ and $h \in \mathcal{S}$.
ii For any pair of morphisms $Y \xrightarrow{f} X, Y \xrightarrow{g} Z$ there exist morphisms $X \xrightarrow{h} W, Z \xrightarrow{i} W$ such that $h \circ f = g \circ i$ and $h \in \mathcal{S}$.
S4. If $f, g : X \to Y$ then there exists $t \in \mathcal{S}$ such that $t \circ f = t \circ g$ if and only if there exists $s \in \mathcal{S}$ such that $f \circ s = g \circ s$

If we have a multiplicative system $\mathcal{S}$ we can invert all arrows in it in the following way: For the new localized category $\mathcal{C}_\mathcal{S}$ take the same objects as in $\mathcal{C}$ and morphisms

$$\mathrm{Hom}_{\mathcal{C}_\mathcal{S}}(X, Y) := \{(X', s, f) \mid X' \in \mathcal{C}, s : X' \to X, f : X' \to Y, s \in \mathcal{S}\}/\sim$$

where two triplets $(X', s, f), (X'', t, g)$ are related if and only if there exist $s' : X''' \to X', t' : X''' \to X''$ such that the following diagram commutes

$$\begin{tikzcd}
 & X & \\
X' \arrow[ur, "s"] \arrow[dr, "f"'] & X''' \arrow[l, "s'"'] \arrow[r, "t'"] \arrow[u, "\in\mathcal{S}"'] & X'' \arrow[ul, "t"'] \arrow[dl, "g"] \\
 & Y &
\end{tikzcd}$$

This may seem unnatural but if one makes an abuse of notation omitting X' and writing (X', s, f) as (s, f) this reminds to the usual localization. Moreover $(s, f) \sim (t, g)$ iff there are $s', t' \in \mathcal{S}$ such that $ss' - tt' = 0, fs' - gt' = 0$ which reminds to the usual definition. Composition is defined in the natural way: $(X', s, f) : X \to Y$ and $(Y', t, g) : Y \to Z$. By S3, there are morphisms $t' : X'' \to X', h : X'' \to Y'$ such that $f \circ t' = t \circ h$ and $t' \in \mathcal{S}$ this means that $s \circ t' \in \mathcal{S}$ by S2. Therefore we can set $(Y', t, g) \circ (X', s, f) := (X'', t' \circ s, g \circ h)$. Although we won't prove it, localization preserves additivity (meaning that if $\mathcal{C}$ is additive so is $\mathcal{C}_\mathcal{S}$) and also full sub-categories (meaning that if $\mathcal{C}'$ is a full sub-category of $\mathcal{C}$ so is it $\mathcal{C}'_\mathcal{S}$ from $\mathcal{C}_\mathcal{S}$). Notice that there is a functor $Q : \mathcal{C} \to \mathcal{C}_\mathcal{S}$ which is the identity on objects and sends a morphism $f : X \to Y$ to $Q(f) := (X, id_X, f)$. In fact $\mathcal{C}_\mathcal{S}$ has the universal localization property:

Proposition 2.8. *Let $\mathcal{C}$ be a category, and $\mathcal{C}_\mathcal{S}$ it's localization by a multiplicative system $\mathcal{S}$. Then $Q(s)$ is an isomorphism for all $s \in \mathcal{S}$. Moreover if $F : \mathcal{C} \to \mathcal{C}'$ is a functor that sends all $s \in \mathcal{S}$ to isomorphisms the F factors through $\mathcal{C}_\mathcal{S}$.*

Proof. First notice that $Q(s) := (X, id_X, s)$ and $(X, s, id_X) \circ (X, id_X, s) = (X, id_X, id_X)$ taking $t' = h = id_X$ in the definition of the composition. For the second assertion if such a factorization exists there must be a functor $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{C}'$ such that $F = G \circ Q$. This means that on objects $F(X) = G(Q(X)) = G(X)$. Now if $(X', s, f) : X \to Y$, since $(X', s, f) \circ Q(s) = (X', s, f) \circ (X', id_{X'}, s) = (X', id'_X, f) = Q(f)$ were we have taken $t' = id_{X'} = h$. Applying G we get $G(X', s, f) \circ F(s) = F(f)$ so we can define $G(X', s, f) := F(f) \circ (F(s)^{-1})$.

Now we return to the triangulated categories. With the extra triangulated structure there is an easy way of constructing multiplicative systems relying on null systems:

Definition 2.9. Let $\mathcal{T}$ be a triangulated category and $\mathcal{N}$ a family of objects in $\mathcal{T}$. We say that $\mathcal{N}$ is a null system if:

N1. $0 \in \mathcal{N}$
N2. $A \in \mathcal{N} \iff A[1] \in \mathcal{N}$.
N3. If two of the tree objects of a distinguished triangle are in $\mathcal{N}$ so is the third.

From a null system we can construct a multiplicative system

$$S(\mathcal{N}) := \{f : A \to B \mid C(f) \in \mathcal{N}\}.$$

Checking that in fact this is a multiplicative system can be a little tedious so we avoid it. A proof can be found in [Sos12]. Returning to our homotopy category $K(\mathcal{A})$, if we take as a null system $\mathcal{N}$ the acyclic objects in $K(\mathcal{A})$, by proposition 2.6 we have that $S(\mathcal{N})$ consist precisely in all quasi-isomorphisms of $K(\mathcal{A})$. Then we define the derived category of $\mathcal{A}$ as $\mathcal{D}(A) := K(\mathcal{A})_{S(\mathcal{N})}$. Although the construction may seem complicated at first sight there is nothing strange here: the objects of $\mathcal{D}(\mathcal{A})$ are complexes and the morphisms are those from $K(\mathcal{A})$ inverting the quasi-isomorphisms. Also it inherits a structure of triangulated category from $K(\mathcal{A})$ thanks to the compatibility of the null system respect to the distinguished triangles where the shift functor is still the same and the distinguished triangles are the ones that come from a triangle in $K(\mathcal{A})$ via the canonical localization functor $Q : K(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. Furthermore this functor becomes exact. We also have the bounded derived category $\mathrm{D^b}(\mathcal{A})$ and an exact functor $Q : K^b(\mathcal{A}) \to \mathrm{D^b}(\mathcal{A})$. we mainly work over this last one.

2.2. **Bridgeland Stability on surfaces.** Before entering into Bridgeland stability we need a little survey on stability over abelian categories. Here since we are on a quite general framework there is no natural choice for a slope function so we simply consider those functions that behave like slope functions. This is one of the main ideas behind Bridgeland stability.

Definition 2.10. Let $\mathcal{A}$ be an abelian category, $K_0(\mathcal{A})$ its Grothendieck group (i.e $\mathbb{Z}\langle\mathcal{A}\rangle/\sim$ where $A \sim B + C$ iff there are $B, C \in \mathcal{A}$ such that $0 \to B \to A \to C \to 0$ is a short exact sequence). A (full numerical) stability function on $\mathcal{A}$ is an additive morphism $Z : K_0(\mathcal{A}) \to \mathbb{C}$ such that

- $\Im Z \geq 0$.
- If $\Im Z = 0$ then $\Re Z < 0$.

From this we call $R := \Im Z$ the generalized rank, $D := -\Re Z$ the generalized degree and $M = D/R$ the generalized slope where, when $R = 0$, we set $M = +\infty$ as in the previous cases.

Now that we have a stability function we can define stability:

Definition 2.11. If $Z : K_0(\mathcal{A}) \to \mathbb{C}$ is a stability function and $E \in \mathcal{A}$ we say that E is semistable if for all nontrivial sub-objects $F \subset E$ we have $M(F) \leq M(E)$. If the inequality holds strictly for all sub-objects then we say that E is stable.

Example 2.12. One easy example of a stability function is taking $\mathcal{A} = \operatorname{Coh}(C)$ and $Z(E) := -\deg E + \sqrt{-1}\operatorname{rk} E$ where C is a smooth projective curve. In this case stability coincides with the classical slope stability that we saw in the first chapter. We can also go further and define

$$\overline{Z}_{\omega,B} := -\omega \operatorname{ch}_1^B(E) + \sqrt{-1}\omega^2 \operatorname{ch}_0^B(E)$$

for $E \in \operatorname{Coh}(X)$ where X is a surface. Note that this is not a stability function because if T is a 0-dimensional supported torsion sheaf then $\operatorname{ch}_0(E) = 0$ because is torsion and $\operatorname{ch}_1(E) = 0$ because is supported in dimension 0. This means that the condition $\Im\overline{Z}_{\omega,B} = 0 \Longrightarrow \Re\overline{Z}_{\omega,B} < 0$ fails. In order to have a well-defined stability function we have to change our abelian category. Instead of taking $\operatorname{Coh}(X)$ we can take $\mathcal{A} := \operatorname{Coh}_{2,1}(X) := \operatorname{Coh}(X)/\operatorname{Coh}(X)_{\leq 0}$. This is the Serre quotient of $\operatorname{Coh}(X)$ by the Serre subcategory $\operatorname{Coh}(X)_{\leq 0}$ of coherent sheaves supported in dimension 0. Here a Serre subcategory means a full subcategory of an abelian category such that for any exact sequence $A \longrightarrow B \longrightarrow C$ where A, C belong to the Serre subcategory, then B also does. The construction of the quotient category is done via localization taking the multiplicative system

$$\mathcal{S} := \{(f : E \to F) \in \operatorname{Coh}(X) \mid \ker f, \operatorname{coker} f \in \operatorname{Coh}(X)_{\leq 0}\}.$$

Then $\operatorname{Coh}_{2,1}(X)$ is nothing but $\operatorname{Coh}(X)$ but we are "identifying" all sheaves that "differ" in a 0-dimensional closed sub-scheme (because we are inverting morphisms that fail to be isomorphisms on a 0-dimensional closed sub-scheme). Now $\overline{Z}_{\omega,B}$ is a well-defined stability function (we see this later but for the moment is not clear that the image lives in the region of the complex plane that we want). Furthermore the stable and semistable objects coincide with slope-stable and slope-semistable sheaves.

One could ask also why we define stability in terms of sub-objects but not in terms of quotient objects. In turns out that they are dual notions:

Lemma 2.13. *Let $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ a stability function. An object $E \in \mathcal{A}$ is (semi)stable if and only if for every $E \twoheadrightarrow Q$, $M(E) < (\leq) M(Q)$.*

Proof. This is a trivial fact from complex numbers: Take $F \subset E$. Then $Z(E) = Z(F) + Z(E/F) = Z(F) + Z(Q) = (a_F + a_Q) + (b_F + b_Q)\sqrt{-1}$. So we have that E is (semi)stable iff $-a_F/b_F < (\leq) - (a_F + a_Q)/(b_F + b_Q)$ holds iff $a_F b_Q > (\geq) a_Q b_F$ holds iff $-a_Q/b_Q < (\leq) - (a_F + a_Q)/(b_F + b_Q)$ holds. ☐

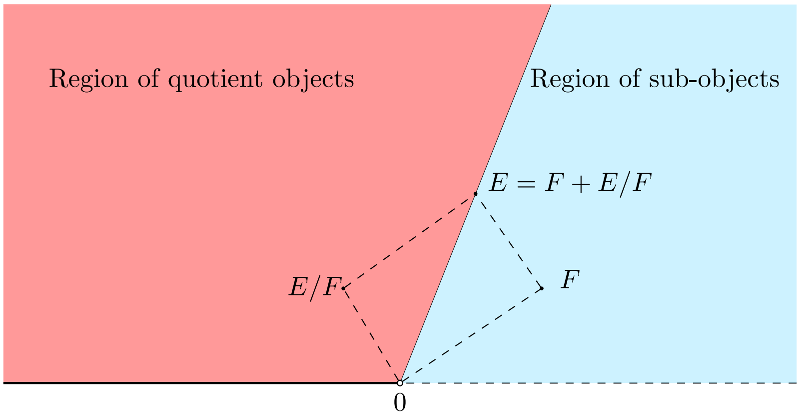
Remark 2.14. The proof of this lemma shows that we could have defined that an object E is (semi)stable satisfies that for any $F \subset E$, $M(F)(<) \leq M(E/F)$. This is because they satisfy axiomatically $a_F b_Q > (\geq) a_Q$ which is equivalent to $M(F) < (\leq) M(E)$ and $M(E) < (\leq) M(Q)$.

This allows us to prove that semi-stability has a significant impact on morphisms:

Lemma 2.15. *Let $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ a stability function. If $A, B \in \mathcal{A}$ are non-zero semistable objects with $M(A) > M(B)$ then* $\operatorname{Hom}(A, B) = 0$.

Proof. If $f : A \to B$ then $\operatorname{Im} f$ is a quotient of A and a sub-object of B. Therefore, by the previous lemma and the semi-stability of B, $M(A) \leq M(Q) \leq M(B)$ which contradicts the hypothesis on the slopes. □

Here there is a very useful geometric intuition: our semistable objects can be represented by semi-lines in the image of Z passing through 0 and $Z(E)$ whose slope is precisely $M(E)$ and the sub-objects and quotient objects can be seen as complex points living in two regions separated by that semi-line.



Recall that in the picture the objects in $K_0(\mathcal{A})$ are identified with their image via Z. Also notice that not all points in a region correspond necessarily to a quotient/sub-object of E. In fact the imaginary part of all sub-objects and quotient objects is bounded above by the imaginary part of $Z(E)$ as a consequence of its additivity.

With this intuition in mind we can pass to the following step: finding a Harder-Narasimhan filtration in our current setting.

Definition 2.16. Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be a stability function. A Harder-Narasimhan filtration for Z and $E \in \mathcal{A}$ is finite collection $\{E_i\}_{i=1}^n$ of objects of $\mathcal{A}$ such that $E_i \subset E_{i+1}, E_0 = 0, E_n = E$, $A_{i+1} := E_{i+1}/E_i$ is semistable for all $i = 0, \ldots, n-1$ and $M(A_1) > \cdots > M(A_n)$.

With all these ingredients we can talk about stability conditions:

Definition 2.17. A stability condition is a pair $(\mathcal{A}, Z)$ where $\mathcal{A}$ is an abelian category and Z a stability function on $\mathcal{A}$ such that for all non-zero object E of $\mathcal{A}$ there exists a Harder-Narasimhan filtration for Z and E.

The fact that the Harder-Narasimhan filtration is unique is not hard to proof. If E is already semistable the only possible filtration is $0 \subset E$ because lemma 2.13 assures that any filtration of the form $0 \subset F \subset E$ must satisfy that $M(E) \leq M(E/F) < M(F)$ which is impossible because of the semi-stability of E. Thus if E is semi-stable the unique possible filtration has to be $0 \subset E_1 = E$ because E_1 being proper leads to contradiction.. Then, by induction we only have to check that if $\{E_i\}_{i=1}^n$ and $\{E_i'\}_{i=1}^n$ are two Harder-Narasimhan filtration of E then $E/E_1 \cong E/E_1'$ or equivalently $E_1 = E_1'$. Notice that both E_1 and E_1' are semi-stable. If

$M(E_1) \neq M(E_1')$, for example $M(E_1) > M(E_1')$ then by lemma 2.15 $\mathrm{Hom}(E_1, A_i') = 0$ for all $i = 1, \dots, n$. Then any morphism $f : E_1 \to E$ is zero in E/E_{n-1}' so $f : E_1 \to E_{n-1}'$ but this is zero in E_{n-1}/E_{n-2} and so on until we reach that $f : E_1 \to E_1'$ that implies $f = 0$ by assumption. This is impossible since $E_1 \subseteq E$ is distinct from zero. Therefore $M(E_1) = M(E_1') > M(A_i')$ for all $i = 2, \dots, n$. Applying the same argument as before we get that $E_1 \subseteq E_1'$ and $E_1' \subseteq E_1$ which leads to $E_1 = E_1'$.

Now we would like a nice condition on $\mathcal{A}$ and Z so that we have always have a Harder-Narasimhan filtration. The nice condition on $\mathcal{A}$ turns out to be the following one.

Definition 2.18. An abelian category $\mathcal{A}$ is noetherian if any object $A \in \mathcal{A}$ has the ascending chain condition. This means that any ascending chain of sub-objects of A

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots A_i \subseteq \cdots \subseteq A$$

reaches an $i \gg 0$ such that $A_i = A_{i+l}$ for all $l \geq 0$.

This condition is good enough because if it holds we don't need to impose any extra condition on Z to get a Harder-Narasimhan filtration for any object. To prove this we proceed in various steps.

Lemma 2.19. *If $Z : K_0(\mathcal{A}) \to \mathbb{C}$ is a stability function on an abelian noetherian category $\mathcal{A}$ and $\Im Z$ is discrete in $\mathbb{R}$, then for any $E \in \mathcal{A}$ there exists a upper bound $D_E \in \mathbb{R}$ for $-\Re Z|_{\{F \subset E\}}$.*

Proof. Recall that $\Im Z = R$ and $-\Re Z = D$. The discreteness hypothesis allows to work via induction on $R(E)$. If $R(E) = 0$, since $R(F)$ with $F \subset E$ is bounded above by $R(E)$, we get that $R(F) = 0$. By additivity it follows that $D(F) \leq D(E)$ so we can take $D_E := D(E)$. Now suppose $R(E) > 0$ and that there is no such a bound. This implies that there is an infinite sequence of sub-objects $\{F_i\}_{i \in \mathbb{N}}$ whose generalized degree D goes to infinity if i goes to infinity. Now if $R(F_i) = R(E)$ then $R(E/F_i) = 0$ and the second condition of stability functions guarantees that $D(E/F_i) \geq 0$. Again by additivity this implies $D(F_i) \leq D(E)$. This means that we can suppose that $R(F_i) < R(E)$ for all i because otherwise the degree is bounded by the degree of E. Notice that the fact that the degree is not bounded above is incompatible with the noetherianity of $\mathcal{A}$ so the idea now is basically construct another sequence $\{F_i'\}$ with the same properties of $\{F_i\}$ but also with $F_i' \subseteq F_{i+1}'$ for all $i \in \mathbb{N}$ contradicting directly the ascending chain condition on E.

