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Credit risk modelling and valuation of CoCo bonds

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Abstract

The main goal of this thesis is to perform a detailed introduction to credit risk modelling and then apply it to do a theoretical study of one particular reduced-form model of credit risk.

The first part of this thesis is aimed to understand what credit risk is (illustrating this concept with some basic examples of credit-risk sensitive financial instruments) and to build an abstract mathematical setting under which we can model and price credit-risk sensitive instruments. In this sense, we present the well-known *risk-neutral valuation formula* and we discuss its validity.

The second part of this thesis is an exposition of the two main approaches to model credit risk: the *structural approach* and the *reduced-form approach*. We focus on the advantages and disadvantages of each approach and we present some particular models. We also derive one useful rewrite of the risk-neutral valuation formula under the reduced-form approach and we apply it to price a *credit default swap*.

The last part of this thesis is focused on studying the reduced-form model built in the article [2] in order to develop pricing formulas for the so-called *contingent convertible bonds* or *CoCos*. Our purpose is to describe this model by adapting it to the abstract setting and notations established in the first two parts of this thesis, to complete the proofs given in [2] by developing those details which are left to the reader and to explain how this model can be implemented in practice.

Contents

1 Introduction

Modern financial instruments can be very complex products involving several kinds of risks. In most mathematical models, the valuation of these instruments is done by modelling the different types of risk which these instruments are subject to. For this reason, risk valuation and management is an essential issue in mathematical finance. One of the most studied types of risk in mathematical literature is *market risk*, which is the risk that arises from fluctuations in some market variables, such as assets prices and volatility. However, many financial contracts are also subject to the possibility of default or decline in the creditworthiness of a counterparty, so the proper valuation of these contracts should also take into account this possibility, which is known as *credit risk*.

In this thesis we are going to do a deep introduction to credit risk in which we will start presenting some mathematical and financial background that is needed to understand this topic, then we will study the main mathematical approaches to model credit risk, and finally we shall focus on the problem of pricing some popular credit-risk sensitive financial instruments under one specific model.

Chapter 2 provides an introduction to the area of credit risk modelling, where we start explaining the concept of *credit risk* with the help of two basic financial instruments: *corporate bonds* and *credit default swaps*. After this, we present some probability theory about general stochastic integration in order to describe the abstract setting used by several quantitative models to price credit-risk sensitive instruments. We also make here a comparison between the two main approaches to model credit risk: the *structural approach* and the *reduced-form approach*.

Chapter 3 deals with the structural approach for credit risk modelling. We describe the simplest classic structural models developed by Merton and Black and Cox and we develop pricing formulas for a corporate zero-coupon bond under the original Merton's model. We also mention how to extend these models and we highlight the drawbacks of this approach.

Chapter 4 deals with the reduced-form approach for credit risk modelling. We explain here the main ideas of reduced-form models, distinguishing two different approaches (the *hazard process approach* and the *martingale approach*) and commenting the advantages and disadvantages of these models. This chapter is specially focused on the hazard process approach, under which we develop pricing formulas for credit-risk sensitive instruments and we apply them to the particular case of a credit default swap.

Chapter 5 is devoted to study one particular reduced-form model (developed in [2]) for pricing some popular credit-risk sensitive instruments called *contingent convertible bonds* or *CoCos*. We first do an introduction about these instruments and a complete description of the model, and then we develop pricing formulas for CoCos and also formulas to calibrate the model with market prices of traded credit default swaps.

In Chapter 6 we summarize the most important results obtained throughout this thesis and we comment some relevant conclusions inferred from these results, along with some continuity proposals for future deeper research on this topic.

Chapter 2 is based on [1] for all the financial theory and it is based on [3] and [8] for all the probability concepts and results about stochastic integration. We refer to [1] and [4] for more information about the theory developed in Chapters 3 and 4. Chapter 5 is mainly based on [2]. Finally, we also recommend [5] for the readers interested in a credit risk book with a good balance between theory and practice and several examples.

2 Introduction to credit risk modelling

As we commented in the previous introduction, credit risk is an important issue to bear in mind for a proper valuation of several financial instruments. This thesis will be devoted to one particular form of credit risk, which is *default risk*. In a financial contract, default risk stands for the possibility that a counterparty will not fulfil a contractual commitment and will default on their liabilities.

In this chapter, we will start introducing two basic credit-risk sensitive instruments that we shall study later in this text: *corporate bonds* and *credit default swaps*. After this, we will present the mathematical framework for credit risk modelling in most quantitative models, including some of the hypothesis and the notation we shall use in the following chapters. We will conclude this chapter with a summary of the two main classes of quantitative models: *structural models* and *reduced-form models*.

2.1 Corporate bonds

Bonds are debt instruments issued by a corporation or a government. More specifically, a bond is a contract between two parties in which the *issuer* or *debtor* commits to make regular specified payments (also called *coupons*) to the *holder* or *creditor* until this contract matures. When this happens, the issuer makes a final bigger cash payment, whose amount is usually known as the *face value* or *par value* of the bond.

Consider a bond with face value $L > 0$ which matures at time $T = T_n > 0$ and pays coupons $c_1, \ldots, c_n \geq 0$ at times $0 < T_1 < \cdots < T_n = T$ respectively. Then, the cash flows received by the holder over the lifetime of the bond are

$$\sum_{i=1}^{n} c_i \cdot \mathbf{1}_{\{T_i\}}(t) + L \cdot \mathbf{1}_{\{T\}}(t), \quad t \in [0, T] \tag{2.1}$$

If a bond does not pay any coupon (only the face value), then we call it a *zero-coupon bond*.

Of course, the previous concept of bond implies a positive profit without risk for the holder of a bond. Thus, in any viable market (without arbitrage opportunities), the holder must pay a specified fee to the issuer when the bond is settled. Therefore, a bond can be seen as a loan made by the holder to the issuer in exchange for the commitment of the issuer to return the debt to the holder in a specified future time with some specified interests.

As it happens with most loans, the creditor pays the initial amount of money and then, all the responsibilities fall on the debtor, since he/she must make sure to make all the specified payments to the creditor at the future specified dates. Hence, the previous idea of bond is just an idealization of the real world, because the issuer of a bond might default on its payments at some future date.

Bonds issued by a government (usually called *Treasury bonds*) are considered to be pretty secure, in the sense that the possibility of default is negligible. For this reason, these bonds are also known as *default-free bonds* or *risk-free bonds*, although the holders of these bonds are exposed to the market risk, of course. Nevertheless, bonds issued by a corporation (usually called *corporate bonds*) are considered to be prone to the possibility of default to a greater or lesser extent, depending on several factors, such as the value and the creditworthiness of the firm. For this reason, these bonds are also known as *defaultable bonds* or *risky bonds*.

Under these considerations, corporate bonds are subject to the default risk, so the specification of the cash flows received by the holder of such a bond is more difficult than the one done in (2.1) for risk-free bonds. In order to model these cash flows, we need to answer two important questions:

- Will the firm default on their liabilities? If so, when will it happen?
- What would happen if a default occurs?

All credit risk models try to find a good answer to the first question, which has to do with how the default event is modeled. All the models we shall study in this thesis define a non-negative random variable τ, known as the *default time* of the bond, which models the time at which the default will happen. Of course, there are many ways of modelling τ, since this variable might depend on several different variables, such as the value and solvency of the firm, other econometric variables of the firm and even some exogenous factors. In Chapters 3 and 4, we shall study the two most popular approaches for the modelling of a default time.

Regarding the second question, typically if a default occurs, then the issuer stops making coupon payments from that moment, and the final payment of the face value at maturity is replaced by a payment of a smaller amount, which we will refer to as the *recovery payment*. The specifications of the amount paid with the recovery payment and how this payment is performed are usually known as the *recovery rules* of the bond. The recovery rules of a bond are not specified in the contract, so they are not known a priori. Hence, these rules must be modeled (like the default time τ).

In several models, the recovery payment is assumed to be a fraction $\delta \in [0,1)$ of the face value of the bond, where δ is called the *recovery rate*, and the payment is assumed to be made at time of default or at the maturity date. However, the modelling of recovery rules can become far more complicated.

To sum up, when a bond is considered to be subject to the risk of default (which is the case of a corporate bond), we need to add two important elements to our model: the default time and the recovery rules. Consider a corporate bond with face value $L > 0$ which matures at time $T = T_n > 0$ and pays coupons $c_1, \dots, c_n \geq 0$ at times $0 < T_1 < \cdots < T_n = T$ respectively. Let us denote by τ and δ the default time and the recovery rate of the bond respectively. Then,

- If the recovery payment is assumed to be made at time of default, the cash flows received by the holder over the lifetime of the bond are

$$\sum_{i=1}^{n} c_i \cdot \mathbf{1}_{\{\tau > T_i\}} \cdot \mathbf{1}_{\{T_i\}}(t) + L \cdot \mathbf{1}_{\{\tau > T\}} \cdot \mathbf{1}_{\{T\}}(t) + \delta L \cdot \mathbf{1}_{\{\tau \leq T\}} \cdot \mathbf{1}_{\{\tau\}}(t), \quad t \in [0,T] \tag{2.2}$$

- If the recovery payment is assumed to be made at the maturity date, the cash flows received by the holder over the lifetime of the bond are

$$\sum_{i=1}^{n} c_i \cdot \mathbf{1}_{\{\tau>T_i\}} \cdot \mathbf{1}_{\{T_i\}}(t) + \big(L \cdot \mathbf{1}_{\{\tau>T\}} + \delta L \cdot \mathbf{1}_{\{\tau\leq T\}}\big) \cdot \mathbf{1}_{\{T\}}(t), \quad t \in [0,T]$$

In this text, we shall focus on the case of a recovery payment of δL at time of default. This recovery scheme is usually called *fractional recovery of par value.*

The knowledge of the cash flows of a defaultable bond will be useful later to price these instruments. If we compare (2.1) and (2.2), we can see that an investor will make a higher or equal profit (higher with positive probability) holding a default-free bond rather than a defaultable bond. For this reason, in practice, defaultable bonds are issued either with a smaller initial fee or with greater coupons than default-free bonds, in order to compensate the investor for the default risk. As a result, the rate of return obtained by the holder of a defaultable bond is larger than the one obtained with an equivalent default-free bond. The difference between these returns is called the *credit spread*, and it is an important element in many credit risk models to price defaultable bonds.

2.2 Credit Default Swaps (CDS's)

In the previous section, we have seen that *corporate bonds* are instruments which are usually subject to the risk of default. Just as there exist derivatives in order to mitigate the market risk attached to the underlying asset (for instance, an European stock option), there also exist derivatives that aim to mitigate the credit risk attached to the underlying asset. These instruments are called *credit derivatives* and their underlying asset is a credit-risk sensitive instrument, such as a defaultable bond. As expected, credit derivatives derive its value from the performance of the underlying asset.

In this text, we shall focus on a particular type of credit derivatives: *credit default swaps* (CDS's). A credit default swap is a credit derivative that offer protection (over a specified time interval) against the default risk of a defaultable bond. More specifically, given a defaultable bond issued by a reference entity, a CDS on that bond is a contract between two parties in which the *protection buyer* commits to make regular specified payments to the *protection seller* and, in exchange, in case the reference entity defaults on its liabilities with regard to the issued bond, the *protection buyer* stops making payments and the *protection seller* commits to give a payoff to the *protection buyer.*

The underlying asset of a CDS is the bond issued by the reference entity. As we can imagine, if the holder of the bond and the protection buyer of the CDS are the same person, then the CDS works as an insurance that protects this investor against the default risk of the bond. Nevertheless, in practice, the protection buyer of a CDS does not need to be the holder of the underlying bond, unlike what happens with a traditional insurance.

Of course, in addition to the default risk attached to the underlying bond, a CDS is also subject to the risk that one of the counterparties defaults on their corresponding payments. However, in practice, this risk has a small impact on the valuation of CDS's, so we will neglect it and we shall only consider the default risk associated to the reference entity.

Consider a defaultable bond with face value $L > 0$ and maturity date $U > 0$ and let us denote by τ and δ the default time and the recovery rate of the bond respectively (remember that we always assume fractional recovery of par value). Now, consider a CDS on this bond giving protection until time $T = T_n \leq U$ and with payment dates $0 < T_1 < \cdots < T_n = T$. The usual conventions about all the payments (the ones we shall adopt) are the following:

- The protection buyer will have to make a cash payment of $\kappa L(T_i - T_{i-1})$ at time T_i if no default has happened yet, for $i = 1, \ldots, n$ (with the convention $T_0 = 0$). Here, κ is the *spread* of the CDS, a parameter which is specified in the contract and it is taken so that the initial value of the CDS is 0.

- The protection seller will have to make a cash payment at time of default (τ) if it happens at time T_n or before. The aim of this payment is to cover the loss caused to the holder of the bond because of the default, so the payment made by the protection seller is usually the face value L of the bond minus the value of the bond at time of default. Hence, since we are assuming a fractional recovery of par value scheme, this payment must be $(1 - \delta)L$.

Therefore, the cash flows of the CDS over the protection time interval from the point of view of the protection buyer are

$$(1-\delta)L \cdot \mathbf{1}_{\{\tau \leq T\}} \cdot \mathbf{1}_{\{\tau\}}(t) - \sum_{i=1}^{n} \kappa L(T_i - T_{i-1}) \cdot \mathbf{1}_{\{\tau > T_i\}} \cdot \mathbf{1}_{\{T_i\}}(t), \quad t \in [0, T] \tag{2.3}$$

Notice that in case of default, a CDS covers the loss of the face value of the bond, but it does not cover the coupons lost at all. Let us also remark that, for the sake of simplicity, we are omitting the accrued interest rate payment: an extra cash payment of $\kappa L(\tau - T_{\beta(\tau)})$ made by the protection buyer at time of default (in case of default), where $\beta(\tau) = i - 1$ if $T_{i-1} \leq \tau < T_i$.

2.3 General framework for credit risk modelling

In the previous sections, we have studied two important default-risk sensitive instruments: corporate bonds and credit default swaps. Now, we are going to present the abstract setting used by several quantitative models for the valuation of general default-risk sensitive instruments, assuming that we already have a model for the default time τ. Finally, we shall briefly discuss the main approaches to model this random variable τ, which will be the main topic of Chapters 3 and 4.

2.3.1 General stochastic integration with respect to a semimartingale

In the following sections and chapters we will often work with stochastic integrals with respect to stochastic processes which are not the Brownian motion, but more general processes. Thus, we are going to do now an introduction about which types of integrals we might come across later.

As usual, $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ shall denote the filtered probability space where all the processes we will consider are defined, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration in $(\Omega, \mathcal{F})$. First of all, let us define a new class of processes:

Definition 2.3.1 (Semimartingale). *An adapted càdlàg stochastic process Y is a semimartingale if it can be descomposed as*

$$Y_t = Y_0 + M_t + A_t, \quad t \geq 0$$

where M is a local martingale, A is a finite variation process and $M_0 = A_0 = 0$.

Remember that a finite variation process is a stochastic process whose paths are functions of bounded variation on each compact interval of $[0, \infty)$ a.s. From now on, we shall assume that a finite variation process is automatically adapted and càdlàg, since we will only work with adapted càdlàg processes. We shall also assume that a local martingale is automatically càdlàg (and adapted, by definition). Then, it follows from the previous definition that every finite variation process and every local martingale is a semimartingale.

Throughout this text, we shall work with 3 types of stochastic integrals:

1. The integral of an adapted process X whose paths are bounded on each compact interval of $[0, \infty)$ a.s, with respect to a finite variation process Y.
2. The integral of an adapted càglàd (left continuous with right limits) process X with respect to a semimartingale Y.
3. The integral of a bounded predictable process X with respect to a semimartingale Y.

The first integral can be defined path by path as a Lebesgue-Stieltjes integral, because the integrator has finite variation and the integrand has bounded paths on compact intervals. Notice that this integral is well defined for càdlàg (resp. càglàd) integrands, since it is known that càdlàg (resp. càglàd) functions are bounded on compact intervals.

The construction of the second integral can be found in detail in [8]. An important result (see [8], Chapter II, Section 5, Theorem 17) is that this integral coincides with the first integral if the integrator is a finite variation process.

The last type of integral is an extension of the second type, which is also developed in [8]. We will not spent time defining the concept of predictability of a stochastic process, but the only thing we need to bear in mind is that every adapted process with càglàd paths is predictable. Furthermore, as expected, any process defined by a deterministic Borel-measurable function is predictable. As expected, this integral also coincides with the first one for finite variation integrators (see [8], Chapter IV, Section 2, Theorem 20).

All these integrals are linear, as desired, and the first integral satisfies the following properties, which we will not prove, because they are well-known results of the Lebesgue-Stieltjes integral:

Proposition 2.3.2. *Let Y be a finite variation process and let X and Z be adapted càdlàg processes. If we define the integral process $I = \left(I_t := \int_0^t X_u dY_u,\ t \geq 0\right)$, then*

1. *I is a finite variation process.*
2. *$\int_0^t Z_u dI_u = \int_0^t Z_u X_u dY_u$ (Associativity).*

3. *If Y is continuous (has continuous paths), then I is continuous.*

4. *If X and Y are bounded and Y is either increasing or decreasing, then I is bounded.*

Now, we are going to prove one version of the integration by parts formula that we shall use later. Before this, let us introduce the concept of quadratic covariation of two semimartingales:

Definition 2.3.3 (**Quadratic covariation**). *Let X and Y be semimartingales. The quadratic covariation of X and Y is the stochastic process $[X,Y] = ([X,Y]_t,\ t \geq 0)$ defined by*

$$[X,Y]_t := X_tY_t - X_0Y_0 - \int_0^t X_{u-}dY_u - \int_0^t Y_{u-}dX_u, \quad t \geq 0$$

where $X_{t-} = \lim_{s\nearrow t} X_s$ and $Y_{t-} = \lim_{s\nearrow t} Y_s$ for every $t > 0$, with the convention $X_{0-} = X_0$ and $Y_{0-} = Y_0$.

In the previous definition, notice that X and Y are semimartingales (in particular, càdlàg processes), so X_- and Y_- are thus well defined adapted càglàd processes. Therefore, the two integrals above are well defined as integrals of the second type, so the previous definition makes sense.

From now on, given a semimartingale X, we will denote by $\Delta X = X - X_-$ the jump process of X. It is known that a càdlàg function has at most countably many jumps, so each path of $\Delta X = X - X_-$ has a finite or countable support a.s. Moreover, since X is càdlàg, we know that each path of $\Delta X = X - X_-$ is bounded on compact intervals a.s.

We refer to [8] (see Chapter II, Section 6, Theorem 28) for the proof of the following Lemma:

Lemma 2.3.4. *Let X be a semimartingale and let Y be a finite variation process. Then,*

$$[X,Y]_t = \sum_{0<s\leq t} \Delta X_s \Delta Y_s, \quad t \geq 0$$

where the previous sum is at most countable for each $\omega \in \Omega$.

Finally, let us state and prove a useful version of the integration by parts formula:

Theorem 2.3.5 (**Integration by parts (I)**). *Let X be a semimartingale and let Y be a finite variation process. Then,*

$$X_tY_t = X_0Y_0 + \int_0^t X_u dY_u + \int_0^t Y_{u-}dX_u, \quad t \geq 0$$

Before proving this formula, let us stress that the first (resp. second) integral above is well defined as an integral of the first (resp. second) type.

Proof. First of all, combining Definition 2.3.3 and Lemma 2.3.4, we obtain

$$X_tY_t = X_0Y_0 + \int_0^t X_{u-}dY_u + \int_0^t Y_{u-}dX_u + \sum_{0<s\leq t} \Delta X_s \Delta Y_s, \quad t \geq 0 \tag{2.4}$$

Notice that the first integral in the previous equation can be thought as an integral of the first type or an integral of the second type, because Y is a finite variation process.

We have argued before that each path of the process ΔX has a finite or countable support and is bounded on compact intervals a.s. Thus, given $t \geq 0$, the integral $\int_0^t \Delta X_u dY_u$ is well defined path by path as a Lebesgue-Stieltjes integral. Furthermore, if we fix $\omega \in \Omega$ and we let S be the (finite or countable) support of the function $\Delta X(\omega)$, then

$$\int_0^t \Delta X_u(\omega) dY_u(\omega) = \int_0^t \sum_{s \in S} \Delta X_s(\omega) \mathbf{1}_{\{s\}}(u) dY_u(\omega) = \sum_{s \in S} \Delta X_s(\omega) \int_0^t \mathbf{1}_{\{s\}}(u) dY_u(\omega) =$$
$$= \sum_{\substack{s \in S \\ s \leq t}} \Delta X_s(\omega) \Delta Y_s(\omega)$$

where the commutation of the sum with the integral sign in the second equality can be justified applying the Dominated convergence theorem (and using Lemma 2.3.4). Hence, it follows that

$$\int_0^t \Delta X_u dY_u = \sum_{0<s\leq t} \Delta X_s \Delta Y_s, \quad t \geq 0 \tag{2.5}$$

so combining (2.4) and (2.5) we conclude that

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X_{u-} dY_u + \int_0^t Y_{u-} dX_u + \sum_{0<s\leq t} \Delta X_s \Delta Y_s = \\ &= X_0 Y_0 + \int_0^t X_{u-} dY_u + \int_0^t Y_{u-} dX_u + \int_0^t \Delta X_u dY_u = \\ &= X_0 Y_0 + \int_0^t \left(X_{u-} + \Delta X_u \right) dY_u + \int_0^t Y_{u-} dX_u = \\ &= X_0 Y_0 + \int_0^t X_u dY_u + \int_0^t Y_{u-} dX_u \end{aligned}$$

as we wanted to prove. □

There is another useful version of the integration by parts formula that we are going to present without proving it. The proof can be found in [3] (see Chapter I, Proposition 4.49).

Theorem 2.3.6 (Integration by parts (II)). *Let X be a semimartingale and let Y be a bounded predictable finite variation process. Then,*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{u-} dY_u + \int_0^t Y_u dX_u, \quad t \geq 0$$

Let us notice that the first (resp. second) integral above is well defined as an integral of the first (resp. third) type. Actually, the boundedness of Y is not needed for the previous result to hold, but this restriction will be enough for our purposes.

Before concluding this introduction to general stochastic integration and returning to the valuation of default-risk sensitive instruments, we are going to present one version of the Girsanov theorem for martingales, which tells how a martingale changes under a change of probability. This theorem will be useful in Chapter 5 to price CoCo bonds.

