ADVANCED MATHEMATICS
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Study of Stochastic differential equations driven by fractional Brownian motion

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ABSTRACT

In this thesis we study and develop in detail the research paper *Differential equations driven by fractional brownian motion* by D. Nualart and A. Rascanu, [7]. It is a landmark paper in which the authors prove the existence and uniqueness of solution to stochastic differential equations driven by fractional Brownian motion of Hurst parameter $H \in (1/2, 1)$. Moreover, they show that, under additional hypothesis, the solution has finite moments of all orders. They take a path-by-path approach given the Hölder-continuity property of the paths of the fractional Brownian motion.

On our part, after a gentle introduction to the fractional integrals and derivatives and to the generalized Stieltjes integral, we fully develop the results and proofs of this paper. Not only that but we insert our own remarks and comment on the obtained results regarding the measurability of the solution. As a result, this thesis could be considered a companion paper intended to the reader interested in this important result but not versed in the foundations of stochastic differential equations.

Contents

1. Introduction

One application in stochastic calculus is to predict behaviours modeled by a differential equation which depend on some indeterminable factors. These behaviours and factors are mainly described with stochastic processes. In the particular framework where these behaviours evolve with time and we cannot give an accurate inference on these behaviours, we have to consider any prediction we obtain for every possible outcome of the indeterminable factors.

In this thesis we study the results in the research paper [7] which constructs a regression model based on a stochastic differential equation under the assumption that the previously mentioned indeterminable factors are fractional Brownian motions of Hurst parameter $H \in (1/2, 1)$. More specifically, given $(\sigma_{i,j})_{i,j}$ and $(b_i)_i$ the named diffusion and drift coefficients respectively and B_t a m-dimensional Brownian motion of Hurst parameter $H \in (1/2, 1)$, we aim to find the path-by-path existence and uniqueness of solution $X_t = \left(X_t^1, \ldots, X_t^d\right)$ to

$$X_t^i = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma_{i,j}(s, X_s) dB_s^j + \int_0^t b_i(s, X_s) ds \ , \quad \forall t \in [0, T], i \in \{1, \ldots, d\}, \tag{1}$$

in some space of functions. Under additional hypothesis, we also present interesting properties on the solution X_t. However, we must first give a meaning to the integrals

$$\int_0^t \sigma_{i,j}(s, X_s) dB_s^j, \tag{2}$$

where the integrator is a fractional Brownian motion.

We have organized the contents in the following way. The first milestone we set is to define the generalized Stieltjes integral. Hence, we first dedicate chapter 4 to introduce the fractional integrals and derivatives, that is, we define the Riemann-Liouville fractional integrals and Weyl operators. We also present some of their properties such as the first and second composition and integration by parts formulas. These notions allow us in the next chapter, 5, to define the generalized Stieltjes integral and prove some of its properties. These properties allow us to verify the additivity property of the generalized Stieltjes integral.

We set the second milestone on finding estimates on the operators $G_t^{(\sigma)}$ and $F_t^{(b)}$ (not defined yet) which involve the drift and diffusion coefficients and the generalized Stieltjes integral. For this purpose we need to consider more regular spaces. In chapter 6 we introduce some fractional Sobolev spaces and verify we can use the generalized Stieltjes integral on them. Then, in chapter 7 we present and prove the previously mentioned estimates involving $G_t^{(\sigma)}$ and $F_t^{(b)}$ defined on these fractional Sobolev spaces.

These estimates in chapter 7 are not useful on their own. Instead, they are tailor-made in order to prove a Theorem on the existence and uniqueness of solution to the differential equation (20). This Theorem is obviously the third milestone, together with Proposition 8.5 both in chapter 8. This proposition, under addition hypothesis, provides a bound for the solution in terms of the integrator function in (20).

Finally, in chapter 9 we use a path-by-path approach so that by applying the results in the previous sections we get the existence and uniqueness of solution to (1), and with Proposition 8.5 we prove the solution has finite moments of all orders. Then, chapter 10 is a summary of what we have achieved.

Even though we have neither defined integral (2) nor introduced the space of functions where the solution exists and is unique, we name the hypothesis set on σ, b that will be used in the following sections. Let's consider $d, m \in \mathbb{N}$ and $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. We take $\sigma\colon \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $b\colon \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ measurable functions with σ differentiable with respect to the last argument such that for almost every $\omega \in \Omega$ and $\forall i \in \{1,\dots,d\}$

$$H_\sigma^1 \quad \begin{cases} \bullet \ \text{(Lipschitz continuity): there exists } M_0 > 0 \text{ with} \\ \quad |\sigma(t,x) - \sigma(t,y)| \le M_0|x-y| \quad \forall t \in [0,T],\ \forall x, y \in \mathbb{R}^d \\ \bullet \ \text{(Local Hölder continuity): there exists } \delta \in (0,1], M_N > 0\ \forall N \ge 0 \text{ with} \\ \quad |\partial_{x_i}\sigma(t,x) - \partial_{y_i}\sigma(t,y)| \le M_N|x-y|^\delta \quad \forall t \in [0,T],\ \forall |x|, |y| \le N \\ \bullet \ \text{(Hölder continuity in time): there exists } \beta \in (0,1], M_0 > 0 \text{ with} \\ \quad |\sigma(t,x) - \sigma(s,x)| + |\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(s,x)| \le M_0|t-s|^\beta \\ \hfill \forall t, s \in [0,T],\ \forall x \in \mathbb{R}^d \end{cases}$$

$$H_\sigma^2 \quad \begin{cases} \bullet \ \text{(Boundedness): there exists } \gamma \in [0,1],\ K_0 > 0 \text{ with} \\ \quad |\sigma(t,x)| \le K_0\big(1 + |x|^\gamma\big) \quad \forall t \in [0,T],\ \forall x \in \mathbb{R}^d \end{cases}$$

$$H_b \quad \begin{cases} \bullet \ \text{(Local Lipschitz continuity): there exists } L_N > 0\ \forall N > 0 \text{ with} \\ \quad |b(t,x) - b(t,y)| \le L_N|x-y| \quad \forall t \in [0,T],\ \forall |x|, |y| \le N \\ \bullet \ \text{(Boundedness): there exists } L_0 > 0,\ \rho \ge 2 \text{ and } b_0 \in L^\rho(0,T;\mathbb{R}^d) \text{ with} \\ \quad |b_0(t,x)| \le L_0|x| + b_0(t) \quad \forall t \in [0,T],\ \forall x \in \mathbb{R}^d \end{cases}$$

2. Motivation and objectives

Stochastic differential equations are very useful as regression models whenever we can express a variation of the predicting variables in terms of a variation in our available variables and a one-dimensional variable ranging a bounded interval (i.e. time). In such models we obtain the distribution of the predicting variables but we must first describe our available variables with stochastic processes and at the same time, have well-defined the stochastic differential equation driven by these same stochastic processes. The goal in research papers such as [7] is to broaden the range of stochastic processes in which a stochastic differential equation can be considered.

We aim to reproduce step-by-step the results in [7] and do an analysis on the obtained results. For the first goal we need to properly introduce the required notions, add some results on their well-definiteness and add detail to the already existing proofs and in some cases, we need to provide the entire proof. As for the second goal, we focus our attention on the measurability of the solution.

3. Background

We expect the reader to be familiarized with function analysis and probability theory. Even so, we give a brief introduction to the fractional Brownian motion. We also recall the Beta and Gamma functions and their most known properties.

3.1. **The fractional Brownian motion.** There are several definitions of fractional Brownian motion (f.B.m). We will use the following one.

Definition 3.1. *A stochastic process $B = \{B_t : t \in I\}$ with I an interval in $\mathbb{R}_+$ and with values in $\mathbb{R}$ is a f.B.m of Hurst parameter $H \in (0,1)$ if it is a Gaussian process with covariance function*

$$C_H(s,t) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

If B denotes a f.B.m of Hurst parameter H, we can easily deduce that $\mathbb{E}\big(|B_t - B_s|^2\big) = |t-s|^{2H}$. In fact, in [1] it is proved that the increments of B are normally distributed and consequently, we have

$$||B_t - B_s||_p = c_p|t-s|^H \tag{3}$$

for t, s in its parameter set.

In chapter 9 we prove that B has λ-Hölder continuous paths for $\lambda \in (0,H)$ for almost every realization. However, its paths are nowhere differentiable. This last fact causes difficulties for taking B the integrator in (2) and that is the reason we need the generalized Stieltjes integral.

The multivariate fractional Brownian motion (m.f.B.m) is a stochastic process with values in $\mathbb{R}^n$ with $n \in \mathbb{N}$ whose components are f.B.m. The m.f.B.m is determined by its covariance matrix and by the fact that it is a Gaussian process. Unless it has independent components and all Hurst parameters are the same, the covariance function becomes difficult to work with. Given that we only work with the m.f.B.m when its components are independent and have the same Hurst parameter, its covariance function is simplified to

$$R_H(s,t) = C_H(s,t) \cdot \mathrm{Id}_n,$$

where Id_n is the identity matrix of dimension n, the output dimension of the m.f.B.m. Then, c_p in equation (3) will depend on n as well, and how it will depend will be determined by the norm we set on $\mathbb{R}^n$.

3.2. **Beta and Gamma functions.** Beta and Gamma functions are well-known in several fields of mathematics. In particular, in the theory of fractional integration. We give the definition of Beta and Gamma functions necessary to define the Riemann-Liouville fractional integrals and the Weyl operators. We also state some properties of these two functions which will be useful to prove future results.

We start defining the Beta function in a more general setting than what we need.

Definition 3.2. *For $z, w \in \mathbb{C}$ with $\Re(z), \Re(w) > 0$, we define the Beta function at z, w with*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt.$$

Proposition 3.3.

(a) The Beta function takes finite values.

(b) $B(z,w) = B(w,z)$ $\forall z,w \in \mathbb{C}$ *with* $\Re(z), \Re(w) > 0$.

Proof.

(a) Taking in account that for $x, y \in \mathbb{R}$ $x^{iy} = e^{iy\log(x)}$ and $|x^{iy}| = 1$ if $x > 0$, we have

$$|B(z,w)| = \Big|\int_0^1 t^{z-1}(1-t)^{w-1}dt\Big| \overset{\substack{z=a+bi\ ,\ w=c+di\ ,\ a,b,c,d\in\mathbb{R}\\ \downarrow}}{=} \Big|\int_0^1 t^{a-1}(1-t)^{c-1}t^{bi}(1-t)^{di}dt\Big| =$$

$$\leq \int_0^1 t^{a-1}(1-t)^{c-1}dt = \int_0^{1/2} t^{a-1}(1-t)^{c-1}dt + \int_{1/2}^1 t^{a-1}(1-t)^{c-1}dt \leq$$

$$\leq \max\{1, 2^{1-c}\}\int_0^{1/2} t^{a-1}dt + \max\{1, 2^{1-a}\}\int_{1/2}^1 (1-t)^{c-1}dt < \infty.$$

(b) It follows from the change of variable $s = 1 - t$.

□

Now we introduce the Gamma function and give minimal results we need of the function.

Definition 3.4. *For* $z \in \mathbb{C}$ *with* $\Re(z) > 0$, *we define the Gamma function at* z *with*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Proposition 3.5.

(a) The Gamma function takes finite values.

(b) The Gamma function restricted to $\mathbb{R}_+$ *takes real and strictly positive values. In particular,* $\Gamma(1) = 1$.

Proof.

(a) Taking in account that for $x, y \in \mathbb{R}$ $x^{iy} = e^{iy\log(x)}$ and $|x^{iy}| = 1$ if $x > 0$, we have

$$|\Gamma(z)| \leq \int_0^\infty t^{\Re(z)-1}e^{-t}dt = \int_0^1 t^{\Re(z)-1}e^{-t}dt + \int_1^\infty t^{\Re(z)-1}e^{-t}dt,$$

where $\int_0^1 t^{\Re(z)-1}e^{-t}dt \lesssim \int_0^1 t^{\Re(z)-1}dt < \infty$ and $\int_1^\infty t^{\Re(z)-1}e^{-t}dt$ is finite integrating by parts $\lfloor \Re(z) - 1 \rfloor + 1$ times.

(b) For $x \in \mathbb{R}_+$, we have

$$\Gamma(x) = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt \overset{\substack{t^{x-1}e^{-t}\geq 0 \text{ in } (1,\infty)\\ \downarrow}}{>} \int_0^1 t^{x-1}e^{-t}dt > \int_0^1 t^{x-1}dt = 1/x > 0,$$

so $\Gamma(x) > 0$. Also, the equality $\Gamma(1) = 1$ follows from the fact that $\int_0^\infty e^{-t}dt = 1$.

□

Finally, we give the most important result which relates the Beta and Gamma functions.

Proposition 3.6. *Let $z, w \in \mathbb{C}$ with $\Re(z), \Re(w) > 0$. Then,*

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

Proof. Using Proposition 3.5, we can apply Fubini-Tonelli Theorem and we have

$$\begin{aligned}\Gamma(z)\Gamma(w) &= \int_0^\infty e^u u^{z-1} du \int_0^\infty e^{-v} v^{w-1} dv = \int_0^\infty \int_0^\infty e^{-(u+v)} u^{z-1} v^{w-1} du dv = \\ &= \int_0^\infty \int_0^1 e^{-s} s^{z+w-1} t^{z-1} (1-t)^{w-1} dt ds = \\ &= \int_0^\infty e^{-s} s^{z+w-1} ds \int_0^1 t^{z-1}(1-t)^{w-1} dt = \Gamma(z+w) B(z,w),\end{aligned}$$

using the change of variables $u = st$ for $t \in (0,1)$ and $v = s(1-t)$ for $s \in (0, \infty)$.

□

4. Fractional integrals and derivatives

We introduce the notions of fractional integrals and derivatives for functions in $L^p(a,b)$ with $p \geq 1$ and (a,b) a bounded interval in $\mathbb{R}$. These concepts and some properties stated will allow us to properly define the generalized Stieltjes integral.

Even though we closely follow [7], we require some results found in [8] and [10], and omit other unnecessary results as well. More specifically, these results are on properties of the Riemann-Liouville integration properties and help us on proving properties of the generalized Stieltjes integral.

We start by defining the Riemann-Liouville fractional (right and left)-sided integrals for functions in $L^p(a,b)$ with $p \geq 1$ and with respect to the Lebesgue measure. We deal with intervals (a,b) of finite measure so $L^p(a,b) \subset L^1(a,b)$ and the following definition does not depend on p.

Definition 4.1. *Let $f \in L^1(a,b)$ and $\alpha > 0$. The Riemann-Liouville fractional integrals are defined as*

$$I^{\alpha}_{a+}(f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy,$$

$$I^{\alpha}_{b-}(f)(x) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)dy,$$

where $(-1)^{\alpha} := e^{-i\pi\alpha}$.

The following result is a strong inequality which bounds the Riemann-Liouville fractional integrals under some settings. This result is needed in order to prove Proposition 4.3. However, we do not add its proof since it is involved and it requires results beyond the scope of this thesis.

Theorem 4.2. *Hardy-Littlewood-Sobolev's inequality*
Let $\alpha \in (0,1)$, $1 < p < 1/\alpha$ and $-\infty \leq a < b \leq \infty$. Then, $\forall f \in L^p(a,b)$

$$||I^{\alpha}_{a+}(f)||_q \leq C||f||_p \quad (resp. I^{\alpha}_{b-}),$$

where C is a constant which only depends on p and q, and $q > 1$ satisfies $1/q = 1/p - \alpha$.

Proof. The proof is given in [3] under more general settings.

□

Proposition 4.3. *If $f \in L^p(a,b)$ with $p \geq 1$, then, both sides of the Riemann-Liouville fractional integrals of f belong to $L^p(a,b)$. In particular, the Riemann-Liouville fractional integrals of f converge for almost all $x \in (a,b)$.*

Proof. First, we prove that $I^{\alpha}_{a+}(L^1(a,b)) \subset L^1(a,b)$. Both functions $f(y)$ and $(x-y)^{\alpha-1}\mathbb{1}_{(a,x)}(y)$ are measurable in $L^1(a,b)$. If $\alpha \neq 1$

$$||I^{\alpha}_{a+}(f)(x)||_1 = \int_a^b \Big|\frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)dy\Big|dx \leq$$

$$\leq \frac{1}{|\Gamma(\alpha)|}\int_a^b\int_a^x (x-y)^{\alpha-1}|f(y)|dydx \overset{\text{Fubini-Tonelli theorem}}{=}$$

$$= \frac{1}{|\Gamma(\alpha)|}\int_a^b |f(y)|\int_y^b (x-y)^{\alpha-1}dxdy =$$

$$= \frac{1}{\alpha|\Gamma(\alpha)|}\int_a^b (b-y)^{\alpha}|f(y)|dy \leq \frac{(b-a)^{\alpha}}{\alpha|\Gamma(\alpha)|}||f||_1 < \infty.$$

Instead, if $\alpha = 1$

$$||I^{\alpha}_{a+}(f)(x)||_1 = \int_a^b \Big|\frac{1}{\Gamma(\alpha)}\int_a^x f(y)dy\Big|dx \leq \frac{1}{|\Gamma(\alpha)|}\int_a^b\int_a^x |f(y)|dydx \leq$$

$$\leq \frac{1}{|\Gamma(\alpha)|}\int_a^b\int_a^b |f(y)|dydx \leq \frac{b-a}{|\Gamma(\alpha)|}||f||_1 < \infty.$$

Now, we take $p > 1$ and we prove that $I^{\alpha}_{a+}(L^p(a,b)) \subset L^p(a,b)$. If $\alpha \geq 1$, we can bound $(x-y)^{\alpha-1}$ with the constant $(b-a)^{\alpha-1}$ and applying Jensen's inequality on the function $|\cdot|^p$, we obtain the result. Instead, if $\alpha \in (0,1)$, we first give the proof when $\alpha < \frac{1}{p}$ and then we extend it to when $\alpha \geq \frac{1}{p}$. Let's take $q > 1$ such that $\frac{1}{q} = \frac{1}{p} - \alpha$ so $1 < p < q < \infty$. Applying Theorem 4.2, we have $||I^{\alpha}_{a+}(f)(x)||_q \leq C_p||f||_p < \infty$ where C_p is a constant that only depends on p. Given that $L^q(a,b) \subset L^p(a,b)$ with Hölder inequality, we have proved $I^{\alpha}_{a+}(L^p(a,b)) \subset L^p(a,b)$ when $\alpha < \frac{1}{p}$. Finally, we extend this result when $\alpha \geq \frac{1}{p}$. If $\alpha \geq \frac{1}{p}$, let's consider $p' = \frac{1}{\alpha+\varepsilon}$ with $\varepsilon \in (0, 1-\alpha)$. Then, $1 < p' = \frac{1}{\alpha+\varepsilon} < \frac{1}{\alpha} \leq p$ so $L^p(a,b) \subset L^{p'}(a,b)$ and applying Theorem 4.2 we obtain

$$||I^{\alpha}_{a+}(f)(x)||_{q'} \leq C_{p'}||f||_{p'} < \infty,$$

where $q' = \frac{p'}{1-\alpha p'}$. However, $q' = \big(\frac{1}{\alpha+\varepsilon}\big)/\big(1 - \frac{\alpha}{\alpha+\varepsilon}\big) = \frac{1}{\varepsilon}$ and we can choose ε small enough such that $q' > p$ and in consequence, $||I^{\alpha}_{a+}(f)(x)||_p < \infty$.

