UNIVERSITAT DE BARCELONA

ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

Cayley Graphs and Endomorphism Monoids

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ABSTRACT

Since its first steps at the hands of Euler, graph theory has gradually become a field of great interest and innovation for the mathematical community.
From its surprising capability to simplify the formulation of applied problems to the rich complexity that some of its natural problems contain, even in finite settings (being the field to see the first computer-assisted proof in mathematics), the list of its merits and uses seems to only grow in length, keeping the promise of attracting research for the times to come.
For the untrained eye, however, it could appear as a branch with few theoretically rich connections to other fields of mathematics aside from topology (via graph embeddings), which is a misconception. In a certain way, disproving this thought is the main focus of this project, as the aim is to show the connections between graph theory and abstract algebra (semigroup and monoid theory specifically), hoping to put both in a more interesting light.
Specifically, we introduce and talk about the basic tools of the field (mainly the Cayley graph construction, and a fairly young generalization of it by Yongwen Zhu as seen in [10]), introduce some recent interesting results in the literature by many authors, mostly by K.Knauer and coauthors (as in references [5], [7], [6]) and try to put together a comprehensive guide to try and understand the main difficulties and ideas used in one of the main lines of work in the field. This can be exemplified in our in-depth study of some families of outerplanar graphs as monoid graphs, or our brief study of $K_4 \sqcup C_5$ as a non monoid but possibly semigroup graph. Both questions were originally raised by K.Knauer and Puig i Surroca in their work referenced in [5].

Contents

1. Introduction

As discussed in the abstract, the fundamental aim of this work is to show the ways in which graph theory is directly related to semigroup theory, and how different graph properties can be deduced from semigroup properties and vice versa. In particular, we are interested in families of graphs that can be generated by either semigroups or monoids or that can give us all monoids of a given class as endomorphism monoids of some of its members.
As is often the case in mathematics, the way in which these two topics relate is via a convenient construction. On the one hand, we have a natural way to assign a semigroup to any given graph (a monoid, specifically) considering the algebraic structure constituted by its set of endomorphisms with respect to usual function composition.
On the other hand, the construction that assigns a graph to a given semigroup, the so-called Cayley graph, is, perhaps, not so direct, but also quite natural and simple to consider. In both cases, however, appearances can be deceiving, as they can be quite complex structures to work with.
In this section, in order to be able to properly work with these concepts, we proceed to lay down the fundamental definitions used in this work.

Definition 1.1. *Given graphs G snd H with vertices $V(G)$, $V(H)$ and edges $E(G)$, $E(H)$; a function $\phi : V(G) \longmapsto V(H)$ is called a graph morphism if, for every $(x, y) \in E(G)$, $(\phi(x), \phi(y)) \in E(H)$*
If $G = H$, we call such a function an endomorphism of the graph in question and, if it is also a bijective function, it is called an automorphism. The corresponding sets of functions, which are monoids with the identity function and the usual function composition, are denoted as $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$, respectively.

Definition 1.2. *Given a graph G, we say it is a digraph when we allow edges to be directed (that is (x, y) and (y, x) define different edges), and we say it is a multigraph when they can be multiple (there can be more than one edge adjacent to the same pair of vertices) or loops (vertices where both endpoints coincide).*

Definition 1.3. *We say that G is colored (by edges or by vertices) when we consider it with a coloring C, which is, essentially, a partition of either $V(G)$ or $E(G)$ where every set that is part of it is called a color. If it is a partition of the set of vertices, we call it a coloring of the vertices, and the same applies to the edges.*

Definition 1.4. *Given a set S, we say that it is a semigroup with a given operation $* : S \times S \longmapsto S$ (of course, we denote $*((x, y))$ as $x * y$) if it is associative, that is, if for every $x, y, z \in S$, $(x * y) * z = x * (y * z)$*

We use the $*$ in order to make emphasis on the operation in this first definition, but usually we will omit the sign (multiplicative notation) or use additive notation if the given semigroup we are working with happens to be commutative.

Definition 1.5. *A semigroup S is called a* monoid *iff it has a neutral element, that is, if there is $e \in S$ such that, for every $x \in S$ satisfies $ex = xe = x$*

It is useful to make the observation that every semigroup can be made into a monoid by attaching to it a new element which acts as a neutral one, as such an element will not conflict with the associativity of the operations in the original set.
In some contexts, this way to look at a semigroup can be advantageous, which motivates the following notation:

Definition 1.6. *Given a semigroup S, we denote by S^1 the following associated monoid:*

$$S^1 := \begin{cases} S & \textit{if } S \textit{ is already a monoid} \\ S \cup \{1\} & \textit{otherwise} \end{cases}$$

where 1 is the aforementioned, new neutral element.

Having introduced the most basic definitions, it is now our objective to introduce the construction that allows us to assign a graph to a given semigroup: the Cayley graph of a semigroup, and explore some of its properties.

Definition 1.7. *Let S be a semigroup (which we will normally take to be finite, although the definition does not need it), and consider a nonempty set $C \subseteq S$*

which we will call the connection set of the graph.
Then, we call the Cayley graph of S with connection set C, which we will denote as $\mathrm{Cay}(S, C)$*, the graph defined by taking S as its set of vertices and* $E := \{(s, sc) \mid s \in S,\ c \in C\}$ *as its set of (directed) edges.*

Although simple at first view, this definition requires some caveats to be made clear before using it: first off, it is obvious that the same type of construction could be made by taking the edges to be the pairs (s, cs) in the same conditions as specified above, that is, that operations could be reversed.
As it is often the case when talking semigroups, the "right translation" definition is the standard one, but the left would be equally valid, considering that it could change the resulting graph.
The second one is to observe that, in order to preserve the maximum amount of algebraic information possible, this graph finds its most natural realization as an edge colored digraph, where edges are directed, possibly multiple or loops and have a color indicated by the element c of the connection set that spawns it. This colored version is usually denoted as $Cay_{col}(S, C)$

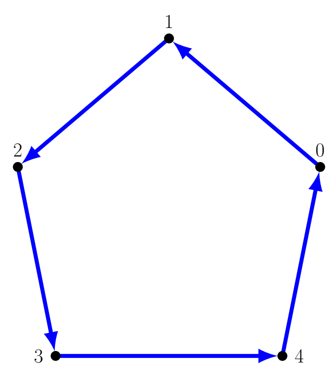

FIGURE 1. A simple picture of $Cay_{col}(\mathbb{Z}/5\mathbb{Z},\ \{1\})$

In some occasions, however, it might be convenient to go the opposite route: taking the colored Cayley digraph of a semigroup and getting rid of all multiple edges, loops and colors. This way, we obtain what is called the underlying simple Cayley graph which, given a connection set C, is denoted as $\underline{Cay}(S, C)$
Looking at the figure above, for instance, we see directly that
$\underline{Cay}(\mathbb{Z}/5\mathbb{Z},\ \{1\}) = C_5$, the cycle with five vertices. It follows immediately

from the described way of obtaining the graph that we lose relevant algebraic information when taking this step, but we gain from a graph theoretical perspective, as it is sometimes convenient to consider the simplest version of a graph. The underlying graph also plays an important role for part of this work.

2. Semigroups and Cayley graphs

2.1. **Basic semigroup theory.** The definition presented in the preceding section is the classic definition of a Cayley graph as it was first presented in 1878 and, in practice, will be the one we will use in order to classify families of graphs as what we will refer to as *monoid graphs*, but there exist modern, more abstract versions of the concept that generalize it and enrich its meaning. We will focus on one of such generalizations, particularly the one found in detail in [10].

Before that, though, we need to see some basic concepts from semigroup theory. They can be found, sometimes in more detail, in references [1] and [4]:

Let S, R be semigroups. A function $f : S \longmapsto T$ is called a semigroup morphism if, of course, it is compatible with the respective operations in the usual way.

If it is injective, it is called a monomorphism; if it is surjective an epimorphism and, in case of both, we are dealing with an isomorphism. The only remarkable difference to note in this more general, noncommutative case, is that we also consider two semigroups essentially equal if there is a bijective function between them that just flips the order of the operation.

Definition 2.1. *Let S, R be semigroups.*
A bijective function $f : S \longmapsto R$ is called an anti-isomorphism iff
$f(ab) = f(b)f(a)$ for every a, b in A and B, respectively.
We say that two semigroups related in such a way are anti-isomorphic.

Definition 2.2. *Let A and B be nonempty subsets of a semigroup S. We can now define an operation of subsets of S as folllows:*
$AB := \{ab|\ a \in A,\ b \in B\}$, which we call the product set of A and B

Definition 2.3. *Let S be a semigroup, $T \subseteq S$ such that it is nonempty. Then, if T is closed with respect to the operation defined in S, that is, $xy \in T$ $\forall x, y \in T$, we say T is a subsemigroup of S.*
An equivalent way to see it is that T must satisfy $TT \subseteq T$ in the terminology we have just introduced.