We shall start presenting some preliminary concepts and properties:

Definition 2.3.7. *The space $\mathcal{H}^2$ is defined as the class of all the square-integrable martingales, that is, all the martingales X such that $\sup_{t\in\mathbb{R}^+}\mathbb{E}_{\mathbb{P}}\left(X_t^2\right)<\infty$.*

We can also define the extended class $\mathcal{H}^2_{loc}$ of all the locally square-integrable martingales (and then, $\mathcal{H}^2\subset\mathcal{H}^2_{loc}$), but we will restrict our research to the martingales in $\mathcal{H}^2$. We refer to [8] (see Chapter IV, Section 2, Theorem 11) for the following important result:

Proposition 2.3.8. *If X is a bounded predictable process and $Y\in\mathcal{H}^2$, then the stochastic process $I=\left(I_t:=\int_0^t X_u dY_u,\ t\geq 0\right)$ is a martingale belonging to $\mathcal{H}^2$.*

Now, let us define the predictable quadratic covariation of two locally square-integrable martingales, which is closely related to the concept of quadratic covariation of two semimartingales:

Definition 2.3.9 (Predictable quadratic covariation). *Given $X,Y\in\mathcal{H}^2_{loc}$, the predictable quadratic covariation of X and Y is the only (up to a null probability set) predictable finite variation process $\langle X,Y\rangle$ with initial value 0 and such that $XY-\langle X,Y\rangle$ is a local martingale.*

The existence and uniqueness of $\langle X,Y\rangle$ is proved in [3] (see Chapter I, Theorem 4.2), so the previous definition makes sense. We also refer to [3] (see Chapter I, Theorem 4.2 and Proposition 4.50) for the proof of the following result:

Proposition 2.3.10. *If $X,Y\in\mathcal{H}^2$, then the processes $XY-\langle X,Y\rangle$, $XY-[X,Y]$ and $[X,Y]-\langle X,Y\rangle$ are martingales.*

Now, imagine that we build in the filtered space $(\Omega,\mathcal{F},\mathbb{F})$ a new probability $\mathbb{Q}$ defined by

$$\mathbb{Q}(A):=\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_A\widetilde{Z}\right),\quad A\in\mathcal{F} \tag{2.6}$$

where $\widetilde{Z}$ is some non-negative random variable such that $\mathbb{E}_{\mathbb{P}}(\widetilde{Z})=1$. Then, $\mathbb{Q}$ is a well defined probability such that $\mathbb{Q}\ll\mathbb{P}$ (that is, $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$) and $\widetilde{Z}$ is the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ ($\widetilde{Z}=d\mathbb{Q}/d\mathbb{P}$). Actually, given two probabilities $\mathbb{P}$ and $\mathbb{Q}$ defined in $(\Omega,\mathcal{F},\mathbb{F})$ and such that $\mathbb{Q}\ll\mathbb{P}$, it follows from Radon-Nikodym theorem that there exists a unique random variable $\widetilde{Z}\geq 0$ with $\mathbb{E}_{\mathbb{P}}(\widetilde{Z})=1$ satisfying (2.6).

Let us consider the density process $Z=(Z_t,\ t\geq 0)$, which is defined as

$$Z_t:=\mathbb{E}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\,\middle|\,\mathcal{F}_t\right)=\mathbb{E}_{\mathbb{P}}\left(\widetilde{Z}\,\middle|\,\mathcal{F}_t\right)\quad t\geq 0 \tag{2.7}$$

The following proposition collects some well-known properties of the density process Z:

Proposition 2.3.11. *Let X be an adapted càdlàg process. Let Y be an $\mathcal{F}_t$-measurable random variable for some $t\geq 0$, which is either bounded or $\mathbb{Q}$-integrable. Then, the following properties hold:*

1. *Z is a martingale under $\mathbb{P}$.*
2. *For every $s\in[0,t]$, $\mathbb{E}_{\mathbb{Q}}(Y\mid\mathcal{F}_s)=\frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}(YZ_t\mid\mathcal{F}_s)$ (Abstract Bayes' rule).*

3. XZ is a martingale under $\mathbb{P}$ if and only if X is a martingale under $\mathbb{Q}$.

We are ready to prove the following version of the Girsanov theorem for martingales:

Theorem 2.3.12 (Girsanov theorem for martingales). *Assume that $\mathbb{P}$ and $\mathbb{Q}$ are two probabilities defined in the filtered space $(\Omega, \mathcal{F}, \mathbb{F})$ such that $\mathbb{Q} \ll \mathbb{P}$ and let Z be the density process given by (2.7). Let X be a martingale under $\mathbb{P}$ with $X_0 = 0$ and assume that $X, Z \in \mathcal{H}^2$. Then, the process $A = (A_t,\ t \geq 0)$ given by*

$$A_t = \int_0^t \frac{1}{Z_{u-}} d\langle X, Z\rangle_u \quad t \geq 0$$

is $\mathbb{Q}$-a.s. well defined and it is a finite variation process. Moreover, if A is predictable and bounded, then the process $\widetilde{X} = X - A$ is a martingale under $\mathbb{Q}$.

Proof. First of all, since $X, Z \in \mathcal{H}^2$, we know from Definition 2.3.9 that $\langle X, Z\rangle$ is a well defined finite variation process. Hence, since Z_- is an adapted càglàd process (because Z is a martingale), we know that the process A is well defined path by path as a Lebesgue-Stieltjes integral (and then, A is a finite variation process) as long as there are no vanishing problems with Z_-. It can be seen that there are no such vanishing problems and A is a $\mathbb{Q}$-a.s. well defined finite variation process with $A_0 = 0$ (see [3], Chapter III, Theorem 3.11).

Now, let us assume that A is predictable and bounded. On the one hand, since $X, Z \in \mathcal{H}^2$, it follows from Proposition 2.3.10 that $XZ - \langle X, Z\rangle$ is a martingale under $\mathbb{P}$. On the other hand, since Z is a semimartingale under $\mathbb{P}$ (because it is a martingale under $\mathbb{P}$) and A is a bounded predictable finite variation process with $A_0 = 0$, we know from Theorem 2.3.6 (integration by parts (II)) that

$$\begin{aligned} A_t Z_t &= \int_0^t Z_{u-} dA_u + \int_0^t A_u dZ_u = \int_0^t Z_{u-} \frac{1}{Z_{u-}} d\langle X, Z\rangle_u + \int_0^t A_u dZ_u = \\ &= \langle X, Z\rangle_t + \int_0^t A_u dZ_u, \quad t \geq 0 \end{aligned}$$

where in the second equality we have applied the definition of A and the associativity of the Lebesgue-Stieltjes integral and the last equality follows from the fact that $\langle X, Z\rangle_0 = 0$. Thus, we can write

$$A_t Z_t - \langle X, Z\rangle_t = \int_0^t A_u dZ_u, \quad t \geq 0$$

and we know from Theorem 2.3.8 that the process $\left(\int_0^t A_u dZ_u,\ t \geq 0\right)$ is a martingale, because $Z \in \mathcal{H}^2$ and A is predictable and bounded by hypothesis. Hence, it follows from the previous equation that $AZ - \langle X, Z\rangle$ is a martingale under $\mathbb{P}$.

Finally, consider the process $\widetilde{X} = X - A$. We have shown that $XZ - \langle X, Z\rangle$ and $AZ - \langle X, Z\rangle$ are martingales under $\mathbb{P}$, so the difference $\widetilde{X}Z = XZ - AZ$ is also a martingale under $\mathbb{P}$, and therefore, we conclude from Proposition 2.3.11 that $\widetilde{X}$ is a martingale under $\mathbb{Q}$, as we wanted to prove. □

2.3.2 Defaultable claims

So far, we have introduced a couple of examples of default-risk sensitive instruments. However, in the real market we can find a large amount of complex financial instruments which are subject to the risk of default, so we need a formal definition of a more general concept that encompasses all these instruments, which is the concept of *defaultable claim*.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space, endowed with some reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Here, $\mathbb{P}$ stands for the *real-world probability* and the filtration $\mathbb{F}$ will be assumed to represent the information available through assets prices and other economic factors. Then, we formally define a *defaultable claim* with maturity date $T > 0$ as a quintuplet $(X, A, \widetilde{X}, Z, \tau)$, where:

- The *promised contingent claim* X is a random variable that represents the payoff received by the owner/holder of the claim at the maturity date T if there is no default prior to or at time T.
- The *promised dividends* A is a process representing the stream of cash flows received by the holder over the lifetime of the claim, prior to default.
- The *recovery claim* $\widetilde{X}$ is a random variable which represents the recovery payoff received by the holder of the claim at the maturity date T, if a default occurs before or at time T.
- The *recovery process* Z is a process representing the recovery payoff received by the holder of the claim at time of default, if a default occurs prior to or at the maturity date T.
- The *default time* τ is a non-negative random variable which models the random time of default of the claim.

The usual assumptions about these random objects that we shall make throughout this text are the following:

- All the random objects introduced above satisfy suitable integrability conditions.
- The process Z is càdlàg and predictable with respect to the filtration $\mathbb{F}$ (that is, $\mathbb{F}$-predictable).
- The random variables X and $\widetilde{X}$ are $\mathcal{F}_T$-measurable.
- The process A is an $\mathbb{F}$-adapted finite variation process with $A_0 = 0$.

In the previous definition, we can notice that $\widetilde{X}$ and Z stand for the recovery payments received by the holder of the claim in case of default, assuming that these payments are made respectively at the maturity date and at time of default. Let us remember that, in this text, we will focus on the case of a recovery payment at default, so we shall often consider that $\widetilde{X} = 0$.

Now, we can already provide formal definitions of *defaultable bond* and *credit default swap*. Of course, we will adapt these definitions based on the notation and the assumptions made in Sections 2.1 and 2.2:

Definition 2.3.13 (Defaultable bond). *Given $U > 0$, a U-maturity defaultable bond with face value $L > 0$, coupon payments $c_1, \dots, c_m \geq 0$ at times $0 < U_1 < \cdots < U_m = U$ and fractional recovery of par value with recovery rate $\delta \in [0, 1)$ is the defaultable claim $(L, A, 0, \delta L, \tau)$, where τ is called the default time of the bond and*

$$A_t = \sum_{i=1}^{m} c_i \cdot \mathbf{1}_{[U_i, U]}(t), \quad t \in [0, U]$$

Definition 2.3.14 (**Credit default swap**). *Given the defaultable bond defined above and given* $0 < T \leq U$*, a* T*-maturity credit default swap (CDS) with spread* κ*, payment dates* $0 < T_1 < \cdots < T_n = T$ *and protection at default on the given bond is the defaultable claim* $(0, \widetilde{A}, 0, (1-\delta)L, \tau)$*, where*

$$\widetilde{A}_t = -\sum_{i=1}^{n} \kappa L(T_i - T_{i-1}) \cdot \mathbf{1}_{[T_i,T]}(t), \quad t \in [0,T]$$

with the convention $T_0 = 0$*.*

As we commented before, the filtration $\mathbb{F}$ usually represents the information available through assets prices and other economic factors. However, we will see that several models also take into account another kind of information: the information about the occurrence of the default time of some defaultable claim. This information is modeled by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ generated by the *default process* $H = \big(H_t := \mathbf{1}_{\{\tau\leq t\}},\ t \geq 0\big)$. Thus, we can consider the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for all $t \geq 0$.

Although we have imposed some $\mathbb{F}$-adaptability and $\mathcal{F}_T$-measurability conditions to the random objects that define a defaultable claim (and we shall keep these assumptions), we will regard $\mathbb{G}$ as the underlying filtration for our market model. It is worth noting that in structural models we usually have that $\mathbb{G} = \mathbb{F}$, but we shall explain this in more detail in the following chapters.

2.3.3 Risk-neutral valuation formula

Once we have already defined the concept of defaultable claim, we need a general formula to price these instruments. The valuation of defaultable claims in a quantitative model is done under the common assumption that the market is viable (free of arbitrage). Thus, we shall assume that there exists a *risk-neutral probability* $\mathbb{P}^*$ (not necessarily unique) under which the price process of any tradeable asset which pays no coupons or dividends follows a $\mathbb{G}$-martingale, when discounted by the *savings account* process $B = (B_t,\ t \geq 0)$ given by

$$B_t := \exp\left(\int_0^t r_u du\right), \quad t \geq 0$$

Here, $r = (r_t,\ t \geq 0)$ represents the short-term interest rate process, which is assumed to be a non-negative càdlàg $\mathbb{F}$-adapted process. Remember that $\mathbb{G} := \mathbb{F} \vee \mathbb{H}$, where the filtration $\mathbb{F}$ stands for the information available through assets prices and $\mathbb{H}$ is the filtration generated by the *default process* $H = \big(H_t := \mathbf{1}_{\{\tau\leq t\}},\ t \geq 0\big)$ associated to the defaultable claim that we consider.

Definition 2.3.15. *The dividend process of a defaultable claim* $(X, A, \widetilde{X}, Z, \tau)$ *with maturity date* $T > 0$ *is the stochastic process* $D = (D_t,\ t \in [0,T])$ *defined by*

$$D_t := \Big(X \cdot \mathbf{1}_{\{\tau>T\}} + \widetilde{X} \cdot \mathbf{1}_{\{\tau\leq T\}}\Big) \cdot \mathbf{1}_{\{T\}}(t) + \int_0^t (1-H_u) dA_u + \int_0^t Z_u dH_u, \quad t \in [0,T]$$

Since H and A are finite variation processes, both integrals in the previous definition are well defined path by path as Lebesgue-Stieltjes integrals. Moreover, it follows from Proposition 2.3.2 that D is a finite variation process over $[0, T]$. Using the definition of H, we see that

$$\begin{aligned} D_t &= \left(X \cdot \mathbf{1}_{\{\tau > T\}} + \widetilde{X} \cdot \mathbf{1}_{\{\tau \le T\}}\right) \cdot \mathbf{1}_{\{T\}}(t) + \int_0^t \mathbf{1}_{\{\tau > u\}} dA_u + \int_0^t Z_u dH_u = \\ &= \left(X \cdot \mathbf{1}_{\{\tau > T\}} + \widetilde{X} \cdot \mathbf{1}_{\{\tau \le T\}}\right) \cdot \mathbf{1}_{\{T\}}(t) + A_{\tau-} \mathbf{1}_{\{\tau \le t\}} + A_t \mathbf{1}_{\{\tau > t\}} + Z_\tau \mathbf{1}_{\{\tau \le t\}}, \quad t \in [0, T] \end{aligned}$$

Thus, we can notice that the dividend process D models the stream of cash flows received by the holder of the defaultable claim. Indeed, it is easy to check that if we compute the cumulative cash flows of a defaultable bond (resp. a CDS) from the cash flows formula (2.2) (resp. (2.3)), then we obtain precisely the process D.

The arbitrage price of a default-free replicable financial instrument can be computed with the well-known *risk-neutral valuation formula*, that is, it is computed as the current value (using the risk-neutral probability) of all the future cash flows associated to this instrument. Hence, the idea is to follow a similar approach for the valuation of defaultable claims, which leads to the following definition:

Definition 2.3.16. *The ex-dividend price process of a defaultable claim* $(X, A, \widetilde{X}, Z, \tau)$ *with maturity date* $T > 0$ *is the stochastic process* $S = (S_t,\ t \in [0, T])$ *defined by*

$$S_t := B_t \cdot \mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1} dD_u \,\middle|\, \mathcal{G}_t\right), \quad t \in [0, T] \tag{2.8}$$

Using the definition of the dividend process D and the associativity of the Lebesgue-Stieltjes integral (see Proposition 2.3.2), we can write the ex-dividend price process as

$$\begin{aligned} S_t &= B_t \cdot \mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1}(1 - H_u) dA_u + \int_t^T B_u^{-1} Z_u dH_u + B_T^{-1}\left(X \cdot \mathbf{1}_{\{\tau > T\}} + \widetilde{X} \cdot \mathbf{1}_{\{\tau \le T\}}\right) \,\middle|\, \mathcal{G}_t\right) = \\ &= B_t \cdot \mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1}(1 - H_u) dA_u + B_\tau^{-1} Z_\tau \cdot \mathbf{1}_{\{t < \tau \le T\}} + B_T^{-1}\left(X \cdot \mathbf{1}_{\{\tau > T\}} + \widetilde{X} \cdot \mathbf{1}_{\{\tau \le T\}}\right) \,\middle|\, \mathcal{G}_t\right) \end{aligned} \tag{2.9}$$

We notice that (2.8) is a variant of the risk-neutral valuation formula, so we shall take (2.8) (or equivalently (2.9)) as the general formula for the valuation of defaultable claims. However, if $\widetilde{X} = 0$ (which is the typical assumption we will often make), it can be checked from (2.9) that $S_t = S_t \cdot \mathbf{1}_{\{\tau > t\}}$, that is, the ex-dividend price process is worth 0 when a default has already happened. Thus, we shall only use these formulas to compute the value of a defaultable claim prior to default, that is, the *pre-default value*.

The formula (2.8) can be formally justified using no-arbitrage arguments (see [1], Chapter 2, Section 2.1). Nevertheless, the validity of this formula presents two problems. The first problem is that we are not assuming that the market is complete, so the risk-neutral probability may not be unique. Hence, (2.8) returns several different values of the ex-dividend price corresponding to all the possible risk-neutral probabilities, and this forces us to fix one of these probabilities as "the right one". In practice, the choice of this probability is done by setting a model for it and calibrating the parameters with real market prices of traded assets.

The second problem follows from the first one and it has to do with the fact that the validity of (2.8) relies on the assumption that the defaultable claim can be replicated by trading in primary default-free assets, which might not be true. Actually, the possibility of replication depends on the modelling of the default time, and in most cases this replication is not possible. In the derivation of (2.8) that is done in [1], this problem is avoided by making the extra assumption of the existence of one particular admissible trading strategy involving the defaultable claim.

Due to these problems, (2.8) is postulated as a definition and not as a proposition, and the validity of this formula should be examined case by case.

2.3.4 Quantitative models: structural approach vs reduced-form approach

We have already introduced the concept of defaultable claim and we have a general formula to price these financial instruments. Before we are able to apply this formula, we need to deal with the original problem about how to model the default time τ and the recovery payments of a defaultable claim. There are several ways to model the default time, but we shall focus on two important approaches: the *structural approach* and the *reduced-form approach*.

The structural approach is the one taken by the *structural models*, which we will study in Chapter 3. These models focus on the modelling of the total value of the reference firm associated to a defaultable claim and they link the default event of this claim to the relative position of the firm's value with respect to a given threshold or barrier. Hence, it is commonly said that the default time is defined *endogenously* within the model and it is linked to the firm's economic fundamentals. The recovery payments are usually also specified as functions of the firm's value.

The reduced-form approach is the one taken by the *reduced-form models*, which we will study in Chapter 4. These models do not model the reference firm's total value, but the default event is specified *exogenously* in terms of a given jump process (and the same happens with the recovery payments). Thus, they do not link the default event to the movements of the firm's value, which allows for an element of surprise. We shall focus on the *hazard process approach*, which focuses on the problem of modelling the conditional distribution of the default time τ with respect to $\mathcal{F}_t$ for every $t \geq 0$ and it derives useful formulas for the evaluation of conditional expectations in order to express the value of a defaultable claim in terms of the conditional distribution of τ.

Both approaches present advantages and disadvantages that we will discuss in the following chapters.

3 Structural approach

In this chapter, we are going to study the well-known structural models of credit risk, which link the default event of a defaultable claim to the value of the reference firm. We will start with the simplest structural model, which is the original Merton's model, and we shall remark all the problems of this model, each of which has given rise to several extensions. After this, we shall do a brief introduction about the first-passage-time models, focusing on the Black and Cox model. We will finish this chapter with some comments about the advantages and disadvantages of most structural models.

3.1 Merton's model

As usual, let us denote by $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ the underlying filtered probability space, where $\mathbb{P}$ stands for the *real-world probability* and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the reference filtration representing the information available through assets prices (the so-called default-free market information). Let us also denote by $\mathbb{P}^*$ the chosen risk-neutral probability (we must fix one, since it might not be unique).

The main assumptions of the original Merton's model are the following:

- The short-term interest rate is constant and equal to $r \geq 0$.
- The total value process $V = (V_t,\ t \geq 0)$ of the firm's assets follows a geometric Brownian motion under $\mathbb{P}^*$ with respect to $\mathbb{F}$. More specifically, V satisfies the stochastic differential equation (SDE)

$$dV_t = V_t\left((r-\kappa)dt + \sigma_V dW_t\right) \tag{3.1}$$

  with $V_0 > 0$, where the constants $\kappa \geq 0$ and $\sigma_V > 0$ represent the *payout ratio* and the *volatility* respectively, and the process $W = (W_t,\ t \geq 0)$ is a standard Brownian motion under $\mathbb{P}^*$ with respect to $\mathbb{F}$.
- The firm issues both equity and debt, and the debt is modeled by a defaultable zero-coupon bond with face value $L > 0$ and maturity date $T > 0$. In other words, the firm has a single liability with the promised payoff L at time T.
- The ability of the firm to return the debt (the face value L) is determined by the firm's value at time T, that is, V_T. More specifically, a default can only happen at time T and the default event is the event $\{V_T < L\}$. In case of default, the recovery payment received by the holders of the bond is V_T.

Under these assumptions, we can notice that the total value process V of the firm is a continuous $\mathbb{F}$-adapted process (in fact, the SDE (3.1) can be solved easily). Now, since this model assumes that the firm has a single liability interpreted as a defaultable zero-coupon bond with face value $L > 0$ and maturity date $T > 0$, we are interested in pricing this bond.

Under the previous assumptions, we can think this defaultable zero-coupon bond as the defaultable claim $(X, A, \widetilde{X}, Z, \tau)$ given by

$$X = L, \quad A = 0, \quad \widetilde{X} = V_T, \quad Z = 0, \quad \tau = T \cdot \mathbf{1}_{\{V_T < L\}} + \infty \cdot \mathbf{1}_{\{V_T \geq L\}} \tag{3.2}$$

Notice that we could also write the default time as $\tau = T \cdot \mathbf{1}_{\{V_T < L\}} + U \cdot \mathbf{1}_{\{V_T \geq L\}}$ for a given $U > T$, since we are only interested in knowing whether or not a default has occurred before or at time T.

Hence, applying the risk-neutral valuation formula (2.9), we obtain the following result:

Proposition 3.1.1. *The pre-default value of a defaultable zero-coupon bond with face value $L > 0$ and maturity date $T > 0$ is given by the process $(D_t,\ t \in [0,T])$ defined by*

$$D_t = V_t e^{-\kappa(T-t)} N\left(-d_1(V_t, T-t)\right) + L e^{-r(T-t)} N\left(d_2(V_t, T-t)\right), \quad t \in [0,T] \tag{3.3}$$

where

$$\begin{aligned} d_1(V_t, T-t) &= \frac{\ln(V_t/L) + \left(r - \kappa + \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V \sqrt{T-t}}, \quad t \in [0,T] \\ d_2(V_t, T-t) &= \frac{\ln(V_t/L) + \left(r - \kappa - \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V \sqrt{T-t}}, \quad t \in [0,T] \\ N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \quad x \in \mathbb{R} \end{aligned}$$

Proof. If we apply the risk-neutral valuation formula (2.9) to the zero-coupon bond given by (3.2) and we use the fact that the short-term interest rate is the constant r, then we obtain

$$\begin{aligned} D_t &= e^{rt} \cdot \mathbb{E}_{\mathbb{P}^*}\left(e^{-rT}\left(L \cdot \mathbf{1}_{\{\tau > T\}} + V_T \cdot \mathbf{1}_{\{\tau \leq T\}}\right) \,\middle|\, \mathcal{G}_t\right) = \\ &= e^{-r(T-t)} \cdot \mathbb{E}_{\mathbb{P}^*}\left(L \cdot \mathbf{1}_{\{V_T \geq L\}} + V_T \cdot \mathbf{1}_{\{V_T < L\}} \,\middle|\, \mathcal{G}_t\right) = \\ &= e^{-r(T-t)} \cdot \mathbb{E}_{\mathbb{P}^*}\left(L \cdot \mathbf{1}_{\{V_T \geq L\}} + V_T \cdot \mathbf{1}_{\{V_T < L\}} \,\middle|\, \mathcal{F}_t\right), \quad t \in [0,T] \end{aligned} \tag{3.4}$$

The last equality follows from the fact that $\mathcal{G}_t = \mathcal{F}_t$ for every $t \in [0,T]$. Indeed, remember that the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for all $t \geq 0$, where $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ is the filtration generated by the default process $H = \left(H_t := \mathbf{1}_{\{\tau \leq t\}},\ t \geq 0\right)$. It follows easily from (3.2) that for every $t \in [0,T]$,

$$H_t = \mathbf{1}_{\{\tau \leq t\}} = \mathbf{1}_{\{T\}}(t) \cdot \mathbf{1}_{\{V_T < L\}}$$

Hence, since V_T is $\mathcal{F}_T$-measurable (because V follows a geometric Brownian motion under $\mathbb{P}^*$ with respect to $\mathbb{F}$), we have that for every $t \in [0,T]$, $\mathcal{H}_t \subset \mathcal{F}_t$ and then $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \mathcal{F}_t$, so (3.4) is valid.

