UNIVERSITAT DE BARCELONA

ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

MATLIS DUALITY, INVERSE SYSTEMS AND CLASSIFICATION OF ARTIN ALGEBRAS

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Abstract

The aim of this project is to study the classification of some families of Artin algebras. In order to do that, we will study some important results of injective modules with the objective to be able to prove Matlis duality.
In particular, we will study the case of Matlis duality when $R = \mathbf{k}[\![x_1, \dots, x_n]\!]$ (the ring of the formal series) with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. With this scenario, we are talking about Macaulay's duality.
Using Macaulay's correspondence, we will be able to study important results as Hilbert functions, essential in the classification of algebras. We will study Gorenstein, level and compressed algebras.
Along the paper, we use SINGULAR [DGPS22] and the library INVERSE-SYST.LIB by Joan Elias [Eli14], which with we will compute some examples seen through the project.

Introduction

In this project, the objective is to study notions about injective modules, Matlis duality and Macaulay's correspondence in order to show how to use Macaulay's duality for classifying Artin algebras.
In the first chapter and until we say otherwise, R is a commutative ring. During the first section, we will define injective modules and some properties of them. Those concepts will be the base in order to study the following theorems and propositions that will be seen in this memory. We will prove the existence of injective hulls of modules. From this result, we deduce the existence of minimal injective resolutions of R-modules.
In the second section, we focus on the case when R is a Noetherian ring and we study some results of injective modules on this particular case. In the last section, we will introduce the concept of Bass numbers and we are going to study the relation between this numbers and the minimal injective resolution of a finite R- module, with R Noetherian.
In the second chapter, we are going to study Matlis duality. It ensures an isomorphism between Artin and Noetherian modules. Given A a R-module, let $(R, \mathfrak{a}, \mathbf{k})$ Noetherian local ring, its dual will be $A^\vee = \mathrm{Hom}_R(A, E)$, E an injective hull of $\mathbf{k}$, by Matlis duality. It was written and proved by Eben Matlis at 1958, [Mat58].
Later, we will study the particular case of Matlis duality when

$$R = \mathbf{k}[\![x_1, \ldots, x_n]\!],$$

the ring of formal series, with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$, called Macaulay's correspondence. We are going to see all the consequences that one can obtain with this result and we will compute some examples. Lastly, using this correspondence, we will focus on Hilbert function and we are going to see how to define it using all the previous knowledge.
In this section, we will see some examples, for instance how to build a

submodule generated by two polynomials over R using contraction, which will be defined.
One of the well-known references in order to study all the knowledge required until this part is *Injective modules* by Sharpe and Vámos, [SV72].
In the fourth chapter, we will study Gorenstein rings, level and compressed algebras and some results of them. First, we focus on the case of Gorenstein rings and we use all the previous sections in order to announce some of the theorems and propositions. We will study the relation between Artin and Gorenstein rings. Once studying level rings, we will talk about Irrabino's Q-decomposition of the associated graded ring of an Artinian s-level local k-algebra.
Moreover, we will study how to achieve isomorphism classes of local algebras using Macaulay's inverse system. We are going to reach an important result, that is that an isomorphism between two Artinian s-level algebras is defined by a matrix. Closing the chapter, we will focus on some results about h-vectors and Artinian 3-level local algebras.
In order to understand and make more tangible some of the results seen in the project, we will finish with a chapter related to Singular, [DGPS22]. In particular, INVERSE-SYST.LIB [Eli14] by J. Elias will be used.
The interesting proposition which says that there exists an isomorphism between some models for A and its inverse system when A is an Artin Gorenstein local $\mathbf{k}$-algebra with Hilbert function $\mathrm{HF}_A = \{1, 3, 3, 1\}$ will be proved via SINGULAR.
Lastly, we select some command ot the library, the ones used previously.

Now, we will fix some basic notations and definitions that we will use along the project.

NOTATIONS:
Let R be a commutative ring with an unit element.
Let A be a module over a ring R, and let S be a subset of A, $x \in A$. Then, we define

- $\mathbf{Ann}_R(x) := \{r \in R \mid rx = 0\}$.
- $\mathbf{Ass}_R(A) := \{\mathfrak{p} \in \mathbf{Spec}(R) \mid \mathfrak{p} = \mathbf{Ann}_R(x) \text{ for some } 0 \neq x \in A\}$.
- $\mathbf{Supp}_R(A) := \{\mathfrak{p} \in \mathbf{Spec}(R) \mid A_\mathfrak{p} \neq 0\}$.

- $(0 :_R A) := \{r \in R \mid rA = 0\} = \mathbf{Ann}_R(A)$

Definition 0.0.1. *Let R be a local ring. A* ***regular sequence*** *$x_1, \ldots, x_t$ on a R-module M is any set of elements such that $(x_1, \ldots, x_t)M \neq M$ and such that x_i is not a zero divisor on $M/(x_1, \ldots, x_{i-1})M$.*

Definition 0.0.2. *Let R be a local ring. We define the* ***depth*** *of a finitely generated module over R as the maximal length of a regular sequence on the module.*

Definition 0.0.3. *Let R be a Noetherian local ring. A finite R-module $A \neq 0$ is a* ***Cohen-Macaulay module*** *if depth A = dim A. If R is a Cohen-Macaulay module, then it is called a* ***Cohen-Macaulay ring****.*

Definition 0.0.4. *Let R be a local ring. We define the* ***system of parameters*** *(s.o.p.) for R of dimension n as the elements $x_1, \ldots, x_n$ which generate any $\mathfrak{m}-$primary ideal.*

Definition 0.0.5. *A local ring R is* ***regular*** *if and only if the maximal ideal is generated by a s.o.p.*

Definition 0.0.6. *A local ring is Cohen-Macaulay iff every system of parameters forms a regular sequence.*

Definition 0.0.7. *Let R be a Noetherian ring, A a R-module finitely generated I and ideal of R. Now, we define the degree of I with respect to A as*

$$G(I, A) = \inf\{i | Ext^i_R(R/I, A) \neq 0\}.$$

If $(R, \mathfrak{m})$ is a Noetherian local ring, then $G(\mathfrak{m}, A)$ can be written as $G(A)$.
If $R = A$, then $G(\mathfrak{m}, A)$ can be written as $gr(\mathfrak{m})$.
When we are talking about ***depth****, we are also referring to this definition.*

Definition 0.0.8 (Artinian)**.** *A ring R is Artinian if it satisfies the descending chain condition for ideals.*

Definition 0.0.9 (Hilbert function)**.** *Let $(A, \mathfrak{m}, \mathbf{k})$ an Artinian local $\mathbf{k}$- algebra. Then, the* ***Hilbert function*** *of A in degree $j \geq 0$ is $HF_A(j) = dim_{\mathbf{k}} \mathfrak{m}^j/\mathfrak{m}^{j+1}$. It is also the Hilbert function of the corresponding associated graded ring $G = \bigoplus_{j \geq 0} \mathfrak{m}^j/\mathfrak{m}^{j+1}$ which is a standard graded $\mathbf{k}$-algebra.*

Definition 0.0.10. *An ideal I is irreducible if $I = J \cap K$ and $I \subsetneq J, K$.*

Definition 0.0.11. *Let S be a regular ring and let $R = S/I$. The codimension of R (or I) is defined by*

$$codim(R) = dim(S) - dim(R) = n - dim(R).$$

Definition 0.0.12. *Let S be a regular ring and let $R = S/I$. R is said to be a complete intersection if I can be generated by $codim(R)$ elements.*

For more definitions and theorems, see *Cohen-Macaulay rings* ([BH98]) by Bruns and Herzog and *Introduction to commutative algebra* ([AM69]) by Atiyah and Macdonald.

Before going on, I want to thank my supervisor Dr. Joan Elias all his help and the way he guided me through all this process. After this project, I know for sure what I want to be.
Per últim, dono les gràcies a tots aquells que considero la meva família, especialment a l'Antonio i la Paqui.

Chapter 1

Injective modules

The aim of this chapter is to study injective modules over a commutative ring R, reviewing the main results of them.
In the first section, we define injective R-modules and we give their main properties. We prove the existence of injective hulls of modules. From this result, we deduce in the second section the existence of minimal injective resolutions of R-modules.
In the third section, we focus on the case when R is a Noetherian ring and we study some results of injective modules on this particular case.
In the last section, we introduce the concept of Bass numbers and we study the relation between this numbers and the minimal injetive resolution of a finite R- module, with R Noetherian.

1.1 First concepts

Definition 1.1.1 (Injective module)**.** *Let R a commutative ring, E an R-module. E is* ***injective*** *if and only if, for all injective morphism $i\colon A \to B$ and for all morphism $f\colon A \to E$, where A and B are R- modules, a morphism $g\colon B \to E$ exists such that the following diagram commute.*

$$\begin{tikzcd}
 & E & \\
0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"'] & B \arrow[ul, dashed, "g"']
\end{tikzcd}$$

We can also define **injective modules** using the following proposition.

Proposition 1.1.2. *Let R be a commutative ring. An R-module E is injective if and only if the functor $Hom_R(\square, E)$ is exact.*

Proof. Assume that the functor $\mathrm{Hom}_R(\square, E)$ is exact. Let

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ k\ } C \longrightarrow 0$$

be a short exact sequence. Then, by assumption

$$0 \longrightarrow \mathrm{Hom}_R(C, E) \xrightarrow{\ k^*\ } \mathrm{Hom}_R(B, E) \xrightarrow{\ i^*\ } \mathrm{Hom}_R(A, E) \longrightarrow 0$$

is also exact.
The key is to remember that the induced map i^* is surjective when i is injective. If $f \in \mathrm{Hom}_R(A, E)$, there exists $g \in \mathrm{Hom}_R(B, E)$ such that $f = i^*(g) = g \circ i$, i injective. This implies that the diagram commutes, therefore E is an injective module.
From left to right, let E be an injective R-module. Then, given $f\colon A \to E$, there exists $g\colon B \to E$ with $g \circ i = f$ just by Definition 1.1.1. If $f \in \mathrm{Hom}_R(A, E)$, $f = g \circ i = i^*(g) \in \mathrm{im}\, i*$. This implies that i^* is surjective. Surjectivity is related to exactness, therefore, $\mathrm{Hom}_R(\square, E)$ is an exact functor. $\square$

Theorem 1.1.3. *Let R be a ring and M a R-module. Then, there exists an injective R-module E and $f\colon M \hookrightarrow E$ monomorphism.*

Proof. Since M is a $\mathbb{Z}$-module, we have that $M \cong \mathbb{Z}^{(I)}/H$ for a suitable subgroup H of $\mathbb{Z}^{(I)}$. Notice that $\mathbb{Z}^{(I)} \subset \mathbb{Q}^{(I)}$ are seen as abelian groups, so $M \subset G = \mathbb{Q}^{(I)}/H$. But $\mathbb{Q}$ is divisible, then G is also divisible. Therefore, $H \hookrightarrow G$, G is an injective abelian group. Now, we have this exact sequence of R-modules

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \longrightarrow E = \mathrm{Hom}_{\mathbb{Z}}(R, G).$$

What is next is to embed M in E. In order to achieve that, it is enough to show that the linear map $f\colon M \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$, $f(m)(r) = rm$ if $r \in R$, is injective. But if $f(m)(r) = 0$ for all $r \in R$, we have that $f(m)(1) = m = 0$. $\square$

Proposition 1.1.4. *Let R be a ring and E be an R-module. The following conditions are equivalent:*

(a) E is injective,

(b) for all injective morphism $i\colon A \to B$ and for all $f\colon A \to E$, where A and B are R-modules, a morphism $g\colon B \to E$ exists such that $f = g \circ i$,

(c) given R-modules $A \subset B$ and a homomorphism $f\colon A \to E$, it can be extended to a morphism $g\colon B \to E$,

(d) let B be an R-module with $E \subset B$. Then, E is a direct summand of B,

(e) the functor $Hom_R(\square, E)$ is exact.

Proof. (a) $\Leftrightarrow$ (e) We have already proved it (Proposition 1.1.2).

(e) $\Rightarrow$ (b) The injective morphism $i\colon A \to B$ induces the homomorphism

$$\mathrm{Hom}_R(i, E)\colon \mathrm{Hom}_R(B, E) \to \mathrm{Hom}_R(A, E),$$

and $\mathrm{Hom}_R(i, E)(g) = g \circ i$ for all $g \in \mathrm{Hom}_R(B, E)$. Because $\mathrm{Hom}_R(i, E)$ is an epimorphism, then $f \in \mathrm{Hom}_R(A, E)$ is of the form $g \circ i$ for some $g \in \mathrm{Hom}_R(B, E)$.

(b) $\Rightarrow$ (e) Let

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ k\ } C \longrightarrow 0$$

be a short exact sequence and i an injective morphism. We want to see that

$$0 \longrightarrow \mathrm{Hom}_R(C, E) \xrightarrow{\ k^*\ } \mathrm{Hom}_R(B, E) \xrightarrow{\ i^*\ } \mathrm{Hom}_R(A, E) \longrightarrow 0 \tag{1.1}$$

is also exact. Notice that $\mathrm{Hom}_R(\square, E)$ is always left exact but it is not always right exact. By assumption, we have that i is an injective morphism, therefore, i^* is a surjective morphism. This implies that the sequence (1.1) is exact.

(b) $\Leftrightarrow$ (c) It is clear that if there is an injective morphism i and f such that there exists a morphism g with $f = g \circ i$, then g extends f.
By the same argument, if g extends f, there is some i injective morphism such that $f = g \circ i$.

(a) $\Rightarrow$(d) Let $E \subset B$ and E injective. Then, we can have the following diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & E \arrow[u, "1_E"] \arrow[r, "i"'] & B \arrow[ul, dashed, "g"'] \end{tikzcd}$$

where i is an injective morphism. E is injective, then g exists and the diagram commutes: $1_E = g \circ i$. Therefore, i splits and there exists some R-module C such that $B = E \bigoplus C$.

(d) $\Rightarrow$ (a) In order to prove this implication, let's recall Theorem 1.1.3.
Now, using this theorem, there is an exact sequence

$$0 \longrightarrow E \longrightarrow B \longrightarrow B^{''} \longrightarrow 0$$

with B injective. By hypothesis, the sequence splits, then $B \cong E \bigoplus B^{''}$.
Let $i\colon E \to B$ be the inclusion, and let $p\colon B \to E$ be the projection. Then

$$\begin{tikzcd} E \arrow[r, "i"] & E \bigoplus B^{''} \arrow[l, bend left, "p"] \end{tikzcd}$$

with $p \circ i = 1_E$.
Notice that B is injective, therefore we can have the following diagram

$$\begin{tikzcd} & E & \\ & B \arrow[u, "p"] & \\ 0 \arrow[r] & M \arrow[u, "f^*"] \arrow[r, "i^*"'] & N \arrow[ul, dashed, "g^*"] \arrow[uul, "h"'] \end{tikzcd}$$

Using (b), given a homomorphism $p \circ f^*\colon M \to E$, there exists a morphism $h\colon N \to E$ such that $p \circ f^* = h \circ i^*$. In this case, $h = p \circ g^*$.
It remains to check that this diagram commutes too. Given $k \in \mathrm{Hom}_R(M, E)$, we get that

$$k = p \circ f^* = p \circ g^* \circ i^* = h \circ i^*.$$

Hence, the diagram commutes and E is injective, as we wanted to prove.
$\square$

Proposition 1.1.5. *Let R a commutative ring and E an R-module. Every direct summand of an injective R-module E is injective.*

Proof. Seen at the proof of (d) $\Rightarrow$ (a) in Proposition 1.1.4. $\square$

Proposition 1.1.6. *Let R be a commutative ring and E an R-module. If $(E_k)_{k\in K}$ is a family of injective R-modules, then $\prod_{k\in K} E_k$ is also an injective R-module.*

Proof. Let's consider the following diagram

$$\begin{tikzcd} & \prod E_k & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"] & B \arrow[ul, dashed, "g"'] \end{tikzcd}$$

Now, consider $p_k\colon E \to E_k$ be the kth projection, then $p_k \circ f\colon A \to E_k$. Notice that, by assumption, E_k is injective, therefore there is $g_k\colon B \to E_k$ such that $g_k \circ i = p_k \circ f$. Now, define $g\colon B \to \prod E_k$ by $g\colon b \to (g_k(b))$ which extends f. Let $a \in A$ such that $b = i(a)$, then

$$g(i(a)) = (g_k(i(a))) = (g_k \circ i)(a) = (p_k \circ f)(a) = f(a)$$

Therefore, the initial diagram commutes and $\prod E_k$ is injective. $\square$

Corollary 1.1.7. *A finite direct sum of injective R-modules is injective.*

Proof. The direct sum of finitely many modules coincides with the direct product and this is the previous corollary. $\square$

The following theorem is a useful tool to check if a module is injective or not.

Theorem 1.1.8 (Baer's criterion)**.** *A R-module E is injective if and only if every R-map $f\colon I \to E$, where I is an ideal in R, can be extended to R.*

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r, "i"'] & R \arrow[ul, dashed, "g"] \end{tikzcd}$$

Proof. Assume that E is an injective R-module. By Proposition 1.1.4 (c), using that $I \subset R$, given the homomorphism $f\colon I \to E$, it can be extended to $g\colon R \to E$.

Now, from right to left. Assume that every R-map $f\colon I \to E$ can be extended to R, I an ideal in R. Let A be a submodule of a R-module B. Consider the following diagram:

$$\begin{array}{ccccc} & & E & & \\ & & {\scriptstyle f}\big\uparrow & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B \end{array}$$

where A is a submodule of B. Consider $\Delta = \{(A', g') \mid A \subseteq A' \subseteq B, g'|_A = f\}$. Notice that $g'|_A = f$ means that $g'\colon A' \to E$ extends f. Therefore, $\Delta \neq \emptyset$ because $(A, f) \in \Delta$.

We must define a partially order in Δ by

$$(A', g') \preceq (A'', g'')$$

This means that $A' \subset A''$ and g'' extends g'. It is easy to see, just by taking the union, that any chain in Δ has an upper bound in Δ. Then Zorn's lemma can be applied and there exists some $(A_0, g_0) \in \Delta$. This is the maximal element in Δ.

We want to see that $A_0 = B$. In order to prove it, assume that there is some $b \in B$ with $b \notin A_0$. Define I the ideal such that

$$I = \{r \in R\colon rb \in A_0\}.$$

Now, define $h\colon I \to E$ by $h(r) = g_0(rb)$. By hypothesis, there is a map $h^*\colon R \to E$ extending h. Finally, define $A_1 = A_0 + \langle b \rangle$ and $g_1\colon A_1 \to E$ by

$$g_1(a_0 + rb) = g_0(a_0) + rh^*(1)$$

where $a_0 \in A_0$ and $r \in R$.

It is remained to show that g_1 is well-defined. If $a_0 + rb = a_0' + r'b$, then $(r - r')b = a_0' - a_0 \in A_0$, therefore $r - r' \in I$, just by definition of the ideal. Hence, $g_0((r - r')b)$ and $h(r - r')$ are defined and

$$g_0(a_0' - a_0) = g_0((r - r')b) = h(r - r') = (r - r')h^*(1).$$

Then, $g_0(a_0') - g_0(a_0) = rh^*(1) - r'h^*(1)$ and $g_0(a_0') + r'h^*(1) = g_0(a_0) + rh^*(1)$, as we want. Clearly, $g_1(a_0) = g_0(a_0)$ for all $a_0 \in A_0$, so g_1 extends

g_0. This contradicts the maximality of (A_0, g_0) because we have seen that $(A_0, g_0) \preceq (A_1, g_1)$. As a consequence, $A_0 = B$, g_0 is a lifting of f and E is injective.

$\square$

Definition 1.1.9. *An R-module A is* ***divisible*** *if for every regular element $r \in R$ and every element $a \in A$, there exists an element $a' \in A$ such that $a = ra'$.*

Example 1.1.10. As $\mathbb{Z}$-module, the additive group of the rationals $\mathbb{Q}$ is divisible.

Corollary 1.1.11. *Let R be a ring and E and R-module.*

(a) If E is injective, then E is divisible.

(b) If R is PID and E is divisible, then E is injective.

Proof. Let E be injective. Then, by Theorem 1.1.8, we can extend the R-map $f\colon I \to E$, where I is an ideal in R, to $g\colon R \to E$. We define $g\colon R \to E$ as $g(r) = re$ with $e \in E$ and $r \in R$. Therefore, E is divisible.
Now, let R be a PID and E divisible. Assume that $f\colon I \to E$ where I is a non zero ideal. By hypothesis of R being PID, $I = (a)$ for some $a \in I$. Since E is divisible, there is some $e \in E$ with $f(a) = ae$. Define $g\colon R \to E$ by $g(s) = se$ for some $s \in R$. It is easy to see that g is an homomorphism and it extends f. That is, if $s = ra \in I$, we have $g(s) = g(ra) = rae = rf(a) = f(ra)$. Therefore, by Theorem 1.1.8, E is injective. $\square$

Definition 1.1.12 (Proper essential extensions)**.** *Let R be a ring and let $A \subset B$ be R-modules. B is an* ***essential extension*** *of A if for any C submodule of B different from 0, one has $C \cap A \neq 0$. An essential extension B of A is called* ***proper*** *if $A \neq B$.*

Proposition 1.1.13. *Let A be a submodule of an R-module B. Then B is an essential extension of A if and only if for every $0 \neq b \in B$, there exists an $r \in R$ such that $0 \neq rb \in A$.*

Proof. Suppose that B is an essential extension of A. Let $0 \neq b \in B$. The left principle ideal $(b) = \{rb | r \in R\}$ is a submodule of B, so $(b) \cap A \neq 0$. Then there exists an $r \in R$ such that $0 \neq rb \in A$.
Conversely, suppose that for every $0 \neq b \in B$, there exists an $r \in R$ such that $0 \neq rb \in A$. Let S be a nonzero submodule of B. Then there exists $0 \neq b \in S \subseteq B$ and there is an $r \in R$ such that $0 \neq rb \in A$. Since S is an R-module, $0 \neq rb \in S$ and $A \cap S \neq 0$. Therefore, B is an essential extension of A. □

Lemma 1.1.14. *Let A, B, C be R-modules such that $A \subseteq B \subseteq C$. Then C is an essential extension of A if and only if C is an essential extension of B and B is an essential extension of A, i.e. essentialness is transitive.*

Proof. Suppose that C is an essential extension of A. Let $0 \neq c \in C$. Then by Proposition 1.1.13, there exists an $r \in R$ such that $0 \neq rc \in A \subseteq B$. Therefore, C is an essential extension of B. Now, let $0 \neq b \in B \subseteq C$. Again by Proposition 1.1.13, there exists an $r' \in R$ such that $0 \neq r'b \in A$. Then, B is an essential extension of A too.
Conversely, suppose that C is an essential extension of B and B is an essential extension of A. Let $0 \neq c \in C$. There exists an $s \in R$ such that $0 \neq sc \in B$ and there exists $t \in R$ such that $0 \neq t(sc) \in A$. Choose $r = ts$ and there exists $r \in R$ such that $0 \neq rc \in A$. Therefore, C is an essential extension of A by Proposition 1.1.13. □

Proposition 1.1.15. *Let R be a ring. An R-module A is injective if and only if it has no proper essential extension.*

Proof. Suppose that A is injective. Let B be a proper extension of A. Then by Proposition 1.1.4, (d), $B = A \bigoplus X$, where $A \cap X = 0$ for some submodule $X \subset B$. Then, A has no proper essential extensions.
Conversely, suppose that A has no proper essential extension. Let B be an extension of A. If $B = A$, then A is a direct summand of B so that A is injective. Assume that B is a proper extension of A. Since B is not an essential extension of A, there exists a submodule X of B, so $X \cap A = 0$. Moreover, by Zorn's lemma, there exists some X_0 which is the maximal submodule of B with respect the property $X \cap A = 0$.