The same result with the right-sided Riemann-Liouville fractional integral follows with the same procedure.

□

Remark 4.4. *The Riemann-Liouville fractional integrals are linear operators on $L^1(a,b)$. In addition, since $L^p(a,b)$ is a vector space for any $p \geq 1$, $I^{\alpha}_{a+}\big(L^p(a,b)\big)$ (resp. I^{α}_{b-}) is a vector space.*

The Riemann-Liouville fractional integral has an interesting property, which we call *the first composition formula.* This formula, will allow us to verify properties of the generalized Stieltjes integral.

Proposition 4.5. *First composition formula*
Let $f \in L^1(a,b)$ and $\alpha, \beta > 0$. Then,

$$I^{\beta}_{a+}\big(I^{\alpha}_{a+}(f)\big) = I^{\alpha+\beta}_{a+}(f)$$

(resp. $I^{\alpha}_{b-}, I^{\beta}_{b-}$ everywhere).

Proof. We complete the proof for the left-sided operator, the right-sided follows with the same argument. We previously proved that $I^{\gamma}_{a+}(L^1(a,b)) \subset L^1(a,b)$, $\forall \gamma > 0$. Hence, $I^{\alpha}_{a+}(f) \in L^1(a,b)$ and we have

$$\begin{aligned}
I^{\beta}_{a+}\big(I^{\alpha}_{a+}(f)\big) &= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^x (x-y)^{\beta-1}\int_a^y (y-z)^{\alpha-1} f(z)dzdy \overset{\text{Fubini-Tonelli Theorem}}{=} \\
&= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^x f(z)\int_z^x (x-y)^{\beta-1}(y-z)^{\alpha-1}dydz \overset{y=z+s(x-z)}{=} \\
&= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^x f(z)\int_0^1 (x-z-s(x-z))^{\beta-1}s^{\alpha-1}(x-z)^{\alpha-1}(x-z)dsdz = \\
&= \frac{1}{\Gamma(\beta)\Gamma(\alpha)}\int_a^x f(z)(x-z)^{\alpha+\beta-1}dz\int_0^1 (1-s)^{\beta-1}s^{\alpha-1}ds = \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\Gamma(\alpha)}B(\alpha,\beta)I^{\alpha+\beta}_{a+}(f) \overset{\text{Proposition 3.6}}{=} I^{\alpha+\beta}_{a+}(f).
\end{aligned}$$

□

When $\alpha \in (0,1)$, we define the (left and right)-sided Weyl operators which we will see later that they are the opposite operators to the Riemann-Liouville fractional integral operators.

Definition 4.6. *Let $f \in I^{\alpha}_{a+}(L^1(a,b))$ (respectively I^{α}_{b-}) and $\alpha \in (0,1)$. Then the left-sided (respectively the right-sided) Weyl derivative of f with parameter α is defined as:*

$$D^{\alpha}_{a+}(f)(x) = \frac{1}{\Gamma(1-\alpha)}\Big(\frac{f(x)}{(x-a)^{\alpha}} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}dy\Big)\mathbb{1}_{(a,b)}(x)$$

(respectively),

$$D^{\alpha}_{b-}(f)(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\Big(\frac{f(x)}{(b-x)^{\alpha}} + \alpha\int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}}dy\Big)\mathbb{1}_{(a,b)}(x).$$

Proposition 4.7. *If $1 \le p < \infty$ and $f = I^{\alpha}_{a+}(\varphi)$ (respectively I^{α}_{b-}) with $\varphi \in L^p(a,b)$, then, the corresponding Weyl operator is in L^p-sense, the function φ.*

Proof. This result is presented as a Theorem in [8], chapter 13.1.

□

Remark 4.8. *The Weyl operators are linear on $I^{\alpha}_{a+}\big(L^p(a,b)\big)$ (resp. I^{α}_{b-}) for any $p \ge 1$.*

The *first composition formula* which involves the Riemann-Liouville fractional integral can be rewritten with the Weyl derivative operators due to Proposition 4.7.

Proposition 4.9. *Second composition formula*
For $p \ge 1$, let $f \in I^{\alpha+\beta}_{a+}\big(L^p(a,b)\big)$ (resp. $I^{\alpha+\beta}_{b-}$) with $\alpha, \beta \in (0,1)$ and $\alpha+\beta < 1$. Then,

$$D^{\alpha}_{a+}\big(D^{\beta}_{a+}(f)\big) = D^{\alpha+\beta}_{a+}(f)$$

(resp. D^{α}_{b-}, D^{β}_{b-} *and* $D^{\alpha+\beta}_{b-}$*).*

Proof. We complete the proof for the left-sided operator, the right-sided follows with the same argument. If $f \in I^{\alpha+\beta}_{a+}\big(L^p(a,b)\big)$, there exists $\psi \in L^p(a,b)$ such that $I^{\alpha+\beta}_{a+}(\psi) = f$. Let's denote $\psi_\alpha = I^{\alpha}_{a+}(\psi)$ and by the first composition formula, $f = I^{\alpha+\beta}_{a+}(\psi) = I^{\beta}_{a+}(\psi_\alpha)$. Then,

$$D^{\alpha}_{a+}\big(D^{\beta}_{a+}(f)\big) = D^{\alpha}_{a+}\big(D^{\beta}_{a+}(I^{\beta}_{a+}(\psi_\alpha))\big) \overset{\text{Proposition 4.7}}{=} D^{\alpha}_{a+}(\psi_\alpha) = D^{\alpha}_{a+}\big(I^{\alpha}_{a+}(\psi)\big) \overset{\text{Proposition 4.7}}{=} \psi,$$

where $\psi = D^{\alpha+\beta}_{a+}(f)$ again due to Proposition 4.7.

□

We present a last property on fractional integration, which proves to be a useful tool when dealing with generalized Stieltjes integrals.

Proposition 4.10. *Integration-by-parts formula*
For $p, q \geq 1$ *with* $\frac{1}{p} + \frac{1}{q} \leq 1$ *and* $\alpha \in (0,1)$*, if* $f \in I^{\alpha}_{a+}\big(L^p(a,b)\big)$ *and* $g \in I^{\alpha}_{b-}\big(L^q(a,b)\big)$*, then,*

$$(-1)^\alpha \int_a^b D^{\alpha}_{a+}(f)(x) g(x) dx = \int_a^b f(x) D^{\alpha}_{b-}(g)(x) dx.$$

Proof. If $f \in I^{\alpha}_{a+}\big(L^p(a,b)\big)$ and $g \in I^{\alpha}_{b-}\big(L^q(a,b)\big)$, there exists $\varphi \in L^p(a,b)$ and $\psi \in L^q(a,b)$ such that $I^{\alpha}_{a+}(\varphi) = f$ and $I^{\alpha}_{b-}(\psi) = g$. With *Proposition* 4.7 $D^{\alpha}_{a+}(f) = \varphi \in L^p(a,b)$ and with *Proposition* 4.3 $g \in L^q(a,b)$. Then, with Hölder inequality, the integral $\int_a^b D^{\alpha}_{a+}(f)(x) g(x) dx$ is well-defined in $L^1(a,b)$. In fact,

$$(-1)^\alpha \int_a^b D^{\alpha}_{a+}(f)(x) g(x) dx = (-1)^\alpha \int_a^b \varphi(x) g(x) dx = (-1)^\alpha \int_a^b \varphi(x) I^{\alpha}_{b-}(\psi)(x) dx =$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^b \varphi(x) \int_x^b (y-x)^{\alpha-1} \psi(y) dy dx \overset{\text{Fubini-Tonelli Theorem}}{=} \frac{1}{\Gamma(\alpha)} \int_a^b \psi(y) \int_a^x (y-x)^{\alpha-1} \varphi(x) dx dy =$$

$$= \int_a^b I^{\alpha}_{a+}(\varphi)(y) \psi(y) dy \overset{\text{Proposition 4.7}}{=} \int_a^b f(y) D^{\alpha}_{b-}(g)(y) dy,$$

as we wanted to prove.

□

Remark 4.11. *The result can be extended to when* $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ *with Theorem 4.2. However, the simplification we present is sufficient for proving properties of the generalized Stieltjes integral.*

5. Generalized Stieltjes integral

We are interested in defining an integral where the integrator is a Hölder-continuous function. Such integral, under more general settings is called the generalized Stieltjes integral.

The generalized Stieltjes integral is defined using the notions on fractional integrals of Riemann-Liouville. However, we will need to add more hypothesis on the functions involved.

Definition 5.1. *Let f be a measurable mapping from (a,b) to $\mathbb{R}$ such that*

$$f(a+) = \lim_{\varepsilon \searrow 0} f(a+\varepsilon) \quad \big(\textit{resp.} \quad f(b-) = \lim_{\varepsilon \searrow 0} f(b-\varepsilon)\ \big)$$

exists and it is finite. Then, we define the function f_{a+} (resp. f_{b-}) as

$$f_{a+}(x) = f(x) - f(a+) \quad \big(\textit{resp.} \quad f_{b-}(x) = f(x) - f(b-)\ \big).$$

Definition 5.2. *Let $f, g\colon (a,b) \to \mathbb{R}$ be Lebesgue-measurable functions such that $f(a+)$, $g(a+), g(b-)$ exist and they are finite. Assume $f_{a+} \in I^{\alpha}_{a+}(L^p(a,b))$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q(a,b))$ with $\alpha \in (0,1)$ and $p, q \geq 1$ such that $1/p + 1/q \leq 1$. Then, the integral of f with respect to g is defined as*

$$\int_a^b f dg := (-1)^{\alpha} \int_a^b D^{\alpha}_{a+}(f_{a+})(x) D^{1-\alpha}_{b-}(g_{b-})(x) dx + f(a+)\big(g(b-) - g(a+)\big),$$

where the integral is with respect to the Lebesgue measure.

In order for this definition to have meaning, we need to check that the integral involved is well-defined and it does not depend on α.

Proposition 5.3. *Under the assumptions in Definition 5.2, the generalized Stieltjes integral is well-defined and takes finite values.*

Proof. By definition of f_{a+} and g_{b-}, there exists $\varphi \in L^p(a,b)$ and $\psi \in L^q(a,b)$ such that $f_{a+} = I^{\alpha}_{a+}(\varphi)$ and $g_{b-} = I^{1-\alpha}_{b-}\big(\psi\big)$. With Proposition 4.7, $\varphi = D^{\alpha}_{a+}(f_{a+})$, $\psi = D^{1-\alpha}_{b-}(g_{b-})$. Applying Hölder inequality, the integral part of $\int_a^b f dg$ satisfies:

$$\Big|(-1)^{\alpha} \int_a^b D^{\alpha}_{a+}(f_{a+})(x) D^{1-\alpha}_{b-}(g_{b-})(x) dx\Big| = \Big| \int_a^b \varphi(x)\psi(x) dx\Big| \leq$$
$$\leq \int_a^b |\varphi(x)| \cdot |\psi(x)| dx \leq ||\varphi||_p ||\psi||_{p'}$$

where $p' \geq 1$ is such that $1/p + 1/p' = 1$. If $p = 1$, $p' = q = \infty$ so $||\varphi||_p||\psi||_{p'} < \infty$. Otherwise, $q \geq p' = \frac{p}{p-1}$ and $||\varphi||_p||\psi||_{p'} < \infty$ since $L^q(a,b) \subset L^{p'}(a,b)$.

□

Proposition 5.4. *The generalized Stieltjes integral does not depend on the parameter α.*

Proof. Let's take $\alpha, \alpha' \in (0,1)$ with $\alpha' > \alpha$. Assume that for $\gamma \in \{\alpha, \alpha'\}$, $f_{a+} \in I^{\gamma}_{a+}\big(L^{p(\gamma)}(a,b)\big)$ and $g_{b-} \in I^{1-\gamma}_{b-}\big(L^{q(\gamma)}(a,b)\big)$ where $p(\gamma), q(\gamma) \geq 1$ and $1/p(\gamma) + 1/q(\gamma) \leq$

1. Then, the generalized Stieltjes integral of f with respect to g is well-defined either taking α or α'. We define $\beta = \alpha' - \alpha > 0$ and we have:

$$
\begin{aligned}
&(-1)^{\alpha'} \int_a^b D_{a+}^{\alpha'}(f_{a+}) D_{b-}^{1-\alpha'}(g_{b-}) dx = \\
&= (-1)^{\alpha}(-1)^{\beta} \int_a^b D_{a+}^{\alpha+\beta}(f_{a+}) D_{b-}^{1-\alpha-\beta}(g_{b-}) dx \overset{\text{Proposition 4.9}}{=} \\
&= (-1)^{\alpha}(-1)^{\beta} \int_a^b D_{a+}^{\beta}\big(D_{a+}^{\alpha}(f_{a+})\big) D_{b-}^{1-\alpha-\beta}(g_{b-}) dx \overset{\text{Proposition 4.10}}{=} \\
&= (-1)^{\alpha} \int_a^b D_{a+}^{\alpha}(f_{a+}) D_{b-}^{\beta}\big(D_{b-}^{1-\alpha-\beta}(g_{b-})\big) dx \overset{\text{Proposition 4.9}}{=} (-1)^{\alpha} \int_a^b D_{a+}^{\alpha}(f_{a+}) D_{b-}^{1-\alpha}(g_{b-}) dx
\end{aligned}
$$

so the generalized Stieltjes integral does not depend on α.

□

The following result simplifies the computation of the generalized Stieltjes integral under more restricting settings.

Proposition 5.5. *Let $\alpha \in (0,1)$ and $p, q \geq 1$ such that $1/p + 1/q \leq 1$. Assume $f, g \colon (a,b) \to \mathbb{R}$ are Lebesgue-measurable functions such that $f(a+), g(a+), g(b-)$ exist and they are finite. If $\alpha p < 1$, then,*

(a) $f \in I_{a+}^{\alpha}\big(L^p(a,b)\big)$ if and only if $f_{a+} \in I_{a+}^{\alpha}\big(L^p(a,b)\big)$,

(b) if $f \in I_{a+}^{\alpha}\big(L^p(a,b)\big)$ and $g_{b-} \in I_{b-}^{1-\alpha}\big(L^q(a,b)\big)$, we can rewrite the generalized Stieltjes integral as

$$
\int_a^b f\, dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha}\big(f\big)(x) D_{b-}^{1-\alpha}\big(g_{b-}\big)(x) dx.
$$

Proof.

(a) The double implication, due to Remark 4.4, is simplified to prove that under $\alpha p < 1$, $f(a+) \in I_{a+}^{\alpha}\big(L^p(a,b)\big)$. If we find $\varphi \in L^p(a,b)$ such that $I_{a+}^{\alpha}(\varphi) = f(a+)$, then, point (a) will be proved.

We consider $\varphi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{f(a+)}{(x-a)^{\alpha}}$ which belongs to $L^p(a,b)$ since $\alpha p < 1$. Then,

$$
\begin{aligned}
I_{a+}^{\alpha}(\varphi) &= \frac{f(a+)}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_a^x \frac{(x-y)^{\alpha-1}}{(y-a)^{\alpha}} dy \overset{z=(y-a)/(x-a)}{=} \frac{f(a+)}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 z^{-\alpha}(1-z)^{\alpha-1} dz = \\
&= f(a+) \frac{B(1-\alpha,\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} \overset{\text{Proposition 3.6}}{=} f(a+)/\Gamma(1) \overset{\text{Proposition 3.5}}{=} f(a+).
\end{aligned}
$$

(b) Due to point (a), both $f, f_{a+} \in I_{a+}^{\alpha}\big(L^p(a,b)\big)$ and with Remark 4.8, $f(a+) \in I_{a+}^{\alpha}\big(L^p(a,b)\big)$. In fact, in point (a) we have seen that $h(x) := \frac{1}{\Gamma(1-\alpha)} \frac{f(a+)}{(x-a)^{\alpha}} \in L^p(a,b)$ with $I_{a+}^{\alpha}(h) = f(a+)$ and applying Proposition 4.7, $D_{a+}^{\alpha}\big(f(a+)\big) =$

$\frac{1}{\Gamma(1-\alpha)}\frac{f(a+)}{(x-a)^\alpha}$. Hence, we can split the integral part in the Stieltjes integral $\int_a^b fdg$

$$\int_a^b fdg = (-1)^\alpha \int_a^b D^\alpha_{a+}(f)D^{1-\alpha}_{b-}(g_{b-})dx - (-1)^\alpha \int_a^b D^\alpha_{a+}(f(a+))D^{1-\alpha}_{b-}(g_{b-})dx + \\ + f(a+)\big(g(b-)-g(a+)\big),$$

where

$$\int_a^b D^\alpha_{a+}\big(f(a+)\big)D^{1-\alpha}_{b-}(g_{b-})dx = \frac{f(a+)}{\Gamma(1-\alpha)}\int_a^b \frac{1}{(x-a)^\alpha}D^{1-\alpha}_{b-}(g_{b-})dx.$$

Now, there exists $\varphi \in L^q(a,b)$ such that $I^{1-\alpha}_{b-}(\varphi) = g_{b-}$ and with Proposition 4.7, $\varphi = D^{1-\alpha}_{b-}(g_{b-})$. However,

$$\frac{f(a+)}{\Gamma(1-\alpha)}\int_a^b (x-a)^{-\alpha}D^{1-\alpha}_{b-}(g_{b-})dx = \frac{f(a+)}{\Gamma(1-\alpha)}\int_a^b (x-a)^{(1-\alpha)-1}\varphi(x)\,dx = \\ = (-1)^{1-\alpha}f(a+)I^{1-\alpha}_{b-}(\varphi)(a) = (-1)^{1-\alpha}f(a+)g_{b-}(a+) = \\ = (-1)^{1-\alpha}f(a+)\big(g(a+)-g(b-)\big).$$

With this equality, we can rewrite the generalized Stieltjes integral as

$$\int_a^b fdg = (-1)^\alpha \int_a^b D^\alpha_{a+}(f)D^{1-\alpha}_{b-}(g_{b-})dx - (-1)^\alpha(-1)^{1-\alpha}f(a+)\big(g(a+)-g(b-)\big) + \\ + f(a+)\big(g(b-)-g(a+)\big) = (-1)^\alpha \int_a^b D^\alpha_{a+}(f)D^{1-\alpha}_{b-}(g_{b-})dx.$$

□

Before jumping to the next section, we give a last property of the generalized Stieltjes integral. We have defined the integral on the whole domain (a,b) but we might be interested in integrating only a sub-interval $(c,d) \subset (a,b)$. We want to define the indefinite generalized Stieltjes integral and show it satisfies the additivity property.