Now, we could solve the SDE (3.1) and use the solution to compute the last expectation in (3.4) in order to deduce (3.3) (see [1], Chapter 2, Proposition 2.3.1). However, we will provide here an alternative proof using no-arbitrage arguments.

Since V_T is $\mathcal{F}_T$-measurable, it follows from (3.4) that

$$\begin{aligned} D_T &= \mathbb{E}_{\mathbb{P}^*}\left(L \cdot \mathbf{1}_{\{V_T \geq L\}} + V_T \cdot \mathbf{1}_{\{V_T < L\}} \,\middle|\, \mathcal{F}_T\right) = L \cdot \mathbf{1}_{\{V_T \geq L\}} + V_T \cdot \mathbf{1}_{\{V_T < L\}} = \min(V_T, L) = \\ &= L - (L - V_T)_+ \end{aligned}$$

Thus, at time T, the value of the bond is equal to L (which we can think as the amount of money in a bank account at time T) minus the value of a European put option on the total value of the firm's assets (assuming that this total value represents a traded asset) with strike L and maturity date T. Then, since the market is viable, it follows that

$$D_t = Le^{-r(T-t)} - P_t, \quad t \in [0,T]$$

where $(P_t,\ t \in [0,T])$ stands for the value of the European put option. Since V satisfies the SDE (3.1), we know from the classic Black-Scholes options valuation formula that

$$P_t = Le^{-r(T-t)} N\left(-d_2(V_t, T-t)\right) - V_t e^{-\kappa(T-t)} N\left(-d_1(V_t, T-t)\right), \quad t \in [0,T]$$

so we conclude that

$$\begin{aligned} D_t &= Le^{-r(T-t)} - P_t = \\ &= Le^{-r(T-t)} - \left[Le^{-r(T-t)} N\left(-d_2(V_t, T-t)\right) - V_t e^{-\kappa(T-t)} N\left(-d_1(V_t, T-t)\right)\right] = \\ &= V_t e^{-\kappa(T-t)} N\left(-d_1(V_t, T-t)\right) + Le^{-r(T-t)}\left(1 - N\left(-d_2(V_t, T-t)\right)\right) = \\ &\underset{\substack{\uparrow \\ N(-x) = 1 - N(x)}}{=} V_t e^{-\kappa(T-t)} N\left(-d_1(V_t, T-t)\right) + Le^{-r(T-t)} N\left(d_2(V_t, T-t)\right), \quad t \in [0,T] \end{aligned}$$

which proves (3.3). □

The previous result provides us with a formula to price a defaultable zero-coupon bond. Although this formula does not seem very cumbersome, the implementation of this formula to compute the current value of a bond relies on our knowledge of all the parameters of the model, including the current total value of the firm, which is usually unknown in practice. There are different approaches to estimate these parameters, and the Itô formula can be used to deduce some useful equations to determine the volatility σ_V and the current value of the firm V_t (see [1], Chapter 2, Section 2.3.2), but we will no go into this issue.

The original Merton's model is the doorway to the world of credit risk modelling, but even though this model puts together the essential ingredients, it is not enough to model the default risk in an accurate way, since the assumptions which it is based on are too strong and unrealistic. There are a number of extensions of this model aimed to relax the underlying assumptions, which take several different approaches, such as considering a stochastic interest rate, introducing jumps to the firm's value process and taking into account other kinds of debt (not only a single zero-coupon bond) with different maturities and face values.

Nevertheless, one of the main drawbacks of the Merton's model is the assumption that a default can only occur at the maturity date. This assumption is not realistic at all, so we need to extend this model in order to bear in mind the possibility of default prior to maturity. This extension will lead us to the well-known *first-passage-time models*, which we shall discuss in the next section.

3.2 First-passage-time models

The first-passage-time models extend the original Merton's model allowing for the possibility of default before the maturity date of the debt instruments issued by the firm. They also give more freedom to model the recovery payments of a defaultable claim in a wide choice of ways. In this section, we are going to do a very brief introduction about these models.

Like the original Merton's model, the first-passage-time models link the default event of a defaultable claim to the total value of the reference firm (this is the common feature of most structural models), but they set the default time as the first time at which the firm's value process $V = (V_t,\ t \geq 0)$ falls below a specified (random or deterministic) barrier process $v = (v_t,\ t \geq 0)$. In other words, the default time is defined as

$$\tau = \inf \{t \geq 0 : V_t \leq v_t\}$$

Different specifications of the processes V and v lead to different models. If we want to price bonds and other defaultable claims under one of these models, we need to know some distributions involving τ. There is a whole section in [1] (see Chapter 3, Section 3.1) devoted to the study of some distributional properties of τ which are important to know in order to price defaultable claims under a first-passage-time model. In the reference text, the processes V and v are assumed to be Itô processes satisfying some SDE's with a specific form and constant parameters (and also satisfying the condition $V_0 > v_0 > 0$), so that the process $Y = (Y_t := \ln(V_t/v_t),\ t \geq 0)$ is given (as a consequence of the Itô formula) by

$$Y_t = Y_0 + \mu t + \sigma W_t, \quad t \geq 0$$

for some constants $\mu \in \mathbb{R}$ and $\sigma > 0$, where $W = (W_t,\ t \geq 0)$ is a standard Brownian motion under the risk-neutral probability $\mathbb{P}^*$ with respect to the reference filtration $\mathbb{F}$.

Under these assumptions, we can write the default time as $\tau = \inf \{t \geq 0 : Y_t \leq 0\}$, and one can find in the previous reference a number of formulas for the marginal and joint distributions of τ and Y under $\mathbb{P}^*$, both the unconditional distributions and the conditional distributions with respect to the σ-fields $\mathcal{F}_t$.

3.2.1 Black and Cox model

One of the most famous first-passage-time models (and also one of the simplest ones) is the Black and Cox model, which extends the original Merton's model in several directions. Although this model deals with several features of the contract of a bond and also deals with the capital structure of the firm, we shall focus on the main mathematical assumptions of the model, which can be summarized as it follows:

- The short-term interest rate is constant and equal to $r \geq 0$.
- The total value process $V = (V_t,\ t \geq 0)$ of the firm's assets follows a geometric Brownian motion under $\mathbb{P}^*$ with respect to $\mathbb{F}$. More specifically, V satisfies the stochastic differential equation (SDE)

  $$dV_t = V_t\left((r - \kappa)dt + \sigma_V dW_t\right) \tag{3.5}$$

  with $V_0 > 0$, where the constants $\kappa \geq 0$ and $\sigma_V > 0$ represent the *payout ratio* and the *volatility* respectively, and the process $W = (W_t,\ t \geq 0)$ is a standard Brownian motion under $\mathbb{P}^*$ with respect to $\mathbb{F}$.
- The default event of a bond with face value $L > 0$ and maturity date $T > 0$ occurs at the first

time at which the firm's value process V falls below the barrier process $v = (v_t,\ t \geq 0)$ given by

$$v_t = \begin{cases} Ke^{-\gamma(T-t)} & \text{if} \quad 0 \leq t < T, \\ L & \text{if} \quad t = T, \\ +\infty & \text{if} \quad t > T \end{cases}$$

for some constants $K > 0$ and $\gamma \geq 0$, so the default time of the bond is $\tau = \inf\{t \geq 0 : V_t \leq v_t\}$ (with the convention $\inf \emptyset = +\infty$).

- The recovery process Z and the recovery claim $\widetilde{X}$ of a bond with maturity date $T > 0$ are proportional to the firm's value process, that is, $\widetilde{X} = \beta_1 V_T$ and $Z_t = \beta_2 V_t$ for all $t \in [0, T]$ and for some constants $\beta_1, \beta_2 \in [0, 1]$.

Under these assumptions, one can deduce valuation formulas for different kinds of bonds (and therefore, valuation formulas for credit derivatives on those bonds) by applying the risk-neutral valuation formula (2.9) and using the known formulas (mentioned before) for the conditional and unconditional distributions of τ and V_T. Notice that the distribution of V_T is determined by the distribution of $Y_T = \ln(V_T / v_T)$ and vice versa, since the process v is assumed to be deterministic.

There is a large number of extensions of the original Black and Cox model, which consider stochastic interest rates, stochastic barriers and jumps in the firm's value process. We refer to [1] (see Chapter 3) for more information about the first-passage-time models.

3.3 Pros and cons

The structural approach is very appealing from the economic point of view, because it links the default events to market fundamentals, so it is commonly thought that the approach taken by these models is reasonable, consistent and far from arbitrary specifications. Moreover, this approach makes the problem of hedging defaultable claims and the study of the optimal capital structure of a firm considerably easier.

Nevertheless, most structural models define the default time as a function of the total value of the firm, and this generates two big problems. The first problem is that, as we had already commented, the total value of a firm is usually difficult to measure accurately (if not impossible), so it can become difficult to do a proper implementation of these models. The second and most important problem is that the default time is modeled as a predictable stopping time with respect to the reference filtration representing the information available through assets prices, which is not a realistic feature, since default events usually come as a surprise in the real world. As a consequence of this disagreement between the model and the real world, the credit spreads predicted by these models for bonds with short maturities are many times significantly different from the spreads observed in the market.

To sum up, although the approach taken by the structural models is consistent from an economic point of the view, the mathematical specification of the default times as predictable stopping times produces discrepancies between the predictions made by these models and the real market observations. The reduced-form models, which we shall study in the next chapter, take a different approach for the modelling of default events aimed to avoid this problematic predictable nature of the default times.

4 Reduced-form approach

In this chapter, we are going to study the reduced-form models of credit risk, usually also known as intensity-based models, which offer an alternative modelling approach to the one taken by structural models, aimed to allow unpredictable default events.

We will start presenting some basic concepts and results within the reduced-form approach, including the concept of *default intensity*, several formulas to evaluate certain relevant conditional expectations and one useful rewrite of the risk-neutral valuation formula to price defaultable claims under this approach. We shall also study how to apply this rewrite of the risk-neutral valuation formula to the particular case of a CDS.

After this, we shall do a brief introduction about an alternative way to price defaultable claims under the reduced-form approach involving some special martingales, and we will briefly explain the importance of this alternative approach. We will finish this chapter with some comments about the advantages and disadvantages of the reduced-form models.

4.1 Hazard process approach

As usual, let us denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the underlying probability space endowed with some reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, where $\mathbb{P}$ stands for the *real-world probability* and $\mathbb{F}$ represents the information available through assets prices (the default-free market information). Let us also denote by $\mathbb{P}^*$ the chosen risk-neutral probability.

Consider a non-negative random variable τ, which will represent the default time associated to some default event, and let us assume that $\mathbb{P}^* \{\tau = 0\} = \mathbb{P}^* \{\tau = +\infty\} = 0$ and $\mathbb{P}^* \{\tau > t\} > 0$ for all $t \geq 0$. As we established in Chapter 2, we will regard the enlarged filtration $\mathbb{G} := \mathbb{F} \vee \mathbb{H}$ as the underlying filtration for our market model, where $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ is the filtration generated by the default process $H = \left(H_t := \mathbf{1}_{\{\tau \leq t\}},\ t \geq 0\right)$, that is, $\mathcal{H}_t = \sigma\left(H_u : 0 \leq u \leq t\right) = \sigma\left(\{\tau \leq u\} : 0 \leq u \leq t\right)$ for all $t \geq 0$.

As we have already mentioned before, in structural models we usually have that $\mathbb{G} := \mathbb{F} \vee \mathbb{H} = \mathbb{F}$, due to the fact that the default time is modeled as a predicable stopping time with respect to $\mathbb{F}$ and then $\mathcal{H}_t \subset \mathcal{F}_t$ for all $t \geq 0$. However, under the reduced-form approach, τ does not need to be predictable or even a stopping time with respect to $\mathbb{F}$. Instead of this, reduced-form models provide an exogenous specification of τ by directly modelling the $\mathbb{F}$-*hazard process* of τ under $\mathbb{P}^*$, which we define as it follows:

Definition 4.1.1. *Consider the stochastic process $F = (F_t,\ t \geq 0)$ given by $F_t = \mathbb{P}^* \{\tau \leq t \mid \mathcal{F}_t\}$ for every $t \geq 0$. Assume that $F_t < 1$ for all $t \geq 0$. Then, the $\mathbb{F}$-hazard process of τ under $\mathbb{P}^*$, denoted by $\Gamma = (\Gamma_t,\ t \geq 0)$, is defined by $\Gamma_t := -\ln(1 - F_t)$ for every $t \geq 0$.*

Notice that if the $\mathbb{F}$-hazard process Γ is well defined (that is, if $F_t < 1$ for all $t \geq 0$), then $\Gamma_0 = 0$ and we can recover F from Γ through the formula $F_t = 1 - e^{-\Gamma_t}$, so the reduced-form models focus on modelling Γ or, equivalently, F. From now on, we will assume that $F_t < 1$ for all $t \geq 0$ and that F is a càdlàg process, so that Γ is a well defined càdlàg process. Notice that F is an $\mathbb{F}$-adapted process (and thus, also Γ) and, for every $0 \leq t \leq s$, we have that $F_t \in [0, 1)$ and

$$\mathbb{E}_{\mathbb{P}^*}\left(F_s \mid \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{P}^*\left\{\tau \leq s \mid \mathcal{F}_s\right\} \mid \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(H_s \mid \mathcal{F}_s\right) \mid \mathcal{F}_t\right) \overset{\text{Tower property } (\mathcal{F}_t \subset \mathcal{F}_s)}{=} \mathbb{E}_{\mathbb{P}^*}\left(H_s \mid \mathcal{F}_t\right) \overset{H_t \leq H_s}{\geq}$$
$$\geq \mathbb{E}_{\mathbb{P}^*}\left(H_t \mid \mathcal{F}_t\right) = \mathbb{P}^*\left\{\tau \leq t \mid \mathcal{F}_t\right\} = F_t$$

so it follows that F is a bounded non-negative submartingale with respect to $\mathbb{F}$ under $\mathbb{P}^*$ (and then, F is a semimartingale, since every submartingale is a semimartingale).

Moreover, if F (and thus, also Γ) is an increasing continuous process, then F (and also Γ) is a continuous finite variation process and in this case, the relation $F_t = 1 - e^{-\Gamma_t}$ implies that $F_t = \int_0^t e^{-\Gamma_u} d\Gamma_u$ for every $t \geq 0$. The continuity of F and Γ is not an unusual assumption among reduced-form models. Actually, there is a broad class of reduced-form models which assume that Γ is an absolutely continuous process (a condition stronger than continuity) of the form $\Gamma_t = \int_0^t \gamma_u du$, for some càdlàg $\mathbb{F}$-adapted process $\gamma = (\gamma_u,\ u \geq 0)$ that is known as the $\mathbb{F}$-*default intensity* of τ under $\mathbb{P}^*$. These models focus on the modelling of γ.

Since the specification of τ is usually done through its default intensity, reduced-form models are also known as *intensity-based models*. There are several ways to model either Γ or γ, giving rise to a large number of different models. Nevertheless, we will not do a list of several different reduced-form models, but we are going to study how to price defaultable claims with the risk-neutral valuation formula once the model has already been set.

This issue is not straightforward at all, because the formula (2.9) let us price defaultable claims by computing the conditional expectation with respect to the σ-fields $\mathcal{G}_t$ of a random object involving τ, but we would need a formula to price defaultable claims by computing the conditional expectation with respect to the σ-fields $\mathcal{F}_t$ of a random object in terms of Γ or γ (instead of τ). In other words, we need a formula that can be applied once we have already set a model for Γ or γ.

Hence, in order to get a useful rewrite of the risk-neutral valuation formula, we shall start proving some technical results to evaluate certain relevant conditional expectations.

4.1.1 Conditional expectations

For the following results, we do not need to assume that τ admits an $\mathbb{F}$-default intensity, but we will only assume that the $\mathbb{F}$-hazard process Γ of τ under $\mathbb{P}^*$ is a well defined càdlàg $\mathbb{F}$-adapted process, as we mentioned before. Let us begin to prove a simple technical Lemma:

Lemma 4.1.2. *The filtration $\mathbb{G}^* = (\mathcal{G}_t^*)_{t \geq 0}$ defined by*

$$\mathcal{G}_t^* := \left\{A \in \mathcal{F} : A \cap \{\tau > t\} = B \cap \{\tau > t\} \text{ for some } B \in \mathcal{F}_t\right\}, \quad t \geq 0$$

satisfies that $\mathbb{G} \subset \mathbb{G}^*$, that is, $\mathcal{G}_t \subset \mathcal{G}_t^*$ for all $t \geq 0$.

Proof. It is easy to check that $\mathbb{G}^*$ is a filtration in $(\Omega, \mathcal{F})$, that is, an increasing family of sub-σ-fields of $\mathcal{F}$. Thus, in order to prove that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \subset \mathcal{G}_t^*$ for all $t \geq 0$, it is enough to see that $\mathcal{F}_t \subset \mathcal{G}_t^*$

and $\mathcal{H}_t \subset \mathcal{G}_t^*$ for all $t \geq 0$. Actually, we know that $\mathcal{H}_t = \sigma(\{\tau \leq u\} : 0 \leq u \leq t)$ for every $t \geq 0$, so it is enough to show that $\mathcal{F}_t \subset \mathcal{G}_t^*$ and $C_t \subset \mathcal{G}_t^*$ for all $t \geq 0$, where $C_t := \{\{\tau \leq u\} : 0 \leq u \leq t\}$. This follows from the fact that if we fix $t \geq 0$ and we take some $A \in \mathcal{F}_t \cup C_t$, then by considering

$$B := \begin{cases} A & \text{if} \quad A \in \mathcal{F}_t, \\ \emptyset & \text{if} \quad A \notin \mathcal{F}_t \end{cases}$$

it is straightforward to check that $B \in \mathcal{F}_t$ and $A \cap \{\tau > t\} = B \cap \{\tau > t\}$, so it follows that $A \in \mathcal{G}_t^*$ and this concludes the proof. □

Now, let us prove an important result to evaluate the conditional expectation $\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}X \mid \mathcal{G}_t\right)$ for a $\mathbb{P}^*$-integrable random variable X by changing the conditioning σ-field for $\mathcal{F}_t$:

Proposition 4.1.3. *Let X be a $\mathbb{P}^*$-integrable random variable. For any $t \geq 0$, we have that*

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}X \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(X \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}X \mid \mathcal{F}_t\right)}{\mathbb{P}^*\left\{\tau > t \mid \mathcal{F}_t\right\}} \tag{4.1}$$

In particular, for any $0 \leq t \leq s$,

$$\mathbb{P}^*\left\{t < \tau \leq s \mid \mathcal{G}_t\right\} = \mathbf{1}_{\{\tau>t\}}\frac{\mathbb{P}^*\left\{t < \tau \leq s \mid \mathcal{F}_t\right\}}{\mathbb{P}^*\left\{\tau > t \mid \mathcal{F}_t\right\}}$$

Proof. The last equation is obtained applying (4.1) to the random variable $X = \mathbf{1}_{\{\tau\leq s\}}$, so let us focus on proving (4.1) for a given $t \geq 0$. The equality $\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}X \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(X \mid \mathcal{G}_t\right)$ follows from the $\mathbb{P}^*$-integrability of X and from the fact that $\mathbf{1}_{\{\tau>t\}}$ is $\mathcal{G}_t$-measurable, since $\{\tau > t\} \in \mathcal{H}_t \subset \mathcal{G}_t$. Regarding the second equality of (4.1), if we denote $C = \{\tau > t\}$, then we can write it (by arranging terms and using the fact that $\mathcal{F}_t \subset \mathcal{G}_t$) as

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X\, \mathbb{P}^*(C \mid \mathcal{F}_t) \mid \mathcal{G}_t\right) = \mathbf{1}_C\, \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right)$$

Hence, in order to prove this equality, we need to show (by definition of conditional expectation) that the random variable $\mathbf{1}_C\, \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right)$ is $\mathcal{G}_t$-measurable (which is true, since $\mathbf{1}_C$ is $\mathcal{H}_t$-measurable and $\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right)$ is $\mathcal{F}_t$-measurable and we know that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$) and that the following identity holds for any $A \in \mathcal{G}_t$:

$$\int_A \mathbf{1}_C X\, \mathbb{P}^*\left(C \mid \mathcal{F}_t\right) d\mathbb{P}^* = \int_A \mathbf{1}_C\, \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right) d\mathbb{P}^*$$

Given $A \in \mathcal{G}_t$, we know from Lemma 4.1.2 that $A \cap C = B \cap C$ for some $B \in \mathcal{F}_t$, so the previous identity can be proved as it follows:

$$\begin{aligned}
\int_A \mathbf{1}_C X\, \mathbb{P}^*\left(C \mid \mathcal{F}_t\right) d\mathbb{P}^* &= \int_{A\cap C} X\, \mathbb{P}^*\left(C \mid \mathcal{F}_t\right) d\mathbb{P}^* = \int_{B\cap C} X\, \mathbb{P}^*\left(C \mid \mathcal{F}_t\right) d\mathbb{P}^* = \\
&= \int_B \mathbf{1}_C X\, \mathbb{P}^*\left(C \mid \mathcal{F}_t\right) d\mathbb{P}^* \underset{\substack{\uparrow \\ \text{Definition of conditional expectation } (B \in \mathcal{F}_t)}}{=} \int_B \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X\, \mathbb{P}^*\left(C \mid \mathcal{F}_t\right) \mid \mathcal{F}_t\right) d\mathbb{P}^* = \\
&= \int_B \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right)\, \mathbb{P}^*\left(C \mid \mathcal{F}_t\right) d\mathbb{P}^* = \int_B \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right)\, \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C \mid \mathcal{F}_t\right) d\mathbb{P}^* =
\end{aligned}$$

$$= \int_B \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C\, \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right) \mid \mathcal{F}_t\right) d\mathbb{P}^* \underset{\substack{\uparrow \\ \text{Definition of conditional expectation } (B \in \mathcal{F}_t)}}{=} \int_B \mathbf{1}_C\, \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right) d\mathbb{P}^* =$$
$$= \int_{B\cap C} \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right) d\mathbb{P}^* = \int_{A\cap C} \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right) d\mathbb{P}^* =$$
$$= \int_A \mathbf{1}_C\, \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_C X \mid \mathcal{F}_t\right) d\mathbb{P}^*$$

Therefore, we conclude that (4.1) holds. □

The next result, which is a consequence of Proposition 4.1.3, provides us with some useful modifications of the formula (4.1) involving the $\mathbb{F}$-hazard process Γ of τ.

Proposition 4.1.4. *Let X be a $\mathbb{P}^*$-integrable random variable and let $0 \leq t \leq s$. Then,*

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} X \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} e^{\Gamma_t} X \mid \mathcal{F}_t\right) \tag{4.2}$$

Moreover, if X is $\mathcal{F}_s$-measurable, then

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} X \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(e^{\Gamma_t - \Gamma_s} X \mid \mathcal{F}_t\right) \tag{4.3}$$

Proof. First, notice that if Y is a $\mathbb{P}^*$-integrable random variable, then it follows from (4.1) that

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}} Y \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}} Y \mid \mathcal{F}_t\right)}{\mathbb{P}^*\left\{\tau > t \mid \mathcal{F}_t\right\}} = \mathbf{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}} Y \mid \mathcal{F}_t\right)}{1 - F_t} =$$
$$= \mathbf{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}} Y \mid \mathcal{F}_t\right) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}} e^{\Gamma_t} Y \mid \mathcal{F}_t\right)$$

where we have used in the last equality the fact that Γ_t is $\mathcal{F}_t$-measurable (because Γ is $\mathbb{F}$-adapted). Then, applying the previous equation to the random variable $Y := \mathbf{1}_{\{\tau>s\}} X$ and using the identity $\mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{\tau>s\}} = \mathbf{1}_{\{\tau>s\}}$, we obtain (4.2).