This operation might not seem so interesting at first glance, as it has no special properties to speak of: even if A and B were to be subsemigroups of S, AB would not have to be a subsemigroup at all.
The only immediate, interesting property of it is that it directly inherits associativity from the fact that the operation defined in S is.
Where this operation finds its theoretical meaning, besides making some definitions and arguments more direct, is in the concept of semigroup ideal.

Definition 2.4. *Let S be a semigroup, T a nonempty subset of S. We then say T is a* left ideal *of S iff $TS \subseteq T$ Analogously, we can also define the concept of* right ideal *of a semigroup.*
If a given set T is both a left and right ideal, we call it a two-sided ideal*.*

Remark 1. *Of course, these distinctions become superfluous when dealing with commutative semigroups, a context where we recover the very familiar concept of ideal.*
In general, any ideal in the usual sense used in ring theory, whether right or left sided, is first and foremost a left or right ideal in the semigroup sense with respect to the "multiplicative" ring operation.

This notion of ideal also allows us to consider a definition of a quotient semigroup, as tends to be the norm in abstract algebra.

Definition 2.5. *Let S be a semigroup, I one of its ideals. We can then define the following relation in S:*
Given two elements x, y, then $x \sim y$ iff either $x = y$ or both $x, y \in I$. This relation is easily checked to be an equivalence relation, but, even more importantly, is also a congruence, that is: given $x, y, z, t \in S$,
$x \sim y$ and $z \sim t \implies xz \sim yt$ This property basically means that the relation is compatible with the semigroup operation, making the quotient set also

a semigroup.
Therefore, we can consider $S/\sim := S/I$ (for short) the quotient semigroup of S by I with the following operation:
$x * y := \begin{cases} xy & \text{if } x, y \text{ is are not elements of } I \\ 0 & \text{otherwise} \end{cases}$, *where 0 refers to the class of $S/\sim$ shared by all the elements of I.*
This construction is called the Rees quotient semigroup *of S by I, or the* Rees factor semigroup*.*

Having introduced the principal notion that will allow us to generalize Cayley graphs (the notion of ideal), we define some basic, sought-after properties in semigroup theory which we may encounter when dealing with them.

Definition 2.6. *Let S be a semigroup. Then, we say that S is* left-cancellative *if, for every triple of elements $a, x, y \in S$, $ax = ay \implies x = y$*
Of course, this same definition can be formulated transposing the order of operations, which gives us the definition of a right-cancellative semigroup*. When a given semigroup is both left-cancellative and right-cancellative, we refer to it simply as* cancellative*.*

Definition 2.7. *Let S be a semigroup. We say that S is* idempotent*, or a* band*, if $a^2 = a$ for every $a \in S$. If the semigroup also happens to be commutative, we refer to it as a* semilattice*.*

Definition 2.8. *Let S be a semigroup. Then, we say that it is* regular *if, for every element $a \in S$, there is an element x satisfying $axa = a$, which we call a* pseudoinverse *of a. This is equivalent to stating that every element a of S has at least one pseudoinverse b in the following sense: b satisfies $aba = a$ and $bab = b$*
When, for every element a, this inverse element is unique, we say that S is an inverse *semigroup.*

Finally, we show some basic results that relate these general properties between them and to the most well-known class of semigroups of all: groups. They can be found in [4].

Proposition 2.1. *Let S be semigroup. Then, S is an inverse semigroup iff it is regular and idempotent elements commute in S.*

Proof. We begin the proof in the usual order. Of course, if S is an inverse semigroup, in particular, it is regular. A particular thing that is useful to note is that, given an element a and its pseudoinverse b, both ab and ba are idempotent elements. This is direct, as:
$(ab)(ab) = (aba)b = ab$, and the proof for the other mentioned element is analogous. That these special kind of idempotents commute is not obvious, however, so we must prove it before. Take a pair of elements x, y, and denote $xx_s = e$ and $yy_s = f$, where x_s denotes the unique pseudoinverse of x. Let us now take the unique pseudoinverse of ef, call it z. Then, we have that $(ef)(fze)(ef) = efzef = ef$ and $(fze)(ef)(fze) = f(zefz)e = fze$, so $(ef)_s = fze$, which means that $z = fze$. This means that z is an idempotent, as $(fze)^2 = f(zefz)e = fze$ It then follows that z is its own pseudoinverse as well and, by uniqueness, $ef = z$. Similarly, we have that fe is idempotent as well (making the correspondent analogous arguments for it), and as $(ef)(fe)(ef) = efef = (ef)^2 = ef$ and $(fe)(ef)(fe) = fefe = (fe)^2 = fe$, which means that fe and ef are pseudoinverses of ef, by uniqueness, it follows immediately that $ef = fe$.
Therefore, it would be enough to prove that every idempotent element of S is of this form, a product of an element and its pseudoinverse. So, consider an element e of S that is idempotent. By definition, we know that there is a unique pseudoinverse for e, call it e_s, which satisfies $e = ee_se$ and such that e is the pseudoinverse of e_s as well. Hence, we have that $e_s = e_see_s = e_s(ee_s) = e_s(e_se) = (e_s)^2e = e_se$, since we know ee_s is an idempotent (and thus its own pseudoinverse) and also $(ee_s)(e_se)(ee_s) = (ee_se)e_s = ee_s$, so e_se is also its pseudoinverse and, then, $e_se = ee_s$. So, this immediately means that e_s is of the desired form, and applying the same treatment to e grants us the equality $e = ee_se = e(e_se) = e(ee_s) = e^2e_s = ee_s$, which finishes the proof of this first part by showing that all idempotents are of this special form and must, in turn, commute.
For the converse result, we suppose that S is regular and all idempotent elements commute. We have to show that pseudoinverses are unique. So, let a

be some element in S, and suppose that it has two different pseudoinverses, b, c, so $aba = a$, $bab = b$, $aca = a$, $cac = c$ and ab, ba, ac, ca are all idempotent elements as previously shown.
Then, we have $b = bab = b(aca)b = (ba)(ca)b = (ca)(ba)b = cabab = c(abab) = c(ab)^2 = c$ because idempotents commute by hypothesis. Therefore, pseudoinverses are unique and S is an inverse semigroup. $\square$

Inverse semigroups are notorious for being quite common among special families of semigroups and pseudoinverses having some of the properties we are used to expect from inverse elements in group theory, mainly due to their uniqueness. These facts make them one of the most natural generalizations of groups one can work with, although the definition of a pseudoinverse in itself is not so natural to grasp at first.

Proposition 2.2. *Let S be a finite semigroup. If it is both cancellative and a monoid, it is a group.*

Proof. The only thing we need to do is reformulate cancellative properties in terms of set operations and functions: S is cancellative iff it is both right and left cancellative, which is equivalent to the functions $L_a : S \longmapsto S$ and $R_a : S \longmapsto S$, defined term by term as $L_a(x) = ax$ and $R_a(x) = xa$, respectively, being injective. Due to S being finite and this fact, we can make a counting argument directly to obtain the equations $aS = S$ and $Sa = S$ for every element a of S. In particular, this means that, for the neutral element e, there are elements b_1 and b_2 such that $ab_1 = e$ and $b_2a = e$, and, of course, playing a little bit with these equations, one can quickly see that it follows that $b_1 = eb_1 = (b_2a)b_1 = b_2(ab_1) = b_2$. $\square$

Proposition 2.3. *Let S be a monoid. Then, S is a group iff it is both cancellative and regular.*

Proof. If S is a group, it is clear that its inverse obviously plays the part of a pseudoinverse and more (hence the name) and it is cancellative.
The converse is also quickly verifiable, as, if we consider an element a of S and its corresponding pseudoinverse, say x, then it is clear from $axa = a$ and

cancellative properties that $ax = e$ and $xa = e$, which makes x the inverse of a in the usual sense. This, of course, implies that S is a group. □

Definition 2.9. *Let S, T be a pair of semigroups. If $S \subseteq T$, we say that T is an extension of S if S is a subsemigroup of T. When S is an ideal of T, we say that it is an ideal extension* [2].

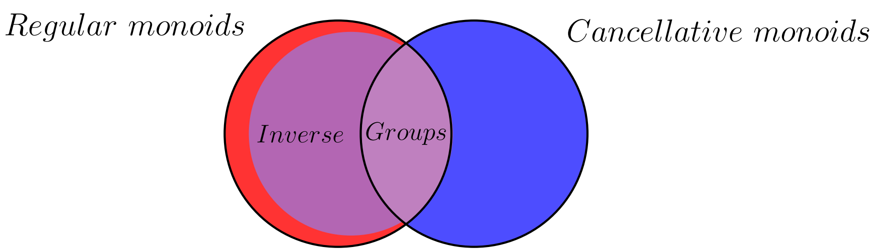

FIGURE 2. A simple diagram depicting relations between exposed properties for monoids.