For F_1' we take an element of $\{F_i\}$ such that $D(F_i) \geq 1$. Now we proceed with the construction via induction. If we already have $F_1', \dots, F_n'$ with $D(F_k') \geq k$ and $R(F_k') < R(E)$ for all $k = 1, \dots, n$. Then there is a short exact sequence

$$0 \longrightarrow F_n' \cap F_i \longrightarrow F_n' \oplus F_i \longrightarrow F_n' + F_i$$

for all i.

There is no problem with the intersection and the sum because thanks to the Freyd-Mitchell embedding theorem we can treat the sequences as modules. By additivity $D(F_n' + F_i) = D(F_n') + D(F_i) - D(F_n' \cap F_i)$. Since $R(F_n') < R(E)$ by induction there is an upper bound of $D(F_n' \cap F_i)$ so $D(F_n' + F_i) \geq n - D_{F_n'} + D(F_i)$. If we take the the limit when i goes to infinity, since $n - D_{F_n'}$

is a fixed quantity we get that $D(F'_n + F_i)$ goes to infinity. Therefore there is an i_0 such that $D(F'_n + F_{i_0}) \geq n+1$. We set $F'_{n+1} := F_{n'} + F_i$ and we are done because we have an ascending chain

$$F'_1 \subset F'_2 \subset \cdots \subset F'_i \subset \cdots \subset E$$

so by the noetherian hypothesis there is a $i \gg 0$ such that $F'_i = F'_{i+l}$ for all $l \geq 0$ with the property that $D(F'_i) = D(F'_{i+l}) \geq i + l$ for all $l \geq 0$ which is obviously impossible.

With this lemma it won't be hard to prove the existence of the Harder-Narasimhan filtration in the discrete case (meaning that R and D are discrete).

Proposition 2.20. *Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ a stability function. Then, if $\mathcal{A}$ is noetherian and R and D are discrete in $\mathbb{R}$, the Harder-Narasimhan in $\mathcal{A}$ respect to Z exists.*

Proof. Take $E \in \mathcal{A}$. Denote by $\mathcal{H}(E)$ the convex hull of $Z(\{F \subset E\})$. By lemma 2.19 $\mathcal{H}(E)$ is bounded from the left by some D_E. Now denote by $\mathcal{H}_L$ the half plane given by $\{a + b\sqrt{-1} \in \mathbb{C} \mid a \geq M(E)b\}$. This is the left half-plane given by the line passing through 0 and $Z(E)$. We can also assume that E is not semistable because otherwise the filtration is trivial. Now we have a convex polygon $\mathcal{P}(E) := \mathcal{H}(E) \cap \mathcal{H}_L$. Note that this convex polygon is bounded and the imaginary part of its vertex is discrete so it must have a finite number of vertices. So our current situation is the one shown in figure 1. Notice that in the figure there are no elements with infinite slope. This is because if this object appears in the filtration it will be the first one to appear, so we can quotient by it and will end up with a filtration where all slopes are finite.

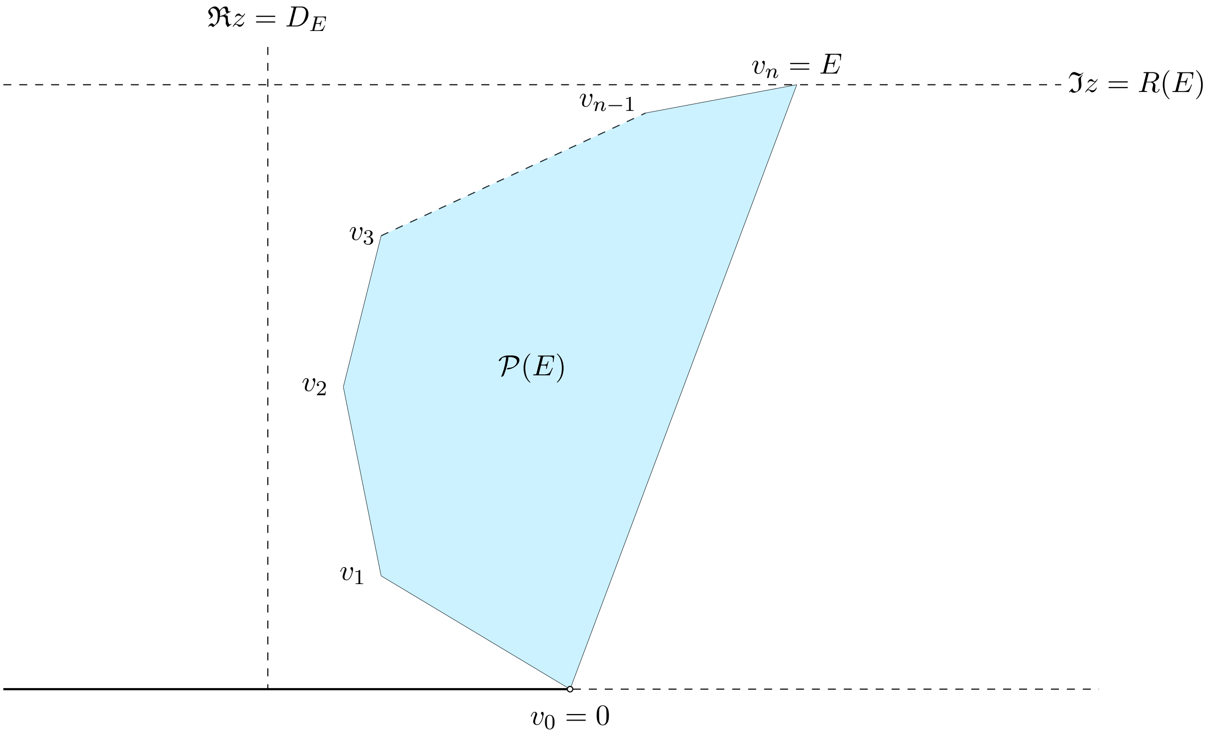


FIGURE 1. Sketch of $\mathcal{P}(E)$

First take v_i the vertexes of $\mathcal{P}(E)$ sorted by increasing imaginary part as shown in figure 1. The discreteness of D assures that the vertices are of the form $v_i = Z(F_i)$ for some $F_i \in K_0(\mathcal{A})$.

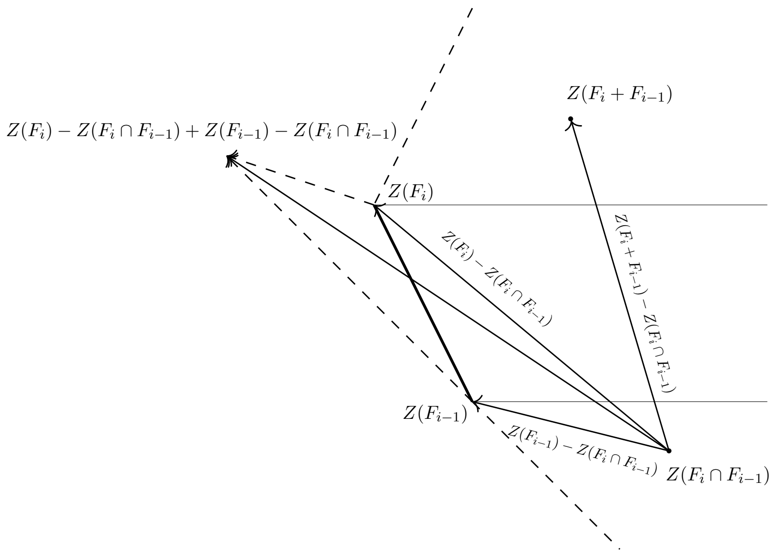


FIGURE 2. Representation in the plane of the situation in equation 2.1

Now we want to see that $0 \subset F_1 \subset \cdots \subset F_{n-1} \subset E$ is a Harder-Narasimhan filtration for E. By definition of $\mathcal{H}(E)$ both $Z(F_i \cap F_{i-1})$ and $Z(F_i + F_{i-1})$ lie in $\mathcal{H}(E)$ for all $i = 0, \dots, n$. In particular, since we have ordered F_i by the generalized rank we have that $R(F_i \cap F_{i-1}) \leq R(F_{i-1}) < R(F_i) \leq R(F_i + F_{i-1})$. Note that we have exact sequence

$$0 \longrightarrow F_i \cap F_{i-1} \longrightarrow F_i \oplus F_{i-1} \longrightarrow F_i + F_{i-1} \longrightarrow 0$$

and by additivity this gives that

$$Z(F_i + F_{i-1}) - Z(F_i \cap F_{i-1}) = Z(F_i) - Z(F_i \cap F_{i-1}) + Z(F_{i-1}) - Z(F_i \cap F_{i-1}). \tag{2.1}$$

From this is clear that $Z(F_i) = Z(F_i + F_{i-1})$ and $Z(F_{i-1}) = Z(F_i \cap F_{i-1})$ because $Z(F_i + F_{i-1})$ is in $\mathcal{H}(E)$ as seen in figure 2.

This implies that $Z(F_{i-1}/(F_i \cap F_{i-1})) = 0$ by additivity but this only cant be if $F_{i-1}/(F_i \cap F_{i-1}) = 0$ or equivalently $F_{i-1} = F_i \cap F_{i-1} \subset F_i$. Now define $G_i := F_i/F_{i-1}$. Consider a sub-object $A' \subset G_i$ with preimage by the projection equals to $A \subset F_i$. Then $Z(A)$ is in $\mathcal{H}(E)$ with $R(F_{i-1}) \leq R(A) \leq R(F_i)$. Then $Z(A) - Z(F_{i-1})$ has slope smaller or equal that $Z(F_i) - Z(F_{i-1})$. This is because the slope of the line that passes through $Z(F_i)$ and $Z(F_{i-1})$ is maximal among the slopes of the lines that pass through $Z(F_{i-1})$ and any other point of $\mathcal{H}(E)$ with generalized rank between $R(F_{i-1})$ and $R(F_i)$. Then $M(A') \leq M(G_i)$ for any $A' \subset G_i$ implying that G_i is semistable. As we said $M(G_i)$ is the slope of the line that passes through $Z(F_i)$ and $Z(F_{i-1})$. Since $\mathcal{P}(E)$ is convex this slopes have to strictly decrease (they can't increase because of convexity and if they are the same there is no vertex). $\square$

Remark 2.21. From this is clear that in the previous example the filtration exists because our $R = \omega^2 \operatorname{ch}_0^B$ is discrete.

Bridgeland Stability enhances the notion of stability in abelian categories to any triangulated category because at the end we will basically work with stability conditions over the heart of a bounded t-structure wich, as we have seen, is abelian.

Definition 2.22. The heart of a bounded t-structure in $\mathrm{D^b}(X)$ is a full additive subcategory $\mathcal{A} \subset \mathrm{D^b}(X)$ such that:

HBT1) For any integer $r < 0$ and $A, B \in \mathcal{A}$ then $\mathrm{Hom}(A, B[r]) = 0$.
HBT2) For any $E \in \mathrm{D^b}(X)$ there are integers $k_1 > \ldots > k_m$, objects $E_1, \ldots, E_{m-1} \in \mathrm{D^b}(X)$, $A_1, \ldots, A_m \in \mathcal{A}$ and a collection of (distinguished) triangles

$$E_{i-1} \longrightarrow E_i \longrightarrow A_i[k_i] \longrightarrow E_{i-1}[1]$$

where we are considering $E_0 = 0$ and $E_m = E$.

As one expects from the previous definition of a Harder-Narasimhan filtration this filtration is also unique and thenthe objects A_i in HBT2) are uniquely determined and functorial. We denote A_i by $H^{k_i}_{\mathcal{A}}(E)$ and we refer to it as the k_i-th cohomology in $\mathcal{A}$. The functoriality of the A_i makes $H^{k_i}_{\mathcal{A}} : \mathrm{D^b}(X) \to \mathcal{A}$ a functor. The first important consequence of the definition is the following one.

Lemma 2.23. *The heart of a bounded t-structure is an abelian category.*

Proof. To define the abelian structure in the heart, it is enough to determine which are the short exact sequences. From this is declare that any exact sequence in $\mathcal{A}$ whose objects belong to $\mathcal{A}$ is an exact triangle in $\mathrm{D^b}(X)$. Since we have a full additive subcategory we only have to check that indeed kernels and cokernels belong to $\mathcal{A}$.

The only remaining thing is to define the kernels and the cokernels. Take $f : A \to B$ a non zero morphism in A. Let $C := C(f)$ be the cone of f and suppose it's not zero otherwise we have an isomorphism and nothing to prove. By HBT2) we have a triangle

$$C_{>0} \longrightarrow C \longrightarrow C_{\leq 0} \longrightarrow C_{>0}[1]$$

with $C_{>0}$ belonging to $\mathcal{A}_{>0}$ and $C_{\leq 0}$ to $\mathcal{A}_{\leq 0}$ (this are categories generated by all $\mathcal{A}[i]$ with $i > 0$ and $i \leq 0$ respectively). This is because the $H^{k_i}(C)[k_i]$ gets mapped into $C[1]$ if $k_i > 0$ and C gets mapped to $H^{k_i}(C)[k_i]$ if $k_i \leq 0$ (recall that E_j are bounded complexes and $H^{k_i}(C)[k_i]$ is the complex with only one non-zero object in the k_i-th position). In fact $C_{>0} = E_s$ such that $k_s > 0$ and $k_{s+1} \leq 0$ and $C_{\leq 0}$is the cone of $C_{>0} \to C$.

Then, since Hom is exact on triangles by proposition 2.3, if $T \in \mathcal{A}$ with $k < 0$ then $\mathrm{Hom}(A[1], T[k]) = \mathrm{Hom}(B, T[k]) = 0$ which leads to $\mathrm{Hom}(C, T[k]) = 0$. Since $\mathrm{Hom}(C_{>0}[1], T[k])$ also vanishes this leads to $\mathrm{Hom}(C_{\leq 0}, T[k]) = 0$. This means that $C_{\leq 0} \in \mathcal{A}$. On the other hand if $F \in \mathcal{A}$ with $k > 1$ then $\mathrm{Hom}(F[k], B) = 0 = \mathrm{Hom}(F[k], A[1])$ so again $\mathrm{Hom}(F[k], C) = 0$. Moreover $\mathrm{Hom}(F[k], C_{\leq 0}[-1]) = 0$ so this means that $\mathrm{Hom}(F[k], C_{>0}) = 0$ so basically $C_{>0} \in \mathcal{A}[1]$ i.e. $C_{>0}[-1] \in \mathcal{A}$.

Hence $\ker(f) = C_{>0}[-1]$ and $\mathrm{coker}(f) = C_{\leq 0}$ which both belong to $\mathcal{A}$ and we're done. ☐

Now we work over the bounded derived category of coherent sheaves of a surface X which is not an abelian category but, as we said before, it is triangulated. Moreover we have also different versions of the stability function which properties that will allow us to equip our set of stability conditions with a structure of complex manifold.

we also define an equivalent concept that will become useful for us because it's related to the heart of a bounded t-structure.

Definition 2.24. A slicing $\mathcal{P}$ of $\mathrm{D^b}(X)$ is a collection of subcategories $\{\mathcal{P}_\phi\}_{\phi\in\mathbb{R}}$ of $\mathrm{D^b}(X)$ such that:

SLC1) $\mathcal{P}_\phi[1] = \mathcal{P}_{\phi+1}$,
SLC2) if $A \in \mathcal{P}_{\phi_1}, B \in \mathcal{P}_{\phi_2}$ with $\phi_1 > \phi_2$ then $\mathrm{Hom}(A, B) = 0$,
SLC3) for all $E \in \mathrm{D^b}(X)$ there are real numbers $\phi_1 > \cdots > \phi_m$ real numbers, $A_1, \ldots, A_m \in \mathcal{P}_{\phi_i}$, $E_1, \ldots, E_{m-1} \in \mathrm{D^b}(X)$ and a collection of triangles

$$E_{i-1} \longrightarrow E_i \longrightarrow A_i \longrightarrow E_{i-1}[1]$$

where $i \in \{1, \ldots, m\}$ with $E_0 = 0$ and $E_m = E$.

We denote ϕ_1 by $\phi^+(E)$ and ϕ_m by $\phi^-(E)$. If $E \in \mathcal{P}_\phi$ we call $\phi(E) := \phi$ the phase of E.

The last property, as well as HBT2), is called the Harder-Narasimhan filtration. We can construct a heart out of a slicing by setting $\mathcal{A} = \mathcal{P}_{]0,1]}$ which is the extension closure of the family of subcategories $\{\mathcal{P}_\phi \ : \ \phi \in]0, 1]\}$ (i.e. if A, B belong to this family any C that sits in a distinguished triangle $A \to C \to B \to A[1]$ is also in extension closure of this family, so in $\mathcal{A}$).