The formal expression $\int_a^b fdg$ is well-defined under the hypothesis in Definition 5.2 but such hypothesis might not be enough to have well-defined $\int_c^d fdg$ as well. Thus, our aim is to see under what additional assumptions, the integral $\int_c^d fdg$ is well-defined and how we can rewrite it in terms of an integral of domain (a,b). We start by giving some results on the corresponding function f.

Theorem 5.6. *Let $\alpha \in (0,1)$, $p \in (1, 1/\alpha)$ and $(a,b) \subset \mathbb{R}$ a finite interval with $(c,d) \subset (a,b)$. Then,*

(a) $f \in I^\alpha_{a+}(L^p(a,b))$ implies $f|_{(c,d)} \in I^\alpha_{c+}(L^p(c,d))$.

(b) $f \in I^\alpha_{c+}(L^p(c,d))$ implies $\mathbb{1}_{(c,d)}f \in I^\alpha_{a+}(L^p(a,b))$.

Proof. The Theorem is stated as a Corollary in [8], chapter 13.3. The proof follows from the previous Theorems and it involves the Riemann-Liouville fractional integral operator on the whole real line.

□

Under the hypothesis of Proposition 5.5 over the interval (a, b) and with Theorem 5.6, we know $f|_{(c,d)} \in I^{\alpha}_{c+}(L^p(c, d))$. If we additionally assume that there exists $g(d-)$ finite and $g_{d-} \in I^{1-\alpha}_{d-}(L^q(c, d))$, then, the integral $\int_c^d f dg$ is well-defined.

Once seen $\int_c^d f dg$ is well-defined we want to write the integral in terms of an integral of domain (a, b).

Theorem 5.7. *Under the hypothesis of Proposition 5.5, if we further assume that there exists $g(d-)$ finite and $g_{d-} \in I^{1-\alpha}_{d-}(L^q(c, d))$, then,*

$$\int_c^d f dg = \int_a^b \mathbb{1}_{(c,d)} f dg. \tag{4}$$

Proof. The Theorem is stated and proved in [10]. Notice that with Theorem 5.6, the left side of equation (4) is well-defined. Also, we can easily write $D^{\alpha}_{c+}(f)$ in terms of $D^{\alpha}_{a+}(\mathbb{1}_{(c,d)} f)$ but we need more involved results in order to write $D^{1-\alpha}_{d-}(g_{d-})$ in terms of $D^{1-\alpha}_{b-}(g_{b-})$ and for equality (4) to hold.

□

6. The generalized Stieltjes integral on fractional Sobolev spaces

We have been working with L^p functions and their image with respect to the Riemann-Liouville fractional integrals. Now, we consider functions in fractional Sobolev spaces which satisfy Hölder continuity properties. Our goal is to check that the indefinite generalized Stieltjes integral is well-defined on these new type of functions.

From now on, we set $\alpha \in (0, 1/2)$, $a = 0$ and $b = T$ for $0 < T < \infty$ since these are the parameters α and intervals we will work with from now on.

We start by introducing these type of function spaces with $W_0^{\alpha,1}(0, T; \mathbb{R}^d)$.

Definition 6.1. *For $T > 0$, $d \in \mathbb{N}$ and $\alpha \in (0, 1/2)$ we define $W_0^{\alpha,1}(0, T; \mathbb{R}^d)$ the set of Lebesgue-measurable functions $f : [0, T] \to \mathbb{R}^d$ such that*

$$||f||_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dy ds < \infty,$$

where $|\cdot|$ is the norm in $\mathbb{R}^d$. In particular, we denote such set by $W_0^{\alpha,1}(0, T)$ when $d = 1$.

Remark 6.2. *$W_0^{\alpha,1}(0, T; \mathbb{R}^d)$ is actually a normed vector space with respect to $||\cdot||_{\alpha,1}$.*

Remark 6.3. *For $T > 0$ and $\alpha \in (0, 1/2)$, $W_0^{\alpha,1}(0, T) \subset I_{0+}^\alpha(L^1(0, T))$.*

With Remark 6.3, notice that condition $f \in W_0^{\alpha,1}(0, T)$ is stronger than $D_{0+}^\alpha(f) \in L^1(0, T)$, that is, not only both components of $D_{0+}^\alpha(f)$ must be in $L^1(0, T)$ but we also take the differences $f(s) - f(y)$ in absolute value.

We define the set $W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^d)$, now, centered at T instead of at 0.

Definition 6.4. *For $T > 0$, $d \in \mathbb{N}$ and $\alpha \in (0, 1/2)$, we consider $W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^d)$ the set of Lebesgue-measurable functions $f : [0, T] \to \mathbb{R}^d$ such that*

$$||f||_{1-\alpha,\infty,T} := \sup_{0 \le s < t \le T} \left\{ \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|f(y) - f(s)|}{(y-s)^{2-\alpha}} dy \right\} < \infty,$$

where $|\cdot|$ is the norm in $\mathbb{R}^d$. In particular, we denote such set by $W_T^{1-\alpha,\infty}(0, T)$ when $d = 1$.

Remark 6.5. *$W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^d)$ is actually a normed vector space with respect to $||\cdot||_{1-\alpha,\infty,T}$.*

Remark 6.6. *For $T > 0$ and $\alpha \in (0, 1/2)$, if $g \in W_T^{1-\alpha,\infty}(0, T)$ and $g(T-)$ exists and it is finite, then, $g_{T-} \in I_{T-}^{1-\alpha}(L^\infty(0, T))$.*

We intent to integrate functions $f \in W_0^{\alpha,1}(0, T; \mathbb{R}^d)$ with respect to some function $g \in W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^d)$. However, the following sections require f to satisfy further conditions so we will consider the space $W_0^{\alpha,\infty}(0, T; \mathbb{R}^d)$ which is a subset of $W_0^{\alpha,1}(0, T; \mathbb{R}^d)$.

We can relate the spaces $W_0^\alpha(0, T; \mathbb{R}^d)$ and $W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^d)$ with the spaces of Hölder-continuous functions. The Hölder-continuity property is essential for defining the indefinite generalized Stieltjes integral and proving results in the following sections. Thus, we introduce these spaces.

Definition 6.7. *For $T > 0$, $d \in \mathbb{N}$ and $\gamma \in (0,1]$, we denote by $C^\gamma(0,T;\mathbb{R}^d)$ the set of Lebesgue-measurable functions $f\colon [0,T] \to \mathbb{R}^d$ such that*

$$||f||_\gamma := ||f||_\infty + \sup_{0\le s<t\le T} \left\{ \frac{|f(t)-f(s)|}{(t-s)^\gamma} \right\} < \infty,$$

where $|\cdot|$ is the norm in $\mathbb{R}^d$. In particular, we denote such set by $C^\gamma(0,T)$ when $d = 1$.

Remark 6.8. *$C^\gamma(0,T;\mathbb{R}^d)$ is actually a normed vector space with respect to $||\cdot||_\gamma$.*

Proposition 6.9. *For $T > 0$, $d \in \mathbb{N}$ and $\alpha \in (0,1/2)$, $W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^d) \subset C^{1-\alpha}(0,T;\mathbb{R}^d)$.*

Proof. We need to check that for $f \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^d)$, $||f||_{1-\alpha} < \infty$. On one hand, we have

$$\sup_{0\le s<t\le T} \left\{ |f(t)-f(s)| \right\} \le T^{1-\alpha} \sup_{0\le s<t\le T} \left\{ \frac{|f(t)-f(s)|}{(t-s)^{1-\alpha}} \right\} \le T^{1-\alpha} \cdot ||f||_{1-\alpha,\infty,T} < \infty,$$

that implies $|f(t)| < \infty$ $\forall t \in [0,T]$ (not just a set of plenty measure) and $||f||_\infty < \infty$.

On the other hand, the second term that appears in $||f||_{1-\alpha}$ is bounded by $||f||_{1-\alpha,\infty,T}$.

□

As an immediate corollary, $g \in W_T^{1-\alpha,\infty}(0,T)$ is continuous, $g(t-)$ exists and it is finite $\forall t \in (0,T]$. In fact, together with Remark 6.6, we can easily check that $g_{t-} \in I_{t-}^{1-\alpha}(L^\infty(0,t))$ and we would like to apply Theorem 5.7 to have the additivity property of the indefinite integral. However, this procedure would require applying Theorem 5.6 to check that $f|_{[0,t]} \in I_{0+}^\alpha(L^1(0,t))$ and $\mathbb{1}_{(0,t)} f \in I_{0+}^\alpha(L^1(0,T))$ but we are taking $p = 1 \not> 1$ in Theorem 5.6.

Instead, we can bypass Theorem 5.6 by checking that $f|_{[0,t]} \in W_0^{\alpha,1}(0,t)$ and $\mathbb{1}_{(0,t)} f \in W_0^{\alpha,1}(0,T)$ which is straightforward. Hence, we can apply Theorem 5.7 and we obtain the additivity of the indefinite integral.

As promised previously, we introduce the set $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$.

Definition 6.10. *For $T > 0$, $d \in \mathbb{N}$ and $\alpha \in (0,1/2)$, we denote $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ the set of Lebesgue-measurable functions $f\colon [0,T] \to \mathbb{R}^d$ such that*

$$||f||_{\alpha,\infty} := \sup_{t\in[0,T]} \left\{ |f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds \right\} < \infty,$$

where $|\cdot|$ is the norm in $\mathbb{R}^d$. In particular, we denote such set by $W_0^{\alpha,\infty}(0,T)$ when $d = 1$.

Remark 6.11. *$W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ is a Banach space with respect to the norm $||f||_{\alpha,\infty}$.*

Proposition 6.12. *For $T > 0$, $d \in \mathbb{N}$ and $\alpha \in (0,1/2)$, $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \subset W_0^{\alpha,1}(0,T;\mathbb{R}^d)$.*

Proof. The result follows from the Hölder inequality.

□

Proposition 6.13. *For $T > 0$, $d \in \mathbb{N}$ and $\alpha \in (0,1/2)$, $C^{1-\alpha}(0,T;\mathbb{R}^d) \subset W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$.*

Proof. It's sufficient to prove that for $f \in C^{1-\alpha}(0,T;\mathbb{R}^d)$, $||f||_{\alpha,\infty} < \infty$. In fact, we can bound $||f||_{\alpha,\infty}$ with $||f||_{1-\alpha}$ multiplied by a constant. That is,

$$\begin{aligned}
||f||_{\alpha,\infty} &= \sup_{t\in[0,T]} \Big\{|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds\Big\} \leq \\
&\leq ||f||_\infty + \sup_{t\in[0,T]} \Big\{ \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds\Big\} = \\
&= ||f||_\infty + \sup_{t\in[0,T]} \Big\{ \int_0^t (t-s)^{-2\alpha} \cdot \frac{|f(t)-f(s)|}{(t-s)^{1-\alpha}} ds\Big\} \leq \\
&\leq ||f||_\infty + \sup_{t\in[0,T]} \Big\{ \int_0^t (t-s)^{-2\alpha} ds \cdot \sup_{s\in[0,t]} \big\{\frac{|f(t)-f(s)|}{(t-s)^{1-\alpha}}\big\}\Big\} \leq \\
&\leq ||f||_\infty + \frac{T^{1-2\alpha}}{1-2\alpha} \sup_{0\leq s\leq t\leq T} \Big\{\frac{|f(t)-f(s)|}{(t-s)^{1-\alpha}}\Big\} \leq \max\{1, \frac{T^{1-2\alpha}}{1-2\alpha}\} \cdot ||f||_{1-\alpha} < \infty.
\end{aligned}$$

□

We have introduced the spaces in Definitions 6.1, 6.4, 6.7 and 6.10, proved in the one-dimensional case their inclusion to more general sets, and shown a way to have the indefinite generalized Stieltjes integral well-defined. From now on, we will mainly work with multi-dimensional functions and so, we need to clarify the results seen so far.

Take $f \in W_0^{\alpha,1}(0,T;\mathbb{R}^d)$ and $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^d)$ with $d \in \mathbb{N}$. It's straightforward to check that all components of f and g belong to $W_0^{\alpha,1}(0,T)$ and $W_T^{1-\alpha,\infty}(0,T)$ respectively, thus, we can extend the Weyl operator in Definition 4.6 to a component-wise operator.

Hence, the results seen in this section hold for every component and it's easy to check that in terms of the indefinite generalized Stieltjes integral,

$$\int_0^t f\, dg = \Big(\int_0^t f_i\, dg_i\Big)_{i=1}^d = (-1)^\alpha \int_0^t D_{0+}^\alpha(f)(s) \star D_{t-}^{1-\alpha}(g_{t-})(s) ds$$

where $\star$ denotes the broadcasting product.

7. A priori estimates

The indefinite generalized Stieltjes can be thought either as a function in time or as an operator with respect to the function we integrate once t is fixed. We present bounds for the indefinite generalized Stieltjes integral in terms of the norm of the functions we integrate.

These bounds are Tailor-made for proving the existence and uniqueness of solutions to a differential equation of the form (20).

However, the generalized Stieltjes integral depends on the integrator function as well. Thus, we before define the notion of $\Lambda_\alpha(g)$ which will appear in these estimates.

Definition 7.1. *For $T > 0$, $m \in \mathbb{N}$, $\alpha \in (0, 1/2)$ and $g \in W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^m)$, we denote*

$$\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0 \le s \le t \le T} \left\{ \left| D_{t-}^{1-\alpha}(g_{t-})(s) \right| \right\}.$$

Remark 7.2. *$\Lambda_\alpha(g)$ is well-defined since all components in g_{t-} always exists in $I_{t-}^{1-\alpha}(L^\infty(0, T))$ with Remark 6.6 and Proposition 6.9. In particular, the indices in the supremum can be written $0 < s \le t < T$ and the value of $\Lambda_\alpha(g)$ does not change.*

Remark 7.3. *Notice that we can upper-bound $\Lambda_\alpha(g)$ with*

$$\Lambda_\alpha(g) \le \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} ||g||_{1-\alpha,\infty,T} < \infty.$$

We can think of the indefinite generalized Stieltjes integral as an operator from $W_0^{\alpha,1}(0, T; \mathbb{R}^d)$ with values in $\mathbb{R}^d$.

Definition 7.4. *For $T > 0$, $d \in \mathbb{N}$, $\alpha \in (0, 1/2)$, $f \in W_0^{\alpha,1}(0, T; \mathbb{R}^d)$, $g \in W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^d)$ and $t \in [0, T]$, we denote*

$$G_t(f) := \int_0^t f\, dg.$$

Remark 7.5. *Under the hypothesis in the definition, we also know $f \in W_0^{\alpha,1}(0, t; \mathbb{R}^d)$, $g \in W_t^{1-\alpha,\infty}(0, t; \mathbb{R}^d)$ and the operator $G_t(f)$ is thus well-defined and coincides with the integral $\int_0^T \mathbb{1}_{(0,t)} f dg$.*

We can check G_t is linear (as an operator) from either $W_0^{\alpha,1}(0, T; \mathbb{R}^d)$ or $W_0^{\alpha,1}(0, t; \mathbb{R}^d)$ to $\mathbb{R}^d$ and with Proposition 7.6, when $d = 1$, bounded.

Proposition 7.6. *For $T > 0$, $\alpha \in (0, 1/2)$, $f \in W_0^{\alpha,1}(0, T)$, $g \in W_T^{1-\alpha,\infty}(0, T)$ and $t \in [0, T]$, we have*

$$|G_t(f)| \le \Lambda_\alpha(g) \cdot ||f||_{\alpha,1}.$$

Proof. With the help of the previous results, we can apply Proposition 5.5 and we can write the indefinite generalized Stieltjes integral as

$$\int_0^t f dg = (-1)^\alpha \int_0^t D_{0+}^\alpha(f)(s) \cdot D_{t-}^{1-\alpha}(g_{t-})(s) ds.$$

Given that $D^{\alpha}_{0+}(f)$ and $D^{1-\alpha}_{t-}(g_{t-})$ belong to $L^1(0,t)$ and $L^{\infty}(0,t)$ respectively, we apply Hölder inequality and we have

$$\Big|\int_0^t f dg\Big| \leq \big\|D^{1-\alpha}_{t-}(g_{t-})\big\|_{\infty}\int_0^t |D^{\alpha}_{0+}(f)(s)|ds \leq \Lambda_{\alpha}(g)\cdot ||f||_{\alpha,1}.$$

□

The first estimate we present under a more general setting than the following ones but we restrict ourselves to one-dimensional functions.

Proposition 7.7. *For $T>0$, $\alpha\in(0,1/2)$, $f\in W^{\alpha,1}_0(0,T)$ and $g\in W^{1-\alpha}_T(0,T)$, if $s,t\in[0,T]$ with $s<t$, we have*

$$|G_t(f)-G_s(f)| \leq \Lambda_{\alpha}(g)\int_s^t\Big(\frac{|f(r)|}{(r-s)^{\alpha}}+\alpha\int_s^r\frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}}dy\Big)dr \tag{5}$$

and

$$|G_t(f)|+\int_0^t\frac{|G_t(f)-G_s(f)|}{(t-s)^{\alpha+1}}ds \leq$$
$$\Lambda_{\alpha}(g)c^{(1)}_{\alpha,T}\int_0^t\big((t-r)^{-2\alpha}+r^{-\alpha}\big)\Big(|f(r)|+\int_0^r\frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}}dy\Big)dr, \tag{6}$$

where $c^{(1)}_{\alpha,T}$ is a constant which only depends on α and T.