2.2. **General properties of generalized Cayley graphs.** With all these concepts on the table, we can now introduce the generalized definition of a Cayley graph and see some of is most notorious properties:

Definition 2.10. *Let S be a semigroup, and T an ideal extension of S in the sense previously stated. Then, if we consider the monoid T^1 and a relation $\rho \subseteq T^1 \times T^1$ (nonempty), we define the generalized Cayley graph of S with respect to the relation ρ, denoted by $\mathrm{Cay}(S, \rho)$ as the graph described by having set of vertices $V(\mathrm{Cay}(S, \rho)) = S$ and set of edges defined the following way:* $E(\mathrm{Cay}(S, \rho)) := \{(a, b) |$ *there is* $(x, y) \in \rho$ *such that* $xay = b\}$

So, the main addition we can observe is that this version of a Cayley graph extends the classical definition basically in the sense that it allows us to add more edges than was previously possible, all while preserving the original set of vertices. Of course, taking some specific choices of ρ allows us to recover special graphs: for $T = S$ and $\rho = \omega := S^1 \times S^1$, we obtain what we call the

[2]Alternative definitions of ideal extensions make use of the Rees factor semigroup and a semigroup Q which is disjoint to S, and say that T is an ideal extension of S by Q if the condition marked above is met and also $T/S \cong Q$

universal Cayley graph for the chosen ideal extension, for $\rho = \{1\} \times C$ for a nonempty subset of S we obtain the classic Cayley graph as we introduced before (given that we choose the right action one) and for $\rho = C \times \{1\}$ the left action one. For the special relations $\omega_l := S^1 \times \{1\}$ and $\omega_r := \{1\} \times S^1$ we obtain what we call the left-universal and right-universal generalized Cayley graphs of S, respectively.

As before, we note that the defined construction makes the most sense when considered as a colored digraph, but even the underlying simple graph gets extended with respect to the original definition. Let us try to illustrate it by way of an example, as exposed in [10]: let N the semigroup of all natural numbers with the usual multiplication, and take S to be the subsemigroup of all even numbers. Then, for every nonempty subset T of S, the graph $\mathrm{Cay}(S, T)$ in the classic sense will only contain edges (a, b) such that $4|b$. With the extended definition, as we can consider relations in a bigger pool of elements and S is an ideal of N, we can take $\mathrm{Cay}(S, N \times N)$, where it is obvious the previous observation does not apply. For example, $(1, 3)$ belongs to the relation and so $(2, 6) \in E(\mathrm{Cay}(S, N \times N))$.

This main tool being introduced, let us explore some of the main ways in which it can transform graph properties into algebraic ones, and the other way around.

Let a be an element of a semigroup S. As it is usual in these settings, it is easy to note that the intersection of any family of (left or right) ideals of S is also a (left or right) ideal of S; that semigroup ideals are closed by intersection. This property immediately leads to the natural definition of "least left, right or both sided ideal containing a" as the intersection of all such sets that contain it. Let us denote them by $L(a)$, $R(a)$ and $J(a)$ for the minimal (by inclusion) left, right and both sided ideals respectively.

Proposition 2.4. *Let a be an element of a semigroup S.*
Then, $L(a) = S^1a$, $R(a) = aS^1$, $J(a) = S^1aS^1$

Proof. First thing of interest to observe is that it is immediate to see that $L(a)$, $R(a)$ and $J(a)$ are left, right and both sided ideals of S respectively, giving directly one of the desired inclusions for each set. For the other ones, we work it by hand: let L_I be a left sided ideal of S containing a. Then, by its stated properties, it will contain $a = 1a$ and sa for every element $s \in S$, and thus it will contain the whole $L(a)$ as well. The same argument can be made in order to prove the inclusions for $R(a)$ and $J(a)$. We show the result as found in [10]. $\square$

Proposition 2.5. *Let S be a regular semigroup, T an ideal extension of S and $\rho \subseteq T^1 \times T^1$ a nonempty relation.*
Then, the following statements hold:
(1) If a, b are such that $(a, b) \in E(\mathrm{Cay}(S, \rho))$, then $J(b) \subseteq J(a)$;
(2) There is a relation $\rho_0 \subseteq S \times S$ such that $\mathrm{Cay}(S, \rho)$ is a subgraph of $\mathrm{Cay}(S, \rho_0)$;

Proof. (1): If a, b are such that (a, b) is in $E(\mathrm{Cay}(S, \rho))$, there is, by definition, $(x, y) \in \rho$ such that $xay = b$. In consequence, given an element $z \in J(b)$, it is of the form s_1bs_2 for $s_1, s_2 \in S$, so $z = s_1bs_2 = s_1xays_2 = (s_1x)a(ys_2)$ and, as S is an ideal of T, s_1x and ys_2 are both in S, so $z \in J(a)$
(2): Let ρ be a nonempty relation $T^1 \times T^1$, and let us consider $\mathrm{Cay}(S, \rho)$. We want to find a relation $\rho_0 \subseteq S \times S$ such that $G := \mathrm{Cay}(S, \rho)$ is a subgraph of $H := \mathrm{Cay}(S, \rho_0)$. As their sets of vertices are both S, this is equivalent to stating that every edge of G is an edge of H. Knowing this, let us consider $(a, b) \in E(\mathrm{Cay}(S, \rho))$. Once again, by definition, we have that there is $(x, y) \in \rho$ such that $xay = b$. At this point, we use that S is a regular semigroup: we know there is at least one element $c \in S$ satisfying $aca = a$ and $cac = c$, so we obtain that $b = xay = xacay = (xa)c(ay) = (xa)cac(ay) = (xac)a(cay)$.
As both a, c are in S and it is an ideal of T, we have that xac and cay are elements of S. In consequence, given an edge of G (a, b), ${x_a}^b, {y_a}^b$ the elements in T^1 such that ${x_a}^b a {y_a}^b = b$ and c_a its pseudoinverse, we only need the set inclusion
$R_G := \{({x_a}^b ac_a, c_a a{y_a}^b) | a \in S \text{ s.t } \exists b \text{ with } (a, b) \in E(\mathrm{Cay}(S, \rho)\} \subseteq \rho_0$
to be true in order for ρ_0 to be a relation with the desired property. Of course, such a thing can be forced, as we could even choose $R_G = \rho_0$, and, clearly, $R_G \subseteq S \times S$; making the proof complete. $\square$

This first property is interesting insofar as it allows us to simplify any relation ρ to one about elements of S in exchange of a reasonable property to be found in practical settings, as many semigroups naturally encountered are regular.

Definition 2.11. *Let S be a semigroup, T an ideal extension of S and $\rho \subseteq T^1 \times T^1$ a nonempty relation. Given an element a of S, we can define the following sets: $\rho(a) := \{xay|\ (x, y) \in \rho\}$, $\rho(a)^1 := \rho(a) \cup \{a\}$, and we call the latter the $\rho-$class of a*

Remark 2. *Given a directed graph G, and a vertex a of $V(G)$, we can consider the sets:*
$\overrightarrow{a} := \{b \in V(G)|\ (a, b) \in E(G)\}$, $\overleftarrow{a} := \{b \in V(G)|\ (b, a) \in V(G)\}$
If the graph G is of the form $\mathrm{Cay}(S, \rho)$ for a semigroup S and ρ as stated before, it is clear that $(a, b) \in E(\mathrm{Cay}(S, \rho)) \iff b \in \overrightarrow{a}$
Therefore, in this context, it is easy to note that $\rho(a) = \overrightarrow{a}$

The following result further relates the edges of the generalized Cayley graph to the $\rho-$class:

Proposition 2.6. *Let T be an ideal extension of a semigroup S, ρ a nonempty subset of the cartesian product of T^1 with itself. Let a, b be elements of S: if $a \neq b$ and $\rho^1(b) \subseteq \rho^1(a)$, then $(a, b) \in E(\mathrm{Cay}(S, \rho))$*

Proof. Let a, b be elements as described in the statement.
As $\rho^1(b) \subseteq \rho^1(a)$ and $a \neq b$, it must be that $b \in \rho(a)$, so, by definition, there is $(x, y) \in \rho$ such that $b = xay$. Of course, by construction of $\mathrm{Cay}(S, \rho)$, this means that $(a, b) \in E(\mathrm{Cay}(S, \rho))$ $\square$

Introducing some special but somewhat reasonable properties for ρ to satisfy, we can obtain far stronger restrictions for the Cayley graph associated to it. We now present some of the most common:

Definition 2.12. *Let S be a semigroup, $\rho \subseteq S \times S$ a nonempty relation. If, given elements a, b, c, d of S, $(a, b) \in \rho$, $(c, d) \in \rho$ always implies that $(ca, bd) \in \rho$, we say that ρ is an inversely compatible relation, or $I-$compatible, for short.*

This first property is just the condition of being compatible with the semigroup operation, but changing the order of the first component. We can see how strong of a condition this already is for the corresponding generalized Cayley graph in the next pair of results:

Definition 2.13. *Let G be a graph. We say that G is edge-transitive if, for every $a, b, c \in V(G)$, $(a, b), (b, c) \in E(G)$ implies that $(a, c) \in E(G)$*