Since $A \cap X_0 = 0$, we now must show that $B = A + X_0$. Assume by contradiction that $B \neq A + X_0$, then $A + X_0 \subset B$. B is not an essential extension of $A + X_0$ since it is not an essential extension of A, using Lemma 1.1.14. Then B/X_0 is not an essential extension of $A + X_0/X_0$. So there exists $X \subset B$ such that $X_0 \subset C$ and $C/X_0 \cap (A + X_0)/X_0 = 0$. Thus, $C \cap (A + X_0) = X_0$, showing that $C \cap A \subset C \cap (A + X_0) = X_0$. Since $C \cap A = A \cap (C \cap A) \subset A \cap X_0 = 0, C \cap A = 0$ and $X_0 \subset C$. This contradicts X_0 being the maximal element. Therefore $B = A \bigoplus X_0$ and M is injective by Proposition 1.1.4. $\square$

Definition 1.1.16 (Injective hull)**.** *Let R a ring and A a R-module. An* ***injective hull of*** *A is an injective module $E_R(A)$ such that $A \subset E_R(A)$ is an essential extension.*

Proposition 1.1.17. *Let R be a ring and let A be a R-module. Then A admits an injective hull. Moreover, if $A \subset I$, with I injective, then a maximal essential extension of A in I is an injective hull of A.*

Proof. By Theorem 1.1.3, we can embed A into an injective module I. Consider $\mathcal{S}$ the set of all essential extensions B with $A \subset B \subset I$. By Zorn's lemma, this set yields to a maximal extension $A \subset E$ such that $E \subset I$. We claim that E has no proper essential extensions and by Proposition 1.1.15, E is injective, it is the injective hull we are looking for.
Now, assume that E has a proper essential extension E'. Since I is injective, there exists $\delta \colon E' \to I$ extending the inclusion $E \subset I$. Suppose $\ker \delta = 0$, $\operatorname{im} \delta \subset I$ is an essential extension of A properly containing E. It is a contradiction with the fact that E is maximal.
Since δ extends the inclusion $E \subset I$ we have $E \cap \ker \delta = 0$ and it contradicts the essentiality of the extension $E \subset E'$. $\square$

One important result is that the injective hull is unique up to isomorphism.

Lemma 1.1.18. *Let R be a ring. Let M, N be R-modules and let $M \to E$ and $N \to E'$ be injective hulls. Then,* *(i) for any R-module map $\varphi \colon M \to N$ there exists an R-module map $\psi \colon E \to E'$ such that*

$$\begin{array}{ccc} M & \longrightarrow & E \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ N & \longrightarrow & E' \end{array}$$

commutes,

(ii) if φ is injective, then ψ is injective,

(iii) if φ is an essential extension, then ψ is an isomorphism,

(iv) if φ is an isomorphism, then ψ is an isomorphism,

(v) if $M \to I$ is an embedding of M into an injective R-module, then there is an isomorphism $I \cong E \bigoplus I'$ compatible with the embeddings of M.

In particular, the injective hull E of M is unique up to isomorphism.

Proof. Part (i) follows from the fact that E' is an injective R-module. Part (ii) follows as $\mathrm{Ker}(\psi) \cap M = 0$ and E is an essential extension of M. Assume φ is an essential extension. Then $E \cong \psi(E) \subset E'$ by (ii) which implies $E' = \psi(E) \bigoplus E''$ because E is injective. Since E' is an essential extension of M, we get $E'' = 0$. As an special case of (iii) we get (iv).
Now, assume $M \to I$ and choose a map $\alpha \colon E \to I$ extending the map $M \to I$. Arguing as before, α is injective. Then $\alpha(E)$ splits off from I. This proves (v). □

Proposition 1.1.19. *Let R be a ring and let A be an R-module, E an injective hull of A, I an injective R-module and $\alpha \colon A \to I$ a monomorphism. Then, there exists a monomorphism $\varphi \colon E \to I$ such that the following diagram is commutative, and i is the inclusion:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"] & E \arrow[dl, "\varphi"] \\ I & \end{tikzcd}$$

Proof. Since I is injective, α can be extended to an homomorphism $\beta \colon E \to I$. We have that $\beta \mid A = \alpha$ and so $A \cap \ker \beta = \ker \alpha = 0$. This extension $A \subset E$ is essential and we even have that $\ker \beta = 0$. Therefore, β is a monomorphism. □

Theorem 1.1.20. *Let R be a Noetherian ring. Let $I \neq 0$ be an injective R-module and let $\mathfrak{p} \in$ Ass I. Then $E(R/\mathbf{p})$ is a direct summand of I. In particular, if I is indecomposable, then*

$$I \cong E(R/\mathbf{p}).$$

Proof. The proof can be found at [BH98], Theorem 3.2.6, (b). □

1.2 Resolutions

Definition 1.2.1. *Let R be a ring and A and R-module. An exact sequence*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

with injective modules I^i is an ***injective resolution of an R-module****.*

Remark 1.2.2. It is not obvious that every module has an injective resolution. However, we can assure the existence of an injective resolution I^i of a R-module A by Theorem 1.1.3. Let $I^0(A) = E_R(A)$ (the injective hull of A) and denote the embedding ∂^{-1}. Now, suppose that the injective resolution has been constructed till the i-th step:

$$0 \longrightarrow I^0(A) \xrightarrow{\partial^0} I^1(A) \xrightarrow{\partial^1} \cdots \longrightarrow I^{i-1}(A) \xrightarrow{\partial^{i-1}} I^i(A)$$

We define then $I^{i+1} = E_R(\text{Coker } \partial^{i-1})$ and ∂^i is defined as the inclusion.

Definition 1.2.3. *Let R be a ring and A an R-module. Then*

$$0 \longrightarrow A \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} \cdots$$

is a minimal injective resolution of A, that is, E^i is the injective hull of Ker $\mathrm{d}^\mathrm{i} =$ Im $\mathrm{d}^{\mathrm{i}-1}$ for all i. In other words, $E^i = E_R(\text{Ker } d^i) = E_R(\text{Im } d^{i-1})$.

Proposition 1.2.4. *Let R be a ring, A an R-module. Every two minimal injective resolutions of A are isomorphic. In particular, if $\dim_R(A) = n$, every minimal injective resolution of A is of lenght n.*

Proof. Let $E^\bullet$ and $E^{\bullet'}$ two minimal injective resolutions of A. By definition, E^0 and $E^{0'}$ are hulls, therefore there exists an isomorphism $E^0 \cong E^{0'}$ which is the identity with respect to A. Then,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E^0 \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & E^n \arrow[d, "\cong"] \\
0 \arrow[r] & A \arrow[r] & E^{0'} \arrow[r] & \cdots \arrow[r] & E^{n'}
\end{tikzcd}$$

commutes. Then

$$\begin{tikzcd}
E^{n-1} \arrow[r] \arrow[d, "\cong"] & E^n \arrow[r] \arrow[d, "\cong"] & \text{Coker } d^{n-1} \cong \text{Im } d^n \arrow[r] \arrow[d, "\psi"] & 0 \\
E^{n-1'} \arrow[r] & E^{n'} \arrow[r] & \text{Coker } d^{n-1'} \cong \text{Im } d^{n'} \arrow[r] & 0
\end{tikzcd}$$

Here, the Five Lemma is used in order to see that ψ is an isomorphism. Now, extending ψ to an isomorphism $E(\text{Im } d^n) \cong E(\text{Im } d^{n'})$, we can prove what we desired by recurrence. □

Proposition 1.2.5. *If $I^\bullet$ is an injective resolution of A, then $E^\bullet$ (the minimal injective resolution) is isomorphic to a direct summand of $I^\bullet$.*

Proof. Let $Y \colon 0 \to A \to I$ be an injective resolution and $X \colon 0 \to A \to E$ an minimal injective resolution.
The comparison theorem let us extend the identity of A to morphisms

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"]
\end{tikzcd}$$

and using Proposition 1.2.4, $h = gf$ is an automorphism of X, $g(fh^{-1}) = \text{Id}_X$. Then, if $Z = \text{Ker } g$

$$0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0$$

Then Z is an injective resolution because $Y \cong Z \bigoplus X$. □

Definition 1.2.6. *The **injective dimension of an R-module** A, denoted by $id_R(A)$, is*

- *$id_R(A) = n$, $A \neq 0$, n the shortest natural for which there exists an injective resolution $I^\cdot$ of A with $I^n = 0$*
- *$id_R(A) = \infty$ if A has no finite injective resolutions.*
- *$id_R(A) = -\infty$ if $A = 0$.*

The injective dimension is a way to define when an R-module A is injective.

Remark 1.2.7. Let A be an R-module, $A \neq 0$. A is injective if and only if $id_R(A) = 0$.

Proposition 1.2.8. *Let $(R, \mathfrak{a}, \mathbf{k})$ be a Noetherian local ring, and A a finite R-module. Then*

$$id_R(A) = \sup\{i \colon Ext^i_R(\mathbf{k}, A) \neq 0\}$$

Proof. The proof can be found in [BH98], Proposition 3.1.14.. □

As a consequence, the following corollary.

Corollary 1.2.9. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring and M a finite R-module. If $x \in \mathfrak{m}$ is an element which is R- and M-regular,then*

$$id_{R/(x)} M/xM = id_R M - 1.$$

1.3 Injective modules over Noetherian rings

Proposition 1.3.1. *Let R be a ring, $\mathfrak{a}$ an ideal of R and A a R-module annihilated by $\mathfrak{a}$. If $E = E_R(A)$,*

$$E_{R/\mathfrak{a}}(A) = \{e \in E \colon \mathfrak{a}e = 0\} = (0 :_E \mathfrak{a})$$

Proof. By hypothesis, A is annihilated by $\mathfrak{a}$ and, by definition, $(0 :_E \mathfrak{a})$ too. Then, both of them can be thought as $R/\mathfrak{a}$-modules. It's clear that $A \subset (0 :_E \mathfrak{a}) \subset E$. All $R/\mathfrak{a}$-submodules of $(0 :_E \mathfrak{a})$ are also a R-submodule of E, then $(0 :_E \mathfrak{a})$ is an essential extension of A. What we need to check is that $(0 :_E \mathfrak{a})$ is injective. Let's consider the following diagram of $R/\mathfrak{a}$-modules:

$$
\begin{tikzcd}
 & (0 :_E \mathfrak{a}) & \\
0 \arrow[r] & N \arrow[u, "f"] \arrow[r, "i"'] & M \arrow[ul, dashed, "g"']
\end{tikzcd}
$$

Does $g = f \circ i$ really exist? Remember that all this modules are $R/\mathfrak{a}$-modules, in particular R-modules too. Then, we can replace $(0 :_E \mathfrak{a})$ by E. E is injective, then we can extend the diagram and make it commutative. The commutativity implies that $\mathrm{Im}(g) \subset (0 :_E \mathfrak{a})$ therefore the original diagram also commutes. $\square$

Then, we have the following corollary:

Corollary 1.3.2. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a local ring and $E = E_R(\mathbf{k})$. Let $\mathfrak{a}$ be an ideal of R. Then:*

(i) $E_{R/\mathfrak{a}}(\mathbf{k}) = (0 :_E \mathfrak{a})$.

(ii) $E = \cup_{t \geq 1} E_{R/\mathfrak{m}^t}(\mathbf{k})$.

Proof. The first statement is a consequence of Proposition 1.3.1. For the second one, we must take into account that every element of $E_R(\mathbf{k})$ is killed by a power or $\mathfrak{m}$. Therefore, $E = \cup_{t \geq 1}(0 :_E \mathfrak{m}^t)$. Now, using the first statement, we get

$$E = \cup_{t \geq 1}(0 :_E \mathfrak{m}^t) = \cup_{t \geq 1} E_{R/\mathfrak{m}^t}(\mathbf{k}).$$

$\square$

Definition 1.3.3. *An R-module A is **decomposable** if there exist non-zero submodules A_1, A_2 of A such that $A = A_1 \bigoplus A_2$. Otherwise, it is **indecomposable**.*

In the following theorem, we will determine the indecomposable injective R-modules of a Noethering ring R. From now on, R is always Noetherian.

Theorem 1.3.4. *Let R be a Noetherian ring.*

(a) For all $\mathfrak{p} \in \mathbf{Spec}\, R$ the module $E(R/\mathfrak{p})$ is indecomposable.

*(b) Let $I \neq 0$ be an injective R-module and let $\mathfrak{p} \in$ **Ass** I. Then $E(R/\mathfrak{p})$ is a direct summand of I.*

Proof. (a) Suppose $E(R/\mathfrak{p})$ is decomposable, there exist non-zero submodules $N_1, N_2 \in E(R/\mathfrak{p})$ such that $N_1 \cap N_2 = 0$. It follows that

$$(N_1 \cap R/\mathfrak{p}) \cap (N_1 \cap R/\mathfrak{p}) = (N_1 \cap N_2) \cap R/\mathfrak{p} = 0.$$

On the other hand, since $R/\mathfrak{p} \subset E(R/\mathfrak{p})$ is an essential extension, we have

$$N_1 \cap R/\mathfrak{p} \neq 0 \neq N_2 \cap R/\mathfrak{p}.$$

This contradicts the fact that $R/\mathfrak{p}$ is a domain.

(b) $R/\mathfrak{p}$ could be considered as a submodule of I since $p \in$ **Ass** I. One can check in Proposition 3.2.4. in [BH98] that there exists an injective hull $E(R/\mathfrak{p})$ of $R/\mathfrak{p}$ such that $E(R/\mathfrak{p}) \subset I$. But $E(R/\mathfrak{p})$ is injective, therefore it is a direct summand of I.

□

Proposition 1.3.5. *If R is a Noetherian ring and $(E_k)_{k\in K}$ is a family of injective R-modules, then $\bigoplus_{k\in K} E_i$ is an injective R-module.*

Proof. By the Baer Criterion, Theorem 1.1.8, it suffices to complete the diagram

$$\begin{tikzcd} & \bigoplus_{k\in K} E_k & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r, "i"'] & R \arrow[ul, dashed] \end{tikzcd}$$

where I is an ideal of R. If $x \in \bigoplus_k E_k$, then $x = (E_k)$, where $e_k \in E_k$; define $\mathbf{Supp}(x) = \{k \in K \colon e_k \neq 0\}$. Since R is Noetherian, I is finitely generated, $I = (a_1, \dots, a_n)$. As any element in $\bigoplus_{k\in K} E_k$, each $f(a_j)$ for $j = 1, \dots, n$ has finite support $\mathbf{Supp}(f(a_j)) \subseteq K$. Then $\cup_{j=1}^n \mathbf{Supp}(f(a_j))$ is a finite set, and $\operatorname{im} f \subseteq \bigoplus_{l\in S} E_l$. Using Corollary 1.1.7, this finite sum is injective. Then, there is an R- map $g' \colon R \to \bigoplus_{l\in S} E_l$ extending f. Composing g' with the inclusion of $\bigoplus_{l\in S} E_l$ into $\bigoplus_{k\in K} E_k$ completes the above diagram. □

Working with Noetherian rings is the key to prove the above proposition. Check Proposition 1.1.6 in order to see that when we are working without Noetherian rings, this proposition is not true.
There is the converse of this proposition, but we are not going to prove it.

Theorem 1.3.6 (Bass-Papp)**.** *If R is a ring for which each direct sum of injective R-modules is an injective module, then R is Noetherian.*

Proof. We show that if R is not Noetherian, then there are an ideal I and an R-map from I to a sum of injectives that cannot be extended to R. Since R is not Noetherian, we have a strictly ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq \cdots$, $I = \cup I_n$. We note that $I/I_n \neq \{0\}$ for all n. By Theorem 1.1.3, we may embed I/I_n in an injective R-module E_n. We claim that $E = \bigoplus_n E_n$ is not injective.
Let $\pi_n \colon I \to I/I_n$ be the natural map. For each $a \in I$, note that $\pi_n(a) = 0$ for large n (because $a \in I_n$ for some n), and so the R-map $f \colon I \to \prod(I/I_n)$ defined by

$$f \colon a \mapsto (\pi_n(a)),$$

does have its image in $\bigoplus_n(I/I_n)$. That is, for each $a \in I$, almost all of the coordinates of $f(a)$ are 0. Composing with the inclusion $\bigoplus(I/I_n) \to \bigoplus E_n = E$, we may regard f as a map $I \to E$. If there is an R-map $g \colon R \to E$ extending f, then $g(1)$ is defined and let's say $g(1) = (e_n)$. Now, choose an index m and choose $a_m \in I$ with $a_m \notin I_m$. Since $a_m \in I_m$, we have $\pi_m(a_m) \neq 0$ and $g(a_m) = f(a_m)$ as nonzero mth coordinate $\pi_m(a_m)$.
But $g(a_m) = a_m g(1) = a_m(e_n) = (a_m e_n)$, then $\pi_m(a_m) = a_m e_m$. It follos that $e_m \neq 0$ for all m, and this contradicts $g(1)$ lying in the direct sum $E = \bigoplus E_n$. □

Theorem 1.3.7. *Let R be a ring and $I = J_1 \cap \cdots \cap J_n$ an irredundant decomposition of the ideal I by ideals J_i. Assume that each $E(R/J_i)$ is indecomposable. Then the natural embedding of R/I into $C = E(R/J_1) \bigoplus \cdots \bigoplus E(R/J_n)$ can be extended to an isomporphism of $E(R/I)$ onto C.*

Proof. The proof can be found in [Mat58], Theorem **2.3**. □

Theorem 1.3.8. *A module M over a ring R is an indecomposable, injective module if and only if $M \cong E(R/J)$ where J is an irreducible ideal of R. In this case, for every $x \neq 0 \in M$, $(0 :_R x)$ is an irreducible ideal and $M \cong E(R/(0 :_R x))$.*

Proof. The proof can be found in [Mat58], Theorem **2.4**. □

Theorem 1.3.9. *Let R be a Noetherian ring. Then every injective R-module has a decomposition as a direct sum of indecomposable, injective submodules.*

Proof. Let M be an injective R-module. Then, by Zorn's lemma, we can find a submodule N of M which is maximal with respect to the property of being a direct sum of indecomposable, injective submodules.
Suppose that $N \neq M$, and by Theorem 1.3.5, N is injective. Hence, there exist a non-zero submodule P of M such that $M = N \bigoplus P$.
Let $x \neq 0 \in P$. Since R is Noetherian, then $(0 :_R x)$ is an intersection of a finite number of irreducible ideals.
Therefore, by Theorem 1.3.7 and Theorem 1.3.8, $E(R/(0 :_{n_R} x))$ is a direct sum of a finite number of indecomposable, injective R-modules. Now, $Rx \cong R/(0 :_R x)$, so we can consider that $E(R/(0 :_R x))$ is imbedded in D. Then, $N \bigoplus E(R/(0 :_R x))$ contradicts the maximality of N, thus $N = M$. This concludes the proof of the theorem. □

Remark 1.3.10. This decomposition is unique. For any $\mathfrak{p} \in \operatorname{Spec} R$, the number of indecomposable summands in the decomposition of M which are isomorphic to $E(R/\mathfrak{p})$ depends only on M and $\mathfrak{p}$.

Proof. Let M be an injective R-module. We will use transfinite recursion to construct $M_\alpha \subset M$ for ordinals α which are direct sums of indecomposable injective R-modules $E_{\beta+1}$, $\beta < \alpha$.
For $\alpha = 0$, we let $M_0 = 0$. Suppose given an ordinal α such that M_α has been constructed. Then M_α is an injective R-module, Proposition 1.3.5. Then, $M \cong M_\alpha \bigoplus M_0$. If $M' = 0$, we are done. If not, M' contains a copy of $R/\mathfrak{p}$ for some $\mathfrak{p}$ prime. Then, M' contains an indecomposable submodule E. Set $M_{\alpha+1} = I_\alpha \oplus E_\alpha$. If α is a limit ordinal and M_β has been constructed for $\beta < \alpha$, then $M_\alpha = \cup_{\beta<\alpha} I_\beta$. Notice that $M_\alpha = \bigoplus_{\beta<\alpha} E_{\beta+1}$. □

Theorem 1.3.11. *Let R be a ring. Indecomposable injective R-modules are of the form $E(R/I)$, I an irreducible ideal of R.*

Proof. Let A be an indecomposable injective. If $A = 0$, then it is enough to take $I = R$. If $A \neq 0$, A contains a submodule $\approx R/I$ and since A is an indecomposable injective, $A = E(R/I)$. If $I = J_1 \cap J_2$ and $J_1/I, J_2/I \neq 0$, then $E(J_1/I) = E(A/I)$ then $J_2/I \cap J_1/I \neq 0$, which is not possible.