In addition, if $f\in W^{\alpha,\infty}_0(0,T)$, then, $G_t(f)\in C^{1-\alpha}(0,T)$ and

$$||G_t(f)||_{1-\alpha}\leq \Lambda_{\alpha}(g)c^{(2)}_{\alpha,T}||f||_{\alpha,\infty}, \tag{7}$$

where $c^{(2)}_{\alpha,T}$ is again a constant which only depends on α and T.

Proof. With the additivity property of the indefinite generalized Stieltjes integral, we have

$$|G_t(f)-G_s(f)| = \Big|\int_s^t f dg\Big|.$$

In the previous section, we proved that for f,g under the hypothesis of this Proposition, we can apply Proposition 5.5, and

$$|G_t(f)-G_s(f)| = \Big|\int_s^t D^{\alpha}_{s+}(f)(r)\cdot D^{1-\alpha}_{t-}(g_{t-})(r)dr\Big| \overset{\text{Hölder inequality}}{\leq}$$
$$\leq ||D^{1-\alpha}_{t-}(g_{t-})||_{\infty}\cdot\int_s^t|D^{\alpha}_{s+}(f)(r)|dr \leq \Lambda_{\alpha}(g)\int_s^t\Big(\frac{|f(r)|}{(r-s)^{\alpha}}+\alpha\int_s^r\frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}}dy\Big)dr,$$

which proves equation (5).

We multiply by $(t-s)^{-\alpha-1}$ and integrate with respect to s in $(0,t)$ at both sides of equation (5) and by monotony,

$$\int_0^t\frac{|G_t(f)-G_s(f)|}{(t-s)^{\alpha+1}}ds \leq \Lambda_{\alpha}(g)\int_0^t(t-s)^{-\alpha-1}\Big(\int_s^t\frac{|f(r)|}{(r-s)^{\alpha}}dr+$$
$$+\alpha\int_s^t\int_s^r\frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}}dy\,dr\Big)ds.$$

We do not know if the right-side of the inequality converges but since all terms are positive, we can still apply Fubini-Tonelli Theorem and split the integral with the additive property. The first term satisfies

$$\int_0^t (t-s)^{-\alpha-1} \int_s^t \frac{|f(r)|}{(r-s)^\alpha} dr\, ds = \int_0^t \int_s^t (t-s)^{-\alpha-1} \frac{|f(r)|}{(r-s)^\alpha} dr\, ds \overset{\overset{0\le s\le r\le t}{\downarrow}}{=}$$
$$= \int_0^t |f(r)| \int_0^r \frac{(t-s)^{-\alpha-1}}{(r-s)^\alpha} ds\, dr \overset{\overset{s=r-(t-r)y}{\downarrow}}{=} \int_0^t |f(r)|(t-r)^{-2\alpha} \int_0^{r/(t-r)} (1+y)^{-\alpha-1} y^{-\alpha} dy\, dr \le$$
$$\le \int_0^t |f(r)|(t-r)^{-2\alpha} B(2\alpha, 1-\alpha)\, dr,$$

meanwhile the second term satisfies

$$\alpha \int_0^t (t-s)^{-\alpha-1} \int_s^t \int_s^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy\, dr\, ds =$$
$$= \alpha \int_0^t \int_s^t \int_s^r (t-s)^{-\alpha-1} \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy\, dr\, ds \overset{\overset{0\le s\le y\le r\le t}{\downarrow}}{=}$$
$$= \alpha \int_0^t \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} \int_0^y (t-s)^{-\alpha-1} ds\, dy\, dr \le \int_0^t \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} (t-y)^{-\alpha} dy\, dr.$$

If we join both estimations, we obtain

$$\int_0^t \frac{|G_t(f)-G_s(f)|}{(t-s)^{\alpha+1}} ds \le$$
$$\le \Lambda_\alpha(g) \Big[\int_0^t |f(r)|(t-r)^{-2\alpha} B(2\alpha, 1-\alpha) dr + \int_0^t \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} (t-y)^{-\alpha} dy\, dr \Big]. \tag{8}$$

Now, once proved equation (5), with $s = 0$

$$|G_t(f)| \le \Lambda_\alpha(g) \int_0^t \Big(\frac{|f(r)|}{r^\alpha} + \alpha \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy \Big) dr$$

and together with equation (8), we have

$$|G_t(f)| + \int_0^t \frac{|G_t(f)-G_s(f)|}{(t-s)^{\alpha+1}} ds \le \Lambda_\alpha(g) \Big[\int_0^t |f(r)| \big(r^{-\alpha} + (t-r)^{-2\alpha} B(2\alpha, 1-\alpha)\big) dr +$$
$$+ \int_0^t \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} \big(\alpha + (t-y)^{-\alpha}\big) dy\, dr \Big].$$

We need to find a constant bound that might only depend on α and T, $c^{(1)}_{\alpha,T}$, so that $\forall y \in [0, r]$ and $r \in (0, t)$

$$\max\Big(r^{-\alpha} + (t-r)^{-2\alpha} B(2\alpha, 1-\alpha)\, ,\, \alpha + (t-y)^{-\alpha} \Big) \le c^{(1)}_{\alpha,T} \big(r^{-\alpha} + (t-r)^{-2\alpha} \big).$$

If we consider $c^{(1)}_{\alpha,T} = \max(B(2\alpha, 1-\alpha), 1) + T^\alpha$, then, the condition above holds and equation (6) follows.

Finally, we assume $f \in W_0^{\alpha,\infty}(0,T)$ and we have to prove $G_t(f) \in C^{1-\alpha}(0,T)$ satisfying equation (7). With Proposition 6.12, $f \in W_0^{\alpha,1}(0,T)$ so $G_t(f)$ is well-defined and with Proposition 7.6, $|G_t(f)| < \infty$ $\forall t \in [0,T]$.

On one hand, we bound the $||G_t(f)||_\infty$ term of $C^{1-\alpha}(0,T)$-norm. From equation (7),

$$||G_t(f)||_\infty \le \Lambda_\alpha(g) \sup_{0\le t\le T} \Big\{ \int_0^t \Big(\frac{|f(r)|}{r^\alpha} + \alpha \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy\Big) dr \Big\} =$$

$$= \Lambda_\alpha(g) \int_0^T \Big(\frac{|f(r)|}{r^\alpha} + \alpha \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy\Big) dr =$$

$$= \Lambda_\alpha(g) \int_0^T r^{-\alpha} \Big(|f(r)| + \alpha r^\alpha \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy\Big) dr \overset{\text{Hölder inequality}}{\le}$$

$$\le \Lambda_\alpha(g) \frac{T^{1-\alpha}}{1-\alpha} \sup_{0\le r\le T} \Big\{ |f(r)| + \alpha r^\alpha \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy \Big\} \le$$

$$\le \Lambda_\alpha(g) \frac{T^{1-\alpha}}{1-\alpha} \max\{1, \alpha T^\alpha\} \cdot ||f||_{\alpha,\infty}.$$

On the other hand, we bound the Hölder term of $C^{1-\alpha}(0,T)$-norm. Applying estimate (5) with $t > s$

$$|G_t(f) - G_s(f)| \le \Lambda_\alpha(g) \int_s^t \Big(\frac{|f(r)|}{(r-s)^\alpha} + \alpha \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy\Big) dr \le$$

$$\le \Lambda_\alpha(g) \int_s^t (r-s)^{-\alpha} \Big(|f(r)| + \alpha (r-s)^\alpha \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy\Big) dr \le$$

$$\overset{\text{Hölder inequality}}{\le} \Lambda_\alpha(g) \int_s^t (r-s)^{-\alpha} dr \sup_{s\le r\le t} \Big\{ |f(r)| + \alpha (r-s)^\alpha \int_0^r \frac{|f(r)-f(y)|}{(r-y)^{\alpha+1}} dy \Big\} \le$$

$$\le \Lambda_\alpha(g) \frac{(t-s)^{1-\alpha}}{1-\alpha} \max\{1, \alpha T^\alpha\} \cdot ||f||_{\alpha,\infty}.$$

Combining both terms of the $C^{1-\alpha}(0,T)$-norm we obtain

$$||G_t(f)||_{1-\alpha} \le \Lambda_\alpha(g) \cdot c^{(2)}_{\alpha,T} \cdot ||f||_{\alpha,\infty}$$

with $c^{(2)}_{\alpha,T} = \frac{\max\{1,\alpha T^\alpha\}}{1-\alpha}\big(1 + T^{1-\alpha}\big)$ which proves the last part of the Proposition.

□

Once defined the notion G_t as an operator on $W_0^{\alpha,1}(0,T)$, we introduce a very similar operator which is closer to what we work with in the following sections. This new operator will focus only on functions in $W_0^{\alpha,\infty}(0,T)$ but instead of integrating some function $f \in W_0^{\alpha,\infty}(0,T)$, we integrate $\sigma(t, f(t))$ where σ satisfies hypothesis $\mathbf{H^1_\sigma}$.

Another important remark is that we will possibly deal with multi-dimensional functions so the previous results must be applied carefully.

Definition 7.8. *For $T > 0$, $d, m \in \mathbb{N}$, $\alpha \in (0, 1/2)$, $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$, $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$ and $t \in [0,T]$, we denote*

$$G_t^{(\sigma)}(f) := \Big(\sum_{j=1}^m \int_0^t \sigma_{i,j}(s, f(s))\, dg_s^j \Big)_{i=1}^d$$

where $\sigma\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ satisfying hypothesis $\mathbf{H^1_\sigma}$ in the introduction with $\beta > \alpha$ and g^j is the j-th component of g.

Proposition 7.10 proves the well-definiteness of $G_t^{(\sigma)}$ as an operator on $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ and gives continuity results with respect to the parameter t. However, we first need to introduce an equivalent norm to the current one on $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$.

Definition 7.9. *For $T>0$, $d\in\mathbb{N}$, $\alpha\in(0,1/2)$, $f\in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ and $\lambda\geq 0$, we define*

$$||f||_{\alpha,\lambda} \coloneqq \sup_{t\in[0,T]}\Big\{e^{-\lambda t}\cdot\Big(|f(t)|+\int_0^t\frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}}ds\Big)\Big\}.$$

We can easily check that the norms $||\cdot||_{\alpha,\infty}$ and $||\cdot||_{\alpha,\lambda}$ are equivalent, and given that $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ is a Banach space with the norm $||\cdot||_{\alpha,\infty}$, $\big(W_0^{\alpha,\infty}(0,T;\mathbb{R}^d),||\cdot||_{\alpha,\lambda}\big)$ is Banach as well.

Proposition 7.10. *For $T>0$, $d,m\in\mathbb{N}$, $f\in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$, $g\in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$ and σ deterministic under the hypothesis $\mathbf{H}^1_\sigma$,*

(a) *$G_t^{(\sigma)}(f)\in C^{1-\alpha}(0,T;\mathbb{R}^d)\subset W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$.*

(b) *There exists $C^{(2)}$ constant with respect to f and g such that*

$$||G_t^{(\sigma)}(f)||_{1-\alpha}\leq\Lambda_\alpha(g)C^{(2)}\big(1+||f||_{\alpha,\infty}\big).$$

(c) *There exists $C^{(3)}$ constant with respect to f and g such that $\forall\lambda\geq 1$*

$$||G_t^{(\sigma)}(f)||_{\alpha,\lambda}\leq\frac{\Lambda_\alpha(g)\cdot C^{(3)}}{\lambda^{1-2\alpha}}\big(1+||f||_{\alpha,\lambda}\big).$$

(d) *If $h\in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ such that $||f||_\infty,||h||_\infty\leq N$, then, there exists $C_N^{(4)}$ constant with respect to f,g,h such that $\forall\lambda\geq 1$*

$$||G_t^{(\sigma)}(f)-G_t^{(\sigma)}(h)||_{\alpha,\lambda}\leq\frac{\Lambda_\alpha(g)C_N^{(4)}}{\lambda^{1-2\alpha}}\big(1+\Delta(f)+\Delta(h)\big)\cdot||f-h||_{\alpha,\lambda}$$

where

$$\Delta(f)=\sup_{r\in[0,T]}\Big\{\int_0^r\frac{|f(r)-f(s)|^\delta}{(r-s)^{\alpha+1}}ds\Big\}.$$

Proof. We prove the Proposition in the simplified case where $d=m=1$.

(a) We first check that $\sigma(t,f(t))\in W_0^{\alpha,\infty}(0,T)$, that is, we must verify that $||\sigma(t,f(t))||_{\alpha,\infty}<\infty$. Applying Lipschitz continuity and Hölder continuity on time properties,

$$|\sigma(r,f(r))|+\int_0^r\frac{|\sigma(r,f(r))-\sigma(s,f(s))|}{(r-s)^{\alpha+1}}ds\leq|\sigma(r,0)|+|\sigma(r,f(r))-\sigma(r,0)|+$$
$$+\int_0^r\frac{|\sigma(r,f(r))-\sigma(r,f(s))|+|\sigma(r,f(s))-\sigma(s,f(s))|}{(r-s)^{\alpha+1}}ds\leq$$
$$\leq|\sigma(0,0)|+|\sigma(r,0)-\sigma(0,0)|+M_0|f(r)|+M_0\int_0^r\frac{|f(r)-f(s)|+(r-s)^\beta}{(r-s)^{\alpha+1}}ds\leq$$
$$\leq|\sigma(0,0)|+M_0r^\beta+M_0\frac{r^{\beta-\alpha}}{\beta-\alpha}+M_0|f(r)|+M_0\int_0^r\frac{|f(r)-f(s)|}{(r-s)^{\alpha+1}}ds.$$

Hence,

$$||\sigma(r,f(r))||_{\alpha,\infty}\leq C+M_0||f||_{\alpha,\infty}<\infty \tag{9}$$

with $C = |\sigma(0,0)| + M_0\big(T^\beta + \frac{T^{\beta-\alpha}}{\beta-\alpha}\big)$. Applying Proposition 7.7, we obtain $G_t^{(\sigma)}(f) \in C^{1-\alpha}(0,T)$.

(b) The proof follows from applying again Proposition 7.7 on $\sigma(t, f(t))$ and equation (9) afterwards so that

$$\begin{aligned}||G_t^{(\sigma)}(f)||_{1-\alpha} &\le \Lambda_\alpha(g) c^{(2)}_{\alpha,T} ||\sigma(t,f(t))||_{\alpha,\infty} \le \Lambda_\alpha(g) c^{(2)}_{\alpha,T}\big(C + M_0 ||f||_{\alpha,\infty}\big) \le \\ &\le \Lambda_\alpha(g) c^{(2)}_{\alpha,T}\big(C + M_0\big)\big(1 + ||f||_{\alpha,\infty}\big) = \Lambda_\alpha(g) C^{(2)}\big(1+||f||_{\alpha,\infty}\big).\end{aligned}$$

(c) Right from the definition of $||\cdot||_{\alpha,\lambda}$,

$$\begin{aligned}
||G_t^{(\sigma)}(f)||_{\alpha,\lambda} &= \sup_{t\in[0,T]} \Big\{ e^{-\lambda t}\cdot\Big(|G_t^{(\sigma)}(f)| + \int_0^t \frac{|G_t^{(\sigma)}(f) - G_r^{(\sigma)}(f)|}{(t-r)^{\alpha+1}} dr\Big)\Big\} \overset{\text{Proposition 7.7}}{\le} \\
&\le \Lambda_\alpha(g) c^{(1)}_{\alpha,T} \sup_{t\in[0,T]} \Big\{ e^{-\lambda t} \int_0^t \big((t-r)^{-2\alpha} + r^{-\alpha}\big) \times \\
&\qquad \Big(|\sigma(r,f(r))| + \int_0^r \frac{|\sigma(r,f(r)) - \sigma(y,f(y))|}{(r-y)^{\alpha+1}} dy\Big) dr \Big\} \le \\
&\overset{\text{Proof in (a)}}{\le} \Lambda_\alpha(g) c^{(1)}_{\alpha,T} \sup_{t\in[0,T]} \Big\{ e^{-\lambda t} \int_0^t \big((t-r)^{-2\alpha} + r^{-\alpha}\big) \times \\
&\qquad \Big(C + M_0 |f(r)| + M_0 \int_0^r \frac{|f(r) - f(y)|}{(r-y)^{\alpha+1}} dy\Big) dr \Big\} = \\
&= \Lambda_\alpha(g) c^{(1)}_{\alpha,T} \sup_{t\in[0,T]} \Big\{ \int_0^t e^{-\lambda(t-r)} \big((t-r)^{-2\alpha} + r^{-\alpha}\big) \times \\
&\qquad e^{-\lambda r}\Big(C + M_0 |f(r)| + M_0 \int_0^r \frac{|f(r) - f(y)|}{(r-y)^{\alpha+1}} dy\Big) dr \Big\} \le \\
&\overset{\text{Hölder inequality}}{\le} \Lambda_\alpha(g) c^{(1)}_{\alpha,T} \sup_{t\in[0,T]} \Big\{ \int_0^t e^{-\lambda(t-r)} \big((t-r)^{-2\alpha} + r^{-\alpha}\big) dr \times \\
&\qquad \sup_{s\in[0,t]} \Big\{ e^{-\lambda s}\Big(C + M_0 |f(s)| + M_0 \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dy\Big)\Big\}\Big\}
\end{aligned}$$

where $\forall t \in [0,T]$,

$$\begin{aligned}
\int_0^t e^{-\lambda(t-r)}\big((t-r)^{-2\alpha} + r^{-\alpha}\big) dr &\overset{x = t-r}{=} \int_0^t e^{-\lambda x}\big(x^{-2\alpha} + (t-x)^{-\alpha}\big) dx \overset{y=\lambda x}{=} \\
&= \lambda^{-1} \int_0^{\lambda t} e^{-y}\big(\lambda^{2\alpha} y^{-2\alpha} + \lambda^\alpha (\lambda t - y)^{-\alpha}\big) dy \le \\
&\le \lambda^{2\alpha-1}\Big(\int_0^{\lambda t} e^{-y} y^{-2\alpha} dy + \int_0^{\lambda t} e^{-y}(\lambda t - y)^{-\alpha} dy\Big) \le \\
&\le \lambda^{2\alpha-1}\Big(\Gamma(1-2\alpha) + \sup_{z>0}\Big\{\int_0^z e^{-y}(z-y)^\alpha dy\Big\}\Big) =: \lambda^{2\alpha-1} c_\alpha. \qquad (10)
\end{aligned}$$

Thus, provided that c_α is finite, we can bound $||G_t^{(\sigma)}(f)||_{\alpha,\lambda}$ with

$$||G_t^{(\sigma)}(f)||_{\alpha,\lambda} \leq \Lambda_\alpha(g)\frac{c_{\alpha,T}^{(1)}c_\alpha}{\lambda^{1-2\alpha}} \sup_{s\in[0,T]} \Big\{e^{-\lambda s}\Big(C + M_0|f(s)| + \\ + M_0\int_0^s \frac{|f(s)-f(y)|}{(s-y)^{\alpha+1}}dy\Big)\Big\} \leq \Lambda_\alpha(g)\frac{c_{\alpha,T}^{(1)}c_\alpha}{\lambda^{1-2\alpha}}\Big(C + M_0||f||_{\alpha,\lambda}\Big) \leq \\ \leq \Lambda_\alpha(g)\frac{c_{\alpha,T}^{(1)}c_\alpha}{\lambda^{1-2\alpha}}(C+M_0)\cdot(1+||f||_{\alpha,\lambda}) =: \Lambda_\alpha(g)\frac{C^{(3)}}{\lambda^{1-2\alpha}}(1+||f||_{\alpha,\lambda}).$$

Finally, we need to prove c_α is finite. We know that the Gamma function takes finite values with Proposition 3.5 so we only need to check that the supremum in (10) is finite. One way to prove it is by splitting the integral domain $[0,z]$ into $[0,\min(z,1)]$ and $[\min(z,1),z]$, then, applying the sub-additivity property of the supremum and bound both supremum but since it is tedious, we do not include the procedure.