This will be our replace for the "abelian category" in the stability condition. Now we want our stability function to factor through a lattice so we can have a nice structure on the set of stability conditions. So we fix a finite rank lattice Λ (that in our case is $K_{\mathrm{num}}(X)$ the Grothendieck numerical group), a surjective group homomorphism $v : K_0(X) \twoheadrightarrow \Lambda$ (that in our case will be the usual projection and we identify $v(E)$ with its numerical Chern character $(\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E)))$ and a norm on $\Lambda_\mathbb{R} := \Lambda \otimes \mathbb{R}$ (but we don't care much which one we choose, because they are all equivalent). With this prerequisites we can define what a Bridgeland stability condition means:

Definition 2.25. A Bridgeland stability condition on $\mathrm{D^b}(X)$ is a pair $\sigma = (\mathcal{P}, Z)$ where:

BS1) $\mathcal{P}$ is a slicing of $\mathrm{D^b}(X)$,
BS2) $Z : \Lambda \to \mathbb{C}$ is an additive morphism called the central charge such that $Z(v(E)) \in \mathbb{R}_{>0}e^{\sqrt{-1}\phi(E)\pi}$ and moreover it has the so called support property i.e.

$$C_\sigma := \inf\left\{\frac{|Z(v(E))|}{\|v(E)\|} \ : \ 0 \neq E \in \mathcal{P}_\phi, \ \phi \in \mathbb{R}\right\} > 0$$

There is an immediate alternative definition using the heart of a bounded t-structure:

Lemma 2.26. *Giving a Bridgeland Stability condition $(\mathcal{P}, Z)$ on $\mathrm{D^b}(X)$ is the same as giving a stability condition $(\mathcal{A}, Z)$ in the sense of definition 2.17 where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathrm{D^b}(X)$ together with the support property*

$$\inf\left\{\frac{|Z(v(E))|}{\|v(E)\|} : 0 \neq E \in \mathcal{A}, E \textit{ semistable}\right\} > 0$$

Proof. If $(\mathcal{P}, Z)$ is a Bridgeland stability condition, taking $\mathcal{A} := \mathcal{P}(]0, 1])$ as we said before gives a heart of a bounded t-structure on $\mathrm{D^b}(X)$ and $Z' : K_0(\mathcal{A}) \to \mathbb{C}$ is given by composing Z with the projection $K_0(\mathcal{A}) \twoheadrightarrow K_{\mathrm{num}}(X)$. This implies that the support property is satisfied because $(\mathcal{P}, Z)$ is a stability condition. To make the other way we just take $\mathcal{P}_\phi = \mathcal{A}[k] \cap Z^{-1}(\mathbb{R}_{>0} \cdot e^{\sqrt{-1}\pi(\phi-k)})$ if $\phi \in]k, k+1]$. This gives a slicing of $\mathrm{D^b}(X)$ satisfying the conditions in definition 2.25. □

The set of stability conditions on a surface X will be denoted by $\mathrm{Stab}(X)$. This set can be endowed with a structure of topological space in the following way. The topology is the coarsest such that the following maps are continuous for any $E \in \mathrm{D^b}(X)$:

$$\begin{array}{rcl} \mathrm{Stab}(X) & \longrightarrow & \mathbb{R} \\ \sigma & \longmapsto & \phi^+_\sigma(E) \\ \sigma & \longmapsto & \phi^-_\sigma(E) \end{array} \qquad \begin{array}{rcl} \mathcal{Z} : \mathrm{Stab}(X) & \longrightarrow & \mathrm{Hom}(\Lambda, \mathbb{C}) \\ \sigma = (\mathcal{A}, Z) & \longmapsto & Z \end{array}.$$

Moreover this topology coincides with the induced by the metric

$$d(\sigma_1, \sigma_2) := \sup_{E \in \mathrm{D^b}(X)} \{ |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)\}|, |\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E)|, \|Z_1 - Z_2\| \}$$

where $\|\cdot\|$ is the induced operator norm on $\mathrm{Hom}(\Lambda, \mathbb{C})$ (this is nothing but $\|Z_1 - Z_2\| = \inf\{c \geq 0 \ : \ \|Z_1 v - Z_2 v\| \leq c\|v\|\}$).

Moreover it can be endowed with a structure of complex manifold as long as it's not empty. This is the main result in [Bri07].

Theorem 2.27 (Bridgeland's deformation theorem)**.** *The map* $\mathcal{Z} : \mathrm{Stab}(X) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C})$ *is a local homeomorphism. In particular,* $\mathrm{Stab}(X)$ *can be endowed with a structure of complex manifold.*

Proof. See [Bri07, Theorem 1.2]. □

A very good way of studying $\mathrm{Stab}(X)$ is through its wall and chamber structure. Given a fixed numerical class $v \in K_{\mathrm{num}}(X)$, the wall and chamber structure tells us that stable objects with class v are the same for all stability conditions in the same chamber and they change when we cross a wall. This behaviour will be very useful at the end of the work. First we introduce the concept of numerical wall

Definition 2.28. Given two non-zero, non-parallel vectors $v_0, w \in \Lambda$ we define a numerical wall $W_w(v_0)$ for V_0 with respect of W as the non empty subset of $\mathrm{Stab}(X)$ given by

$$W_w(v_0) := \left\{ (\mathcal{A}, Z) \in \mathrm{Stab}(X) \ : \ \frac{\Re Z}{\Im Z}(v_0) = \frac{\Re Z}{\Im Z}(w) \right\}$$

Despite that we see in detail the behaviour of the walls in the case of the (α, β)-plane latter (and is the only case that we are going to need) we have a more general result that tells us how the walls behave in general.

Proposition 2.29. *Let* $v_0 \in K_{num}(X)$ *in the numerical Grothendieck group, and* $S \subset v^{-1}(v_0)$*. Then we have a collection of walls* $W^S_w(v_0)$ *with* $\omega \in K_{num}(X)$ *such that:*

(1) Every wall in $W^S_w(v_0)$ *is a 1-codimensional real closed sub-manifold with boundary.*
(2) $W^S_w(v_0)$ *is locally finite.*
(3) For every stability condition $(\mathcal{A}, Z)$ *on a wall of* $W^S_w(v_0)$ *there is an integer* $k \in \mathbb{Z}$ *such that* $F_w \hookrightarrow E_{v_0}$ *in* $\mathcal{A}[k]$ *with* $v(F_w) = w$ *and* $E_{v_0} \in S$*.*

(4) If $C \subset \mathrm{Stab}(X)$ *is a connected component of* $\mathrm{Stab}(X) - (\bigcup_{w \in K_{num}(X)} W_w^S(v_0))$ *and* $\sigma_1, \sigma_2 \in C$ *then any* $E_{v_0} \in S$ *is* σ_1*-stable iff is* σ_2*-stable.*

Proof. See [BM11, Proposition 3.3]. □

This proposition basically shows that the numerical walls defined above are the possible places where our stable objects change but, locally, there are only finitely many places where this really occurs. Moreover explains that this change happens due to the existence of a morphism i.e. a destabilizing object inside our stable object.

3. Bogomolov inequality and existence of stability conditions

Now, our next goal is basically to show that Bridgeland stability conditions exist on surfaces. For this we introduce first the concept of tilting. If X is a surface, we saw in Example 2.12 a stability function on $\mathrm{Coh}(X)$ that wasn't a stability condition because of the existence of sheaves supported in 0-dimensional schemes.

In general this will be a problem and, since $\mathrm{Coh}(X)$ is our canonical heart of $\mathrm{D^b}(X)$, we would like a nice way to modify our heart to obtain a better behaved one respect stability functions. In fact, if X is a smooth surface, one can prove that there is no stability function Z such that $(\mathrm{Coh}(X), Z)$ is a stability condition where the skyscrapers are stable (see also Lemma lem:hastobetilting). So, we need to construct other hearts. The best known technique to construct new hearts out of a given one, is via tilting.

The first step for tilting is getting a torsion pair.

Definition 3.1. A torsion pair on an abelian category $\mathcal{A}$ is a pair of full additive sub-categories $(\mathcal{F}, \mathcal{T})$ such that:

TP1) for any $F \in \mathcal{F}, T \in \mathcal{T}$ we have $\mathrm{Hom}(T, F) = 0$,
TP2) any $E \in \mathcal{A}$ fits in a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$

where $F \in \mathcal{F}$ and $T \in \mathcal{T}$.

It's not hard to see that the T and F in the exact sequence of TP2) are unique. If we have two such short exact sequences $0 \to T^1 \to E \to F^1 \to 0$ and $0 \to T^{-1} \to E \to F^{-1} \to 0$ then, taking $\mathrm{Hom}(-, F^{\pm 1})$, and $\mathrm{Hom}(T^{\pm 1}, -)$, we have that $\mathrm{Hom}(F^{\mp 1}, F^{\pm 1}) \cong \mathrm{Hom}(E, F^{\pm 1})$ and $\mathrm{Hom}(T^{\mp 1}, T^{\pm 1}) = \mathrm{Hom}(T^{\pm 1}, E)$ by TP1). This, together with the fact that $E \to F^{\pm 1}$ and $T^{\pm 1} \to E$ are epimorphisms and monomorphisms respectively, leads to isomorphisms $F^1 \cong F^{-1}$ and $T^1 \cong T^{-1}$.

Example 3.2. Our main example of a torsion pair in $\mathrm{Coh}(X)$, as the names indicates, will be the category of torsion sheaves on X, that we denote by $\mathcal{T}$, and the category of torsion-free sheaves on X, that we denote by $\mathcal{F}$ as done in Lemma 1.5 but except in this case we can't assure that it will be locally free in general.

TP1) follows easy from the fact that torsion sections must be mapped to torsion sections but the torsion free sheaves don't have torsion sections beside the zero section. TP2) is just the fact

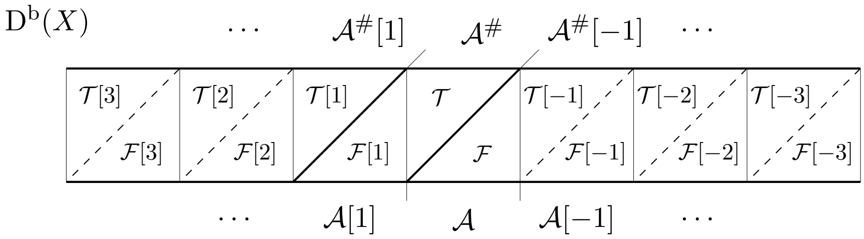


FIGURE 3. Representation of the tilting process

that we can take the torsion sub-sheaf of any coherent sheaf (which belongs to $\mathcal{T}$) and hence its quotient sheaf will belongs to $\mathcal{F}$.

Before we move into the proper concept of tilting in this context we need a little lemma that relates the shift functor, the Hom functor and the Ext functor.

Lemma 3.3. *Let $A, B \in \mathcal{A}$ an abelian category with enough injectives. Then there are natural isomorphisms*

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(A, B[i]) \simeq \mathrm{Ext}^i_{\mathcal{A}}(A, B)$$

where clearly A, B in $\mathrm{D^b}(A)$ are the complexes with A and B in the 0-th position and zero elsewhere.

Proof. Take an injective resolution $\{I^k\}_{k\geq 0}$ of B. The key fact is based on the observation that, since $D^+(\mathcal{A})$ are complexes with zeros in negative positions up to cohomology every complex is isomorphic to its injective resolution. In particular, without entering in much detail, the inclusion functor $\iota : K^+(\mathcal{I}_{\mathcal{A}}) \to D^+(\mathcal{A})$ gives an equivalence of categories between the category of positive complexes on the full subcategory of injective objects of $\mathcal{A}$ and $D^+(\mathcal{A})$. This allows us to define the right derived functor of $\mathrm{Hom}_{D^+(\mathcal{A})}(A, -)$ as $R\,\mathrm{Hom}(A, -) = Q_{Ab} \circ \mathrm{Hom}_{K^+(\mathcal{A})}(A, -) \circ \iota^{-1} : D^+(\mathcal{A}) \to D^+(Ab)$ (where $Q_{Ab} : K^+(Ab) \to D^+(Ab)$ is the localization functor as in Proposition 2.8). Then $R\,\mathrm{Hom}(A, B)$ is the complex $(\mathrm{Hom}(A, I^k))_{k\geq 0}$ and $\mathrm{Ext}^i(A, B)$ is its cohomology. A morphism f belongs to $\ker(\mathrm{Hom}(A, I^k) \to \mathrm{Hom}(A, I^{k+1}))$ if and only if defines a morphism $f : A \to I^\bullet[i]$ (otherwise it does not commute) and it's homotopically zero iff comes from a morphism of $\mathrm{Hom}(A, I^{k-1})$ (in that case we have a morphism $g : A \to I^{k-1}$ and we take the homotopy $s^k = g$ and zero otherwise). This means that the morphisms killed by homotopy in $K(\mathcal{A})$ are those killed by cohomology in Ext meaning that $\mathrm{Ext}^i(A, B) \simeq \mathrm{Hom}_{K(\mathcal{A})}(A, I^\bullet[i])$. Now since $I^\bullet$ is a complex of injectives we have that $\mathrm{Hom}_{K(\mathcal{A})}(A, I^\bullet[i]) \simeq \mathrm{Hom}_{D(\mathcal{A})}(A, I^\bullet)$ (this needs a little work because we need to see that quasi-isomorphisms induce isomorphisms between the $\mathrm{Hom}_{K(\mathcal{A})}$'s but a proof can be found in [Huy06, Lemma 2.38]). Since $I^\bullet[i] \cong B[i]$ we're done. □

Once we have a torsion $(\mathcal{T}, \mathcal{F})$ pair basically every element of the heart is an extension of an element of $\mathcal{T}$ and an element of $\mathcal{F}$. Basically, what we tilt (or lean) is the torsion pair (see figure 3 and the following lemma).

Lemma 3.4. *(Happel-Reiten-Smalø) Let X be a smooth projective surface. Let $(\mathcal{F}, \mathcal{T})$ be a torsion pair on a heart $\mathcal{A}$ on $\mathrm{D^b}(X)$. We define*

$$\mathcal{A}^{\#} := \{E \in \mathrm{Coh}(X) \ : \ H^0_{\mathcal{A}}(E) \in \mathcal{T},\ H^{-1}_{\mathcal{A}}(E) \in \mathcal{F}\ H^i_{\mathcal{A}}(E) = 0 \textit{ otherwise } \}.$$

Then this is a heart of a bounded t-structure on $\mathrm{D^b}(X)$*.*

Proof. We only have to check that HBT1) and HBT2) hold. First take $E, E' \in \mathcal{A}^{\#}$. We want to show that $\mathrm{Hom}(E, E'[i]) = 0$ for all $i < 0$. By definition of $\mathcal{A}^{\#}$ and HBT2) of $\mathcal{A}$ we have exact triangles

$$F_E[1] := H^{-1}_{\mathcal{A}}(E)[1] \to E \to H^0_{\mathcal{A}}(E) =: T_E,$$
$$F_{E'}[1] := H^{-1}_{\mathcal{A}}(E')[1] \to E' \to H^0_{\mathcal{A}}(E') =: T_{E'}.$$

Taking Hom functors and applying Lemma 3.3 we have a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & \mathrm{Hom}(T_E, F_{E'}[i+1]) \arrow[l] \arrow[d] & \mathrm{Hom}(E, F_{E'}[i+1]) \arrow[l] \arrow[d] & \mathrm{Hom}(F_E[1], F_{E'}[i+1]) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \mathrm{Hom}(T_E, E'[i]) \arrow[l] \arrow[d] & \mathrm{Hom}(E, E'[i]) \arrow[l] \arrow[d] & \mathrm{Hom}(F_E[1], E'[i]) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \mathrm{Hom}(T_E, T_{E'}[i]) \arrow[l] \arrow[d] & \mathrm{Hom}(E, T_{E'}[i]) \arrow[l] \arrow[d] & \mathrm{Hom}(F_E[1], T_{E'}[i]) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

If $i < 0$ then $\mathrm{Hom}(T_E, T_{E'}[i]) = \mathrm{Hom}(F_E[1], T_{E'}[i]) = \mathrm{Hom}(F_E[1], F_{E'}[i+1]) = 0$ because of HBT1) in $\mathcal{A}$. This implies that also $\mathrm{Hom}(F_E[1], E'[i]) = \mathrm{Hom}(E, T_{E'}[i]) = 0$ and hence $\mathrm{Hom}(E, E'[i]) \cong \mathrm{Hom}(T_E, F_{E'}[i+1])$ for all $i < 0$. If $i < -1$ then $\mathrm{Hom}(T_E, F_{E'}[i+1]) = 0$ because of HBT1) meanwhile if $i = -1$ we have that $\mathrm{Hom}(T_E, F_{E'}) = 0$ because $(\mathcal{F}, \mathcal{T})$ is a torsion pair so we are done with HBT1).

HBT2) it's not hard to prove since we already have for any $E \in \mathrm{D^b}(X)$ a collection of triangles

$$E_{i-1} \longrightarrow E_i \longrightarrow A_i[k_i] \longrightarrow E_{i-1}[1].$$

Since $(\mathcal{F}, \mathcal{T})$ is a torsion pair on $\mathcal{A}$ for each A_i there are $T_i \in T, F_i \in F$ such that

$$0 \longrightarrow T_i \xrightarrow{f} A_i \xrightarrow{g} F_i \longrightarrow 0$$

is exact. Now we have a morphism from A_i to the complex $A'_i := \ldots \to 0 \to T_i \xrightarrow{f} A_i \xrightarrow{0} A_i \xrightarrow{g} F_i \to 0 \ldots$ centered at the second A_i and we can replace A_i by this complex in the filtration. Notice that the cohomology of this complex in $\mathcal{A}$ is $H^{-1}_{\mathcal{A}}(A'_i) = F_i$ and $H^0_{\mathcal{A}}(A'_i) = T_i$ so $A'_i \in \mathcal{A}^{\#}$. □

Whenever we have an abelian category $\mathcal{A}$ we call $\mathcal{B}$ a tilt of $\mathcal{A}$ if there is some torsion pair on $\mathcal{A}$ such that $\mathcal{A}^{\#} = \mathcal{B}$. The following lemma is a very useful condition to decide if a heart of $\mathrm{D^b}(X)$ is a tilt of another one.