Now, assume that X is $\mathcal{F}_s$-measurable and let us prove (4.3). Since $\mathcal{F}_t \subset \mathcal{F}_s$ (because $0 \leq t \leq s$), the random variables Γ_t and $\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} e^{\Gamma_t} X \mid \mathcal{F}_t\right)$ are $\mathcal{F}_s$-measurable. Hence, applying (4.2), we obtain

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} X \mid \mathcal{G}_t\right) \overset{\substack{(4.2)\\ \downarrow}}{=} \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} e^{\Gamma_t} X \mid \mathcal{F}_t\right) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} e^{\Gamma_t} X \mid \mathcal{F}_t\right) \mid \mathcal{F}_s\right) \overset{\substack{\text{Tower property}\\ \downarrow}}{=}$$
$$= \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} e^{\Gamma_t} X \mid \mathcal{F}_s\right) \mid \mathcal{F}_t\right) =$$
$$= \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>s\}} \mid \mathcal{F}_s\right) e^{\Gamma_t} X \mid \mathcal{F}_t\right) =$$
$$= \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\mathbb{P}^*\left\{\tau > s \mid \mathcal{F}_s\right\} e^{\Gamma_t} X \mid \mathcal{F}_t\right) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left((1 - F_s) e^{\Gamma_t} X \mid \mathcal{F}_t\right) =$$
$$= \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(e^{\Gamma_t - \Gamma_s} X \mid \mathcal{F}_t\right)$$

where we have used the $\mathcal{F}_s$-measurability of Γ_t and X in the fourth equality. Thus, this proves (4.3). □

The following proposition will be useful, later in this text, to compute the part of the value of a defaultable claim that corresponds to the recovery process (that is, the current value of the recovery payoff at time of default). The complete proof can be found in [1] (see Chapter 5, Section 5.1.1, Proposition 5.1.1 and Corollary 5.1.3), but we shall only sketch the main ideas.

Proposition 4.1.5. *Let $Z = (Z_u,\ u \geq 0)$ be a bounded $\mathbb{F}$-predictable process. Then, for any $0 \leq t \leq s \leq \infty$,*

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\leq s\}} Z_\tau \,\middle|\, \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}^*}\left(\int_t^s Z_u dF_u \,\middle|\, \mathcal{F}_t\right) \tag{4.4}$$

Furthermore, if F (and thus, also Γ) is an increasing continuous process, then, for any $0 \leq t \leq s \leq \infty$,

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\leq s\}} Z_\tau \,\middle|\, \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\int_t^s Z_u e^{\Gamma_t - \Gamma_u} d\Gamma_u \,\middle|\, \mathcal{F}_t\right) \tag{4.5}$$

Proof. The proof of (4.4) is done by assuming first that Z is a stepwise $\mathbb{F}$-predictable process, that is, that $Z_u = \sum_{i=0}^n Z_{t_i} \mathbf{1}_{\{t_i<u\leq t_{i+1}\}}$ for $t < u \leq s$, where $t = t_0 < \cdots < t_{n+1} = s$ and Z_{t_i} is an $\mathcal{F}_{t_i}$-measurable random variable for $i = 0, 1, \ldots, n$. After this, the general case can be proved by approximating a general bounded $\mathbb{F}$-predictable process by a sequence of bounded stepwise $\mathbb{F}$-predictable processes.

Notice that the integral appearing in (4.4) is defined as an integral of the third type (within the list of integrals described in Chapter 2). If F is an increasing continuous process, then F (and also Γ) is a finite variation process, so the integral mentioned above can be thought as an integral of the first type, which means that it is well defined path by path as a Lebesgue-Stieltjes integral. Moreover, since F is an increasing continuous process, we also know that $F_t = \int_0^t e^{-\Gamma_u} d\Gamma_u$ for every $t \geq 0$. Therefore, applying the associativity of the Lebesgue-Stieltjes integral (see Proposition 2.3.2), we can write (4.4) as

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\leq s\}} Z_\tau \,\middle|\, \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} e^{\Gamma_t}\, \mathbb{E}_{\mathbb{P}^*}\left(\int_t^s Z_u e^{-\Gamma_u} d\Gamma_u \,\middle|\, \mathcal{F}_t\right)$$

In this formula, we can put e^{Γ_t} inside the conditional expectation (because Γ_t is $\mathcal{F}_t$-measurable) and we can also put it inside the integral, since this integral is well defined path by path. Hence, we obtain (4.5). □

The boundedness of Z among the assumptions of the previous proposition is not a necessary condition, and it can be relaxed. However, this is not a weird assumption in practice, because the recovery payoff Z of a defaultable claim at time of default is limited (by obvious reasons) by the ability of the firm to redeem its debt, which at the same time is bounded by the highest value of the firm over a finite time interval, and in the real world we can always find a reasonable (large enough) upper bound for the value of a firm over a specific finite time period. In any case, for practical purposes, we will always assume that the recovery payoff Z satisfies the necessary conditions for Proposition 4.1.5 to hold.

The next result will be useful to compute the part of the value of a defaultable claim coming from the promised dividends prior to default.

Proposition 4.1.6. *Let $A = (A_u,\ u \geq 0)$ be a bounded $\mathbb{F}$-adapted finite variation process. Then, for any $0 \leq t \leq s$,*

$$\mathbb{E}_{\mathbb{P}^*}\left(\int_t^s (1 - H_u) dA_u \,\middle|\, \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{P}^*}\left(\int_t^s e^{\Gamma_t - \Gamma_u} dA_u \,\middle|\, \mathcal{F}_t\right) \tag{4.6}$$

Proof. Let us fix $0 \leq t \leq s$. Notice that the integral $\int_t^s (1-H_u)dA_u$ is well defined path by path as a Lebesgue-Stieltjes integral, since H is a bounded $\mathbb{F}$-adapted càdlàg process and A is a finite variation process. Consider now the process $\widetilde{A} = \left(\widetilde{A}_u,\ u \geq 0\right)$ defined by $\widetilde{A}_u := A_u - A_t$ for all $u \geq 0$. Over the time interval $[t, s]$, we know that $\widetilde{A}$ is a bounded $\mathbb{F}$-adapted finite variation process. Moreover, we can see that

$$\begin{aligned}\int_t^s (1-H_u)dA_u &= \int_t^s \mathbf{1}_{\{\tau>u\}}dA_u = \mathbf{1}_{\{t<\tau\leq s\}}\int_t^s \mathbf{1}_{\{\tau>u\}}dA_u + \mathbf{1}_{\{\tau>s\}}\int_t^s \mathbf{1}_{\{\tau>u\}}dA_u = \\ &= \mathbf{1}_{\{t<\tau\leq s\}}\int_t^s \mathbf{1}_{\{\tau>u\}}dA_u + \mathbf{1}_{\{\tau>s\}}\int_t^s dA_u = \\ &= \mathbf{1}_{\{t<\tau\leq s\}}\left(A_{\tau-} - A_t\right) + \mathbf{1}_{\{\tau>s\}}\left(A_s - A_t\right) = \\ &= \mathbf{1}_{\{t<\tau\leq s\}}\widetilde{A}_{\tau-} + \mathbf{1}_{\{\tau>s\}}\widetilde{A}_s\end{aligned}$$

Since the process $\widetilde{A}$ is càdlàg (because it is a finite variation process), bounded and $\mathbb{F}$-adapted over the time interval $[t, s]$, we know that the random variable $\widetilde{A}_s$ is $\mathcal{F}_s$-measurable and the process $\widetilde{A}_-$ is bounded, càglàd and $\mathbb{F}$-adapted over $[t, s]$ (and then, also $\mathbb{F}$-predictable). Hence, applying formulas (4.3) and (4.4) to the random objects $\widetilde{A}_s$ and $\widetilde{A}_-$ respectively and using the fact that Γ is $\mathbb{F}$-adapted, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}\left(\int_t^s (1-H_u)dA_u \,\middle|\, \mathcal{G}_t\right) &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{t<\tau\leq s\}}\widetilde{A}_{\tau-} + \mathbf{1}_{\{\tau>s\}}\widetilde{A}_s \,\middle|\, \mathcal{G}_t\right) = \\ &= \mathbf{1}_{\{\tau>t\}}e^{\Gamma_t}\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^s \widetilde{A}_{u-}dF_u + e^{-\Gamma_s}\widetilde{A}_s \,\middle|\, \mathcal{F}_t\right)\end{aligned} \tag{4.7}$$

Let us consider the stochastic process $G = (G_u,\ u \geq 0)$ defined by $G_u = 1 - F_u = e^{-\Gamma_u}$ for all $u \geq 0$, which is usually called the $\mathbb{F}$-*survival process* of τ under $\mathbb{P}^*$. Then, G is a semimartingale (because F is too) and we can write the random object inside the last expectation of (4.7) as

$$\int_t^s \widetilde{A}_{u-}dF_u + e^{-\Gamma_s}\widetilde{A}_s = -\int_t^s \widetilde{A}_{u-}dG_u + G_s\widetilde{A}_s \tag{4.8}$$

Now, since G is a semimartingale and $\widetilde{A}$ is a finite variation process such that $\widetilde{A}_t = 0$, it follows from Theorem 2.3.5 (integration by parts (I)) that

$$G_s\widetilde{A}_s = \int_t^s G_u d\widetilde{A}_u + \int_t^s \widetilde{A}_{u-}dG_u \tag{4.9}$$

where the first integral is well defined path by path as a Lebesgue-Stieltjes integral. Finally, combining (4.7), (4.8) and (4.9), we obtain

$$\mathbb{E}_{\mathbb{P}^*}\left(\int_t^s (1-H_u)dA_u \,\middle|\, \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}e^{\Gamma_t}\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^s G_u d\widetilde{A}_u \,\middle|\, \mathcal{F}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(\int_t^s e^{\Gamma_t-\Gamma_u}dA_u \,\middle|\, \mathcal{F}_t\right)$$

where the last equality follows from the $\mathcal{F}_t$-measurability of Γ_t and from the associativity of the Lebesgue-Stieltjes integral (notice that we can write $\widetilde{A}_u = \int_t^u dA_v$ for $u \in [t, s]$). Therefore, this proves (4.6) for every $0 \leq t \leq s$. □

As it happened with the process Z in Proposition 4.1.5, the boundedness of A in Proposition 4.1.6 is not a necessary condition, but is not a weird assumption in practice, since we can assume that the promised dividends of a defaultable claim over a specific finite time period are bounded by a large enough upper bound. In any case, for practical purposes, we will always assume that the promised dividends process A satisfies the necessary conditions for Proposition 4.1.6 to hold.

4.1.2 Intensity-based valuation of defaultable claims

Once we have already proved some needed technical results, we want to determine a useful rewrite of the risk-neutral valuation formula (2.9) to compute the pre-default value of a defaultable claim $(X, A, 0, Z, \tau)$ (as usual, $\widetilde{X} = 0$) in terms of the $\mathbb{F}$-hazard process Γ of τ under $\mathbb{P}^*$.

As before, we shall assume that Γ is a well defined càdlàg $\mathbb{F}$-adapted process and we will also keep the assumptions made in Section 2.3.2 for the random objects X, A, Z and τ of a defaultable claim. Moreover, we shall make the following extra assumptions in order that we can apply the technical results proved before:

- The processes A and Z are bounded.
- The process A is either increasing or decreasing.

As we argued before, the boundedness of A and Z can be relaxed, but it is not a weird assumption. The same happens with the second assumption above: it is not needed, but it is a sufficient condition which will be enough for our practical purposes, since this condition is fulfilled when the defaultable claim is a bond or a CDS.

Let us remark that the inverse savings account process $\left(B_t^{-1},\ t \geq 0\right)$ is bounded, because in this text the short-term interest rate process r is assumed to be non-negative. However, if we allowed r to take negative values, we should impose some boundedness assumption to r (from below) to ensure that we can apply the previous technical results to some specific processes. Let us now state and prove the following rewrite of the risk-neutral valuation formula:

Theorem 4.1.7. *Under the assumptions made above, the ex-dividend price process $S = (S_t,\ t \in [0,T])$ of a defaultable claim $(X, A, 0, Z, \tau)$ with maturity date $T > 0$ admits the following representation:*

$$S_t = \mathbf{1}_{\{\tau > t\}} G_t^{-1} B_t \, \mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1} \left(G_u dA_u - Z_u dG_u \right) + G_T B_T^{-1} X \,\middle|\, \mathcal{F}_t \right), \quad t \in [0,T] \tag{4.10}$$

where $G = (G_u,\ u \geq 0)$ is the $\mathbb{F}$-survival process of τ under $\mathbb{P}^*$. Moreover, if F (and thus, also Γ) is an increasing continuous process, then

$$S_t = \mathbf{1}_{\{\tau > t\}} B_t \, \mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1} e^{\Gamma_t - \Gamma_u} \left(dA_u + Z_u d\Gamma_u \right) + e^{\Gamma_t - \Gamma_T} B_T^{-1} X \,\middle|\, \mathcal{F}_t \right), \quad t \in [0,T] \tag{4.11}$$

If, in addition, τ admits an $\mathbb{F}$-default intensity $\gamma = (\gamma_u,\ u \geq 0)$ under $\mathbb{P}^*$, then

$$S_t = \mathbf{1}_{\{\tau > t\}} \mathbb{E}_{\mathbb{P}^*}\left(\int_t^T e^{-\int_t^u (r_v + \gamma_v) dv} \left(dA_u + Z_u \gamma_u du \right) + X e^{-\int_t^T (r_v + \gamma_v) dv} \,\middle|\, \mathcal{F}_t \right), \quad t \in [0,T] \tag{4.12}$$

Proof. It follows from the risk-neutral valuation formula (2.9) that $S_t = I_t + J_t + K_t$ for every $t \in [0,T]$, where

$$\begin{aligned} I_t &= B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1}(1-H_u)dA_u \,\middle|\, \mathcal{G}_t\right) \\ J_t &= B_t\,\mathbb{E}_{\mathbb{P}^*}\left(B_\tau^{-1} Z_\tau \mathbf{1}_{\{t<\tau\le T\}} \,\middle|\, \mathcal{G}_t\right) \\ K_t &= B_t\,\mathbb{E}_{\mathbb{P}^*}\left(B_T^{-1} X \mathbf{1}_{\{\tau>T\}} \,\middle|\, \mathcal{G}_t\right) \end{aligned}$$

First of all, since the process $\left(B_t^{-1},\ t \ge 0\right)$ is càdlàg and $\mathbb{F}$-adapted and A is a finite variation process, we can define the process $\widetilde{A} = \left(\widetilde{A}_t := \int_0^t B_u^{-1} dA_u,\ t \ge 0\right)$ path by path as a Lebegue-Stieltjes integral. Moreover, since A and $\left(B_t^{-1},\ t \ge 0\right)$ are bounded and A is either increasing or decreasing by hypothesis, it follows from Proposition 2.3.2 that $\widetilde{A}$ is a bounded $\mathbb{F}$-adapted finite variation process. Thus, applying Proposition 4.1.6 to the process $\widetilde{A}$, we obtain

$$\begin{aligned} I_t = B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1}(1-H_u)dA_u \,\middle|\, \mathcal{G}_t\right) = B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T (1-H_u)d\widetilde{A}_u \,\middle|\, \mathcal{G}_t\right) = \\ = \mathbf{1}_{\{\tau>t\}} B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^s e^{\Gamma_t-\Gamma_u} d\widetilde{A}_u \,\middle|\, \mathcal{F}_t\right) = \mathbf{1}_{\{\tau>t\}} B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^s B_u^{-1} e^{\Gamma_t-\Gamma_u} dA_u \,\middle|\, \mathcal{F}_t\right), \quad t \in [0,T] \end{aligned}$$

where we have used the associativity of the Lebesgue-Stieltjes integral in the second and the last equalities. Since Γ_t is $\mathcal{F}_t$-measurable, we can equivalently write

$$I_t = \mathbf{1}_{\{\tau>t\}} G_t^{-1} B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1} G_u dA_u \,\middle|\, \mathcal{F}_t\right), \quad t \in [0,T]$$

Now, on the other hand, we know that $\left(B_t^{-1},\ t \ge 0\right)$ is defined path by path as the inverse exponential of the classical Lebesgue integral of the càdlàg $\mathbb{F}$-adapted process r (the short-term interest rate). Then, it follows from Proposition 2.3.2 that $\left(B_t^{-1},\ t \ge 0\right)$ is a continuous $\mathbb{F}$-adapted finite variation process, so in particular it is $\mathbb{F}$-predictable.

Thus, we have that Z and $\left(B_t^{-1},\ t \ge 0\right)$ are bounded $\mathbb{F}$-predictable processes, and then so is the product of them, so if we apply Proposition 4.1.5 to the product of these processes we obtain

$$\begin{aligned} J_t = B_t\,\mathbb{E}_{\mathbb{P}^*}\left(B_\tau^{-1} Z_\tau \mathbf{1}_{\{t<\tau\le T\}} \,\middle|\, \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} e^{\Gamma_t} B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1} Z_u dF_u \,\middle|\, \mathcal{F}_t\right) = \\ = -\mathbf{1}_{\{\tau>t\}} G_t^{-1} B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1} Z_u dG_u \,\middle|\, \mathcal{F}_t\right), \quad t \in [0,T] \end{aligned}$$

where we have used the identity $G_u = 1 - F_u = e^{-\Gamma_u}$ in the last equality. Furthermore, if F is an increasing continuous process, then

$$J_t = \mathbf{1}_{\{\tau>t\}} B_t\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1} e^{\Gamma_t-\Gamma_u} Z_u d\Gamma_u \,\middle|\, \mathcal{F}_t\right), \quad t \in [0,T]$$

Finally, since B_T^{-1} and X are $\mathcal{F}_T$-measurable random variables (satisfying also suitable integrability conditions), it follows from Proposition 4.1.4 that

$$K_t = B_t\,\mathbb{E}_{\mathbb{P}^*}\left(B_T^{-1} X \mathbf{1}_{\{\tau>T\}} \,\middle|\, \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}} B_t\,\mathbb{E}_{\mathbb{P}^*}\left(e^{\Gamma_t-\Gamma_T} B_T^{-1} X \,\middle|\, \mathcal{F}_t\right), \quad t \in [0,T]$$

or equivalently,

$$K_t = \mathbf{1}_{\{\tau > t\}} G_t^{-1} B_t \, \mathbb{E}_{\mathbb{P}^*} \left(G_T^{-1} B_T^{-1} X \mid \mathcal{F}_t \right), \quad t \in [0, T]$$

Therefore, if we add the expressions obtained for I_t, J_t and K_t, we obtain (4.10) and (4.11). The formula (4.12) follows from (4.11), making the substitutions $B_t = \exp\left(\int_0^t r_u du\right)$ and $\Gamma_t = \int_0^t \gamma_u du$ and applying the associativity of the Lebesgue-Stieltjes integral. □

So far in this text we have assumed that the underlying filtration for our market model is given by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ represents the information available through assets prices and $\mathbb{H}$ represents the information about the occurrence of the default time associated to some default event. Nevertheless, in some situations we can access more information (for instance, information about other default events), so in this case we will have that $\mathbb{F} \vee \mathbb{H} \subset \mathbb{G}$ but the equality might not hold. Actually, we will come across this situation in Chapter 5.

The problem is that if we only impose that $\mathbb{F} \vee \mathbb{H} \subset \mathbb{G}$, then the valuation formulas obtained with Theorem 4.1.7 do not hold in general. However, when we want to price financial instruments, sometimes we are only interested in the price at time 0, and it is easy to check that the valuation formulas obtained with Theorem 4.1.7 are valid to compute the value of a defaultable claim $(X, A, 0, Z, \tau)$ at time 0 as long as $\mathbb{F} \vee \mathbb{H} \subset \mathbb{G}$ and the following conditions are satisfied:

- $\mathcal{G}_0$ is the trivial σ-field, that is, $\mathcal{G}_0 = \{\Omega, \emptyset\}$.
- All the assumptions made in Theorem 4.1.7 for the random objects X, A, Z and τ are fulfilled.

4.1.3 Valuation of a CDS

Now, we are going to apply the valuation formulas obtained with Theorem 4.1.7 to price a CDS at time 0. Actually, as we have explained in Chapter 2, the value of a CDS depends on a parameter κ, which is called the spread of the CDS, and κ is adjusted so that the value of the CDS at time 0 is 0. Thus, what we want is to find a formula for this spread κ.

Let $0 < T \leq U$ and consider a T-maturity CDS with spread κ, payment dates $0 < T_1 < \cdots < T_n = T$ and protection at default on a U-maturity defaultable bond with face value L and fractional recovery of par value with recovery rate $\delta \in [0, 1)$. Remember that, by Definition 2.3.14, this CDS is the defaultable claim $(0, A, 0, (1-\delta)L, \tau)$, where

$$A_t = -\sum_{i=1}^{n} \kappa L (T_i - T_{i-1}) \cdot \mathbf{1}_{[T_i, T]}(t), \quad t \in [0, T] \tag{4.13}$$

with the convention $T_0 = 0$, and τ is the default time of the underlying bond, which is a random variable such that $\mathbb{P}^* \{\tau = 0\} = \mathbb{P}^* \{\tau = +\infty\} = 0$ and $\mathbb{P}^* \{\tau > t\} > 0$ for all $t \geq 0$.

As usual, $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is the underlying filtration for our market model, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the reference filtration representing the information available through assets prices (the default-free market information) and $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ is the filtration generated by the default process $H = \left(H_t := \mathbf{1}_{\{\tau \leq t\}},\ t \geq 0\right)$.

Let us consider the stochastic process $F = (F_t,\ t \geq 0)$ given by $F_t = \mathbb{P}^* \{\tau \leq t \mid \mathcal{F}_t\}$ for every $t \geq 0$ and let $G = (G_t,\ t \geq 0)$ be the $\mathbb{F}$-survival process of τ under $\mathbb{P}^*$, which is defined by $G_t = 1 - F_t$ for all $t \geq 0$. In order to find a formula for the spread κ, we shall make the following assumptions:

- $\mathbb{F} \vee \mathbb{H} \subset \mathbb{G}$ (the equality is not needed) and $\mathcal{G}_0$ is the trivial σ-field.
- As usual, the short-term interest rate $r = (r_t,\ t \geq 0)$ is a non-negative càdlàg $\mathbb{F}$-adapted process (the non-negativity can be relaxed, but this restriction will be enough for our purposes).
- The process H is independent of the filtration $\mathbb{F}$ under $\mathbb{P}^*$, that is, the random variable H_t is independent of the σ-field $\mathcal{F}_s$ for every $s, t \geq 0$ under $\mathbb{P}^*$. As a consequence,

  $$F_t = \mathbb{P}^* \left\{ \tau \leq t \mid \mathcal{F}_t \right\} = \mathbb{E}_{\mathbb{P}^*} \left(H_t \mid \mathcal{F}_t \right) = \mathbb{E}_{\mathbb{P}^*} \left(H_t \right) = \mathbb{P}^* \left\{ \tau \leq t \right\}, \quad t \geq 0$$

  so F is the cumulative distribution function of τ and, in particular, F is an increasing càdlàg function taking values in $[0, 1)$.
- F is an absolutely continuous function of the form $F_t = \int_0^t f(u)du$, for some Lebesgue integrable function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$. Actually, it is not difficult to check that this is equivalent to saying that τ admits an $\mathbb{F}$-default intensity $\gamma = (\gamma(t),\ t \geq 0)$ under $\mathbb{P}^*$ and, in this case, the relations $\gamma(t) = f(t)/(1 - F_t)$ and $f(t) = \gamma(t) e^{-\int_0^t \gamma(u)du}$ hold for all $t \geq 0$.

Under these assumptions, we can prove the following result, which gives us a formula for the spread κ of our CDS in terms of the probability density function f of τ (in fact, we could also derive a formula in terms of the default intensity γ using the relations between f and γ described above).