(d) Even though the operator $G_t^{(\sigma)}$ is not necessarily linear, G_t is and right from the definition of $||\cdot||_{\alpha,\lambda}$,

$$||G_t^{(\sigma)}(f) - G_t^{(\sigma)}(h)||_{\alpha,\lambda} = ||G_t\big(\sigma(\cdot,f)-\sigma(\cdot,h)\big)||_{\alpha,\lambda} = \\ = \sup_{t\in[0,T]}\Big\{e^{-\lambda t}\cdot\Big(\big|G_t\big(\sigma(\cdot,f)-\sigma(\cdot,h)\big)\big| + \\ + \int_0^t \frac{\big|G_t\big(\sigma(\cdot,f)-\sigma(\cdot,h)\big) - G_s\big(\sigma(\cdot,f)-\sigma(\cdot,h)\big)\big|}{(t-s)^{\alpha+1}}ds\Big)\Big\} \leq$$

$$\underset{\text{Proposition 7.7}}{\leq} \Lambda_\alpha(g)c_{\alpha,T}^{(1)}\sup_{t\in[0,T]}\Big\{e^{-\lambda t}\int_0^t\big((t-r)^{-2\alpha}+r^{-\alpha}\big)\cdot\Big(|\sigma(t,f(t))-\sigma(t,h(t))| + \\ + \int_0^r \frac{\big|\sigma(t,f(t))-\sigma(t,h(t))-\sigma(s,f(s))+\sigma(s,h(s))\big|}{(t-s)^{\alpha+1}}ds\Big)dr\Big\}.$$

In the item (c) of the proof, we obtained a similar expression so following the same procedure and using the properties in $\mathbf{H}_\sigma^\mathbf{1}$, we obtain

$$||G_t^{(\sigma)}(f) - G_t^{(\sigma)}(h)||_{\alpha,\lambda} \leq \Lambda_\alpha(g)\frac{c_{\alpha,T}^{(1)}c_\alpha}{\lambda^{1-2\alpha}}\sup_{t\in[0,T]}\Big\{e^{-\lambda t}\cdot\Big(M_0|f(t)-h(t)| + \\ + \int_0^t \frac{\big|\sigma(t,f(t))-\sigma(t,h(t))-\sigma(s,f(s))+\sigma(s,h(s))\big|}{(t-s)^{\alpha+1}}ds\Big)\Big\}.$$

Now, using the Local Hölder continuity and Hölder continuity in time properties of σ, we know $\partial_x\sigma$ is continuous in $[0,T]\times\mathbb{R}$ and we can apply the Mean Value

Theorem so as to obtain

$$
\begin{aligned}
&\sigma(t,f(t))-\sigma(t,h(t))-\sigma(s,f(s))+\sigma(s,h(s))=\\
&\quad=\big(f(t)-h(t)\big)\int_0^1\partial_x\sigma\big(t,h(t)+\theta(f(t)-h(t))\big)d\theta+\\
&\quad+\big(h(s)-f(s)\big)\int_0^1\partial_x\sigma\big(s,h(s)+\theta(f(s)-h(s))\big)d\theta=\\
&\quad=\big(f(t)-h(t)\big)\int_0^1\Big[\partial_x\sigma\big(t,h(t)+\theta(f(t)-h(t))\big)\\
&\qquad\qquad-\partial_x\sigma\big(s,h(s)+\theta(f(s)-h(s))\big)\Big]d\theta+\\
&\quad+\big(f(t)-h(t)+h(s)-f(s)\big)\int_0^1\partial_x\sigma\big(s,h(s)+\theta(f(s)-h(s))\big)d\theta
\end{aligned}
$$

Consequently,

$$
|\sigma(t,f(t))-\sigma(t,h(t))-\sigma(s,f(s))+\sigma(s,h(s))|\le\big|f(t)-h(t)\big|\times
$$

$$
\times\int_0^1\big|\partial_x\sigma\big(t,h(t)+\theta(f(t)-h(t))\big)-\partial_x\sigma\big(s,h(s)+\theta(f(s)-h(s))\big)\big|d\theta+ \tag{11}
$$

$$
+\big|f(t)-h(t)+h(s)-f(s)\big|\int_0^1\big|\partial_x\sigma\big(s,h(s)+\theta(f(s)-h(s))\big)\big|d\theta, \tag{12}
$$

where the integral in (11) can be bounded applying the Local Hölder continuity and Hölder continuity in time properties of σ, and the integral in (12) can be bounded applying the Lipschitz continuity property of σ on the definition of derivative. Thus, obtaining

$$
\begin{aligned}
&|\sigma(t,f(t))-\sigma(t,h(t))-\sigma(s,f(s))+\sigma(s,h(s))|\le M_0|f(t)-h(t)+h(s)-f(s)|+\\
&+M_0|f(t)-h(t)|\cdot|t-s|^\beta+M_N|f(t)-h(t)|\cdot\big(|h(t)-h(s)|^\delta+|f(t)-f(s)|^\delta\big).
\end{aligned}
$$

Finally, putting the results together we have

$$
\begin{aligned}
&||G_t^{(\sigma)}(f)-G_t^{(\sigma)}(h)||_{\alpha,\lambda}\le\\
&\le\Lambda_\alpha(g)\frac{c^{(1)}_{\alpha,T}c_\alpha}{\lambda^{1-2\alpha}}\sup_{t\in[0,T]}\Big\{e^{-\lambda t}\Big(M_N|f(t)-h(t)|\int_0^t\frac{|h(t)-h(s)|^\delta+|f(t)-f(s)|^\delta}{(t-s)^{\alpha+1}}ds+\\
&+M_0(1+\frac{t^{\beta-\alpha}}{\beta-\alpha})\cdot|f(t)-h(t)|+M_0\int_0^t\frac{|f(t)-h(t)+h(s)-f(s)|}{(t-s)^{\alpha+1}}ds\Big)\Big\}\le\\
&\le\Lambda_\alpha(g)\frac{c^{(1)}_{\alpha,T}c_\alpha}{\lambda^{1-2\alpha}}\big((1+\frac{T^{\beta-\alpha}}{\beta-\alpha})M_0+M_N(\Delta(f)+\Delta(h))\big)||f-h||_{\alpha,\lambda}
\end{aligned}
$$

and by taking $C^{(4)}=c^{(1)}_{\alpha,T}c_\alpha(M_0+M_N)\cdot(1+\frac{T^{\beta-\alpha}}{\beta-\alpha})$, the estimate in (d) holds.

□

The results on the generalized Stieltjes integral and more specifically the operator $G_t^{(\sigma)}$ will prove useful to have well-defined the integrals (2).

Now, we give some estimates on the drift coefficient. These estimates will be very similar to the estimates obtained on the diffusion coefficient but fortunately, much easier to prove since we integrate with respect to time and the functions involved are Lebesgue-integrable.

Definition 7.11. *For $T > 0$, $d \in \mathbb{N}$ and $f \in L^1(0,T;\mathbb{R}^d)$, we define*

$$F_t(f) \coloneqq \int_0^t f(s)ds$$

where the integral operator is applied component-wise.

Proposition 7.12. *For $T > 0$, $d \in \mathbb{N}$, $\alpha \in (0, 1/2)$ and $f\colon [0,T] \to \mathbb{R}^d$ measurable with*

$$\sup_{t\in[0,T]} \left\{ \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds \right\} < \infty,$$

then, $F_t(f) \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ and

$$|F_t(f)| + \int_0^t \frac{|F_t(f) - F_s(f)|}{(t-s)^{\alpha+1}} ds \leq C_{\alpha,T} \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds \tag{13}$$

with $C_{\alpha,T} = T^\alpha + \alpha^{-1}$.

In addition, if $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$, then, $F_t(f) \in C^1(0,T;\mathbb{R}^d)$,

$$|F_t(f) - F_s(f)| \leq (t-s) \cdot ||f||_\infty \tag{14}$$

and

$$||F_t(f)||_{\alpha,\infty} \leq C'_{\alpha,T} ||f||_\infty, \tag{15}$$

with $C'_{\alpha,T} = C_{\alpha,T} \cdot \frac{T^{1-\alpha}}{1-\alpha}$.

Proof. We start by proving equation (13). From the hypothesis on f, we can check $f \in L^1(0,T;\mathbb{R}^d)$ so we can apply F_t on f $\forall t \in [0,T]$. Hence,

$$|F_t(f)| + \int_0^t \frac{|F_t(f) - F_s(f)|}{(t-s)^{\alpha+1}} ds \leq \int_0^t |f(s)|ds + \int_0^t (t-s)^{-\alpha-1} \int_s^t |f(r)|dr\, ds \overset{\text{Fubini-Tonelli Theorem}}{=}$$
$$= \int_0^t |f(s)|ds + \int_0^t |f(r)| \int_0^r (t-s)^{-\alpha-1} ds\, dr = \int_0^t |f(r)|dr +$$
$$+ \alpha^{-1} \int_0^t |f(r)| \cdot \big((t-r)^{-\alpha} - t^{-\alpha}\big) dr \leq \int_0^t |f(r)|dr + \alpha^{-1} \int_0^t |f(r)| \cdot (t-r)^{-\alpha} dr =$$
$$= \int_0^t |f(r)| \cdot (t-r)^{-\alpha} \cdot \big(T^\alpha + \alpha^{-1}\big) dr,$$

which proves (13) and implies $F_t(f) \in W_0^{\alpha,T}(0,T;\mathbb{R}^d)$.

For the second part of the Proposition, we use the fact that $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \subset L^\infty(0,T;\mathbb{R}^d)$ and applying Hölder inequality on $|F_t(f) - F_s(f)|$ we get (14) and consequently, $F_t(f) \in C^1(0,T;\mathbb{R}^d)$.

As for equation (15),

$$||F_t(f)||_{\alpha,\infty} = \sup_{t\in[0,T]} \left\{ |F_t(f)| + \int_0^t \frac{|F_t(f) - F_s(f)|}{(t-s)^{\alpha+1}} ds \right\} \overset{(13)}{\leq}$$
$$\leq C_{\alpha,T} \sup_{t\in[0,T]} \left\{ \int_0^t \frac{|f(s)|}{(t-s)^\alpha} ds \right\} \overset{\text{Hölder inequality}}{\leq} C_{\alpha,T} \frac{T^{1-\alpha}}{1-\alpha} ||f||_\infty = C'_{\alpha,T} ||f||_\infty.$$

□

We introduce a similar operator to F_t involving the drift coefficient b satisfying $\mathbf{H_b}$.

Definition 7.13. *For $T>0$, $d\in\mathbb{N}$, $f\in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ and $t\in[0,T]$, we denote*

$$F_t^{(b)}(f):=\int_0^t b(s,f(s))ds,$$

where $b\colon[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies hypothesis $\mathbf{H_b}$ in the introduction.

Applying the Boundedness property of b, we check the operator $F_t^{(b)}$ has values in $\mathbb{R}^d$ $\forall t\in[0,T]$ even for functions in $L^1(0,T;\mathbb{R}^d)$.

Proposition 7.14. *For $T>0$, $d\in\mathbb{N}$, $\alpha\in(0,1/2)$, $f\in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ and b deterministic under hypothesis $\mathbf{H_b}$ with $\rho=1/\alpha$,*

(a) $F_t^{(b)}(f)\in C^{1-\alpha}(0,T;\mathbb{R}^d)$.

(b) There exists $d^{(1)}$ constant with respect to f such that

$$||F_t^{(b)}(f)||_{1-\alpha}\le d^{(1)}(1+||f||_\infty).$$

(c) There exists $d^{(2)}$ constant with respect to f such that $\forall\lambda\ge1$,

$$||F_t^{(b)}(f)||_{\alpha,\lambda}\le\frac{d^{(2)}}{\lambda^{1-2\alpha}}(1+||f||_{\alpha,\lambda}).$$

(d) If $h\in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ such that $||f||_\infty,||h||_\infty\le N$, then, there exists d_N constant with respect to f,h such that $\forall\lambda\ge1$,

$$||F_t^{(b)}(f)-F_t^{(b)}(h)||_{\alpha,\lambda}\le\frac{d_N}{\lambda^{1-2\alpha}}||f-h||_{\alpha,\lambda}.$$

Proof. We prove the Proposition in the simplified case where $d=1$.

(a) As for the first term of $C^{1-\alpha}(0,T;\mathbb{R})$-norm, applying the Boundedness property of b,

$$||F_t^{(b)}(f)||_\infty=\sup_{t\in[0,T]}\left\{\Big|\int_0^t b(s,f(s))ds\Big|\right\}\le\int_0^T|b(s,f(s))|ds\le$$
$$\le L_0\int_0^T|f(s)|ds+\int_0^T b_0(s)ds<\infty \tag{16}$$

and as for the second term, again applying the Boundedness property of b,

$$|F_t^{(b)}(f)-F_s^{(b)}(f)|\le\int_s^t b(r,f(r))dr\le L_0\int_s^t|f(r)|dr+\int_s^t b_0(r)dr\overset{\text{Hölder inequality}}{\le}$$
$$\le L_0(t-s)||f||_\infty+(t-s)^{1-\alpha}||b_0||_{L^{1/\alpha}}\le(t-s)^{1-\alpha}\cdot\big(T^\alpha L_0||f||_\infty+B_{0,\alpha}\big), \tag{17}$$

where $B_{0,\alpha}=||b_0||_{L^{1/\alpha}}$.

(b) Using equations (16) and (17) in (a),

$$\begin{aligned}||F_t^{(b)}(f)||_{1-\alpha}&\le L_0T||f||_\infty+T^{1-\alpha}B_{0,\alpha}+T^\alpha L_0||f||_\infty+B_{0,\alpha}=\\&=(1+T^{1-\alpha})\cdot(B_{0,\alpha}+L_0T^\alpha||f||_\infty)\le\\&\le(1+T^{1-\alpha})\cdot(B_{0,\alpha}+L_0T^\alpha)\cdot(1+||f||_\infty)=:d^{(1)}(1+||f||_\infty).\end{aligned}$$

(c) We want to apply Proposition 7.12 so we must fulfill the requirements on $b(t,f(t))$. Applying the Boundedness property of b,

$$\sup_{t\in[0,T]}\Big\{\int_0^t \frac{|b(s,f(s))|}{(t-s)^\alpha}ds\Big\} \le \sup_{t\in[0,T]}\Big\{\int_0^t \frac{L_0|f(s)|+b_0(s)}{(t-s)^\alpha}ds\Big\} \overset{\text{Hölder inequality}}{\le}$$

$$\le L_0 \sup_{t\in[0,T]}\Big\{\int_0^t \frac{|f(s)|}{(t-s)^\alpha}ds\Big\} + \sup_{t\in[0,T]}\Big\{||b_0||_{L^{1/\alpha}}\cdot\Big(\int_0^t (t-s)^{-\alpha/(1-\alpha)}ds\Big)^{1-\alpha}\Big\} \le$$

(18)

$$\overset{\text{Hölder inequality}}{\le} L_0\frac{T^{1-\alpha}}{1-\alpha}||f||_\infty + B_{0,\alpha}T^{1-2\alpha}\Big(\frac{1-\alpha}{1-2\alpha}\Big)^{1-\alpha} < \infty,$$

where $B_{0,\alpha} \coloneqq ||b_0||_{L^{1/\alpha}}$.

Hence, we can apply Proposition 7.12 and we have

$$|F_t^{(b)}(f)| + \int_0^t \frac{|F_t^{(b)}(f)-F_s^{(b)}(f)|}{(t-s)^{\alpha+1}}ds \le C_{\alpha,T}\int_0^t \frac{|b(s,f(s))|}{(t-s)^\alpha}ds \overset{(18)}{\le}$$

(19) $$\le C_{\alpha,T}\Big(L_0\int_0^t \frac{|f(s)|}{(t-s)^\alpha}ds + B_{0,\alpha}\Big(\int_0^t (t-s)^{-\alpha/(1-\alpha)}ds\Big)^{1-\alpha}\Big),$$

which implies

$$||F_t^{(b)}(f)||_{\alpha,\lambda} \le C_{\alpha,T}L_0\sup_{t\in[0,T]}\Big\{e^{-\lambda t}\int_0^t \frac{|f(s)|}{(t-s)^\alpha}ds\Big\} + \\ + C_{\alpha,T}B_{0,\alpha}\Big(\frac{1-\alpha}{1-2\alpha}\Big)^{1-\alpha}\sup_{t\in[0,T]}\big\{e^{-\lambda t}t^{1-2\alpha}\big\}.$$

On one hand,

$$\sup_{t\in[0,T]}\Big\{e^{-\lambda t}\int_0^t \frac{|f(s)|}{(t-s)^\alpha}ds\Big\} \le \sup_{t\in[0,T]}\Big\{\frac{|f(t)|}{e^{\lambda t}}\Big\}\sup_{t\in[0,T]}\Big\{\int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha}ds\Big\},$$

with

$$\int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha}ds \overset{r=\lambda(t-s)}{=} \int_0^{\lambda t} e^{-r}\lambda^{\alpha-1}r^{-\alpha}dr \le \lambda^{\alpha-1}\Gamma(1-\alpha),$$

and on the other hand,

$$\sup_{t\in[0,T]}\big\{e^{-\lambda t}t^{1-2\alpha}\big\} = e^{-\lambda t}t^{1-2\alpha}\Big|_{t=\lambda^{-1}(1-2\alpha)} = e^{2\alpha-1}\frac{(1-2\alpha)^{1-2\alpha}}{\lambda^{1-2\alpha}}.$$

Consequently, we have

$$||F_t^{(b)}(f)||_{\alpha,\lambda} \le C_{\alpha,T}L_0\lambda^{\alpha-1}\Gamma(1-\alpha) + \frac{C_{\alpha,T}B_{0,\alpha}}{(\lambda e)^{1-2\alpha}}\frac{(1-\alpha)^{1-2\alpha}}{(1-2\alpha)^\alpha} \le \frac{d^{(2)}}{\lambda^{1-2\alpha}}(1+||f||_{\alpha,\lambda})$$

with $d^{(2)} = C_{\alpha,T}\big(L_0\Gamma(1-\alpha) + B_0e^{2\alpha-1}(1-\alpha)^{1-2\alpha}(1-2\alpha)^{-\alpha}\big)$ since $\lambda \ge 1$.

