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Abstract

This paper focuses on testing the stability of the unconditional variance when the stochastic processes may have heavy-tailed distributions. Finite sample distributions that depend both on the effective sample size and the tail index are approximated using Extreme Value distributions and summarized using response surfaces. A modification of the Iterative Cumulative Sum of Squares (ICSS) algorithm to detect the presence of multiple structural breaks is suggested, adapting the algorithm to the tail index of the underlying distribution of the process. We apply the algorithm to eighty absolute log-exchange rate returns, finding evidence of (i) infinite variance in about a third of the cases, (ii) finite changing unconditional variance for another third of the time series - totalling about one hundred structural breaks - and (iii) finite constant unconditional variance for the remaining third of the time series.

JEL Classification: C12, C22.

Keywords: CUMSUMQ test, Unconditional variance, Multiple structural changes, Heavy tails, Generalized Extreme Value distribution.

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Generalized Extreme Value Approximation to the CUMSUMQ Test for Constant Unconditional Variance in Heavy-Tailed Time Series*

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Abstract

This paper focuses on testing the stability of the unconditional variance when the stochastic processes may have heavy-tailed distributions. Finite sample distributions that depend both on the effective sample size and the tail index are approximated using Extreme Value distributions and summarized using response surfaces. A modification of the Iterative Cumulative Sum of Squares (ICSS) algorithm to detect the presence of multiple structural breaks is suggested, adapting the algorithm to the tail index of the underlying distribution of the process. We apply the algorithm to eighty absolute log-exchange rate returns, finding evidence of (i) infinite variance in about a third of the cases, (ii) finite changing unconditional variance for another third of the time series – totalling about one hundred structural breaks – and (iii) finite constant unconditional variance for the remaining third of the time series.

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1 Introduction

Testing for constant unconditional variance in time series has been considered a relevant issue, specially in financial time series analysis. Formal procedures based on CUMSUMQ-type statistics have been proposed, among others, by Inclan and Tiao (1994), Loretan and Phillips (1994), Andreou

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and Ghysels (2002), Ho and Wan (2002), Sansó et al. (2004), Deng and Perron (2008), Kapetanios (2009), Han et al. (2010), Xu (2013, 2015), Jentsch and Rao (2015), the last one for multivariate time series.

Loretan and Phillips (1994) show that the asymptotic distribution of CUMSUMQ-type test statistics depends on the index of tail thickness (α) of the distribution of the innovations. If $\alpha \geq 4$ and the fourth order moment of the innovations exists, the limit distribution is a standard Brownian bridge on [0,1]. If $2 \leq \alpha < 4$, the limit distribution depends on functionals of Lévy processes. Finally, when $\alpha < 2$ the unconditional variance is not finite and, then, it does not make sense to test for its constancy. Moreover, when $\alpha \leq 2$, Loretan and Phillips (1994) reveal that the statistic to test for constant variance is inconsistent. Hence, there is a discontinuity in the limit distribution of CUMSUMQ-type statistics depending on α.

Inclan and Tiao (1994) proposed an algorithm – the *so-called* Iterative Cumulative Sum of Squares, ICSS hereafter – for detecting several changes in the unconditional variance. This algorithm is based on a CUMSUMQ test statistic that in the limit converges to the supremum of a Brownian bridge under the assumption that the innovations are Gaussian independent. Sansó et al. (2004) suggested implementing the ICSS algorithm using a modified CUMSUMQ statistic that uses a non-parametric estimation of the long-run fourth order moment. The modified statistic is robust to non-Gaussianity and to some persistence in the conditional variance. Notwithstanding, the existence of the fourth order moment is still required in order to get a Brownian bridge as a limit distribution. Sansó et al. (2004) also considered the fact that the ICSS algorithm is computed with varying sample sizes so that the use of asymptotic critical values may distort the results. To avoid this limitation, they suggested the use of sample-size adapted critical values summarized in a response surface.

The ICSS algorithm, which relies on the CUMSUMQ test as a statistic to decide if there are structural breaks in the variance, has been extensively applied in empirical finance, specially to check the constancy of the unconditional volatility, an analysis that is carried out before the conditional variance is modelled. Some recent examples include Kartsonakis-Mademlis and Dritsakis (2020), Malik (2021), Apostolakis et al. (2022), Ngene and Mungai (2022), Souffargi and Boubaker (2022), Cevik et al. (2023) and Luo et al. (2023), among others.

The aim of the paper is twofold. First, we show that when $\alpha = 4$ the CUMSUMQ test can be written as a sequence of independent rescaled maximums of Brownian excursions, which can be shown to belong to the maximal domain of attraction of the Gumbel distribution. Based on this result, we use the Generalized Extreme Value (GEV) distribution to approximate the distribution of the CUMSUMQ statistic for different sample sizes and values of the tail index. This approximation allows us to compute p-values and any desired quantile. Monte Carlo experiments evidence good finite sample properties of this approximation. Second, we propose an automatic procedure – the *so-called* Modified ICSS (MICSS) – to test the null hypothesis of constant unconditional variance against the alternative hypothesis of multiple unknown structural breaks. The proposal is based, first, on the ICSS algorithm of Inclan and Tiao (1994) and, second, on the results of Loretan and

Phillips (1994), which provides a unified framework that considers the specific value of the tail index when performing the statistical inference. In this procedure, the critical values used in each step are adapted both to the sample size and to the index of tail thickness α. Finally, the MICSS algorithm is applied to eighty daily absolute log-exchange rate returns, finding evidence of (i) infinite variance in about one third of the time series, (ii) constant finite unconditional variance for another third of the time series, and (iii) changing unconditional variance for the remaining third of the time series. To ease empirical implementations, an R library to compute the algorithm is available from the authors upon request.

The structure of the paper is as follows. In Section 2 we describe the details concerning the stochastic processes of interest, present the test statistic and its limiting distribution. Section 3 proposes the GEV approximation that allows us to adapt the implementation of the CUSUMQ statistic to the sample size and tail index. Section 4 details the iterative algorithm that is used to detect multiple structural changes in unconditional variance and details the sequential testing procedure that is applied. Section 5 estimates response surfaces to approximate the percentiles of interest and p-values of the CUSUMQ statistic, while Section 6 conducts Monte Carlo simulations to assess the finite sample performance of the approach. Section 7 illustrates the implementation of the procedure analysing the evolution of daily exchange rates for eighty currencies. Finally, Section 8 concludes.

2 Preliminaries

We shall make use of the same assumptions of Loretan and Phillips (1994).

Assumption A1:

1. $\{\varepsilon_t\}_{t=1}^{T}$ is an *iid* sequence of innovations whose tail behaviour is of the asymptotic Pareto-Lévy form

$$\begin{aligned} P(\varepsilon > x) &= pC^{\alpha}x^{-\alpha}(1+o(1)), \quad x>0 \\ P(\varepsilon > -x) &= qC^{\alpha}x^{-\alpha}(1+o(1)), \quad x>0, \end{aligned}$$

   as $x \to \infty$, $p \geq 0$ and $q \geq 0$ satisfy $p+q=1$, $C>0$ is a scale dispersion parameter and α is the maximal moment exponent of the distribution.[1]

2. Centering condition: If $\alpha > 1$, we require $E(\varepsilon) = 0$. If $\alpha = 1$, then $\varepsilon =_d -\varepsilon$.

3. The observed time series is generated by

$$y_t = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}, \tag{1}$$

[1] Absolute moments of ε of order less than α are finite, while all higher-order moments are infinite.

with θ_j satisfying the summability condition $\sum_{j=1}^{\infty} j\,|\theta_j|^c$ for $0 < c \leq \alpha$ and $c \leq 1$.[2]

As shown by Loretan and Phillips (1994), it is convenient to approximate (1) by a finite AR(k) process

$$y_t = \mu + \sum_{j=1}^{k} \phi_j y_{t-j} + \varepsilon_t, \tag{2}$$

with all characteristic roots of the autoregressive polynomial lying inside the unit circle. Then, the ordinary least-squares (OLS) estimated residuals $\hat{\varepsilon}_t$ from (2) are used to compute the CUSUMQ-type statistic of the form

$$\eta_T(r) = T^{-1/2}\hat{\omega}_4^{-1/2}\left(S_{[rT]} - \frac{[rT]}{T}S_T\right),$$

where, throughout this paper, $[\cdot]$ denotes the integer part, $r \in [0,1]$, $S_t = \sum_{j=1}^{t} \hat{\varepsilon}_j^2$ and $\hat{\omega}_4$ is a consistent estimator of the long-run fourth order moment of $\hat{\varepsilon}_t$. The paper suggests estimating ω_4 using the proposal in Andrews and Monahan (1992) with the boundary rule of Sul, Phillips and Choi (2005) and the Bartlett window, although other consistent estimators available in the literature might be used.

The null hypothesis of constant unconditional variance can be tested using the statistic given by

$$\kappa_T = \sup_{r\in[0,1]} |\eta_T(r)|. \tag{3}$$

Proposition 3 and subsequent discussion in Loretan and Phillips (1994) establish that, under Assumption A1, if $\alpha \geq 4$, then $\kappa_T \Rightarrow \sup_{r\in[0,1]} |B(r)|$, and if $2 < \alpha < 4$, then

$$\kappa_T \Rightarrow \sup_{r\in[0,1]} \left| \frac{U_{\alpha/2}(r) - rU_{\alpha/2}(1)}{\sqrt{\int_0^1 \left(dU_{\alpha/2}\right)^2}} \right|, \tag{4}$$

where $\Rightarrow$ stands for weak convergence on $D[0,1]$, the space of CADLAG functions on the $[0,1]$ interval, $B(r) = W(r) - rW(1)$ is a standard Brownian bridge, $W(r)$ is a standard Brownian motion and $U_{\alpha/2}(r)$ is a stable-Lévy process on $r \in [0,1]$.

Let us now consider the asymptotic behaviour of the κ_T statistic under the alternative hypothesis of changing unconditional variance. In this case, the parameter C in Assumption A1 is not constant over time, that is,

$$H_a : C^{(1)} \neq C^{(2)}, \tag{5}$$

where $C^{(1)}$ is the value of parameter C for the first fraction of the sample, $t = 1, ..., [uT]$, $u \in (0,1)$, and $C^{(2)}$ is that of the second part of the sample, $t = [\pi T] + 1, ..., T$, $\pi \in (0,1)$, with $u < \pi$. If $\alpha > 2$, (5) implies that the variance is not constant. Proposition 4 in Loretan and Phillips (1994) establishes that under Assumption A1 and (5):

[2]Stationary ARMA processes fulfil this condition.

1. If $\alpha \geq 4$, then $\kappa_T = O_p\left(T^{1/2}\right)$;

2. If $2 < \alpha < 4$, then $\kappa_T = O_p\left(T^{1-2/\alpha}\right)$;

3. If $\alpha \leq 2$, then $\kappa_T = O_p(1)$.

From (4), it follows that the parameter α plays a key role in the inference on the constancy of the unconditional variance. If $\alpha \geq 4$, the limit distribution of κ_T under the null hypothesis is based on a Brownian bridge, whereas under the alternative hypothesis κ_T diverges at the rate $O_p\left(T^{1/2}\right)$. When $2 < \alpha < 4$, the limiting distribution of the κ_T test statistic under the null hypothesis is given by a standardized stable-Lévy bridge, whereas under the alternative hypothesis κ_T diverges at an $O_p\left(T^{1-2/\alpha}\right)$ rate – i.e., a slower rate than in the former case. Finally, for $\alpha \leq 2$, the test is inconsistent. Furthermore, note that when $\alpha < 2$ the limiting distribution has not a finite variance and, hence, it does not make sense to test for its constancy.