Lemma 3.5. *Let* $\mathcal{A}, \mathcal{B}$ *be two hearts of a bounded t-structure on* $\mathrm{D^b}(X)$ *such that* $\mathcal{A} \subset \langle \mathcal{B}, \mathcal{B}[1] \rangle$ *(this is the smallest subcategory generated by extensions of* $\mathcal{B}$ *and* $\mathcal{B}[1]$*). Then* $\mathcal{A}$ *is a tilt of* $\mathcal{B}$*.*

Proof. Take first $\mathcal{T} = \mathcal{A} \cap \mathcal{B}$ and $\mathcal{F} = \mathcal{A}[-1] \cap \mathcal{B}$. If $T \in \mathcal{T}$ and $F \in \mathcal{F}$ then $\mathrm{Hom}(T, F) = 0$ because of HBT1) from $\mathcal{A}$. For HBT2) take $E \in \mathcal{B}$, then for any $A \in \mathcal{A}$ we have that $\mathrm{Hom}(A, E[-1]) = 0$ because $\mathcal{A} = \langle \mathcal{B}, \mathcal{B}[1] \rangle$. Since $\mathcal{A}$ is a heart of a bounded t-structure E is isomorphic to a complex generated by shifts of $\mathcal{A}$ but the vanishing Hom imply that this complex is generated by $\mathcal{A}_{\leq 0} = \langle \mathcal{A}, \mathcal{A}[-1], \mathcal{A}[-2], \ldots \rangle$ the negative extension closure of $\mathcal{A}$. We also have $\mathrm{Hom}(E, A[-2]) = 0$ and by the same argument $E \in \mathcal{A}_{\geq -1} = \langle \mathcal{A}[-1], \mathcal{A}, \mathcal{A}[1], \ldots \rangle$. This means that $E \in \mathcal{A}_{\leq 0} \cap \mathcal{A}_{\geq -1} = \langle \mathcal{A}, \mathcal{A}[-1] \rangle$ and hence $\mathcal{B} \subset \langle \mathcal{A}, \mathcal{A}[-1] \rangle$. From this, TP2) follows immediately.

Then with this torsion pair it's clear that $\mathcal{A} \subset \mathcal{B}^{\#}$ because $\mathcal{A} \subset \langle \mathcal{B}, \mathcal{B}[1] \rangle$ so the cohomology with values in $\mathcal{B}$ of any object in $\mathcal{A}$ must vanish except on the positions $i = 0, -1$. Also if $E \in \mathcal{B}^{\#}$ then $E \in \langle \mathcal{F}[1], \mathcal{T} \rangle \subset \mathcal{A}$ (in fact $\mathcal{B}^{\#} = \langle \mathcal{F}[1], \mathcal{T} \rangle$ always holds). $\square$

Now we return to our main path: existence of Bridgeland conditions on surfaces.

Let X be a smooth projective surface over $\mathbb{C}$ and a fixed ample divisor class $\omega \in NS(X) \otimes \mathbb{R}$ and another (not necessarily ample) divisor class $B \in NS(X) \otimes \mathbb{R}$. From this choice we want to construct a Bridgeland stability condition depending on ω and B so at the end we will have a family of stability conditions parameterized by ω and B. The first step is to construct a good heart. Recall that we defined

$$\mu_{\omega,B}(E) := \frac{\omega \operatorname{ch}_1^B(E)}{\omega^2 \operatorname{ch}_0^B(E)}$$

the twisted slope stability in $\mathrm{Coh}(X)$. As we saw in the Example 2.12 this is not a stability function on $\mathrm{Coh}(X)$ but we can use it to construct a torsion pair on $\mathrm{Coh}(X)$ so that we can have a well-defined stability function on the tilting of $\mathrm{Coh}(X)$ at this torsion pair. That is, let

$$\mathcal{T}_{\omega,B} := \{E \in \mathrm{Coh}(X) \ : \text{all semistable } F \subset E \text{ with } \mu_{\omega,B}(F) > 0\},$$

$$\mathcal{F}_{\omega,B} := \{E \in \mathrm{Coh}(X) \ : \text{all semistable } F \subset E \text{ with } \mu_{\omega,B}(F) \leq 0\}.$$

Indeed, $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ is a torsion pair: TP1) is true because $\mu_{\omega,B}(T) > 0 \leq \mu_{\omega,B}(F)$ and we can use Lemma 2.15; and TP2) because we have a Harder-Narasimhan filtration for any $E \in \mathrm{Coh}(X)$ and we can consider the greatest sub-object $E_i \subseteq E$ in the Harder-Narasimhan filtration such that $\mu_{\omega,B}(E_i/E_{i-1}) > 0$. Then, clearly $E_i \in \mathcal{T}_{\omega,B}$, while $E/E_i \in \mathcal{F}_{\omega,B}$, because all the Harder-Narasimhan factors of E/E_i have non-positive $\mu_{\omega,B}$-slope.

Then we can take the tilted heart respect to this torsion pair.

Definition 3.6. We define $\mathrm{Coh}^{\omega,B} = \langle \mathcal{F}_{\omega,B}[1], \mathcal{T}_{\omega,B} \rangle$ the tilted heart of $\mathrm{Coh}(X)$ by the torsion pair defined above.

We can define a stability function over $\mathrm{Coh}^{\omega,B}(X)$ as follows:

Definition 3.7. For any $E \in \mathrm{D}^{\mathrm{b}}(X)$ we define

$$Z_{\omega,B}(E) := (-\operatorname{ch}_2^B(E) + \frac{\omega^2}{2} \cdot \operatorname{ch}_0^B(E)) + \sqrt{-1}\omega \cdot \operatorname{ch}_1^B(E).$$

The associated slope function will be denoted by

$$\nu_{\omega,B}(E) := \frac{\operatorname{ch}_2^B(E) - \frac{\omega^2}{2} \cdot \operatorname{ch}_0^B(E)}{\omega \cdot \operatorname{ch}_1^B(E)}$$

Now our goal is basically to see that $(\mathrm{Coh}^{\omega,B}, Z_{\omega,B})$ is a Bridgeland stability condition.

The first step is checking that, in fact, $Z_{\omega,B}$ is a stability function, this is, it maps $\mathrm{Coh}^{\omega,B}(X)$ into the region of $\mathbb{C}$ that we want. For this purpose we need two theorems: the Hodge index theorem and the Bogomolov inequality.

Theorem 3.8 (Hodge index theorem)**.** *Let ω be an ample divisor on X and D another non-zero divisor with $\omega \cdot D = 0$. Then $D^2 < 0$*

Proof. (Sketch) We argue by contradiction. Suppose $D^2 \geq 0$. We have two possibilities: if $D^2 > 0$, then take $\omega' := D + n\omega$. Since ω is ample there must be some $k \geq 0$ such that the line bundle associated to $D + k\omega$ is globally generated and an $l \geq 0$ such that $l\omega$ is ample. Then for $n = k + l$ we have that ω' is ample. Moreover $\omega' \cdot D = D^2 > 0$. By the Riemann-Roch theorem (and Serre duality),

$$h^0(X, mD) + h^0(X, K_X - mD) \geq \frac{m^2 D^2}{2} - \frac{mD \cdot K_X}{2} + \chi(\mathcal{O}_X),$$

and since $\omega' \cdot D > 0$, then $h^0(X, K_X - mD) = 0$ for m large enough. Indeed, if not, $(K_X - mD)\omega' \geq 0$ which contradicts $\omega' \cdot D > 0$.

Thus, for m large enough mD is effective. This yields to $mD \cdot \omega > 0$ because the ampleness of ω allows to embed mD in some projective space. Since mD is effective, it will cut effectively an hyperplane of this projective space leading to $\omega \cdot D > 0$, which contradicts our hypothesis.

If $D^2 = 0$ then, since D is non-zero there must exists some divisor E such that $D \cdot E \neq 0$. Then if we take $E' = (\omega^2)E - (E \cdot \omega)\omega$ we have $D \cdot E' = (\omega^2)D \cdot E \neq 0$ and $E' \cdot \omega = 0$. Then we define $D' = nD + E'$. Therefore $D' \cdot \omega = 0$ and $D'^2 = n^2 D^2 + 2nD \cdot E' + E'^2 = 2nD \cdot E' + E'^2$. Since this quantity depends on n we can find an integer n with $D'^2 > 0$ and, by the first case, we get a contradiction again.

For the Bogomolov inequality we have to introduce our main actors: the discriminants of a sheaf.

Definition 3.9. Let $\omega, B \in NS(X) \otimes \mathbb{R}$ with ω ample. We define the following functions.

- The discriminant function $\Delta := (\mathrm{ch}_1)^2 - 2\,\mathrm{ch}_2 \cdot \mathrm{ch}_0$.
- The ω-discriminant $\overline{\Delta}_\omega^B := (\omega \cdot \mathrm{ch}_1)^2 - 2\omega^2 \cdot \mathrm{ch}_2 \cdot \mathrm{ch}_0$.
- The (ω, B, C_ω)-discriminant $\Delta_{\omega,B}^C := \Delta + C_\omega(\omega \cdot \mathrm{ch}_1)^2$.

The constant C_ω that appears in the (ω, B, C_ω)-discriminant is a constant such that $C_w(w \cdot D)^2 + D^2 \geq 0$ for all effective divisors D. This constant exists because ω is ample and the Nakai-Moishezon criterion grants that for any effective divisor D we have $\omega \cdot D > 0$. Since ω lives in $H^2(X, \mathbb{R})$, given a norm $\|\cdot\|$ on $H^2(X, \mathbb{R})$, we have that the ample cone (which is the cone generated by the ample divisors) is open. Since ample divisors is open there is a constant B_ω such that $B_\omega(\omega \cdot D) \geq \|D\|$ and a positive constant $D^2 \leq A\|D\|^2$ because $D^2 > 0$ for all D effective. This implies that $AB_\omega^2(\omega \cdot D)^2 \geq D^2$ so it's enough to take $C_\omega := AB_\omega^2$.

Now our goal is basically to prove the Bogomolov inequality that basically states that the discriminant of a $\mu_{\omega,B}$-semistable torsion free sheaf is non negative.

Theorem 3.10 (Bogomolov inequality). *Let X be a smooth projective surface, $\omega \in NS(X)\otimes\mathbb{R}$ with ω ample and E a $\mu_{\omega,B}$-semistable sheaf. Then*

$$\Delta(E) \geq 0.$$

Proof. Notice that the stability of any sheaf respect to $\mu_{\omega,B}$ doesn't depend on B since it only adds a constant term. So one can suppose $B = 0$. If we take the double dual of E we end up with $E^{\vee\vee} = \operatorname{Hom}(\operatorname{Hom}(E, \mathcal{O}), \mathcal{O})$. Note that this is locally free since E is torsion free. Also $E \hookrightarrow E^{\vee\vee}$ in the natural way. Since both have the same dimension $E^{\vee\vee}/E$ is supported in dimension zero and we have an exact sequence

$$0 \longrightarrow E \hookrightarrow E^{\vee\vee} \twoheadrightarrow E^{\vee\vee}/E \longrightarrow 0.$$

Since the Chern character is additive we have that $\operatorname{ch}_2(E^{\vee\vee}) = \operatorname{ch}_2(E)+\operatorname{ch}_2(E^{\vee\vee}/E) = \operatorname{ch}_2(E)-l(E^{\vee\vee}/E)$ and $\operatorname{ch}_i(E) = \operatorname{ch}_i(E^{\vee\vee})$ for $i = 0, 1$ because $\operatorname{ch}_i(E^{\vee\vee}/E) = 0$. This leads to

$$\frac{\Delta(E)}{\operatorname{ch}_0(E)} = \frac{\Delta(E^{\vee\vee})}{\operatorname{ch}_0(E)} + 2l(E^{\vee\vee}/E).$$

This holds if and only if

$$\Delta(E) = \Delta(E^{\vee\vee}) + 2\operatorname{ch}_0(E)l(E^{\vee\vee}/E).$$

This implies that $\Delta(E) \geq \Delta(E^{\vee\vee})$ so we can assume E is locally free. Since $c_i(E^\vee) = -c_i(E)$ (the Chern classes) we have that $\operatorname{ch}_i(E) = (-1)^i \operatorname{ch}_i(E^\vee)$. Now that E is locally free we have that $\mathcal{E}nd(E) \cong E^\vee \otimes E$ so

$$\begin{aligned}
\operatorname{ch}_0(\mathcal{E}nd(E)) &= \operatorname{ch}_0(E)\operatorname{ch}_0(E^\vee) = \operatorname{ch}_0(E)^2,\\
\operatorname{ch}_1(\mathcal{E}nd(E)) &= \operatorname{ch}_0(E)\operatorname{ch}_1(E^\vee) + \operatorname{ch}_1(E)\operatorname{ch}_0(E^\vee) = 0, \text{ and}\\
\operatorname{ch}_2(\mathcal{E}nd(E)) &= \operatorname{ch}_0(E)\operatorname{ch}_2(E^\vee) + 2\operatorname{ch}_1(E)\operatorname{ch}_1(E^\vee) + \operatorname{ch}_2(E)\operatorname{ch}_0(E^\vee)\\
&= 2\operatorname{ch}_0(E)\operatorname{ch}_2(E) - 2\operatorname{ch}_1(E)^2 = 2\Delta(E)
\end{aligned}$$

Therefore we end up with

$$\frac{\Delta(\mathcal{E}nd(E))}{2\operatorname{ch}_0(E)^2} = \frac{1}{2}\operatorname{ch}_2(\mathcal{E}nd(E)) = \Delta(E).$$

So we can also assume that $\operatorname{ch}_1(E) = 0$.

By the Nakai-Moishezon criterion, there is a large enough k such that $k\omega^2 > k\omega \cdot K_X$ such that $k\omega$ is linearly equivalent to a smooth curve C. We can take the exact sequence

$$0 \longrightarrow S^nE \otimes \mathcal{O}(-C) \longrightarrow S^nE \longrightarrow S^nE|_C \longrightarrow 0$$

where S^nE is the n-th symmetric power of E. Also by Riemann-Roch we have that $\chi(X, S^nE) = \operatorname{ch}_0(E)^n\chi(X, \mathcal{O}_X) + n\operatorname{ch}_0(E)^{n-1}\operatorname{ch}_2(E)$ for all $n > 0$. The exact sequence above lets us bound above $h^0(X, S^nE)$ by $h^0(C, S^nE|_C)$ and this is bounded above by $\gamma_E \operatorname{ch}_0(E)^n$ where $\gamma_E \in \mathbb{R}_{>0}$ doesn't depend on n by Riemann-Roch (since $S^nE|_C$ is defined over a smooth curve the second Chern character vanishes). By Serre Duality

$$h^2(X, S^nE) = h^0(X, S^nE^* \otimes \omega_X) \leq h^0(C, S^nE^* \otimes \omega_X|_C) = \chi(C, S^nE^* \otimes \omega_X|_C) \leq \delta_E \operatorname{ch}_0(E)^n$$

for some $\delta_E > 0$ that doesn't depend on n either. Then we have

$$\frac{\chi(X, S^n E)}{\operatorname{ch}_0(E)^n} = \chi(X, \mathcal{O}_X) + n\frac{\operatorname{ch}_2(E)}{\operatorname{ch}_0(E)} \leq \gamma_E + \delta_E$$

for all n. Since neither γ_E, nor δ_E, depend on n and $\chi(X, \mathcal{O}_X)$ is fixed this means that $\operatorname{ch}_2(X)/\operatorname{ch}_0(X) \leq 0$ but this is only possible if and only if $\operatorname{ch}_2(E) \leq 0$. Since $\Delta(E) = -2\operatorname{ch}_0(E)\operatorname{ch}_2(E)$ because we saw that E can be chosen such that $\operatorname{ch}_1(E) = 0$ we have that $\Delta(E) \geq 0$.

Proposition 3.11. *The group homomorphism $Z_{\omega,B}$ is a stability function on* $\operatorname{Coh}^{\omega,B}(X)$*.*

Proof. Notice that $\Im Z_{\omega,B} = \omega \cdot \operatorname{ch}_1^B$ so, if $E \in \operatorname{Coh}^{\omega,B}(X)$, then by definition there are some $T \in \mathcal{T}_{\omega,B}$ and $F \in \mathcal{F}_{\omega,B}$ such that

$$0 \longrightarrow F[1] \longrightarrow E \longrightarrow T \longrightarrow 0.$$

Then since $\omega \cdot \operatorname{ch}_1^B(T) > 0$ and $\omega \cdot \operatorname{ch}_1^B(F) \leq 0$ by additivity $\Im Z_{\omega,B}(E) = \omega \cdot \operatorname{ch}_1^B(E) = \omega \cdot \operatorname{ch}_1^B(T) - \omega \cdot \operatorname{ch}_1^B(F) \geq 0$.

Moreover, if $\Im Z_{\omega,B}(E) = 0$, this means that T is supported in dimension 0 and $\mu_{\omega,B}(F) = 0$, so F needs also to be $\mu_{\omega,B}$-semistable in order to belong to $\mathcal{F}_{\omega,B}$.

Thus, on the one hand, $\Re Z_{\omega,B}(T) = -\operatorname{ch}_2^B(T) < 0$ because T is a zero-dimensional supported torsion sheaf so $\operatorname{ch}_2^B(T) = length(Supp(T)) > 0$.