Reciprocally, let I be an irreducible ideal of R. Then (0) is irreducible in R/I. If $E(A/I) = K_1 \bigoplus K_2$ with $K_1 \cap K_2 = 0$, this implies $(K_1 \cap R/I) \cap (K_2 \cap R/I) = 0$, where, for example, $K_1 \cap R/I = 0$. If $K_1 \neq 0$, it will be a contradiction with $R/I \subset E(A/I)$ being essential. Then $E(R/I)$ is indecomposable. □

Theorem 1.3.12. *Let R be a Noetherian ring. If I is an irreducible ideal and $r(I) = P, E(R/I) = E(R/P)$. Then there exists a bijection between Spec R and isomorphism classes of indecomposable injective R-modules different from 0, given by*

$$P \longleftrightarrow E(R/P)$$

Proof. $P \in \mathbf{Ass}_R(R/I)$ implies $R/P \subset R/I \subset E(R/I)$ is an indecomposable injective, then by Theorem 1.3.11, the map is surjective.

If $P, Q \in \text{Spec } R$ and $E(R/P) \overset{\varphi}{\approx} E(R/Q)$ (since $\varphi(R/P) \cap R/Q \neq 0$), there exists $x \neq 0$, $x \in R/Q \cap \varphi(R/P)$. The map is injective because P and Q are prime and $Q = \mathbf{Ann}(x) = P$. □

Lemma 1.3.13. *Let R be a ring, $A \neq 0$ an indecomposable injective R-module. Then $\text{End}_R(A)$ is a local ring.*

Proof. It must be seen that no bijective endomorphisms are ideals. If $\varphi \in \text{End}_R$ is not bijective, then $\varphi\psi$ is not either for all $\psi \in \text{End}_R(A)$. In other hand, Ker $\varphi \neq 0$, otherwise $\varphi(A) \subset A$ is injective and $\varphi(A) = A$ because A is indecomposable and φ injective. Then, if φ, ψ are not bijective, using Ker $\varphi \subset A$,

$$\text{Ker } (\varphi + \psi) \supset \text{Ker } \varphi \cap \text{Ker } \psi \neq 0,$$

then $\varphi + \psi$ is not bijective either. □

1.4 Bass numbers

Theorem 1.4.1. *Let R be a Noetherian ring. Then, every injective R-module can be written uniquely except isomorphism by*

$$I = \bigoplus_{\mathfrak{p} \in Spec\ R} E(R/\mathfrak{p})^{(c_{\mathfrak{p}})}$$

for some cardinals $c_{\mathfrak{p}}$.

Proof. It is direct from Theorem 1.3.9, Theorem 1.3.12, Theorem 1.3.13. $\square$

Definition 1.4.2 (Bass number)**.** *Let R be a Noetherian ring, A a finite R-module, and $E^{\bullet}(A)$ the minimal injective resolution of A. Then*

$$E^i(A) \cong \bigoplus_{\mathfrak{p} \in \mathbf{Spec}\ R} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p},A)}.$$

μE represents the direct sum of μ copies of E. Using Theorem 1.4.1 and Proposition 1.2.4, $\mu_i(\mathfrak{p}, A)$ is well defined.

The Bass numbers have an interpretation in terms of the minimal injective resolution of A.
One may ask if there is a way to compute an explicit formula of all μ_i. First, we will need some previous results in order to prove the desired one.

Corollary 1.4.3. *Let R be a Noetherian ring, S a multiplicative system of R, A a R-module. Then,*

(a) If $\mathfrak{p} \in Spec\ R$, $\mathfrak{p} \cap S = \emptyset$, then $\mu_i((p), A) = \mu_i(S^{-1}\mathfrak{p}, S^{-1}A)$,

(b) $id_{S^{-1}R}(S^{-1}A) \leq id_R(A)$,

(c) $id_R(A) = \sup_{Spec\ R} id_{\ R_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \sup_{Max\ R} id_{R_{\mathfrak{m}}}(A_{\mathfrak{m}})$.

Proof. In order to prove (a) and (b), we must know that S^{-1} preserves minimal injective resolutions.
Now, let's prove (c). Let $\mathrm{id}_R(A) \geq n$ and

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^n \longrightarrow \cdots$$

an injective minimal resolution and let $\mathrm{id}_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}) < n$, $\forall \mathfrak{p} \in \mathrm{Spec}\, R$. Using that S^{-1} preserves minimal injective resolutions, $(E^n)_{\mathfrak{p}} = 0, \forall \mathfrak{p}$ and $E^n = 0$ but this doesn't make sense.
The same argument can be followed in order to prove it for Max R instead of Spec R. $\square$

Proposition 1.4.4. *Let R be a Noetherian ring, A a finite R-module and $\mathfrak{p} \in$* **Spec** *R. The finite number*

$$\mu_i(\mathfrak{p}, A) = dim_{\mathbf{k}(\mathfrak{p})} Ext^i_{R_{\mathfrak{p}}}(\mathbf{k}(\mathfrak{p}), A_{\mathfrak{p}})$$

is called the $i-th$ ***Bass number*** *of A with respect to $\mathfrak{p}$.*

Proof. By Corollary 1.4.3, let $\mathfrak{p} = \mathfrak{m}$ a maximal ideal and R local. It must be seen that $\mu_i(\mathfrak{m}, A) = \dim_{\mathbf{k}} \mathrm{Ext}^i_R(\mathbf{k}, A)$, with $\mathbf{k} = R/\mathfrak{m}$. Let

$$0 \longrightarrow A \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$$

a minimal injective resolution of A. $\mathrm{Ext}^{\bullet}_R(\mathbf{k}, A)$ is the cohomology of

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(\mathbf{k}, E^0) & \longrightarrow & \mathrm{Hom}_R(\mathbf{k}, E^1) & \longrightarrow & \cdots \\ & & \Big| \simeq & & \Big| \simeq & & \\ 0 & \longrightarrow & (0 : \mathfrak{m})_{E^0} & \longrightarrow & (0 : \mathfrak{m})_{E^1} & \longrightarrow & \cdots \end{array}$$

Let $N_i = (0 : \mathfrak{m})_{I_0}$. Then $N_i \subset C_{i-1} = \mathrm{Ker}\, d_i$, because $C_{i-1} \subset E(C_{i-1}) = I_1$ is essential and $\forall x \in I_i$, $x \neq 0$, has a multiple $rx \in C_{i-1}$, $rx \neq 0$, $r \in R$. If $\mathfrak{m}x = 0$, then $x \in C_{i-1}$ because A is local. Then $d_i(N_i) = 0$, where $N_i = \mathrm{Ext}^i_R(\mathbf{k}, A)$, $\forall i \geq 0$.
It is enough to see that $\mu_i(\mathfrak{m}, A) = \dim_{\mathbf{k}} N_i$. An element $r \in \mathfrak{m} - \mathfrak{p}$ doesn't cancel elements $\neq 0$ in R/P, neither in $E(R/P)$, where

$$N_i = \{x \in \bigoplus_{\mathfrak{p}} E(R/P)^{\mu_i(\mathfrak{p}, A)} | \mathfrak{m}x = 0\} = \{x \in E(R/\mathfrak{m})^{\mu_i(\mathfrak{m}, A)} | \mathfrak{m}x = 0\}$$

The elements of $E(R/\mathfrak{m})$ cancelled by $\mathfrak{m}$ are exactly $R/\mathfrak{m}$, then

$$N_i = (R/\mathfrak{m})^{\mu_i(\mathfrak{m}, A)}.$$

Then,

$$\mu_i(\mathfrak{m}, A) = \dim_{\mathbf{k}} N_i = \dim_{\mathbf{k}} \mathrm{Ext}^i_R(\mathbf{k}, A)$$

$\square$

Corollary 1.4.5. *Let R a Noetherian ring, A a finitely generated R-module. Then* $\mu_i(\mathfrak{p}, A) < \infty$, $\forall i \geq 0$, $\forall \mathfrak{p} \in$ *Spec R.*

Proof. $\mathrm{Ext}^i_{R_\mathfrak{p}}(\mathbf{k}(\mathfrak{p}), A_\mathfrak{p})$ is finitely generated as $R_\mathfrak{p}$-module, then it is $\mathbf{k}(\mathfrak{p})$-space of finite dimension. □

Corollary 1.4.6. *Let R be a Noetherian local ring, A a R-module finitely generated. Then*

$$G(A) = \inf\{i | \mu_i(\mathfrak{m}, A) \neq 0\}$$

Proof. This is by definition of degree, Definition 0.0.7. □

As a consequence of this corollary:

Corollary 1.4.7. *Let R be a Noetherian local ring. Then*

$$R \text{ is Cohen-Macaulay} \Leftrightarrow \mu_i(\mathfrak{m}, R) = 0,\ \forall i < \ \dim R$$

Proof. From left to right, let R be a Cohen-Macaulay ring. By Definition 0.0.3 and Definition 0.0.7, we know that $\dim R = \operatorname{depth} R = \mathrm{G}(R)$. Therefore,

$$G(R) = \inf\{i | \mu_i(\mathfrak{m}, R) \neq 0\} = \dim R$$

and $\mu_i(\mathfrak{m}, R) = 0$, $\forall i < \dim R$.
From right to left, let $\mu_i(\mathfrak{m}, R) = 0$, $\forall i < \dim R$. This implies that $\mu_i(\mathfrak{m}, R) \neq 0$ when $i = \dim R$. By Definition 0.0.7, $G(R) = \dim R$ and using Definition 0.0.3, R is a Cohen-Macaulay ring. □

What we have obtained in Corollary 1.4.6 is a lower bound of i such that $\mu_i \neq 0$. Our objective now is to find a upper bound.

Lemma 1.4.8. *Let R be a Noetherian ring, A a f. g. R-module,* $P \subset Q$ *prime ideals of R. If* $s = h(Q/P)$, *then*

$$\mu_i(P, A) \neq 0 \Rightarrow \mu_{i+s}(Q, A) \neq 0$$

Proof. It is enough if we prove for $s = 1$ and, localizing on $\mathbb{Q}$, we can suppose R to be local and $Q = \mathfrak{m}$. Let $a \in \mathfrak{m}$, $a \notin P$. Let $B = A/P$ and $C = B/\bar{a}B = A/(P, a)$. Since $a \notin z(B)$, there is the following exact sequence

$$0 \longrightarrow B \xrightarrow{.a} B \longrightarrow C \longrightarrow 0 \ .$$

Consequently,

$$\mathrm{Ext}^i_A(B, R) \xrightarrow{.a} \mathrm{Ext}^i_A(B, R) \longrightarrow \mathrm{Ext}^{i+1}_A(C, R)$$

By Proposition 1.4.4, $\mu_i(P, A) \neq 0$ implies $(\mathrm{Ext}^i_R(B, A))_P \neq 0$. Hence $\mathrm{Ext}^i_R(B, A)$ is finitely generated, the sequence implies $\mathrm{Ext}^{i+1}_R(C, A)$ using Nakayama's lemma.
The objective is to see that this implies $\mathrm{Ext}^{i+1}_R(\mathbf{k}, A) \neq 0$. Since C has annulator (P, M) which is $\mathfrak{m}$- primary, $l_R(C) < \infty$. By recurrence of $l_R(C)$, let's see that $\mathrm{Ext}^j_R(\mathbf{k}, A) = 0$ implies $\mathrm{Ext}^j_R(C, A) = 0$, which is direct if $l_R(C) = 1$ (this means $C \approx \mathbf{k}$). Let $l_R(C) = r$ and

$$0 = C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_r = C$$

a sequence with coefficients $\approx \mathbf{k}$. The exact sequence

$$0 \longrightarrow C_{r-1} \longrightarrow C_r \longrightarrow \mathbf{k} \longrightarrow 0$$

implies

$$\begin{array}{ccccc} \mathrm{Ext}^j_R(\mathbf{k}, A) & \longrightarrow & \mathrm{Ext}^j_R(C_r, A) & \longrightarrow & \mathrm{Ext}^j_R(C_{r-1}, A) \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

and $\mathrm{Ext}^j_R(C, A) = 0$.
Since $\mathrm{Ext}^{i+1}_R(C, A) \neq 0$, then $\mu_{i+1}(\mathfrak{m}, A) = \dim_{\mathbf{k}} \mathrm{Ext}^{i+1}_R(\mathbf{k}, A) \neq 0$. □

Lemma 1.4.9. *Let R be a Noetherian local ring, A a f.g. R-module different from 0. Then*

$$dim_R\ A \leq \sup\{i | \mu_i(\mathfrak{m}, R) \neq 0\} = id_R\ (A)$$

Proof. Let P be a prime ideal of $\mathbf{Supp}(A)$ such that $\dim A = \dim M/P = s$. Since $P \in \mathbf{Ass}_R(A)$, $\mu_0(P,A) \neq 0$. By Lemma 1.4.8, $\mu_s(\mathfrak{m},A) \neq 0$, then $\dim A \leq \sup\{i|\mu_i(\mathfrak{m},A) \neq 0\}$.
If $\mathrm{id}_R(A) = \infty$, for every j there exists some prime ideal P such that $\mu_j(P,A) \neq 0$. By the lemma, $\mu_{j+r}(\mathfrak{m},A) \neq 0$ if $r = h(\mathfrak{m}/P)$, and this guarantees that $\sup\{i|\mu_i(\mathfrak{m},A) \neq 0\} = \infty$.
If $\mathrm{id}_R(A) = n < \infty$, then $\mu_i(\mathfrak{m},A) = 0$ if $i > \mathrm{id}_R(A)$ and $\mu_n(P,A) \neq 0$ for some P. But P must be equal to $\mathfrak{m}$, otherwise the lemma will be $\mu_i(\mathfrak{m},A) \neq 0$ for some $i > n$. Hence, $\mathrm{id}_R = \sup\{i|\mu_i(\mathfrak{m},A) \neq 0\}$. $\square$

The result of the previous results is

$$G(A) = \inf\{i|\mu_i(\mathfrak{m},A) \neq 0\} \leq \dim A \leq \sup\{i|\mu_i(\mathfrak{m},A) \neq 0\} = \mathrm{id}_R(A)$$

Lemma 1.4.10. *Let R be a Noetherian local ring, N a f.g R-module with $pd_R(N) = s < \infty$. Then $Ext^s_R(N,M) \neq 0$ for all f. g. R-module $M \neq 0$.*

Proof. If N' is the $s-1$-th syzygies module of N, then $\mathrm{Ext}^s_R(N,M) \approx \mathrm{Ext}^1_R(N',M)$ and $\mathrm{pd}_A(N') = 1$. Suppose $\mathrm{Ext}^s_R(N,M) = 0$ then

$$0 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow N' \longrightarrow 0$$

a minimal projective sequence of N'. Then L_1, L_0 are f.g. free and $L_1 \subset \mathfrak{m}L_0$. Hence, the following exact sequence

$$\mathrm{Hom}_R(L_0,M) \longrightarrow \mathrm{Hom}_R(L_1,M) \longrightarrow \mathrm{Ext}^1_R(N',M) = 0$$

Therefore, if $x \in M$, there exists $\varphi \in \mathrm{Hom}_R(L_1,M)$ such that $x \in \mathrm{Im}\ \varphi$ and $\psi \in \mathrm{Hom}_R(L_0,M)$ such that $\psi|L_1 = \varphi$, where

$$\mathrm{Im}\ \varphi = \psi(L_1) \subset \mathfrak{m}\psi(L_0) \subset \mathfrak{m}M$$

and then $x \in \mathfrak{m}M$. What we have obtained is $M = \mathfrak{m}M$ and by Nakayama's lemma, we achieve a contradiction with $M = 0$. $\square$

Now, let's see a restrictive theore for $\mathrm{id}_R(A)$.

Theorem 1.4.11. *Let R be a Noetherian local ring, $A \neq 0$ a f.g. R-module. If $id_R(A) < \infty$, then $id_R(A) = G(R)$.*

Proof. Let $r = \mathrm{id}_R(A)$, $s = G(R)$. Let $a_1, \dots, a_s$ a R-succession and $B = r/(a_1, \dots, a_s)$. We have $\mathrm{dp}_R(B) = s$ and $G(B) = 0$, therefore B contains a copy of $\mathbf{k} = R/\mathfrak{m}$. Using Lemma 1.4.9, $\mathrm{Ext}^r_R(\mathbf{k}, A) \neq 0$, therefore the exactness of

$$\mathrm{Ext}^r_R(B, A) \longrightarrow \mathrm{Ext}^r_R(\mathbf{k}, B) \longrightarrow \mathrm{Ext}^{r+1}_R(B/\mathbf{k}, A) = 0$$

implies $\mathrm{Ext}^r_R(B, A) \neq 0$, where $s \leq \mathrm{pd}_R(B) = r$ □

As a result of Lemma 1.4.9 and Theorem 1.4.11, the following theorem.

Theorem 1.4.12. *Let $(R, \mathfrak{a}, \mathbf{k})$ be a Noetherian local ring, and let A be a finite R-module of finite injective dimension. Then*

$$dim\ A \leq id\ A = depth R.$$

Theorem 1.4.13. *Let R be a Noetherian local ring, A a f. g. R-module. If $id_R(A) = \infty$, then $\mu_i(\mathfrak{m}, A) \neq 0$, $\forall i \geq dim\ R$.*

Proof. We will prove it by recurrence of $n = \dim R$. If $n = 0$ and $\mu_i(\mathfrak{m}, A) = 0$, in an injective minimal resolution $0 \longrightarrow M \longrightarrow I^{\cdot}$ we will have $I_i = 0$ and this implies $\mathrm{id}_R(A) < i$. If $n > 0$, we will distinguish two cases:

(a) There exists a prime $P \neq \mathfrak{m}$ such that $\mathrm{id}_{R_p} = \infty$. By recurrence, $\mu_j(P, A) \neq 0, \forall j \geq \dim R_p$. If $h(\mathfrak{m}/P) = r$ and $i \geq \dim R$, $j = i - r \geq \dim R - h(\mathfrak{m}/P) \geq h(P)$, then $\mu_{i-r}(P, A) \neq 0$. By Lemma 1.4.8, then $\mu_i(\mathfrak{m}, A) \neq 0$.

(b) If $\mathrm{id}_{R_p}(A_p) < \infty$ for all $P \neq \mathfrak{m}$. By Theorem 1.4.11

$$i \geq \dim R > h(P) \geq G(R_p) = \mathrm{id}_{R_p}(A_p)$$

where $\mu_i(P, A) = 0, \forall P \neq \mathfrak{m}$. Then $\mu_i(\mathfrak{m}, A) \neq 0$, otherwise $\mathrm{id}_R(A) < i$ and this does not make sense.

□

Chapter 2

Matlis duality

The aim of this chapter is to study Matlis duality over $(R, \mathfrak{m}, \mathbf{k})$ a Noetherian local ring and some consequences of this result.
In the first section, we enumerate the necessary definitions and theorems in order to be able to study Matlis duality with R a Noetherian local ring.
In the second section, we focus on a particular case when $R = \mathbf{k}[\![x_1, \ldots, x_n]\!]$, the ring of formal series, with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. When we are studying Matlis duality in this ring, it is called Macaulay's correspondence. We see all the consequences that one can obtain with this result and we compute some examples. Lastly, using this correspondence, we focus on Hilbert function and we see how to define it using all the previous knowledge.

2.1 Matlis duality

Let $(R, \mathfrak{a}, \mathbf{k})$ be a Noetherian local ring. Given an R-module A, we are going to study the functor which takes its dual $A^\vee$ with respect the injective hull E of $\mathbf{k}$. Let's define formally what is the functor $\vee$.

Definition 2.1.1. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a local ring. Given an R-module M the Matlis dual of M is $M^\vee = Hom_R(M, E_R(\mathbf{k}))$. We write $(-)^\vee = Hom_R(-, E_R(\mathbf{k}))$, which is a contravariant exact functor form the category R-mod into itself.*

The concept of socle degree will be used in the following proposition.

Definition 2.1.2 (Socle degree). *If $(A, \mathfrak{m}, \mathbf{k})$ is an Artinian local k-algebra, there exists an integer s such that $\mathfrak{m}^s \neq 0$ and $\mathfrak{m}^{s+1} = 0$. The socle degree of A is that* *integer s and A is said to be s-level of type τ if*

$$Soc(A) := (0 :_a \mathfrak{m}) = \mathfrak{m}^s \cong Hom_R(\mathbf{k}, A) \text{ and } dim_k\, Soc(A) = \tau.$$

Example 2.1.3. Let $(R, \mathfrak{m})$ be a local ring and N be an R-module such that every element of N is killed by a power of $\mathfrak{m}$. Then $\mathrm{Soc}(N) \subseteq N$ is an essential extension. If $n \in N$ is a nonzero element, let t be the smallest integer such that $\mathfrak{m}^t n = 0$. Then $\mathfrak{m}^{t-1}n \subseteq \mathrm{Soc}(N)$ and $\mathfrak{m}^{t-1}n$ contains a nonzero multiple of n.