Theorem 4.1.8. *Under the previous assumptions, the spread κ of the CDS considered above can be computed as*

$$\kappa = \frac{(1-\delta) \int_0^T P(0,u) f(u) du}{\sum_{i=1}^n P(0,T_i) G_{T_i} (T_i - T_{i-1})} = \frac{(1-\delta) \int_0^T P(0,u) f(u) du}{\sum_{i=1}^n P(0,T_i) \left(1 - \int_0^{T_i} f(u)du\right) (T_i - T_{i-1})} \tag{4.14}$$

where, for every $t \geq 0$, $P(0,t) = \mathbb{E}_{\mathbb{P}^*} \left(B_t^{-1} \right)$ is the value at time 0 of a default-free zero-coupon bond with face value 1 and maturity date t.

Proof. We know that the CDS that we have considered is the defaultable claim $(X, A, 0, Z, \tau)$, where $X = 0$, $Z = (1-\delta)L$ and A is given by (4.13). It is easy to check that these random objects satisfy all the hypothesis of Theorem 4.1.7. Then, since $\mathbb{F} \vee \mathbb{H} \subset \mathbb{G}$ and $\mathcal{G}_0$ is the trivial σ-field, we can apply (4.10) to compute the price of the CDS at time $t = 0$. Since we know that this price is 0, it follows that

$$0 = \left(\mathbf{1}_{\{\tau > t\}} G_t^{-1} B_t \, \mathbb{E}_{\mathbb{P}^*} \left(\int_t^T B_u^{-1} \left(G_u dA_u - Z_u dG_u \right) + G_T B_T^{-1} X \,\middle|\, \mathcal{F}_t \right) \right) \Bigg|_{t=0} =$$

$$= \mathbb{E}_{\mathbb{P}^*} \left(\int_0^T B_u^{-1} \left(G_u dA_u + Z_u dF_u \right) + G_T B_T^{-1} X \right)$$

where we have used the fact that $\mathbb{P}^* \left\{ \tau = 0 \right\} = 0$, $G_0 = B_0 = 1$ and $\mathcal{F}_0$ is the trivial σ-field.

Hence, using the expressions for X, Z and A, using the fact that $F_t = \int_0^t f(u)du$ for all $t \geq 0$ and applying the associativity of the Lebesgue-Stieltjes integral, we can see that

$$\begin{aligned}
0 &= \mathbb{E}_{\mathbb{P}^*}\left(\int_0^T B_u^{-1}\left(G_u dA_u + Z_u dF_u\right) + G_T B_T^{-1} X\right) = \\
&= \mathbb{E}_{\mathbb{P}^*}\left(\int_0^T B_u^{-1} G_u dA_u\right) + (1-\delta) L\, \mathbb{E}_{\mathbb{P}^*}\left(\int_0^T B_u^{-1} dF_u\right) = \\
&= -\kappa L\, \mathbb{E}_{\mathbb{P}^*}\left(\sum_{i=1}^n B_{T_i}^{-1} G_{T_i}(T_i - T_{i-1})\right) + (1-\delta) L\, \mathbb{E}_{\mathbb{P}^*}\left(\int_0^T B_u^{-1} f(u) du\right) = \\
&= -\kappa L \sum_{i=1}^n \mathbb{E}_{\mathbb{P}^*}\left(B_{T_i}^{-1}\right) G_{T_i}(T_i - T_{i-1}) + (1-\delta) L \int_0^T \mathbb{E}_{\mathbb{P}^*}\left(B_u^{-1}\right) f(u) du
\end{aligned} \tag{4.15}$$

In the last equality we have used the fact that G and f are deterministic functions and we have applied Fubini-Tonelli theorem to the second term (we can apply it because the integrand is jointly measurable and non-negative).

Now, it is well known that the value at time 0 of a default-free zero-coupon bond with face value 1 and maturity date $t \geq 0$ is given by $P(0,t) = \mathbb{E}_{\mathbb{P}^*}\left(B_t^{-1}\right)$. Actually, such a bond can be seen as the defaultable claim $(1, 0, 0, 0, \infty)$, and then the previous formula follows from (2.9). Thus, using this formula and isolating κ in (4.15), we obtain (4.14), as desired. □

4.2 Martingale approach

Let us go back to the abstract setting established at the beginning of Section 4.1, that is, let us assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is the reference filtration representing the default-free market information and $\mathbb{H}$ is the filtration generated by the default process H associated to the default time τ.

In the previous section, we have studied how to price a defaultable claim in terms of the $\mathbb{F}$-hazard process of τ under $\mathbb{P}^*$. This is the most typical approach in a reduced-form model, and it is usually called the *hazard process approach*, but in this section we will briefly introduce an alternative approach which is also very popular among the reduced-form models: the *martingale approach*. Let us start with an important definition:

Definition 4.2.1. *A càdlàg $\mathbb{F}$-predictable increasing process $\Lambda = (\Lambda_t,\ t \geq 0)$ such that $\Lambda_0 = 0$ is called an $\mathbb{F}$-martingale hazard process of τ under $\mathbb{P}^*$ if and only if the process $\widetilde{M} = \left(\widetilde{M}_t = H_t - \Lambda_{t\wedge\tau},\ t \geq 0\right)$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$.*

It can be proved that the $\mathbb{F}$-martingale hazard process Λ of τ under $\mathbb{P}^*$ exists and it is unique until time τ (see [1], Chapter 6, Section 6.1).

Now, let us consider the process $F = (F_t,\ t \geq 0)$ given by $F_t = \mathbb{P}^*\{\tau \leq t \mid \mathcal{F}_t\}$ for every $t \geq 0$. If F is an increasing continuous process and $F_t < 1$ for all $t \geq 0$, then the $\mathbb{F}$-hazard process Γ of τ under $\mathbb{P}^*$ is a well defined continuous process and $\Lambda = \Gamma$ (see [1], Chapter 6, Proposition 6.2.1), that is, Γ is the $\mathbb{F}$-martingale hazard process of τ under $\mathbb{P}^*$. Hence, in this case, the process $\widetilde{M} = \left(\widetilde{M}_t = H_t - \Gamma_{t\wedge\tau},\ t \geq 0\right)$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$, and we could derive useful pricing and hedging results for defaultable claims by applying only the martingale condition of $\widetilde{M}$ (without needing to use the fact that $\Gamma_t = -\ln(1 - F_t)$

for all $t \geq 0$).

However, if we relax some of the hypothesis established above for F (for instance, if F is not increasing or it is not continuous), then Λ and Γ might not be the same process (and thus, Γ would not be the $\mathbb{F}$-martingale hazard process of τ under $\mathbb{P}^*$). In fact, in some situations the $\mathbb{F}$-hazard process Γ could be difficult to find or even it might not be well defined. For this reason, sometimes it is easier or more preferable to work with the $\mathbb{F}$-martingale hazard process Λ (instead of with Γ) and take advantage of the martingale condition of the process $\widetilde{M} = \left(\widetilde{M}_t = H_t - \Lambda_{t\wedge\tau},\ t \geq 0\right)$ in order to price and hedge defaultable claims. This alternative approach is known as the *martingale approach*, and we refer to [1] (see Chapters 5, 6 and 8) for more information about this topic.

4.3 Pros and cons

The reduced-form approach gives an exogenous specification of the default time by providing a model for the conditional probability of default, which is usually calibrated with current market data or historical data. Thus, unlike what happens with structural models, under the reduced-form approach, the default time does not need to be a predictable stopping time with respect to the reference filtration representing the default-free market information, which is a very appealing and realistic feature that allows for an element of surprise in the occurrence of the default event. Moreover, since the default event is not tied to the total value of the firm's assets, the reduced-form approach gives more freedom to model this default event (there are a number of ways to model the hazard process of a default time).

From an economic point of view, the structural approach might seem more reasonable and attractive, because the total value of the firm's assets is usually not included in a reduced-form model. Nevertheless, let us remark that, within the class of reduced-form models, we can also find the so-called hybrid models, which take a reduced-form approach to model the default event but at the same time they incorporate the total value of the firm (or other economic factors) into the modelling of the hazard process of the default time.

To sum up, the reduced-form approach is very appealing because of the unpredictable nature of the default times, and although these models do not link directly the default times to market fundamentals, this approach offers more freedom to model the hazard process of a default time, so actually we can incorporate some economic factors into the modelling of this hazard process.

5 A reduced-form model for the valuation of CoCo bonds

In this chapter, we are going to study in detail the reduced-form model built in [2] for pricing some popular credit-risk sensitive instruments which are called *contingent convertible bonds* or simply *CoCos*. We shall start introducing these financial instruments and making a complete description of the model. Then, we will apply the theory of the previous chapters to develop pricing formulas for CoCos under this model. Finally, we will develop formulas to calibrate the parameters of the model with market spreads of traded CDS's.

5.1 Contingent convertible bonds (CoCos)

A contingent convertible bond (also called CoCo) is a bond that converts into equity shares or a cash payment if a pre-specified trigger event occurs. There exist several versions of this instrument, but we will focus on the two most relevant:

- *Standard CoCos*, which convert into a specified number of shares of the issuing firm's stock if the trigger event occurs (the term "standard" used to refer to this kind of CoCos has been arbitrarily chosen by us).
- *Write-down CoCos*, which convert into a specified cash payment if the trigger event occurs.

As we can see, classical CoCos give the holders the right to take over the firm's assets if some (usually undesirable) event happens. These contracts became very popular during the financial crisis of 2007-08. As we did with CDS's, we are omitting the accrued interest rate payment of a CoCo: an extra payment made by the issuing firm at time of conversion (if the CoCo converts), which is proportional to the distance between the conversion time and the date of the last coupon payment prior to conversion.

The line that distinguishes CoCos from classical bonds is very thin, but we can notice three main differences. First of all, the trigger event of a CoCo does not need to be the same as the default event of a classical bond. The latter represents the situation in which the firm is not able to redeem its debt (and this is usually linked to the possibility of bankruptcy), but a CoCo might convert before this situation occurs. Actually, the trigger event of a CoCo usually depends on the relative position of a certain accounting ratio with respect to a given threshold, or on the decision of a supervisory authority or on the combination of several factors. In any case, the trigger event of a CoCo usually occurs before or at time of firm's default.

Another difference is that CoCos are usually subordinated to classical bonds, which means that, in case of default, CoCos are ranked lower in priority for repayment compared to classical bonds, that is, they have a lower priority in the repayment hierarchy. For this reason, in several models, if the firm goes bankrupt at time of conversion, then the cash payment made by a write-down CoCo is usually assumed to be 0 (the CoCo evaporates).

Finally, unlike classical bonds, the recovery payment of a standard CoCo (the price of the CoCo at time of conversion) relies on the performance of the firm's stock price, since standard CoCos convert into shares. For this reason, the valuation of CoCos requires a model for the firm's stock price.

In the next section, we will study a particular reduced-form model for pricing CoCos and, under this model, we shall provide formal definitions of *standard CoCo* and *write-down CoCo* as defaultable claims.

5.2 Description of the model

In this section, we are going to describe a particular reduced-form model for pricing CoCos. The model that we will present is the one built in [2] but with two differences. On the one hand, for the sake of simplicity, we shall omit the accrued interest rate payments of CDS's and CoCos, which are considered in [2]. On the other hand, in [2], the calibration of the parameters is done by fitting the model to market spreads of CDS's with different maturities, assuming that the payment dates of all CDS's lie on an equally spaced grid $0 < s_1 < s_2 < \dots$. However, we will offer more freedom in the choice of CDS's payment dates, allowing them to depend on the CDS to which they correspond.

Let us start considering the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for our market model, where $\mathbb{P}$ stands for the real-world probability. Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the underlying filtration for our model representing all the observable information and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the filtration representing the default-free market information. We will specify these filtrations later, but for now it is enough to establish that $\mathbb{F} \subset \mathbb{G}$ and $\mathbb{G}$ supports the traded assets of the model, which are supposed to be the following:

- A savings account (for instance, a bank account), whose value is given by the process $B = (B_t,\ t \geq 0)$ defined by
$$B_t := \exp\left(\int_0^t r_u du\right), \quad t \geq 0$$
where $r = (r_t,\ t \geq 0)$ represents the short-term interest rate process.
- K default-free zero-coupon bonds ($K \in \mathbb{N}$) with face value 1 and maturity dates $0 < T_1 < \cdots < T_K$ (with the convention $T_0 = 0$). There might be bonds with other maturities in the market, but only the prices of these ones are directly observable. However, later we will model the prices of bonds with other maturities using interpolation (as an estimation), so we can assume that $\mathbb{G}$ supports traded default-free zero-coupon bonds with any maturity date $\leq T_K$, because we can estimate their price from the observable bond prices. As usual, given $0 \leq t \leq s \leq T_K$, we denote by $P(t, s)$ the value at time t of a default-free zero-coupon bond with face value 1 and maturity date s.
- The firm's stock, whose value is given by the process $S = (S_t,\ t \geq 0)$.
- K CDS's with maturity dates $0 < T_1 < \cdots < T_K$ on defaultable bonds issued by the firm (these bonds are also traded assets, but they are not important for our purposes). More specifically, for each $k \in \{1, \dots, K\}$, there is a T_k-maturity CDS with spread κ_k (at time 0), payment dates $0 < s_1^{(k)} < \cdots < s_{d_k}^{(k)} = T_k$ and protection at default on a U_k-maturity defaultable bond (for some $U_k \geq T_k$) with some face value (which is not relevant) and fractional recovery of par value with recover rate $\delta \in [0, 1)$. Notice that the recovery rate is the same for all the CDS's and, moreover, we shall assume that all these CDS's and bonds have the same default time, that is, they are subject to the same default event.
- One CoCo (standard or write-down) that promises to pay coupons $c_1, \dots, c_n \geq 0$ at times $0 < t_1 < \cdots < t_n = T$ (with $T_{K-1} < T \leq T_K$) and the face value $L > 0$ at the maturity date T. If a pre-specified trigger event occurs before or at time T, then the CoCo converts into $R \geq 0$ shares

of the issuing firm's stock (standard CoCo) or into $R \geq 0$ units of currency (write-down CoCo) provided that the default event of the CDS's has not occured yet. If the firm defaults at time of conversion, then the CoCo evaporates.

As usual in a viable market model, we assume that there exists a risk-neutral probability measure $\mathbb{P}^*$ under which the price process of any tradeable asset which pays no coupons or dividends follows a $\mathbb{G}$-martingale, when discounted by the savings account process B. Now, we are going to specify $\mathbb{F}$ and $\mathbb{G}$ and give a model under $\mathbb{P}^*$ for the traded assets described above in a way that is consistent with the previous assumption.

First of all, it follows from the previous assumption that the value at time t of a traded default-free zero-coupon bond with face value 1 and maturity date s is given by

$$P(t,s) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^s r_u du} \,\middle|\, \mathcal{G}_t\right) \tag{5.1}$$

Indeed, such a bond does not pay coupons or dividends, so the discounted price process $\left(B_u^{-1}P(u,s),\ u \in [0,s]\right)$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ by hypothesis. Hence, combining this with the fact that $P(s,s) = 1$, we obtain

$$e^{-\int_0^t r_u du} P(t,s) = B_t^{-1} P(t,s) = \mathbb{E}_{\mathbb{P}^*}\left(B_s^{-1} P(s,s) \mid \mathcal{G}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^s r_u du} \,\middle|\, \mathcal{G}_t\right)$$

Arranging terms and using the $\mathbb{G}$-adaptability of r, we obtain (5.1). Remember that actually we only have market data for the prices of default-free zero-coupon bonds with maturity dates $T_1, \ldots, T_K$, so we shall assume that the prices of bonds with other maturities $< T_K$ can be computed by interpolation. More specifically, given $t \in [0, T_K)$ with $T_{m-1} \leq t < T_m$, we set

$$P(t,s) = \begin{cases} 1 & \text{if} \quad s = t, \\ \exp\left(-f_m(s-t)\right) & \text{if} \quad s \in (t, T_m], \\ P(t, T_{k-1}) \exp\left(-f_k(s - T_{k-1})\right) & \text{if} \quad s \in (T_{k-1}, T_k] \text{ for some } k \in \{m+1, \ldots, K\} \end{cases} \tag{5.2}$$

where

$$f_k = \begin{cases} -\dfrac{\log\left(P(t,T_m)\right)}{T_m - t} & \text{if} \quad k = m, \\ \\ -\dfrac{\log\left(P(t,T_k)\right) - \log\left(P(t,T_{k-1})\right)}{T_k - T_{k-1}} & \text{if} \quad k \in \{m+1, \ldots, K\} \end{cases} \tag{5.3}$$

Notice that the previous interpolation is based on the assumption that given $t \in [0, T_K)$ with $T_{m-1} \leq t < T_m$,

- The continuously compounded forward rate contracted at t for the interval $[t,s]$ is constant as a function of s over $(t, T_m]$.
- For any $k \in \{m+1, \ldots, K\}$, the continuously compounded forward rate contracted at t for the interval $[T_{k-1}, s]$ is constant as a function of s over $(T_{k-1}, T_k]$.

By doing the previous interpolation of bond prices, we do not need to specify the dynamics of the short-term interest rate r, but we will only assume that it is càdlàg and non-negative. As we shall see later, this interpolation usually simplify the computations. However, let us notice that in the following sections we will develop general pricing formulas without applying this interpolation (in order to keep a certain generality), and once these formulas are obtained we will perform this interpolation as a particular case.

Now, we set $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ as the filtration generated by the processes r and $W = (W_t,\ t \geq 0)$, that is, $\mathcal{F}_t = \sigma(r_u, W_u : 0 \leq u \leq t)$ for all $t \geq 0$, where W is a Brownian motion under $\mathbb{P}^*$ satisfying one of the following conditions:

- W and r are independent under $\mathbb{P}^*$.
- r is adapted to the filtration $\mathbb{F}^W$ generated by W (and then, $\mathbb{F} = \mathbb{F}^W$).

This filtration $\mathbb{F}$ represents the default-free market information, so in order to obtain the enlarged filtration $\mathbb{G}$ from $\mathbb{F}$, we need to add the information about the occurrence of the conversion time τ of the CoCo and the default time θ of the CDS's. Thus, we are going to model these random times.

Let N be a standard Poisson process under $\mathbb{P}^*$ independent of the filtration $\mathbb{F}$ (in particular, independent of r and W). Now, let us consider some Lebesgue integrable function $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$ and we shall assume that the conversion time τ is the first jump time of the time-changed Poisson process $(N_{\Lambda_t},\ t \geq 0)$, where $\Lambda_t := \int_0^t \lambda_u du$ for all $t \geq 0$. Hence, τ is independent of $\mathbb{F}$ under $\mathbb{P}^*$ (because so is N) and then, for every $t \geq 0$,

$$\mathbb{P}^*\{\tau \leq t \mid \mathcal{F}_t\} = \mathbb{P}^*\{\tau \leq t\} = 1 - \mathbb{P}^*\{\tau > t\} = 1 - \mathbb{P}^*\{N_{\Lambda_t} = 0\} = 1 - e^{-\Lambda_t} = 1 - e^{-\int_0^t \lambda_u du}$$

so λ represents the conversion intensity of the CoCo under $\mathbb{P}^*$ (actually, λ will be the $\mathbb{F}$-default intensity of τ under $\mathbb{P}^*$, once we define the CoCo as a defaultable claim).

On the other hand, we shall assume that the default time θ of the CDS's is given by

$$\theta = \mathbf{1}_{\{\xi=0\}}\tau + \mathbf{1}_{\{\xi=1\}}\nu$$

where ξ is a Bernoulli random variable independent of N and $\mathbb{F}$ under $\mathbb{P}^*$ with distribution $\mathbb{P}^*\{\xi = 0\} = \alpha$ and $\mathbb{P}^*\{\xi = 1\} = 1 - \alpha$ for some $\alpha \in [0, 1]$, and ν is the second jump time of the time-changed Poisson process $(N_{\Lambda_t^\beta},\ t \geq 0)$ for some $\beta \geq 0$, where

$$\lambda_t^\beta := \mathbf{1}_{\{\tau>t\}}\lambda_t + \mathbf{1}_{\{\tau\leq t\}}\beta\lambda_t, \quad t \geq 0$$
$$\Lambda_t^\beta := \int_0^t \lambda_u^\beta du, \quad t \geq 0$$

In other words, the firm will default at time of conversion of the CoCo (that is, $\theta = \tau$) with probability α, and if this does not occur, then the default event of the CDS's will come with intensity $\beta\lambda_t$ thenceforth. In particular, notice that $\tau \leq \theta$, which is consistent with the fact that the trigger event of a CoCo usually occurs before or at time of firm's default, as we explained in the previous section.

We assume that the parameters α and β are known (they can be specified arbitrarily or empirically) and λ is constant and equal to $l_k \geq 0$ on the interval $(T_{k-1}, T_k]$ for $k = 1, \ldots, K$. This latter assumption about λ is aimed to let us calibrate λ with the market spreads of the K traded CDS's, as we will discuss in detail in the last section of this chapter. Nevertheless, as it happens with the issue of bond prices interpolation, in the following sections we shall work with a general function λ and we will study the case where λ is piecewise constant as a particular case.

As we have argued before, $\mathbb{G}$ must contain the information about occurrence of τ and θ, so we set $\mathbb{G} := \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{F}^M$, where $\mathbb{H}^1 = \left(\mathcal{H}_t^1\right)_{t\geq 0}$, $\mathbb{H}^2 = \left(\mathcal{H}_t^2\right)_{t\geq 0}$ and $\mathbb{F}^M = \left(\mathcal{F}_t^M\right)_{t\geq 0}$ are the filtrations generated by the processes $H^1 = \left(H_t^1 := \mathbf{1}_{\{\tau\leq t\}},\ t\geq 0\right)$, $H^2 = \left(H_t^2 := \mathbf{1}_{\{\theta\leq t\}},\ t\geq 0\right)$ and $M = \left(M_t := \zeta H_t^1,\ t\geq 0\right)$ respectively, with $\zeta := (1+\gamma)\xi - 1$ for some constant $\gamma > -1$. The choice of γ and the inclusion of $\mathbb{F}^M$ within $\mathbb{G}$ have to do with the way in which we will model the firm's stock price, which is the only thing left to deal with to finish the description of our model.

We shall assume that the firm's stock price process S satisfies the following SDE for $t \leq \tau\wedge T$ (that is, in $\{t\leq \tau\wedge T\}$):

$$dS_t = S_{t-}\left((r_t - q - [\gamma - \alpha(1+\gamma)]\,\lambda_t)dt + \sigma dW_t + \zeta dH_t^1\right)$$

with $S_0 > 0$, where the constants $q, \sigma \in \mathbb{R}$ represent the dividend rate and the volatility of the stock respectively. Thus, S follows a geometric Brownian motion under $\mathbb{P}^*$ with a jump at conversion. Notice that if the firm defaults at time of conversion, then S jumps to $S_\tau = 0$, whilst if the firm survives the conversion, then S jumps to $S_\tau = (1+\gamma)S_{\tau-}$. We assume that the parameters q, σ and γ are known (they can be specified arbitrarily or empirically).