In the other hand, we can use the Bogomolov's inequality to obtain that $2\operatorname{ch}_0^B(F) \cdot \operatorname{ch}_2^B(F) \leq \operatorname{ch}_1^B(F)^2$, but since $\omega \cdot \operatorname{ch}_1^B(F) = 0$ with ω ample, the Hodge index theorem implies that $\operatorname{ch}_1^B(F)^2 \leq 0$. Therefore, $2\operatorname{ch}_0^B(F) \cdot \operatorname{ch}_2^B(F) \leq \operatorname{ch}_1^B(F)^2 \leq 0$ so $\operatorname{ch}_2^B(F) \leq 0$. This leads to

$$\Re Z_{\omega,B}(F) = \frac{\omega^2}{2}\operatorname{ch}_0^B(F) - \operatorname{ch}_2^B(F) \geq 0.$$

Therefore

$$\Re Z_{\omega,B}(E) = \Re Z_{\omega,B}(T) - \Re Z_{\omega,B}(F) < 0.$$

□

Now that we have seen that $Z_{\omega,B}$ is indeed a stability function on the tilted heart $\operatorname{Coh}^{\omega,B}$ we want to verify that the pair $(\operatorname{Coh}^{\omega,B}, Z_{\omega,B})$ satisfies the other properties needed to be a Bridgeland stability condition: the existence of the Harder-Narasimhan filtrations and the support property. We'll prove the first property via Proposition 2.20 showing that $\operatorname{Coh}^{\omega,B}(X)$ is a noetherian category.

Lemma 3.12. $\operatorname{Coh}^{\omega,B}(X)$ *is noetherian.*

Proof. Recall that $\Im Z_{\omega,B}$ is discrete because B is rational and then the images of ch_0^B and ch_1^B are discrete. In fact if $\omega \cdot B = \frac{p}{q}$ then $\omega \cdot \operatorname{ch}_1 \in \frac{1}{q}\mathbb{Z}_{\geq 0}$.

Suppose $M \in \operatorname{Coh}^{\omega,B}(X)$ with an infinite filtration

$$0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_l \subsetneq \cdots \subset M$$

with $\Im Z_{\omega,B}(A_l) = 0$ for all l. We can do this because in some point $\Im Z_{\omega,B}(A_k)$ has to take a fixed value for all $k \gg 0$, so we start from this A_k and mod out M by it.

Define $Q_l := M/A_l$ so we have an exact sequence

$$0 \longrightarrow A_l \longrightarrow M \longrightarrow Q_l \longrightarrow 0.$$

Since we are in $\mathrm{Coh}^{\omega,B}(X)$, applying the cohomology functor in $\mathrm{Coh}(X)$ gives an exact sequence

$$0 \to H^{-1}(A_l) \to H^{-1}(M) \to H^{-1}(Q_l) \to H^0(A_l) \to H^0(M) \to H^0(Q_l) \to 0.$$

This gives two sequences

$$H^0(M) = H^0(Q_0) \twoheadrightarrow H^0(Q_1) \twoheadrightarrow H^0(Q_2) \twoheadrightarrow \cdots$$
$$0 = H^{-1}(A_0) \hookrightarrow H^{-1}(A_1) \hookrightarrow H^{-1}(A_2) \hookrightarrow \cdots \hookrightarrow H^{-1}(M).$$

because H^0 and H^{-1}. preserve surjective an injective morphisms respectively from $\mathrm{Coh}^{\omega,B}(X)$ to $\mathrm{Coh}(X)$. Since this two sequences are now in $\mathrm{Coh}(X)$ which is noetherian they stabilize at some $k \gg 0$. The we can also assume that $H^0(Q_l) = H^0(Q_{l+1}),\ H^{-1}(A_l) = H^{-1}(A_{l+1})$ for all l. This means that $U := H^{-1}(M)/H^{-1}(A_l)$ is constant and $\ker(H^0(M) \twoheadrightarrow H^0(Q_l)) = 0$ because $H^0(M) = H^0(Q_l)$ for all $l \geq 0$.

We can suppose $H^0(M) = H^0(Q_l)$ because if $H^0(M) \twoheadrightarrow H^0(Q_1) \twoheadrightarrow H^0(Q_2) \twoheadrightarrow \cdots$ stabilizes at k then we take the chain

$$A_k \subset A_{k+1} \subset \cdots \subset M$$

that stabilizes if and only if

$$0 \subset A_{k+1}/A_k \subset A_{k+2}/A_k \subset M/A_k = Q_k$$

stabilizes. This new sequence has $(M/A_k)/(A_{k+l}/A_k) \cong Q_l$ so $H^0(M/A_k) = H^0(Q_l)$ for all l.

So this gives a new exact sequence

$$0 \longrightarrow U \longrightarrow H^{-1}(Q_l) \longrightarrow H^0(A_l) \longrightarrow 0.$$

Also consider $B_l := A_l/A_{l-1}$. The short exact sequence associated to this quotient gives the exact sequence

$$0 \to H^{-1}(A_{l-1}) \to H^{-1}(A_l) \to H^{-1}(B_l) \to H^0(A_{l-1}) \to H^0(A_l) \to H^0(B_l) \to 0.$$

Notice that $H^{-1}(A_l) \cong H^{-1}(A_{l-1})$ by assumption so the morphism $H^{-1}(A_l) \to H^{-1}(B_l)$ has to be the zero morphism which implies that $H^{-1}(B_l) \to H^0(A_{l-1})$ is injective but $H^{-1}(B_l)$ is torsion-free and $H^0(A_{l-1})$ is torsion (because $\Im Z_{\omega,B}(A_{l-1}) = 0$) so $H^{-1}(B_l)$ must be 0. Since $\Im Z_{\omega,B}(A_l) = 0$ this means that $\Im Z_{\omega,B}(H^{-1}(A_l)) = 0$ and also that $\Im Z_{\omega,B}(M) = \Im Z_{\omega,B}(Q_l)$, together with $H^0(M) = H^0(Q_l)$ imply that $\Im Z_{\omega,B}(H^{-1}(M)) = \Im Z_{\omega,B}(H^{-1}(Q_l))$. Since $\Im Z_{\omega,B}(U) = \Im Z_{\omega,B}(H^{-1}(M)) - \Im Z_{\omega,B}(H^{-1}(A_l)) = \Im Z_{\omega,B}(H^{-1}(M))$ this implies that $\Im Z_{\omega,B}(H^0(A_l)) = \Im Z_{\omega,B}(H^{-1}(Q_l)) - \Im Z_{\omega,B}(U) = 0$ which is the same as $\omega \cdot \mathrm{ch}_1^B(H^0(A_l)) = 0$. Since we also have that $\omega^2 \mathrm{ch}_0^B(H^0(A_l)) = 0$ this implies that $H^0(A_l) = 0$. Therefore $H^{-1}(A_l)[1] \cong A_l$ meaning that the sequence stabilizes and we are done. $\square$

Now that we have proved that $(\mathrm{Coh}^{\omega,B}(X), Z_{\omega,B})$ is a stability condition we denote it by $\sigma_{\omega,B}$. We only have to check the support property and we're (almost) done. Before we need a couple lemmas.

Lemma 3.13. *If $E \in \mathrm{Coh}^{\omega,B}(X)$ is $\sigma_{\alpha\omega,B}$-semistable for all $\alpha \gg 0$ then one of the following conditions holds:*

(1) $H^{-1}(E) = 0$ and $H^0(E)$ is a torsion-free $\mu_{\omega,B}$-semistable.
(2) $H^{-1}(E) = 0$ and $H^0(E)$ is torsion.
(3) $H^{-1}(E)$ is $\mu_{\omega,B}$-semistable and $H^0(E)$ is zero or torsion supported in dimension zero.

Proof. Notice that we can compute $\sigma_{\alpha\omega,B}$-stability with $\nu_{\alpha\omega,B}$ or $\frac{2\nu_{\alpha\omega,B}}{\alpha}$ because this change doesn't affect which objects are stable or semistable at all. Also note that

$$\lim_{\alpha\to\infty} \frac{2\nu_{\alpha\omega,B}}{\alpha}(E) = \lim_{\alpha\to\infty} \frac{\frac{2\operatorname{ch}_2(E)}{\alpha^2} - \omega^2 \operatorname{ch}_0(E)}{\omega \cdot \operatorname{ch}_1(E)} = \frac{-1}{\mu_{\omega,B}(E)}.$$

By definition of the tilted heart, E fits into an exact sequence

$$0 \longrightarrow F[1] \longrightarrow E \longrightarrow T \longrightarrow 0$$

where $F \in \mathcal{F}_{\omega,B}$ and $T \in \mathcal{T}_{\omega,B}$.

If $\omega \cdot \operatorname{ch}_1^B(E) = 0$ then $\omega \cdot \operatorname{ch}_1^B(F) = \omega \cdot \operatorname{ch}_1^B(T) = 0$ which imply that T is 0 or it's a torsion sheaf supported in dimension zero and F is 0 or a $\mu_{\omega,B}$-semistable torsion-free sheaf (because there can only be one HN-factor it has infinite slope) so this case is true and leads to cases (2) or (3) depending on the different possibilities for T and F.

If $\omega \cdot \operatorname{ch}_1^B(E) > 0$ and $\operatorname{ch}_0^B(E) \geq 0$ then $\frac{-1}{\mu_{\omega,B}(E)} < 0$. By definition of $\mathcal{F}_{\omega,B}$ we have $\mu_{\omega,B}(F[1]) \leq 0$ so $\frac{-1}{\mu_{\omega,B}(F[1])} \geq 0$. If F was not zero, since E is $\sigma_{\alpha\omega,B}$-semistable for all $\alpha \gg 0$ this would mean that $\nu_{\alpha\omega,B}(F) \leq \nu_{\alpha\omega,B}(E)$ for $\alpha \gg 0$ but this is impossible because when α tends to ∞ the limits above give a contradiction. So F must be zero and therefore $E \in \mathcal{T}_{\omega,B}$. If E is torsion we're done because this leads again to case (2). If E is not torsion, but it is not slope semistable (in particular, if it has torsion), by definition of $\mathcal{T}_{\omega,B}$, there is some $A \in \mathcal{T}_{\omega,B}$ that destabilizes E i.e. $\mu_{\omega,B}(A) > \mu_{\omega,B}(E) > 0$. This implies that $\frac{-1}{\mu_{\omega,B}(E)} < \frac{-1}{\mu_{\omega,B}(A)}$ contradicting the $\sigma_{\alpha\omega,B}$-stability of E for $\alpha \gg 0$. So, we are in case (1).

If $\omega \cdot \operatorname{ch}_1^B(E) > 0$ and $\operatorname{ch}_0^B(E) < 0$, we have two cases. If $\omega \cdot \operatorname{ch}_1^B(T) = 0$, then $T \in \mathcal{T}_{\omega,B}$ which implies that $\operatorname{ch}_0^B(T) = 0$. If $\omega \cdot \operatorname{ch}_1^B(T) > 0$, then $\frac{-1}{\mu_{\omega,B}(T)} < 0$ together with the $\sigma_{\omega,B}$-stability of E, which has $\frac{-1}{\mu_{\omega,B}(E)} > 0$, this gives $T = 0$ as before. In both cases, we will be in case (3) if we can show that F is a $\mu_{\omega,B}$-semistable torsion-free sheaf. If this isn't the case, we must have a destabilizing object $A \in \mathcal{F}_{\omega,B}$ such that $\mu_{\omega,B}(A) > \mu_{\omega,B}(F)$. Then there is an injection $A[1] \hookrightarrow E$ that leads to $\frac{-1}{\mu_{\omega,B}(A[1])} > \frac{-1}{\mu_{\omega,B}(E)}$ that directly contradicts the $\sigma_{\alpha\omega,B}$-stability of E for large enough α. $\square$

The discriminant begin positive helped us in our goal of proving that $Z_{\omega,B}$ is a stability function and now the ω-discriminant and the (ω, B, C_ω)-discriminant will prove (directly) the support property. In fact the support property of a stability condition $(\mathcal{P}, Z)$ is equivalent to giving a quadratic form Q on $\lambda_{\mathbb{R}}$ which is semi-positive definite for semistable objects in $\mathcal{P}$ and negative definite on $\ker Z$. This quadratic form, as theorem 3.14 states, $\overline{\Delta}_H^B$. So we want again a "Bogomolov inequality" for the ω-discriminant and the (ω, B, C_ω)-discriminant, but now, for $\sigma_{\omega,B}$-semistable objects!

The proof of the following theorem can be found in [BMS16]. The idea is to use Lemma 3.13 to prove it when E is $\sigma_{\alpha\omega,B}$-semistable for all $\alpha > 0$. If not, that is, if there exists an α such that E is $\sigma_{\alpha\omega,B}$-semistable, when B is rational, the idea is to use induction on $\omega \operatorname{ch}_1^B(E)$ to prove at the same time that the "wall-crossing" is well-behaved and that the discriminants are non negative.

The fact that we need to prove the wall-crossing in parallel, makes this proof quite complicated, so we won't give the prove here. Moreover, since the aim of this thesis is to show a very different technique that could be useful in some cases to improve this bound for $\overline{\Delta}^B_\omega$, this may also justify that we skip this proof.

Theorem 3.14 (Bogomolov inequality for σ-semistable objects)**.** *Let X be a smooth projective surface over $\mathbb{C}$ and $\omega, B \in NS(X) \otimes \mathbb{R}$ with ω begin ample as usual. For any $\sigma_{\omega,B}$-semistable sheaf E we have*

$$\Delta^C_{\omega,B}(E) \geq 0 \text{ and } \overline{\Delta}^B_\omega(E) \geq 0.$$

The following lemma tells us why $\mathrm{Coh}^{\omega,B}(X)$ is the right heart to work with.

Lemma 3.15. *If $\sigma = (\mathcal{A}, Z_{\omega,B})$ is a stability condition satisfying the support property and all skyscraper sheaves are stable of phase one (i.e. $\mathbb{C}(x) \in \mathcal{P}(1)$) then $\mathcal{A} = \mathrm{Coh}^{\omega,B}(X)$.*

Proof. The first step is showing that if $E \in \mathcal{A}$ then all $H^i(E)$ vanish except for $i = -1, 0$ and that $H^{-1}(E)$ is always torsion free. In any case, E is an iterated extension of stable objects so we can suppose that E is stable without problems. Since skyscraper sheaves have phase one, they only have $H^0 \neq 0$, so they satisfy the claim and we can assume is not one of them. The idea now is to use the following result: if X is a smooth projective variety, any $E \in \mathrm{D^b}(X)$ with $\mathrm{Ext}^i(E, \mathbb{C}(x)) = 0$ for all $x \in X$, and $i \notin \{0, 1, 2, \ldots, s\}$ for some positive integer $s \in \mathbb{Z}$ is isomorphic to a complex $F^\bullet$ of locally free sheaves such that $F^i = 0$ if $i \notin \{0, -1, -2, \ldots, -s\}$. Suppose then E is not a skyscraper sheaf. By Serre duality, for any $x \in \mathbb{C}$ we have

$$\mathrm{Hom}(E, \mathbb{C}(x)[i]) \cong \mathrm{Ext}^i(E, \mathbb{C}(x)) \cong \mathrm{Ext}^i(E, \mathbb{C}(x) \otimes \omega_X) \cong \mathrm{Ext}^{2-i}(\mathbb{C}(x), E) = \mathrm{Hom}(\mathbb{C}(x), E[2-i]).$$

Since both $E, \mathbb{C}(x) \in \mathcal{A}$, HBT1) implies that the previous vector spaces vanish for all $i \neq 0, 1, 2$. Furthermore, if $i = 2$ we have $\mathrm{Hom}(\mathbb{C}(x), E) = 0$ because $\nu_{\omega,B}(\mathbb{C}(x)) = +\infty > \nu_{\omega,B}(E)$ because if $\nu_{\omega,B}(E) = +\infty$ this means that E is supported in dimension zero and stability implies that has to be supported at a unique point and hence it must be a skyscraper sheaf contradicting our previous assumption. Thus, the above stated result works for $s = 1$ and we can assume that E is a two term complex of locally free sheaves in positions 0 and -1. The cohomology of the complex $G^{-1} \to G^0$ is the cohomology of E so we only have the 0th and -1th cohomology and $H^{-1}(E) = \ker(G^{-1} \to G)$ is torsion free.

Now we have $\mathcal{A} \subset \langle \mathrm{Coh}(x), \mathrm{Coh}(X)[1] \rangle$ and by Lemma 3.5, $\mathcal{A} = \langle \mathcal{T}, \mathcal{F}[1] \rangle$ where $\mathcal{T} := \mathrm{Coh}(X) \cap \mathcal{A}$ and $\mathcal{F} := \mathrm{Coh}(X) \cap \mathcal{A}[-1]$. Thus we have to see that in fact $\mathcal{T} = \mathcal{T}_{\omega,B}$ and $\mathcal{F} = \mathcal{F}_{\omega,B}$ but, in fact, it is enough to show that $\mathcal{T}_{\omega,B} \subset \mathcal{T}$ and $\mathcal{F}_{\omega,B} \subset \mathcal{F}$.

Consider $E \in \mathrm{Coh}(X)$ slope semistable. By the definition of a torsion pair there is an exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0.$$

We've seen that $F = H^{-1}(F)$ has to be torsion free so, if E is torsion, then $F = 0$ and $E = T$. If E is torsion free, and since $F[1], T \in \mathcal{A}$, we have $\mu_{\omega,B}(F) \leq 0 \leq \mu_{\omega,B}(T)$. This directly contradicts the stability of E (because if E is stable the $\mu_{\omega,B}(E) < \mu_{\omega,B}(F)$) unless F or T vanishes. In any case $E \in \mathcal{T}$ or $E \in \mathcal{F}$.