Definition 2.1.4. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring, and M a finite nonzero R-module of depth t. The number $t(M) = dim_{\mathbf{k}} Ext^t_R(\mathbf{k}, M)$ is called the type of M.*

Lemma 2.1.5. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring, M a finite R-module and x a maximal M-sequence. Then $t(M) = dim_{\mathbf{k}} Soc(M/xM)$.*

Proof. In order to proof the equivalences between both definitions of the type of M, let's recall the following lemma:

Lemma 2.1.6. *Let R be a ring, M, N be R-modules and $x = x_1, \ldots, x_n$ a M-sequence in* **Ann** *N. Then*

$$Hom_R(N, M/xM) \cong Ext^n_R(N, M).$$

The prove of the above lemma can be found at [BH98], Lemma 1.2.4.
Using the lemma, we have the following isomorphism:

$$\mathrm{Hom}_R(\mathbf{k}, M/xM) \cong \mathrm{Ext}^t_R(\mathbf{k}, M)$$

and by Definition 2.1.2, $\mathrm{Soc}(M/xM) \cong \mathrm{Hom}_R(\mathbf{k}, M)$. Therefore,

$$\dim_{\mathbf{k}} \mathrm{Ext}^t_R(\mathbf{k}, M) = \dim_{\mathbf{k}} \mathrm{Soc}(M/xM)$$

as we wanted to prove. □

Lemma 2.1.7. *Let R be a Noetherian ring, $\mathfrak{p} \in$* **Spec** *R, and A a finite R-module. Then $Ass\ A = Ass\ E(A)$, in particular $\{\mathfrak{p}\} = Ass\ E(R/\mathfrak{p})$ and $\mathbf{k}(\mathfrak{p}) \cong Hom_{R_{\mathfrak{p}}}(\mathbf{k}(\mathfrak{p}), E(R/\mathfrak{p})_{\mathfrak{p}})$.*

Proof. The proof can be found at [BH98], Lemma 3.2.7. □

Proposition 2.1.8. *Let $(R, \mathfrak{m}, \mathbf{k})$ be a Noetherian local ring, E the injective hull of $\mathbf{k}$, A an R-module of finite lenght ($\ell(A) < \infty$). Then:*

(a) $Ext^i_R(\mathbf{k}, E) = \begin{cases} \mathbf{k} & \text{for } i = 0 \\ 0 & \text{for } i > 0 \end{cases}$,

(b) $\ell(A) = \ell(A^\vee)$,

(c) The canonical homomorphism $A \to A^{\vee\vee}$ *is an isomorphism,*

(d) $\mu(A) := dim(A/\mathfrak{m}A) = t(A^\vee)$ *and* $t(A) = \mu(A^\vee)$,

(e) If R is Artinian, then E is a finite faithful R-module satisfying:

(i) $\ell(E) = \ell(R)$,

(ii) The canonical homomorphism

$$\begin{array}{ccc} R & \to & End_R(E) \\ a & \mapsto & \varphi_a \end{array},$$

where $\varphi_a(x) = ax$ *for all* $x \in E$*, is an isomorphism,*

(iii) $t(E) = 1$ *and* $\mu(E) = t(R)$.

Conversely, any finite faithful R-module of type 1 *is isomorphic to E.*

Remark 2.1.9. If the local ring of Proposition 2.1.8 is an Artinian ring, then equivalent to (b): $\ell_R(E_R(\mathbf{k})) = \ell_R(R) < \infty$.

Proof. (a) Using that E is injective, then $\mathrm{Ext}^i_R(\mathbf{k}, E) = 0$ for $i > 0$. Using Lemma 2.1.7, $\mathrm{Hom}_R(\mathbf{k}, E) \cong k$. Then, when $i = 0$, $\mathrm{Ext}^0_R(\mathbf{k}, E) = \mathbf{k}$.

(b) We are going to do induction on $\ell(A)$. If $\ell(A) = 1 = \ell(A/\mathfrak{m}) = \ell((A/\mathfrak{m})^\vee) = \ell(A^\vee)$. If $\ell(A) > 1$, it has some proper submodule $U \subsetneq A$ with which we can build an exact sequence as $0 \to U \to A \to W \to 0$ with $\ell(U) < \ell(A)$ and $\ell(W) < \ell(A)$. Using that E is injective, we have the following dual exact sequence; $0 \to W^\vee \to N^\vee \to A^\vee \to 0$. Then $\ell(A^\vee) = \ell(W^\vee) + \ell(U^\vee) = \ell(W) + \ell(U) = \ell(A)$ because we can apply the induction hypothesis to U and W.

(c) Once again, we are going to do induction on $\ell(A)$. If $\ell(A) = 1$, then $A \cong R/\mathfrak{m}$ and then $A^{\vee\vee} \cong (R/\mathfrak{m})^{\vee\vee} \cong (R/\mathfrak{m})^{\vee} \cong (R/\mathfrak{m}) \cong A$ using (a). Then, it is only remaind to prove that the canonical morphism

$$\alpha \colon R/\mathfrak{m} \to \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{m}, E), E)$$

is not the zero morphism.

Now, let $x \in E$, $x \neq 0$, be a socle element of E. Then, it exists $\varphi \in \mathrm{Hom}_R(R/\mathfrak{m}, E)$ with $x = \varphi(1) = \alpha(1)(\varphi)$.

Suppose that $\ell(A) > 1$. Then, we have again an exact sequence $0 \to U \to A \to W \to 0$ with $\ell(U) < \ell(A)$ and $\ell(A) > \ell(W)$. We can have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & A & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U^{\vee\vee} & \longrightarrow & A^{\vee\vee} & \longrightarrow & W^{\vee\vee} & \longrightarrow & 0.
\end{array}$$

By induction hypothesis, the outer vertical morphisms are isomorphism. Using the Snake Lemma, we get that $A \to A^{\vee\vee}$ is an isomorphism.

(d) The module $(A/\mathfrak{m}A)^{\vee}$ is the kernel of the linear map $A^{\vee} \to (\mathfrak{m}A)^{\vee}$ which assigns to every $\varphi \in A^{\vee}$ its restrictions to $\mathfrak{m}A$. Then $\varphi \in (A/\mathfrak{m}A)^{\vee}$ if and only if $\mathfrak{m}\varphi(A) = \varphi(\mathfrak{m}A) = 0$. In other words,

$$(A/\mathfrak{m}A)^{\vee} = \{\varphi \in A^{\vee} \colon \mathfrak{m} \cdot \varphi = 0\} = \mathrm{Soc}\, A^{\vee}$$

Then, we get $\mu(A) = \dim_k A/\mathfrak{m}A = \dim_k (A/\mathfrak{m}A)^{\vee} = \dim_k \mathrm{Soc}\, A^{\vee} = t(A^{\vee})$.

The second equality follows from the first by (c).

(e) We have seen that $\ell(E) = \ell(R^{\vee}) = \ell(R) < \infty$, by (b). In particular, E is a finite R-module. Using (c), the canonical homomorphism $\alpha \colon R \to \mathrm{Hom}_R(\mathrm{Hom}_R(R, E), E)$ is an isomorphism. If we identify $\mathrm{Hom}_R(R, E)$ with E, α identifies with the canonical homomorphism $R \to \mathrm{End}_R(E)$. A module whose endomorphism ring is R is necesarrily faithful. Lastly, $\mu(E) = \mu(R^{\vee}) = t(R)$ and $t(E) = t(R^{\vee}) = \mu(R) = 1$ are consequences of (d).

The other way around, if A is a faithful finitely generated R-module of type 1, then $\mu(A^{\vee}) = 1$. In other words, $A^{\vee} \cong R/I$ for some ideal I.

Therefore, $A \cong A^{\vee\vee} \cong \mathrm{Hom}_R(R/I, E)$. Using that A is faithful, $I = 0$ and $A \cong R^{\vee} \cong E$.

□

Remark 2.1.10. This previous proposition may be seen as the Matlis duality theorem for finite Artinian modules.

In order to prove a more general theorem, some notation should be established and a lemma must be recalled.

- R-mod.Noeth: the category of Noetherian R-modules.
- R-mod.Artin: the category of Artinian R-modules.

Lemma 2.1.11. *Let $(R, \mathfrak{a}, \mathbf{k})$ be a complete Noetherian local ring and $E = E_R(\mathbf{k})$. Then*

(a) $R^{\vee} \cong E$ and $E^{\vee} \cong R$.

(b) For every R-module A the natural map $A \to A^{\vee\vee}$ induce isomorphisms $R \to R^{\vee\vee}$ and $E \to E^{\vee\vee}$.

Proof. (a) It is well known that $R^{\vee} = \mathrm{Hom}_R(R, E) \cong E$. Let's prove $E^{\vee} \cong R$. Assume that R is Artinian. Consider the map $\theta \colon R \to E^{\vee} = \mathrm{Hom}_R(E, E)$ which sends an element $r \in R$ to the homothety defined by r. Since $\ell(R) = \ell(E^{\vee})$, by Proposition 2.1.8, it is left to prove that θ is injective. Suppose that $rE = 0$. Then, by Corollary 1.3.2, $E_{R/(r)}(\mathbf{k}) = (0 :_E (r)) = E$ and, by the same argument, $\ell(E) = \ell(R/(r))$. This implies that $\ell(R) = \ell(R/(r))$, then $r = 0$.

Assume now that R is Noetherian and complete. We consider the map $\theta \colon R \to E^{\vee} = \mathrm{Hom}_R(E, E)$ as above, we wil prove that θ is an isomorphism. Let's write $R_t = R/\mathfrak{m}^t$ for each t. By Corollary 1.3.2, $E_t := E_{R_t}(\mathbf{k}) = (0 :_E \mathfrak{m}^t)$. Let $\varphi \in \mathrm{Hom}_R(E, E) = E^{\vee}$. It is clear that $\varphi(E_t) \subset E_t$ and thus $\varphi \in \mathrm{Hom}_{R_t}(E_t, E_t)$. Since R_t is Artinian, we have φ is a homothety defined by an element $r_t \in R_t$. The fact $E_t \subset E_{t+1}$ implies that $r_t - r_{t+1} \in \mathfrak{m}^t$ for all $t \geq 1$. In consequence, $r = (r_t)_t \in R$ and $r_t = r + \mathfrak{m}^t$ for all $t \geq 1$. We claim that φ is given by multiplication by r. This follows from the fact that $E = \cup_t E_t$ and that $\varphi(e) = r_t e$ for all $e \in E_t$. Moreover, r is uniquely determined by φ, and we conclude that φ is bijective.

(b) We consider the natural homomorphism

$$\gamma\colon M \to M^{\vee\vee} = \operatorname{Hom}_R(\operatorname{Hom}_R(M,E),E)$$

given by $\gamma(m)(\varphi) = \varphi(m)$. First, we prove that $\gamma\colon R \to R^{\vee\vee}$ is an isomorphism. This map is the composition of the two isomorphisms given in part (a), $R \cong E^{\vee} \cong (R^{\vee})^{\vee}$. In fact, if $r \in R$, the map $R \cong E^{\vee}$ sends r to multiplication by r, $h_r\colon E \to E$. Now the map $E^{\vee} \cong (R^{\vee})^{\vee}$ sends h_r to α_r defined by $\alpha_r(\varphi) = h_r(\varphi(1)) = \varphi(r)$, so $\alpha_r = \gamma(r)$. The case of E is analogous to this one.

□

Theorem 2.1.12 (Matlis duality)**.** *Let $(R, \mathfrak{a}, \mathbf{k})$ be a complete Noetherian local ring, $E = E_R(\mathbf{k})$ and let A be a R-module. Then:*

(a) If A is Noetherian, then $A^{\vee}$ is Artinian.

(b) If A is Artinian, then $A^{\vee}$ is Noetherian.

(c) If A is either Noetherian or Artinian, then $A^{\vee\vee} \cong A$.

(d) The functor $(-)^{\vee}$ is a contravariant, additive and exact functor.

(e) The functor $(-)^{\vee}$ is an anti-equivalence between R-mod.Noeth and R-mod.Artin (between R-mod.Artin and R-mod.Noeth too). It holds $(-)^{\vee} \circ (-)^{\vee}$ is the identity functor of R-mod.Noeth (resp. R-mod.Artin).

Proof. (a) Let's consider a presentation of A

$$R^m \to R^n \to A \to 0.$$

We know that $(-)^{\vee}$ is exact, therefore

$$0 \to A^{\vee} \to (R^n)^{\vee} \to (R^m)^{\vee}$$

Then, $A^{\vee}$ can be seen as a submodule of $(R^n)^{\vee} \cong (R^{\vee})^n \cong E^n$, by the Lemma 2.1.11. Since E is Artinian, E^n and $A^{\vee}$ are also Artinian. Applying the functor $(-)^{\vee}$ again, we get

$$\begin{tikzcd}
(R^m)^{\vee\vee} \arrow[r] & (R^n)^{\vee\vee} \arrow[r] & A^{\vee\vee} \arrow[r] & 0 \\
R^m \arrow[r] \arrow[u] & R^n \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

whose rows are exact. We have seen that $R \to R^{\vee\vee}$ is an isomorphism at Proposition 2.1.8, then $A \cong A^{\vee\vee}$. Therefore, we have proved (c) when A is Noetherian.

(b) We have seen that $A \hookrightarrow E^n$ for some $n \in \mathbb{N}$. E is Artinian, so is E^n/A. Then $E^n/A \hookrightarrow E^m$ for some $m \in \mathbb{N}$. In consequence, we have the following exact sequence

$$0 \to A \to E^n \to E^m$$

As we have been doing, we apply now $(-)^\vee$:

$$(E^m)^\vee \to (E^n)^\vee \to A^\vee \to 0$$

and $A^\vee$ can be seen as a quotient of $(E^n)^\vee \cong (E^\vee)^n \cong (R^n)$. This implies that $A^\vee$ is Noetherian.

Following the steps done in the Noetherian case, we apply $(-)^\vee$ to the last exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A^{\vee\vee} \arrow[r] & (E^n)^{\vee\vee} \arrow[r] & (E^m)^{\vee\vee} \\
0 \arrow[r] & A \arrow[r] \arrow[u] & E^n \arrow[r] \arrow[u] & E^m \arrow[u]
\end{tikzcd}$$

Again, $E \to E^{\vee\vee}$ is an isomorphism, therefore $A \cong A^{\vee\vee}$.

(c) Seen in the previous statements.

(d) This is a consequence of the previous statements.

□

2.2 Macaulay's correspondence

As we have seen, Matlis duality is defined for Noetherian local ring R. One of the consequences of this result is Macaulay's duality. This particular case was seen after Matlis and it is really helpful in order to understand a "simpler" case of Matlis duality.
In this section, we will see an example of how to build a submodule generated by two polynomials over R using contraction, which will be defined.

Let $\mathbf{k}$ an arbitrary field. Let $R = \mathbf{k}[\![x_1, \ldots, x_n]\!]$ be the ring of the formal series with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ and let $S = \mathbf{k}[y_1, \ldots, y_n]$ a polynomial ring. We denote by $\mathfrak{m} = (y_1, \ldots, y_n)$ the homogeneous maximal ideal of S.
We can find a relation between S and R. R is an S-module with the standard product. But S can be considered as R-module by derivation or by contraction.
If $\mathrm{char}(\mathbf{k}) = 0$, we define by **derivation** the R-module structure of S by

$$\begin{array}{rcl} R \times S & \to & S \\ (x^\alpha, y^\beta) & \mapsto & x^\alpha \circ y^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha} & \beta \geq \alpha \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{2.1}$$

where $\forall \alpha, \beta \in \mathbb{N}^n$, $\alpha! = \prod_{i=1}^n \alpha_i!$.
If $\mathrm{char}(\mathbf{k}) \geq 0$, the R-module structure of S is defined by **contraction**,

$$\begin{array}{rcl} R \times S & \to & S \\ (x^\alpha, y^\beta) & \mapsto & x^\alpha \circ y^\beta = \begin{cases} y^{\beta-\alpha} & \beta \geq \alpha \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{2.2}$$

with $\alpha, \beta \in \mathbb{N}^n$.

Proposition 2.2.1. *Let $\mathbf{k}$ be a field. There is a R-module homomorphism*

$$\begin{array}{rccc} \sigma\colon & (S, der) & \to & (S, cont) \\ & y^\alpha & \mapsto & \alpha! y^\alpha \end{array}$$

If $char(\mathbf{k}) = 0$, σ is an isomorphism of R-modules.

Proof. First, let's prove that this homomorphism exists. It is only required to show that $\sigma(x^\alpha \circ y^\beta) = x^\alpha \circ \sigma(y^\beta)$. This is:

$$\begin{array}{rcl} \sigma(x^\alpha \circ y^\beta) & \underset{(2.1)}{=} & \sigma\left(\frac{\beta!}{(\beta-\alpha)!} y^{\beta-\alpha}\right) = \frac{\beta!}{(\beta-\alpha)!}(\beta-\alpha)! y^{\alpha-\beta} \\ & = & \beta! y^{\beta-\alpha} \underset{(2.2)}{=} x^\alpha \circ \sigma(y^\beta) \end{array}$$

If $\mathrm{char}(\mathbf{k}) = 0$, then the inverse of σ is $y^\alpha \to (1/\alpha!) y^\alpha$. Therefore, we have an isomorphism. $\square$

Given a family of polynomials $F_j, j \in J$, we denote by $\langle F_j, j \in J\rangle$ the submodule of S generated by F_j, $j \in J$. In other words, it is the $\mathbf{k}$-vector subspace of S generated by $x^\alpha \circ F_j, j \in J, \alpha \in \mathbb{N}^n$.
Let's see an example of how we can compute it.

Example 2.2.2. Consider $R = \mathbf{k}[\![x,y]\!]$, $f = x^3, g = y^3$. Consider $\operatorname{char}(\mathbf{k}) \geq 0$. Then (by contraction or derivation)

$$\begin{aligned}\langle x^3, y^3\rangle_R &= \langle x^0 \circ x^3, y^0 \circ y^3, x^1 \circ x^3, xy^1 \circ y^3, x^2 \circ x^3, y^2 \circ y^3, \\ &\quad\; x^3 \circ x^3, y^3 \circ y^3\rangle_{\mathbf{k}} \\ &\underset{(2.2)}{=} \langle x^3, y^3, x^2, y^2, x, y, 1\rangle_{\mathbf{k}}\end{aligned}$$

Example 2.2.3. Now, we will see that computing $\langle -, -\rangle_R$ is different if we compute it by derivation or contraction. Consider again $R = \mathbf{k}[\![x,y]\!]$, $f = x^3$, $g = y^2 + x^2y$. First, let's compute $\langle f, g\rangle$ by derivation.

$$\begin{aligned}\langle f, g\rangle &= \langle x^0 \circ x^3, x^1 \circ x^3, x^2 \circ x^3, x^3 \circ x^3, \\ &\quad\; x^0 \circ (y^2 + x^2y), x^1 \circ (y^2 + x^2y), x^2 \circ (y^2 + x^2y), \\ &\quad\; y^0 \circ (y^2 + x^2y), y^1 \circ (y^2 + x^2y), y^2 \circ (y^2 + x^2y), \\ &\quad\; xy \circ (y^2 + x^2y), x^2y \circ (y^2 + x^2y)\rangle \\ &= \langle x^3, 3x^2, 3!x, 3!, y^2 + x^2y, 2xy, 2y, y^2 + x^2y, 2y + x^2, 2!, 2x, 2!\rangle \\ &= \langle x^3, 3x^2, 3!x, 3!, 2xy, 2y, y^2 + x^2y, 2y + x^2, 2x, 2!\rangle \\ &= \langle x^3, 3x^2, 6x, 2xy, 2y, y^2 + x^2y, 2y + x^2, 2x, 1\rangle\end{aligned}$$

Now, by contraction.

$$\begin{aligned}\langle f, g\rangle &= \langle x^0 \circ x^3, x^1 \circ x^3, x^2 \circ x^3, x^3 \circ x^3, \\ &\quad\; x^0 \circ (y^2 + x^2y), x^1 \circ (y^2 + x^2y), x^2 \circ (y^2 + x^2y), \\ &\quad\; y^0 \circ (y^2 + x^2y), y^1 \circ (y^2 + x^2y), y^2 \circ (y^2 + x^2y), \\ &\quad\; xy \circ (y^2 + x^2y), x^2y \circ (y^2 + x^2y)\rangle \\ &= \langle x^3, x^2, x, 1, y^2 + x^2y, xy, y, y^2 + x^2y, y + x^2, 1, x, 1\rangle \\ &= \langle x^3, x^2, x, 1, xy, 2y, y^2 + x^2y, 1y + x^2, x, 1\rangle \\ &= \langle x^3, x^2, x, xy, y, y^2 + x^2y, y + x^2, x, 1\rangle\end{aligned}$$

Now, we can compute the injective hull of the residue field of a power series ring.

Theorem 2.2.4. *Let $R = \mathbf{k}[\![x_1, \ldots, x_n]\!]$ be the n-dimensional power series ring over a field $\mathbf{k}$. Then, if $char(\mathbf{k}) = 0$,*

$$E_R(\mathbf{k}) \cong (S, der) \cong (S, cont).$$

If char$(\mathbf{k}) > 0$,

$$E_R(\mathbf{k}) \cong (S, cont)$$

Proof. In order to simplify notation, $E_R(\mathbf{k}) = E$. From Corollary 1.3.2, we get $E = \cup_{i\geq 0} E_{R/\mathfrak{m}_R^i}(\mathbf{k})$. Now, the problem is reduced to the computation of $E_{R/\mathfrak{m}_R^i}(\mathbf{k}) \subset E$.
Notice that $S_{\leq i-1} := \{f \in S \mid \deg(f) \leq i-1\} \subset S$ is an sub-R-module of S with respect derivation or contraction and $S_{\leq i-1}$ is annihilated by m_R^i. Then, $S_{\leq i-1}$ is an $R/\mathfrak{m}_R^i$-module. The extension $\mathbf{k} \subset S_{\leq i-1}$ is essential despite the char$(\mathbf{k})$. By Proposition 1.1.17, there is $L \cong E_{R/\mathfrak{m}_R^i}(\mathbf{k})$ such that $\mathbf{k} \subset S_{\leq i-1} \subset L \cong E_{R/\mathfrak{m}_R^i}(\mathbf{k})$. Now, by Remark 2.1.9,

$$\ell_{R/\mathfrak{m}_R^i}(E_{R/\mathfrak{m}_R^i}(\mathbf{k})) = \ell_{R/\mathfrak{m}_R^i}(R/\mathfrak{m}_R^i) = \ell_{R/\mathfrak{m}_R^i}(S_{\leq i-1})$$

Then, $S_{\leq i-1} \cong E_{R/\mathfrak{m}_R^i}(\mathbf{k})$. Hence $E_R(\mathbf{k}) \cong \cup_{i\geq 0} S_{\leq i-1} = S$. □

Let $I \subset R$ be an ideal, then $(R/I)^\vee$ is the sub R-module of S with the notation $I^\perp$ and $I^\perp = \{g \in S \mid I \circ g = 0\}$. This is the **Macaulay's inverse system of** I.
Let A be a sub-R-module of S, then the dual $A^\vee$ is an ideal of R that we already denote by $(S/A)^\perp$,

$$A^\perp = \{f \in R \mid f \circ g = 0 \text{ for all } g \in A\}.$$

$A^\perp$ can be denoted as $\mathrm{Ann}_R(A)$.
As a particular case of Matlis duality (Proposition 2.1.12) there is the following proposition.