Now, notice that the previous SDE can be written as

$$dS_t = S_{t-}\left((r_t - q - \lambda_t\,\mathbb{E}_{\mathbb{P}^*}(\zeta))dt + \sigma dW_t + dM_t\right) \tag{5.4}$$

so S is adapted to $\mathbb{G}$ until time $\tau\wedge T$ as desired, because r, W and M are $\mathbb{G}$-adapted processes by construction. Then, one can use the Itô formula to prove that the process $\widetilde{S} = \left(\widetilde{S}_t,\ t\geq 0\right)$ defined by $\widetilde{S}_t := \exp\left(-\int_0^t r_u du + qt\right) S_t$ for all $t\geq 0$ satisfies the following SDE for $t\leq\tau\wedge T$:

$$d\widetilde{S}_t = \widetilde{S}_{t-}\left(\sigma dW_t + dM_t - \lambda_t\,\mathbb{E}_{\mathbb{P}^*}(\zeta)\,dt\right) \tag{5.5}$$

At the end of this section we will prove that $\widetilde{S}$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ until time $\tau\wedge T$, that is, the process $\left(\widetilde{S}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$. In the particular case where $q=0$ (the stock pays no dividends), this means that the price process of the firm's stock is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ until time $\tau\wedge T$, when discounted by B. Thus, this property is consistent with the initial assumption that we put to the model. Let us remark that we only need the dynamics of S until time $\tau\wedge T$ in order to price the CoCo of our model, so after this time S can be specified arbitrarily in a way that S is $\mathbb{G}$-adapted and $\widetilde{S}$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$.

Once we have already specified the basic elements and assumptions of the model, we can provide formal definitions of *standard CoCo* and *write-down CoCo* as defaultable claims, within the context of our model:

Definition 5.2.1 (Standard CoCo). *Given $T > 0$, a T-maturity standard CoCo with face value*

$L > 0$, coupon payments $c_1, \dots, c_n \geq 0$ at times $0 < t_1 < \cdots < t_n = T$ and recovery of $R \geq 0$ shares of the firm's stock is the defaultable claim $(L, A, 0, RS, \tau)$, where τ is called the conversion time of the CoCo and

$$A_t = \sum_{i=1}^{n} c_i \cdot \mathbf{1}_{[t_i, T]}(t), \quad t \in [0, T]$$

Definition 5.2.2 (Write-down CoCo). *Given $T > 0$, a T-maturity write-down CoCo with face value $L > 0$, coupon payments $c_1, \dots, c_n \geq 0$ at times $0 < t_1 < \cdots < t_n = T$ and recovery of $R \geq 0$ units of currency is the defaultable claim $(L, A, 0, \mathbf{1}_{\{\xi=1\}}R, \tau)$, where τ is called the conversion time of the CoCo and*

$$A_t = \sum_{i=1}^{n} c_i \cdot \mathbf{1}_{[t_i, T]}(t), \quad t \in [0, T]$$

In the previous definitions, S is the firm's stock price process, which evolves as (5.4) until time $\tau \wedge T$, and ξ is the Bernoulli random variable defined before.

Finally, we are going to prove that $\widetilde{S}$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ until time $\tau \wedge T$, but let us first present a technical Lemma that will be also useful for computations in the next section:

Lemma 5.2.3. *Let X be a $\mathbb{P}^*$-integrable random variable. Then, for any $t \geq 0$,*

$$\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}X \mid \mathcal{G}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}X \mid \mathcal{F}_t \vee \mathcal{H}_t^1\right)$$

Proof. The idea of the proof consists of noticing that for $t < \tau$ (that is, in the set $\{t < \tau\}$), the information of $\mathcal{H}_t^2$ and $\mathcal{F}_t^M$ is contained in $\mathcal{H}_t^1$, because the random variables H_u^2 and M_u are equal to H_u^1 for $0 \leq u \leq t$ (this is straightforward to check, bearing in mind that $\theta \geq \tau$ by construction). Hence, in $\{t < \tau\}$, the σ-fields $\mathcal{H}_t^2$ and $\mathcal{F}_t^M$ do not add extra information to the one contained in $\mathcal{F}_t \vee \mathcal{H}_t^1$. Therefore,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}X \mid \mathcal{G}_t\right) &= \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(X \mid \mathcal{G}_t\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(X \mid \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \mathcal{H}_t^2 \vee \mathcal{F}_t^M\right) = \\ &= \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(X \mid \mathcal{F}_t \vee \mathcal{H}_t^1\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}X \mid \mathcal{F}_t \vee \mathcal{H}_t^1\right)\end{aligned}$$

where the first and last equalities follow from the $\mathbb{P}^*$-integrability of X and from the fact that $\mathbf{1}_{\{\tau>t\}}$ is $\mathcal{H}_t^1$-measurable, so this ends the proof. □

Proposition 5.2.4. *$\widetilde{S}$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ until time $\tau \wedge T$, that is, the process $\left(\widetilde{S}_{t\wedge\tau\wedge T},\ t \geq 0\right)$ is $\mathbb{G}$-martingale under $\mathbb{P}^*$.*

Proof. We know that $\widetilde{S}$ satisfies the SDE (5.5) until time $\tau \wedge T$. If we consider the process $\widetilde{M} = \left(\widetilde{M}_t,\ t \geq 0\right)$ defined by $\widetilde{M}_t := M_t - \mathbb{E}_{\mathbb{P}^*}(\zeta)\Lambda_t = \zeta H_t^1 - \mathbb{E}_{\mathbb{P}^*}(\zeta)\Lambda_t$ for all $t \geq 0$, then the SDE (5.5) can be written as

$$d\widetilde{S}_t = \widetilde{S}_{t-}\left(\sigma dW_t + d\widetilde{M}_t\right)$$

where W and $\widetilde{M}$ are independent processes under $\mathbb{P}^*$ ($\widetilde{M}$ depends on the random objects N and ξ, which are assumed to be independent of W under $\mathbb{P}^*$). Thus, if we prove that W and $\widetilde{M}$ are $\mathbb{G}$-martingales under $\mathbb{P}^*$ until time $\tau \wedge T$, then this will prove the statement of this proposition.

On the one hand, since W is a Brownian motion under $\mathbb{P}^*$, we know that W is an $\mathbb{F}^W$-martingale under $\mathbb{P}^*$. Actually, W is also an $\mathbb{F}\vee\mathbb{H}^1$-martingale under $\mathbb{P}^*$, because the filtration $\mathbb{F}\vee\mathbb{H}^1$ is obtained by enlarging $\mathbb{F}^W$ with extra information which is independent of W under $\mathbb{P}^*$. Then, since $\tau\wedge T$ is a stopping time with respect to $\mathbb{F}\vee\mathbb{H}^1$ (because $\{\tau\wedge T\leq t\}\in\mathcal{H}_t^1\subset\mathcal{F}_t\vee\mathcal{H}_t^1$ for all $t\geq 0$), it follows that the stopped process $(W_{t\wedge\tau\wedge T},\ t\geq 0)$ is an $\mathbb{F}\vee\mathbb{H}^1$-martingale under $\mathbb{P}^*$. Hence, the process $(W_{t\wedge\tau\wedge T},\ t\geq 0)$ is also a $\mathbb{G}$-martingale under $\mathbb{P}^*$, because it is $\mathbb{G}$-adapted and for any $0\leq t\leq s$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}\left(W_{s\wedge\tau\wedge T}-W_{t\wedge\tau\wedge T}\mid\mathcal{G}_t\right)&=\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\left(W_{s\wedge\tau\wedge T}-W_{t\wedge\tau\wedge T}\right)\mid\mathcal{G}_t\right)=\\&=\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\left(W_{s\wedge\tau\wedge T}-W_{t\wedge\tau\wedge T}\right)\mid\mathcal{F}_t\vee\mathcal{H}_t^1\right)=\\&=\mathbb{E}_{\mathbb{P}^*}\left(W_{s\wedge\tau\wedge T}-W_{t\wedge\tau\wedge T}\mid\mathcal{F}_t\vee\mathcal{H}_t^1\right)=0\end{aligned}$$

where the second equality follows from Lemma 5.2.3 and the last equality follows from the fact that $(W_{t\wedge\tau\wedge T},\ t\geq 0)$ is an $\mathbb{F}\vee\mathbb{H}^1$-martingale under $\mathbb{P}^*$. Therefore, this proves that W is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ until time $\tau\wedge T$.

On the other hand, since $(N_{\Lambda_t},\ t\geq 0)$ is a Poisson process with intensity function Λ with respect to $\mathbb{F}^N$ under $\mathbb{P}^*$, we know that the process $\widetilde{N}=\left(\widetilde{N}_t:=N_{\Lambda_t}-\Lambda_t,\ t\geq 0\right)$ is an $\mathbb{F}^{\widetilde{N}}$-martingale under $\mathbb{P}^*$, where $\mathbb{F}^N=\left(\mathcal{F}_t^N\right)_{t\geq 0}$ and $\mathbb{F}^{\widetilde{N}}=\left(\mathcal{F}_t^{\widetilde{N}}\right)_{t\geq 0}$ are the filtrations generated by N and $\widetilde{N}$ respectively (it is clear that $\mathbb{F}^N=\mathbb{F}^{\widetilde{N}}$). We can see that $\tau\wedge T$ is a stopping time with respect to $\mathbb{F}^{\widetilde{N}}$, because for every $t\geq 0$,

$$\{\tau\wedge T\leq t\}=\{\tau\leq t\}\cup\{T\leq t\}=\{N_{\Lambda_t}\geq 1\}\cup\{T\leq t\}=\left\{\widetilde{N}_t\geq 1-\Lambda_t\right\}\cup\{T\leq t\}\in\mathcal{F}_t^{\widetilde{N}}$$

so it follows that the stopped process $\left(\widetilde{N}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is an $\mathbb{F}^{\widetilde{N}}$-martingale under $\mathbb{P}^*$. But now, notice that for all $t\geq 0$,

$$\begin{aligned}\widetilde{N}_{t\wedge\tau\wedge T}&=N_{\Lambda_{t\wedge\tau\wedge T}}-\Lambda_{t\wedge\tau\wedge T}=\mathbf{1}_{\{\tau\leq t\wedge T\}}-\int_0^{t\wedge\tau\wedge T}\lambda_u du=\mathbf{1}_{\{\tau\leq t\wedge T\}}-\int_0^{t\wedge(\tau-)\wedge T}\lambda_u du=\\&=\mathbf{1}_{\{\tau\leq t\wedge T\}}-\int_0^{t\wedge T}\mathbf{1}_{\{\tau>u\}}\lambda_u du=H_{t\wedge T}^1-\int_0^{t\wedge T}(1-H_u^1)\lambda_u du\end{aligned}\tag{5.6}$$

where we have used the continuity of the classical Lebesgue integral in the third equality, so the $\mathbb{F}^{\widetilde{N}}$-martingale $\left(\widetilde{N}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is $\mathbb{H}^1$-adapted. Then, since $\mathbb{H}^1\subset\mathbb{F}^{\widetilde{N}}$ (because τ is a stopping time with respect to $\mathbb{F}^{\widetilde{N}}$) and the $\mathbb{F}^{\widetilde{N}}$-martingale $\left(\widetilde{N}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is $\mathbb{H}^1$-adapted, it follows that $\left(\widetilde{N}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is an $\mathbb{H}^1$-martingale under $\mathbb{P}^*$. Actually, it is also an $\mathbb{F}\vee\mathbb{H}^1$-martingale under $\mathbb{P}^*$, because the filtration $\mathbb{F}\vee\mathbb{H}^1$ is obtained by enlarging $\mathbb{H}^1$ with extra information which is independent of N under $\mathbb{P}^*$ (and then, also independent of $\mathbb{H}^1$). Hence, the process $\left(\widetilde{N}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is also a $\mathbb{G}$-martingale under $\mathbb{P}^*$, because it is $\mathbb{G}$-adapted and for any $0\leq t\leq s$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^*}\left(\widetilde{N}_{s\wedge\tau\wedge T}-\widetilde{N}_{t\wedge\tau\wedge T}\,\middle|\,\mathcal{G}_t\right)&=\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\left(\widetilde{N}_{s\wedge\tau\wedge T}-\widetilde{N}_{t\wedge\tau\wedge T}\right)\,\middle|\,\mathcal{G}_t\right)=\\&=\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\left(\widetilde{N}_{s\wedge\tau\wedge T}-\widetilde{N}_{t\wedge\tau\wedge T}\right)\,\middle|\,\mathcal{F}_t\vee\mathcal{H}_t^1\right)=\\&=\mathbb{E}_{\mathbb{P}^*}\left(\widetilde{N}_{s\wedge\tau\wedge T}-\widetilde{N}_{t\wedge\tau\wedge T}\,\middle|\,\mathcal{F}_t\vee\mathcal{H}_t^1\right)=0\end{aligned}\tag{5.7}$$

where the second equality follows from Lemma 5.2.3 and the last equality follows from the fact that

$\left(\widetilde{N}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is an $\mathbb{F}\vee\mathbb{H}^1$-martingale under $\mathbb{P}^*$.

Finally, since $\left(\widetilde{N}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$, it follows that so is $\left(\widetilde{M}_{t\wedge\tau\wedge T},\ t\geq 0\right)$. Indeed, $\left(\widetilde{M}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is $\mathbb{G}$-adapted (because $\widetilde{M}$ is $\mathbb{G}$-adapted and $\tau\wedge T$ is a stopping time with respect to $\mathbb{G}$) and for any $0\leq t\leq s$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}\left(\widetilde{M}_{s\wedge\tau\wedge T}-\widetilde{M}_{t\wedge\tau\wedge T}\,\middle|\,\mathcal{G}_t\right) &= \mathbb{E}_{\mathbb{P}^*}\left(\zeta\left(H^1_{s\wedge T}-H^1_{t\wedge T}\right)\mid\mathcal{G}_t\right)-\mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\Lambda_{s\wedge\tau\wedge T}-\Lambda_{t\wedge\tau\wedge T}\mid\mathcal{G}_t\right)=\\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\zeta\left(H^1_{s\wedge T}-H^1_{t\wedge T}\right)\mid\mathcal{G}_t\right)-\mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\Lambda_{s\wedge\tau\wedge T}-\Lambda_{t\wedge\tau\wedge T}\mid\mathcal{G}_t\right)=\\
&= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\zeta\left(H^1_{s\wedge T}-H^1_{t\wedge T}\right)\mid\mathcal{F}_t\vee\mathcal{H}^1_t\right)-\mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\Lambda_{s\wedge\tau\wedge T}-\Lambda_{t\wedge\tau\wedge T}\mid\mathcal{G}_t\right)=\\
&= \mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\left(H^1_{s\wedge T}-H^1_{t\wedge T}\right)\mid\mathcal{F}_t\vee\mathcal{H}^1_t\right)-\mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\Lambda_{s\wedge\tau\wedge T}-\Lambda_{t\wedge\tau\wedge T}\mid\mathcal{G}_t\right)=\\
&= \mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}\left(H^1_{s\wedge T}-H^1_{t\wedge T}\right)\mid\mathcal{G}_t\right)-\mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\Lambda_{s\wedge\tau\wedge T}-\Lambda_{t\wedge\tau\wedge T}\mid\mathcal{G}_t\right)=\\
&= \mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(H^1_{s\wedge T}-H^1_{t\wedge T}\mid\mathcal{G}_t\right)-\mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\Lambda_{s\wedge\tau\wedge T}-\Lambda_{t\wedge\tau\wedge T}\mid\mathcal{G}_t\right)=\\
&= \mathbb{E}_{\mathbb{P}^*}\left(\zeta\right)\mathbb{E}_{\mathbb{P}^*}\left(\widetilde{N}_{s\wedge\tau\wedge T}-\widetilde{N}_{t\wedge\tau\wedge T}\,\middle|\,\mathcal{G}_t\right)=0
\end{aligned}$$

where we have applied (5.6), (5.7) and Lemma 5.2.3, and we have used the fact that ζ is independent of N and $\mathbb{F}$ under $\mathbb{P}^*$ (because ζ is a deterministic function of ξ).

Therefore, we have shown that W and $\widetilde{M}$ are $\mathbb{G}$-martingales under $\mathbb{P}^*$ until time $\tau\wedge T$, so this concludes the proof. □

5.3 CoCo pricing

Once we have already described the model and provided formal definitions of CoCos as defaultable claims, we want to develop pricing formulas to compute the pre-default value of the CoCo introduced in the previous section in terms of the parameters of the model and the observable information of the market. Let us start deriving a general pricing formula for the case of a general deterministic conversion intensity λ and without interpolating bond prices:

Theorem 5.3.1. *The pre-default value of the CoCo is given by the process $(C_t,\ t\in[0,T])$ defined by*

$$C_t = C^1_t + C^2_t + C^3_t,\quad t\in[0,T]$$

where

$$C^1_t = \mathbf{1}_{\{\tau>t\}}\sum_{i\,:\,t_i>t} c_i P(t,t_i)e^{-(\Lambda_{t_i}-\Lambda_t)} \tag{5.8}$$

$$C^2_t = \mathbf{1}_{\{\tau>t\}} L P(t,T)e^{-(\Lambda_T-\Lambda_t)} \tag{5.9}$$

$$C^3_t = \mathbf{1}_{\{\tau>t\}} R S_t(1-\alpha)(1+\gamma)\int_t^T e^{-q(u-t)}\lambda_u e^{-(1-\alpha)(1+\gamma)(\Lambda_u-\Lambda_t)}du \tag{5.10}$$

or

$$C^3_t = \mathbf{1}_{\{\tau>t\}} R(1-\alpha)\int_t^T P(t,u)\lambda_u e^{-(\Lambda_u-\Lambda_t)}du \tag{5.11}$$

if the CoCo is write-down.

Proof. As usual, we consider that the pre-default value of the CoCo is given by the ex-dividend price process $(C_t,\ t \in [0,T])$ of the CoCo. Applying the risk-neutral valuation formula (2.9) to the defaultable claims considered in Definition 5.2.1 and Definition 5.2.2, it is not difficult to check that $C_t = C_t^1 + C_t^2 + C_t^3$ for every $t \in [0,T]$, where

$$\begin{aligned} C_t^1 &:= \sum_{i\,:\,t_i>t} c_i\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^{t_i} r_u du}\mathbf{1}_{\{\tau>t_i\}}\,\middle|\,\mathcal{G}_t\right) \\ C_t^2 &:= L\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^{T} r_u du}\mathbf{1}_{\{\tau>T\}}\,\middle|\,\mathcal{G}_t\right) \end{aligned}$$

and

$$C_t^3 := \begin{cases} R\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^{\tau} r_u du}S_\tau\mathbf{1}_{\{t<\tau\le T\}}\,\middle|\,\mathcal{G}_t\right) & \text{if the CoCo is standard,} \\ R\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^{\tau} r_u du}\mathbf{1}_{\{\xi=1\}}\mathbf{1}_{\{t<\tau\le T\}}\,\middle|\,\mathcal{G}_t\right) & \text{if the CoCo is write-down.} \end{cases}$$

Thus, we need to prove that C_t^1, C_t^2 and C_t^3 fulfill formulas (5.8), (5.9), (5.10) and (5.11). First of all, notice that for any $0 \le t \le s \le T_K$,

$$\begin{aligned} \mathbf{1}_{\{\tau>t\}}P(t,s) &= \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^s r_u du}\,\middle|\,\mathcal{G}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}e^{-\int_t^s r_u du}\,\middle|\,\mathcal{G}_t\right) = \\ &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}e^{-\int_t^s r_u du}\,\middle|\,\mathcal{F}_t \vee \mathcal{H}_t^1\right) = \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^s r_u du}\,\middle|\,\mathcal{F}_t \vee \mathcal{H}_t^1\right) = \\ &= \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^s r_u du}\,\middle|\,\mathcal{F}_t\right) \end{aligned} \tag{5.12}$$

where we have applied (5.1) in the first equality, we have applied Lemma 5.2.3 in the third equality and we have used the independence between $\mathbb{F}$ and $\mathbb{H}^1$ under $\mathbb{P}^*$ and the $\mathbb{F}$-adaptability of r in the last equality.

Now, on the one hand, we can see that for any $0 \le t \le s \le T_K$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^s r_u du}\mathbf{1}_{\{\tau>s\}}\,\middle|\,\mathcal{G}_t\right) &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}e^{-\int_t^s r_u du}\mathbf{1}_{\{\tau>s\}}\,\middle|\,\mathcal{G}_t\right) = \\ &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau>t\}}e^{-\int_t^s r_u du}\mathbf{1}_{\{\tau>s\}}\,\middle|\,\mathcal{F}_t \vee \mathcal{H}_t^1\right) = \\ &= \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^s r_u du}\mathbf{1}_{\{\tau>s\}}\,\middle|\,\mathcal{F}_t \vee \mathcal{H}_t^1\right) = \\ &= \mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(e^{-(\Lambda_s-\Lambda_t)}e^{-\int_t^s r_u du}\,\middle|\,\mathcal{F}_t\right) = \\ &= \mathbf{1}_{\{\tau>t\}}e^{-(\Lambda_s-\Lambda_t)}\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^s r_u du}\,\middle|\,\mathcal{F}_t\right) = \\ &= \mathbf{1}_{\{\tau>t\}}P(t,s)e^{-(\Lambda_s-\Lambda_t)} \end{aligned}$$

where we have applied Lemma 5.2.3 in the second equality, we have applied Proposition 4.1.4 to the $\mathcal{F}_s$-measurable random variable $e^{-\int_t^s r_u du}$ in the fourth equality, we have used the fact that Λ is a deterministic function in the fifth equality and we have applied (5.12) in the last equality. Hence, if we apply the result of the previous equation taking $s = T$ and also taking $s = t_i$ for each $i \in \{1,\dots,n\}$ such that $t_i > t$, then we easily deduce (5.8) and (5.9).

On the other hand, if the CoCo is write-down, then for every $t \in [0,T]$,

$$C_t^3 = R\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^{\tau} r_u du}\mathbf{1}_{\{\xi=1\}}\mathbf{1}_{\{t<\tau\le T\}}\,\middle|\,\mathcal{G}_t\right) = R\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^{\tau} r_u du}\mathbf{1}_{\{\xi=1\}}\mathbf{1}_{\{t<\tau\le T\}}\,\middle|\,\mathcal{F}_t \vee \mathcal{H}_t^1\right) =$$

$$= R\,\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\xi=1\}}\right)\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^\tau r_u du}\mathbf{1}_{\{t<\tau\leq T\}}\;\middle|\;\mathcal{F}_t\vee\mathcal{H}_t^1\right) =$$
$$= R(1-\alpha)B_t\,\mathbb{E}_{\mathbb{P}^*}\left(B_\tau^{-1}\mathbf{1}_{\{t<\tau\leq T\}}\;\middle|\;\mathcal{F}_t\vee\mathcal{H}_t^1\right) = R(1-\alpha)B_t\mathbf{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T B_u^{-1}e^{-(\Lambda_u-\Lambda_t)}d\Lambda_u\;\middle|\;\mathcal{F}_t\right) =$$
$$= \mathbf{1}_{\{\tau>t\}}R(1-\alpha)\,\mathbb{E}_{\mathbb{P}^*}\left(\int_t^T e^{-\int_t^u r_v dv}\lambda_u e^{-(\Lambda_u-\Lambda_t)}du\;\middle|\;\mathcal{F}_t\right) =$$
$$= \mathbf{1}_{\{\tau>t\}}R(1-\alpha)\int_t^T \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^u r_v dv}\;\middle|\;\mathcal{F}_t\right)\lambda_u e^{-(\Lambda_u-\Lambda_t)}du =$$
$$= \mathbf{1}_{\{\tau>t\}}R(1-\alpha)\int_t^T P(t,u)\lambda_u e^{-(\Lambda_u-\Lambda_t)}du$$

where we have applied Lemma 5.2.3 in the second equality, we have used the fact that ξ is independent of N and $\mathbb{F}$ under $\mathbb{P}^*$ in the third equality, we have applied Proposition 4.1.5 to the inverse savings account process $\left(B_u^{-1},\ u\geq 0\right)$ in the fifth equality (because Λ is a continuous increasing function and $\left(B_u^{-1},\ u\geq 0\right)$ is a continuous bounded $\mathbb{F}$-adapted process, so in particular it is $\mathbb{F}$-predictable) and we have used the associativity of the Lebesgue-Stieltjes integral in the sixth equality. In the last two equalities, we have applied Fubini-Tonelli theorem (because the integrand is jointly measurable and non-negative), we have used the fact that λ and Λ are deterministic functions and we have applied (5.12). Thus, this proves (5.11) in the case where the CoCo is write-down.