If $\omega \cdot \mathrm{ch}_1^B(E) > 0$ by stability and the previous observation $E \in \mathcal{T}$ and if $\omega \cdot \mathrm{ch}_1^B(E) < 0$ also by stability and the previous observation $E \in \mathcal{F}$. So we're reduced to the case $\omega \cdot \mathrm{ch}_1^B(E) = 0$. In this case E must belong to $\mathcal{F}$ so we're going to suppose that it belongs to $\mathcal{T}$ and argue by contradiction. If $E \in \mathcal{T}$ then $Z_{\omega,B}(E) < 0$ (it has no imaginary part) and E is $\sigma_{\omega,B}$-semistable. Recall that we always have the following exact sequence for a skyscraper sheaf

$$0 \longrightarrow \mathfrak{m}_x \longrightarrow \mathcal{O}_X \longrightarrow \mathbb{C}(x) \longrightarrow 0.$$

Since tensoring is right-exact we have a surjection $\mathcal{O}_X \otimes F \cong F \twoheadrightarrow \mathbb{C}(x) \otimes F \cong \mathbb{C}(x)^{\mathrm{rk}(F)} \twoheadrightarrow \mathbb{C}(x)$ for an coherent sheaf F and $x \in \mathrm{supp}(F)$. In particular we have $E \twoheadrightarrow \mathbb{C}(x)$ for some $x \in \mathrm{supp}(E)$ (recall that E is a coherent sheaf because it belongs to $\mathcal{T}$). Since $\mathbb{C}(x)$ is stable of slope $+\infty$ this surjection is in $\mathcal{A}$ and by abelianity so does the kernel F. Then $F \in \mathrm{Coh}(X) \cap \mathcal{A} = \mathcal{T}$ and $Z_{\omega,B}(F) + Z_{\omega,B}(\mathbb{C}(x)) = Z(E)$ but $Z_{\omega,B}(\mathbb{C}(x)) = -1$ because it's supported in one unique point and has length 1 so $Z_{\omega,B}(F) = Z_{\omega,B}(E) + 1$. The we can repeat this process until $Z_{\omega,B}(F) \geq 0$ which is impossible because $Z_{\omega,B}$ is a stability function. Therefore $E \in \mathcal{F}$ and we're done. $\square$

The last step before moving into the final part is a nice description of the slice of the stability manifold $\mathrm{Stab}(X)$ where we're going to work in. This slice is usually called the (α, β)-plane.

Definition 3.16. Let $H \in NS(X)$ an ample divisor class and $B_0 \in NS(X) \otimes \mathbb{Q}$. We define the (α, β)-plane as the subset of $\mathrm{Stab}(X)$ of stability conditions of the form $\sigma_{\alpha H, B_0 + \beta H}$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$.

The structure of the (α, β)-plane is as follows.

Proposition 3.17. *Given a fixed class $v \in K_{num}(X)$ we have that:*

1. *All numerical walls are either semicircles centered at the β-axis or vertical rays.*
2. *Two different numerical walls can't intersect.*
3. *The hyperbola $\Re Z_{\alpha,\beta}(v) = 0$ intersects all semicircular walls at their top points (here there is an abuse of notation where $\alpha H, B_0 + \beta H$ gets replaced by α, β).*
4. *If $\mathrm{ch}_0^\beta(v) \neq 0$ there is a unique numerical vertical wall defined by*
$$\beta = \mu_{H,0}(v).$$
5. *If $\mathrm{ch}_0^\beta(v) \neq 0$ all semicircular walls to either side of the vertical wall are strictly nested.*
6. *If $\mathrm{ch}_0^\beta(v) = 0$ there are only strictly nested semicircular walls.*
7. *If a wall is an actual wall at a single point, then it's an actual wall everywhere along the numerical wall. Here by actual wall it means that stable objects change when trespassing the numerical wall.*

Note that the walls described in the previous result divide the (α, β)-plane in chambers separated by real codimension 1 submanifolds (the walls) and (semi)stable objects doesn't change inside the chambers. Then the last property tells us that if (semi)stable objects change at a point of a numerical wall, then they change along the whole wall. Thus, we see numerical walls as "candidates" to actual walls that separate different chambers such that, within this chamber (semi)stable objects don't change.

Proof. Recall that numerical walls we're defined as

$$W_w(v) := \left\{ (\mathcal{A}, Z) \in \mathrm{Stab}(X) \ : \ \frac{\Re Z}{\Im Z}(v) = \frac{\Re Z}{\Im Z}(w) \right\}.$$

If we intersect with the (α, β)-plane the equality becomes

$$\frac{-2\,\mathrm{ch}_2^\beta(v) + \alpha^2 H^2\,\mathrm{ch}_0^\beta(v)}{2\alpha H\,\mathrm{ch}_1^\beta(v)} = \frac{-2\,\mathrm{ch}_2^\beta(w) + \alpha^2 H^2\,\mathrm{ch}_0^\beta(w)}{2\alpha H\,\mathrm{ch}_1^\beta(w)}.$$

Developing the twisted Chern characters on the left hand side, we have

$$\frac{-2\,\mathrm{ch}_2(v) + 2(B_0 + \beta H)\,\mathrm{ch}_1(v) - (B_0 + \beta H)^2\,\mathrm{ch}_0(v) + \alpha^2 H^2\,\mathrm{ch}_0(v)}{2\alpha H\,\mathrm{ch}_1(v) - 2\alpha H (B_0 + \beta H)\,\mathrm{ch}_0(v)}.$$

Notice that we can suppose $B_0 \cdot H = 0$ because otherwise we can replace B_0 with $B_0 - \frac{B_0 \cdot H}{H^2} H$ and the result will be the same because this is only a translation in the direction of H. With this in mind, if $NS(X) \otimes \mathbb{Q}$ has dimension 1 then $\mathrm{ch}_1(v) = y_v H$. If $NS(X) \otimes \mathbb{Q}$ has dimension strictly higher than 1 we have an orthogonal decomposition $NS(X) \otimes \mathbb{Q} = \langle H \rangle \oplus \langle B_0 \rangle \oplus \langle H, B_0 \rangle^\perp$ so $\mathrm{ch}_1(v) = y_v H + z_v B_0 + \tilde{H}_v$. We can suppose then that $\mathrm{ch}_1(v)$ is in all cases of this form. Changing $\mathrm{ch}_1(v)$ in the last expression we get

$$\frac{-2\,\mathrm{ch}_2(v) + 2 z_v B_0^2 + 2 y_v \beta H^2 + (\alpha^2 H^2 - B_0^2 - \beta^2 H^2)\,\mathrm{ch}_0(v)}{2\alpha (y_v - \beta\,\mathrm{ch}_0(v)) H^2}.$$

Since we get the same expression for w, if we rewrite the equation by moving everything to the right hand side and collect terms in α and β in the numerator we get

$$\begin{aligned} \alpha^2 H^2 (\mathrm{ch}_0(v) y_w - \mathrm{ch}_0(w) y_v) + \beta^2 H^2 (\mathrm{ch}_0(v) y_w - \mathrm{ch}_0(w) y_v) - \\ -2\beta (B_0^2 (z_w\,\mathrm{ch}_0(v) - z_v\,\mathrm{ch}_0(w)) + \mathrm{ch}_2(v)\,\mathrm{ch}_0(w) - \mathrm{ch}_2(w)\,\mathrm{ch}_0(v)) + \\ +2(\mathrm{ch}_2(w) y_v - \mathrm{ch}_2(v) y_w) + B_0^2 (2(z_v y_w - z_w y_v) + (\mathrm{ch}_0(w) y_v - \mathrm{ch}_0(v) y_w)) = 0. \end{aligned}$$

Now we have two cases that depend on $\mathrm{ch}_0(v) y_w - \mathrm{ch}_0(w) y_v$. Notice that $\mathrm{ch}_0(v) y_w - \mathrm{ch}_0(w) y_v = 0$ if and only if $H^2\,\mathrm{ch}_0(v) \cdot H\,\mathrm{ch}_1(v) = H^2\,\mathrm{ch}_0(w) \cdot H\,\mathrm{ch}_1(w)$ if and only if $\mu_{H,0}(v) = \mu_{H,0}(w)$. So, if $\mu_{H,0}(v) \neq \mu_{H,0}(w)$ we can divide by $H^2(\mathrm{ch}_0(v) y_w - \mathrm{ch}_0(w) y_v)$ and we get

$$\alpha^2 + \beta^2 - 2\beta A + B = 0,$$

where

$$A = \frac{B_0^2 (z_w\,\mathrm{ch}_0(v) - z_v\,\mathrm{ch}_0(w)) + \mathrm{ch}_2(v)\,\mathrm{ch}_0(w) - \mathrm{ch}_2(w)\,\mathrm{ch}_0(v))}{H^2 (\mathrm{ch}_0(v) y_w - \mathrm{ch}_0(w) y_v)}$$

and

$$B = \frac{2(\mathrm{ch}_2(w) y_v - \mathrm{ch}_2(v) y_w) + B_0^2 (2(z_v y_w - z_w y_v) + (\mathrm{ch}_0(w) y_v - \mathrm{ch}_0(v) y_w))}{H^2 (\mathrm{ch}_0(v) y_w - \mathrm{ch}_0(w) y_v)}.$$

Thus the numerical wall in the (α, β)-plane is:

$$W_w(v) \ = \{(\alpha, \beta) \in]0, +\infty[\times \mathbb{R} \ : \ \alpha^2 + (\beta - A)^2 = A^2 - B\}.$$

The radius of this semicircle is $R = \sqrt{A^2 - B}$ when this happens to be a positive real number and the center is $(0, A)$.

On the other hand, if $\mu_{H,0}(v) = \mu_{H,0}(w)$ then the equation is

$$\beta = \frac{\operatorname{ch}_2(w)y_v - \operatorname{ch}_2(v)y_w + B_0^2((z_v y_w - z_w y_v) + (\operatorname{ch}_0(w)y_v - \operatorname{ch}_0(v)y_w)}{B_0^2(z_w \operatorname{ch}_0(v) - z_v \operatorname{ch}_0(w)) + \operatorname{ch}_2(v)\operatorname{ch}_0(w) - \operatorname{ch}_2(w)\operatorname{ch}_0(v))}.$$

If $\operatorname{ch}_0(v) \neq 0$ we can isolate $y_w = \frac{\operatorname{ch}_0(w)y_v}{\operatorname{ch}_0(v)}$ and we get

$$\beta = \frac{\operatorname{ch}_2(w)\operatorname{ch}_0(v)y_v - \operatorname{ch}_2(v)\operatorname{ch}_0(w)y_v + B_0^2((z_v \operatorname{ch}_0(w)y_v - z_w y_v \operatorname{ch}_0(v))}{(B_0^2(z_w \operatorname{ch}_0(v) - z_v \operatorname{ch}_0(w)) + \operatorname{ch}_2(v)\operatorname{ch}_0(w) - \operatorname{ch}_2(w)\operatorname{ch}_0(v)))\operatorname{ch}_0(v)} = \frac{-y_v}{\operatorname{ch}_0(v)} = \mu_{H,0}(v).$$

Notice that in general if $\operatorname{ch}_0(v) \neq 0$

$$\frac{2y_v A}{\operatorname{ch}_0(v)} + B = \frac{-B_0^2(2z_v - \operatorname{ch}_0(v))(\operatorname{ch}_0(w)y_v - y_w \operatorname{ch}_0(v)) + 2\operatorname{ch}_2(v)(\operatorname{ch}_0(w)y_v - y_w \operatorname{ch}_0(v))}{\operatorname{ch}_0(v)H^2(\operatorname{ch}_0(v)y_w - \operatorname{ch}_0(w)y_v)}.$$

If $\mu_{H,0}(v) \neq \mu_{H,0}(w)$ then we can divide by $\operatorname{ch}_0(w)y_v - y_w \operatorname{ch}_0(v)$ and we have

$$\frac{2y_v A}{\operatorname{ch}_0(v)} + B = \frac{B_0^2(2z_v - \operatorname{ch}_0(v)) - 2\operatorname{ch}_2(v)}{\operatorname{ch}_0(v)H^2} =: C.$$

This shows that dependence of A and B and vice-versa is completely independent of ω if $\operatorname{ch}_0(v) \neq 0$. If $\operatorname{ch}_0(v) = 0$ then the center is

$$A = \frac{-B_0^2 z_v \operatorname{ch}_0(w) + \operatorname{ch}_2(v)\operatorname{ch}_0(w)}{-H^2 \operatorname{ch}_0(w)y_v} = \frac{B_0^2 z_v - \operatorname{ch}_2(v)}{H^2 y_v}$$

that neither depends on w (y_v doesn't vanish unless v is the class of some zero dimensional supported sheaf in which case there are no numerical walls). Thus in this last case the center is fixed so increasing w leads to bigger semicircles with same center so they are nested. If $\operatorname{ch}_0(v) \neq 0$ then consider $f_\pm(A) = A \pm R = A \pm \sqrt{(A - \frac{y_v}{\operatorname{ch}_0(v)})^2 - \frac{y_v^2}{\operatorname{ch}_0(v)^2} - C}$. Now we derive this functions we get $f'_\pm(A) = 1 \pm \frac{A + \frac{y_v}{\operatorname{ch}_0(v)}}{R}$ Also $\frac{y_v^2}{\operatorname{ch}_0(v)^2} + C$ measures the difference between the center and the radius. In fact

$$\begin{aligned}
\frac{y_v^2}{\operatorname{ch}_0(v)^2} + C &= \frac{1}{\operatorname{ch}_0(v)H^2}\left(B_0^2(2z_v \operatorname{ch}_0(v) - \operatorname{ch}_0(v)^2) + y_v^2 H_v^2 - 2\operatorname{ch}_2(v)\operatorname{ch}_0(v)\right) \\
&= \frac{1}{\operatorname{ch}_0(v)H^2}\left(-B_0^2(\operatorname{ch}_0(v) - z_v)^2 + y_v^2 H_v^2 + z_v^2 B_0^2 - 2\operatorname{ch}_2(v)\operatorname{ch}_0(v)\right) \\
&= \frac{1}{\operatorname{ch}_0(v)H^2}\left(-B_0^2(\operatorname{ch}_0(v) - z_v)^2 + \operatorname{ch}_1(v)^2 - \tilde{H}_v^2 - 2\operatorname{ch}_2(v)\operatorname{ch}_0(v)\right) \\
&= \frac{1}{\operatorname{ch}_0(v)H^2}\left(-B_0^2(\operatorname{ch}_0(v) - z_v)^2 + \Delta(v) - \tilde{H}_v^2\right)
\end{aligned}$$

By the Hodge index theorem, since B_0 and $\tilde{H}_v$ are orthogonal to H which is ample then $B_0^2 \leq 0$ and $\tilde{H}_v^2 \leq 0$. This implies that $\frac{y_v^2}{\operatorname{ch}_0(v)^2} + C \geq 0$ (when v is the class of a semistable sheaf) and then $R \leq \left|A + \frac{y_v}{\operatorname{ch}_0(v)}\right|$ always. This meas that $|f'_\pm(A)| \geq 0$ i.e. the amounts $A + R$ and $A - R$ grow and decrease together with the radius an hence they are nested.

We have seen (1), (2), (4), (5) and (6) so far. For (3) we have that the hyperbola $\Re Z_{\alpha,\beta} = 0$ is

$$-2\operatorname{ch}_2(v) + 2z_v B_0^2 + 2y_v \beta H^2 + (\alpha^2 H^2 - B_0^2 - \beta^2 H^2)\operatorname{ch}_0(v) = 0.$$

Isolating α and β we get

$$\operatorname{ch}_0(v)H^2\left(\alpha^2-(\beta-\frac{y_v}{\operatorname{ch}_0(v)})^2\right)+\frac{y_v^2H^2}{\operatorname{ch}_0(v)}+-2\operatorname{ch}_2(v)+2z_vB_0^2-B_0^2\operatorname{ch}_0(v)=0.$$

Moving all free terms to the right hand side we get

$$\operatorname{ch}_0(v)H^2\left(\alpha^2-(\beta-\frac{y_v}{\operatorname{ch}_0(v)})^2\right)=\frac{y_v^2H^2+-2\operatorname{ch}_2(v)\operatorname{ch}_0(v)+2z_vB_0^2\operatorname{ch}_0(v)-B_0\operatorname{ch}_0(v)^2}{\operatorname{ch}_0(v)}.$$

Using the same trick as before we have

$$\alpha^2-(\beta-\frac{y_v}{\operatorname{ch}_0(v)})^2=\frac{-B_0^2(\operatorname{ch}_0(v)-z_v)^2+\Delta(v)-\tilde{H}_v^2}{\operatorname{ch}_0(v)^2H^2}=:D.$$

Therefore we have the following two equations so far

$$\alpha^2+(\beta-A)^2=A^2-B=\left(A-\frac{y_v}{\operatorname{ch}_0(v)}\right)^2-D,$$

$$\alpha^2-(\beta-\frac{y_v}{\operatorname{ch}_0(v)})^2=D.$$

The top point of the semicircles is (RH,B_0+AH) so if they belong to the hyperbola we're done with (3). This is clear since $R^2-(A-\frac{y_v}{\operatorname{ch}_0(v)})^2=D=R^2-(A-\frac{y_v}{\operatorname{ch}_0(v)})^2$.

For (7) we have to check that, if a wall $W_w(v)$ has some $(\alpha_0H,B_0+\beta_0H)$ and an $E\in\operatorname{Coh}^{\alpha_0,\beta_0}(X)$ with numerical class v for which there exists an $F\subset E$ such that $\nu_{\alpha_0,\beta_0}(F)=\nu_{\alpha_0,\beta_0}(E)$ then for all $(\alpha H,B_0+\beta H)\in W_w(v)$ we have $\nu_{\alpha,\beta}(F)=\nu_{\alpha,\beta}(E)$ where $w=v(F)$. This means that if a numerical wall is an actual wall in a point (i.e. (semi)stable objects really change when crossing the wall) then is an actual wall in all points (i.e. (semi)stable objects change at all points of the wall). Then E was $\nu_{\alpha,\beta}$-stable in one side of the wall but not on the other side. By Proposition 2.29, at any point of the wall E loses his stability because at the other side of the wall is not $\nu_{\alpha,\beta}$-stable anymore. This implies that it must exist an $F\subset E$ at any point that causes this destabilization. □

This description has very strong consequences in the number of walls that one can find in the (α,β)-plane. This consequences could be also deduced from Proposition 2.29, but the following way will be convenient for our study of the (α,β)-plane. In particular semistable sheaves in the (α,β)-plane are controlled by the H-discriminant.