Therefore, and to properly use κ_T, it is very important to determine the value of the tail index α. Hill (1975) and Hall (1990) propose to estimate the upper tail using

$$\hat{\alpha}_s^{(u)} = -\left(\ln \hat{\varepsilon}_{T-s} - \frac{1}{s}\sum_{j=1}^{s} \ln \hat{\varepsilon}_{T-j+1}\right)^{-1}, \tag{6}$$

where $\hat{\varepsilon}_1 \leq \hat{\varepsilon}_2 \leq \cdots \leq \hat{\varepsilon}_T$ are the sorted OLS estimated residuals. It is worth noting that, given that we are working with the squares of (zero mean) stochastic processes, we are only interested in the upper tail of the distribution and do not need to care about the skewness of the distribution. Therefore, the estimator given in (6) is computed using the absolute values of the stochastic process, which is denoted as $\hat{\alpha}_H$. We have also considered the regression-based tail index estimator proposed by Nicolau and Rodrigues (2019) – henceforth, $\hat{\alpha}_{NR}$ – which is shown to have better finite sample properties than the estimators in Hill (1975) and Hall (1990). Finally, we propose the use of t-ratio test statistics to test (i) the null hypothesis that $H_0 : \alpha \geq 4$ against the alternative hypothesis that $H_a : \alpha < 4$ – hereinafter, the t-ratio test statistic associated with this hypotheses is denoted as $\varphi_{\alpha\geq4}$ – and (ii) the null hypothesis that $H_0 : \alpha \leq 2$ against the alternative hypothesis that $H_a : \alpha > 2$ – hereafter, $\varphi_{\alpha\leq2}$ refers to the t-ratio test statistic associated to this hypotheses.

3 Generalized Extreme Value approximation of the CUSUMQ statistic

The limit distribution of the κ_T statistic is related to random extremes, which suggests that the natural candidate under which the limiting distribution is embedded is the GEV class of distributions, which is the only non-degenerate max-stable distribution – note that the GEV class of distributions has as particular cases the Gumbel, Weibull and Fréchet distributions. It is also known as a von-Mises family of distributions of maxima. Although results concerning the GEV limit distribution of maxima are obtained for sequences of random variables, and not for random functions, actually it is well known that, for a standard Brownian bridge, $\Pr\left(\sup_{r\in[0,1]} B(r) > u\right) = e^{-2u^2}$, which corresponds to a Rayleigh distribution with scale parameter $\sigma = 0.5$, which in turn is a Weibull distribution with shape parameter $\gamma = 2$. Hence, here is an example of the supremum of a random functional converging to a GEV. Further, in finite sample the statistic is computed as $\kappa_T = \max\{v_1, \ldots, v_T\}$, where $v_t = T^{-1/2}|w_t|$, $w_t = \hat{\omega}_4^{-1/2}\left(S_t - \frac{t}{T}S_T\right)$, and it is well known that, for *iid* sequences of random variables $X_1, \ldots, X_T$, there exist sequences $\{a_T\}$ and $\{b_T\}$, $a_T > 0$, $b_T \in \mathbb{R}$, such that

$$\frac{\max(X_1, ..., X_T) - b_T}{a_T} \xrightarrow{d} G,$$

if and only if G is max-stable – see, for instance, Nair et al. (2022), Theorem 7.5 – which in turn is a GEV distribution, and where $\xrightarrow{d}$ denotes convergence in distribution. This implies that, for large samples, $\max(X_1, ..., X_T) \sim a_T G + b_T$. Unfortunately, $\{v_t\}_{t=1}^T$ is not *iid* in general, although the following proposition shows that it is possible to write the supremum of the absolute value of a Brownian bridge as the sequence of independent variables, and that these variables belong to the maximal domain of attraction of the Gumbel distribution – i.e., a GEV distribution. To reach this result, we shall make use of Brownian excursions, that is, processes defined as $B_t^{ex} := (X_t : 0 \leq t \leq 1 \mid X_t > 0 \text{ for } 0 < t < 1 \text{ and } X_1 = 0)$, where X_t is a Brownian motion.

Proposition 1 *Let $0 < t_1 < \cdots < t_n < T$ be the time periods in which $w_t = \hat{\omega}_4^{-1/2}\left(S_t - \frac{t}{T}S_T\right)$, $S_t = \sum_{j=1}^t \hat{\varepsilon}_j^2$, $t = \{1, ..., T\}$, crosses zero. Then:*

1. *$T^{-1/2}n \sim \sqrt{\frac{2}{\pi}}R$ where R is a random variable with a standard Rayleigh distribution.*
2. *For large T, the expected number of zero-crossings is $E(n) = T^{1/2}$.*
3. *We may write $\sup_r |B_t| = \sup\left(\sqrt{\frac{t_1}{T}}M_1^+, \sqrt{\frac{t_2-t_1}{T}}M_2^+, ..., \sqrt{\frac{t_n-t_{n-1}}{T}}M_n^+, \sqrt{\frac{T-t_n}{T}}M_{n+1}^+\right)$, where B_t is a standard Brownian bridge and M_i^+, $i = 1, ..., n+1$, denote a sequence of independent maximums of Brownian excursions.*
4. *M_i^+, $i = 1, ..., n+1$, belong to the maximal domain of attraction of the Gumbel distribution.*

The proof is given in the appendix. As can be seen, the (expected) number of times that the Brownian bridge w_t crosses zero is of order of magnitude $O(T^{1/2})$ and a Brownian excursion is generated every time that this Brownian bridge crosses zero – note that these Brownian excursions are mutually independent. Moreover, the distribution of the supremum of Brownian excursions belongs to the maximal domain of attraction of the Gumbel distribution. Furthermore, Corollary 3.2 in Durrett and Iglehart (1977) proofs that $E\left(M_i^+\right) = \sqrt{\pi/2}$. Finally, note that the previous results apply to any statistic that converges to a Brownian bridge, not just the ones built as in this case.

Hence, when $\alpha = 4$, the statistic $\kappa_T = \sup_{r\in[0,1]} |\eta_T(r)|$ can be written as a sequence of independent maximums of Brownian excursions. When $2 < \alpha < 4$ the third result of the previous proposition will have the form

$$\sup_t \left| L_{\alpha/2}(t) \right| = \sup \left(\sqrt{\frac{t_1}{T}} M^+_{\alpha/2,1}, \sqrt{\frac{t_2 - t_1}{T}} M^+_{\alpha/2,2}, ..., \sqrt{\frac{t_n - t_{n-1}}{T}} M^+_{\alpha/2,n}, \sqrt{\frac{T - t_n}{T}} M^+_{\alpha/2,n+1} \right)$$

where $L_{\alpha/2}(r) := U_{\alpha/2}(r) - rU_{\alpha/2}(1)$, and $M^+_{\alpha/2,i}$, $i = 1, ..., n+1$, denote a sequence of uncorrelated maximums of Levy excursions, defined in a similar way as a Brownian excursion, but using stable-Lévy processes. Based on these results, we conjecture that the GEV distributions could provide a reasonable approximation to the supremum of the absolute value of a Brownian bridge, i.e.,

$$\kappa_T \sim GEV(\lambda_T, \delta_T, \gamma_T), \tag{7}$$

where λ is the location parameter, $\delta > 0$ is the scale parameter and γ is the shape parameter of the GEV distribution. In particular, the cumulative distribution function of $GEV(\lambda_T, \delta_T, \gamma_T)$ when $\gamma \neq 0$ is given by

$$F(x) = \exp\left\{ -\left[1 + \gamma \left(\frac{x - \lambda}{\delta} \right) \right]^{-1/\gamma} \right\}, \tag{8}$$

for $1 + \gamma(x - \lambda)/\delta > 0$. For $\gamma > 0$, the cumulative distribution function of $GEV(\lambda_T, \delta_T, \gamma_T)$ corresponds to the Fréchet distribution, for $\gamma < 0$ it is the Weibull and, finally, when $\gamma = 0$ it belongs to the Gumbel family

$$F(x) = \exp\left[-\exp\left(-\frac{x - \lambda}{\delta} \right) \right],$$

for $-\infty < x < \infty$. In Section 5 we use this result to approximate critical values and p-values for given T and α.

4 The MICSS algorithm

The MICSS algorithm that is suggested in this paper is based on the ICSS algorithm of Inclan and Tiao (1994) with some previous steps added to determine the index of tail thickness α and, according to it, the distribution to be used in each step. The implementation of the algorithm makes extensive use of response surfaces that are computed in Section 5 to approximate the critical values for given T and α. The algorithm consists of the following steps:

Algorithm 1 *MICSS algorithm*

1. *Estimate (2) by OLS and obtain the estimated residuals* $\hat{\varepsilon}_t$.

2. *Estimate* $\hat{\alpha}_H$ *or* $\hat{\alpha}_{NR}$.

3. *Test the null hypothesis* $H_0 : \alpha \geq 4$ *against the alternative hypothesis* $H_a : \alpha < 4$ *with statistic* $\varphi_{\alpha \geq 4}$. *If the null hypothesis is not rejected, set* $\hat{\alpha} = 4$ *and proceed to step 5.*

4. *Test the null hypothesis $H_0 : \alpha \leq 2$ against the alternative hypothesis $H_a : \alpha > 2$ with statistic $\varphi_{\alpha \leq 2}$. If the null hypothesis is not rejected, we can conclude that the unconditional variance is not finite and stop here. Otherwise, proceed to step 5.*

5. *Run the ICSS algorithm of Inclan and Tiao (1994), but using the κ_T statistic and the GEV approximation, which depends both on T and $\hat{\alpha}$, to compute the critical values and p-values for the κ_T statistic.*

To obtain better approximations when applying the statistics that have been proposed in this paper, the next section considers the computation of critical values adapted to T and α, which are summarized with the estimation of response surfaces to calculate the corresponding critical values and p-values.

5 Response surfaces for critical values and p-values

Let us first describe the data generating process (DGP) that is used to simulate the distribution of the κ_T statistic, from which critical values and p-values are computed. Provided that κ_T is defined upon the squared residuals and, hence, the asymmetry of the distribution is of no concern here, the random variable e_t, $t = 1, \ldots, T$, is generated as follows:

1. The α-stable distribution is defined for $\alpha \in (0, 2]$ and since we need values for $\alpha' \in (2, 4]$, we follow Chambers et al. (1976) and generate u_t, $t = 1, \ldots, T$, as an α-stable distribution with $\alpha = \alpha'/2$.

2. Then, we set $e_t = sign\,(u_t)\sqrt{|u_t|}$, to which the ICSS algorithm is applied.

The computation of κ_T requires a consistent estimation of the long-run fourth order moment – i.e., an estimator of the long-run variance of the squared residuals. As mentioned above, we follow the proposal in Andrews and Monahan (1992) with the Bartlett window and the boundary rule defined in Sul, Phillips and Choi (2005). The order of the autoregressive model of the prewhitening step is selected by the Bayesian information criterion (BIC) allowing for up to $k_{\max} = \left[12\,(T/100)^{1/4}\right]$ lags. The design of the Monte Carlo experiment has specified twenty-nine finite sample values $T \in$ {26, 28, 30, 32, 35, 40, 45, 50, 55, 60, 65, 70, 80, 90, 100, 110, 120, 140, 160, 180, 200, 300, 400, 500, 600, 700, 800, 900, 1000}, twenty values of $\alpha \in$ {2.1, 2.2,..., 3.9, 4}, and 40,000 replications are conducted for each pair of T and α values – this amounts 580 experiments.

With the 40,000 observations of κ_T computed for each experiment for given α and T, and considering (7), we have estimated a GEV distribution of the type given in (8) and stored the three estimated values of parameters of the GEV distribution.[3] Let us denote by $\hat{\lambda}\,(T, \alpha)$ the location parameter, by $\hat{\delta}\,(T, \alpha)$ the scale parameter and by $\hat{\gamma}\,(T, \alpha)$ for the shape parameter of the

[3]We have used the R package 'evd' – see Stephenson (2022).

estimated GEV distribution. With the 580 different estimates of each parameter, we have fitted response surfaces of the form

$$\hat{\theta}\left(T_i,\alpha_i\right) = \beta_\infty + \beta_1\frac{1}{T_i} + \beta_2\frac{1}{T_i^2} + \beta_3\frac{1}{T_i^3} + \beta_k\frac{1}{k_{\max,i}} + \beta_{k,T}\frac{k_{\max,i}}{T_i} + \beta_{\alpha,1}\alpha_i + \beta_{\alpha,2}\alpha_i^2 + \beta_{\alpha,3}\alpha_i^3 + \beta_{\alpha,T}\frac{\alpha}{T_i} + \epsilon_i \quad (9)$$

where $\theta \in \{\lambda, \delta, \gamma\}$. Note that the previous expression tends to $\beta_\infty + \beta_{\alpha,1}\alpha_i + \beta_{\alpha,2}\alpha_i^2 + \beta_{\alpha,3}\alpha_i^3$ as $T \to \infty$, which could be considered the corresponding value of the asymptotic distribution. In this way, we are able to compute the location, scale and shape parameter for any given T and $\alpha \in [2.1, 4]$, which can be used to approximate either the desired quantiles – x_p, $p \in (0,1)$ – to get critical values

$$x_p = \begin{cases} \lambda + \frac{\delta}{\gamma}\left[(-\ln p)^{-\gamma} - 1\right] & \gamma \neq 0 \\ \lambda - \delta\ln(-\ln p) & \gamma = 0 \end{cases},$$

or the cumulative probability function given by (8) that will deliver the corresponding p-value.