Proposition 2.7. *Let S be a semigroup, T an ideal extension of it and ρ a nonempty subset of $T^1 \times T^1$ as usual. If ρ is $I-$compatible, then $\mathrm{Cay}(S, \rho)$ is edge-transitive.*

Proof. The argument is fairly straightforward: let us consider a pair of edges of $\mathrm{Cay}(S, \rho)$ of the desired form (a, b) (b, c) and see that (a, c) has to be an edge as well. By definition of the generalized Cayley graph, $(a, b), (b, c)$ being edges of it means that there are $(x, y), (z, t) \in \rho$ such that $xay = b$ and $zbt = c$, respectively, which immediately means that $c = zxayt = (zx)a(yt)$ As ρ is $I-$compatible, it is clear that $(zx, yt) \in \rho$ and, in consequence, (a, c) is also an edge of $\mathrm{Cay}(S, \rho)$ □

As strong as this result is, it is also reasonable enough to encounter, as there are some egregious examples of $I-$compatible relations. For instance, it is clear that, given $A, B \subseteq S$ subsemigroups of S, $\rho = A \times B$ is a subset of $T^1 \times T^1$ for any ideal extension T of S that is clearly $I-$compatible. This also happens for the usual relations $S^1 \times S^1$, $S \times S$, $\{1\} \times S$, $S \times \{1\}$, $S^1 \times \{1\}$ and $\{1\} \times S^1$ and even $\{1\} \times C$ if C is a subsemigroup of S.
In a way, Cayley graphs that are not edge transitive basically come from connection sets that do not inherit algebraic structure from S.
$I-$compatibility also offers us a converse version of Prop.2.6.

Proposition 2.8. *Let S be a semigroup, T an ideal extension of it and ρ a relation with the usual properties which is also $I-$compatible. Then, $(a, b) \in E(\mathrm{Cay}(S, \rho))$ for elements a, b of S implies $\rho(a) \subseteq \rho(b)$ and $\rho(b)^1 \subseteq \rho(a)^1$.*

Proof. By definition, (a, b) being an edge of $\mathrm{Cay}(S, \rho)$ means that there is $(x, y) \in \rho$ such that $xay = b$, so we know that $b \in \rho(a)^1$ already. Let us

now consider an element $c \in \rho(b)^1$ such that $c \neq b$. By the remark we made previously, c is in $\rho(b)$ and $\rho(b) = \overrightarrow{b}$ for $\mathrm{Cay}(S, \rho)$, so $(b, c) \in E(\mathrm{Cay}(S, \rho))$. By the previous result, as we know that ρ is $I-$compatible, it is clear that the Cayley graph associated to ρ is edge-transitive, from which it quickly follows that $(a, c) \in E(\mathrm{Cay}(S, \rho))$ and, so, $c \in \rho(a) \subseteq \rho(a)^1$
Of course, this also means that $\rho(b) \subseteq \rho(a)$ □

In order to reach a context in which these two conditions are completely equivalent, however, we would have to ask one last property to be satisfied.

Definition 2.14. *Let T be an ideal extension of a semigroup S and $\rho \subseteq T^1 \times T^1$ which is nonempty. Given an element a of S, we say that a is stable under ρ if $a \in \rho(a)$, that is, iff $\rho(a) = \rho(a)^1$*
If this is satisfied for every element, we say that S is stable under ρ.

Of course, one could consider the question of how much of a natural property this really is, but it turns out to be quite reasonable. For example, given S that is regular, if we take $T = S$ and ρ is such that
$D \subseteq \rho$, where $D := \{(a, a) |\ a \in S\}$, S is clearly stable under ρ.
As a consequence of these two results and this definition, we can finally state:

Corollary 2.1. *Let S be a semigroup, T an ideal extension of it. If $\rho \subseteq T^1 \times T^1$ is a nonempty relation which is $I-$compatible and such that S is stable under ρ, the following are equivalent for any pair a, b of elements of S:*
(1) $(a, b) \in E(\mathrm{Cay}(S, \rho))$;
(2) $\rho(b) \subseteq \rho(a)$;
(3) $\rho(b)^1 \subseteq \rho(a)^1$;

Proof. (1) $\Longrightarrow$ (2) is Prop.2.8, (2) $\Longrightarrow$ (3) is ensured by the property of S being stable under ρ and (3) $\Longrightarrow$ (1) is Prop.2.6. □

Before ending this subsection, we return to the relations that define classic Cayley graphs as seen at the introductory section, $\rho = \{1\} \times C$ for C a nonempty subset of the semigroup S, which we will denote just as C, and prove some important results about the endomorphism monoid of a Cayley graph in relation to the original semigroup used to span it.
In order to state and prove them, we need a relevant definition:

Definition 2.15. *Let S be a semigroup, $C \subseteq S$ a nonempty connection set, and let $\phi : S \longmapsto S$ be a function such that $\phi(xc) = \phi(x)c$, for all $x \in S$, $c \in C$. Then, it is clear that ϕ is a graph endomorphism of* $\mathrm{Cay}(S, C)$*, and we say that it is a graph endomorphism that preserves colors, or a color morphism of the Cayley graph. If ϕ is bijective, we say that it is a color automorphism of* $\mathrm{Cay}(S, C)$*. We denote the sets of all color morphisms of a given Cayley graph as* $\mathrm{ColEnd}(Cay(S, C))$ *and* $\mathrm{ColAut}(\mathrm{Cay}(S, C))$*, respectively.*

Remark 3. *Although this definition corresponds to the classical setting, it is clear that it is an easy concept to extend to generalized Cayley graphs. Given S a semigroup, T an ideal extension of it and $\phi : T^1 \longmapsto T^1$ a function, for it to induce a color morphism of* $\mathrm{Cay}(S, \rho)$ *(its restriction to S) for a nonempty $\rho \subseteq T^1 \times T^1$, it would suffice that $\phi(xay) = x\phi(a)y$ for all $a \in S$ and all $(x, y) \in \rho$. We could denote the corresponding sets of color morphisms and bijective color morphisms as* $\mathrm{ColEnd}(S, \rho)$ *and* $\mathrm{ColAut}(S, \rho)$*, following the notation seen in the definition.*

We now see the main two results for the classic construction as stated in [5], [7] that solidify Cayley graphs as tools to represent monoids as endomorphism groups of graphs:

Proposition 2.9. *Let S be a semigroup, $C \subseteq S$ a nonempty connection set. Then, the function defined by mapping every $s \in S$ to the monoid morphism $\phi_s : S \longmapsto S$ defined by $\phi_s(t) = st$ for every $t \in S$ is a homomorphism from S to* $\mathrm{End}(Cay_{col}(S, C)) = \mathrm{ColEnd}(\mathrm{Cay}(S, C))$

Proof. We can begin by proving the equation $\mathrm{End}(Cay_{col}(S, C)) = \mathrm{ColEnd}(\mathrm{Cay}(S, C))$. Let $f \in \mathrm{End}(Cay_{col}(S, C))$. Then, by definition of the set, f is a graph morphism that preserves colors. Therefore, for $(a, b) \in E(Cay_{col}(S, C))$, $(f(a), f(b))$ must also be an edge of $\mathrm{Cay}(C, S)$. If the color of (a, b) was $c \in C$, then we have that $ac = b$ and it must be that $f(a)c = f(b)$. As this argument is true for every $c \in C$ and $a \in S$, given that (a, ac) will always be an edge of the Cayley graph, it follows that $f \in Col\mathrm{End}(S, C)$. The other inclusion follows quickly, as, given $f \in Col\mathrm{End}(\mathrm{Cay}(S, C))$, $f(ac) = f(a)c$ for every a in S and c in the connection set. Then, if we consider an edge (a, b) of the

colored Cayley graph, there is $c \in C$ such that $ac = b$ and, in consequence, $f(ac) = f(b) \iff f(a)c = f(b)$, making $(f(a), f(b))$ an edge of the Cayley graph with the same color, which means that $f \in \mathrm{End}(Cay_{col}(S, C))$
Let us now tackle the main content of the proposition: left-multiplication morphisms as endomorphisms of the corresponding colored Cayley graph, and the mapping that assigns them as a semigroup morphism.
Take $s \in S$, and consider $\phi_s : S \longmapsto S$ as defined. We only have to see that it is a morphism that assigns edges to edges and preserves colors. If we consider $(a, b) \in E(\mathrm{Cay}(S, C))$, we have, once again, that there is $c \in C$ such that $ac = b$. Then, $\phi_s(ac) = \phi_s(b) \iff sac = sb$, so (sa, sb) is also an edge of $Cay_{col}(S, C)$ with the same color c, so $\phi_s \in End(\mathrm{Cay}_{col}(S, C))$
Finally, seeing that $\Phi : S \longmapsto \mathrm{End}(Cay_{col}(S, C))$ defined by $\Phi(s) = \phi_s$ is a semigroup morphism is immediate, as the operation in the endomorphism set of the Cayley graph is function composition and it is clear that $\phi_{st} = \phi_s \circ \phi_t$ for any pair of elements s, t of S. $\square$

This result admits a sort of a strengthened converse, in the sense that we have to ask a bit more of the connection set and S for it to be true: specifically, we need that C is such that $< C >= S$, a set of generators of S. Although we have not defined this concept in the context of semigroup theory, it can be easily done using the same idea as when working with groups: for a given subset A of S, $< A >$ is defined as the intersection of all subsemigroups of S that contain A, or the least subsemigroup of S in the inclusion sense that contains it. We need to impose that S is a monoid as well.