Lemma 3.18. *Let Q be a quadratic form over a real vector space V and a linear map $Z:V\to\mathbb{C}$ such that* $\ker Z$ *is negative semi-definite respect to Q. For any ray ρ in $\mathbb{C}$ starting at the origin we define*

$$C_\rho^+:=Z^{-1}(\rho)\cap\{v\in V\mid Q(v)\geq 0\}.$$

Then we have that

(1) *if $\omega_1,\omega_2\in C_\rho^+$, then $Q(\omega_1,\omega_2)\geq 0$,*
(2) *C_ρ^+ is a convex cone,*
(3) *for any $\omega,\omega_1,\omega_2\in C_\rho^+$ with $\omega=\omega_1+\omega_2$ the inequality $Q(\omega_1)+Q(\omega_2)\leq Q(\omega)$ holds and $Q(\omega_1)=Q(\omega)$ implies $Q(\omega_1)=Q(\omega_2)=Q(\omega_1,\omega_2)=0$,*
(4) *if* $\ker Z$ *is negative definite with respect to Q any vector $\omega\in C_\rho^+$ with $Q(\omega)=0$ generates an extremal ray of C_ρ^+.*

Proof. If $\omega_1, \omega_2 \in C_\rho^+$ non zero there is some $\lambda > 0$ such that $Z(\omega_1 - \lambda\omega_2) = 0$. This gives $0 \leq Q(\omega_1 - \lambda\omega_2) = Q(\omega_1) + \lambda^2 Q(\omega_2) - 2\lambda Q(\omega_1, \omega_2)$. Since $Q(\omega_1), Q(\omega_2) \geq 0$, this leads to $0 \geq -2\lambda Q(\omega_1, \omega_2)$ that implies $Q(\omega_1, \omega_2) \geq 0$ proving (1) and (2). This also implies the first part of (3) because $Q(\omega) = Q(\omega_1) + 2Q(\omega_1, \omega_2) + Q(\omega_2) \geq 0$ and this also implies the second part of (3). For the last one assume $\ker Z$ is negative definite and take $\omega \in C_\rho^+$ with $Q(\omega) = 0$ and assume is not extremal. This means that there are two linearly independent $\omega_1, \omega_2 \in C_\rho^+$ such that $\omega = \omega_1 + \omega_2$. By (3) $Q(\omega_1) = Q(\omega_2) = Q(\omega_1, \omega_2) = 0$ and hence there is some positive λ such that $Z(\omega_1 - \lambda\omega_2) = 0$ but $\omega_1 - \lambda\omega_2 \neq 0$. This is impossible because $\ker Z$ is negative definite that means $0 > Q(\omega_1 - \lambda\omega_2) = 0$. □

Lemma 3.19. *Let $v \in K_{num}(X)$ be a non-zero class such that $\overline{\Delta}_H^{B_0}(v) \geq 0$. For a given rational number β_0 there are only finitely many (actual) walls intersecting the line $\beta = \beta_0$.*

Proof. Any wall comes from an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

in $\mathrm{Coh}^{H,\beta}(X)$ (here there is again an abuse of notation $\mathrm{Coh}^{H,\beta}(X) = \mathrm{Coh}^{H,B_0+\beta H}(X)$). Let $(R, C, D) := (H^2 \operatorname{ch}_0^\beta(E), H \cdot \operatorname{ch}_1^\beta(E), \operatorname{ch}_2^\beta(E))$, $(r, c, d) := (H^2 \operatorname{ch}_0^\beta(F), H \cdot \operatorname{ch}_1^\beta(F), \operatorname{ch}_2^\beta(F))$, and (r', c', d') in the case of G. Since $\beta \in \mathbb{Q}$ we have that r, c and d are discrete in $\mathbb{R}$ and bounding them will imply finitness. By additivity we have that $0 \leq c \leq C$. If $C = 0$ then $c = 0$ and, on the other hand, $C = H \cdot \operatorname{ch}_1(E) + H \cdot (B_0 + \beta H) \operatorname{ch}_0(E) = H \cdot \operatorname{ch}_1(E) + \beta H^2 \operatorname{ch}_0(E)$ which implies $\beta = \mu_{H,0}(E)$ is the unique vertical line and we have nothing to prove. Suppose $C \neq 0$ and define $\Delta := C^2 - 2RD$ (this is the H-discriminant of E). By Bogomolov's inequality (Theorem 3.14) $c^2 - 2rd \geq 0$ because we can assume without lost of generality that F is $\sigma_{\alpha,\beta}$-semistable. On a wall we have that F, E and G have the same $\nu_{\alpha,\beta}$-slope so the must lie in the same ray. By Lemma 3.18.(3), since $F + G = E$, $\Delta \geq c^2 - 2rd \geq 0$. By this we get $\frac{c^2}{2} \geq rd \geq \frac{c^2 - \Delta}{2}$. Since c is an integer we have finitely many possibles values for rd unless one of them vanishes. If $R = r = 0$ then $\nu_{\alpha,\beta}(E) = \nu_{\alpha\beta}(F)$ if and only if $Dc - Cd = 0$. Dc is nothing else than

$$\begin{aligned} &\operatorname{ch}_2(E) H \cdot \operatorname{ch}_1(F) - \operatorname{ch}_2(E)\beta H^2 \operatorname{ch}_0(F) - \beta H^2 \operatorname{ch}_1(E) \operatorname{ch}_1(F) + \beta^2 H \cdot \operatorname{ch}_1(E) \operatorname{ch}_0(F) \\ &+ (B_0^2 + \beta^2 H^2) \operatorname{ch}_0(E) H \cdot \operatorname{ch}_1(F) - \beta(B_0^2 + \beta^2 H^2) \operatorname{ch}_0(E) \operatorname{ch}_0(F). \end{aligned}$$

Symmetric terms and terms with ch_0 disappear in $Dc - Cd$ so

$$Dc - Cd = \operatorname{ch}_2(E) H \cdot \operatorname{ch}_1(F) - \operatorname{ch}_2(F) H \cdot \operatorname{ch}_1(E) = 0$$

which doesn't depend on α and β so there is no wall.

If $r = 0, R \neq 0$ and $D \neq d$ we have that $R = r$ and $d' \neq 0$. Using the same argument as before we have that $\frac{c'^2}{2} \geq r'd' \geq \frac{c'^2 - \Delta}{2}$ so there are finitely many possibilities for r' and d' and hence for r and d too. If $r = 0$ and $D = d$ we're done because c is already bounded by C. If $D = d = 0$ we have that $\nu_{\alpha,\beta}(E) = \nu_{\alpha\beta}(F)$ if and only if $Rc - Cr = 0$ that neither depends on (α, β) so this doesn't define a wall. If $d = 0$ and $D \neq 0$ and $R - r \neq$ we're in the same case as the inequality above for G and if $d = 0$ and $R = r$ we have finitely many possibilities. □

The main consequence of this lemma is the following:

Corollary 3.20. *Let $v \in K_{num}(X)$ a non-zero class with positive H-discriminant. Then the semicircular walls in the (α, β)-plane are bounded from above. Moreover the walls with respect* *to v are locally finite (i.e. any compact subset of the upper half-plane intersects a finite number of walls).*

3.1. **Improvements of the Bogomolov inequality.** We have stated two Bogomolov inequalities so far. The first one, Theorem 3.10 is used basically to prove that the group homomorphism $Z_{\omega,B}$ is a stability function on the tilted heart $\mathrm{Coh}^{\omega,B}(X)$ (also the characterization Lemma 3.13 which is the basis for the second Bogomolov inequality).

The second one, Theorem 3.14, which we haven't prove it, gives directly the support property. Note that the support property is what allows us to define the wall and chamber structure and the finiteness of wall intersecting a given line $\beta = \beta_0$ in the (α, β)-plane. Moreover, by Lemma 3.13, an object which is $\sigma_{\alpha\omega,B}$-semistable for $\alpha \gg 0$, has the discriminant of $\mu_{\omega,B}$-semistable sheaf. And viceversa, a $\mu_{\omega,B}$-semistable sheaf is $\sigma_{\alpha\omega,B}$-semistable for α big enough Hence, the second Bogomolov inequality implies the first one. Moreover, we can see the Bogomolov inequality as a bound of the quotient $\frac{\mathrm{ch}_2(E)}{H^2 \mathrm{ch}_0(E)}$ in terms of the quotient $\frac{H \mathrm{ch}_1(E)}{H^2 \mathrm{ch}_0(E)}$. Thus, if we are able to improve this bound (for some particular surface X), we will be able to describe a larger region of $\mathrm{Stab}(X)$, than just the (α, β)-plane.

Trying to solve this question is our final goal. Starting from the seminal results by Bridgeland, it is known that one can improve the Bogomolov inequality in the case of K3 surfaces (basically Riemann-Roch makes the work), and it has been seen in [Liu22] and [Li19] that the answer is true in some surfaces constructed as complete intersections. The bounds given in the articles by C. Li and S. Liu are very accurate because they need them to extract consequences for the existence of stability condition on specific types of Calabi-Yau threefolds. Our aim is to get a more general improvement of the Bogomolov inequality, even if it's clearly not be the best possible.

Before anything, we rephrase a little bit our problem so we have a nice starting point. This rephrasing basically consists in considering another parametrization of the (α, β)-plane as it is done in the articles [Li19, Liu18]. In fact, we try to follow the same techniques they've used as far as we can.

We need a change of coordinates in the (α, β)-plane, which doesn't change the condition of being $\nu_{\alpha,\beta}$-(semi)stable. We consider $(\alpha, \beta) \mapsto (\frac{1}{2}(\alpha^2 + \beta^2), \beta)$.

Now, the upper half-plane (that is (α, β) such that $\alpha > 0$) become the upper region of the parabola $\alpha' = \frac{1}{2}\beta^2$. In the representations of the (α, β)-plane, as done before, the convention is to swap the coordinates so the parabola stands vertically.

First we consider a new slope function based on the stability function we have in $\mathrm{Coh}^{\beta}(X)$.

Definition 3.21. For any $E \in \mathrm{Coh}^{\alpha,\beta}(X)$ we define the tilt slope function

$$\mu_{\alpha,\beta,H}(E) := \frac{\mathrm{ch}_2(E) - \alpha H^2 \mathrm{ch}_0(E)}{H \cdot \mathrm{ch}_1^{\beta}(E)}$$

setting $\mu_{\alpha,\beta} = +\infty$ when $H \cdot \mathrm{ch}_1^{\beta}(E) = 0$ (here there is also an abuse of notation when omitting the ample divisor $H \in NS(X)$). As any other slope function E is $\mu_{\alpha,\beta}$-(semi)stable if for any $F \subset E$ we have $\mu_{\alpha,\beta}(F) < (\leq)\mu_{\alpha,\beta}(E/F)$.

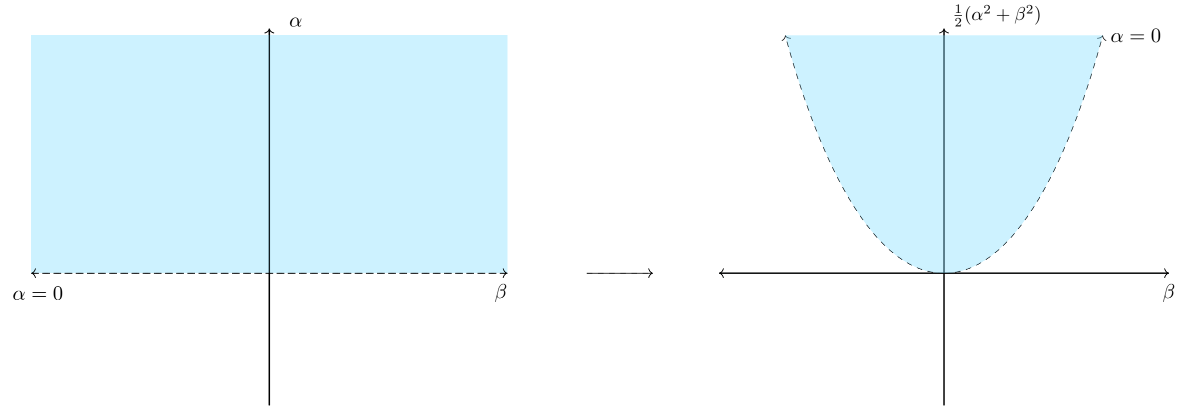


FIGURE 4. (α, β)-plane compared to $(\frac{1}{2}(\alpha^2+\beta^2), \beta)$-plane

We have relation between this tilt slope stability function and our previous $\nu_{\alpha,\beta}$ after changing coordinates because

$$\mu_{\frac{1}{2}(\alpha^2+\beta^2),\beta}(E) = \frac{2\operatorname{ch}_2(E) - \alpha^2 H^2 \operatorname{ch}_0(E) - \beta^2 H^2 \operatorname{ch}_0(E)}{2H\cdot \operatorname{ch}_1^\beta(E)} =$$
$$= \frac{2\operatorname{ch}_2(E) - 2\beta H\cdot\operatorname{ch}_1(E) + \beta^2 H^2\operatorname{ch}_0(E) - \alpha^2 H^2\operatorname{ch}_0(E) - 2\beta^2 H^2 \operatorname{ch}_0(E) + 2\beta H\cdot \operatorname{ch}_1(E)}{2H\cdot\operatorname{ch}_1^\beta(E)}$$
$$= \frac{2\operatorname{ch}_2^\beta(E) - \alpha^2\operatorname{ch}_0(E) - 2\beta H\cdot \operatorname{ch}_1^\beta(E)}{2H\cdot\operatorname{ch}_1^\beta(E)} = \alpha\nu_{\alpha,\beta}(E) - \beta.$$

We also rewrite the H-discriminant as

$$\Delta_H(E) := \left(\frac{H\cdot \operatorname{ch}_1(E)}{H^2\operatorname{ch}_0(E)}\right)^2 - 2\frac{\operatorname{ch}_2(E)}{H^2\operatorname{ch}_0(E)}$$

and define the point

$$p_H(E) = \left(\frac{H\cdot\operatorname{ch}_1(E)}{H^2\operatorname{ch}_0(E)}, \frac{\operatorname{ch}_2(E)}{H^2\operatorname{ch}_0(E)}\right) \in \mathbb{R}^2$$

Notice that there is a relation between our change of variables, the rewriting of the H-discriminant and these points: the parabola given by $\Delta_H = 0$ is the same as $\alpha' = \frac{1}{2}\beta^2$ but now the $p_H(E)$ of a sheaf E that have positive H-discriminant is below the parabola i.e. $\frac{\operatorname{ch}_2(E)}{H^2\operatorname{ch}_0(E)} \leq \frac{1}{2}\left(\frac{H\cdot\operatorname{ch}_1(E)}{H^2\operatorname{ch}_0(E)}\right)^2$. We can do this comparison because of the following lemma:

Lemma 3.22. *If E is $\mu_{\alpha'_0,\beta_0}$-(semi)stable for some $\alpha'_0 > \frac{1}{2}\beta_0^2$ then E is $\mu_{\alpha',\beta}$-(semi)stable for any point (α', β) from the line that joins $p_H(E)$ and (α'_0, β_0) with $\alpha' > \frac{1}{2}\beta^2$. This is E is $\mu_{\alpha',\beta}$-(semi)stable for any (α', β) such that*

$$\det\begin{pmatrix} 1 & \alpha' & \beta \\ 1 & \alpha'_0 & \beta_0 \\ H^2\operatorname{ch}_0(E) & \operatorname{ch}_2(E) & H\cdot\operatorname{ch}_1(E)\end{pmatrix} = 0$$

Proof. It's clear that the semicircular walls we had in the (α, β)-plane have now become lines because $\alpha^2 + (\beta - A)^2 = R^2 = 2\alpha' - 2\beta A = -B$ and vice-versa. Therefore the line that joins $p_H(E)$ with (α'_0, β_0) is a part of a semicircle in the (α, β)-plane. Note that the unique vertical wall passes through the second coordinate of $p_H(E)$ that doesn't lie above the parabola, so

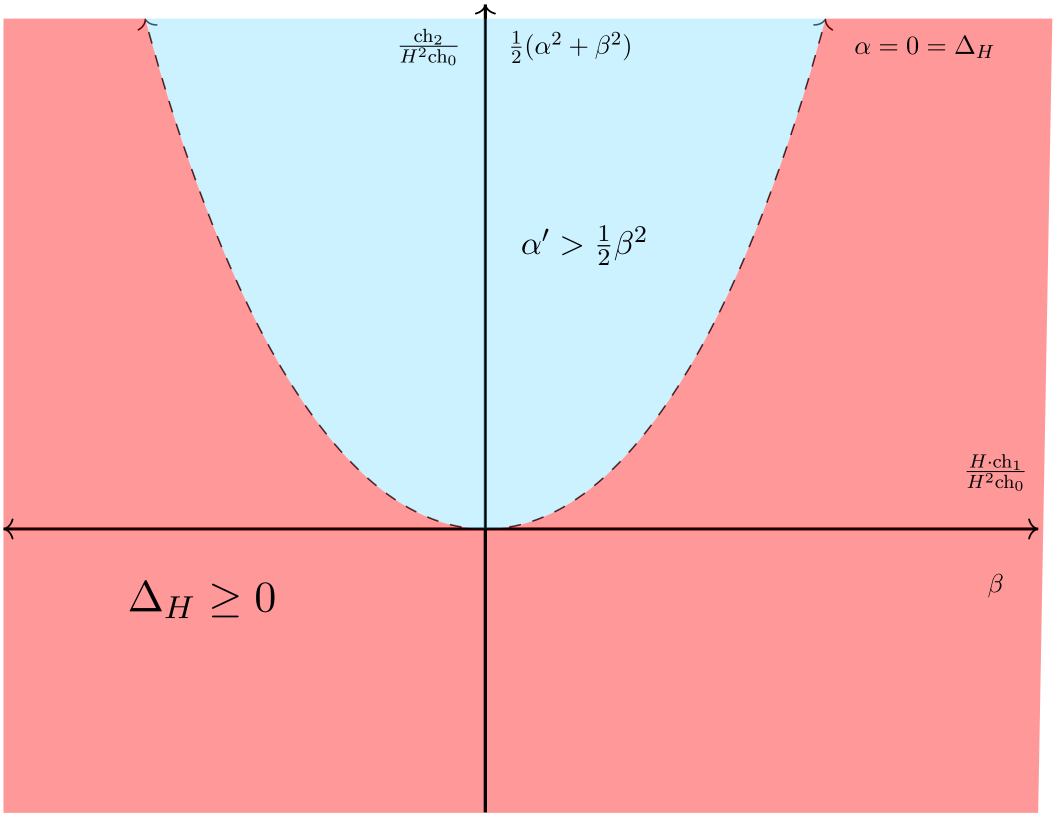


FIGURE 5. Division of $\mathbb{R}^2$ given by the (α', β)-plane and the H-discriminant with the double coordinate system

there are two possibilities for the segment of the line between $p_H(E)$ and (α'_0, β_0) in the upper part of the parabola:

either it is a numerical wall or it's inside a chamber. In a wall E can't be $\mu_{\alpha'_0\beta_0}$-stable but can be $\mu_{\alpha'_0,\beta'_0}$-semistable by the definition of a wall. Thus along the wall we can't have stability on any point but we can have semistability in which case holds in the whole wall. If not we know that between walls stability doesn't change an then we're done. □

Figures 5 and 6 represent our current situation. Figure 5 shows how the plane has been divided in two regions separated by the parabola given by the H-discriminant. Figure 6 show how our walls look in this new setting and their relation between the walls of the class of E and the point $p_H(E)$.