6 Monte Carlo simulation experiments

Throughout this section, the long-run variance of the squared residuals is estimated as described above. We first check the accuracy of the GEV approximation to the empirical distribution of κ_T that has been suggested in the paper. Next, we focus on the empirical size and power of κ_T. The nominal size is set at the 5% significance level.

6.1 The GEV approximation

The stochastic process e_t is simulated according to the DGP described in 5. Figure 1 shows the empirical and the estimated GEV-based densities of κ_T for $\alpha = 3.5$ and $T = 100$.[4] As can be seen, the estimated GEV-based density smooths the empirical one. A formal analysis to compare both density functions can be performed by computing the Kolmogorov-Smirnov statistic. The empirical distributions of κ_T for the pairs of values $(\alpha, T) \in \{(4, 100), (3.5, 100), (2.75, 200)\}$ are generated using 500 replications, which are compared with the GEV-based ones. After conducting this exercise 500 times, and using the p-value of the Kolmogorov-Smirnov statistic, it is possible to compute the number of times that the null hypothesis is rejected at the 1%, 2.5%, 5% and 10% significance levels. The empirical rejection frequencies of the null hypothesis of the Kolmogorov-Smirnov statistic are close to the nominal ones – see Table 1. Similar results for other values of α and T have been obtained, which evidences that the approximation of the empirical distributions using GEV distributions is adequate.

Next, three response surfaces of the form (9) were fitted to the estimated values $\hat{\lambda}(T, \alpha)$ for the

[4] The estimated parameters are $\hat{\lambda}(100, 3.5) = 0.717$ for the location (0.718 when using the response surfaces), $\hat{\delta}(100, 3.5) = 0.211$ for the scale (0.210 for the response surfaces) and $\hat{\gamma}(100, 3.5) = 0.101$ for the shape (0.101).

location, $\hat{\delta}(T,\alpha)$ for the scale and $\hat{\gamma}(T,\alpha)$ for the shape parameters. For the location and scale parameter, the R^2 are above 0.995. For the shape parameter, $R^2 = 0.986$. Figure 2 depicts the densities of the distributions resulting from the response surfaces for $T = 30$ (small sample) and $T = 900$ (large sample) and $\alpha \in \{2.1, 4\}$ – i.e., the two limit cases considered in the simulation experiments. For small T, the right tail has an important amount of probability, which tends to concentrate around the mode of the distribution as T increases. It is worth noting that the right tail of the distribution seems to be affected by changes in α only marginally, whereas the effect on the mode is important.

6.2 Empirical size

We have conducted simulation experiments where sequences of *iid* observations from an α-stable distribution for different values of $\alpha \in \{2.4, 2.8, 3.2, 4\}$ and $T \in \{50, 75, 100, 200, 300, 500\}$ are generated as described in Section 5. The analysis uses three different sets of critical values for the κ_T statistic: (i) the ones obtained from the response surface of Sansó et al. (2004) that assumes $\alpha \geq 4$ – denoted as kappa2 in the figures – (ii) the critical values obtained from the GEV approximation provided in the present paper for a given α – denoted as kappa3 – and, (iii) the critical values from the GEV approximation, but imposing $\alpha = 4$ – denoted as kappa3_a4.

Figure 3 shows how the empirical size varies as T increases with $\alpha \in \{2.4, 2.8, 3.2, 4\}$. For $\alpha = 4$, in which case kappa3=kappa3_a4, the empirical size of the tree options almost equals the nominal one. However, there are mild over-size distortions when $T \leq 100$ as α decreases when using critical values not adapted to α, namely kappa2 and kappa3_a4. This deterioration is more severe for low values of the tail index α. Figure 4 shows the empirical size as α varies for different values of T. As above, rejection frequencies of kappa3 are close to the nominal size. Further, over-size distortions are observed for the κ_T statistic for small values of T and α when the computation of the critical values does not consider the value of α.

6.3 Empirical power

We have conducted a simulation experiment where sequences of *iid* observations from an α-stable distribution for different values of $\alpha \in \{2.4, 2.8, 3.2, 4\}$ and $T \in \{50, 100, 200, 300\}$ are generated as described in Section 5. The simulation experiment specifies one structural break that affects the variance of the stochastic process in the middle of the sample ($\pi = 0.5$). The standard deviation of the stochastic process for the first regime is set at $\sigma_1 = 1$, whereas we consider two magnitudes for the standard deviation of the stochastic process for the second regime, i.e., $\sigma_2 \in \{1.5, 2\}$. Figure 5 summarizes the empirical power of κ_T for $\sigma_2 = 1.5$. For $\alpha = 4$, the empirical power that is obtained is similar regardless of the critical value option that is used, although the GEV-based ones report a slightly higher power. Further, it is worth noting that the empirical power is close to 1 for $T > 100$. For $\alpha < 4$, there is a deterioration of the empirical power of the procedure that uses the critical values adapted to the value of α with respect to ones that impose $\alpha = 4$ – namely, kappa2 and kappa3_a4 – although this might be a consequence of the over-size distortions shown by the latter.

Note also that the low empirical power that is obtained for $\alpha = 2.4$ – which shows a slow increase as T gets large – is an expected consequence since $\kappa_T = O_p\left(T^{1-2/\alpha}\right)$ when $2 < \alpha < 4$.

Except for small values of T, the empirical power of κ_T increases with α for a given T. Figure 6 evidences a non-monotonic behaviour of the empirical power as α increases for $T = 50$, although this feature disappears as T increases. In general, the performance of kappa3 is encompassed by kappa2 and kappa3_a4. Similar results are obtained for $\sigma_2 = 2$. Figure 7 shows that there is an important increase of power as the magnitude of the change in the variance increases, regardless of α. Figure 8 depicts the empirical power as α varies for different values of T, which evidences that the power is reduced for low values of α.

7 Empirical application

To illustrate the procedures developed in this paper, we have analysed daily exchange rates returns of 80 currencies with respect to the US dollar. The tail properties of exchange rate returns have been studied by Hols and de Vries (1991), Koedijk et al. (1992), Loretan and Phillips (1994) – who also studied the stability of the unconditional variance – Ibragimov et al. (2010), Hartmann et al. (2010), Ibragimov et al. (2013), and Nicolau and Rodrigues (2019), among others. The wide scope of the study covers the main economies of the planet as well as emerging markets.[5] This data set is similar to that of Nicolau and Rodrigues (2019) although it covers more currencies (they analysed 74 currencies) and a longer period, from January 1994 to July 2023 (they used the period January 1999 to May 2016). Not all time series start in January 1994, so that the initial day for each variable is indicated in Table 2.[6,7]

Table 2 collects the results of applying the MICSS algorithm designed in Section 4 to determine the tail index of the pre-whitened log-returns of the exchange rates, $\Delta \ln x_{i,t}$, where $x_{i,t}$ is the nominal exchange rate of a given currency against the US dollar, $i = 1, \ldots, 80$, $t =, \ldots, T$. The analysis has been conducted for both the Hill (1975), $\hat{\alpha}_H$, and the Nicolau and Rodrigues (2019), $\hat{\alpha}_{NR}$, estimators, which were computed using the absolute value of the pre-whitened log-returns. The non-rejection of the null hypothesis that $\alpha \leq 2$ is indicated with the symbol ∞ in the column that collects the number of breaks detected by the algorithm. To be specific, the null hypothesis that $\alpha \leq 2$ cannot be rejected using both estimators in 29 currencies, namely: Albania, Algeria, Argentine, Bahrain, Bangladesh, Bolivia, Bulgaria, Ecuador, Egypt, Fiji, Ghana, Indonesia, Kazakhstan, Kenya, Malaysia, Malta, Mauritania, Namibia, New Guinea, Nigeria, Oman, Pakistan,

[5] The currencies/countries considered are: Albania, Algeria, Argentine, Australia, Bahrain, Bangladesh, Bolivia, Botswana, Brazil, Brunei, Bulgaria, Burundi, Canada, Chile, China, Colombia, Czech, Denmark, Ecuador, Egypt, Euro, Fiji, Finland, French Guinea, Gambia, Ghana, Hong Kong, Hungary, Iceland, India, Indonesia, Israel, Japan, Jordan, Kazakhstan, Kenya, Kuwait, Lebanon, Malawi, Malaysia, Mauritania, Mauritius, Mexico, Morocco, Mozambique, Namibia, New Guinea, New Zealand, Nigeria, Norway, Oman, Pakistan, Paraguay, Peru, Philippines, Poland, Qatar, Romania, Russia, Samoa, Saudi Arabia, Singapore, South Africa, South Korea, Special Drawing Rights (SDR), Sri Lanka, Sweden, Switzerland, Taiwan, Tanzania, Thailand, Tunisia, Turkey, United Arab Emirates (UAE), Uganda, Ukraine, United Kingdom, Uruguay, Venezuela, Vietnam, Zambia.

[6] Not all the series start in January 1999, so that the initial day of each variable is indicated in Table 2.

[7] We are very grateful to Paulo Rodrigues for providing us with the database.

Russia, Saudi Arabia, Sri Lanka, Thailand, Turkey, UAE and Venezuela, most of them emerging economies. Further, the use of $\hat{\alpha}_{NR}$ leads detecting infinite variance for South Korea, whereas $\hat{\alpha}_H$ also finds infinite variance for six additional cases: China, Hong Kong, Jordan, Lebanon, Uruguay and Zambia. Therefore, we have found evidence of infinite variance for 35 out of 80 log-exchange rate returns – for 29 of which the same conclusion is reached regardless of the $\hat{\alpha}$ estimator that is used. Hence, more than a third of the log-exchange rate returns would show infinite variance.

Several countries (11), specially in Africa, show estimated values of the tail index well above 4 with both estimators: Burundi, French Guinea, Gambia, Malawi, Mozambique, Paraguay, Qatar, Samoa, Tanzania, Uganda and Vietnam. For Denmark, the Euro, Morocco, Sweden and Tunisia, the null hypothesis that $\alpha \geq 4$ cannot be rejected with $\hat{\alpha}_{NR}$, whereas the opposite is found for with $\hat{\alpha}_H$. Hence, according to the results with this last estimator, the reaming 35 log-exchange rate returns have a tail index between 2 and 4.

The log-exchange rate returns with finite variance and no structural breaks according to the results that are obtained with both $\hat{\alpha}$ estimators are 20, namely: Australia, Brazil, Brunei, Burundi, Canada, Denmark, Euro, Gambia, Hungary, Iceland, Japan, Kuwait, Mexico, Mozambique, New Zealand, Singapore, Sweden, Taiwan, Tunisia and United Kingdom, mainly though not exclusively developed countries. Moreover, the MICSS algorithm that is implemented with $\hat{\alpha}_H$ does not find any structural break for South Korea, which has infinite variance according to the MICSS algorithm that is based on $\hat{\alpha}_{NR}$ – in what follows, MICSS($\hat{\alpha}_{NR}$) algorithm. Furthermore, with this last implementation, no structural breaks have been found for Lebanon, Uruguay and Zambia, where infinite variance cannot be rejected when using $\hat{\alpha}_H$ – from now on, MICSS($\hat{\alpha}_H$) algorithm.