Proposition 2.10. *Let S be a monoid, $C \subseteq S$ such that $< C >= S$. Then, the mapping $\Phi : S \longmapsto \mathrm{End}(Cay_{col}(S, C))$ defined as before, by the equation $\Phi(s) = \phi_s \ \forall s \in S$, is a monoid isomorphism.*

Proof. It is clear that we can use the previous proposition as a lemma of sorts for a considerable part of the proof. We already know that ϕ_s is a color morphism of $\mathrm{Cay}(S, C)$ for every $s \in S$, and that Φ is a mapping that preserves operations from S to $\mathrm{End}(Cay_{col}(S, C))$, so it suffices to check that it maps the neutral element of S to the identity graph endomorphism and its bijectivity. Let us proceed: let 1 be the neutral element of S, then $\Phi(1) = \phi_1$; the function

that maps any element $t \in S$ to $1t = t$, so, of course, $\phi_1 = Id$ and we have a monoid morphism. To check injectivity, we need only consider a pair s, t of elements satisfying that $s \neq t$. If it were the case that $\phi_s = \phi_t$, then $sa = ta$ $\forall a \in S$ and, in particular, for $a = 1$, $s1 = t1 \iff s = t$
Finally, we tackle surjectivity, where the fact that C is a generating subset of S plays an important role. Let $f \in \mathrm{End}(Cay_{col}(S, C))$, and let us consider $e = f(1)$. Given $a \in S$, as $< C >= S$, there are elements $c_1, ..., c_n \in C$ for a nonzero $n \in \mathbb{N}$ such that $a = \prod_{i=1}^{n} c_i$. For us to use this property to prove that $f = \phi_e$, we only need to consider the equation given by
$\Phi(e)(a) = \phi_e(a) = ea = f(1)a = f(1)\prod_{i=1}^{n} c_i = f(1\prod_{i=1}^{n} c_i) = f(a)$, where the second to last step is justified by the fact that c_i is in C for every $i \in \{1, 2, ...n\}$ and $f \in \mathrm{End}(Cay_{col}(S, C))$ □

In the general case, for T an ideal extension of S and ρ some nonempty relation as usual, we lose this direct link between the semigroup operation and the set of color endomorphisms of the corresponding Cayley graph, as, no matter if we try it with left or right multiplication, the element $s \in S$ we choose will need special properties for edges to be sent to edges and for colors to be preserved. For example, a naive condition to ask for that would be sufficient is the following:

Proposition 2.11. *Let S be a semigroup, T an ideal extension of it; ρ a relation as we have defined before. Then, if $\rho := l_\rho \times r_\rho$ for some pair of subsets of T^1, if $s \in S$ is such that $sx = xs$ $\forall x \in l_\rho$, then $\phi_s \in Col\mathrm{End}(\mathrm{Cay}(S, \rho))$*

Proof. Of course, ϕ_s defines a mapping from S to S, we only have to see that it sends edges to edges preserving colors. But this is obvious, as, if $(a, b) \in E(\mathrm{Cay}(S, C))$, $b = xay$ for $(x, y) \in \rho$, then $\phi_s(b) = sb = sxay = xsay = x(sa)y$, sending (a, b) to (sa, sb) and preserving the color (x, y) □

In order to end the subsection, we see one last relation between a common graph property and an important one in semigroup theory. Let us state the necessary definition and prove it:

Definition 2.16. *Let G be a directed graph. We say that it is* strongly-connected *if, for every pair of vertices x, y, there is a directed path joining*

x to y; that is, a finite set of directed edges with starting point x and ending point y.

Proposition 2.12. *Let S be a finite semigroup, $\mathrm{Cay}(S, C)$ its Cayley graph corresponding to some empty subset C. Then, if $\mathrm{Cay}(S, C)$ is strongly-connected, S is left-cancellative.*

Proof. Let s be an element of S. Since it is clear that $SC \subseteq S$, then $sSC \subseteq sS$ by associativity of the subset operation. This means that, if $sS \neq S$, there are no edges in $\mathrm{Cay}(S, C)$ that go from a vertex in sS to $S \setminus sS$, thus creating a contradiction with the fact that every pair of vertices x, y should be connected by a directed path, as we could choose $x \in sS$ and $y \in S \setminus sS$ in order to disprove it. In consequence, it must be that $sS = S$. As this implies $|sS| = |S|$ and they are both finite, it must be that, for $a, b \in S$ such that $a \neq b$, then $sa \neq sb$; which is equivalent to say that $sa = sb \implies a = b$, as we wanted to show. □

2.3. **Categorical properties of Cayley graphs.** Let us now tackle generalized Cayley graphs with a categorical mindset for a bit, so as to mathematically justify its good properties as a way to obtain a graph from a semigroup.
Let us remind the reader for context that a category (usually denoted in general as $\mathcal{C}$) is a pair given by a class (in the set-theory sense) of objects, which we usually denote by $Obj(\mathrm{C})$ and, for every pair of objects $A, B \in Obj(\mathcal{C})$, a class $Hom(A, B)$ of morphisms equipped with an associative composition rule $Hom(A, B) \times Hom(B, C) \longmapsto Hom(B, C)$ for any three objects A, B, C which also has an identity element in $Hom(A, A)$ for every A. In the usual language of category theory, we want to prove that the Cayley graph construction assigns graphs to semigroups somewhat reasonably, preserving categoric structure. Such special mappings are described by the notion of a *functor*, which is a categorical mapping F connecting a pair of categories $\mathcal{C}, \mathcal{D}$; sending objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and morphisms of $Hom(A, B)$ to morphisms $Hom(F(A), F(B))$, sending the identity elements to identity elements and preserving the composition law in the original sense (a *covariant* functor) or reversing it (*contravariant*).
We first take $T = S$ as the ideal extension for simplicity.

Let $\mathcal{G}_{dir}$ be the category of directed graphs, with directed graphs as objects and directed graph morphisms filling the homonymous role. In the same way, we refer to the category of semigroups as $\mathcal{S}$. We assess the basic categorical properties of the two main constructions we have mentioned in this work:

Proposition 2.13. *Let $\mathcal{G}_{dir}$, $\mathcal{S}$ be the previously mentioned categories. Then, if we consider $\omega := S^1 \times S^1$, $\omega_l := S^1 \times \{1\}$ and $\omega_r := \{1\} \times S^1$, then the mappings from $\mathcal{S}$ to $\mathcal{G}_{dir}$ given by the assignation of Cayley graphs* $\mathrm{Cay}(S^1, \omega)$, $\mathrm{Cay}(S^1, \omega_l)$, $\mathrm{Cay}(S^1, \omega_r)$, *which we will name as C_ω, C_{ω_l} and C_{ω_r}, respectively, are covariant, faithful functors.*

Proof. Given a semigroup S, it is clear that $C_\omega(S)$, $C_{\omega_l}(S)$ and $C_{\omega_r}(S)$ are directed graphs by construction, so we only have to check that the mapping works for morphisms and respects composition. Let N, L be semigroups, and let us consider the maps $f : M \longmapsto N$, $g : N \longmapsto L$, being semigroup morphisms, and let C_ρ refer to any of the aforementioned mappings, while ρ_M is the symbol used to refer to any of the previously stated relations as a subset of $M \times M$. Then, $C_\rho(f)$ is the function that takes every vertex of $\mathrm{Cay}(M, \rho_M)$ to its image by f in $\mathrm{Cay}(N, \rho_N)$. As f is a semigroup morphism, if (a, b) is an edge of $\mathrm{Cay}(N, \rho_N)$, as this means that $xay = b$ for $(x, y) \in \rho_N$, then $f(b) = f(x)f(a)f(y)$ and, of course, $f(\rho_M) \subseteq \rho_N$ for any of the defined values of ρ, so $(f(a), f(b))$ is an edge of $\mathrm{Cay}(N, \rho_N)$, making $C_\rho(f)$ a directed graph morphism.
Finally, as for morphism composition, $C_\rho(g \circ f)$ is the graph morphism that maps a of M to $g(f(a))$ of L, while mapping every edge (a, b) of $\mathrm{Cay}(M, \rho_M)$ to $(g(f(a)), g(f(b)))$ of $\mathrm{Cay}(L, \rho_L)$, and so it is direct to note that
$C_\rho(g \circ f) = C_\rho(g) \circ C_\rho(f)$, as we wanted to see.
Concerning faithfulness, it follows directly when one notes that, if f, g are semigroup morphisms such that $C_\rho(f) = C_\rho(g)$, in particular they must induce the same functions over the corresponding sets of vertices, and so $f = g$ $\square$

In fact, more generally, we could imitate the categorical construction presented in [7] and repeat the same arguments (taking a small modification into account) so that we obtain a version of this proposition which is as general as possible.