Finally, it only remains to prove (5.10) when the CoCo is standard, so let us now assume that the CoCo is standard. Since the process $\left(\widetilde{S}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ is $\mathbb{G}$-martingale under $\mathbb{P}^*$ (see Proposition 5.2.4), we know that the non-negative random variable $\widetilde{S}_{\tau\wedge T}/\widetilde{S}_0$ satisfies the condition $\mathbb{E}_{\mathbb{P}^*}(\widetilde{S}_{\tau\wedge T}/\widetilde{S}_0) = 1$. Hence, we can define a new probability $\overline{\mathbb{P}}$ in the filtered space $(\Omega,\mathcal{F},\mathbb{G})$ by setting $d\overline{\mathbb{P}}/d\mathbb{P}^* = \widetilde{S}_{\tau\wedge T}/\widetilde{S}_0$. In particular, we know that $\overline{\mathbb{P}}\ll\mathbb{P}^*$, and if we consider the density process $Z=(Z_t,\ t\geq 0)$ defined by

$$Z_t := \mathbb{E}_{\mathbb{P}^*}\left(\frac{d\overline{\mathbb{P}}}{d\mathbb{P}^*}\;\middle|\;\mathcal{G}_t\right)\qquad t\geq 0$$

then it follows from the martingale condition of the process $\left(\widetilde{S}_{t\wedge\tau\wedge T},\ t\geq 0\right)$ that for every $t\geq 0$,

$$Z_t = \mathbb{E}_{\mathbb{P}^*}\left(\frac{\widetilde{S}_{\tau\wedge T}}{\widetilde{S}_0}\;\middle|\;\mathcal{G}_t\right) = \frac{\mathbb{E}_{\mathbb{P}^*}\left(\widetilde{S}_{T\wedge\tau\wedge T}\;\middle|\;\mathcal{G}_t\right)}{\widetilde{S}_0} = \frac{\widetilde{S}_{t\wedge\tau\wedge T}}{\widetilde{S}_0} \tag{5.13}$$

Now, let us fix $t\in[0,T]$ and consider the random variable $Y := e^{-q(\tau-t)}\mathbf{1}_{\{t<\tau\leq T\}}$. It is easy to check that Y is bounded and $\mathcal{G}_T$-measurable (it is $\mathcal{H}_T^1$-measurable and we know that $\mathcal{H}_T^1\subset\mathcal{G}_T$). Then, using the definition of the process $\widetilde{S}$, applying the abstract Bayes' rule (Proposition 2.3.11) to the random variable Y and using (5.13), we can see that

$$C_t^3 = R\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^\tau r_u du}S_\tau\mathbf{1}_{\{t<\tau\leq T\}}\;\middle|\;\mathcal{G}_t\right) = RS_t\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^\tau r_u du}\frac{S_\tau}{S_t}\mathbf{1}_{\{t<\tau\leq T\}}\;\middle|\;\mathcal{G}_t\right) =$$
$$= RS_t\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-q(\tau-t)}\frac{\widetilde{S}_\tau}{\widetilde{S}_t}\mathbf{1}_{\{t<\tau\leq T\}}\;\middle|\;\mathcal{G}_t\right) = RS_t\,\mathbb{E}_{\mathbb{P}^*}\left(e^{-q(\tau-t)}\frac{\widetilde{S}_{\tau\wedge T}}{\widetilde{S}_{t\wedge\tau\wedge T}}\mathbf{1}_{\{t<\tau\leq T\}}\;\middle|\;\mathcal{G}_t\right) =$$
$$= RS_t\frac{1}{Z_t}\,\mathbb{E}_{\mathbb{P}^*}\left(YZ_T\mid\mathcal{G}_t\right) = RS_t\,\mathbb{E}_{\overline{\mathbb{P}}}\left(Y\mid\mathcal{G}_t\right) = RS_t\,\mathbb{E}_{\overline{\mathbb{P}}}\left(e^{-q(\tau-t)}\mathbf{1}_{\{t<\tau\leq T\}}\;\middle|\;\mathcal{G}_t\right) \tag{5.14}$$

Thus, in order to compute this latter expectation, we need to determine first the conditional distribution of τ with respect to $\mathcal{G}_t$ under $\overline{\mathbb{P}}$. Let us consider the processes $X^1 = \left(X^1_s,\ s \geq 0\right)$ and $X^2 = \left(X^2_s,\ s \geq 0\right)$ defined by

$$X^1_s := \widetilde{N}_{s\wedge\tau\wedge T} = N_{\Lambda_{s\wedge\tau\wedge T}} - \Lambda_{s\wedge\tau\wedge T} = H^1_{s\wedge T} - \Lambda_{s\wedge\tau\wedge T}, \quad s \geq 0$$
$$X^2_s := \widetilde{M}_{s\wedge\tau\wedge T} = M_{s\wedge\tau\wedge T} - \mathbb{E}_{\mathbb{P}^*}(\zeta)\,\Lambda_{s\wedge\tau\wedge T} = \zeta H^1_{s\wedge T} - \mathbb{E}_{\mathbb{P}^*}(\zeta)\,\Lambda_{s\wedge\tau\wedge T}, \quad s \geq 0$$

On the one hand, in the proof of Proposition 5.2.4 we saw that X^1 and X^2 are (càdlàg) $\mathbb{G}$-martingales under $\mathbb{P}^*$. Actually, notice also that X^1 and X^2 are bounded processes, which follows from the fact that H^1 and ζ are bounded and $\Lambda_{s\wedge\tau\wedge T}$ is bounded for all $s \geq 0$ by $\int_0^T |\lambda_u| du < \infty$. Hence, since X^1 and X^2 are bounded $\mathbb{G}$-martingales under $\mathbb{P}^*$, it follows that $X^1, X^2 \in \mathcal{H}^2$.

On the other hand, we can also show that $Z \in \mathcal{H}^2$. Indeed, it follows from (5.13) that Z is a $\mathbb{G}$-martingale under $\mathbb{P}^*$ (because so is the process $(\widetilde{S}_{s\wedge\tau\wedge T},\ s \geq 0)$), so it only remains to prove that Z is square-integrable. Remember that $\widetilde{S}$ satisfies the SDE (5.5) until time $\tau \wedge T$ with initial condition $S_0 > 0$, so at time of conversion $\widetilde{S}$ jumps to $\widetilde{S}_\tau = \xi(1+\gamma)\widetilde{S}_{\tau-}$, where γ is a known constant and ξ takes values in $\{0, 1\}$. Then, it is not difficult to check that, until time $\tau \wedge T$, $\widetilde{S}$ is non-negative and bounded (from above) by the process $\widehat{S}$ satisfying the SDE (5.5) without the jump term (dM) and with initial condition $\max(1, 1+\gamma)\cdot S_0$. This process $\widehat{S}$ is thus a geometric Brownian motion with constant volatility and with a deterministic drift coefficient given by a Lebesgue integrable function, so it can be proved that $\sup_{s\in[0,T]} \mathbb{E}_{\mathbb{P}^*}\left(\widehat{S}^2_s\right) < \infty$ and then $\left(\widetilde{S}_{s\wedge\tau\wedge T},\ s \geq 0\right)$ belongs to $\mathcal{H}^2$. Therefore, we conclude that $Z \in \mathcal{H}^2$.

Now, we know that Z is a semimartingale (because $Z \in \mathcal{H}^2$) and X^1 is a finite variation process (because it is by definition the difference between two increasing processes), so it follows from Lemma 2.3.4 that

$$[X^1, Z]_s = \sum_{0<u\leq s} \Delta X^1_u \Delta Z_u, \quad s \geq 0$$

Notice that the process X^1 has only one jump at time τ (an increasing jump of length 1). Hence, applying (5.13) and using the fact that $\widetilde{S}_\tau = \xi(1+\gamma)\widetilde{S}_{\tau-}$, we can see that for every $s \geq 0$,

$$\begin{aligned}
[X^1, Z]_s &= \sum_{0<u\leq s} \Delta X^1_u \Delta Z_u = \mathbf{1}_{\{\tau\leq s\}} \Delta X^1_\tau \Delta Z_\tau = \mathbf{1}_{\{\tau\leq s\}} \left(Z_\tau - Z_{\tau-}\right) = \mathbf{1}_{\{\tau\leq s\}} \frac{\widetilde{S}_{\tau\wedge T} - \widetilde{S}_{(\tau-)\wedge T}}{\widetilde{S}_0} = \\
&= \mathbf{1}_{\{\tau\leq s\wedge T\}} \frac{\widetilde{S}_\tau - \widetilde{S}_{\tau-}}{\widetilde{S}_0} = H^1_{s\wedge T} \frac{\xi(1+\gamma)\widetilde{S}_{\tau-} - \widetilde{S}_{\tau-}}{\widetilde{S}_0} = \frac{\zeta H^1_{s\wedge T}\widetilde{S}_{\tau-}}{\widetilde{S}_0} = \frac{M_{s\wedge T}\widetilde{S}_{\tau-}}{\widetilde{S}_0} = \\
&= \frac{1}{\widetilde{S}_0}\int_0^{s\wedge T} \widetilde{S}_{u-} dM_u = \frac{1}{\widetilde{S}_0}\int_0^{s\wedge\tau\wedge T} \widetilde{S}_{u-} dM_u = \frac{1}{\widetilde{S}_0}\int_0^{s} \widetilde{S}_{u-} dM_{u\wedge\tau\wedge T}
\end{aligned}$$

where the penultimate equality follows from the fact that, after time τ, the process M is constant.

Since $X^1, Z \in \mathcal{H}^2$, we know that predictable quadratic covariation of X^1 and Z under $\mathbb{P}^*$ is well defined. Now, we are going to show that $\left\langle X^1, Z\right\rangle = I$, where $I = (I_s,\ s \geq 0)$ is the process defined by

$$I_s := \frac{1}{\widetilde{S}_0}\int_0^{s\wedge\tau\wedge T} \widetilde{S}_{u-} d\left(\mathbb{E}_{\mathbb{P}^*}(\zeta)\,\Lambda_u\right) = \frac{1}{\widetilde{S}_0}\int_0^{s} \widetilde{S}_{u-} d\left(\mathbb{E}_{\mathbb{P}^*}(\zeta)\,\Lambda_{u\wedge\tau\wedge T}\right), \quad s \geq 0$$

First of all, since $\widetilde{S}_-$ is a $\mathbb{G}$-adapted càglàd process (at least until time $\tau \wedge T$), Λ is an increasing function and $\tau \wedge T$ is a stopping time with respect to $\mathbb{G}$, it follows that I is well defined path by path as a Lebesgue-Stieltjes integral and it is $\mathbb{G}$-adapted. Then, I is a $\mathbb{G}$-adapted finite variation process. Moreover, applying the associativity of the Lebesgue-Stieltjes integral, we can see that

$$I_s = \frac{\mathbb{E}_{\mathbb{P}^*}(\zeta)}{\widetilde{S}_0} \int_0^{s\wedge\tau\wedge T} \widetilde{S}_{u-}\lambda_u du = \frac{\mathbb{E}_{\mathbb{P}^*}(\zeta)}{\widetilde{S}_0} \int_0^{s} \widetilde{S}_{u-}\lambda_u d\left(u\wedge\tau\wedge T\right), \quad s \geq 0 \tag{5.15}$$

so I is defined path by path as a classical Lebesgue integral. Therefore, I is a continuous $\mathbb{G}$-adapted process, so in particular it is $\mathbb{G}$-predictable. Now, notice that for all $s \geq 0$,

$$[X^1, Z]_s - I_s = \frac{1}{\widetilde{S}_0} \int_0^s \widetilde{S}_{u-} dM_{u\wedge\tau\wedge T} - \frac{1}{\widetilde{S}_0} \int_0^s \widetilde{S}_{u-} d\left(\mathbb{E}_{\mathbb{P}^*}(\zeta)\Lambda_{u\wedge\tau\wedge T}\right) = \frac{1}{\widetilde{S}_0} \int_0^s \widetilde{S}_{u-} dX^2_u$$

Then, since $\widetilde{S}_-$ is a $\mathbb{G}$-adapted càglàd process and $X^2 \in \mathcal{H}^2$, it follows from the previous equation that $[X^1, Z] - I$ is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$ (see for instance [8], Chapter II, Section 5, Theorem 20). Furthermore, we know from Proposition 2.3.10 that $X^1 Z - [X^1, Z]$ is a $\mathbb{G}$-martingale under $\mathbb{P}^*$, so we conclude that $X^1 Z - I$ is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$.

To sum up, we have seen that I is a $\mathbb{G}$-predictable finite variation process such that $I_0 = 0$ and $X^1 Z - I$ is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$. Thus, we conclude that $\langle X^1, Z\rangle = I$ (because of the uniqueness of the predictable quadratic covariation). Now, since $X^1, Z \in \mathcal{H}^2$ and $X^1_0 = 0$, it follows from the Girsanov theorem for martingales (Theorem 2.3.12) that the process $A = (A_s,\ s \geq 0)$ given by

$$A_s = \int_0^s \frac{1}{Z_{u-}} d\left\langle X^1, Z\right\rangle_u = \int_0^s \frac{\widetilde{S}_0}{\widetilde{S}_{(u-)\wedge\tau\wedge T}} dI_u \quad s \geq 0$$

is $\overline{\mathbb{P}}$-a.s. well defined and it is a finite variation process. Actually, A is $\overline{\mathbb{P}}$-a.s. well defined path by path as a Lebesgue-Stieltjes integral, since I is a finite variation process. Then, using (5.15) and applying the associativity of the Lebesgue-Stieltjes integral, we obtain

$$\begin{aligned} A_s &= \int_0^s \frac{\widetilde{S}_0}{\widetilde{S}_{(u-)\wedge\tau\wedge T}} dI_u = \frac{\mathbb{E}_{\mathbb{P}^*}(\zeta)}{\widetilde{S}_0} \int_0^s \frac{\widetilde{S}_0}{\widetilde{S}_{(u-)\wedge\tau\wedge T}} \widetilde{S}_{u-}\lambda_u d\left(u\wedge\tau\wedge T\right) = \\ &= \mathbb{E}_{\mathbb{P}^*}(\zeta) \int_0^{s\wedge\tau\wedge T} \frac{\widetilde{S}_{u-}}{\widetilde{S}_{(u-)\wedge\tau\wedge T}} \lambda_u du = \mathbb{E}_{\mathbb{P}^*}(\zeta) \int_0^{s\wedge\tau\wedge T} \lambda_u du = \mathbb{E}_{\mathbb{P}^*}(\zeta)\Lambda_{s\wedge\tau\wedge T}, \quad s \geq 0 \end{aligned}$$

Hence, A is $\mathbb{G}$-predictable and bounded (because $(\Lambda_{s\wedge\tau\wedge T},\ s \geq 0)$ is a $\mathbb{G}$-adapted continuous process bounded by $\int_0^T |\lambda_u| du < \infty$ for all $s \geq 0$), so it follows from Girsanov theorem for martingales (Theorem 2.3.12) that the process $\widetilde{X} := X^1 - A$ is a $\mathbb{G}$-martingale under $\overline{\mathbb{P}}$. But now, notice that

$$\widetilde{X}_s = X^1_s - A_s = N_{\Lambda_{s\wedge\tau\wedge T}} - \Lambda_{s\wedge\tau\wedge T} - \mathbb{E}_{\mathbb{P}^*}(\zeta)\Lambda_{s\wedge\tau\wedge T} = N_{\Lambda_{s\wedge\tau\wedge T}} - \Lambda^*_{s\wedge\tau\wedge T}, \quad s \geq 0$$

where

$$\begin{aligned} \lambda^*_s &:= \left(1 + \mathbb{E}_{\mathbb{P}^*}(\zeta)\right)\lambda_s = (1-\alpha)(1+\gamma)\lambda_s, \quad s \geq 0 \\ \Lambda^*_s &:= \int_0^s \lambda^*_u du, \quad s \geq 0 \end{aligned}$$

so the following conditions hold:

- The process $(N_{\Lambda_s},\ s \geq 0)$ is a $\mathbb{G}$-adapted point process, that is, a càdlàg $\mathbb{G}$-adapted increasing process with initial value 0 that takes values in $\mathbb{N} \cup \{0\}$ and whose jumps are equal to 1.

- Λ^* is a $\mathbb{G}$-adapted increasing continuous process (and then, also $\mathbb{G}$-predictable) with initial value 0 and such that $\widetilde{X} = (N_{\Lambda_{s\wedge\tau\wedge T}} - \Lambda^*_{s\wedge\tau\wedge T},\ s \geq 0)$ is a $\mathbb{G}$-martingale under $\overline{\mathbb{P}}$. Thus, Λ^* is the $\mathbb{G}$-compensator of $(N_{\Lambda_s},\ s \geq 0)$ under $\overline{\mathbb{P}}$ until time $\tau \wedge T$ (see [3], Chapter I, Theorem 3.17).

- Λ^* is a deterministic function.

Hence, it follows from the martingale characterization of Poisson processes (see [3], Chapter II, Theorem 4.5) that $N_{\Lambda_{s\wedge\tau\wedge T}} = N^*_{s\wedge\tau\wedge T}$ for all $s \geq 0$, where $N^* = (N^*_s,\ s \geq 0)$ is a Poisson process with intensity function Λ^* with respect to $\mathbb{G}$ under $\overline{\mathbb{P}}$. Let us denote by τ^* the first jump time of N^* and fix $t \in [0, T]$. On the one hand, we know that conditioned on $\{t < \tau^*\} = \{t < \tau\}$, the conditional density of τ^* with respect to $\mathcal{G}_t$ under $\overline{\mathbb{P}}$ is the function $\lambda^*_s e^{-(\Lambda^*_s - \Lambda^*_t)}$. On the other hand, since the processes N^* and $(N_{\Lambda_s},\ s \geq 0)$ are equal until time $\tau \wedge T$ and τ is the first jump time of $(N_{\Lambda_s},\ s \geq 0)$, we know that $\tau^* = \tau$ in the set $\{\tau \leq T\} = \{\tau^* \leq T\}$. Therefore, applying (5.14), we obtain

$$
\begin{aligned}
C^3_t &= RS_t\, \mathbb{E}_{\overline{\mathbb{P}}}\left(e^{-q(\tau - t)} \mathbf{1}_{\{t<\tau\leq T\}} \,\middle|\, \mathcal{G}_t \right) = \mathbf{1}_{\{\tau>t\}} RS_t\, \mathbb{E}_{\overline{\mathbb{P}}}\left(e^{-q(\tau^* - t)} \mathbf{1}_{\{t<\tau^*\leq T\}} \,\middle|\, \mathcal{G}_t \right) = \\
&= \mathbf{1}_{\{\tau>t\}} RS_t \int_t^T e^{-q(u-t)} \lambda^*_u e^{-(\Lambda^*_u - \Lambda^*_t)} du = \\
&= \mathbf{1}_{\{\tau>t\}} RS_t (1-\alpha)(1+\gamma) \int_t^T e^{-q(u-t)} \lambda_u e^{-(1-\alpha)(1+\gamma)(\Lambda_u - \Lambda_t)} du
\end{aligned}
$$

so this proves (5.10) in the case where the CoCo is standard. □

Once we have already developed a general pricing formula for the CoCo of our model, we are going to study the particular case where λ is constant and equal to $l_k \geq 0$ on the interval $(T_{k-1}, T_k]$ for $k = 1, \ldots, K$ and where the prices of bonds with any maturity $< T_K$ can be computed from the prices of bonds with maturities $T_1, \ldots, T_K$ by interpolation, as described in the previous section.

Under these assumptions, C^1_t and C^2_t can be easily computed with formulas (5.8) and (5.9) by applying the interpolation formulas (5.2) and (5.3) and using the fact that for every $s \in [0, T_K]$,

$$
\Lambda_s = \sum_{k=1}^{k_s - 1} l_k \left(T_k - T_{k-1} \right) + l_{k_s} \left(s - T_{k_s - 1} \right)
$$

where $T_{k_s-1} < s \leq T_{k_s}$. Moreover, for the computation of C^3_t we can use the following corollary, which can be obtained by applying the previous expression for Λ and the interpolation described in (5.2) and (5.3) to the formulas (5.10) and (5.11) (thus, we shall not prove it):

Corollary 5.3.1.1. *Fix $t \in [0, T)$ with $T_{m-1} \leq t < T_m$ for some $m \in \{1, \ldots, K\}$. For each $k \in \{m, \ldots, K\}$, let f_k be the value given by (5.3). Let us also define the times $T^*_{m-1} := t$, $T^*_K := T$ and $T^*_k := T_k$ for $k = m, \ldots, K-1$. Then, under the assumptions made above, the following statements hold:*

1. *If the CoCo of our model is standard, then (5.10) can be written as*

$$C_t^3 = \mathbf{1}_{\{\tau>t\}} R S_t (1-\alpha)(1+\gamma) \sum_{k=m}^{K} l_k \exp\left(-\sum_{j=m}^{k-1} \widetilde{m}_j \left(T_j^* - T_{j-1}^*\right)\right) v_k^{(1)}$$

where for every $k \in \{m, \dots, K\}$, $\widetilde{m}_k = q + (1-\alpha)(1+\gamma) l_k$ and

$$v_k^{(1)} = \begin{cases} \dfrac{1 - e^{-\widetilde{m}_k (T_k^* - T_{k-1}^*)}}{\widetilde{m}_k} & \text{if} \quad \widetilde{m}_k \neq 0, \\ T_k^* - T_{k-1}^* & \text{if} \quad \widetilde{m}_k = 0 \end{cases}$$

2. *If the CoCo of our model is write-down, then (5.11) can be written as*

$$C_t^3 = \mathbf{1}_{\{\tau>t\}} R (1-\alpha) \sum_{k=m}^{K} l_k P\left(t, T_{k-1}^*\right) \exp\left(-\sum_{j=m}^{k-1} l_j \left(T_j^* - T_{j-1}^*\right)\right) v_k^{(2)}$$

where for every $k \in \{m, \dots, K\}$, $m_k = f_k + l_k$ and

$$v_k^{(2)} = \begin{cases} \dfrac{1 - e^{-m_k (T_k^* - T_{k-1}^*)}}{m_k} & \text{if} \quad m_k \neq 0, \\ T_k^* - T_{k-1}^* & \text{if} \quad m_k = 0 \end{cases}$$

5.4 Calibration of the conversion intensity

If we want to apply the pricing formulas derived in the previous secion, first we need to specify the parameters of the model. In this section, we shall assume that the parameters $\alpha, \beta, \gamma, q, \sigma$ and δ are known and we will study how to determine λ from the market spreads of the K traded CDS's of our model. To do so, we shall start developing a general pricing formula for the initial value of the traded CDS's (that is, a formula for the spreads $\kappa_1, \dots, \kappa_K$) for the case of a general deterministic conversion intensity λ and without interpolating bond prices. After this, we will adapt this formula to the particular case where λ is piecewise constant and bond prices can be computed by interpolation, and finally we will briefly explain how these formulas can be used to calibrate λ.