For log-exchange rate returns for which evidence of finite variance is found, the MICSS($\hat{\alpha}_H$) algorithm detects 25 cases in which the unconditional variance is not constant, whereas MICSS($\hat{\alpha}_{NR}$) detects 28. Overall, it represents about 55% of the cases with finite variance for the two tail index estimators that have been considered. All the log-exchange rate returns with structural breaks in the unconditional variance detected by the MICSS($\hat{\alpha}_H$) algorithm have also been characterized as stochastic processes with changing unconditional variance by the MICSS($\hat{\alpha}_{NR}$) algorithm. These are (figures between parentheses indicate the number of structural breaks): Botswana (2 with $\hat{\alpha}_H$, 3 with $\hat{\alpha}_{NR}$), Chile (2), Colombia (2), Czech (1), French Guinea (2), India (6), Israel (3), Malawi (11), Mauritius (1), Morocco (8 with $\hat{\alpha}_H$, 9 with $\hat{\alpha}_{NR}$), Norway (1), Paraguay (1), Peru (1), Philippines (8), Poland (2), Qatar (6), Romania (1), Samoa (3), South Africa (2), Special Drawing Rights (12), Switzerland (2), Tanzania (2), Uganda (6), Ukraine (3 with $\hat{\alpha}_H$, 5 with $\hat{\alpha}_{NR}$) and Vietnam (8). Additionally, MICSS($\hat{\alpha}_{NR}$) leads detecting structural breaks for China (6), Hong Kong (1) and Jordan (1), where the hypothesis of non-finite variance cannot be rejected when using the $\hat{\alpha}_H$ estimator for these three countries.

To sup up, a total of 109 structural breaks in the unconditional variance of the log-exchange rate returns have been found when using the MICSS($\hat{\alpha}_{NR}$) algorithm, and 97 with the MICSS($\hat{\alpha}_H$) one. Tables 3 to 7 present detailed results for MICSS($\hat{\alpha}_{NR}$), and Figure 9 shows some illustrative examples. Although most of the changes in the variance of the exchange rate are country specific, some of them seem to be common to several countries. For example, COVID pandemic represents the only structural change that has been detected in Norway and Peru, and could also be associated with some of the structural breaks in China, Israel and Qatar. On the other hand, the invasion of Ukraine by Russia has implied a structural change in the variance of the log-exchange rate returns of Ukraine. Moreover, the only structural break in Hong Kong log-exchange rate returns variance could be related to the 2003 SARS epidemic, during which the territory experienced an important economic downturn; the one in Jordan could be related to the Israel-Jordan Treaty of Peace signed on October 26th, 1994; and that of Paraguay to the impeachment proceedings against President Lugo and the rise of Horacio Cartes as new President in 2013, to mention few cases.

8 Concluding remarks

We have shown that the GEV family of distributions provide accurate approximations to the distribution of the supremum of the CUMSUMQ test statistic of constant unconditional variance in finite samples. Moreover, the distribution of the CUMSUMQ test statistic also depends on the tail index of the underlying distribution, and the GEV approximation implicitly incorporates this feature. Response surfaces to approximate the location, scale and shape parameters that define GEV distributions have been estimated in the paper. These response surfaces allow computing critical values or p-values for the CUMSUMQ test statistic that is proposed in the paper for any sample size or tail index within the range $2 < \alpha \leq 4$. The simulation exercise that has been performed suggests that the CUMSUMQ test statistic shows good size and non-negligible power, specially for the Gaussian case ($\alpha = 4$).

The CUMSUMQ statistic can be easily implemented within the ICSS algorithm which tests the constancy of the unconditional variance for different portions of the data and, hence, different number of observations. The paper proposes a modification of the ICSS algorithm that considers the tail index of the time series under investigation and, depending on it, adjusts the critical values to be used in the inference stage. To this end, an R library is available upon request.

We have applied the modified ICSS algorithm to eighty log-exchange rate returns against the US dollar, finding evidence of infinite variance in slightly more than one third of cases. Further, structural changes have been detected in around a third of cases, totalling about a hundred of structural breaks. Slightly less than a third of the variables do not present instability in the unconditional variance.

Finally, it is worth noting that the approach suggested here may also be applied to other statistics that are computed as sequences of maxima.

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A Mathematical appendix

To proof Proposition 1 we shall make use of the following three lemmas.

Lemma 1 *1. Let X be standard a **Brownian motion** and $\{(t_{i-1}, t_i)\}_{i\in I}$ be pairwise disjoint subintervals of $\mathbb{R}^+$. Then, the processes $\left\{B_s^i\right\}_{s\in[0,t_i-t_{i-1}]}$ defined by*

$$B_s^i = X_{t_{i-1}+s} - X_{t_{i-1}} - \frac{s}{t_i - t_{i-1}}\left(X_{t_i} - X_{t_{i-1}}\right)$$

are independent Brownian bridges, and are independent of X_t over the range $t \in \mathbb{R}^+ \setminus \bigcup_i (t_{i-1}, t_i)$.

*2. Let X be standard a **Brownian bridge** on interval $[0,T]$ and $\{(t_{i-1}, t_i)\}_{i\in I}$ be pairwise disjoint subintervals of $[0,T]$. Then, the processes $\left\{B_s^i\right\}_{s\in[0,t_i-t_{i-1}]}$ defined by*

$$B_s^i = X_{t_{i-1}+s} - X_{t_{i-1}} - \frac{s}{t_i - t_{i-1}}\left(X_{t_i} - X_{t_{i-1}}\right) \tag{10}$$

are independent Brownian bridges, and are independent of X_t over the range $t \in [0,T] \setminus \bigcup_i (t_{i-1}, t_i)$.

Proof. The proof is almost trivial (see also Lemmas 5 and 6 in https://almostsuremath.com/2021/03/29/brownian-bridges/). As the processes are joint normal, it is sufficient that there is zero covariance between them:

$$\begin{aligned} E\left(B_s^i B_{s'}^{i'}\right) &= E\left[\left(X_{t_{i-1}+s} - X_{t_{i-1}} - \frac{s}{t_i - t_{i-1}}\left(X_{t_i} - X_{t_{i-1}}\right)\right)\right. \\ &\quad \left.\left(X_{t_{i'-1}+s'} - X_{t_{i'-1}} - \frac{s'}{t_{i'} - t_{i'-1}}\left(X_{t_{i'}} - X_{t_{i'-1}}\right)\right)\right] \\ &= 0 \end{aligned}$$

for $i \neq i'$ because all the cross-products have zero expectation. For the second results, let $X_t = Y_t - \frac{t}{T} Y_T$, where Y is a Brownian motion. Then, substituting in (10)

$$\begin{aligned} B_s^i &= Y_{t_{i-1}+s} - \frac{t_{i-1}+s}{T} Y_T - Y_{t_{i-1}} + \frac{t_{i-1}}{T} Y_T - \frac{s}{t_i - t_{i-1}}\left(Y_{t_i} - \frac{t_i}{T} Y_T - Y_{t_{i-1}} + \frac{t_{i-1}}{T} Y_T\right) \\ &= Y_{t_{i-1}+s} - Y_{t_{i-1}} - \frac{s}{t_i - t_{i-1}}\left(Y_{t_i} - Y_{t_{i-1}}\right) + \frac{t_{i-1} - t_{i-1} - s}{T} Y_T + \frac{s}{t_i - t_{i-1}} \frac{t_i - t_{i-1}}{T} Y_T \\ &= Y_{t_{i-1}+s} - Y_{t_{i-1}} - \frac{s}{t_i - t_{i-1}}\left(Y_{t_i} - Y_{t_{i-1}}\right) \end{aligned}$$

and result 1 applies. ■

Lemma 2 *Let X be a Brownian motion on $[0,1]$, $B_t^{ex} = (X_t : 0 \leq t \leq 1 \mid X_t > 0$ for $0 < t < 1$ and $X_1 = 0)$ be a Brownian excursion and $M^+ = \sup_{0 \leq t \leq 1} B_t^{ex}$, then*

$$\Pr\left(M^+ \leq u\right) = 1 + 2\sum_{k=1}^{\infty}\left(1 - (2ku)^2\right) e^{-2(ku)^2}.$$

Proof. See Proposition 3.1 in Durrett and Iglehart (1977). ■

Lemma 3 *The distribution function F with right endpoint $x_F \leq \infty$ belongs to the maximal domain of attraction of the Gumbel distribution if and only if there exists $z < x_F$, such that F has representation*

$$\overline{F}(x) = c(x) \exp\left\{-\int_z^x \frac{\beta(t)}{g(t)} dt\right\}$$

for $x \in (z, x_F)$, where $\overline{F}(x) = 1 - F(x)$ is the survival function, $\lim_{x \to x_F} c(x) = c \in (0, \infty)$, $\lim_{x \to x_F} \beta(x) = 1$ and g is a positive absolutely continuous function satisfying $\lim_{x \to x_F} g'(x) = 0$.

Proof. Embrechts et al. (2003), Theorem 3.3.26, page 142. ■

Proof of Proposition 1

Let $w_t = \hat{\omega}_4^{-1/2}\left(S_t - \frac{t}{T} S_T\right)$, $S_t = \sum_{j=1}^t \hat{\varepsilon}_j^2$, and

$$K_T(z) = T^{-1/2} \sum_{t=1}^T \left(\mathbf{1}\left[w_{t-1} \leq z, w_t > z\right] + \mathbf{1}\left[w_{t-1} > z, w_t \leq z\right]\right)$$

where $\mathbf{1}[\cdot]$ is the indicator function that takes value 1 when the conditions between brackets holds true and zero otherwise, be the normalized number of level crossings of the Brownian bridge w_t. Using García and Sansó's (2006) Lemmas 3 and 4, when $T \to \infty$, then $K_T(0) \xrightarrow{d} cR$, where $c = E|z| = \sqrt{\frac{2}{\pi}}$, the expectation of the absolute value of a standard Gaussian distribution, and R is a random variable with (standard) Rayleigh distribution, so that the first result of Proposition 1 is proven. Using this result and the fact that $E(R) = \sqrt{\frac{\pi}{2}}$, then, for large T we have that $E(n) = T^{1/2} c E(R) = T^{1/2}\sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}} = T^{1/2}$ and the second result is proven.

Each time the Brownian bridge crosses zero, there is the start of a new Brownian bridge until it crosses zero next time. Let $0 < t_1 < \ldots < t_n < T$ be the moments the Brownian bridge crosses zero. Then, we may write (10) as $B_s^i = X_{t_{i-1}+s}$, $s \in [0, t_i - t_{i-1}]$, where $X_{t_{i-1}+s}$ itself is a Brownian bridge, witch is independent of $B_r^j = X_{t_{j-1}+r}$, $r \in [0, t_j - t_{j-1}]$. Lemma 1 shows that they are independent and so are their absolute values. That is, $\{B_s^i\}_{s\in[0,t_i-t_{i-1}]}$, $0 < t_1 < \ldots < t_n < T$, is a sequence of independent Brownian bridges and $\{|B_s^i|\}_{s\in[0,t_i-t_{i-1}]}$, $0 < t_1 < \ldots < t_n < T$, is also and independent sequence. By a scaling argument and the scale invariance of the Brownian bridge, $\{((t_i - t_{i-1})/T)^{-1/2}|B_r^i|\}_{r\in[0,1]}$, $i = 1, ..., n+1$, is a sequence of $n+1$ Brownian excursions, $\{B_{i,r}^{ex}\}_{r\in[0,1]}$, with $B_{i,r}^{ex} = ((t_i - t_{i-1})/T)^{-1/2}|B_r^i|$, $r \in [0,1]$. Let $M_i^+ = \sup_{0\le r\le 1} B_{i,r}^{ex}$, and $s_i \in [0, t_i - t_{i-1}]$. Then, we may write,

$$\begin{aligned}
\sup_t |B_t| &= \sup\left(\sqrt{\frac{t_1/T}{t_1/T}}\left|B_{s_1}^1\right|, \sqrt{\frac{(t_2-t_1)/T}{(t_2-t_1)/T}}\left|B_{s_2}^2\right|, \ldots, \sqrt{\frac{(t_n-t_{n-1})/T}{(t_n-t_{n-1})/T}}\left|B_{s_n}^n\right|, \sqrt{\frac{1-t_n/T}{1-t_n/T}}\left|B_{s_{n+1}}^{n+1}\right|\right) \\
&= \sup\left(\sqrt{\frac{t_1}{T}}B_{1,r}^{ex}, \sqrt{\frac{t_2-t_1}{T}}B_{2,r}^{ex}, \ldots, \sqrt{\frac{t_n-t_{n-1}}{T}}B_{n,r}^{ex}, \sqrt{1-\frac{t_n}{T}}B_{n+1,r}^{ex}\right) \\
&= \sup\left(\sqrt{\frac{t_1}{T}}M_1^+, \sqrt{\frac{t_2-t_1}{T}}M_2^+, \ldots, \sqrt{\frac{t_n-t_{n-1}}{T}}M_n^+, \sqrt{1-\frac{t_n}{T}}M_{n+1}^+\right),
\end{aligned}$$

for $t \in [0,T]$, and the third result of Proposition 1 is proven.