Let us consider the category $Sg\rho$, with its objects being the pairs (S, ρ), ρ a relation defined for the fixed ideal extension $T = S$ of S and its morphisms being the subset of all semigroup morphisms $f : S \longmapsto R$ defined by the extra property $\{(f(x), f(y))|\ (x, y) \in \rho_1\} \subseteq \rho_2$ for $(R, \rho_2) \in Obj(Sg\rho)$. Then, using similar arguments as before (with the additional hypothesis for the corresponding morphisms guaranteeing that edges go to edges), it follows easily that the following statement holds:

Corollary 2.2. *$Cay : Sg\rho \longmapsto \mathcal{G}_{dir}$ defined by $Cay((S, \rho_1)) := \mathrm{Cay}(S, \rho_1)$ and $Cay(f)$ the corresponding induced graph morphism as before for $f \in Hom((S, \rho_1), (R, \rho_2))$ for a given pair of objects, is also a faithful, covariant functor.*

It could be tempting to try and treat our second object, the mapping from $\mathcal{G}$ to $\mathcal{M}$ (the subcategory of $\mathcal{S}$ corresponding to monoids and monoid morphisms) defined by taking a graph G to its endomoprhism monoid (or even to its automorphism monoid) in the same way, but this has remarkably worse categorical properties due to the difficulties of assigning a monoid morphism between $\mathrm{End}(G)$ and $\mathrm{End}(H)$ to a graph morphism $f : G \longmapsto H$.
In fact, no matter what assignment we use for morphisms, there is no functor $F : \mathcal{G}_{dir} \longmapsto \mathcal{M}$ such that takes object G to monoid $End(G)$. We now present a proof of this fact due to Keith Kearnes, as seen in his answer here. The example sparked from a question made by the author in the posted page when trying to test the functoriality of such mappings.

Proposition 2.14. *Let $\mathcal{G}_{dir}$ be the category of directed graphs. There is no functor $F : \mathcal{G}_{dir} \longmapsto \mathcal{M}$ such that sends every $G \in Obj(\mathcal{G}_{dir})$ either to the monoid $\mathrm{End}(G)$ or the group $\mathrm{Aut}(G)$.*

Proof. This proof is based in the concept of a *retract*. Given objects A, B in a category, we say that A is a retract of B if there exist morphisms $i \in Hom(A, B)$ (which we call a section) and $r \in Hom(B, A)$ (which we call a retraction) such that $r \circ i = id_A$. As we are working with categories where morphisms are set functions, this also implies that such an i must be injective and such an r surjective. Let us suppose that some functor F as specified

existed, and let us consider the discrete graphs in 3 and 5 vertices, respectively (the graphs of three and five vertices with no edges), which we can note as D_3 and D_5. We can now make two important observations. Due to a functor preserving category morphism composition (except the order of the operation), retracts are either preserved (for covariant functors) or inverted, in the sense that now $F(B)$ is a retract of $F(A)$ (for contravariant functors). Moreover, as a functor will send a directed graph morphism $f \in Hom(D_3, D_5)$ to a monoid morphism $F(f) : \mathrm{End}(D_3) \longmapsto \mathrm{End}(D_5)$, $F(f)$ will send invertible elements to invertible elements, inducing also a morphism
$F(f) : \mathrm{Aut}(D_3) \longmapsto \mathrm{Aut}(D_5)$ taking the restriction. Therefore, as in both graphs any vertex can be interchanged by every other bijectively as a graph morphism (there are no edges to preserve), we have that $\mathrm{Aut}(D_3) = S_3$ and $\mathrm{Aut}(D_5) = S_5$. Due to all these observations, as D_3 is a retract of D_5 trivially, if our functor were to be covariant, S_3 should also be a retract of S_5, but this is not possible. The reason why is the following: S_3 can be seen as a subgroup of S_5 but, for it to be a group retract, it would have to have a *normal complement*, a normal subgroup K of S_5 such that $S_3K = S_5$ and $S_3 \cap K = \{Id\}$. But there are only three normal subgroups of S_5, which are S_5 itself, A_5 and $< Id >$. The first two cases have an obviously non trivial intersection with S_3, and the last one does not satisfy the property of being a complement. If F were contravariant, it would be even quicker to disprove: it would mean that S_5 is a retract of S_3, which is clearly not the case just by cardinals alone. $\square$

We also prove the auxiliary result from group theory we used:

Proposition 2.15. *Let G, H be groups. Then, G is a retract of H in the group category iff $H \cong GK$, for K a complement of G which is a normal subgroup of G.*

Proof. We begin with the easiest implication: if $H \cong GK$ in these conditions, we can take $i : G \longmapsto H$ to be the inclusion of G in GK, and we take as r the function such that $r|_G = Id$ and $r(k) = e$, the neutral element of G, for every $k \in K$. This, of course, respects operations, and so is a group morphism, clearly meaning that G is a retract of H, as $r \circ i = Id_G$.
As for the other implication, it is clear that, if G is a retract of H, G is

isomorphic to $i(G)$, and thus can be seen as a subgroup of H. We then propose the most obvious candidate to provide a decomposition: we want to prove that $H \cong GKer(r)$. It is also clear that, as $r \circ i = Id_G$, any $g \in G$ not being the neutral element cannot belong to $Ker(r)$, and, so, their intersection is just the neutral element. Moreover, the equation $G((H \setminus G) \cup \{e\}) = H$ holds with the product of subsets, as any element of H belongs to it. We have to see, though, that $K := (H \setminus G) \cup \{e\}$ has group structure. As we have observed before, elements that are not e in G do not go to it through r, so $Ker(r) \subseteq K$. For the other inclusion, it suffices to observe that $\frac{H}{Ker(r)} \cong G$ by the first isomorphism theorem. Then, if $h \in K$ were not in $Ker(r)$, it would be such that $[h] = [g]$ for an element of G which is not the neutral, and so $hg^{-1} \in G$, which would mean $h \in G$ and then $h = e$, a contradiction. Therefore, $K \subseteq Ker(r)$ and we have that $H \cong GK$. Of course, due to being the kernel of a morphism, K is a normal subgroup of H. $\square$

This, in a way, albeit being a bit intuition-defying, confirms that the Cayley graph assignments are natural and more well-behaved than assigning a monoid to a graph thorugh its endomorphisms, even if it would appear to be the other way around from a more naive perspective.

3. Semigroup and Monoid (di)graphs

3.1. **Semigroup and monoid digraphs.** Up to this point, we have become familiarized with the Cayley graph constructions (both normal and generalized) and semigroups, and the most basic ways in which they are correlated. We now want to see some of the direct applications of the construction, and explore, among the different classes of graphs, which special ones that can be obtained via the Cayley construction from a semigroup or a monoid and what conditions they must fulfill, as well as reviewing some special examples that do not admit such representations. We will work with the (different) concepts of semigroup or monoid digraphs and semigroup and monoid graphs.
In order to begin properly, we first lay down some basic definitions relevant to the context.

Definition 3.1. *Let G be a directed graph. We say that it is a monoid digraph (respectively a semigroup digraph) when it can be obtained from some monoid (respectively, semigroup) S and some connection set C via the Cayley construction, that is* $\mathrm{Cay}(S, C) = G$

As we have observed before, this is another case of a concept fairly easy to extrapolate to the context of generalized Cayley graphs.

Definition 3.2. *Let G be a directed graph. We say that G is a generalized monoid (resp. semigroup) digraph if there are S a monoid (resp. a semigroup), T an ideal extension of S and $\rho \subseteq T^1 \times T^1$ nonempty such that G can be obtained using the generalized Cayley graph construction; that is,* $G = \mathrm{Cay}(S, \rho)$

As we can see, these definitions follow the general trend of this work, mainly talking about properties of directed graphs. In contexts when directions and colors are negligible, as in many topological applications, we can restrict ourselves to working with simple graphs, which makes it natural to adopt the notion of a monoid or semigroup (simple) graph.

Definition 3.3. *Let G be a simple graph. We say that it is a monoid graph (respectively semigroup) if there is S a monoid or semigroup and a nonempty connection set C such that G is obtainable as the result of taking the underlying simple graph of the corresponding Cayley directed graph; that is,* $G = \underline{Cay}(S, C)$

Apart from the obvious name similarity, these conditions are quite different: not only for the fact that they refer to different classes of graphs, but also in the magnitude of what they ask for; in mathematical "strength". It is quite easier to obtain contradicitions that make a directed graph not representable as a Cayley graph of a monoid or a semigroup than to ask for a simple graph to not admit any of the many directed representations available. So, in this sense, one can say that a simple graph not being either a monoid or a semigroup graph is quite a remarkable property, it must be structurally incompatible with the corresponding operation somehow. We will assess this topic again in more formal terms in future parts of this section.