(d) In section (c) we proved we can apply Proposition 7.12 on $b(t,f(t))$ when $f \in L^\infty(0,T)$, in particular, when $f \in W_0^{\alpha,\infty}(0,T)$. Consequently, also applying the

Local Lipschitz Continuity property of b,

$$||F_t^{(b)}(f) - F_t^{(b)}(h)||_{\alpha,\lambda} \leq C_{\alpha,T} \sup_{t\in[0,T]} \left\{ e^{-\lambda t} \int_0^t \frac{|b(s,f(s)) - b(s,h(s))|}{(t-s)^\alpha} ds \right\} \leq$$

$$\leq C_{\alpha,T} L_N \sup_{t\in[0,T]} \left\{ \int_0^t e^{-\lambda(t-s)} e^{-\lambda s} \frac{|f(s) - h(s)|}{(t-s)^\alpha} ds \right\} \leq$$

$$\leq C_{\alpha,T} L_N \sup_{t\in[0,T]} \left\{ e^{-\lambda t} |f(t) - h(t)| \right\} \sup_{t\in[0,T]} \left\{ \int_0^t e^{-\lambda(t-s)} (t-s)^{-\alpha} ds \right\} \leq$$

$$\leq C_{\alpha T} L_N ||f-h||_{\alpha,\lambda} \lambda^{\alpha-1} \Gamma(1-\alpha) = \frac{d_N}{\lambda^{1-\alpha}} ||f-h||_{\alpha,\lambda},$$

with $d_N = C_{\alpha,T} L_N \Gamma(1-\alpha)$.

□

8. Deterministic differential equations involving the generalized Stieltjes integral

For almost every realization $\omega \in \Omega$, the spectral set, we propose the following differential equation expecting a solution $x\colon [0,T] \to \mathbb{R}^d$ and with σ, b deterministic under some hypothesis

$$x(t) = x_0 + F_t^{(b)}(x) + G_t^{(\sigma)}(x)\ , \quad \forall t \in [0,T]. \tag{20}$$

Notice that ordinary differential equations do not contain the $G_t^{(\sigma)}$ term. In this section, we prove the existence and uniqueness of solutions to (20) on $W_0^{\alpha,T}(0,T;\mathbb{R}^d)$. In addition, if σ satisfies the Boundedness property, we bound the solution x in terms of the norm $||\cdot||_{\alpha,\infty}$.

However, we first state and prove some results required to prove the existence of solutions and properties on theses solutions. The first result we present is a Banach fixed point Theorem.

Lemma 8.1. *Let (X,ρ) be a complete metric space, ρ_0, ρ_1, ρ_2 equivalent metrics to ρ and $\mathcal{L}\colon X \to X$ such that*

- *(a) there exist $r_0 > 0$, $x_0 \in X$ so that $\mathcal{L}(B_0) \subset B_0$ for $B_0 = \{x \in X\colon \rho_0(x_0,x) \leq r_0\}$,*
- *(b) there exist $\varphi\colon (X,\rho) \to [0,+\infty]$ lower semi-continuous and $C_0, K_0 \geq 0$ constant such that*
  - $\mathcal{L}(B_0) \subset N_\varphi(C_0)$
  - $\rho_1(\mathcal{L}(x), \mathcal{L}(y)) \leq K_0 \rho_1(x,y) \quad \forall x,y \in N_\varphi(C_0) \cap B_0$

  for $N_\varphi(a) = \{x \in X\colon \varphi(x) \leq a\}$.
- *(c) there exists $a \in (0,1)$ such that*
$$\rho_2(\mathcal{L}(x), \mathcal{L}(y)) \leq a\rho_2(x,y) \quad x,y \in \mathcal{L}(B_0).$$

Then, there exists $x^* \in \mathcal{L}(B_0) \subset X$ such that $x^* = \mathcal{L}(x^*)$.

Proof. With hypothesis in (a), given $x_0 \in X$ and $r_0 > 0$, we consider the sequence $\{x_n\}_{n=0}^\infty \subset X$ with $x_{n+1} = \mathcal{L}(x_n)$ $\forall n \geq 0$. Due to (a), $\{x_n\}_{n=1}^\infty \subset \mathcal{L}(B_0)$ and with (b), there exists $C_0 \geq 0$ constant with $\varphi(x_n) \leq C_0$ $\forall n \geq 1$. Also, with (c),

$$\rho_2(x_{n+1}, x_n) = \rho_2(\mathcal{L}(x_n), \mathcal{L}(x_{n-1})) \leq a\rho_2(x_n, x_{n-1}) \leq \cdots \leq a^n \rho_2(x_1, x_0),$$

so

$$\rho_2(x_{n+p}, x_n) \leq \rho_2(x_{n+p}, x_{n+p-1}) + \cdots + \rho_2(x_{n+1}, x_n) \leq \rho_2(x_1,x_0) a^n \sum_{j=0}^{p-1} a^j =$$
$$= \rho_2(x_1,x_0)\, a^n \frac{1-a^p}{1-a} \leq \rho_2(x_1,x_0) \frac{a^n}{1-a} \xrightarrow{n\to\infty} 0.$$

Given that (X,ρ) is complete, $\rho \sim \rho_2$ and that B_0 is closed with respect to ρ_0, then, B_0 is closed with respect to ρ and there exists $x^* \in B_0$ such that $x_n \to x^*$ with respect to ρ. Finally, with the lower semi-continuity property of φ

$$C_0 \geq \liminf_{n\to\infty} \varphi(x_n) \geq \liminf_{\substack{x \in \mathcal{L}(B_0) \\ x \to x^*}} \varphi(x) \geq \varphi(x^*),$$

so $\forall n \in \mathbb{N}$, $x_n, x^* \in B_0 \cap N_\varphi(C_0)$ and applying (b), there exists $K_0 \geq 0$ such that

$$\rho_1(\mathcal{L}(x_n), \mathcal{L}(x^*)) \leq K_0 \rho_1(x_n, x^*) \xrightarrow{n\to\infty} 0.$$

Thus, $\forall n \in \mathbb{N}$

$$\begin{aligned}\rho(x^*, \mathcal{L}(x^*)) &\leq \lim_{n\to\infty} \left\{\rho(x^*, \mathcal{L}(x_n)) + \rho(\mathcal{L}(x_n), \mathcal{L}(x^*))\right\} \overset{\substack{\rho_1 \sim \rho \\ \downarrow}}{\leq} \\ &\leq \lim_{n\to\infty} \rho(x^*, \mathcal{L}(x_n)) + K \lim_{n\to\infty} \rho_1(\mathcal{L}(x_n), \mathcal{L}(x^*)) \leq \\ &\leq \lim_{n\to\infty} \rho(x^*, x_{n+1}) + K_0 \cdot K \lim_{n\to\infty} \rho_1(x_n, x^*) = 0,\end{aligned}$$

so $\mathcal{L}(x^*) = x^*$.

□

For Lemma 8.1 to guarantee the existence of solutions in (20), we require the operator Δ, defined in Proposition 7.10, to be lower semi-continuous in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$. The following Lemma proves it.

Lemma 8.2. *For $T >$, $\alpha \in (0, 1/2)$, $\delta \in (0, 1]$, the operator Δ is lower semi-continuous in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$.*

Proof. We consider the operator $\Delta_r \colon W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \to [0, +\infty]$ with $r \in [0,T]$ fixed defined by

$$\Delta_r(u) := \int_0^r \frac{|u(r) - u(s)|^\delta}{(r-s)^{\alpha+1}} ds \quad \forall u \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$$

and prove it is lower semi-continuous, that is, $\forall u_0 \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$,

$$\liminf_{u\to u_0} \Delta_r(u) \geq \Delta_r(u_0).$$

Given that $||\cdot||_\infty \leq ||\cdot||_{\alpha,\infty}$, convergence in $||\cdot||_{\alpha,\infty}$ implies uniform convergence for functions in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ in $[0,T]$ and applying Fatou's Lemma, we have

$$\liminf_{u\to u_0} \int_0^r \frac{|u(r) - u(s)|^\delta}{(r-s)^{\alpha+1}} ds \geq \int_0^r \liminf_{u\to u_0} \frac{|u(r) - u(s)|^\delta}{(r-s)^{\alpha+1}} ds = \int_0^r \frac{|u_0(r) - u_0(s)|^\delta}{(r-s)^{\alpha+1}} ds.$$

Hence Δ_r is lower semi-continuous.

Finally, we can write the operator Δ as $\sup_{r\in[0,T]} \Delta_r$ and using the fact that the pointwise supremum of lower semi-continuous functions is lower semi-continuous, we obtain Δ is lower semi-continuous.

□

Theorem 8.3. *Let $T > 0$, $d, m \in \mathbb{N}$, $\alpha \in (0, 1/2)$, $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$ and σ, b be deterministic under hypothesis $\mathbf{H}_\sigma^\mathbf{1}$ and $\mathbf{H_b}$ with $\rho = \alpha^{-1}$, $\beta, \delta \in (0, 1]$ and $\alpha < \min\{1/2, \beta, \delta/(1+\delta)\}$.*

Then, the differential equation (20) where $G_t^{(\sigma)}$ has g as the Stieltjes integrator, has a unique solution $x(t) \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$. In fact, $x(t) \in C^{1-\alpha}(0,T;\mathbb{R}^d)$.

Proof. In Propositions 7.10 and 7.14 we proved that if $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$, then, $G_t^{(\sigma)}(f), F_t^{(b)}(f) \in C^{1-\alpha}(0,T;\mathbb{R}^d)$ respectively. Thus, $x(t) \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ a solution to (20) is also in $C^{1-\alpha}(0,T;\mathbb{R}^d)$.

Now, we prove uniqueness for solutions in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$. If $x, \tilde{x} \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ are two solutions to (20), we consider $N \in \mathbb{N}$ with $||x||_{1-\alpha}, ||\tilde{x}||_{1-\alpha} \leq N$. Given that $||\cdot||_\infty \leq ||\cdot||_{1-\alpha}$ and $||\cdot||_{\alpha,\infty} \sim ||\cdot||_{\alpha,\lambda}$ $\forall \lambda \geq 1$, we can apply Proposition 7.10 and 7.14 and we have $\lambda \geq 1$

$$\begin{aligned} ||x-\tilde{x}||_{\alpha,\lambda} &\leq ||F_t^{(b)}(x) - F_t^{(b)}(\tilde{x})||_{\alpha,\lambda} + ||G_t^{(\sigma)}(x) - G_t^{(\sigma)}(\tilde{x})||_{\alpha,\lambda} \leq \\ &\leq \frac{d_N}{\lambda^{1-\alpha}}||x-\tilde{x}||_{\alpha,\lambda} + \frac{\Lambda_\alpha(g) C_N^{(4)}}{\lambda^{1-2\alpha}}\big(1 + \Delta(x) + \Delta(\tilde{x})\big)||x-\tilde{x}||_{\alpha,\lambda}. \end{aligned} \tag{21}$$

Since $||x||_{1-\alpha}, ||\tilde{x}||_{1-\alpha} \leq N$, if $r, s \in [0,T]$, then, $|x(r) - x(s)| \leq N|r-s|^{1-\alpha}$ and

$$\Delta(x) + \Delta(\tilde{x}) \leq 2N \sup_{r\in[0,T]} \Big\{ \int_0^r \frac{(r-s)^{\delta(1-\alpha)}}{(r-s)^{\alpha+1}} ds \Big\} = 2N \frac{T^{\delta-\alpha(1+\delta)}}{\delta - \alpha(1+\delta)} =: C_N.$$

Hence,

$$||x-\tilde{x}||_{\alpha,\lambda} \leq \Big(\frac{d_N}{\lambda^{1-\alpha}} + (1+C_N)\frac{\Lambda_\alpha(g) C_N^{(4)}}{\lambda^{1-2\alpha}}\Big)||x-\tilde{x}||_{\alpha,\lambda} =: K_{\lambda,\alpha,g,N} \cdot ||x-\tilde{x}||_{\alpha,\lambda},$$

and by taking $\lambda \geq 1$ large enough so that $K_{\lambda,\alpha,g,N} \leq 1$, we conclude $||x-\tilde{x}||_{\alpha,\lambda} = 0$ so $x = \tilde{x}$ in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$-norm which implies $x = \tilde{x}$ point-wise in $[0,T]$.

Finally, we prove the existence of solution in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$. We consider $(W_0^{\alpha,\infty}(0,T;\mathbb{R}^d), \rho)$ the metric space with ρ the metric induced by the norm $||\cdot||_{\alpha,\infty}$. Such space is complete with ρ since it is Banach with $||\cdot||_{\alpha,\infty}$ and in this complete metric space, we take $\mathcal{L}\colon W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \to W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ defined by

$$\mathcal{L}(u)(t) = x_0 + F_t^{(b)}(u) + G_t^{(\sigma)}(u) \quad \forall t \in [0,T]\,,\ \forall u \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d).$$

With Propositions 7.10 and 7.14, $\forall \lambda \geq 1$

$$\begin{aligned} ||\mathcal{L}(u)||_{\alpha,\lambda} &\leq |x_0| + ||F_t^{(b)}(u)||_{\alpha,\lambda} + ||G_t^{(\sigma)}(u)||_{\alpha,\lambda} \leq \\ &\leq |x_0| + \frac{1 + ||u||_{\alpha,\lambda}}{\lambda^{1-2\alpha}}\big(d^{(2)} + \Lambda_\alpha(g) C^{(3)}\big), \end{aligned}$$

and if we take $\lambda_0 \geq 1$ satisfying

$$\lambda_0^{1-2\alpha} \geq 2\big(d^{(2)} + \Lambda_\alpha(g) C^{(3)}\big) \leq \big(d^{(2)} + \Lambda_\alpha(g) C^{(3)}\big)\frac{3 + 2|x_0|}{2 + |x_0|},$$

then, $||\mathcal{L}(u)||_{\alpha,\lambda_0} \leq 2(1+|x_0|)$ whenever $||u||_{\alpha,\lambda_0} \leq 2(1+|x_0|)$. Hence, by taking $B_0 = \{u \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \colon ||u||_{\alpha,\lambda_0} \leq 2(1+|x_0|)\}$ condition (a) in Lemma 8.1 holds with ρ_0 the metric induced by the norm $||\cdot||_{\alpha,\lambda_0}$. In addition, $\forall u \in B_0$

$$||u||_{\alpha,\infty} \leq e^{\lambda_0 T}||u||_{\alpha,\lambda_0} \leq 2e^{\lambda_0 T}(1+|x_0|).$$

With Lemma 8.2 we know Δ is lower semi-continuous and in consequence, $\varphi_c := C \cdot (1/2 + \Delta)$ with $C \geq 0$ is lower semi-continuous as well. If we take $u \in \mathcal{L}(B_0)$, there exists $\overline{u} \in B_0$ with $u = \mathcal{L}(\overline{u})$ and $u \in C^{1-\alpha}(0,T;\mathbb{R}^d)$. Applying Propositions 7.10 and 7.14,

$$\begin{aligned}||u||_{1-\alpha} &\leq |x_0| + ||F_t^{(b)}(\overline{u})||_{1-\alpha} + ||G_t^{(\sigma)}(\overline{u})||_{1-\alpha} \leq \\ &\leq |x_0| + d^{(1)}(1+||\overline{u}||_\infty) + \Lambda_\alpha(g)C^{(2)}(1+||\overline{u}||_{\alpha,\infty}) \leq \\ &\leq |x_0| + \big(d^{(1)} + \Lambda_\alpha(g)C^{(2)}\big)\cdot\big(1+2e^{\lambda_0 T}(1+|x_0|)\big) =: C_2.\end{aligned}$$

That is, there exists a common bound C_2 for every function $u \in \mathcal{L}(B_0)$. Consequently

$$\begin{aligned}\Delta(u) &= \sup_{r\in[0,T]}\Big\{\int_0^r \frac{|u(r)-u(s)|^\delta}{(r-s^{\alpha+1})}ds\Big\} \leq \\ &\leq \sup_{r\in[0,T]}\Big\{\int_0^r C_2^\delta(r-s)^{(1-\alpha)\cdot(\delta-1)}ds\Big\} \leq C_2\frac{T^{\delta-\alpha(1+\delta)}}{\delta-\alpha(1+\delta)} =: C_4.\end{aligned}$$

In addition, we choose $N_0 \in \mathbb{N}$ with $N_0 \geq 2e^{\lambda_0 T}(1+|x_0|)$ and apply the same argument used in (21). That is, $\forall u, v \in B_0$ and $\lambda \geq 1$

$$\begin{aligned}||\mathcal{L}(u)-\mathcal{L}(v)||_{\alpha,\lambda} &\leq \Big(\frac{d_{N_0}}{\lambda^{1-\alpha}} + \frac{\Lambda_\alpha(g)C^{(4)}_{N_0}}{\lambda^{1-2\alpha}}\big(1+\Delta(u)+\Delta(v)\big)\Big)\cdot||u-v||_{\alpha,\lambda} \leq \\ &\leq \big(d_{N_0} + \Lambda_\alpha(g)C^{(4)}_{N_0}\big)\frac{1+\Delta(u)+\Delta(v)}{\lambda^{1-2\alpha}}||u-v||_{\alpha,\lambda} = \\ &=: \frac{C_1}{\lambda^{1-2\alpha}}\big(1+\Delta(u)+\Delta(v)\big)||u-v||_{\alpha,\lambda}.\end{aligned}$$

We consider the function $\varphi = \varphi_{C_1} = C_1(1/2+\Delta)$ and we want to check that assumption (b) in Lemma 8.1 holds. On one hand, given that $\varphi(\mathcal{L}(B_0)) \in [0, C_1(1/2+C_2)]$, we take $C_0 = C_1(1/2+C_2)$ and the first item in (b) is satisfied. On the other hand, we take $\lambda_1 = 1$ and we know that for $u, v \in B_0 \cap N_\varphi(C_0)$

$$\begin{aligned}||\mathcal{L}(u)-\mathcal{L}(v)||_{\alpha,\lambda_1} &\leq C_1\big(1+\Delta(u)+\Delta(v)\big)||u-v||_{\alpha,\lambda_1} \leq \\ &\leq \big(\varphi(u)+\varphi(v)\big)||u-v||_{\alpha,\lambda_1} \leq 2C_0||u-v||_{\alpha,\lambda_1} =: K_0||u-v||_{\alpha,\lambda_1},\end{aligned} \tag{22}$$

and the second item in (b) is satisfied with ρ_1 the metric induced by $||\cdot||_{\alpha,\lambda_1}$. Thus, assumption (b) in Lemma 8.1 is satisfied.