Finally, using lemma 2 we may write

$$\begin{aligned}
\overline{F}(u) &= 1 - \Pr\left(M_0^+ \le u\right) \\
&= -2\sum_{k=1}^{\infty}\left(1 - (2ku)^2\right)e^{-2(ku)^2}.
\end{aligned}$$

For large u

$$\begin{aligned}
\overline{F}(u) &= 8u^2 e^{-2u^2} + o\left(u^2 e^{-2u^2}\right) \\
&\approx 8e^{-2u^2+\ln u^2} \\
&= 8e^2 \exp\left\{\ln u - u^2\right\} \\
&= 8e^{2-\left(z^2-\ln z\right)} \exp\left\{\ln u - u^2 + \left(z^2 - \ln z\right)\right\} \\
&= 8e^{2-\left(z^2-\ln z\right)} \exp\left\{-\int_z^u \left(2t - \frac{1}{t}\right) dt\right\} \\
&= 8e^{2-\left(z^2-\ln z\right)} \exp\left\{-\int_z^u 2t\left(1 - \frac{1}{2t^2}\right) dt\right\} \\
&= c\exp\left\{-\int_z^u \frac{\beta(t)}{g(t)} dt\right\}
\end{aligned}$$

where $0 < z < u < \infty$, $c = 8e^{2-\left(z^2-\ln z\right)} > 0$, $\beta(t) = \left(1 - \frac{1}{2t^2}\right)$, $\lim_{t\to\infty} \beta(t) = 1$, $g(t) = (2t)^{-1}$, $t > 0$, is a positive absolutely continuous function with $g'(t) = -1/\left(2t^2\right)$, and $\lim_{t\to\infty} g'(t) = 0$. Thus, according to Lemma 3, F belongs to the maximal domain of attraction of the Gumbel distribution. ■

B Tables and figures

Table 1: Rejection frequencies of the Kolmogorov-Smirnov test: empirical vs. GEV-based distributions, $T = 100$

	Nominal size			
	0.10	*0.05*	*0.025*	*0.01*
$\alpha = 4$	0.110	0.060	0.030	0.014
$\alpha = 3.5$	0.094	0.034	0.018	0.010
$\alpha = 2.75$	0.141	0.080	0.044	0.018

Table 2: Estimation of the tail index and number of breaks in the unconditional variance for exchange rates

		Hill estimator				Nicolau-Rodrigues estimator			
Country	**First obs.**	$\hat{\alpha}_H$	$\varphi_{\alpha\geq 4}$	$\varphi_{\alpha\leq 2}$	Breaks	$\hat{\alpha}_{NR}$	$\varphi_{\alpha\geq 4}$	$\varphi_{\alpha\leq 2}$	Breaks
Albania	1998/12/2	1.003	-18.939	-12.599	∞	0.927	-54.508	-19.038	∞
Algeria	1998/12/2	0.726	-20.687	-16.096	∞	0.695	-79.101	-31.229	∞
Argentine	1994/1/3	1.350	-18.383	-9.018	∞	1.588	-29.908	-5.111	∞
Australia	1994/1/3	2.862	-7.897	11.973	0	3.128	-5.493	7.109	0
Bahrain	1994/1/3	1.366	-18.264	-8.797	∞	1.524	-31.947	-6.137	∞
Bangladesh	1994/1/3	1.131	-19.919	-12.071	∞	1.432	-35.224	-7.789	∞
Bolivia	1994/1/3	1.442	-17.734	-7.738	∞	1.759	-25.055	-2.698	∞
Botswana	1994/1/3	2.811	-8.251	11.265	2	3.097	-5.751	6.992	3
Brazil	1994/7/1	2.525	-10.152	7.227	0	2.856	-7.823	5.848	0
Brunei	1994/1/3	2.618	-9.592	8.583	0	2.905	-7.428	6.135	0
Bulgaria	1998/12/2	0.460	-17.022	-14.808	∞	0.360	-169.686	-76.439	∞
Burundi	1998/12/2	7.193	20.210	65.739	0	7.961	8.502	12.795	0
Canada	1994/1/3	2.943	-7.338	13.090	0	3.470	-3.012	8.343	0
Chile	1994/1/3	2.955	-7.252	13.263	2	3.426	-3.300	8.199	2
China	1994/1/3	1.820	-15.115	-2.499	∞	2.306	-14.384	2.600	6
Colombia	1994/1/3	2.708	-8.968	9.832	2	3.121	-5.531	7.052	2
Czech	1994/12/12	3.138	-5.885	15.543	1	3.526	-2.606	8.399	1
Denmark	1994/1/3	3.480	-3.606	20.554	0	4.255	1.180	10.439	0
Ecuador	1994/12/12	0.718	-22.365	-17.471	∞	1.006	-56.868	-18.881	∞
Egypt	1994/12/9	1.503	-17.042	-6.789	∞	1.563	-30.254	-5.425	∞
Euro	1994/1/3	3.473	-3.661	20.445	0	4.252	1.165	10.432	0
Fiji	1998/12/2	0.850	-19.939	-14.561	∞	0.292	-193.129	-88.973	∞
French Guinea	1998/12/2	7.944	24.965	75.248	2	8.713	9.227	13.143	2
Gambia	1998/12/2	9.887	37.264	99.845	0	10.959	10.971	14.124	0
Ghana	1997/5/27	1.473	-16.488	-6.881	∞	1.474	-31.807	-6.623	∞
Hong Kong	1994/1/3	1.935	-14.334	-0.900	∞	2.383	-13.355	3.165	1
Hungary	1994/1/3	3.040	-6.665	14.437	0	3.441	-3.192	8.234	0
Iceland	1997/5/27	2.686	-8.577	8.961	0	2.650	-9.459	4.558	0
India	1994/1/3	2.382	-11.225	5.299	6	2.988	-6.643	6.489	6
Indonesia	1994/1/3	1.497	-17.339	-6.965	∞	1.656	-27.783	-4.081	∞
Israel	1994/1/3	2.809	-8.271	11.225	3	3.365	-3.717	7.984	3
Japan	1994/1/3	2.908	-7.577	12.613	0	3.399	-3.482	8.108	0
Jordan	1994/1/3	2.021	-13.712	0.289	∞	2.233	-15.538	2.048	1
Kazakhstan	1998/12/2	0.752	-18.639	-14.321	∞	0.300	-185.933	-85.433	∞
Kenya	1997/5/27	1.731	-14.811	-3.506	∞	2.025	-17.988	0.225	∞
Kuwait	1994/1/3	2.280	-11.940	3.886	0	2.328	-14.180	2.778	0
Lebanon	1994/1/3	1.542	-17.051	-6.354	∞	2.284	-14.832	2.452	0
Malawi	1998/12/2	6.845	18.008	61.335	11	6.547	6.487	11.580	11

Notes: ∞ indicates that the variance is infinite.

Table 2: Estimation of the tail index and number of breaks in the unconditional variance for exchange rates (continued)

		Hill estimator				Nicolau-Rodrigues estimator			
Country	**First obs.**	$\widehat{\alpha}$	$\varphi_{\alpha\geq 4}$	$\varphi_{\alpha\leq 2}$	Breaks	$\widehat{\alpha}$	$\varphi_{\alpha\geq 4}$	$\varphi_{\alpha\leq 2}$	Breaks
Malaysia	1994/1/3	1.885	-14.683	-1.599	∞	1.973	-20.275	-0.270	∞
Mauritania	1997/5/27	1.359	-17.244	-8.373	∞	1.581	-28.311	-4.902	∞
Mauritius	1997/2/4	2.563	-9.438	7.391	1	2.963	-6.517	6.056	1
Mexico	1994/1/3	2.350	-11.431	4.851	0	2.251	-15.298	2.193	0
Morocco	1994/12/12	3.177	-5.620	16.073	8	3.836	-0.827	9.288	9
Mozambique	1998/12/2	8.792	30.333	85.985	0	10.691	10.584	13.748	0
Namibia	2004/1/13	0.346	-12.257	-11.098	∞	0.264	-202.930	-94.289	∞
New Guinea	1994/1/3	1.821	-15.114	-2.478	0	2.050	-18.742	0.478	∞
New Zealand	1994/1/3	3.090	-6.315	15.137	0	3.315	-4.069	7.814	0
Nigeria	1994/1/3	1.588	-16.744	-5.721	∞	1.699	-26.733	-3.499	∞
Norway	1994/1/3	3.038	-6.680	14.406	1	3.456	-3.100	8.299	1
Oman	1994/1/3	1.795	-15.288	-2.844	∞	1.802	-24.054	-2.172	∞
Pakistan	1994/1/3	1.420	-17.911	-8.055	∞	1.581	-30.063	-5.208	∞
Paraguay	1998/12/2	6.502	15.827	56.952	1	7.189	7.722	12.565	1
Peru	1994/1/3	2.324	-11.632	4.503	1	2.888	-7.592	6.065	1
Philippines	1994/1/3	2.312	-11.707	4.336	8	2.311	-14.424	2.656	8
Poland	1995/1/4	2.846	-7.873	11.548	2	3.250	-4.464	7.437	2
Qatar	1998/12/2	8.606	29.155	83.628	6	10.801	10.897	14.102	6
Romania	1997/5/27	2.742	-8.211	9.693	1	2.974	-6.406	6.084	1
Russia	1996/3/7	1.678	-15.501	-4.299	∞	1.689	-25.953	-3.490	∞
Samoa	2004/1/13	6.618	14.754	52.046	3	5.712	4.354	9.440	3
Saudi Arabia	1994/1/3	1.209	-19.348	-10.965	∞	1.377	-37.401	-8.883	∞
Singapore	1994/1/3	2.618	-9.592	8.583	0	2.905	-7.428	6.135	0
South Africa	1994/1/3	3.119	-6.114	15.539	2	3.548	-2.510	8.600	2
South Korea	1994/1/3	2.158	-12.754	2.187	0	2.112	-17.592	1.046	∞
SDR	1994/1/3	3.342	-4.569	18.630	12	3.829	-0.879	9.416	12
Sri Lanka	1994/1/3	1.523	-17.181	-6.613	∞	1.732	-25.787	-3.045	∞
Sweden	1994/1/3	3.314	-4.763	18.241	0	3.740	-1.372	9.163	0
Switzerland	1994/1/3	3.342	-4.569	18.628	2	3.660	-1.831	8.952	2
Taiwan	1994/1/3	2.490	-10.479	6.809	0	2.865	-7.815	5.954	0
Tanzania	1998/12/2	6.914	18.443	62.203	2	7.335	7.615	12.182	2
Thailand	1994/1/3	2.018	-13.753	0.244	∞	2.003	-19.631	0.033	∞
Tunisia	1997/5/27	3.346	-4.273	17.570	0	3.851	-0.721	8.927	0
Turkey	1994/1/3	2.063	-13.430	0.872	∞	2.062	-18.521	0.595	∞
UAE	1994/12/12	1.610	-16.288	-5.318	∞	1.728	-25.464	-3.049	∞
Uganda	1998/12/2	9.107	31.741	88.342	6	5.298	3.748	9.523	6
Ukraine	1998/12/2	3.602	-2.520	20.277	3	2.279	-11.819	1.917	5
UK	1994/1/3	3.277	-5.018	17.731	0	3.536	-2.592	8.572	0
Uruguay	1994/12/2	1.892	-14.374	-1.472	∞	2.237	-15.276	2.051	0
Venezuela	1994/1/3	1.094	-20.174	-12.580	∞	1.352	-38.649	-9.463	∞
Vietnam	1998/12/2	7.740	23.150	71.059	8	7.064	6.933	11.459	8
Zambia	1997/5/27	1.993	-13.102	-0.089	∞	2.341	-13.115	2.696	0

Notes: ∞ indicates that the variance is infinite. SDR stands for Special Drawing Rights.