As before, this second definition is also easy to expand to a more general setting:

Definition 3.4. *Let G be a simple graph. We say that G is a generalized monoid (resp. semigroup) graph if there are S a monoid (resp. a semigroup), T an ideal extension of S and $\rho \subseteq T^1 \times T^1$ nonempty such that G can be obtained as the underlying simple graph of the corresponding generalized Cayley graph; that is, $G = \underline{\mathrm{Cay}}(S, \rho)$*

In order to begin exploring these concepts, we will see in which conditions some simple families of graphs satisfy the property of being generalized semigroup digraphs, using results found in [10], and we see what properties the semigroups that produce them need to have.
The first class of graphs we want to review is that of linear graphs, which, in essence, are directed graphs that can be easily drawn as a representation of a total order. We introduce them in a more rigorous sense: let $(Y, \leq)$ be a partially ordered set (also called a poset), a set where we can define some order relation. We say that two given elements $a, b \in Y$ are *comparable* iff either $a \leq b$ or $b \leq a$, and we say that the order relation is a *total order* if every pair of elements is comparable. We can also define the following relevant sets:

Definition 3.5. *Let $(Y, \leq)$ be a poset, a an element of Y. Let us consider the sets:*
$a \uparrow := \{b \in Y |\ b \geq a\}$; $a \downarrow := \{b \in Y |\ b \leq a\}$, which are sometimes refered to as the lower closure and upper closure of a, respectively.

In this context, we have tools to properly define linear graphs. As we have just said in an informal way, linear graphs are relation (directed) graphs associated to total orders, which essentially means they are constructed via the order relation in the following way: G_Y and $V(G_Y) := Y$,
$E(G_Y) := \{(a, b) \in Y \times Y |\ a \geq b\}$
Due to the properties of a total order, we can characterize them in terms of graph theoretical properties in the following way:

Proposition 3.1. *Let G be a directed graph. Then, G is a linear graph iff the following properties are satisfied:*

(1) G has no multiple edges;
(2) For any $a \in V(G)$, (a,a) is an edge of G;
(3) Given vertices a, b, then either $(a,b) \in E(G)$ or $(b,a) \in E(G)$;
(4) If a pair of vertices a, b are such that $a \neq b$, then $(a,b) \in E(G)$ implies $(b,a) \notin E(G)$;
(5) G is edge-transitive;

Proof. We prove the characterization in the usual order. Take G a linear graph. (1) is satisfied precisely because of antisymmetry, as two different a, b elements will not be such that $a \leq b$ and $b \leq a$ at the same time. (2) is direct because of the defintion of total order relation, and (3) is just symmetry. Finally, (4) is again due to antisymmetry of the order relation and (5) is just transitivity translated into the graph theoretical context.
Conversely, if we have such a graph G, we can take define the following relation over the set of vertices of G: given $a, b \in V(G)$, $a \geq b \iff (a,b) \in E(G)$ This is an order relation thanks to the properties that G satisfies, and it is a total order due to property (3). Obviously, with this order constructed *ad hoc*, G is its relation graph and, in consequence, is a linear graph. $\square$

This is an interesting characterization in terms of practical use of the concept, and it will serve as a useful lemma in order to prove the result that tells us in which circumstances exactly a linear graph can be represented via the generalized Cayley construction. Let us see it, as presented in [10]:

Proposition 3.2. *Let S be a semigroup, T an ideal extension of it and $\rho \subseteq T^1 \times T^1$ which is nonempty. If ρ is $I-$compatible and such that S is stable under ρ, the following conditions are equivalent:*
(1) $\mathrm{Cay}(S,\rho)$ is a linear graph;
(2) There exists a linear order defined over S such that $\rho(a) = a\downarrow$ for $a \in S$;
(3) There exists a linear order defined over S such that, for every $a \in S$, $\rho(a)^1 = a\downarrow$;

Proof. To begin the proof, let us remark a pair of properties we have thanks to our special hypotheses: thanks to Prop.2.7, we know that $\mathrm{Cay}(S,\rho)$ is edge-transitive and, due to Corollary 3.1, that $\rho(a) = \rho^1(a)$ for every $a \in S$. With that being said, let us now tackle the implications themselves: (1) $\implies$ (2)

can be proven using the same trick as before: we define the relation in $S \times S$, which we name $\geq$, such that $a \geq b \iff (a, b) \in E(\mathrm{Cay}(S, \rho))$. Once again, due to the Cayley graph being linear, we have that $\geq$ is a total order relation, and then, by construction, for a given element $a \in S$, it follows that $a \downarrow = \overrightarrow{a} = \rho(a)$, where this second equation is a fact noted in the beginning of the generalized Cayley graph section.

As for $(2) \implies (1)$, making the same observation as before, if we take this relation $\geq$ behaving as hypothesized, we have that $a \downarrow = \overrightarrow{a} = \rho(a)$ for any $a \in S$ again, which means that $a \geq b$ iff (a, b) is an edge of the Cayley graph. Due to this being an order relation by hypothesis, it is clear that $\mathrm{Cay}(S, \rho)$ must be a linear graph.

$(2) \iff (3)$ follows immediately from the hypothesis that S is stable under ρ, as noted in the beginning of the proof. $\square$

Reinterpreting this result gives us the desired characterization:

Corollary 3.1. *Let G be a linear graph. Then, G is a generalized semigroup digraph iff the following objects exist: a semigroup S, T an ideal extension of it and a relation $\rho \subseteq T^1 \times T^1$ such that ρ is $I-$compatible, S is stable under ρ and S admits a total order relation compatible with the relation in the way we have seen before, satisfying $\rho(a) = a \downarrow$ for any $a \in S$*

If we restrict ourselves to the classic case, that is, if we take $\rho = \{1\} \times C$ for some subset of S, we can use what we know to see how the characterization looks in this setting: we know that ρ is $I-$compatible iff C is a subsemigroup of S, and S is stable under ρ iff $a \in aC$ for every $a \in S$, which means that, for a linear graph to be a semigroup digraph in specific, we need S, C semigroups with these special properties and a total order such that $aC = a \downarrow$ for any element. The second special family of directed graphs we want to study in this sense are directed complete graphs with loops, that is, graphs G such that $E(G) = \{(x, y)|\ x, y \in V(G)\}$. In order to prove the characterization of these types of graphs, though, we have to introduce a new, albeit simple, condition.

Definition 3.6. *Let S be a semigroup. We say that S is left-simple if, for every element $a \in S$, $aS = S$. Analogously, we can define the notion of a* *right-simple semigroup. These conditions are equivalent to the statement that S has no proper left-ideals (resp. right ideals).*

Definition 3.7. *Let S be a semigroup. We say that S is simple if S has no proper ideals. This is equivalent to stating that $SaS = S$ for every $a \in S$.*

Remark 4. *Let us note that, contrary to usual intuition in semigroup matters, being a simple semigroup is a weaker condition to either being left-simple or right-simple, as a semigroup could have both left-ideals and right-ideals that are proper but that were not both-sided.*

In the context of generalized Cayley graphs, we can introduce a generalized notion of simplicity in close relation to a given $\rho \subseteq T^1 \times T^1$

Definition 3.8. *Let S be a semigroup, T an ideal extension of it and $\rho \subseteq T^1 \times T^1$ a nonempty relation. Then, we say that S is $\rho-$simple iff $\rho^1(a) = \rho^1(b)$ for every $a, b \in S$. It follows directly from this definition that it is equivalent to $\rho^1(a) = S \ \forall a \in S$*

Remark 5. *Choosing $\rho \in \{\omega_l, \omega_r, \omega\}$ allows us to recover the previous definitions of left-simple, right-simple and simple semigroup, respectively. Of course, this means that this definition generalizes the classic semigroup property.*

This concept directly characterizes generalized Cayley graphs that are complete graphs with loops, as we can see in the following result:

Proposition 3.3. *Let S be a semigroup, T an ideal extension of it and $\rho \subseteq T^1 \times T^1$ a nonempty relation Then, the following conditions are equivalent:*
(1) $\mathrm{Cay}(S, \rho)$ *is a complete graph with loops;*
(2) S is ρ-simple;

Proof. The implication (1) $\implies$ (2) is immediate, as, if the generalized Cayley graph is complete with loops, there has to be an edge for every pair of vertices, which means that, for every $a, b \in S$, there is $(x, y) \in \rho$ such that $xay = b$, including the case $a = b$. Therefore, it is clear that $\rho(a) = S$ for every element a, making the semigroup ρ-simple.
The converse implication follows the same, almost-trivial reasoning, as S being

ρ-simple quickly implies that $(a,b) \in E(\mathrm{Cay}(S,\rho))$ for every $a,b \in S$, making it a complete graph with loops with vertice set S. $\square$

Once again, thinking in terms of the classic construction, this is equivalent to the also intuitive conclusion that $\mathrm{Cay}(S,C)$ is a complete graph with loops iff $aC = S$ for every $a \in S$.