Proposition 2.2.5 (Macaulay's duality)**.** *Let $R = \mathbf{k}[\![x_1, \ldots, x_n]\!]$ be the n-dimensional power series ring over a field $\mathbf{k}$. Let $S = \mathbf{k}[y_1, \ldots, y_n]$. If A is a submodule of S, then $A^\perp = (0 :_R A)$ and $I^\perp = (0 :_S I)$ for an ideal $I \subset R$.*

Let's do an example in order to see how Macaulay's duality works.

Example 2.2.6. Let $F = y^3 + xy + x^2 \in R = \mathbf{k}[\![x, y]\!]$ be a polynomial. Now, we consider R-module structure of $S = \mathbf{k}[x, y]$ defined by contraction $\circ$.
First, let's build $\langle F \rangle$. As it is said before, we should construct it by contraction. Then,

$$\begin{aligned}
\langle F \rangle &= \langle x^0 \circ F, x^1 \circ F, x^2 \circ F, xy \circ F, y^0 \circ F, y \circ F, y^2 \circ F, y^3 \circ F \rangle \\
&= \langle F, y + x, 1, 1, F, y^2 + x, y, 1 \rangle \\
&= \langle F, y^2 + x, y + x, y, 1 \rangle
\end{aligned}$$

By definition, $\langle F\rangle^{\perp} = \{f \in R | f \circ g = 0 \; \forall g \in \langle F\rangle\}$. By Macaulay's duality, this will be equal to $I \subset R$.
Then, we have that $I = \mathbf{Ann}_R(\langle F\rangle) = (xy - y^3, x^2 - xy)$. Let's check it:

$$\begin{aligned} xy - y^3 \circ F &= 1 - 1 = 0 \\ x^2 - xy \circ F &= 1 - 1 = 0 \end{aligned}$$

Example 2.2.7. Now, a similar example but with $n = 3$. Let $F = z^2 + y^2 + xz + x^3 \in R = \mathbf{k}[\![x,y,z]\!]$. We consider R-module structure of $S = \mathbf{k}[x,y,z]$ defined by contraction $\circ$. Now,

$$\begin{aligned} \langle F\rangle &= \langle x^0 \circ F, x \circ F, x^2 \circ F, x^3 \circ F, y^0 \circ F, y \circ F, y^2 \circ F, \\ &\quad xz \circ F, z^0 \circ F, z \circ F, z^2 \circ F\rangle \\ &= \langle F, z + x^2, x, 1, F, y, 1, 1, F, z, 1\rangle \\ &= \langle F, z + x^2, x, y, z, 1\rangle \end{aligned}$$

Then, $I = \mathbf{Ann}_R(\langle F\rangle) = (xz - x^3, xz - y^2, xz - z^2)$.

Let's study the Hilbert function.

Definition 2.2.8. *Let $A = R/I$ be an Artin quotient of R, we denote $\mathfrak{n} = \mathfrak{m}/I$. The Hilbert function of $A = R/I$ is*

$$HF_A(i) = dim_{\mathbf{k}}\left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right).$$

Once we have defined $HF_A(j)$, we can define the h**- vector** of A. It is the sequence $(h_0, \ldots, h_s)$, where s is the socle degree of A and $h_j = HF_A(j)$ for each $j = 1, \ldots, s$.
The multiplicity of A is the integer $e(A) := \dim_{\mathbf{k}}(A) := \dim_{\mathbf{k}} I^{\perp}$ using Macaulay's duality. Notice that $s(A)$ is the last integer such that $\mathrm{HF}_A(i) \neq 0$ and that $e(A) = \sum_{i=0}^{s} \mathrm{HF}_A(i)$. The embedding dimension of A is $\mathrm{HF}_A(1)$. The associated graded ring to A is the standard graded $\mathbf{k}$-algebra ring $\mathrm{gr}_{\mathfrak{n}}(A) = \bigoplus_{i\geq 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}$. Notice that the Hilbert function of A and its associated graded ring $\mathrm{gr}_{\mathfrak{n}}(A)$ agree. We denote by I^* the homogeneous ideal of S generated by the initial forms of the elements I. It is well known that $\mathrm{gr}_{\mathfrak{n}}(A) \cong S/I^*$ as graded $\mathbf{k}$-algebras, in particular $\mathrm{gr}_{\mathfrak{n}}(A_i) \cong (S/I^*)_i$ for all $i \geq 0$.
The $\mathbf{k}$-vector space of polynomials of S of degree less or equal to i is denoted by $S_{\leq i}$ and we consider the following $\mathbf{k}$-vector space

$$\left(I^{\perp}\right)_i := \frac{I^{\perp} \cap S_{\leq i} + S_{<i}}{S_{<i}}. \tag{2.3}$$

Proposition 2.2.9. *For all $i \geq 0$ it holds*

$$HF_A(i) = dim_{\mathbf{k}} \left(I^{\perp}\right)_i.$$

Proof. Let's consider the following natural exact sequence of R-modules

$$0 \longrightarrow \frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}} \longrightarrow \frac{A}{\mathfrak{n}^{i+1}} \longrightarrow \frac{A}{\mathfrak{n}^i} \longrightarrow 0.$$

Dualizing this sequence we get

$$0 \to (I + \mathfrak{m}^i)^{\perp} \to (I + \mathfrak{m}^{i+1})^{\perp} \to \left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right)^{\vee} \to 0.$$

We get the following sequence of $\mathbf{k}$-vector spaces:

$$\left(\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}\right)^{\vee} \cong \frac{(I + \mathfrak{m}^{i+1})^{\perp}}{(I + \mathfrak{m}^i)^{\perp}} = \frac{I^{\perp} \cap S_{\leq i}}{I^{\perp} \cap S_{\leq i-1}} \cong \frac{I^{\perp} \cap S_{\leq i} + S_{<i}}{S_{<i}}.$$

From Proposition 2.1.8 we get the claim. □

Now, we are going to consider the following map:

$$\begin{array}{rcccc} \langle | \rangle & : & R \times S & \to & \mathbf{k} \\ & & (F, G) & \mapsto & (F \circ G)(0) \end{array} \tag{2.4}$$

We denote by I^* the homogeneous ideal of S generated by the initial forms of the element I and it is called initial ideal of I.

Proposition 2.2.10. *(a) $\langle | \rangle$ is a bilinear non-degenerate map of $\mathbf{k}$-vector spaces.*

(b) If I is an ideal of R, then

$$I^{\perp} = \{G \in S | \langle I | G \rangle = 0\}.$$

(c) $\langle | \rangle$ induces a bilinear non-degenerate map of $\mathbf{k}$- vector spaces

$$\overline{\{|\}} \colon \frac{R}{I} \times I^{\perp} \to \mathbf{k}.$$

(d) We have an isomorphism of $\mathbf{k}$-vector spaces:

$$\left(\frac{S}{I^*}\right)_i \cong (I^{\perp})_i$$

for all $i \geq 0$.

In order to denote the duality defined by exact pairing $\overline{\{|\}}$, we will use $*$. Notice that $(R/I)^* \cong I^\perp$. If $\underline{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ is a integer n-pla we denote by $\delta_{\underline{i}}(G)$, $G \in S$, the derivative of G with respect to $y_1^{i_1} \cdots y_n^{i_n}$, $\delta_{\underline{i}}(G) = (x_1^{i_1} \cdots x_n^{i_n}) \circ G$.
Let $\Omega = \{\omega_i\}$ be the canonical basis of $R/\mathfrak{m}^{s+1}$ as a $\mathbf{k}$- vector space consisting of the standard monomials x^α ordered by the deg-lex order with $x_1 > \cdots > x_n$ and, then the dual basis with respect to $*$ is the basis $\Omega^* = \{\omega_i^*\}$ of $S_{\leq j}$ where

$$(x^\alpha)^* = \frac{1}{\alpha!} y^\alpha.$$

In fact $\omega_j \circ \omega_i^* = \overline{\{\omega_j | \omega_i\}} = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

Example 2.2.11. The aim of this example is to compute the Hilbert formula via (2.3). Let's use the examples Example 2.2.6 and Example 2.2.7.
Starting with Example 2.2.6, we got

$$\begin{aligned} I^\perp = \langle F \rangle &= \langle x^0 \circ F, x^1 \circ F, x^2 \circ F, xy \circ F, y^0 \circ F, y \circ F, y^2 \circ F, y^3 \circ F \rangle \\ &= \langle F, y + x, 1, 1, F, y^2 + x, y, 1 \rangle \\ &= \langle F, y^2 + x, y + x, y, 1 \rangle \end{aligned}$$

and $I = (xy - y^3, x^2 - xy)$. In order to compute HF, let's use (2.3) and counting the elements of differents degree in $I^\perp$, we get $\mathrm{HF}_A = \{1, 2, 1, 1\}$. Let's study in more detail this vector.
The element $\mathrm{HF}_A(0) = 1$ because we have one element of degree 0 in $I^\perp$. If we look at the elements of degree 1 in $I^\perp$, we get 2 ($x + y$ and y), then $\mathrm{HF}_A(1) = 2$. By the same argument, $\mathrm{HF}_A(2) = 1$ and $\mathrm{HF}_A(3) = 1$.
Now, in order to compute $e(A)$, we only have to count how many elements are in $I^\perp$. Therefore, $e(A) = 5$. Last but not least, in order to compute $s(A)$ (the socle degree), we only have to count the elements of HF vector minus 1. Then, $s(A) = 3$.
Now, with Example 2.2.7. With the same argument as before, we have $\mathrm{HF_A} = \{1, 2, 3, 1\}$, $e(A) = 7$ and $s(A) = 3$.

Chapter 3

Gorenstein, level and compressed algebras

In this chapter, we study Gorenstein rings, level and compressed algebras and some results of them.

First, we focus on the case of Gorenstein rings and we use all the previous sections in order to announce some of the theorems and propositions. We study the relation between Artin and Gorenstein rings.

Gorenstein rings were introduced by Grothendiech in his 1961 seminar which was published by Harsthorne [Har67] in 1967. The duality property of singular plane curves was studied by Gorenstein [Gor52] in 1952 and this is why those ring have this name. The zero-dimensional case had been studied by Macaulay [Mac34] in 1934. The concept of Gorenstein rings was publicized by Serre [Ser61] in 1961 and Bass [Bas63] in 1963.

In the second section, we study level rings and we talk about Irrabino's Q-decomposition of the associated graded ring of an Artinian s-level local $\mathbf{k}$-algebra.

In the next section, we study how to achieve isomorphism classes of local algebras using Macaulay's inverse system. We reach an important result, that is that an isomorphism between two Artinian s-level algebras is defined by a matrix.

Lastly, we finish the chapter studying graded compressed level local algebras. Here, we study the different relations between all those concepts.

In order to end the section, we focus on some results about h-vectors and Artinian 3-level local algebras.

3.1 Gorenstein rings

Definition 3.1.1 (Gorenstein ring). *A Noetherian local ring R is a* ***Gorenstein ring*** *if $id_R R < \infty$. A Noetherian ring is a Gorenstein ring if its localization at every maximal ideal is a Gorenstein local ring.*

Proposition 3.1.2. *Let R be a Noetherian ring. Suppose x is an R-regular sequence. If R is Gorenstein, then so is $R/(x)$. If R is local, the converse holds too.*

Proof. It is a consequence of Corollary 1.2.9. □

Proposition 3.1.3. *Let $(R, \mathfrak{a}, \mathbf{k})$ be a Noetherian local ring. Then,*

$$R \text{ is regular } \Rightarrow R \text{ is a complete intersection}$$

$$\Rightarrow R \text{ is Gorenstein} \Rightarrow R \text{ is Cohen-Macaulay}$$

Proof. It's trivial to see that R regular implies that R is a complete intersection. It is a direct consequence from Definition 0.0.5 and Definition 0.0.12.
From the definition of regular, the projective dimension of an R-module is finite, therefore $\mathrm{Ext}^i_R(\mathbf{k}, R) = 0$ for some $i \gg 0$. Then, $\mathrm{Ext}^{i-1}_R(\mathbf{k}, R) \neq 0$ and, from Proposition 1.2.8, the injective dimension is finite. Hence, from Definition 3.1.1, R is Gorenstein.
We have seen that every regular ring is Gorenstein. From Proposition 3.1.2, every $R/(x)$ Gorenstein implies that R is Gorenstein too. Therefore, every complete intersection is a Gorenstein ring.
From Lemma 1.4.9 and Theorem 1.4.11, Gorenstein implies Cohen-Macaulay. □

There is also a relation between Gorenstein rings and Bass numbers.

Theorem 3.1.4. *Let R be a Noetherian local ring, $n = \dim R$. The following conditions are equivalent:*

(a) R is Gorenstein,

(b) $\mu_i(\mathfrak{m}) = \begin{cases} 0 & i < n \\ 1 & i = n \end{cases},$

(c) R_P *is Gorenstein for all* $P \in$ *Spec* R,

(d) $\mu_i(P) = \begin{cases} 0 & i \neq h(P) \\ 1 & i = h(P) \end{cases}$ *for all* $P \in$ *Spec* R,

(e) $\mu_i(\mathfrak{m}) = 0$ *for some* $i > n$.

Proof. The proof can be found at [Sil81b], 2.31. □

Theorem 3.1.5. *Let* $(R, \mathfrak{m}, \mathbf{k})$ *be a Noetherian local ring. The following conditions are equivalent:*

(a) R *is a Gorenstein ring,*

(b) R *is a Cohen-Macaulay ring of type 1.*

Proof. Let x be a maximal R-sequence. By Proposition 3.1.2, R is Gorenstein if and only if $R/(x)$ is. Hence, we may assume that R is Artin. By Lemma 2.1.6, we know that is equivalent to study R or $R/(x)$.
Let R be a Gorenstein Artin ring. We want to prove that R is of type 1. R being Gorenstein implies $\mathrm{id}_R < +\infty$ and by Theorem 1.4.12, $\mathrm{id}_R R =$ depth $R = 0$. This is because, as a consequence of R being Artin, depth $R = 0$. Therefore, we get that R is injective as R-module by Remark 1.2.7. Since R is local, it is indecomposable as an R-module. Then Ann $R = \{\mathfrak{m}\}$ (because we are in a local ring), and we have $R \cong E_R(\mathbf{k})$ from Theorem 1.1.20. From Lemma 2.1.7, R is from type 1. This is because we have to compute the $\dim_R \mathbf{k}$, which is equal to 1.
Now, let R be Artin of type 1. From Proposition 2.1.8, (e), we get that R is Gorenstein. The reason is that R is isomorphic to E and E is the injective hull of $\mathbf{k}$ which has finite injective dimension. Therefore, R is Gorenstein. □

In the following two propositions, Proposition 3.1.6 and Proposition 3.1.7, given an R-module A, we denote by $\mu(A)$ the minimal number of generators of A. It is not the notation used in Proposition 2.1.8.

Proposition 3.1.6. *Let $A = R/I$ be an Artinian local ring. Then*

$$Soc(A)^\vee = \frac{I^\perp}{\mathfrak{m} \circ I^\perp}$$

In particular, the Cohen-Macaulay type of A is

$$t(A) = dim_{\mathbf{k}}(I^\perp / \mathfrak{m} \circ I^\perp) = \mu_R(I^\perp)$$

Proof. Consider an exact sequence of R-modules

$$0 \to \mathrm{Soc}(A) = (0 :_A \mathfrak{n}) \to A \stackrel{(x_1,\dots,x_n)}{\longrightarrow} A^n,$$

If we take the dual, then

$$(I^\perp)^n \stackrel{\sigma}{\to} I^\perp \to \mathrm{Soc}(A)^\vee \to 0$$

where $\sigma(f_1, \dots, f_n) = \sum_{i=1}^n x_i \circ f_i$. Hence

$$\mathrm{Soc}(A)^\vee = \frac{I^\perp}{(x_1, \dots, x_n) \circ I^\perp} = \frac{I^\perp}{m \circ I^\perp}.$$

From Lemma 2.1.5 $t(A) = \dim_{\mathbf{k}}(\mathrm{Soc}(A))$ and from Lemma 2.1.8 (b), this is equal to $\dim_{\mathbf{k}}(\mathrm{Soc}(A)^\vee) = \mu_R(I^\perp)$. □

Proposition 3.1.7. *Let I be an $\mathfrak{m}$-primary ideal of R. Then, $A = R/I$ is Gorenstein of socle degree s if and only if $I^\perp$ is a cyclic R-module generated by a polynomial of degree s.*

Proof. First, let's assume A is Gorenstein of socle degree s. Then $\mathrm{Soc}(A) = \mathfrak{n}^s = \mathfrak{m}^s + I/I$ and

$$\mathrm{Soc}(A)^\vee = \frac{I^\perp}{I^\perp \cap S_{\leq s-1}}.$$

It is easy to understand how this equality works if one looks at the proof of Proposition 2.2.9. From Theorem 3.1.5, $t(A) = 1$ and from Proposition 3.1.6, we obtain $\mu(I^\perp) = 1$. Therefore, $I^\perp$ is generated by one polynomial. If it is generated by one polynomial, R-module is cyclic. By Proposition 3.1.6,

$$\mathfrak{m} \circ I^\perp = I^\perp \cap S_{\leq s-1}. \tag{3.1}$$

Let F be a polynomial such that $0 \neq \overline{F} \in \frac{I^{\perp}}{m \circ I^{\perp}}$. Then, by (3.1), $\overline{F} \in \frac{I^{\perp}}{I^{\perp} \cap S_{\leq s-1}}$ and therefore, deg $F \geq s$. We also now that $\operatorname{Soc}(A) = s$, therefore $\operatorname{HF}_A(s) \neq 0$ and $\operatorname{HF}_A(s+1) = 0$. This implies that deg $F \leq s$. Then, deg $F = s$.
Now, assume that $I^{\perp}$ is a cyclic R-module generated by a polynomial of degree s. Hence, F_1 is the minimal system of generators of $I^{\perp}$, $t(A) = 1$. By Theorem 3.1.5, A is Gorenstein. □

Example 3.1.8. We are going to do an example in order to see one implication of Proposition 3.1.7.
Let $A = R/I$ be Artin with $R = \mathbf{k}[\![x]\!]$. Then, any ideal of R is of the form $I = (x^n), n \geq 1$.
Now, let's compute its socle degree. We know, by definition, that the socle degree is the maximum integer j such that $\mathfrak{n}^j \neq 0$, where $\mathfrak{n} = \mathfrak{m}/I$ and $\mathfrak{m}$ is a maximal ideal of R.
In this case, $j = n-1$ because $n-1$ is the greatest exponent such that $\mathfrak{n}^j \neq 0$.
Consider $\mathbf{k}$ a field with $\operatorname{char}(\mathbf{k}) \geq 0$. We are going to compute $I^{\perp}$ by contraction. Therefore, we must compute all $g \in S$ such that $I \circ g = 0$. But, by definition of contraction, in order to have $I \circ g = 0$, the only possible g is equal to y^{β}, with $\beta < n$. Consequently,

$$I^{\perp} = \langle y^{n-1} \rangle$$

which is cyclic and hence $A = \mathbf{k}[\![x]\!]/(x^n)$ is Gorenstein.

Proposition 3.1.9. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n and let $S = R/I$ be a quotient of R with dimension d. Let*

$$0 \longrightarrow F_k \xrightarrow{\alpha_k} F_{k-1} \xrightarrow{\alpha_{k-1}} \dots \longrightarrow F_0 \longrightarrow S \longrightarrow 0$$

be a minimal free R-resolution of S. Then R/I is Gorenstein iff $k = n-d$ and $F_k \cong R$.

Proof. If R is a regular local ring and M a finitely generated module, then

$$\operatorname{depth}(M) + \operatorname{pd}_R(M) = \dim(R),$$

where $\operatorname{pd}_R(M)$ is the projective dimension of M. This is the well-known Auslander-Buchsbaum's Formula.