Table 3: Breaks in the unconditional variance using MICSS($\hat{\alpha}_{NR}$)

Botswana				
Start	End	Var	κ_T	p-value
1994-01-03	2001-11-28	$3.42 \cdot 10^{-5}$	2.256	0.002
2001-11-29	2009-11-11	$8.32 \cdot 10^{-5}$	2.184	0.002
2009-11-12	2016-12-14	$3.77 \cdot 10^{-5}$	1.627	0.017
2016-12-15	2023-07-14	$2.39 \cdot 10^{-5}$		
Chile				
Start	End	Var	κ_T	p-value
1994-01-03	2008-01-04	$2.31 \cdot 10^{-5}$	1.553	0.024
2008-01-07	2019-10-15	$4.66 \cdot 10^{-5}$	1.475	0.031
2019-10-16	2023-07-14	$9.48 \cdot 10^{-5}$		
China				
Start	End	Var	κ_T	p-value
1994-01-03	1995-05-23	$1.27 \cdot 10^{-7}$	1.413	0.037
1995-05-24	2005-06-24	$3.88 \cdot 10^{-9}$	3.230	0.000
2005-06-27	2015-07-15	$1.09 \cdot 10^{-6}$	2.157	0.003
2015-07-16	2017-08-01	$3.62 \cdot 10^{-6}$	1.881	0.007
2017-08-02	2020-12-09	$7.16 \cdot 10^{-6}$	1.867	0.007
2020-12-10	2022-03-24	$2.80 \cdot 10^{-6}$	2.650	0.001
2022-03-25	2023-07-14	$1.54 \cdot 10^{-5}$		
Colombia				
Start	End	Var	κ_T	p-value
1994-01-03	1998-08-17	$1.14 \cdot 10^{-5}$	1.714	0.012
1998-08-18	2007-07-23	$2.93 \cdot 10^{-5}$	1.856	0.008
2007-07-24	2023-07-14	$6.96 \cdot 10^{-5}$		
Czech				
Start	End	Var	κ_T	p-value
1994-12-12	2012-09-25	$3.22 \cdot 10^{-5}$	1.545	0.024
2012-09-26	2023-07-14	$1.42 \cdot 10^{-5}$		
French Guinea				
Start	End	Var	κ_T	p-value
1998-12-02	2002-03-26	0.151	2.147	0.001
2002-03-27	2005-01-03	0.049	3.094	0.000
2005-01-04	2023-07-14	0.218		
Hong Kong				
Start	End	Var	κ_T	p-value
1994-01-03	2003-09-18	$3.22 \cdot 10^{-8}$	1.579	0.021
2003-09-19	2023-07-14	$1.42 \cdot 10^{-7}$		
Israel				
Start	End	Var	κ_T	p-value
1994-01-03	2007-04-23	$1.35 \cdot 10^{-5}$	2.074	0.003
2007-04-24	2009-08-12	$6.35 \cdot 10^{-5}$	2.902	0.000
2009-08-13	2020-02-25	$1.73 \cdot 10^{-5}$	1.613	0.018
2020-02-26	2023-07-14	$3.95 \cdot 10^{-5}$		

Table 4: Breaks in the unconditional variance using MICSS($\hat{\alpha}_{NR}$)

India				
Start	End	Var	κ_T	p-value
1994-01-03	1995-08-28	$1.03 \cdot 10^{-6}$	2.447	0.001
1995-08-29	1996-05-08	$3.97 \cdot 10^{-5}$	1.899	0.006
1996-05-09	1998-08-17	$8.44 \cdot 10^{-6}$	1.513	0.025
1998-08-18	2004-03-15	$1.26 \cdot 10^{-6}$	2.137	0.003
2004-03-16	2008-07-15	$8.89 \cdot 10^{-6}$	2.088	0.003
2008-07-16	2013-12-06	$3.26 \cdot 10^{-5}$	2.782	0.000
2013-12-09	2023-07-14	$9.60 \cdot 10^{-6}$		
Jordan				
Start	End	Var	κ_T	p-value
1994-01-03	1995-01-04	$2.75 \cdot 10^{-6}$	1.390	0.042
1995-01-05	2023-05-31	$5.29 \cdot 10^{-7}$		
Malawi				
Start	End	Var	κ_T	p-value
1998-12-02	2000-06-02	0.204	2.121	0.001
2000-06-05	2001-05-03	0.068	3.587	0.000
2001-05-04	2003-01-03	0.224	3.694	0.000
2003-01-06	2005-02-24	0.102	2.125	0.001
2005-02-25	2006-08-10	0.183	2.315	0.000
2006-08-11	2007-11-30	0.073	2.075	0.001
2007-12-03	2008-08-25	0.175	1.494	0.021
2008-08-26	2011-04-27	0.243	1.693	0.008
2011-04-28	2015-05-11	0.207	2.980	0.000
2015-05-12	2019-08-06	0.089	3.056	0.000
2019-08-07	2021-07-29	0.224	2.581	0.000
2021-07-30	2023-07-14	0.143		
Mauritius				
Start	End	Var	κ_T	p-value
1997-02-04	2007-12-03	$5.15 \cdot 10^{-6}$	1.752	0.011
2007-12-04	2023-07-06	$2.57 \cdot 10^{-5}$		
Morocco				
Start	End	Var	κ_T	p-value
1994-12-12	2008-01-14	$1.89 \cdot 10^{-5}$	2.213	0.001
2008-01-15	2009-04-01	$6.15 \cdot 10^{-5}$	1.528	0.017
2009-04-02	2010-04-29	$2.54 \cdot 10^{-5}$	1.799	0.004
2010-04-30	2011-11-10	$3.92 \cdot 10^{-5}$	1.699	0.007
2011-11-11	2012-09-25	$2.44 \cdot 10^{-5}$	1.585	0.013
2012-09-26	2013-11-06	$1.56 \cdot 10^{-5}$	2.141	0.001
2013-11-07	2014-10-28	$7.81 \cdot 10^{-6}$	2.066	0.001
2014-10-29	2016-06-27	$2.16 \cdot 10^{-5}$	1.782	0.005
2016-06-28	2022-01-25	$1.56 \cdot 10^{-6}$	2.372	0.000
2022-01-26	2023-07-14	$1.89 \cdot 10^{-5}$		

Table 5: Breaks in the unconditional variance using MICSS($\hat{\alpha}_{NR}$)

Norway				
Start	End	Var	κ_T	p-value
1994-01-03	2020-02-27	$4.88 \cdot 10^{-5}$	2.086	0.003
2020-02-28	2023-07-14	$8.96 \cdot 10^{-5}$		

Paraguay				
Start	End	Var	κ_T	p-value
1998-12-02	2013-06-14	0.222	2.399	0.0003
2013-06-17	2023-06-13	0.023		

Peru				
Start	End	Var	κ_T	p-value
1994-01-03	2020-02-25	$8.70 \cdot 10^{-6}$	2.072	0.004
2020-02-26	2023-07-14	$2.86 \cdot 10^{-5}$		

Philippines				
Start	End	Var	κ_T	p-value
1994-01-03	1997-07-04	$1.32 \cdot 10^{-5}$	2.527	0.001
1997-07-07	1998-06-17	$2.46 \cdot 10^{-4}$	2.456	0.001
1998-06-18	2001-08-07	$4.78 \cdot 10^{-5}$	1.637	0.016
2001-08-08	2007-05-10	$6.83 \cdot 10^{-6}$	2.941	0.000
2007-05-11	2009-02-25	$2.49 \cdot 10^{-5}$	1.351	0.046
2009-02-26	2013-09-27	$1.27 \cdot 10^{-5}$	2.411	0.001
2013-09-30	2021-06-07	$6.04 \cdot 10^{-6}$	1.433	0.033
2021-06-08	2022-07-19	$1.07 \cdot 10^{-5}$	1.679	0.015
2022-07-20	2023-07-14	$1.98 \cdot 10^{-5}$		

Poland				
Start	End	Var	κ_T	p-value
1995-01-04	2008-08-01	$3.60 \cdot 10^{-5}$	1.975	0.005
2008-08-04	2012-09-25	$1.69 \cdot 10^{-4}$	1.716	0.012
2012-09-26	2023-07-14	$4.37 \cdot 10^{-5}$		

Qatar				
Start	End	Var	κ_T	p-value
1998-12-02	2007-03-05	0.234	1.885	0.003
2007-03-06	2015-03-18	0.207	2.201	0.001
2015-03-19	2018-06-15	0.144	2.119	0.001
2018-06-18	2019-09-09	0.232	1.768	0.005
2019-09-10	2020-06-02	0.128	1.336	0.046
2020-06-03	2021-12-03	0.200	2.066	0.001
2021-12-06	2023-07-14	0.010		

Romania				
Start	End	Var	κ_T	p-value
1997-05-27	2017-02-01	$5.77 \cdot 10^{-5}$	1.482	0.029
2017-02-02	2023-07-14	$2.08 \cdot 10^{-5}$		

Table 6: Breaks in the unconditional variance using MICSS($\hat{\alpha}_{NR}$)

Samoa				
Start	End	Var	κ_T	p-value
2004-01-13	2007-06-29	0.170	3.287	0.000
2007-07-02	2014-04-18	0.068	4.538	0.000
2014-04-21	2019-05-30	0.197	3.115	0.000
2019-05-31	2023-07-14	0.127		
South Africa				
Start	End	Var	κ_T	p-value
1994-01-03	2001-10-11	$3.64 \cdot 10^{-5}$	2.089	0.003
2001-10-12	2009-12-23	$1.45 \cdot 10^{-4}$	1.441	0.036
2009-12-24	2023-07-14	$9.27 \cdot 10^{-5}$		
Special Drawing Rights				
Start	End	Var	κ_T	p-value
1994-01-03	1996-02-29	$1.48 \cdot 10^{-5}$	2.301	0.000
1996-03-01	2002-06-17	$6.70 \cdot 10^{-6}$	2.212	0.001
2002-06-18	2005-11-03	$9.72 \cdot 10^{-6}$	2.083	0.001
2005-11-04	2008-03-10	$5.31 \cdot 10^{-6}$	1.827	0.004
2008-03-11	2009-03-19	$2.20 \cdot 10^{-5}$	1.699	0.007
2009-03-20	2010-09-20	$9.13 \cdot 10^{-6}$	1.435	0.028
2010-09-21	2011-11-14	$1.26 \cdot 10^{-5}$	3.235	0.000
2011-11-15	2013-11-06	$5.81 \cdot 10^{-6}$	2.449	0.000
2013-11-07	2014-12-02	$2.55 \cdot 10^{-6}$	2.531	0.000
2014-12-03	2017-01-30	$8.91 \cdot 10^{-6}$	2.853	0.000
2017-01-31	2022-02-22	$3.34 \cdot 10^{-6}$	1.341	0.048
2022-02-23	2022-11-30	$1.01 \cdot 10^{-5}$	1.428	0.029
2022-12-01	2023-07-14	$5.69 \cdot 10^{-6}$		
Switzerland				
Start	End	Var	κ_T	p-value
1994-01-03	2015-08-26	$4.87 \cdot 10^{-5}$	1.796	0.008
2015-08-27	2022-02-17	$1.88 \cdot 10^{-5}$	1.838	0.006
2022-02-18	2023-07-14	$3.61 \cdot 10^{-5}$		
Tanzania				
Start	End	Var	κ_T	p-value
1998-12-02	2003-01-27	0.235	4.026	0.000
2003-01-28	2011-10-21	0.083	3.778	0.000
2011-10-24	2023-07-14	0.198		