Let us start proving the following general pricing formula for the CDS's spreads:

Theorem 5.4.1. *For each $k \in \{1, \dots, K\}$, the spread κ_k (at time 0) of the T_k-maturity CDS of our model can be computed as*

$$\kappa_k = \frac{(1-\delta) \displaystyle\int_0^{T_k} P(0,u) f(u) du}{\displaystyle\sum_{i=1}^{d_k} P(0, s_i^{(k)})\, G_{s_i^{(k)}} (s_i^{(k)} - s_{i-1}^{(k)})} = \frac{(1-\delta) \displaystyle\int_0^{T_k} P(0,u) f(u) du}{\displaystyle\sum_{i=1}^{d_k} P(0, s_i^{(k)}) \left(1 - \int_0^{s_i^{(k)}} f(u) du\right) (s_i^{(k)} - s_{i-1}^{(k)})} \tag{5.16}$$

with the convention $s_0^{(k)} = 0$, where for every $t \geq 0$,

$$f(t) = \begin{cases} \alpha \lambda_t e^{-\Lambda_t} + (1-\alpha) \frac{\beta}{\beta-1} \lambda_t \left(e^{-\Lambda_t} - e^{-\beta \Lambda_t}\right) & \text{if} \quad \beta \neq 1, \\ \alpha \lambda_t e^{-\Lambda_t} + (1-\alpha) \lambda_t \Lambda_t e^{-\Lambda_t} & \text{if} \quad \beta = 1 \end{cases} \tag{5.17}$$

and

$$G_t = \begin{cases} \alpha e^{-\Lambda_t} + \frac{(1-\alpha)}{\beta-1}\left(\beta e^{-\Lambda_t} - e^{-\beta\Lambda_t}\right) & \text{if} \quad \beta \neq 1, \\ \alpha e^{-\Lambda_t} + (1-\alpha)(1+\Lambda_t) e^{-\Lambda_t} & \text{if} \quad \beta = 1 \end{cases} \tag{5.18}$$

Proof. Remember that all the K traded CDS's are supposed to have the same default time θ, which is given by

$$\theta = \mathbf{1}_{\{\xi=0\}}\tau + \mathbf{1}_{\{\xi=1\}}\nu$$

Let us consider the process $F = (F_t,\ t \geq 0)$ given by $F_t = \mathbb{P}^*\{\theta \leq t \mid \mathcal{F}_t\}$ for every $t \geq 0$ and let $G = (G_t,\ t \geq 0)$ be the $\mathbb{F}$-survival process of θ under $\mathbb{P}^*$, which is defined by $G_t = 1 - F_t$ for all $t \geq 0$. Now, fix $k \in \{1, \dots, K\}$. The idea to prove (5.16) will consist of checking that the four assumptions of Theorem 4.1.8 are fulfilled, deriving formulas for the probability density function f of θ and for the $\mathbb{F}$-survival process G of θ and then applying the formula (4.14).

First of all, we know that $\mathbb{F} \vee \mathbb{H}^2 \subset \mathbb{G}$ and $\mathcal{G}_0$ is the trivial σ-field, so the first condition of Theorem 4.1.8 is satisfied. The second condition (about the short-term interest rate r) also holds by the construction of our model. Moreover, since $\theta = \mathbf{1}_{\{\xi=0\}}\tau + \mathbf{1}_{\{\xi=1\}}\nu$ and we know that ξ and N are independent of $\mathbb{F}$ under $\mathbb{P}^*$, it follows that the process H^2 is independent of $\mathbb{F}$ under $\mathbb{P}^*$, so the third assumption of Theorem 4.1.8 is fulfilled. This latter condition implies (as we saw in the previous section) that F (and then, also G) is a deterministic function and it is the cummulative distribution function of θ under $\mathbb{P}^*$.

Now, we are going to compute F. First of all, using the fact that H^2 is independent of $\mathbb{F}$ under $\mathbb{P}^*$, that ξ and N are independent under $\mathbb{P}^*$, that τ and ν are the first jump time of the process $(N_{\Lambda_t},\ t \geq 0)$ and the second jump time of the process $\left(N_{\Lambda_t^\beta},\ t \geq 0\right)$ respectively and using the fact that $\tau \leq \nu$ by construction, we can see that

$$\begin{aligned} F_t &= \mathbb{P}^*\{\theta \leq t \mid \mathcal{F}_t\} = \mathbb{E}_{\mathbb{P}^*}\left(H_t^2 \mid \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}^*}\left(H_t^2\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\xi=0\}}\mathbf{1}_{\{\tau\leq t\}} + \mathbf{1}_{\{\xi=1\}}\mathbf{1}_{\{\nu\leq t\}}\right) = \\ &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\xi=0\}}\mathbf{1}_{\{\tau\leq t\}} + \mathbf{1}_{\{\xi=1\}}\mathbf{1}_{\{\nu\leq t\}}\mathbf{1}_{\{\tau\leq t\}}\right) = \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau\leq t\}} - \mathbf{1}_{\{\xi=1\}}\mathbf{1}_{\{\nu> t\}}\mathbf{1}_{\{\tau\leq t\}}\right) = \\ &= \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\tau\leq t\}}\right) - \mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\xi=1\}}\right)\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\{\nu>t\}}\mathbf{1}_{\{\tau\leq t\}}\right) = 1 - e^{-\Lambda_t} - (1-\alpha)\,\mathbb{P}^*\{\nu > t, \tau \leq t\} = \\ &= 1 - e^{-\Lambda_t} - (1-\alpha)\,\mathbb{P}^*\left\{N_{\Lambda_t^\beta} = 1, \tau \leq t\right\} = \\ &= 1 - e^{-\Lambda_t} - (1-\alpha)\,\mathbb{P}^*\left\{N_{\int_0^\tau \lambda_u du + \int_\tau^t \beta\lambda_u du} - N_{\int_0^\tau \lambda_u du} = 0, \tau \leq t\right\} \quad t \geq 0 \end{aligned}$$

Let us fix $t \geq 0$ and consider the stochastic processes $X = (X_s,\ s \geq 0)$ and $Y = (Y_s,\ s \geq 0)$ defined by

$$Y_s := \begin{cases} N_{\int_0^s \lambda_u du + \int_s^t \beta\lambda_u du} - N_{\int_0^s \lambda_u du} & \text{if} \quad s \in [0,t], \\ 0 & \text{if} \quad s > t \end{cases}$$

$$X_s := \mathbf{1}_{\{Y_s=0\}}, \quad s \geq 0$$

Then, notice that $F_t = 1 - e^{-\Lambda_t} - (1-\alpha)\,\mathbb{E}_{\mathbb{P}^*}\left(X_\tau \mathbf{1}_{\{\tau\leq t\}}\right)$. In order to compute this latter expectation, let us consider a sequence of partitions $\mathcal{P}^m = \{0 = t_{0,m} < t_{1,m} < \cdots < t_{k_m,m} = t\}$ of $[0,t]$ with norm $|\mathcal{P}^m| := \max_{i\in\{1,\dots,k_m\}}(t_{i,m} - t_{i-1,m})$ going to 0 when $m \to \infty$, and also consider the sequence of random variables $(X^{(m)})_{m\in\mathbb{N}}$ defined by

$$X^{(m)} := \sum_{i=1}^{k_m} X_{t_{i,m}} \mathbf{1}_{\{t_{i-1,m}<\tau\leq t_{i,m}\}}, \quad m \in \mathbb{N}$$

Notice that all the random variables of this sequence are non-negative and bounded by 1. Moreover, it is easy to check that the sequence of random variables $(X^{(m)}\mathbf{1}_{\{\tau\leq t\}})_{m\in\mathbb{N}}$ converges to the random variable $X_\tau \mathbf{1}_{\{\tau\leq t\}}$ a.s. Indeed, given $\omega \in \Omega$,

- If $\tau(\omega) > t$, then $X^{(m)}(\omega)\mathbf{1}_{\{\tau(\omega)\leq t\}} = 0 \xrightarrow[m\to\infty]{} 0 = X_{\tau(\omega)}\mathbf{1}_{\{\tau(\omega)\leq t\}}$.
- If $\tau(\omega) = t$, then $X^{(m)}(\omega)\mathbf{1}_{\{\tau(\omega)\leq t\}} = X_t(\omega) \xrightarrow[m\to\infty]{} X_t(\omega) = X_{\tau(\omega)}\mathbf{1}_{\{\tau(\omega)\leq t\}}$.
- If $\tau(\omega) < t$, then the sample path $Y(\omega)$ is a càdlàg function that takes values in the discrete set $\mathbb{N}\cup\{0\}$, so there exists $\varepsilon > 0$ such that the function $Y(\omega)$ is constant on the time interval $[\tau(\omega), \tau(\omega)+\varepsilon)$. Then, the sample path $X(\omega)$ is also constant on $[\tau(\omega), \tau(\omega)+\varepsilon)$. Hence, since $|\mathcal{P}^m| \xrightarrow[m\to\infty]{} 0$, for m large enough we will have that $t_{i_m-1,m} < \tau(\omega) \leq t_{i_m,m} < \tau(\omega)+\varepsilon$ and then $X^{(m)}(\omega) = X_{t_{i_m,m}}(\omega) = X_{\tau(\omega)}(\omega)$. Therefore, $X^{(m)}(\omega)\mathbf{1}_{\{\tau(\omega)\leq t\}} \xrightarrow[m\to\infty]{} X_{\tau(\omega)}\mathbf{1}_{\{\tau(\omega)\leq t\}}$.

Thus, it follows from the dominated convergence theorem that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^*}\left(X_\tau \mathbf{1}_{\{\tau\leq t\}}\right) = \lim_{m\to\infty}\mathbb{E}_{\mathbb{P}^*}\left(X^{(m)}\mathbf{1}_{\{\tau\leq t\}}\right) = \lim_{m\to\infty}\mathbb{E}_{\mathbb{P}^*}\left(\sum_{i=1}^{k_m} X_{t_{i,m}}\mathbf{1}_{\{t_{i-1,m}<\tau\leq t_{i,m}\}}\right) = \\
&= \lim_{m\to\infty}\sum_{i=1}^{k_m}\mathbb{E}_{\mathbb{P}^*}\left(\mathbf{1}_{\left\{Y_{t_{i,m}}=0\right\}}\mathbf{1}_{\{t_{i-1,m}<\tau\leq t_{i,m}\}}\right) = \lim_{m\to\infty}\sum_{i=1}^{k_m}\mathbb{P}^*\left\{Y_{t_{i,m}}=0, t_{i-1,m}<\tau\leq t_{i,m}\right\} = \\
&= \lim_{m\to\infty}\sum_{i=1}^{k_m}\mathbb{P}^*\left\{N_{\Lambda_{t_{i,m}}+\beta(\Lambda_t-\Lambda_{t_{i,m}})} - N_{\Lambda_{t_{i,m}}} = 0, N_{\Lambda_{t_{i-1,m}}} = 0, N_{\Lambda_{t_{i,m}}} - N_{\Lambda_{t_{i-1,m}}} > 0\right\} = \\
&= \lim_{m\to\infty}\sum_{i=1}^{k_m}\mathbb{P}^*\left\{N_{\Lambda_{t_{i,m}}+\beta(\Lambda_t-\Lambda_{t_{i,m}})} - N_{\Lambda_{t_{i,m}}} = 0\right\}\mathbb{P}^*\left\{N_{\Lambda_{t_{i-1,m}}} = 0\right\}\mathbb{P}^*\left\{N_{\Lambda_{t_{i,m}}} - N_{\Lambda_{t_{i-1,m}}} > 0\right\} = \\
&= \lim_{m\to\infty}\sum_{i=1}^{k_m} e^{-\beta\left(\Lambda_t-\Lambda_{t_{i,m}}\right)}e^{-\Lambda_{t_{i-1,m}}}\left(1-e^{-\left(\Lambda_{t_{i,m}}-\Lambda_{t_{i-1,m}}\right)}\right) = \\
&= \lim_{m\to\infty}\sum_{i=1}^{k_m} -e^{-\beta\left(\Lambda_t-\Lambda_{t_{i,m}}\right)}\left(e^{-\Lambda_{t_{i,m}}} - e^{-\Lambda_{t_{i-1,m}}}\right) = \int_0^t -e^{-\beta(\Lambda_t-\Lambda_s)}d\left(e^{-\Lambda_s}\right)
\end{aligned}$$

where we have used the distribution of N and the fact that N has independent increments under $\mathbb{P}^*$ (because N is a Poisson process under $\mathbb{P}^*$), and the last equality follows from the characterization of the Lebesgue-Stieltjes integral for continuous integrands (that is, the Riemann-Stieltjes integral) as the limit of approximating sums (see [8], Chapter I, Section 7, Theorem 53, and see also [7], Chapter 12). If we put together all the results obtained so far and we apply the associativity of the Lebesgue-Stieltjes integral (using the fact that $1-e^{-\Lambda_s} = \int_0^s \lambda_u e^{-\Lambda_u}du$ for all $s \geq 0$), then we obtain

$$\begin{aligned}
F_t &= 1 - e^{-\Lambda_t} - (1-\alpha)\,\mathbb{P}^*\left\{N_{\int_0^\tau \lambda_u du + \int_\tau^t \beta\lambda_u du} - N_{\int_0^\tau \lambda_u du} = 0, \tau \leq t\right\} = \\
&= 1 - e^{-\Lambda_t} - (1-\alpha)\,\mathbb{P}^*\left\{Y_\tau = 0, \tau \leq t\right\} = 1 - e^{-\Lambda_t} - (1-\alpha)\,\mathbb{E}_{\mathbb{P}^*}\left(X_\tau \mathbf{1}_{\{\tau\leq t\}}\right) =
\end{aligned}$$

$$= 1 - e^{-\Lambda_t} - (1-\alpha)\int_0^t -e^{-\beta(\Lambda_t-\Lambda_s)} d\left(e^{-\Lambda_s}\right) = 1 - e^{-\Lambda_t} - (1-\alpha)\int_0^t e^{-\beta(\Lambda_t-\Lambda_s)} d\left(1-e^{-\Lambda_s}\right) =$$

$$= 1 - e^{-\Lambda_t} - (1-\alpha)\int_0^t e^{-\beta(\Lambda_t-\Lambda_s)}\lambda_s e^{-\Lambda_s} ds = 1 - e^{-\Lambda_t} - (1-\alpha)e^{-\beta\Lambda_t}\int_0^t e^{(\beta-1)\Lambda_s}\lambda_s ds$$

so we conclude that:

- If $\beta = 1$, then for every $t \geq 0$,

$$F_t = 1 - e^{-\Lambda_t} - (1-\alpha)e^{-\Lambda_t}\int_0^t \lambda_s ds = 1 - e^{-\Lambda_t} - (1-\alpha)\Lambda_t e^{-\Lambda_t} =$$
$$= 1 - \alpha e^{-\Lambda_t} - (1-\alpha)(1+\Lambda_t)e^{-\Lambda_t}$$

- If $\beta \neq 1$, then for every $t \geq 0$,

$$F_t = 1 - e^{-\Lambda_t} - (1-\alpha)e^{-\beta\Lambda_t}\int_0^t e^{(\beta-1)\Lambda_s}\lambda_s ds = 1 - e^{-\Lambda_t} - \frac{(1-\alpha)}{\beta-1}e^{-\beta\Lambda_t}\left(e^{(\beta-1)\Lambda_t}-1\right) =$$
$$= 1 - e^{-\Lambda_t} - \frac{(1-\alpha)}{\beta-1}\left(e^{-\Lambda_t}-e^{-\beta\Lambda_t}\right) = 1 - \alpha e^{-\Lambda_t} - \frac{(1-\alpha)}{\beta-1}\left(\beta e^{-\Lambda_t}-e^{-\beta\Lambda_t}\right)$$

Hence, it is clear that the $\mathbb{F}$-survival process G of θ is given by (5.18) and it is easy to check that F is an absolutely continuous function of the form $F_t = \int_0^t f(u)du$ for the Lebesgue integrable function f given by (5.17). In particular, this means that f is the probability density function of θ under $\mathbb{P}^*$. Therefore, the last assumption of Theorem 4.1.8 is fulfilled, so it follows from Theorem 4.1.8 that the spread κ_k can be computed with the formula (5.16), as we wanted to prove. □

Now, let us assume that λ is constant and equal to $l_k \geq 0$ on the interval $(T_{k-1}, T_k]$ for $k = 1, \ldots, K$ and that the prices of bonds with any maturity $< T_K$ can be computed from the prices of bonds with maturities $T_1, \ldots, T_K$ by interpolation, as described in Section 5.2. Under these assumptions, the denominator of (5.16) can be easily computed by applying the interpolation formulas (5.2) and (5.3) and using the fact that for every $s \in [0, T_K]$,

$$\Lambda_s = \sum_{k=1}^{k_s-1} l_k\left(T_k - T_{k-1}\right) + l_{k_s}\left(s - T_{k_s-1}\right)$$

where $T_{k_s-1} < s \leq T_{k_s}$.

On the other hand, the computation of the integral in the numerator of (5.16) seems more tricky, but we can obtain a nicer formula for this integral by applying the previous expression for Λ and the interpolation described in (5.2) and (5.3). If we do this, we can obtain the following formula for the CDS's spreads (we shall not prove it):

Corollary 5.4.1.1. *For each $k \in \{1, \ldots, K\}$, let f_k be the value obtained by applying (5.3) with $t = 0$ and $m = 1$. Then, under the assumptions made above, for each $k \in \{1, \ldots, K\}$, the spread κ_k (at time*

0) of the T_k-maturity CDS of our model can be computed as

$$\kappa_k = \frac{(1-\delta)\displaystyle\sum_{j=1}^{k} l_j P\left(0, T_{j-1}\right)\left(\alpha v_j + (1-\alpha) w_j\right)}{\displaystyle\sum_{i=1}^{d_k} P(0, s_i^{(k)})\, G_{s_i^{(k)}}\,(s_i^{(k)} - s_{i-1}^{(k)})} \tag{5.19}$$

where for every $j \in \{1, \ldots, K\}$, $\Delta T_j = T_j - T_{j-1}$, $m_j = f_j + l_j$, $\widetilde{m}_j = f_j + \beta l_j$,

$$v_j = \begin{cases} \exp\left(-\displaystyle\sum_{i=1}^{j-1} l_i \Delta T_i\right) \dfrac{1 - e^{-m_j \Delta T_j}}{m_j} & \textit{if} \quad m_j \neq 0, \\ \exp\left(-\displaystyle\sum_{i=1}^{j-1} l_i \Delta T_i\right) \Delta T_j & \textit{if} \quad m_j = 0 \end{cases}$$

$$\widetilde{v}_j = \begin{cases} \exp\left(-\displaystyle\sum_{i=1}^{j-1} \beta l_i \Delta T_i\right) \dfrac{1 - e^{-\widetilde{m}_j \Delta T_j}}{\widetilde{m}_j} & \textit{if} \quad \widetilde{m}_j \neq 0, \\ \exp\left(-\displaystyle\sum_{i=1}^{j-1} \beta l_i \Delta T_i\right) \Delta T_j & \textit{if} \quad \widetilde{m}_j = 0 \end{cases}$$

and

$$w_j = \begin{cases} \dfrac{\beta}{\beta - 1}\left(v_j - \widetilde{v}_j\right) & \textit{if} \quad \beta \neq 1, \\ \dfrac{1}{m_j}\left[v_j\left(l_j + m_j \displaystyle\sum_{i=1}^{j-1} l_i \Delta T_i\right) - l_j \Delta T_j \exp\left(-f_j \Delta T_j - \displaystyle\sum_{i=1}^{j} l_i \Delta T_i\right)\right] & \textit{if} \quad \beta = 1 \textit{ and } m_j \neq 0, \\ \left(\displaystyle\sum_{i=1}^{j-1} l_i \Delta T_i \Delta T_j + l_j \frac{(\Delta T_j)^2}{2}\right) \exp\left(-\displaystyle\sum_{i=1}^{j-1} l_i \Delta T_i\right) & \textit{if} \quad \beta = 1 \textit{ and } m_j = 0 \end{cases}$$

If we assume that the parameters α, β and δ are known and that we can observe the market prices of bonds with maturities $T_1, \ldots, T_K$, then notice that for each $k \in \{1, \ldots, K\}$, the spread κ_k given by (5.19) only depends on $l_1, \ldots, l_k$. Hence, we can specify the parameters $l_1, \ldots, l_K$ that define λ from the market spreads $\kappa_1, \ldots, \kappa_K$. Indeed, if we substitute $\kappa_1, \ldots, \kappa_k$ in (5.19) for the market values of these spreads, then we obtain a system of n equations depending on $l_1, \ldots, l_K$ in which we can solve each equation in order recursively. Of course the unknowns $l_1, \ldots, l_K$ cannot be isolated, so each equation must be solved by using numerical methods.

The other parameters of our model can be estimated, for instance, from historical data about default events and market data about other traded assets, and once all the parameters of the model have been specified, the CoCo can be priced by applying the formulas developed in the previous section.

6 Conclusions

This thesis has been a deep introduction to credit risk along with the study of one particular reduced-form model.

In Chapter 2 we have introduced the concept of credit risk with the help of two financial instruments which are subject to this kind of risk: corporate bonds and credit default swaps (CDS's). However, we have noticed that there is a wide range of credit-risk sensitive instruments (besides bonds and CDS's) and we have defined the concept of defaultable claim, which encompasses all these instruments. These instruments can be priced with the well-known risk-neutral valuation formula, which requires some general stochastic integration theory to be established. Nevertheless, we have seen that this formula can be problematic in an incomplete market, since there might exist more than one risk-neutral probability and then this formula returns several different values, so we must fix one of these probabilities. Due to this and other reasons, we have postulated this formula as a definition.

In Chapter 3 we have studied the structural approach for credit risk modelling, focusing on the original Merton's model and the original Black and Cox model. For future research, we propose to study in depth some extensions of these two models. As commented before, the structural approach is very appealing from the economic point of view, because it links the default events to the total value of the firm's assets and other economic fundamentals, but this feature is at the same time the main drawback of this approach, since this causes the default times to be predictable stopping times, which is not realistic. Moreover, structural models can become difficult to implement (because the total value of a firm is difficult to estimate) and they often present discrepancies between the credit spreads predicted by the model and the spreads observed in the market. For this reason, for practical implementations, we suggest opting for an hybrid model, that is, a reduced-form model with default intensity depending on economic factors linked to the value of the firm's assets.

In Chapter 4 we have studied the reduced-form approach for credit risk modelling, focusing specially on the hazard process approach. Reduced-form models give an exogenous specification of the default time by providing a model for the conditional probability of default, so this avoids the problematic predictable nature of the default times that is present in structural models. We have developed a useful rewrite of the risk-neutral valuation formula to price a defaultable claim under the hazard process approach. For future research, we propose to investigate how to derive pricing formulas in a model with several (dependent or independent) default times. It would also be a good idea to study how to tackle the problem of pricing and hedging defaultable claims under the martingale approach.

In Chapter 5 we have introduced CoCos, which are bonds that convert into equity shares or a cash payment if a pre-specified trigger event occurs. Then, we have described in detail one particular reduced-form model for pricing CoCos, where we assume that the firm's stock price process follows a geometric Brownian motion with a jump at conversion, we also assume that the conversion time of traded CoCos and the default time of traded CDS's are the first two jump times of a time-changed Poisson process and we also assume that the conversion intensity is a deterministic function. We have developed pricing formulas for CoCos and also formulas to calibrate the model with market prices of traded CDS's, and then we have studied the particular case where the conversion intensity is piecewise constant and bond prices can be interpolated in a specific way. We can notice that, even under this model with the simplest assumptions (deterministic and piecewise constant conversion intensity, constant parameters, interpolation of bond prices...), the pricing and calibration formulas obtained are difficult to derive (we have needed to apply Girsanov theorem for martingales, stochastic integration theory and some properties of the predictable quadratic covariation) and they are not easy to implement (they require numerical methods). For future research, we propose to implement these formulas to real data of some firm. We could also try to extend this simple model by taking also into account the accrued payment of a CoCo and the possibility of stochastic coupons, and we could also consider in our model a stochastic conversion intensity, variable parameters, a more complex model for the stock price and even we could add to the model more stochastic factors involving other sources of risk. Moreover, it would be interesting to use other financial instruments for the calibration of the model (not only CDS's), and one could also try to establish a model for the dynamics of the short-term interest rate process instead of interpolating bond prices. Finally, in this chapter we have focused on the problem of pricing CoCos and calibrating the model, so for future research we suggest focusing on the problem of how to hedge a CoCo by using other traded assets.

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