Using this formula with $S = M$, we obtain that S is Cohen-Macaulay if and only if $\operatorname{depth}(S) = \dim(S)$ by Definition 0.0.3. This happens if and only if $k = n - d$.
The module $\operatorname{Ext}_R^{n-d}(S, R)$ can be computed from the free R-resolution of S. We simply apply $\operatorname{Hom}_R(-, R)$ to the resolution and take homology. The $n - d$ homology is the cokernel of the transpose of α_k from $F_{k-1}^* \to F_k^*$, where $(-)^* = \operatorname{Hom}_R(-, R)$. If S is Gorenstein, this is free of rank 1 by definition. The minimality of the resolution together with Nakayama's lemma shows that the minimal number of generators of $\operatorname{Ext}_R^{n-d}(S, R)$ is precisely the rank of F_k. Hence it must be rank 1.
Conversely, suppose that the rank of F_k is 1. Since I kills S, it also kills $\operatorname{Ext}_R^{n-d}(S, R)$. It follows that $\operatorname{Ext}_R^{n-d}(S, R)$ is isomorphic to R/J for some ideal $J \supseteq I$. However, the vanishing of the other Ext groups gives that the transposed complex

$$0 \to F_0^* \to \cdots \to F_k^*$$

is actually acyclic, and hence a free R- resolution of $\operatorname{Ext}_R^{n-d}(S, R)$. Therefore,

$$\operatorname{Ext}_R^{n-d}(\operatorname{Ext}_R^{n-d}(S, R), R) \cong S$$

by dualizing the complex back, and now the same reasoning shows that J kills S, i.e. $J \subseteq I$. Thus $\operatorname{Ext}_R^{n-d}(S, R) \cong S$ and S is Gorenstein. $\square$

3.2 Level

Let $A = R/I$ be an Artinian s-level local $\mathbf{k}$-algebra, and let $G = \bigoplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}$ be its associated graded ring. It is well known that the associated graded ring G of A can be presented as the quotient of the polynomial ring P by I^*, the initial ideal of I.
Now, define for each $a \in \{0, \ldots, s+1\}$ and for every $i \geq 0$, the following ideals of G:

$$C(a) = \bigoplus_{i \geq 0} C(a)_i$$

whose homogeneous components can be described as

$$C(a)_i = \frac{(0 : \mathfrak{m}^{s+1-a-i}) \cap \mathfrak{m}^i}{(0 : \mathfrak{m}^{s+1-a-i}) \cap \mathfrak{m}^{i+1}} \subseteq G_i.$$

As a consequence of this definition

$$G = C(0) \supseteq C(1) \supseteq \cdots \supseteq C(s) = 0.$$

Define the successive quotients

$$Q(a) = \frac{C(a)}{C(a+1)}.$$

Then

$$\{Q(a) \colon a = 0, \dots, s-1\}$$

is called Irrabino's Q- decomposition of the associated graded ring G. Since the Hilbert function of A and $G(A)$ agree, we have the Iarrobino's Shell decomposition of HF_A, which is

$$\mathrm{HF}_A = \sum_{a=0}^{s-1} \mathrm{HF}_{Q(a)}$$

Proposition 3.2.1. *If A is Artin Gorenstein, then $Q(a)$ is a reflexive $G(A)$-module:*

$$\mathrm{Hom}_{\mathbf{k}}(Q(a)_i, \mathbf{k}) \cong Q(a)_{s-a-i} \text{ for } i = 0, \dots, s-a.$$

In particular, $HF_{Q(a)}$ is a symmetric function with respect to $\frac{s-a}{2}$.

Proposition 3.2.2. *Let $(A, \mathfrak{m}, \mathbf{k})$ be an Artinian s-level local algebra of type τ. Then $Q(0) = G/C(1)$ is the unique Artinian graded s-level quotient of G of type τ up to isomorphism.*

Proof. First, we are going to prove that $Q(0)$ is s-level. Having $\mathfrak{m}^{i+1} \subseteq (0 \colon \mathfrak{m}^{s-i})$:

$$\begin{aligned} Q(0) &= \frac{\bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}}{\bigoplus_{i\geq 0}(0 \colon \mathfrak{m}^{s-i}) \cap \mathfrak{m}^i/(0 \colon \mathfrak{m}^{s-i}) \cap \mathfrak{m}^{i+1}} \\ &= \frac{\bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}}{\bigoplus_{i\geq 0}(0 \colon \mathfrak{m}^{s-i}) \cap \mathfrak{m}^i/\mathfrak{m}^{i+1}} \simeq \bigoplus_{i \geq 0} \frac{\mathfrak{m}^i}{(0 \colon \mathfrak{m}^{s-i}) \cap \mathfrak{m}^i} \end{aligned}$$

Also $(0 \colon Q(0)_1) = Q(0)_s = G_s$. In fact, $Q(0)_s \subseteq (0 \colon Q(0)_1)$. Conversely, let $\overline{a} \in Q(0)_i$ be such that $\overline{a} \in \mathrm{Soc}(Q(0))$ with $i < s$. Then $a \in ((0 \colon \mathfrak{m}^{s-i-1}) \colon \mathfrak{m}) = (0 \colon \mathfrak{m}^{s-i})$, that is $\overline{a} = 0$ and then, $Q(0)$ is s-level. Then $Q(0)$ is the unique Artinian graded s-level quotient of G of type τ from classical facts. Indeed, an s-level standard graded algebra only depends on its homogeneous component of degree s, and in this case $Q(0)_s = G_s$ does not depend on the decomposition. $\square$

In the following results, let $\underline{f} = \{f_1, \dots, f_\tau\}$ and $\underline{F}$ is the set $\{F_1, \dots, F_\tau\}$ of the corresponding leading terms of $\underline{f}$.
Also, $A_{\underline{f}} := R/\mathrm{Ann}_R(\underline{f})$ and $A_{\underline{F}} := R/\mathrm{Ann}_R(\underline{F})$.

Proposition 3.2.3. *Let $A_{\underline{f}}$ be an Artinian s-level local $\mathbf{k}$-algebra of type τ. Then*

$$Q(0) \simeq A_{\underline{F}}.$$

Proof. The proof can be found at Proposition 2.4 in [Ste14]. □

Proposition 3.2.4. *Let $A_{\underline{f}}$ be an Artinian s-level local $\mathbf{k}$-algebra of type τ. The following facts are equivalent:*

(i) G *is s-level of type* τ,

(ii) $G \simeq Q(0)$,

(iii) $C(1) = 0$,

(iv) $C(a) = 0, \forall a \geq 1$,

(v) $Q(a) = 0, \forall a \geq 1$.

Proof. Using Proposition 3.2.2, $Q(0) = G/C(1)$ is the only graded s-level quotient of G of type τ. Then, (i) implies (ii), which is equivalent to (iii). Being $C(1) \supseteq C(2) \supseteq \cdots \supseteq C(s) = 0$, (iii) implies (iv). Using the definition of the $Q(a)'s$, if $C(a) = 0$ for all $a \geq 1$, then (v) holds. Lastly, using Proposition 3.2.2, (v) implies (i). □

Proposition 3.2.5. *Let $A_{\underline{f}}$ be an Artinian s-level local algebra of type τ. The following facts are equivalent:*

(a) $A_{\underline{f}}$ *is graded,*

(b) $A_{\underline{f}} \simeq Q(0)$,

(c) $\langle \underline{f} \rangle_R \simeq \langle \underline{F} \rangle_R$ *as R-modules.*

Proof. $(a) \Rightarrow (b)$ If $A_{\underline{f}}$ is graded, then G is level of type τ. $Q(0)$ is the only level quotient of G, up to isomorphism, by Proposition 3.2.2. Then $A_{\underline{f}} \simeq G \simeq Q(0)$.
$(b) \Rightarrow (c)$ It is an immediate consequence of Proposition 3.2.3.
$(c) \Rightarrow (a)$ Being $\langle \underline{f} \rangle_R \simeq \langle \underline{F} \rangle_R$ one has $A_{\underline{f}} \simeq Q(0) = G/C(1)$. Then, $\mathrm{HF}_{A_{\underline{f}}} = \mathrm{HF}_{G/C(1)}$. Moreover, there exists an epimorphism $G \xrightarrow{\pi} G/C(1)$ given by the natural projection on the quotient. By definition, $\mathrm{HF}_{A_{\underline{f}}} = \mathrm{HF}_G$ and so $\mathrm{HF}_G = \mathrm{HF}_{G/C(1)}$. Therefore, $C(1) = 0$ and $A_{\underline{f}} \simeq G$ is graded. $\square$

Definition 3.2.6 (Compressed level). *We call compressed level all Artinian local algebras with maximal Hilbert function.*

3.3 Inverse system for Artinian Level Algebras

In this section, we are going to study the problem of the isomorphism classes of local algebras using Macaulay's inverse system. In order to see that, we follow the approach used by De Stefani [Ste14] and by Elias and Rossi [ER12] but for Gorenstein algebras.

Given an Artinian local $\mathbf{k}$-algebra A, denote by $Aut(A)$ the group of the automorphisms of A as a $\mathbf{k}$-algebra and $Aut_{\mathbf{k}}(A)$ the group of the automorphism of A as a $\mathbf{k}$-vector space. The automorphisms of the power series ring $(R, \mathfrak{m}, \mathbf{k})$ as a $\mathbf{k}$-algebra are well known. Recall, for example, the following well known theorem.

Theorem 3.3.1 (Inverse Function Theorem). *Let $\mathbf{k}$ be a field, and let $f_1, \ldots, f_n \in \mathbf{k}[[x_1, \ldots, x_n]]$, satisfying $f_1(0) = \cdots = f_n(0) = 0$. Then the $\mathbf{k}$-algebra homomorphism*

$$\varphi\colon \mathbf{k}[[x_1, \ldots, x_n]] \to \mathbf{k}[[x_1, \ldots, x_n]],$$

defined by $\varphi(x_i) = f_i$ is an isomorphism if and only if det $\left(\frac{\partial f_i}{\partial x_j}(0)\right) \neq 0$.

Let $\mathfrak{m} = (x_1, \ldots, x_n)$, those automorphisms act as a replacement of x_i by z_i $i = 1, \ldots, n$, such that $\mathfrak{m} = (x_1, \ldots, x_n) = (z_1, \ldots, z_n)$. Our object of study are Artinian local algebras $A = R/I$ with socle degree s but, since $\mathfrak{m}^{s+1} \subseteq I$, we can restrict to the $\mathbf{k}$-algebra automorphism of $R/\mathfrak{m}^{s+1}$ induced by the projection $\pi\colon R \to R/\mathfrak{m}^{s+1}$. Clearly, $Aut(R/\mathfrak{m}^{s+1}) \subseteq Aut_{\mathbf{k}}(R/\mathfrak{m}^{s+1})$.

Let $E = \{e_i\}$ be the canonical basis of $R/\mathfrak{m}^{s+1}$ as a $\mathbf{k}$-vector space consisting of the standard monomials x^α ordered by the deg-lex order with $x_1 > \cdots > x_n$. The dual basis of E with respect to the perfect pairing $\langle\ ,\ \rangle$ as in (2.4) is the basis $E^* = \{e_i^*\}$ of $P_{\leq s}$, where

$$(x^\alpha)^* = \frac{1}{\alpha!}y^\alpha.$$

In fact, $e_i^*(e_j) = \langle e_j, e_i^*\rangle = \delta_{ij}$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. This is straightforward from the definition of $\langle\ ,\ \rangle$. For the case $\langle e_i, e_i^*\rangle$, by definition,

$$\langle e_i, e_i^*\rangle = (e_i \circ e_i^*)(0) = (x^\alpha \circ \frac{1}{\alpha!}y^\alpha)(0) = \frac{1}{\alpha!}\frac{\alpha!}{(\alpha-\alpha)!}y^0 = 1.$$

When $j \neq i$,

$$\langle e_j, e_i^*\rangle = (e_j \circ e_i^*)(0) = \left(x^\alpha \circ \frac{1}{\beta!}y^\beta\right)(0) = \frac{1}{\beta!}\frac{\beta!}{(\alpha-\beta)!}y^{\beta-\alpha} = 0.$$

Hence for any $\varphi \in Aut(R/\mathfrak{m}^{s+1})$, we can associate a matrix $M(\varphi)$ with respect to the basis E of size $l = dim_{\mathbf{k}}(R/\mathfrak{m}^{s+1}) = \binom{n+s}{s}$. We have the following natural sequnce of morphisms of groups:

$$Aut(R) \xrightarrow{\pi} Aut(R/\mathfrak{m}^{s+1}) \xrightarrow{\sigma} Aut_{\mathbf{k}}(R/\mathfrak{m}^{s+1}) \xrightarrow{\rho_E} Gl_r(\mathbf{k}).$$

Given I and J ideals of R such that $\mathfrak{m}^{s+1} \subseteq I, J$, there exists a $\mathbf{k}$-algebra isomorphism

$$\varphi\colon R/I \to R/J$$

if and only if φ is canonically induced by a $\mathbf{k}$-algebra automorphism of $R/\mathfrak{m}^{s+1}$ sending $I/\mathfrak{m}^{s+1}$ to $J/\mathfrak{m}^{s+1}$. In particular, φ is an isomorphism of $\mathbf{k}$-vector spaces. Dualizing

$$\varphi^*\colon (R/J)^* \to (R/I)^*$$

is an isomorphism of the $\mathbf{k}$-vector subspaces $(R/I)^* \simeq I^\perp$ and $(R/J)^* \simeq J^\perp$ of $P_{\leq s}$.

Definition 3.3.2. *We define ${}^tM(\varphi)$ as the matrix associated to φ^* with respect to the dual basis E^* of $P_{\leq s}$.*

The above diagram can be completed by the following commutative diagram, which will help us to visualize the setting:

$$\begin{array}{ccc} Aut_{\mathbf{k}}(R/\mathfrak{m}^{s+1}) & \xrightarrow{\rho_E} & Gl_r(\mathbf{k}) \\ \Big\downarrow{*} & & \Big\downarrow{{}^t()} \\ Aut_{\mathbf{k}}(P_{\leq s}) & \xrightarrow{\rho_{E^*}} & Gl_r(\mathbf{k}) \end{array}$$

Denote by $\mathfrak{R}$ the subgroup of $Aut_{\mathbf{k}}(P_{\leq s})$ (those are the automorphism of $P_{\leq s}$ as a $\mathbf{k}$-vector space) represented by the matrices ${}^tM(\varphi)$ of $Gl_r(\mathbf{k})$ with $\varphi \in Aut(R/\mathfrak{m}^{s+1})$.

Theorem 3.3.3. *The classification, up to analytic isomorphism, of the Artinian local* $\mathbf{k}$*-algebras of multiplicity d, socle degree s and embedding dimension n is equivalent to the classification, up to the action of $\mathfrak{R}$, of the* $\mathbf{k}$*-vector subspaces of $P_{\leq s}$ of dimension d, stable by derivations and containing $P_{\leq 1} = \mathbf{k}[y_1, \dots, y_n]_{\leq 1}$.*

Through all this approach, we study at first what Elias and Rossi proved at [ER12]. In the case of Gorenstein algebras, given $\varphi \in \mathrm{Aut}(R)$, one has

$$\varphi(A_f) = A_g \text{ if and only if } (\varphi^*)^{-1}(\langle f \rangle_R) = \langle g \rangle_R,$$

where

$$A_f = R/\mathrm{Ann}_R(f).$$

As a consequence,

$$\varphi(A_f) = A_g \text{ if and only if } (\varphi^*)^{-1}(\langle f \rangle_R) = u \circ g,$$

with f and g polynomials of the same degree and u a unit in R.
Our objective is to generalize this result in the case of level algebras. In this scenario, we have

$$\varphi(A_{\underline{f}}) = A_{\underline{g}} \text{ if and only if } (\varphi^*)^{-1}(\langle f_1, \dots, f_\tau \rangle_R) = \langle g_1, \dots, g_\tau \rangle_R.$$

In analogy with the Gorenstein case, we get the following result.

Lemma 3.3.4. *Let $\varphi \in Aut(R)$. The following are equivalent*

(a) $\varphi(A_{\underline{f}}) = A_{\underline{g}}$,

(b) There exists $B \in Gl_\tau(R)$ *such that* ${}^t((\varphi^*)^{-1}(f_1), \ldots, (\varphi^*)^{-1}(f_\tau)) = B \circ {}^t (g_1, \ldots, g_\tau)$.

Before continuing, let's recall an important well known result that is going to be useful to understand what is happening next.

Proposition 3.3.5. *There is a one-to-one correspondence between zero dimensional ideals of* R *such that* R/I *is s-level of type* τ *and* R*-submodules of* P *generated by* τ *polynomials of degree* s *having linearly independent forms of degree* s*. The correspondence is defined as follows:*

$$\left\{ \begin{array}{c} I \subseteq R \textit{ such that } R/I \\ \textit{is Artinian level of type} \\ \tau \textit{ and socle degree } s \end{array} \right\} \overset{1-1}{\leftrightarrow} \left\{ \begin{array}{c} M \subseteq P \textit{ submodule generated by} \\ \tau \textit{ polynomials of degree} \\ s \textit{ with l.i. forms of degree } s \end{array} \right\}$$

$$\begin{array}{rcl} I & \rightarrow & I^\perp \\ Ann_R(M) & \leftarrow & M. \end{array}$$

This proposition is a generalization of Proposition 3.1.7 when studying Artinian level algebras of type τ.

Let $B \in Gl_\tau(R/\mathfrak{m}^{s+1})$, an invertible element of $R/\mathfrak{m}^{s+1}$. The corresponding action of this matrix in $(P_{\leq s})^\tau$ is an isomorphism of $\mathbf{k}$-vector spaces. Let $l = \dim_{\mathbf{k}} P_{\leq s}$, then there exists a matrix $N(B) \in Gl_{l\tau}(\mathbf{k})$ associated to the action of B on $(P_{\leq s})^\tau$ with respect to the basis $(E^*)^{\oplus \tau}$ of $(P_{\leq s})^\tau$.

Let $f_1, \ldots, f_\tau \in P_{\leq s}$. If $f_i = b_{i1}e_1^* + \cdots + b_{il}e_l^* \in P_{\leq s}$ for all $i = 1, \ldots, \tau$, we denote the row vector of the coefficients of the τ polynomials with respect to the basis E^* by

$$[\underline{f}]_{E^*} = (b_{11}, \ldots, b_{1l}, \ldots, b_{\tau 1}, \ldots, b_{\tau l}).$$

Now, we can generalize the result proved by Elias and Rossi in [ER12] which was stated for Gorenstein algebras.

Proposition 3.3.6. *Two Artinian s-level algebras* $A_{\underline{f}}$ *and* $A_{\underline{g}}$ *of type* τ *are isomorphic if and only if there exists* $\varphi \in Aut_a(R/\mathfrak{m}^{s+1})$ *and there is an invertible matrix* $B \in Gl_\tau(R/frm^{s+1})$ *such that*

$$[\underline{g}]_{E^*}({}^tN(B)M(\varphi^{\oplus \tau})) = [\underline{f}]_{E^*}$$

where $M(\varphi^{\oplus \tau})$ *is the matrix associated to the homomorphism which consists of* τ *copies of* φ.

This result allows us to translate the difficult problem of the classification of level local algebras up to isomorphism into a problem of linear algebra.

3.4 Graded compressed level local algebras

Elias and Rossi proved in [ER12] that given A_f a Gorenstein algebra with h-vector $H = (1, m, n, 1)$, then A_f is graded if and only if H is admissible as the h-vector of a graded Gorenstein algebra. In fact, if A has h-vector $(1, m, n, 1)$, then $n \leq m$, and A is graded if and only if the equality holds.
At the end of the chapter, we will see that if A is 3-level of type τ with h-vector $(1, m, n, \tau)$, then $n \leq \tau m$. When the equality holds, we have the following definition.

Definition 3.4.1. *Let A be a 3-level of type τ with h-vector $(1, m, n, \tau)$. When $n = \tau m$, then A is called compressed level or just compressed.*

The first objective of this section is to prove that if A_f is a compressed level local algebra of socle degree 3, then it is graded. In order to study that, some previous definitions are required. Let's see them.
Let $F \in P_3$ be a form of degree three. Then:

1. $\Delta_1(F)$ is the matrix $m \times \binom{m+1}{2}$ which jth row is the vector of the coefficients of $\partial_j F$ ordered following lex order on the dual basis
$$\left\{ \frac{x_1^2}{2}, x_1 x_2, \ldots, \frac{x_m^2}{2} \right\}$$
of P_2,

2. $\Delta_2(F)$ is the matrix $\frac{m+1}{2} \times m$ which jth row is the vector of the coefficients of $\partial_{\underline{s}} F$ ordered following lex order on $P_1 = \{x_1, \ldots, x_m\}$. $\underline{s} \in \mathbb{N}^m$, $|s| = 2$, is such that $x^{\underline{s}}$ is the jth element of $\mathbb{T}_m^2 := \{x_1^2, x_1 x_2, \ldots, x_m^2\}$ in the lex order.

Basically, $\Delta_1(F)$ and $\Delta_2(F)$ are the matrices of the coefficients (with respect to the dual bases) of the first and the second derivatives of F.

Lemma 3.4.2. *Let $F \in P_3$ be a form of degree three. Then*
$$\Delta_1(F) = {}^t\Delta_2(F).$$

Proof. Let $r = 1, \dots, m$. Set $\delta_r \in \mathbb{N}^m$ the vector which entries are all zero, except for position r, in which there is 1. Set $F = \sum_{|\underline{i}|=3} \alpha_{\underline{i}} \frac{x^{\underline{i}}}{\underline{i}!}$ in the dual basis.
Notice that

$$\partial_r F = \sum_{|\underline{j}|=2} \beta_{\underline{j}} \frac{x^{\underline{j}}}{\underline{j}!} = \sum_{|\underline{j}|=2} \alpha_{\underline{j}+\delta_r} \frac{x^{\underline{j}}}{\underline{j}!}$$

Also

$$\partial_{\underline{s}} F = \sum_{k=1}^{m} \gamma_{\delta_k} x_k = \sum_{k=1}^{m} \alpha_{\delta_k + \underline{s}} x_k.$$

The coefficient of $\frac{x^{\underline{s}}}{\underline{s}!}$ in $\partial_r F$ is exactly the coefficient of x_r in $\partial_{\underline{s}} F$, which is $\alpha_{\delta_r + \underline{s}}$. This means $\Delta_1(F) = {}^t\Delta_2(F)$. $\square$

Using the previous lemma, $\Delta_1(F)$ and $\Delta_2(F)$ have the same information about F and we can consider only $\Delta(F) := \Delta_1(F)$. Let $A_{\underline{F}}$ be a graded level algebra with h-vector $(1, m, n, \tau)$. We can define

$$\Delta(\underline{F}) = \begin{pmatrix} \Delta(F_1) \\ \hline \vdots \\ \hline \Delta(F_\tau) \end{pmatrix}$$

which is a $\tau m \times \binom{m+1}{2}$ matrix. The rank of the matrix is

$$\mathrm{rk}(\Delta(\underline{F})) = n, \tag{3.2}$$

and $\Delta(\underline{F})$ has maximal rank if and only if

$$n = \min \left\{ \tau m, \binom{m+1}{2} \right\}$$

if and only if $A_{\underline{F}}$ is compressed.

Lemma 3.4.3. *Let $A_{\underline{f}}$ be a compressed level local algebra of socle degree three. Then $Q(0) = A_{\underline{F}}$ is compressed.*

Proof. The proof can be found at [Ste14], Lemma 3.2. $\square$

Now, we are prepared to prove the following theorem.