Finally, given that $\mathcal{L}(B_0) \subset B_0 \cap N_\varphi(C_0)$ we can repeat the procedure in (22) with any $\lambda \geq 1$ and we have that $\forall u, v \in \mathcal{L}(B_0)$

$$||\mathcal{L}(u)-\mathcal{L}(v)||_{\alpha,\lambda} \leq \frac{C_1(1+2C_4)}{\lambda^{1-2\alpha}}||u-v||_{\alpha,\lambda}$$

and if we take $\lambda_2 \geq 1$ such that $2C_1(1+2C_4) \leq \lambda_2^{1-2\alpha}$, then,

$$||\mathcal{L}(u)-\mathcal{L}(v)||_{\alpha,\lambda_2} \leq \frac{1}{2}||u-v||_{\alpha,\lambda_2} \quad \forall u, v \in \mathcal{L}(B_0),$$

and condition (c) in Lemma 8.1 is satisfied with ρ_2 the metric induced by the norm $||\cdot||_{\alpha,\lambda_2}$.

Given that $\rho_0, \rho_1, \rho_2, \rho$ are equivalent metrics in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$, we can apply Lemma 8.1 to obtain that there exists $x^* \in \mathcal{L}(B_0) \subset W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ such that $x^* = \mathcal{L}(x^*)$.

□

Once proved the existence and uniqueness of solution of (20), we give a bound on the solution. For this purpose, we present Proposition 8.5 which requires the following lemma.

Lemma 8.4. *Gronwall inequality*

For $\theta \in [0,1)$, $a, b \geq 0$ and $f\colon [0,\infty) \to [0,\infty)$ continuous such that $\forall t \geq 0$,

$$f(t) \leq a + bt^{\theta} \int_0^t (t-s)^{-\theta} s^{-\theta} f(s) ds,$$

then, $\forall t \geq 0$,

$$f(t) \leq a d_{\alpha} \exp(c_{\alpha} t b^{1/(1-\theta)}),$$

with $c_{\alpha} = 2\Gamma(1-\theta)^{1/(1-\theta)}$ and $d_{\alpha} = 4e^2 \frac{\Gamma(1-\theta)}{1-\theta}$.

Proof. The proof can be found in [7], in the Appendix.

□

Proposition 8.5. *Let $T > 0$, $d, m \in \mathbb{N}$, $\alpha \in (0, 1/2)$, $g \in W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^m)$ and σ, b be deterministic under hypothesis $\mathbf{H}^{\mathbf{1}}_{\sigma}$, $\mathbf{H}^{\mathbf{2}}_{\sigma}$ and $\mathbf{H_b}$ with $\rho = \alpha^{-1}$, $\beta, \delta \in (0,1]$, $\gamma \in [0,1]$ and $\alpha < \min\{1/2, \beta, \delta/(1+\delta)\}$. Then, the solution to (20), x, satisfies*

$$||x||_{\alpha,\infty} \leq C_1 e^{C_2 \Lambda_{\alpha}(g)^{\kappa}},$$

where

$$\kappa = \begin{cases} \frac{1}{1-2\alpha} & \text{if } \gamma = 1 \\ > \frac{\gamma}{1-2\alpha}, & \text{if } \gamma \in [\frac{1-2\alpha}{1-\alpha}, 1) \\ \frac{1}{1-\alpha}, & \text{if } \gamma \in [0, \frac{1-2\alpha}{1-\alpha}) \end{cases} \tag{23}$$

and C_1, C_2 are constant with respect to x and g.

Proof. We consider

$$h(t) = |x(t)| + \int_0^t \frac{|x(t) - x(s)|}{(t-s)^{\alpha+1}} ds$$

and the goal is to reach inequality

$$h(t) \leq C(1 + \Lambda_{\alpha}(g)) \cdot \Big(1 + \int_0^t \big((t-s)^{-\varepsilon(\gamma)} + s^{-\alpha}\big) h(s) ds\Big) \tag{24}$$

where $\varepsilon(\gamma) \in (0,1)$ and C a constant with respect to g, x, t. Once we reach inequality (24), we will apply Lemma 8.4 to the reach the inequality we want to prove.

Given that $x_t = F_t^{(b)}(x) + G_t^{(\sigma)}(x)$, applying the triangular inequality we have

$$h(t) \leq |x_0| + |F_t^{(b)}(x)| + |G_t^{(\sigma)}(x)| + \int_0^t \frac{|F_t^{(b)}(x) - F_s^{(b)}(x)|}{(t-s)^{\alpha+1}} ds + \int_0^t \frac{|G_t^{(\sigma)}(x) - G_s^{(\sigma)}(x)|}{(t-s)^{\alpha+1}} ds$$

and we find bounds for these terms which depend on h.

First, using Proposition 7.7 with $s=0$ and applying properties of σ,

$$
\begin{aligned}
|G_t^{(\sigma)}(x)| &\le \Lambda_\alpha(g)\Big(\int_0^t \frac{|\sigma(r,x(r))|}{r^\alpha}dr + \alpha\int_0^t\int_0^r \frac{|\sigma(r,x(r))-\sigma(s,x(s))|}{(r-s)^{\alpha+1}}ds\,dr\Big) \le \\
&\le \Lambda_\alpha(g)\Big(K_0\int_0^t \frac{1+|x(r)|^\gamma}{r^\alpha}dr + \alpha M_0\int_0^t\int_0^r \frac{(r-s)^\beta+|x(r)-x(s)|}{(r-s)^{\alpha+1}}ds\,dr\Big) \le \\
&\le \Lambda_\alpha(g)\Big(K_0\frac{T^{1-\alpha}}{1-\alpha} + \frac{\alpha M_0 T^{\beta-\alpha+1}}{(\beta-\alpha)\cdot(\beta-\alpha+1)} + \\
&\qquad + \int_0^t \Big[K_0\frac{|x(r)|^\gamma}{r^\alpha} + \frac{\alpha M_0 T^\alpha}{r^\alpha}\int_0^r \frac{|x(r)-x(s)|}{(r-s)^{\alpha+1}}ds\Big]dr\Big) \le \\
&\le K^{(1)}\Lambda_\alpha(g)\Big(1+\int_0^t \Big[|x(r)|^\gamma + \int_0^r \frac{|x(r)-x(s)|}{(r-s)^{\alpha+1}}ds\Big]r^{-\alpha}dr\Big),
\end{aligned}
$$

with $K^{(1)} = K_0(1+\frac{T^{1-\alpha}}{1-\alpha}) + \alpha M_0(T^\alpha + \frac{T^{\beta-\alpha+1}}{(\beta-\alpha)\cdot(\beta-\alpha+1)})$.

Now, from equation (8), applying properties of σ,

$$
\begin{aligned}
&\int_0^t \frac{|G_t^{(\sigma)}(x)-G_s^{(\sigma)}(x)|}{(t-s)^{\alpha+1}}ds \le \\
&\le \Lambda_\alpha(g)\Big(B(2\alpha,1-\alpha)\int_0^t \frac{|\sigma(r,x(r))|}{(t-r)^{2\alpha}}dr + \int_0^t\int_0^r \frac{|\sigma(r,x(r))-\sigma(s,x(s))|}{(r-s)^{\alpha+1}(t-s)^\alpha}ds\,dr\Big) \le \\
&\le \Lambda_\alpha(g)\Big(\int_0^t \Big[B(2\alpha,1-\alpha)K_0\frac{1+|x(r)|^\gamma}{(t-r)^{2\alpha}} + M_0\int_0^r \frac{(r-s)^\beta+|x(r)-x(s)|}{(r-s)^{\alpha+1}(t-s)^\alpha}ds\Big]dr\Big) \le \\
&\overset{0\le s\le r\le t}{\le} \Lambda_\alpha(g)\Big(B(2\alpha,1-\alpha)K_0\frac{T^{1-2\alpha}}{1-2\alpha} + M_0\int_0^t\int_0^r \frac{(r-s)^{\beta-\alpha-1}}{(t-r)^\alpha}ds\,dr + \\
&\qquad + \big(B(2\alpha,1-\alpha)K_0+M_0\big)\int_0^t \Big[\frac{|x(r)|^\gamma}{(t-r)^{2\alpha}} + (t-r)^{-\alpha}\int_0^r \frac{|x(r)-x(s)|}{(r-s)^{\alpha+1}}ds\Big]dr\Big) \le \\
&\le K^{(2)}\Lambda_\alpha(g)\Big(1+\int_0^t \Big[\frac{|x(r)|^\gamma}{(t-r)^{2\alpha}} + (t-r)^{-\alpha}\int_0^r \frac{|x(r)-x(s)|}{(r-s)^{\alpha+1}}ds\Big]dr\Big),
\end{aligned}
$$

with $K^{(2)} = B(2\alpha,1-\alpha)K_0(1+\frac{T^{1-2\alpha}}{1-2\alpha}) + M_0(1+\frac{T^{1-2\alpha+\beta}}{(\beta-\alpha)\cdot(1-\alpha)})$.

And finally, from equation (19), applying the Boundedness property of b,

$$
\begin{aligned}
|F_t^{(b)}(x)| &+ \int_0^t \frac{|F_t^{(b)}(x)-F_s^{(b)}(x)|}{(t-s)^{\alpha+1}}ds \le \\
&\le C_{\alpha,T}\Big(L_0\int_0^t \frac{|x(r)|}{(t-r)^\alpha}dr + B_{0,\alpha}\Big(\int_0^t (t-r)^{-\alpha/(1-\alpha)}dr\Big)^{1-\alpha}\Big) \le \\
&\le K^{(3)}\Big(1+\int_0^t \frac{|x(r)|}{(t-r)^\alpha}dr\Big),
\end{aligned}
$$

with $K^{(3)} = C_{\alpha,T}(L_0 + B_{0,\alpha}T^{1-2\alpha}(\frac{1-\alpha}{1-2\alpha})^{1-\alpha})$.

Thus,

$$
\begin{aligned}
h(t) &\le |x_0| + \big(\Lambda_\alpha(g)(K^{(1)}+K^{(2)}) + K^{(3)}\big)\cdot\Big(1+\int_0^t \frac{|x(r)|^\gamma}{r^\alpha}dr + \int_0^t \frac{|x(r)|^\gamma}{(t-r)^{2\alpha}}dr + \\
&+ \int_0^t \frac{|x(r)|}{(t-r)^\alpha}dr + \int_0^t \big(r^{-\alpha}+(t-r)^{-\alpha}\big)\int_0^r \frac{|x(r)-x(s)|}{(r-s)^{\alpha+1}}ds\,dr\Big). \qquad (25)
\end{aligned}
$$

We obtain different estimates on $h(t)$ depending on the value of γ. First, if $\gamma = 1$, $h(t)$ can be further bounded by

$$h(t) \leq |x_0| + 2\big(\Lambda_\alpha(g)(K^{(1)} + K^{(2)}) + K^{(3)}\big) \cdot \Big(1 + \int_0^t |x(r)| \cdot (r^{-\alpha} + (t-r)^{-2\alpha})\, dr + \\ + \int_0^t \big(r^{-\alpha} + (t-r)^{-2\alpha}\big) \int_0^r \frac{|x(r) - x(s)|}{(r-s)^{\alpha+1}} ds dr\Big) \leq \\ \leq C(1 + \Lambda_\alpha(g))\Big(1 + \int_0^t \big(r^{-\alpha} + (t-r)^{-2\alpha}\big) h(r) dr\Big),$$

with $C = 2\big(K^{(1)} + K^{(2)} + K^{(3)}\big)$. That is, for $\gamma = 1$, we have obtained equation (24) with $\varepsilon(\gamma) = 2\alpha$.

Now, if $\gamma \in [\frac{1-2\alpha}{1-\alpha}, 1)$, we need to apply Hölder inequality on the terms with $|x(r)|^\gamma$ in equation (25). On one hand, for $\delta^{(1)} \in (0, 2\alpha)$ with $\delta^{(1)} < \gamma$ and $2\alpha - \delta^{(1)} < 1 - \gamma$,

$$\int_0^t \frac{|x(r)|^\gamma}{(t-r)^{2\alpha}} dr \leq \Big(\int_0^t \frac{|x(r)|}{(t-r)^{\delta^{(1)}/\gamma}} dr\Big)^\gamma \cdot \Big(\int_0^t (t-r)^{(-2\alpha+\delta^{(1)})/(1-\gamma)} dr\Big)^{1-\gamma},$$

so we take $\delta^{(1)} \in \big(2\alpha + \gamma - 1, \min\{2\alpha, \gamma\}\big)$ which is a non-empty interval with positive values and applying the inequality $x^\gamma \leq 1 + x$ for $x \geq 0$ and $\gamma \in [0, 1]$, we obtain

$$\int_0^t \frac{|x(r)|^\gamma}{(t-r)^{2\alpha}} \leq K^{(4)}\Big(1 + \int_0^t \frac{|x(r)|}{(t-r)^{\delta^{(1)}/\gamma}} dr\Big),$$

with $K^{(4)} = \Big(\int_0^t (t-r)^{(-2\alpha+\delta^{(1)})/(1-\gamma)} dr\Big)^{1-\gamma}$.

On the other hand, for $\delta^{(2)} = \alpha\gamma$, we have

$$\int_0^t \frac{|x(r)|^\gamma}{r^\alpha} dr \leq \Big(\int_0^t \frac{|x(r)|}{r^\alpha} dr\Big)^\gamma \cdot \Big(\int_0^t r^{(\delta^{(2)}-\alpha)/(1-\gamma)} dr\Big)^{1-\gamma} \leq K^{(5)}\Big(1 + \int_0^t \frac{|x(r)|}{r^\alpha} dr\Big),$$

with $K^{(5)} = \Big(\frac{T^{1-\alpha}}{1-\alpha}\Big)^{1-\gamma}$

Thus, equation (25) leads to

$$h(t) \leq |x_0| + C(1 + \Lambda_\alpha(g))\Big(1 + \int_0^t \big(r^{-\alpha} + (t-r)^{-\delta^{(1)}/\gamma}\big) h(r) dr\Big),$$

with $C = \big(K^{(1)} + K^{(2)} + K^{(3)}\big) \cdot \big(1 + K^{(4)} + K^{(5)}\big)$ since $\delta^{(1)}/\gamma > \alpha$. That is, for $\gamma \in [\frac{1-2\alpha}{1-\alpha}, 1)$, we have obtained equation (24) with $\varepsilon(\gamma) = \delta^{(1)}/\gamma$.

Finally, if $\gamma \in [0, \frac{1-2\alpha}{1-\alpha})$, we need to apply Hölder inequality again on the terms with $|x(r)|^\gamma$ in equation (25) provided that $\gamma > 0$. Otherwise, these terms can be bounded by a constant. Following the same procedure as before, now taking $\delta^{(1)} = \alpha\gamma$ and $\delta^{(2)} = \alpha\gamma$, we have

$$h(t) \leq |x_0| + C(1 + \Lambda_\alpha(g)) \cdot \Big(1 + \int_0^t \big(r^{-\alpha} + (t-r)^{-\alpha}\big) h(r) dr\Big),$$

with $C = \big(K^{(1)} + K^{(2)} + K^{(3)}\big) \cdot \big(1 + K^{(4)} + K^{(5)}\big)$. If $\gamma = 0$, then, the same estimate holds but with another constant C. That is, for $\gamma \in [0, \frac{1-2\alpha}{1-\alpha})$, we have obtained equation (24) with $\varepsilon(\gamma) = \alpha$.

Once reached inequality (24) with $\varepsilon(\gamma) \in \{2\alpha, \delta^{(1)}/\gamma, \alpha\}$, given that $\varepsilon(\gamma) \geq \alpha$, we have

$$h(t) \leq |x_0| + C(1+\Lambda_\alpha(g)) \cdot \Big(1 + \int_0^t \big(r^{-\alpha} + (t-r)^{-\varepsilon(\gamma)}\big) h(r) dr\Big) =$$

$$= |x_0| + C(1+\Lambda_\alpha(g)) \cdot \Big(1 + \int_0^t (t-r)^{-\varepsilon(\gamma)} r^{-\varepsilon(\gamma)} \big[r^{\varepsilon(\gamma)} + (t-r)^{\varepsilon(\gamma)} r^{\varepsilon(\gamma)-\alpha}\big] h(r) dr\Big) \leq$$

$$\leq |x_0| + C(1+\Lambda_\alpha(g))\Big(1 + (1+T^{\varepsilon(\gamma)-\alpha}) \int_0^t (t-r)^{-\varepsilon(\gamma)} r^{-\varepsilon(\gamma)} t^{\varepsilon(\gamma)} h(r) dr\Big).$$

Finally, $h(t)$ is continuous since $x(t) \in C^{1-\alpha}(0,T;\mathbb{R}^d)$ and if we apply Lemma 8.4, we obtain

$$||x||_{\alpha,\infty} = \sup_{t\in[0,T]} \{h(t)\} \leq$$

$$\leq \big(|x_0| + C(1+\Lambda_\alpha(g))\big) d_\alpha \exp\big(c_\alpha T [C(1+T^{\varepsilon(\gamma)-\alpha}) \cdot (1+\Lambda_\alpha(g))]^{1/(1-\varepsilon(\gamma))}\big), \tag{26}$$

with $d_\alpha = 4e^2 \frac{\Gamma(1-\varepsilon(\gamma))}{1-\varepsilon(\gamma)}$ and $c_\alpha = 2\Gamma(1-\varepsilon(\gamma))^{1/(1-\varepsilon(\gamma))}$. Given that $\forall y \geq 0$ and $p \geq 1$, there exist $C_p^{(1)}, C_p^{(2)} \geq 0$ constant with respect to y such that $(1+y) \leq e^{(1+y)}$ and $e^{(1+y)^p} \leq C_p^{(1)} e^{C_p^{(2)} y^p}$, equation (26) yields

$$||x||_{\alpha,\infty} \leq C_1 \exp(C_2 \Lambda_\alpha(g)^{1/(1-\varepsilon(\gamma))})$$

for C_1, C_2 constant with respect to x and g, which concludes the proof.