Table 7: Breaks in the unconditional variance using MICSS($\hat{\alpha}_{NR}$)

Ukraine				
Start	End	Var	κ_T	p-value
1998-12-02	2003-06-06	0.087	1.332	0.049
2003-06-09	2008-10-20	0.061	2.116	0.003
2008-10-21	2012-04-19	0.108	1.342	0.047
2012-04-20	2015-07-10	0.143	3.218	0.000
2015-07-13	2022-02-16	0.055	2.615	0.001
2022-02-17	2023-07-14	0.205		
Uganda				
Start	End	Var	κ_T	p-value
1998-12-02	2002-06-04	0.157	3.127	0.000
2002-06-05	2006-02-17	0.050	1.868	0.000
2006-02-20	2011-11-25	0.089	2.574	0.000
2011-11-28	2013-01-29	0.195	2.729	0.000
2013-01-30	2016-11-17	0.091	2.545	0.000
2016-11-18	2018-05-08	0.180	1.624	0.017
2018-05-09	2023-07-14	0.121		
Vietnam				
Start	End	Var	κ_T	p-value
1998-12-02	2001-04-17	0.225	2.592	0.000
2001-04-18	2003-09-04	0.157	3.330	0.000
2003-09-05	2006-07-06	0.061	1.373	0.042
2006-07-07	2009-11-26	0.095	2.858	0.000
2009-11-27	2011-02-01	0.209	1.758	0.005
2011-02-02	2012-09-27	0.141	2.340	0.000
2012-09-28	2015-07-22	0.212	3.551	0.000
2015-07-23	2021-02-10	0.104	1.447	0.029
2021-02-11	2023-07-14	0.063		

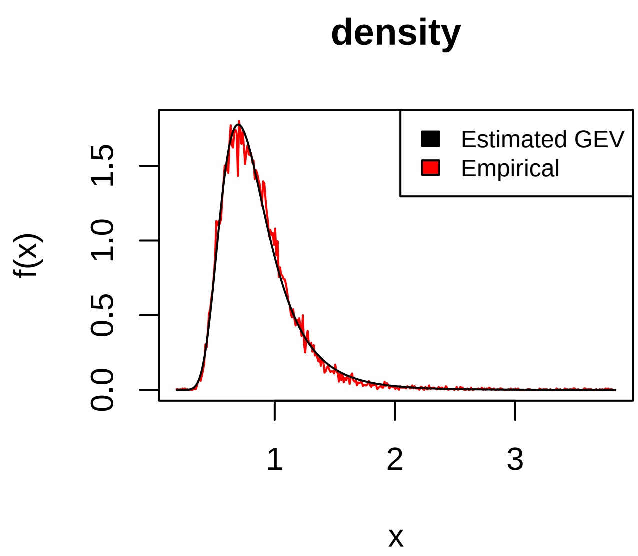

Figure 1: Empirical and GEV-based distributions of the κ_T statistic for $\alpha = 3.5$ and $T = 100$