Theorem 3.4.4. *Let $A_{\underline{f}}$ be a compressed level local algebra of type τ and socle degree 3. Then $A_{\underline{f}}$ is graded.*

Proof. First of all, let's notice some result. If $P_{\leq 1} = \{g \in P\colon \deg g \leq 1\} \subseteq \langle f_1, \dots, f_\tau \rangle_R$, we can assume that $f_i = F_i + Q_i$, $i = 1, \dots, \tau$ with F_i and Q_i forms of degree 3 and 2, respectively. Let $(1, m, n, \tau)$ be the h-vector of $A_{\underline{f}}$. Then

$$n = \min\left\{\tau m, \binom{m+1}{2}\right\}$$

because $A_{\underline{f}}$ is compressed. Suppose $n = \binom{m+1}{2}$. Then, using that the rank is equal to n, $\dim_{\mathbf{k}}(R_1 \circ \langle \underline{f} \rangle_R) = \binom{m+1}{2}$ and $P_2 \subseteq \langle f_1, \dots, f_\tau \rangle_R$ too. In this case,

$$\langle f_1, \dots, f_\tau \rangle_R = \langle F_1 + Q_1, \dots, F_\tau + Q_\tau \rangle_R = \langle F_1, \dots, F_\tau \rangle_R,$$

and $A_{\underline{f}}$ is graded by Proposition 3.2.5 (c).

Let $n = \tau m$. Using Proposition 3.3.6, we want to see that there exists $\varphi \in \mathrm{Aut}_a(R/\mathcal{M}^4)$ such that

$$[\underline{F}]_{E^*} M(\varphi) = [\underline{f}]_{E^*} = [\underline{F} + \underline{Q}]_{E^*}.$$

Consider $\varphi\colon R/\mathcal{M}^4 \to R/\mathcal{M}^4$ the $\mathbf{k}$-algebra automorphism with the identity as Jacobian defined on the variables as

$$\varphi(x_h) := x_h + \sum_{|\underline{i}|=2} a^h_{\underline{i}} x^{\underline{i}} \quad h = 1, \dots, m,$$

with $a^h_{\underline{i}} \in \mathbf{k}$ for each $|\underline{i}| = 2, h = 1, \dots, m$ and $\underline{a} := (a^h_{\underline{i}}\colon |\underline{i}| = 2, h = 1, \dots, m)$ is a row of size $m\binom{m+1}{2}$. The matrix $M(\varphi)$ associated to φ is an element of $Gl_r(\mathbf{k})$, $r = \binom{m+3}{4}$, with respect to the basis E of $R/\mathcal{M}^4$ ordered by deg-lexicographic order. The matrix $M(\varphi)$ has the following form

$$M(\varphi) = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & I_m & 0 & 0 \\ \hline 0 & D & I_{\binom{m+1}{2}} & 0 \\ \hline 0 & 0 & B & I_{\binom{m+2}{3}} \end{array}\right)$$

where for all $t \geq 1$, I_t denotes the $t \times t$ identity matrix. The first block of columns corresponds to the image $\varphi(1) = 1$. The second block of columns corresponds to the image of $\varphi(x_i)$, $i = 1, \ldots, m$. The third one corresponds to the image of $\varphi(x^{\underline{i}})$, where $|\underline{i}| = 2$. Finally, the fourth one corresponds to the image of $\varphi(x^{\underline{i}})$ with $|\underline{i}| = 3$, which is the identity matrix. Then, D is the $\binom{m+1}{2} \times m$ matrix defined by the coefficients of the degree two monomials of $\varphi(x_i)$, $i = 1, \ldots, m$ and B is a $\binom{m+2}{3} \times \binom{m+1}{2}$ matrix defined by the coefficients of the degree three monomials appearing in $\varphi(x^{\underline{i}})$ with $|\underline{i}| = 2$. Clearly, $M(\varphi)$ is determined by D and the entries of B are linear forms in the variables $a^h_{\underline{i}}$, with $|\underline{i}| = 2$, $h = 1, \ldots, m$. Let's write $F_1, \ldots, F_\tau$ in the dual basis E^*:

$$F_j = \sum_{|\underline{i}|=3} \alpha^j_{\underline{i}} \frac{x^{\underline{i}}}{\underline{i}!} \qquad j = 1, \ldots, \tau$$

and $Q_1, \ldots, Q_\tau$:

$$Q_j = \sum_{|\underline{i}|=2} \beta^j_{\underline{i}} \frac{x^{\underline{i}}}{\underline{i}!} \qquad j = 1, \ldots, \tau.$$

Let $\alpha^j := (\alpha^j_{\underline{i}} \colon |\underline{i}| = 3)$ a row vector of size $\binom{m+2}{3}$ and $\beta^j := (\beta^j_{\underline{i}} \colon |\underline{i}| = 2)$ a row vector of size $\binom{m+1}{2}$. We must solve the following linear system:

$$[\alpha^j]_{E^*} B = [\beta^j]_{E^*} \qquad j = 1, \ldots, \tau,$$

or equivalently,

$$[\underline{\alpha}]_{E^*} B^{\bigoplus \tau} = [\underline{\beta}]_{E^*},$$

where $\underline{\alpha} = (\alpha^j \colon j = 1, \ldots, \tau)$ is a row vector of size $\tau \binom{m+2}{3}$ and $\underline{\beta} = (\beta^j \colon j = 1, \ldots, \tau)$ is a vector of size $\tau \binom{m+1}{2}$ and

$$B^{\bigoplus \tau} = \left(\begin{array}{c} B \\ \hline B \\ \hline \vdots \\ \hline B \end{array} \right)$$

is a $\tau \binom{m+2}{3} \times \binom{m+1}{2}$ matrix. In [ER12], Elias and Rossi prove that for each $j = 1, \ldots, \tau$ there exist a $\binom{m+1}{2} \times m \binom{m+1}{2}$ matrix M_j such that

$$[\alpha^j]_{E^*} B = \underline{a}^t M_j$$

where $\underline{a}$ are the coefficients defining the automorphism φ. Then, we have to solve

$$\underline{a}^t M = [\beta]_{E^*}$$

with

$$M = \begin{pmatrix} M_1 \\ \hline M_2 \\ \hline \vdots \\ \hline M_\tau \end{pmatrix}$$

is a $\tau \binom{m+1}{2} \times m \binom{m+1}{2}$ matrix.

For each $j = 1, \dots, \tau$, recall $\Delta(F_j)$. In [ER12], it is proved that the matrices M_j have the following upper-diagonal structure:

$$M_j = \left(\begin{array}{c|c|c|c|c} M_j^1 & * & \dots & * & * \\ \hline 0 & M_j^2 & \dots & * & * \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \dots & M_j^{m-1} & * \\ \hline 0 & 0 & \dots & 0 & M_j^m \end{array}\right)$$

where M_j^l is a $(m-l+1) \times \binom{m+1}{2}$ matrix, $l = 1, \dots, m$, such that the 1st row of M_j^1 equals two times the 1st column of $\Delta(F_j)$, the tth row of M_j^1 equals the tth row of $\Delta(F_j)$, $t = 2, \dots, m$ and the 1st row of M_j^l equals two times the lth column of $\Delta(F_j)$, for $l = 2, \dots, m$, the tth row of M_j^l equals two times the lth column of $\Delta(F_j)$, for $j = 2, \dots, m$ and the lth row of M_j^l equals the $(l+t-1)th$ column of $\Delta(F_j)$, $t = 2, \dots, m-l+1$, $l = 2, \dots, m$.

Elias and Rossi show that M_j has maximal rank because $\Delta(F_j)$ has maximal rank, then the system $\underline{a}^t M_j = [\beta^j]_{E^*}$ is compatible and this completes their proof. In our generalization to level algebras we have to prove that there is a solution $\underline{a}$ which satisfies $\underline{a}^t M_j = [\beta^j]_{E^*}$ for all $j = 1, \dots, \tau$, and we have to show that the matrix M has maximal rank. But in this case:

$$M = \left(\begin{array}{c|c|c|c|c}
M_1^1 & * & \dots & * & * \\ \hline
0 & M_1^2 & \dots & * & * \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & \dots & M_1^{m-1} & * \\ \hline
0 & 0 & \dots & 0 & M_1^m \\ \hline\hline
\vdots & \vdots & \vdots & \vdots & \vdots \\ \hline\hline
M_\tau^1 & * & \dots & * & * \\ \hline
0 & M_\tau^2 & \dots & * & * \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & \dots & M_\tau^{m-1} & * \\ \hline
0 & 0 & \dots & 0 & M_\tau^m
\end{array}\right)$$

Then, the rank of M is linked to the rank of $\Delta(\underline{f})$. But we assumed that A_f is compressed, and hence $Q(0) = A_{\underline{F}}$ is compressed by Lemma 3.4.3. So the matrix $\Delta(\underline{F})$ has maximal rank equal to τm by (3.2). This means that the system is compatible and completes the proof. □

Now, let's see some results about h-vectors.
As the last part of this section, let's recall some important results related to h-vectors. The main result that will be proved is the following one.

Theorem 3.4.5. *Let $H = (1, m, n, \tau)$ be an h-vector. Then H is the h-vector of an Artinian level local algebra if and only if $n \leq \tau m$.*

The proof of this theorem will be divided in three parts.

Proposition 3.4.6. *Let $H = \{1, m, n, \tau\}$ be an h-vector. If $m > n \geq \tau > 0$, then H is the h-vector of an Artinian 3-level local algebra.*

Proof. Consider the following τ polynomials:

$$\begin{array}{lll}
f_1 := x_1^3, & f_2 := x_2^3, & f_3 := x_3^3, \\
\cdots, & f_{\tau-1} := x_{\tau-1}^3, & f_\tau := x_\tau^3 + \cdots + x_n^3 + x_{n+1}^2 + \cdots + x_m^2.
\end{array}$$

Using Proposition 2.2.9, it is easy to prove that

$$\begin{cases} \mathrm{HF}_{A_{\underline{f}}}(1) = \dim_{\mathbf{k}}(\langle x_1, \dots, x_m \rangle_{\mathbf{k}}) = m \\ \mathrm{HF}_{A_{\underline{f}}}(2) = \dim_{\mathbf{k}}(\langle x_1^2, \dots, x_n^2 \rangle_{\mathbf{k}}) = n \\ \mathrm{HF}_{A_{\underline{f}}}(3) = \tau, \end{cases}$$

and hence H is the h-vector of $A_{\underline{f}}$. □

Proposition 3.4.7. *Let $H = (1, m, n, \tau)$ be an h-vector. If*

$$\max\{\tau, m\} \leq n \leq \tau m,$$

then H is the h-vector of an Artinian 3-level graded algebra.

Proof. For each $j = 0, \dots, [\frac{m}{2}]$ we define

$$\mathscr{D}_j := \{x_i^2 x_{i+j} \colon i = 1, \dots, m\},$$

where whenever an index of a variable is $l \geq m+1$ one has to read $l - m$ (for example x_{m+1} is x_1). We equip each $\mathscr{D}_j$ with the following order $\preceq_j$:

$$x_i^2 x_{i+1} \preceq x_k^2 x_{k+j} \Leftrightarrow i \leq k.$$

Denote by $\#A$ the cardinality of a finite set A. Observe that $\mathscr{D}_i \cap \mathscr{D}_j = \emptyset$ if $i \neq j$ and $\#\mathscr{D}_j = m$ for all j. Set $\mathscr{D} = \cup_j \mathscr{D}_j$. We deduce that

$$\#\mathscr{D} = m\left(\left[\frac{m}{2}\right] + 1\right) \geq \binom{m+1}{2}.$$

Using the terms of the set $\mathscr{D}$, we want now to show τ forms of degree three $f_1, \dots, f_\tau \in P = k[x_1, \dots, x_m]$ with disjoint supports such that

$$\#\mathbf{Supp}\{f_1, \dots, f_\tau\} = \dim_{\mathbf{k}}(\langle \partial f_1, \dots, \partial f_\tau \rangle_R) = n.$$

Since H is an h-vector, we have

$$n = \mathrm{HF}(2) \leq \binom{m+1}{2}.$$

By hypothesis, $n \leq \tau m$. Therefore $n \leq \min\left\{\binom{m+1}{2}, \tau m\right\}$. This shows that we can satisfy the request $\#\mathbf{Supp}\{f_1, \dots, f_\tau\} = n$, since $n \leq \#\mathscr{D}$.

Let $\tilde{n} := n - \tau$. This is a positive number because $n \geq \tau$ by assumption. Then there exist $h, l \in \mathbb{N}$ such that

$$\tilde{n} = h(m-1) + l \;\text{ with }\; 0 \leq l < m - 1.$$

Notice that

$$h(m-1) \leq h(m-1) + l = \tilde{n} = n - \tau \leq \tau m - \tau = \tau(m-1). \tag{3.3}$$

Now, if $m = 1$, the h-vector must be $(1, 1, 1, 1)$. In order to compute that recall that by assumption, we have $\max\{\tau, 1\} \leq n \leq \tau$ and it implies that $\tau = n$ and $n = \mathrm{HF}(2) \leq \binom{2}{2} = 1$, therefore $n = 1 = \tau$. Hence, we have $\tilde{n} = h = 0 < 1 = \tau$. If $m > 1$, we can divide by $(m-1)$ and we always get $\tau \geq h$. Let's consider the following different cases:

(a) If $\tau = h$, then define

$$f_j := \sum_{i=1}^{m} x_i^2 x_{i+j-1} \quad \text{for } j = 1, \ldots, h.$$

For all $j = 1, \ldots, h$ the monomials that appear in the support of f_j are the elements of the set $\mathscr{D}_{j-1}$. Also, being $\tau = h$, the inequalities in (3.3) become equalities. Then

$$\begin{cases} n - \tau = \tau m - \tau \\ h(m-1) = h(m-1) + l \end{cases} \Rightarrow \begin{cases} n = \tau m \\ l = 0. \end{cases}$$

Now,

$$V^{(1)} := \langle \partial f_1, \ldots, \partial f_h \rangle_{\mathbf{k}} = \langle x_i x_{i+j} : i = 1, \ldots, m \; j = 0, \ldots, h-1 \rangle_{\mathbf{k}},$$

and then

$$\begin{cases} \mathrm{HF}_{A_{\underline{f}}}(1) = \dim_{\mathbf{k}}(\langle x_1, \ldots, x_m \rangle)_{\mathbf{k}}) = m \\ \mathrm{HF}_{A_{\underline{f}}}(2) = \dim_{\mathbf{k}} V^{(1)} = hm = \tau m = n \\ \mathrm{HF}_{A_{\underline{f}}}(3) = h = \tau. \end{cases}$$

(b) If $\tau = h + 1$, then

$$n = n - \tau + h + 1 = \tilde{n} + h + 1 = h(m-1) + l + h + 1 = hm + l + 1.$$

Define

$$f_{h+1} := \sum_{i=1}^{l+1} x_1^2 x_{i+h}$$

and consider $\underline{f} = \{f_1, \dots, f_h, f_{h+1}\}$, where $f_1, \dots, f_h$ are defined as in (a). Set $V^{(2)} := \langle \partial f_{h+1} \rangle_{\mathbf{k}}$, then one has:

$$\begin{cases} \mathrm{HF}_{A_{\underline{f}}}(1) &= \dim_{\mathbf{k}}(\langle x_1, \dots, x_m \rangle_{\mathbf{k}}) = m \\ \mathrm{HF}_{A_{\underline{f}}}(2) &= \dim_{\mathbf{k}}(V^{(1)} \bigoplus V^{(2)}) = (hm) + (l+1) = n \\ \mathrm{HF}_{A_{\underline{f}}}(3) &= h + 1 = \tau \end{cases}$$

(c) If $h + 2 \leq \tau \leq h + m - l$, then we work again with the same set of polynomials defined in (a) and (b):

$$\begin{cases} f_{h+2} &:= x_{l+2}^2 x_{l+h+2} \\ \dots & \\ f_\tau &:= x_{l+\tau-h}^2 x_{l+\tau} \end{cases}$$

Then, we set

$$V_1^{(3)} := \langle \partial f_{h+2}, \dots, \partial f_\tau \rangle_{\mathbf{k}} = \langle x_i x_{i+h} : i = l+2, \dots, l+\tau-h \rangle_{\mathbf{k}}.$$

We get

$$\begin{cases} \mathrm{HF}_{A_{\underline{f}}}(1) &= \dim_{\mathbf{k}}(\langle x_1, \dots, x_m \rangle_{\mathbf{k}}) = m \\ \mathrm{HF}_{A_{\underline{f}}}(2) &= \dim_{\mathbf{k}}(V^{(1)} \bigoplus V^{(2)} \bigoplus V_1^{(3)}) = (hm) + (l+1) + (\tau - h - 1) \\ &= h(m-1) + l + \tau = \tilde{n} + \tau = n \\ \mathrm{HF}_{A_{\underline{f}}}(3) &= \tau \end{cases}$$

(d) Let $\tau > h + m - l$, consider the set of polynomials $\{f_1, \dots, f_{h+m-l}\}$ already defined in previous items. Take $\tau = h + m - l$ the maximum in the range of (c).

Then we have

$$\begin{cases} f_{h+2} &:= x_{l+2}^2 x_{l+h+2} \\ \dots & \\ f_{h+m-l} &:= x_m^2 x_h \end{cases}$$

which are polynomials that involve all the terms of $\mathscr{D}_h$ not considered in f_{h+1}. The strategy is to define $f_{h+m-l+1}, \dots, f_\tau$ each one as an element of $\mathscr{D}_j$, with $j > h$, following the orders $\preceq_j$. In particular, if we start from $\mathscr{D}_{h+1}$, we pick all the elements in this set before passing $\mathscr{D}_{h+2}$ and so on. Define $\tilde{n} := n - (h+1)m$. Notice that $\tilde{n} := h(m-1) + l + \tau - (h+1)m =$

$\tau - (h + m - l) > 0$ and we can write $\tilde{n} = mr + s$, where $r, s \in \mathbb{N}$ and $0 \leq s < m$. We obtain:

$$\begin{cases} \left.\begin{array}{l} f_{h+1} := x_{l+2}^2 x_{l+h+2} \\ \cdots \\ f_{h+m-l} := x_m^2 x_h \end{array}\right\} \text{we complete } \mathscr{D}_h \\ \left.\begin{array}{l} f_{h+m-l+1} := x_1^2 x_{h+2} \\ \cdots \\ \cdots \end{array}\right\} \text{we complete } \mathscr{D}_{h+1} \\ \cdots \\ \cdots \\ \left.\begin{array}{l} \cdots \\ \cdots \\ f_{\tau-s} := x_m^2 x_{h+r} \end{array}\right\} \text{we complete } \mathscr{D}_{h+r} \end{cases}$$

With these, we define

$$V_2^{(3)} ; = \langle \partial f_{h+2}, \ldots, \partial f_{\tau-s} \rangle_{\mathbf{k}} = \langle x_i x_{i+j} : i = 1, \ldots, m;\ j = h, \ldots, h + r \rangle_{\mathbf{k}}.$$

Let $\underline{f} = \{f_1, \ldots, f_{h+1}, f_{h+2}, \ldots, f_{\tau-s}\}$we get

$$\begin{cases} \mathrm{HF}_{A_{\underline{f}}}(1) &= \dim_{\mathbf{k}}(\langle x_1, \ldots, x_m \rangle_{\mathbf{k}}) = m \\ \mathrm{HF}_{A_{\underline{f}}}(2) &= \dim_{\mathbf{k}}(V^{(1)} \bigoplus V^{(2)} \bigoplus V_2^{(3)}) \\ &= (hm) + (l+1) + (m - (l+1) + rm) \\ &= m(h+1) + \tilde{n} - s = n - s \\ \mathrm{HF}_{A_{\underline{f}}}(3) &= \tau - s \end{cases}$$

If $s = 0$, we have the required τ polynomials. If $s > 0$ just define the last s polynomials following the above strategy which is setting each one of them as a term of $\mathscr{D}_{h+r+1}$ (with respect to $\preceq_{h+r+1}$):

$$\begin{cases} f_{\tau-s+1} &:= x_1^2 x_{h+r+2} \\ \cdots & \\ f_\tau &:= x_s^2 x_{h+r+s+1} \end{cases}$$

We can decide

$$V^{(4)} := \langle \partial f_{\tau-s+1}, \ldots, \partial f_\tau \rangle_{\mathbf{k}} = \langle x_i x_{i+h+r+1} : i = 1, \ldots, s \rangle_{\mathbf{k}}$$

and we get

$$\begin{cases} \mathrm{HF}_{A_{\underline{f}}}(1) &=& \dim_{\mathbf{k}}(\langle x_1,\dots,x_m\rangle_{\mathbf{k}}) = m \\ \mathrm{HF}_{A_{\underline{f}}}(2) &=& \dim_{\mathbf{k}}(V^{(1)} \bigoplus V^{(2)} \bigoplus V_2^{(3)} \bigoplus V^{(4)}) = n-s+s=n \\ \mathrm{HF}_{A_{\underline{f}}}(3) &=& \tau \end{cases}$$

Once again, $A_{\underline{f}}$ has h-vector H.