□

9. Stochastic integrals and differential equations with respect to the fractional Brownian motion

We want to apply the results seen so far on stochastic processes with a path-by-path approach. That is, for almost every $\omega \in \Omega$, we consider a differential equation of the form (20). Notice that the resulting solution $X\colon \Omega \times [0,T] \to \mathbb{R}^d$ will be measurable in $[0,T]$ for almost every $\omega \in \Omega$ but it could happen that X is not $\mathcal{F}$-measurable $\forall t \in [0,T]$. For such purpose, we will prove its measurability in $\Omega \times [0,T]$ which will suffice.

Let's consider $B = \{B_t \colon t \in [0,T]\}$ a f.B.m of Hurst parameter $H \in (1/2, 1)$, defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In order to use B as the integrator in the generalized Stieltjes integral, we need to check that the paths of B belong to $W_T^{1-\alpha,\infty}(0,T)$ for some $\alpha \in (0, 1/2)$.

We have this result in Proposition 9.3. However, we need first two lemmas.

Lemma 9.1. *Garsia-Rademich-Rumsey inequality*

For $T > 0$, $d \in \mathbb{N}$, $p \leq 1$, $\alpha > 1/p$ and $f\colon [0,T] \to \mathbb{R}^d$, we have $\forall t, s \in [0,T]$

$$|f(t) - f(s)|^p \leq C_{\alpha,p} |t-s|^{\alpha p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p + 1}} dx\, dy,$$

with the convention $0/0 = 0$.

Proof. The proof can be found in [2].

□

Lemma 9.2. *For $T > 0$, let $B = \{B(t) : t \in [0,T]\}$ be a f.B.m of Hurst parameter $H \in (0,1)$. Then, $\forall \varepsilon \in (0, H)$ there exists a positive random variable $\eta_{\varepsilon,T}$ with $\mathbb{E}(\eta_{\varepsilon,T}^p) < \infty$ $\forall p \in [1, \infty)$ such that $\forall t, s \in [0,T]$*

$$|B(t) - B(s)| \leq \eta_{\varepsilon,T} |t-s|^{H-\varepsilon} \quad (a.e.).$$

Proof. Applying Lemma 9.1 with $\alpha = H - \varepsilon/2$ and $p = 2/\varepsilon$, we have $\forall t, s \in [0,T]$

$$|B(t) - B(s)|^{2/\varepsilon} \leq C_{H,\varepsilon} |t-s|^{2(H-\varepsilon)/\varepsilon}\, \xi,$$

where

$$\xi = \int_0^T \int_0^T \frac{|B(x) - B(y)|^{2/\varepsilon}}{|x-y|^{2H/\varepsilon}} dx\, dy\,.$$

Now, applying Minkowski's integral inequality on $\xi^{\varepsilon/2}$ with $p = q\varepsilon/2$ where $q \geq 2/\varepsilon$, we obtain

$$||\xi^{\varepsilon/2}||_q^q = \mathbb{E}\Big(\Big[\int_0^T \int_0^T \frac{|B(x) - B(y)|^{2/\varepsilon}}{|x-y|^{2H/\varepsilon}} dx\, dy\Big]^{q\varepsilon/2}\Big) \leq$$

$$\leq \Big(\int_0^T \int_0^T \frac{||B(x) - B(y)||_q^{2/\varepsilon}}{|x-y|^{2H/\varepsilon}} dx\, dy\Big)^{q\varepsilon/2} \overset{\overset{(3)}{\downarrow}}{=} \Big(\int_0^T \int_0^T c_q\, dx\, dy\Big)^{q\varepsilon/2} = c_q^{q\varepsilon/2} \cdot T^{q\varepsilon}.$$

Hence, ξ has finite moments of all orders and by taking $\eta_{\varepsilon,T} = C_{H,\varepsilon} \xi^{\varepsilon/2}$, we conclude the proof.

□

Proposition 9.3. *For $T > 0$, we consider a f.B.m $B = \{B(t)\colon t \in [0,T]\}$ of Hurst parameter $H \in (1/2, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\alpha \in (1 - H, 1/2)$. Then, B has its paths in $W_T^{1-\alpha,\infty}(0,T)$ $\mathbb{P}$-almost surely.*

Proof. Applying Lemma 9.2 with $\varepsilon = \frac{\alpha+H-1}{2}$, there exists $\eta_{\varepsilon,T}$ a positive random variable with finite moments of all orders such that for almost every $\omega \in \Omega$

$$||B(t)||_{1-\alpha,\infty,T} = \sup_{0\le s<t\le T} \Big\{ \frac{|B(t)-B(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|B(y)-B(s)|}{(y-s)^{2-\alpha}} dy \Big\} \le$$

$$\le \eta_{\varepsilon,T} \sup_{0\le s<t\le T} \Big\{ (t-s)^{(H+\alpha-1)/2} + \int_s^t (y-s)^{(H+\alpha-3)/2} \Big\} \overset{H+\alpha-3<-2}{<} \infty.$$

□

In fact, we also need the estimates computed involving the operator Λ_α in the deterministic approach. However, we will need the set of $\Lambda_\alpha(g)$ for g any path of B. Thus, we will define the following random variable.

Definition 9.4. *For $T > 0$, we consider $B = \{B(t)\colon t \in [0,T]\}$ a f.B.m of Hurst parameter $H \in (1/2, 1)$ and $\alpha \in (1 - H, 1/2)$. We denote by $G\colon \Omega \to \mathbb{R}$ the random variable*

$$G = \Lambda_\alpha(B) = \frac{1}{\Gamma(1-\alpha)} \sup_{0\le s\le t\le T} \big\{ |D_{t-}^{1-\alpha}(B_{t-})(s)| \big\}.$$

The random variable G is measurable with respect to $\mathcal{F}$ and with Proposition 9.3 and Remark 7.3 it takes finite values almost surely. The following Proposition gives us more properties on G.

Proposition 9.5. *For $T > 0$, we consider $B = \{B(t)\colon t \in [0,T]\}$ a f.B.m of Hurst parameter $H \in (1/2, 1)$ and $\alpha \in (1 - H, 1/2)$. Then, $\forall p \in [1, \infty)$*

$$\mathbb{E}\big(\sup_{0\le s\le t\le T} \big\{ |D_{t-}^{1-\alpha}(B_{t-})(s)|^p \big\} \big) < \infty.$$

Proof. Given that if $\varepsilon \in (0, \alpha + H - 1)$, then, $\varepsilon < H$ with $\alpha + H - 1 > 0$, we can apply Lemma 9.2 with this ε and we know there exists $\eta_{\varepsilon,T}$ a random variable with finite moments of all orders such that $\forall s, t \in [0,T]$ with $s < t$

$$|D_{t-}^{1-\alpha}(B_{t-})(s)| \le \frac{1}{\Gamma(\alpha)} \Big(\frac{|B(s)-B(t)|}{(t-s)^{1-\alpha}} + (1-\alpha) \int_s^t \frac{|B(s)-B(y)|}{(y-s)^{2-\alpha}} dy \Big) \le$$

$$\le \frac{\eta_{\varepsilon,T}}{\Gamma(\alpha)} \Big((t-s)^{H-\varepsilon+\alpha-1} + (1-\alpha) \int_s^t (y-s)^{H-\varepsilon+\alpha-2} dy \Big) =$$

$$= \frac{(H-\varepsilon)\cdot(t-s)^{H-\varepsilon+\alpha-1}}{(H-\varepsilon+\alpha-1)\cdot\Gamma(\alpha)} \eta_{\varepsilon,T}.$$

Thus, for $p \ge 1$

$$\mathbb{E}\big(\sup_{0\le s\le t\le T} \big\{ |D_{t-}^{1-\alpha}(B_{t-})(s)|^p \big\} \big) \le \Big| \frac{(H-\varepsilon)\cdot T^{H-\varepsilon+\alpha-1}}{(H-\varepsilon+\alpha-1)\Gamma(\alpha)} \Big|^p \cdot \mathbb{E}(|\eta_{\varepsilon,T}|^p) < \infty.$$

□

Remark 9.6. *Under the hypothesis in Proposition 9.5, given that*

$$\Big(\sup_{0\le s\le t\le T}\big\{|D^{1-\alpha}_{t-}(B_{t-})(s)|\big\}\Big)^p \le \sup_{0\le s\le t\le T}\big\{\max\{1,|D^{1-\alpha}_{t-}(B_{t-})(s)|^p\}\big\},$$

we know G has finite moments of all orders.

On the integrator side, we have verified that we can apply the results in the previous sections. Now, we specify what type of functions we are going to integrate so that the resulting generalized Stieltjes integral is well-defined.

Let's consider $u = \{u_t \colon t \in [0,T]\}$ a stochastic process defined in $(\Omega,\mathcal{F},\mathbb{P})$ with paths in $W_0^{\alpha,1}(0,T)$ with $\alpha \in (1-H, 1/2)$ almost surely. Then, the generalized Stieltjes integral $\int_0^T u_s dB_s$ is well-defined and in fact, with Proposition 7.6

$$\Big|\int_0^T u_s dB_s\Big| \le G||u||_{\alpha,1} \quad (a.e.).$$

However, we do not need u to be measurable with respect to $\mathcal{F}$ for such integral to be well-defined almost surely. Therefore, the following Theorem guarantees the path-by-path existence and uniqueness of solution to (1) and under some additional conditions, the solution has finite moments of all orders.

Before stating the following Theorem, we remark that in this section we have considered B to be a one-dimensional f.B.m. for simplicity. However, all results in this section still hold for the m.f.B.m. of independent components and the same Hurst parameter $H \in (1/2, 1)$. In the multivariate case, the constants might depend on its dimension, and how it depends is determined by the norm in $\mathbb{R}^d$.

Theorem 9.7. *For $T > 0$, $d, m \in \mathbb{N}$, we consider X_0 a $\mathbb{R}^d$ random vector and $B = \{B_t \colon t \in [0,T]\}$ a m-dimensional fractional Brownian motion of Hurst parameter $H \in (1/2, 1)$ and with independent components, with X_0, B_t defined in a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let's take σ, b satisfying hypothesis $\mathbf{H}^{\mathbf{1}}_\sigma$ and $\mathbf{H_b}$ with $\beta > 1-H$ and $\delta > 1/H - 1$.*

If $\alpha \in (1-H, \alpha_0)$ and $\rho \ge 1/\alpha$, then, there exists a unique solution $X \in L^0(\Omega,\mathcal{F},\mathbb{P}; W_0^{\alpha,\infty}(0,T;\mathbb{R}^d))$ to the following stochastic differential equation

$$X^i_t = X^i_0 + \sum_{j=1}^m \int_0^t \sigma_{i,j}(s, X_s) dB^j_s + \int_0^t b_i(s, X_s) ds \quad \forall t \in [0,T], \forall i \in \{1,\dots,d\}$$

for almost every $\omega \in \Omega$, where $\alpha_0 = \min\{1/2, \beta, \delta/(1+\delta)\}$ and

$$L^0(\Omega,\mathcal{F},\mathbb{P}; W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)) = \{X \colon \Omega \to W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)\}/\{X = Y \ (a.e.)\}.$$

In this case, for almost every $\omega \in \Omega$, $X_t(\omega) \in C^{1-\alpha}(0,T;\mathbb{R}^d)$.

Moreover, if hypothesis $\mathbf{H}^{\mathbf{2}}_\sigma$ is also satisfied, $\alpha \in (1-H, \min\{\alpha_0, \frac{2-\gamma}{4}\})$ and M_N, L_N, K_0, b_0 do not depend on $\omega \in \Omega$ almost surely, then, the solution X satisfies

$$\mathbb{E}\big(||X||^p_{\alpha,\infty}\big) < \infty \quad \forall p \ge 1.$$

Proof. As we have seen before in this section, the paths of B are $\mathbb{P}$-almost surely in $W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$. Then, for almost every $\omega \in \Omega$, we can apply Theorem 8.3 with

$g = B(\omega)$ and so, we obtain $x_t(\omega)$ a deterministic solution to (20). This yields to the unique solution

$$\begin{aligned} X\colon \Omega &\longrightarrow W_0^{\alpha,\infty}(0,T;\mathbb{R}^d) \\ \omega &\longmapsto x_t(\omega) \end{aligned}$$

to equation (1) in $L^0(\Omega,\mathcal{F},\mathbb{P};W_0^{\alpha,\infty}(0,T;\mathbb{R}^d))$. In particular, with Theorem 8.3 $X_t(\omega) \in C^{1-\alpha}(0,T;\mathbb{R}^d)$ for almost every $\omega \in \Omega$. This proves the first part of the Theorem.

As for the second part, we first prove the solution X is $\mathcal{F}\times\mathcal{B}([0,T])$-measurable. In the first part of the proof we have applied Theorem 8.3 which proves the existence of solutions using Lemma 8.1. For almost every $\omega \in \Omega$, $X_t(\omega)$ is the $||\cdot||_{\alpha,\infty}$-limit of $\{y_n(\omega)\}_{n=0}^{\infty}$ where $y_{n+1}(\omega) = \mathcal{L}(\omega, y_n(\omega))$ $\forall n \geq 0$ and

$$\mathcal{L}(\omega,y) \coloneqq x_0(\omega) + F_t^{(b(\omega))}(y) + G_t^{(\sigma(\omega))}(y).$$

Taking $y_0 = X_0$ which is $\mathcal{F}\times\mathcal{B}([0,T])$-measurable since it is constant in time, $\{y_n\}_{n\geq 0}$ are measurable as well. Consequently, X is $\mathcal{F}\times\mathcal{B}([0,T])$-measurable.

Now, with this result we can check the solution has finite moments of all orders. Applying Proposition 8.5 and taking into account that M_N, L_N, K_0, b_0 in $\mathbf{H}^{\mathbf{1}}_{\sigma}, \mathbf{H}^{\mathbf{2}}_{\sigma}, \mathbf{H_b}$ are constant with respect to $\omega\in\Omega$ almost surely, we know that there exist $C_1, C_2 \geq 0$ constants such that

$$||X||_{\alpha,\infty} \leq C_1 \exp(C_2 G^{\kappa}) \quad (a.e.),$$

where κ is given by (23). This inequality yields $\forall p \geq 1$

$$\mathbb{E}(||X||^p_{\alpha,\infty}) \leq C_1 \mathbb{E}(\exp(C_2 p G^{\kappa})).$$

Applying Fernique's Theorem, whenever $\kappa < 2$

$$\mathbb{E}(||X||^p_{\alpha,\infty}) \leq C_1 \mathbb{E}(\exp(C_2 p G^{\kappa})) < \infty \quad p \geq 1. \tag{27}$$

Therefore, the solution X has finite moments of all orders.

Finally, we need to check that $\kappa < 2$ so that (27) holds. Given that $\alpha < \min\{1/2, \frac{2-\gamma}{4}\}$, we have $\kappa < 2$ for all possible values of $\gamma \in [0,1]$.

□

Remark 9.8. *Under the hypothesis of Theorem 9.7, we know there exists a unique solution $X \in L^0(\Omega,\mathcal{F},\mathbb{P};W_0^{\alpha,\infty}(0,T;\mathbb{R}^d))$. Following the proof of the Theorem, we know X is measurable in $\Omega\times[0,T]$ which implies $X(\cdot,t)$ is $\mathcal{F}$-measurable and $\{X_t\colon t\geq 0\}$ is a stochastic process with trajectories in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ almost surely.*

Hence, there exists a unique solution to (1) in the set of stochastic processes of parameter set $[0,T]$ with values in $\mathbb{R}^d$ and with trajectories in $W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ almost surely.

10. Conclusions

We have achieved our goal of reviewing and making accessible the research paper [7]. Below, we specify what tasks were done.

In chapter 4 we introduce the notions of Riemann-Liouville fractional integral and Weyl derivative. We have proved they are well-defined and some results we need afterwards. This introduction to the fractional integrals and derivatives is briefly mentioned in [7] so we have included some results in [8, 10, 3, 4], providing the proof when possible.

[7] defines the generalized Stieltjes integral and gives some results related to such integral, however, without their proof. In chapter 5 we define the generalized Stieltjes integral and check it is well-defined, we leave out the unused results, and provide the proof for the used ones. We have devoted a lot of time to these tasks, in particular, to check the additivity property of the generalized Stieltjes integral, using [8].

In the following chapter, 6, we consider the fractional Sobolev spaces and prove that the generalized Stieltjes integral is well-defined on them.

In chapter 7, we do not add additional results with respect to [7]. However, we give exhaustive proofs on the presented estimates.

The corresponding chapter 8 in [7] consists of stating and proving Theorem 8.3 and Proposition 8.5. However, it makes use of some results from the appendix. In our thesis, we move these results from the appendix to chapter 8 as lemmas and prove them as long as they are in the scope of this thesis. In particular, we follow step-by-step the proofs of Theorem 8.3 and Proposition 8.5 providing the steps taken for granted.

Finally, in chapter 9 we give an extended explanation on how the results in the previous sections can be applied on the fractional Brownian motion, providing the proof for such statements. Then, we state and prove the main Theorem, 9.7, and add a Remark on the measurability of the solution.

As a personal conclusion, I have realized that papers like [7] can not be self-contained and at the same time have a manageable length: either the length or the completeness must suffer. Also, it seems to me that a difficult and relatively recent topic like stochastic calculus needs of a very careful approach from the part of researchers to avoid missing parts in their complex constructions.

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