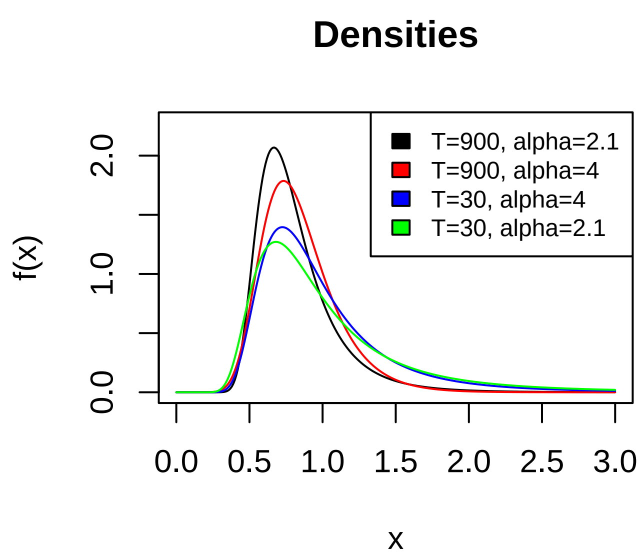

Figure 2: Densities of the κ_T statistic

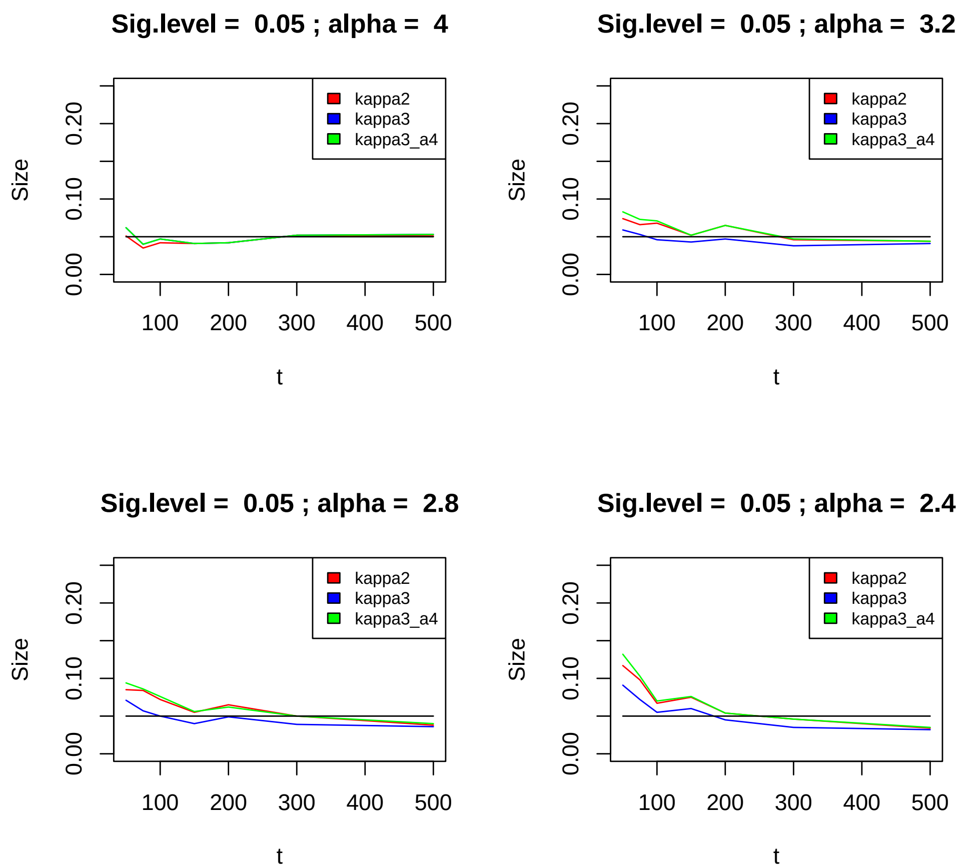

Figure 3: Empirical size for given α values

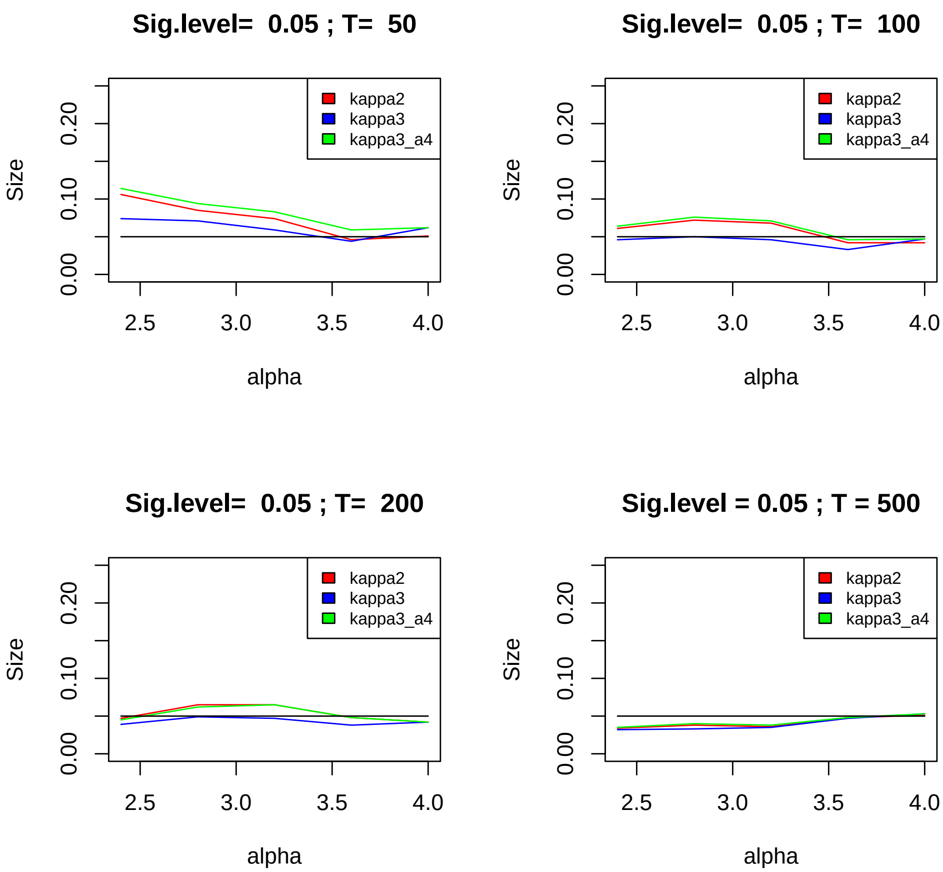

Figure 4: Empirical size for fixed values of T

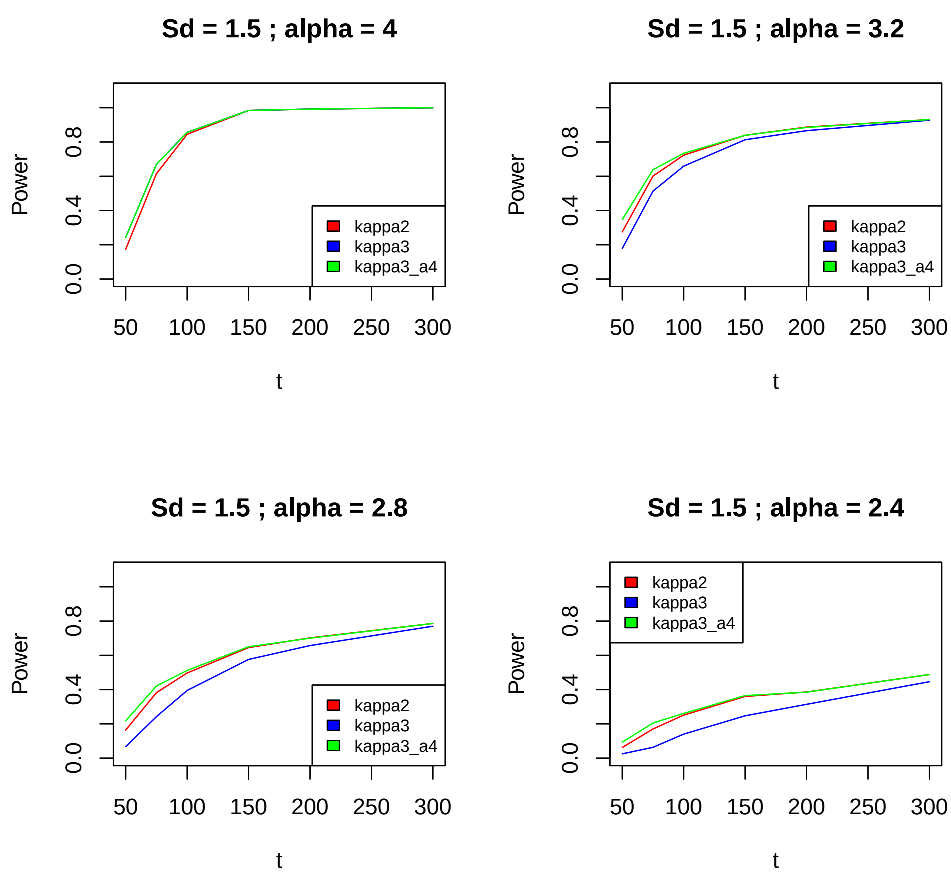

Figure 5: Empirical power for given α values and $\sigma_2 = 1.5$

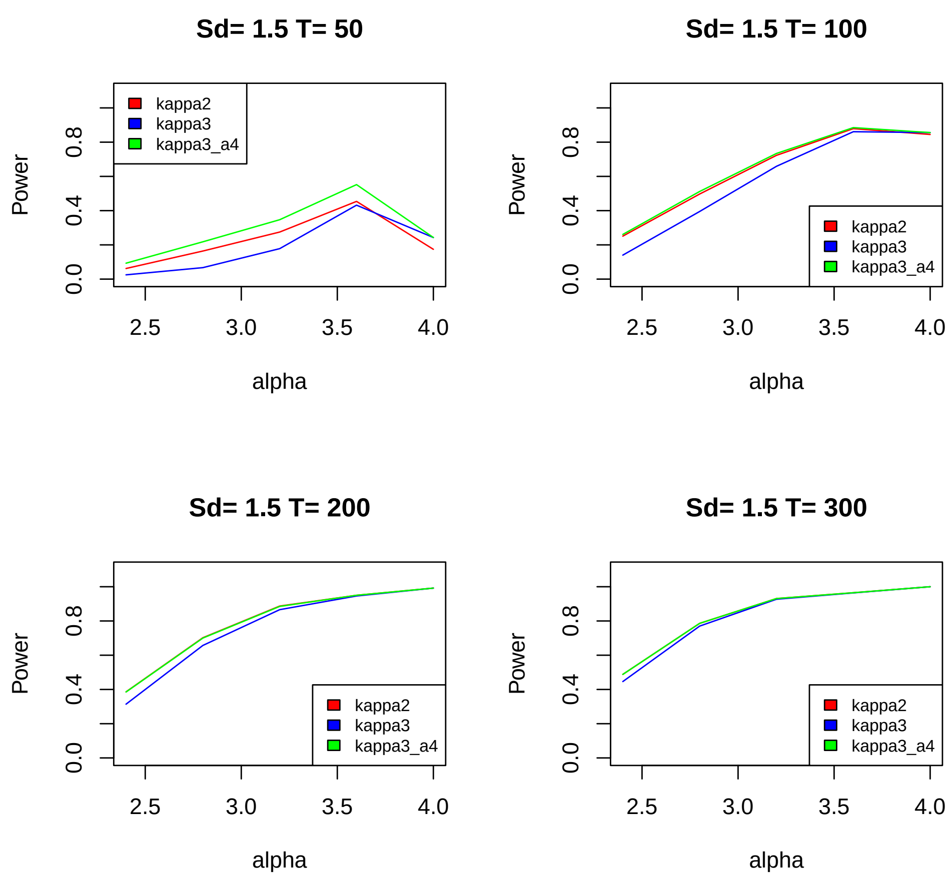

Figure 6: Empirical power for given T values and $\sigma_2 = 1.5$

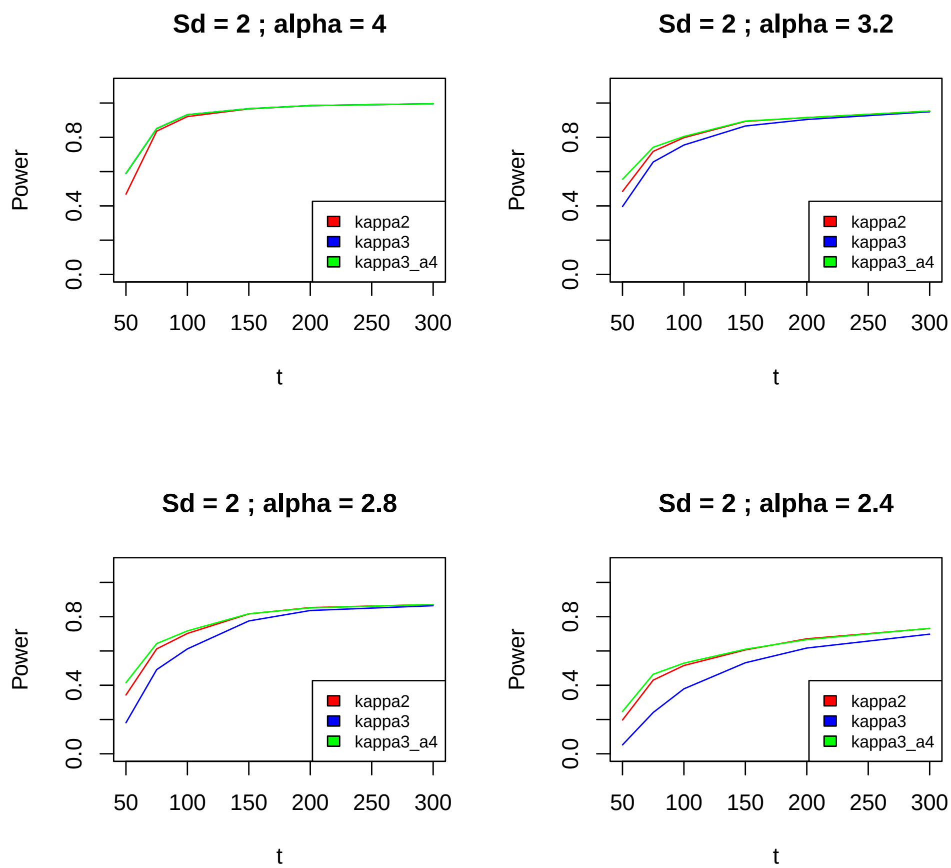

Figure 7: Empirical power for given α values and $\sigma_2 = 2$

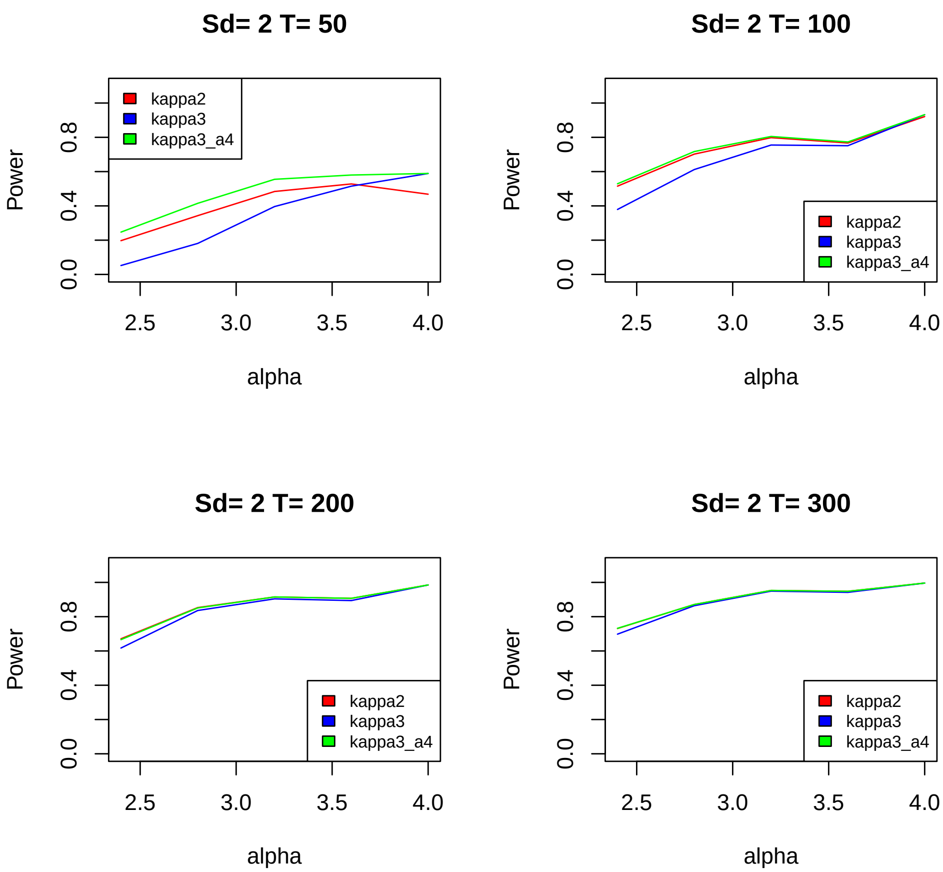

Figure 8: Empirical power for given T values and $\sigma_2 = 2$

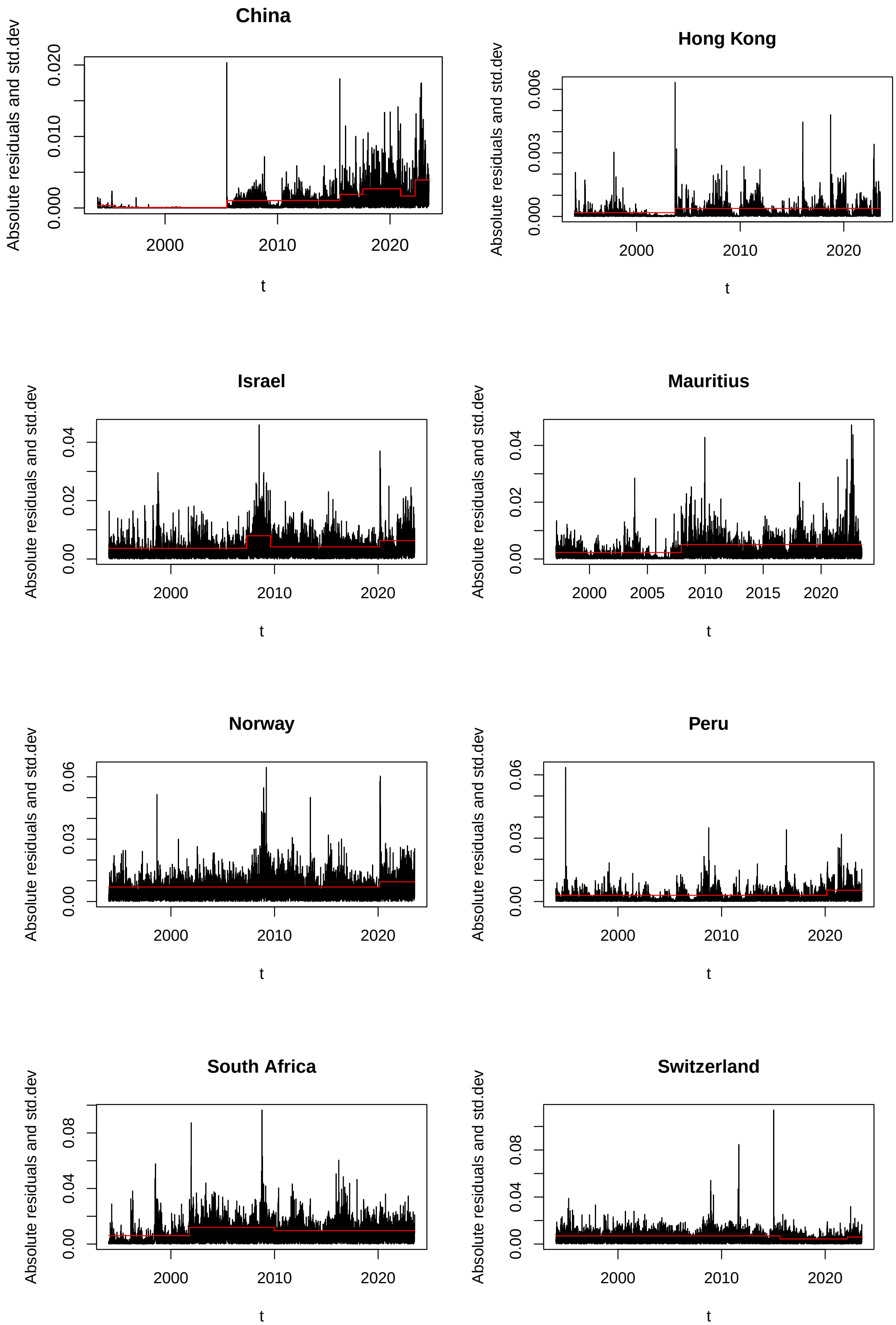

Figure 9: Estimated changes in the unconditional variance of exchange rates

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