$\square$

Proposition 3.4.8. *Let $H = (1, m, n, \tau)$ be an h-vector. If $n < \tau$, then H is the h-vector of an Artinian 3-level local algebra.*

Proof. Since H is an h-vector, it has to satisfy Macaulay's Theorem for the Hilbert function. One can check this theorem in [Sta78] . The following notation is from [Sta78] and [BH98]. To sum up, if

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

with $n_i > n_{i-1} > \cdots < n_j \geq j \geq 1$, define

$$h^{\langle i\rangle} = \binom{n_i+1+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}.$$

Then, by Theorem 2.2. from [Sta78]

$$\tau = \mathrm{HF}_A(3) \leq \mathrm{HF}_A(2)^{\langle 2\rangle} = n^{\langle 2\rangle}.$$

There exists $l \in \mathbb{N}, l \geq 1$ and $h \in \{0, 1, \dots, l\}$ such that

$$n = \binom{l+1}{2} + h.$$

Then

$$n^{\langle 2\rangle} = \binom{l+2}{3} + \binom{h+1}{2}.$$

Moreover, again for the fact that H is an h-vector, it has to be

$$\binom{l+1}{2} \leq \binom{l+1}{2} + h = n \leq m^{\langle 1\rangle} = \binom{m+1}{2},$$

and so $m \geq l$ and $m = l$ only when $h = 0$.
Let $\mathbb{T}_l^n := \mathbf{Supp}((x_1 + \cdots + x_l)^n) = \{x_1^n, x_1^{n-1}x_2, \ldots, x_l^n\}$. Notice that $\#\mathbb{T}_l^n = \binom{l+n-1}{n}$. For every $f \in R$:

$$f \cdot \mathbb{T}_l^n := \{ft \colon t \in \mathbb{T}_l^n\}.$$

Assume $h = 0$. Then $\tau \leq \binom{t+2}{3}$ because H is an h-vector. It will be enough to define τ forms of degree three $F_1, \ldots, F_\tau$ as the first τ monomials in $\mathbb{T}_l^3$ w.r.t. lex order. Notice that $\#\mathbb{T}_l^3 = \binom{l+2}{3}$. We are choosing τ distinct monomials of degree three which generate $n = \binom{l+1}{2}$ linearly independent forms of degree two. We have

$$\begin{cases} \mathrm{HF}_{A_{\underline{F}}}(1) &= \dim_{\mathbf{k}}(\langle x_1, \ldots, x_l \rangle_{\mathbf{k}}) = k \\ \mathrm{HF}_{\underline{F}}(2) &= \dim_{\mathbf{k}} \mathbb{T}_l^2 = \binom{l+1}{2} = n \\ \mathrm{HF}_{A_{\underline{F}}}(3) &= \dim_{\mathbf{k}}(\langle F_1, \ldots, F_\tau \rangle_{\mathbf{k}}) = \tau \end{cases}$$

Having $\mathrm{HF}_A(1) = m \geq l$, we set

$$f_1 := F_1 + \sum_{i=l+1}^{m} x_i^2, \quad f_j := F_j \quad \text{for all } j = 2, \ldots, \tau$$

and then, $A_{\underline{f}}$ has h-vector $(1, m, \binom{l+1}{2}, \tau) = (1, m, n, \tau)$.
Now, let $n = \binom{l+1}{2} + h$, $0 < h \leq l$ and

$$\#\mathbb{T}_l^2 = \binom{l+1}{2}.$$

Then, we consider $n = \binom{l+1}{2} + h$ forms of degree three $F_1, \ldots, F_n$ as follows. Each of the first $\binom{l+1}{2}$ is selected to be a different monomial in the set $x_1 \cdot \mathbb{T}_l^2 = \{x_1^3, x_1^2x_2, \ldots, x_1x_l^2\}$ with lex order:

$$F_1 := x_1^3, \qquad F_2 := x_1^2x_2, \qquad \ldots \qquad F_{\binom{l+1}{2}} := x_1x_l^2.$$

We define the h as it follows

$$F_{\binom{l+1}{2}+1} := x_{l+1}^3, \quad F_{\binom{l+1}{2}+2} := x_l x_{l+1}^2, \quad \ldots \quad F_n := x_{h-1}x_{l+1}^2.$$

These are $n = \binom{l+1}{2} + h$ linearly independent forms of degree three whose derivates generate a $\mathbf{k}$-vector space of dimension exactly $n = \binom{l+1}{2} + h$. In fact:

$$\begin{aligned} &\dim_{\mathbf{k}}(\langle \partial F_1, \ldots, \partial F_n \rangle_{\mathbf{k}}) \\ &= \dim_{\mathbf{k}}(\langle \mathbb{T}_l^2 \rangle_{\mathbf{k}} \bigoplus \langle x_{l+1}^2, x_1x_{l+1}, \ldots, x_{h-1}x_{l+1} \rangle_{\mathbf{k}}) \ = \binom{l+1}{2} + h = n. \end{aligned}$$

Set $\underline{F} = \{F_1, \dots, F_n\}$, then one has

$$\begin{cases} \mathrm{HF}_{A_{\underline{F}}}(1) & = & \dim_{\mathbf{k}}(\langle x_1, \dots, x_{l+1}\rangle_{\mathbf{k}}) = l+1 \\ \mathrm{HF}_{A_{\underline{F}}}(2) & = & \dim_{\mathbf{k}}(\langle \mathbb{T}_l^2\rangle \bigoplus \langle x_{l+1}^2, x_1x_{l+1}, \dots, x_{h-1}x_{l+1}\rangle_{\mathbf{k}}) = n \\ \mathrm{HF}_{A_{\underline{F}}}(3) & = & \dim_{\mathbf{k}}(\langle \mathbb{T}_l^3\rangle_{\mathbf{k}} \bigoplus \langle x_{l+1}^3, x_1x_{l+1}^2, \dots, x_{h-1}x_{l+1}^2\rangle_{\mathbf{k}}) = n \end{cases}$$

By assumption and Macaulay's Theorem,

$$n < \mathrm{HF}_A(3) = \tau \leq \binom{l+2}{3} + \binom{h+1}{2}.$$

We consider now $\tau - n$ new polynomials. Notice that

$$\begin{aligned} \tau - n & \leq & \binom{l+2}{3} + \binom{h+1}{2} - n = \binom{l+1}{3} + \binom{h}{2} \\ & = & \#[(\mathbb{T}_l^3 \backslash (x_1 \cdot \mathbb{T}_l^2)) \cup (x_{l+1} \cdot \mathbb{T}_{j-1}^2)]. \end{aligned}$$

We can define $F_{n+1}, \dots, F_\tau$ each one as a monomial in $[\mathbb{T}_l^3 \backslash (x_1 \cdot \mathbb{T}_l^2)) \cup (x_{l+1} \cdot \mathbb{T}_{h-1}^2)]$, chosen with respect to any order. These new $\tau - n$ forms does not modify the value of $\mathrm{HF}_A(2)$, in fact

$$\langle \partial F_{n+1}, \dots, \partial F_\tau\rangle_{\mathbf{k}} \subseteq \langle \mathbb{T}_l^2\rangle \bigoplus \langle x_{l+1}^2, x_1x_{l+1}, \dots, x_{h-1}x_{l+1}\rangle_{\mathbf{k}} = \langle \partial F_1, \dots, \partial F_n\rangle_{\mathbf{k}}.$$

Being $\mathrm{HF}_A(1) = m \geq l+1$, set:

$$f_1 := F_1 + \sum_{i=l+2}^{m} x_l^2 \quad f_j := F_j \quad \text{for all } j = 2, \dots, \tau$$

With this choices $A_{\underline{f}}$ has h-vector H. $\square$

Example 3.4.9. Let $H = (1, 4, 5, 6)$. In this case, $n = 5 < t = 6$ then, by Proposition 3.4.6, we are in a non-graded example. But, if we consider:

$$g_1 := x_1^3, g_2 := x_1^2x_2, g_3 := x_1x_2^2, g_4 := x_2^3, g_5 := x_3^3, g_6 := x_4^3$$

the h-vector of $A_{\underline{g}}$ is $H = (1, 4, 5, 6)$ and $A_{\underline{g}}$ is a graded level algebra.

The following proposition will be proved in Chapter 4.

Proposition 3.4.10. *Let A be an Artin Gorenstein local* $\mathbf{k}$*-algebra with Hilbert function* $HF_A = \{1,3,3,1\}$*. Then A is isomorphic to one and only one of the following quotients of* $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$*:*

Model for $A = R/I$	Inverse system F	Geometry of $C = V(F) \subset \mathbb{P}^2_{\mathbf{k}}$
(x_1^2, x_2^2, x_3^2)	$x_1x_2x_3$	Three independent lines
$(x_1^2, x_1x_3, x_3x_2^2, x_2^3, x_3^2 + x_1x_2)$	$x_2(x_1x_2 - x_3^2)$	Conic and a tangent line
$(x_1^2, x_2^2, x_3^2 + 6x_1x_2)$	$x_3(x_1x_2 - x_3^2)$	Conic and a non-tangent line
$(x_3^2, x_1x_2, x_1^2 + x_2^2 - 3x_1x_3)$	$x_2^2x_3 - x_1^2(x_1 + x_3)$	Irreducible nodal cubic
$(x_3^2, x_1x_2, x_1x_3, x_2^3, x_1^3 + 3x_2^2x_3)$	$x_2^2x_3 - x_1^3$	Irreducible cuspidal cubic
$(x_3^2, x_1^3 + 3x_2^2x_3, x_1x_3, x_2^2 - x_2x_3 + x_3^2, x_1x_2)$	$W(0)$	Elliptic curve $j = 0$
$(x_2^2 + x_1x_3, x_1x_2, x_1^2 - 3x_3^2)$	$W(1728)$	Elliptic curve $j = 1728$
$I(j) = (x_2(x_2 - 2x_1), H_j, G_j)$	$W(j)$	Elliptic curve with $j \neq 0, 1728$

with

$$H_j = 6jx_1x_2 - 144(j - 1728)x_1x_3 + 72(j - 1728)x_2x_3 - (j - 1728)^2x_3^2,$$

and

$$G_j = jx_1^2 - 12(j - 1728)x_1x_3 + 6(k - 1728)x_2x_3 + 144(j - 1728)x_3^2,$$

$I(j_1) \cong I(j_2)$ *if and only if* $j_1 = j_2$*.*

Chapter 4

Working with Singular

The main objective of this chapter is to show how to compute some of the results seen in the project using SINGULAR, [DGPS22]. In order to achieve this purpose, we use INVERSE-SYST.LIB library, by Joan Elias [Eli14].
First, we compute some examples of Macaulay's correspondence. We will check some of the results seen in Chapter 2.
Then, in the second section, we prove a Proposition seen in Chapter 3, which says that there exists an isomorphism betwwen some models for A and its inverse system when A is an Artin Gorenstein local $\mathbf{k}$-algebra with Hilbert function $\mathrm{HF}_A = \{1, 3, 3, 1\}$.
In the last section, we select some command ot the library, the ones used previously.

4.1 Macaulay's correspondence

Our first studied function is going to be one that computes if the quotient $A = R/I$ is Artin or not. This is the function `isAG(I)` and it returns -2 if the quotient is not Artin, -1 if A is Artin but not Gorenstein and if A is an Artin Gorenstein ring, it returns the socle degree of the ring. The function `socle` and `cmType` compute the socle and the Cohen-Macaulay type of A.
Let's compute the Example 3.1.8 via Singular.

```
>ring r=0,(x), ds;
>ideal i=x^20;
>isAG(i);
19
```

In this case, we let $n = 20$. As we expected, we are dealing with an Artin Gorenstein ring and the function returns 19 which is the socle degree of the ring.
Let's see some examples where the output is different.

```
>ring r=0, (x(1..3)), ds;
>ideal i=x(1)^2+x(2)^3, x(2)^4;
>isAG(i);
-2
//Therefore, the quotient is not Artin
>ideal i=x(1)^2 + x(2)^3, x(2)^4+x(1)^2,
            x(3)^2+x(1)*x(2),x(1)*x(2)^2*x(3);
>isAG(i);
-1
//The quotient is Artin but not Gorenstein
>socle(i);
_[1]=x(1)^2
_[2]=x(1)*x(2)+x(3)^2
_[3]=x(2)^3
_[4]=x(2)^2*x(3)
_[5]=x(1)*x(3)^2
_[6]=x(2)*x(3)^2
_[7]=x(3)^3
>cmType(i);
3
```

We can compute the R-module structure of S by contraction ((2.2)) or derivation ((2.1)). Computing again an example seen before, let's check some results of Example 2.2.3.

```
>ring r=0, (x(1..2)), ds;
>ideal G=x(2)^2+x(1)^2*x(1);
>diff(x(1)^2, G);
_[1,1]=6*x(1)
>contract(x(1)^2, G);
_[1,1]=x(1)
>ideal F=x(1)^3;
>diff(x(1)^3, F);
```

```
_[1,1]=6
>contract(x(1)^3, F);
_[1,1]=1
>diff(x(2)^1, G);
_[1,1]=2*x(2)
>contract(x(2)^1, G);
_[1,1]=x(2)
```

One of the most important results that we had seen through this project is Macaulay's correspondence. Let's use some functions of this library to check the results that we have obtained before. We are going to check Example 2.2.6. As it is said in the example, we computed it by contraction. Now, we are going to compute the results by derivation. Through all the process, we will obtained similiar results but with coefficients. At the end, we will achieve the initial f too.

```
>ring r=0, (x(1), x(2)), ds;
>ideal F= x(2)^3+x(1)*x(2)+x(1)^2,
            x(2)^2+x(1), x(2)+x(1), x(2), 1;
>ideal j=idealAnnG(F);
>j;
j[1]=6*x(1)*x(2)-x(2)^3
j[2]=x(1)^2-2*x(1)*x(2)
//Same as the example but with coefficients
>ideal iv=invSyst(j);
>iv;
iv[1]=x(1)^2+x(1)*x(2)+x(2)^3
//The initial polynomial F
```

Now, without coefficients.

```
>ring r=0, (x(1),x(2)), ds;
>ideal F=x(2)^3+x(1)*x(2)+x(1)^2, x(2)^2+x(1), x(2)+x(1), x(2), 1;
>ideal j=idealAnnNC(F);
>j;
j[1]=x(1)^2-x(1)*x(2)
j[2]=x(1)*x(2)-x(2)^3
j[3]=x(2)^4
//we obtain the same with the following formula:
```

```
>ideal j=idealAnnGNC(x(2)^3+x(1)*x(2)+x(1)^2);
>j;
j[1]=x(1)^2-x(1)*x(2)
j[2]=x(1)*x(2)-x(2)^3
j[3]=x(2)^4
>ideal iv=invSystNC(j);
>iv;
iv[1]=x(1)^2+x(1)*x(2)+x(2)^3
```

With that example, we have checked that we can we achieve the identity map with `idealAnn` $\circ$ `invSyst`.

4.2 Artin Gorenstein rings with Hilbert function $\{1,3,3,1\}$

In this section, we are going to classify Artin Gorenstein local rings with Hilbert function $\{1,3,3,1\}$ by using the Legendre equation of an elliptic curve.
We are going to use the following theorem and we are going to assume that $\text{char}(\mathbf{k}) \neq 2$.

Theorem 4.2.1. *The classification of Artin Gorenstein local* $\mathbf{k}$*-algebras with Hilbert function* $HF_A = \{1,n,n,1\}$ *is equivalent to the proective classification of the hypersurfaces* $V(F) \subset \mathbb{P}^{n-1}_{\mathbf{k}}$ *where* F *is a degree three non degenerate form in* n *variables.*

Assume that $n = 3$. We know that any plane elliptic cubic curve $C \subset \mathbb{P}^2_{\mathbf{k}}$ is defined by a Weierstrass' equation

$$W_{a,b} := x_2^2x_3 = x_1^3 + ax_1x_3^2 + bx_3^3$$

with $a, b \in \mathbf{k}$ such that $4a^3 + 27b^2 \neq 0$. The j invariant of C is

$$j(a,b) = 1728\frac{4a^3}{4a^3 + 27b^2}$$

Two plane elliptic cubic curves $C_i = V(W_{a_i,b_i}) \subset \mathbb{P}^2_{\mathbf{k}}$, $i = 1,2$, are projectively isomorphic if and only if $j(a_1,b_1) = j(a_2,b_2)$.

We denote by $W(j)$ the following elliptic curves with j as moduli: $W(0) = x_2^{''}x_3 + x_2x_3^2 - x_1^3$, $W(1728) = x_2^2x_3 - x_1x_3^2 - x_1^3$, and for $j \neq 0, 1728$

$$W(j) = (j-1728)(x_2^2x_3 + x_1x_2x_3 - x_1^3) + 36x_1x_3^2 + x_3^3.$$

More detailed information about j-invariants can be seen at [Har93].
Using `inverse-syst.lib`, we are going to prove:

Proposition 4.2.2. *Let A be an Artin Gorenstein local $\mathbf{k}$-algebra with Hilbert function $HF_A = \{1,3,3,1\}$. Then A is isomorphic to one and only one of the following quotients of $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$:*

Model for $A = R/I$	Inverse system F	Geometry of $C = V(F) \subset \mathbb{P}^2_{\mathbf{k}}$
(x_1^2, x_2^2, x_3^2)	$x_1x_2x_3$	Three independent lines
$(x_1^2, x_1x_3, x_3x_2^2, x_2^3, x_3^2 + x_1x_2)$	$x_2(x_1x_2 - x_3^2)$	Conic and a tangent line
$(x_1^2, x_2^2, x_3^2 + 6x_1x_2)$	$x_3(x_1x_2 - x_3^2)$	Conic and a non-tangent line
$(x_3^2, x_1x_2, x_1^2 + x_2^2 - 3x_1x_3)$	$x_2^2x_3 - x_1^2(x_1 + x_3)$	Irreducible nodal cubic
$(x_3^2, x_1x_2, x_1x_3, x_2^3, x_1^3 + 3x_2^2x_3)$	$x_2^2x_3 - x_1^3$	Irreducible cuspidal cubic
$(x_3^2, x_1^3 + 3x_2^2x_3, x_1x_3, x_2^2 - x_2x_3 + x_3^2, x_1x_2)$	$W(0)$	Elliptic curve $j = 0$
$(x_2^2 + x_1x_3, x_1x_2, x_1^2 - 3x_3^2)$	$W(1728)$	Elliptic curve $j = 1728$
$I(j) = (x_2(x_2 - 2x_1), H_j, G_j)$	$W(j)$	Elliptic curve with $j \neq 0, 1728$

with

$$H_j = 6jx_1x_2 - 144(j-1728)x_1x_3 + 72(j-1728)x_2x_3 - (j-1728)^2x_3^2,$$

and

$$G_j = jx_1^2 - 12(j-1728)x_1x_3 + 6(k-1728)x_2x_3 + 144(j-1728)x_3^2,$$

$I(j_1) \cong I(j_2)$ if and only if $j_1 = j_2$.

It is easy to check the corresponding inverse system F for the first 7 models using `idealAnn`. Let't see some cases.

```
>ring r=0, (x(1..3)), ds;
>ideal i=x(1)^2, x(2)^2, x(3)^2;
>ideal iv=invSyst(i);
>iv;
```

```
iv[1]=x(1)*x(2)*x(3)
>ideal i=x(1)^2, x(1)*x(3), x(3)*x(2)^2, x(2)^3, x(3)^2+x(1)*x(2);
>ideal iv=invSyst(i);
>iv;
iv[1]=x(1)*x(2)^2-x(2)*x(3)^2
```

Assume that $j \neq 0, 1728$. Let $J(j)$ be the ideal $\langle W(j) \rangle^{\perp}$. It is simple to show that $\mathrm{HF}_{R/J(j)} = \{1, 3, 3, 1\}$.

```
>def r=workringc(0,3);
>setring r;
>r;
// coefficients: QQ(c(1))
// number of vars : 3
//        block   1 : ordering ds
//                  : names    x(1) x(2) x(3)
//        block   2 : ordering C
>ideal i=weierstrassp();
>i;
i[1]=(-c(1)+1728)*x(1)^3+(c(1)-1728)*x(1)*x(2)*x(3)
        +(c(1)-1728)*x(2)^2*x(3)+36*x(1)*x(3)^2+x(3)^3
>ideal q=idealwp();
>q;
q[1]=(6*c(1))*x(1)*x(2)+(-144*c(1)+248832)*x(1)*x(3)+
          (72*c(1)-124416)*x(2)*x(3)+c(1)^2
          +3456*c(1)-2985984)*x(3)^2
q[2]=(c(1))*x(1)^2+(-12*c(1)+20736)*x(1)*x(3)+(6*c(1)-10368)
          *x(2)*x(3)+(144*c(1)-248832)*x(3)^2
q[3]=-2*x(1)*x(2)+x(2)^2
//checking that q is contained in i^\perp
>diff(q, i);
_[1,1]=0
_[2,1]=0
_[3,1]=0
>division(maxideal(4),q);
```

From `division(maxideal(4),q)`, we get that the denominators of the coefficient of the matrix Q are constant polynomials or polynomials with roots in $\{0, 1728\}$. Then, for all $j = c(1) \neq 0, 1728$, we get that $m^4 \subset q$, so q is an Artin ideal. Notice that for all $j = c(1) \in \mathbf{k}$, $q = I(j)$ and $p = \langle W(j) \rangle$. $I(j)$ is a homogeneous complete intersection ideal because it is generated by three homogeneous elements. In particular, $I(j)$ is a homogeneous Artin Gorenstein ideal, therefore $\mathrm{HF}_{R/I(j)}$ is symmetric. The generators of $I(j)$ are three homogeneous forms of degree two, so $\mathrm{HF}_{R/I(j)} = \{1, 3, 3, 1\}$. Since $I(j) \subset J(j) = \langle W(j) \rangle^{\perp}$, and $\mathrm{HF}_{R/I(j)} = \mathrm{HF}_{R/J(j)} = \{1, 3, 3, 1\}$, we get $I(j) = J(j) = \langle W(j) \rangle^{\perp}$ and therefore, $I(j) = \langle W(j) \rangle^{\perp}$.

4.3 Important commands

In this section, we are going to select some commands of the library done by Elias [Eli15].

```
MACAULAY INVERSE SYSTEM CORRESPONDENCE WITH COEFFICIENTS

invSystG(ideal J)
returns the inverse system of J; J is Artin Gorenstein
invSyst(J)
returns the inverse system of J; J is Artin
idealAnnG(ideal f)
returns the Artin Gorenstein ideal with inverse system f
idealAnn(I)
returns the Artin  ideal with inverse system I

MACAULAY INVERSE SYSTEM CORRESPONDENCE WITH NO COEFFICIENTS

invSystGNC(ideal J)
returns the inverse system of J; J is Artin Gorenstein
invSystNC(J)
returns the inverse system of J; J is Artin
idealAnnGNC(poly f)
returns the Artin Gorenstein ideal with inverse system f
idealAnnNC(I)
returns the Artin ideal with inverse system I

ELLIPTIC CURVES
```

```
weiertrassp()
returns the ideal generated by weierstrass equation of
the elliptic curve with moduli j=c(1), c(1) is a parameter.
```

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