But this parabola is not exactly a parabola always: in some particular cases there are places where Bogomolov's inequality can be improved so the upper part of the parabola can be "slightly enlarged". We will explain a technique to improve Bogomolov's inequality that was introduced in [Fey22, Li19]. The main idea is to use the so called Clifford's inequalities for certain types of curves. If we find a Clifford inequality for a certain curve in our surface such that we now how our sheaf changes when we restrict it to the curve, we can bound its Euler Characteristic which leads into an improvement of the Bogomolov's inequality.

The Clifford type inequalities aim to bound the dimension of the 0-th cohomology of the sheaf on a curve. One useful technique to obtain them is via the cone of the canonical map from

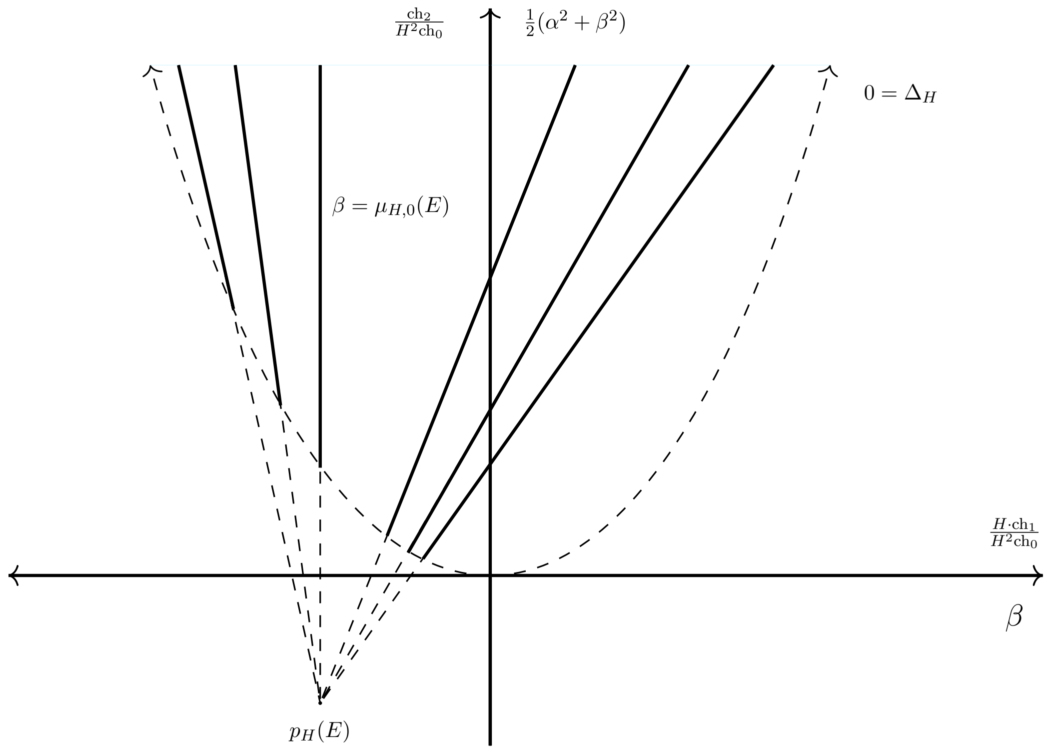


FIGURE 6. Walls for the class of $v(E)$ in the (α', β)-plane

$\mathcal{O}_X$ twisted by a lineal subspace of $\mathrm{Hom}(\mathcal{O}_X, E)$ to E, where E is our sheaf. This produces a new coherent sheaf with the same first and second Chern character of E and with 0-th Chern character $\mathrm{ch}_0(E) - \hom(\mathcal{O}_X, E)$. This allows one to bound $\hom(\mathcal{O}_X, E)$ via the Hizerbuch-Riemann-Roch theorem applied to the cone and E. Clifford's type inequalities are named after the generalized Clifford's index theorem:

Theorem 3.23. *Let C be a projective smooth curve of genus $g \geq 2$ and E a semistable vector bundle over C with $\mu(E) \in [0, 2g-2]$. Then*

$$\frac{h^0(E)}{\mathrm{ch}_0(E)} \leq \frac{1}{2}\mu(E) + 1$$

Then a Clifford type inequality is an inequality of the form

$$\frac{h^0(F)}{\mathrm{ch}_0(F)} \leq f(\mu)\mu + g(\mu)$$

for any semistable locally free sheaf F over a projective smooth curve C of genus $g \geq 2$ in our surface X linearly equivalent to mH where m is a positive integer. The functions f and g are real functions. This inequalities also hold for $\mu \in I \subseteq [0, 2g-2]$ and improve the bound given by the generalized Clifford's index theorem.

We restrict to the case $C \in |mH|$ because we want to use Feyzbakhsh's restriction lemma:

Lemma 3.24 (Feyzbakhsh)**.** *Let (X,H) be a polarized surface and let E be a coherent sheaf in $\mathrm{Coh}^{H,0}(X)$. Suppose there are $\alpha > 0$ and $m \in \mathbb{Z}_{>0}$ such that $E(-mH)[1]$ is in $\mathrm{Coh}(X)$, both E and $E(-mH)[1]$ are $\nu_{\alpha,0}$-tilt stable with $\nu_{\alpha,0}(E) = \nu_{\alpha,0}(E(-mH)[1])$. Then for any smooth irreducible curve $C \in |mH|$ we have that $F|_C$ is μ_H-slope semistable on C. Moreover $\mathrm{ch}_0(F|_C) = \mathrm{ch}_0(F),\ \mathrm{ch}_1(F|_C) = mH\,\mathrm{ch}_1(F)$.*

Proof. A proof can be found in Feyzbakhsh's original paper [Fey22, Corollary 4.3] □

Then the argument go as follows: we begin with $E \in \mathrm{D}^{\mathrm{b}}(X)$ with $\frac{H\,\mathrm{ch}_1(E)}{H^2\,\mathrm{ch}_0(E)} \in]0,1[$ which is $\mu_{\alpha,0}$-stable or $\mu_{\alpha',1}$-stable for some $\alpha > 0$ or $\alpha' > \frac{1}{2}$. Take $C \in |mH|$ a smooth irreducible projective curve. If we prove that E satisfies the conditions in Feyzbakhsh restriction Lemma 3.24, we would have that $E|_C$ has $\mathrm{ch}_0(E|_C) = \mathrm{ch}_0(F)$ and $\mathrm{ch}_1(E|_C) = mH\,\mathrm{ch}_1(F)$ we have that $\mu := \mu_{H,0}(E|_C) = mH^2\mu_{H,0}(E) =: mH^2\mu'$.

Furthermore we can suppose that $\frac{H\cdot\mathrm{ch}_1(F)}{H^2\,\mathrm{ch}_0(F)} \in]0,\frac{1}{2}]$. If not we can work with $F^\vee(H)$ because $\mathrm{ch}_1(F^\vee(H)) = -\mathrm{ch}_1(F) + H\,\mathrm{ch}_0(F)$. Hizerbuch-Riemann-Roch we have

$$\mathrm{ch}_2(E) - H\,\mathrm{ch}_1(E) + \chi(\mathcal{O}_X)\,\mathrm{ch}_0(E) = \chi(\mathcal{O}_X, E) \le \hom(\mathcal{O}_X, E) + \hom(\mathcal{O}_X, E[2]) =$$

$$\hom(\mathcal{O}_X, E) + \hom(\mathcal{O}_X, E^\vee(2H)) \le \hom(\mathcal{O}_C, E|_C) + \hom(\mathcal{O}_C, E^\vee(2H)|_C) \le$$

$$\le \frac{1}{H^2}(f_E(\mu) - f_E(2H^2-\mu))\mu + g_E(\mu) - g_E(2H^2-\mu) + 2H^2 f_E(2H^2-\mu)$$

as long as $\mu, 2H^2 - \mu \in [0, 2g-2]$.

Notice that this may not work in all cases because if $2H^2$ is too big could have $2H^2 - \mu \notin [0,2g-2]$. It could also happen that $2H^2$ is too small so we can only bound for $\mu \in [0,2H^2] \cup [2g-2-2H^2, 2g-2]$.

Dividing everything by $H^2\,\mathrm{ch}_0(E)$ we get

$$\frac{\mathrm{ch}_2(E)}{H^2\,\mathrm{ch}_0(E)} \le (f_E(\mu) - f_E(2H^2-\mu))\mu + g_E(\mu) - g_E(2H^2-\mu) + 2f_E(2H^2-\mu) + \frac{\mu - m\chi(\mathcal{O}_X)}{mH^2} \tag{3.1}$$

Now the argument goes by contradiction in order to prove that E satisfies the hypothesis on Feyzbakhsh's restriction lemma. Take $E \in \mathrm{D}^{\mathrm{b}}(X)$ with $\mu' \in]0,1[$ which is $\mu_{\alpha,0}$-stable or $\mu_{\alpha',1}$-stable for some $\alpha > 0$ or $\alpha' > \frac{1}{2}$ violating the above inequality. Thus, since we are trying to improve the bound of Theorem 3.14, we want to stay as close as possible to the parabola $\Delta_H = 0$ so we take E to have the lowest H-discriminant possible among those that violate the inequality.

Since we're dealing with stability (not semistability) we need the Jordan-Hölder filtration which is the same as the Harder-Narasimhan but for semistable sheaves. In this filtration the quotients are required to be stable and there is no uniqueness in general in counterpart to Harder-Narasimhan. Constructing a Jordan-Holder filtration is very easy: if you semistable sheaf already stable we're done. If not there must be some sub-sheaf with the same slope. Taking the smallest one with this property gives the first factor of the filtration. Then one can repeat with the quotient of the sheaf and this sub-sheaf (that has the same slope). This stable sheaf is the quotient of the second factor by the first one.

If E becomes $\mu_{\alpha,0}$-strictly semistable (i.e semistable but not stable) for some $\alpha > 0$ (or $\mu_{\alpha',1}$-strictly semistable for some $\alpha' > \frac{1}{2}$) we can take the Jordan-Hölder filtration because it's semistable. Now, we need the set

$$\left\{ \left(\mu', \frac{\mathrm{ch}_2(E)}{H^2\,\mathrm{ch}_0(E)}\right) \in \mathbb{R}^2 \;\middle|\; \left(\mu', \frac{\mathrm{ch}_2(E)}{H^2\,\mathrm{ch}_0(E)}\right) \text{ satisfies the inequality 3.1} \right\}$$

to be convex. If not we can sacrifice a little of improvement in order to have the convexity. If $\mu' \in]0, \frac{1}{2}]$, thanks to convexity, the line passing through $(\alpha, 0)$ and $p_H(E)$ (or $(\alpha', 1)$ and $p_H(E)$) has the segment $]0, \mu']$ inside the convex set. This implies that there is some Jordan-Hölder factor E_i also violating the inequality. By [BMS16, Corollary 3.10] $\Delta_H(E_i) < \Delta_H(E)$ (recall that $\Delta_H(E) > 0$ since we're outside the possible improved Bogomolov inequality) contradicting the minimality assumption.

If E becomes $\mu_{\alpha,0}$-strictly semistable at the vertical wall $\beta_0 = \mu'$ and some $\alpha > \frac{1}{2}\beta_0^2$ we may assume that $E \in \mathrm{Coh}^{H,\beta_0}(X)$ (because stability grants that E belongs to some shift of the tilted heart so we can assume that the shift is the tilted heart itself). If there is no Jordan-Hölder factor with torsion then there is some μ_{α,β_0}-stable factor E_i such that $p_H(E_i) = p_H(E_i)$ which implies $\Delta_H(E_i) = \Delta_H(E)$. Moreover the openness of stability conditions implies that E_i has to be $\mu_{\alpha,0}$-stable and $\mu_{1,\alpha}$-stable for large values of α and α'. If E has a torsion Jordan-Hölder factor E_i, it must have positive second Chern character. Since $0 < \mathrm{ch}_0(E_i) \leq \mathrm{ch}_0(E_i)$ since we assumed that $E \in \mathrm{Coh}^{H,\beta}(X)$ there is some factor E_j with the same slope than E and $\Delta_H(E) \leq \Delta(E_j)$. Again, the openness of stability condition implies that for large α and α' we have that E_j is both $\mu_{\alpha,0}$-stable and $\mu_{\alpha',1}$-stable. At the end the conclusion is that there is no problem assuming that E was both $\nu_{\alpha,0}$-stable and $\mu_{\alpha',1}$-stable for all $\alpha > 0$ and $\alpha' > \frac{1}{2}$.

Now we take the line that joins $p_H(E)$ with $p_H(E(-mH)[1])$. If $p_H(E) = (a, b)$, then we have $p_H(E(-mH)[1]) = (a - mb + m, b - m)$. This line cuts the parabola in two intersecting the lines $\beta = 0$ and $\beta = -1$ at some point. As an extra condition, we must assure that the intersection points $(\alpha_0, 0)$ and $(\alpha_1, -1)$ fall inside the convex set. This implies that $E(-mH)[1]$ is also both $\mu_{\alpha,0}$-stable and $\mu_{\alpha',1}$-stable. This allows us to apply Feyzbakhsh's restriction lemma and, since the inequalities were obtained from it, the contradiction will follow naturally. □

Conclusions

We have seen that Bridgeland stability conditions exist for surfaces but this is just the top of the iceberg. We still have a lot of open questions: how does $\mathrm{Stab}(X)$ look on an arbitrary surface? Do stability conditions exist for any variety of arbitrary dimension? When can we improve the Bogomolov inequality? This are very hard questions and they are not expected to be answered soon. The importance of stability in the construction of moduli spaces has impact in a lot of different fields. For example moduli spaces of Bridgeland stable objects are the only complete families known of polarized hyperkhäler manifolds in any even dimension. Thus there is a lot of work to be done in the next years which may have several consequences in other areas.

The last technique we saw aimed to tackle [BMS16, Conjecture 4.1], in the case of some specific Calabi-Yau threefolds, rewritten in [Liu22, Conjecture 2.7] after the change of variables as follows:

Conjecture 3.25. *Let X be a smooth projective threefold and $H \in N^1(X)$ an ample class. If $E \in \mathrm{D}^{\mathrm{b}}(X)$ is $\nu_{\alpha,\beta,H}$-tilt semistable for some $\alpha > \frac{1}{2}\beta^2$ then*

$$\begin{aligned} Q_{\alpha,\beta}(E) :=& (2\alpha - \beta^2)(H^2 \cdot \mathrm{ch}_1(E) - H^3 \,\mathrm{ch}_0(E) H \cdot \mathrm{ch}_2(E)) + 4(H \cdot \mathrm{ch}_2^{\beta H}(E))^2 \\ &- 6H^2 \cdot \mathrm{ch}_1^{\beta H}(E)\,\mathrm{ch}_2^{\beta H}(E) \geq 0. \end{aligned}$$

This would imply the existence of stability conditions together with the support property for any smooth projective manifold X as it is shown in [BMS16]. Unfortunately a counterexample was given in [Sch17], which lead to a corrected conjecture [BM22, Conjecture 4.7].

After seeing that an improvement on Bogomolov-Geiseker's type inequality could lead to the existence of stability condition on threefolds some work was done and it's conjectured in [FLZ22] that this improvement may be possible for all regular surfaces. Since Li's work [Li19] was done for a complete intersection of a quintic hypersurface and a quadratic hypersurface inside $\mathbb{P}^4$ we're studying if this may be also possible for complete intersections of quadratic and quartic hypersurfaces and quadratic and sextic hypersurfaces inside $\mathbb{P}^4$. Since both are regular we expect a positive answer, but we don't have enough evidence yet. This is the reason why we didn't include this research part in this memoir.

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