Following our line of work for this section, we continue by seeing when another simple, this time more general class of graphs are monoid or semigroup digraphs in the sense of the classic Cayley construction: the class of 1-outregular graphs. From now on, we will consider only finite graphs.

In order to know what we are referring to, we recover some concepts of previous sections. In introductory graph theory (that which deals with simple graphs), we define the *degree* of a vertex as the number of different edges that have it as an endpoint. In the context of directed graphs, then, we can obtain two different concepts following the same idea, those of indegree and outdegree. Given a directed graph G and a vertex a, we define the *outdegree* of a (usually denoted as $deg(a)^+$ as the number (or cardinal, if dealing with infinite graphs) of directed edges coming out of a, that is, having this element as its starting point. The *indegree* is defined analogously (and denoted as $Deg(a)^-$) taking the directed edges that have a as their endpoint. In the same vein, we can define the directed graph equivalents of the minimal degree δ and the maximum degree D, which we usually denote as δ^+, D^+ and δ^-, D^- for outdegree and indegree, respectively. This also leads us naturally to similar concepts for regularity (outregularity and inregularity), which quickly gives us the meaning of the term 1-outregular: a directed graph is 1-outregular iff the outdegree of every vertex is 1.

Before tackling the properties of this graph family, though, we see a characterization of monoid digraphs which will be useful from now on. The results of this part are due to K.Knauer and Puig i Surroca and can be found in [5].

Proposition 3.4. *Let G be a directed graph. Then, G is monoid digraph if, and only if, there is some vertex $e \in V(G)$ and a submonoid $M \leq \mathrm{End}(G)$ such that satisfy the following property: for every $x \in V(G)$, there is a unique*

$\phi_x \in M$ such that $\phi_x(e) = x$ and, for every edge (x, y), there is an edge (e, c) such that $\phi_x(c) = y$.

Proof. Let us begin with the first implication. If G is a monoid digraph, then $G = \mathrm{Cay}(S, C)$ for some monoid S and some connection set C. Then, it is clear that the neutral element of S, e, will play the special part that we need to be played. As for the monoid M, using Prop.2.9, we know that we can take $M := \{\phi_s |\ s \in S\}$ where the functions are the monoid morphisms defined by left multiplication by s. Then, it is clear that for every $x \in S$, $\phi_x(e) = xe = x$ and that ϕ_x is the only morphism in M satisfying this. Moreover, for every edge (a, b) of G, we must have that there is an element $c \in C$ that satisfies $ac = b$, and so $\phi_a(c) = b$ by definition.
The converse implication is where the difficulty lies, as we now have to see that, what appears to be a technical condition for G to satisfy, is sufficient for it to be a monoid digraph. Let us consider M the submonoid we have by hypothesis, and e the special vertex. By the property of e with respect to edges in G, it is clear that the connection set we have to choose is
$C := \{\phi \in M |\ (e, \phi(e)) \in E(G)\}$, and we want to see that $G \cong \mathrm{Cay}(M, C)$, via the function $f : V(G) \longmapsto M$ defined by $f(x) = \phi_x$. By hypothesis, this function is injective, and it is also surjective, as, given $\phi \in M$, $\phi = \phi_{\phi(e)}$ because of the aforementioned properties. We only have to check that $(x, y) \in E(G)$ iff $(\phi_x, \phi_y) \in E(\mathrm{Cay}(M, C))$. In order to prove it, let us consider (x, y) an edge of G. By hypothesis, there is some edge (e, c) such that $\phi_x(c) = y = \phi_x(\phi_c(e))$, so $\phi_y = \phi_x \circ \phi_c$ by uniqueness, and $\phi_c \in C$ by construction of C, so $(\phi_x, \phi_y) \in E(\mathrm{Cay}(M, C))$. Conversely, if (ϕ_x, ϕ_y) is an edge of the corresponding Cayley graph, there is $\phi \in C$ such that $\phi_x \circ \phi = \phi_y$. As $\phi \in C$ and $\phi_{\phi(e)} = \phi$ for every element of M, we have that $(e, \phi(e))$ is in $E(G)$. As $(e, \phi(e))$ is sent to $(x, \phi_x(\phi(e))) = (x, y)$ by ϕ_x, and it is a graph endomorphism, we have that (x, y) must also be in $E(G)$, completing the proof. $\square$

With this important characterization as a tool, we are prepared to study the family of 1-outregular graphs. In order to simplify our arguments, we first note the observation that, if a 1-outregular digraph is a semigroup digraph of the form $\mathrm{Cay}(S, C)$, then we can reduce C to any of its elements.

Remark 6. *Let S be a semigroup, $G = \mathrm{Cay}(S, C)$ its Cayley graph for some connection set C. Then, G is 1-outregular iff $G = \mathrm{Cay}(S, \{a\})$ for some $a \in C$.*

The result is direct and does not require an in-depth proof. It is clear that a Cayley graph of this form will be 1-outregular, and that a Cayley graph, in order to be 1-outregular, needs to have a singleton as its connection set.
Another useful information when dealing with these families of graphs is that, as the ones we are going to deal with are finite, every connected component $\mathcal{C}$ of them must have exactly one cycle Z (a path with starting point coincident with its ending point), which could be a loop (if it were not a unique cycle, there would be at least one element with outdegree greater than 1 in $\mathcal{C}$), which, when they are semigroup graphs, will correspond to the property that any element of a finite semigroup has finite order, that is, for every $s \in S$, there are i, j, $i \neq j$ positive integers such that $s^i = s^j$. It is clear as well that, if every finite, strongly-connected 1-outregular digraph has exactly one cycle, that every vertex will have a directed path (a sequence of directed edges) which eventually ends on a vertex of the cycle, and a unique shortest one. If we denote for $v \in V(G)$ the length of this shortest directed path to the cycle as $l(v)$ and, in the context of a general 1-outregular digraph, we define for a connected component $\mathcal{C}$ the quantities $l(\mathcal{C}) := max\{l(v)|\ v \in \mathcal{C}\}$ and $Z(\mathcal{C})$ the length of the corresponding cycle, we obtain an important characterization of monoid 1-outregular graphs in terms of these parameters. Let us see it:

Theorem 3.1. *Let G a 1-outregular digraph. Then, G is a monoid digraph if, and only if, there is a connected component $\mathcal{C}$ of G such that $Z(\mathcal{D})|Z(\mathcal{C})$ and $l(\mathcal{D}) \leq l(\mathcal{C})$ for every other connected component D of G.*

Proof. In order to prove the first implication, let us suppose that G is a monoid digraph. Then, by the previous remark, $G = \mathrm{Cay}(S, \{a\})$ for some element $a \in S$. Let us name as $\mathcal{C}$ the connected component of G where the neutral element, e, is. As we have noted before, as G is finite, a will have finite order, which is essentially equivalent to say that there is a minimal pair of positive integers k, h such that $a^{k+h} = a^k$, that is, that makes the set of elements $e, a, ..., a^k, a^{k+h-1}$ different pairwise. In consequence, it is clear that $l(\mathcal{C}) \geq k$ and $Z(\mathcal{C}) = h$. Let us now consider another connected component $\mathcal{D}$ of G. If

$Z(\mathcal{D}) := q$ did not divide h, we would have that k and $k+h$ would be different modulo q, and so, for $x \in \mathcal{D}$, it would happen that $xa^k \neq xa^{k+h}$, which, of course, would be a contradiction.

As for the other condition, if there were $x \in \mathcal{D}$ such that $l(\mathcal{D}) = l(x) = p$, with $p \geq k+1$, we would have that xa^k is not an element of the cycle of $\mathcal{D}$. Meanwhile, we know that $h = nq$ for some positive integer n, and we can consider the natural number $g := min\{m \in \mathbb{N}|\ xa^{k+mq}$ is in the cycle of $\mathcal{D}\}$, which must exist, as the defined set of positive integers is nonempty. If $g \geq n$, it is clear that either xa^{k+h} is not in the cycle of $\mathcal{D}$ yet (and then they are different because of the different exponent) or it is a point of the cycle, while xa^k is a point out of it. In consequence, $xa^k \neq xa^{k+h}$, obtaining a contradiction. If $g < n$, then $xa^{k+gq} = xa^{k+h}$ are the same element in the cycle, as the difference of their exponents is divisible by q. In any case, we have that xa^k is not in the cycle while xa^{k+h} is, so we have again that they must be different elements. Therefore, $l(\mathcal{D}) \leq k \leq l(\mathcal{C})$.

The converse implication requires more work in comparison, and is the part where Zelinka's original proof found its troubles. Let us consider that G is a 1-outregular graph with a special, connected component satisfying the mentioned properties. Given x, y vertices of G, consider $d(x, y)$ the length of the shortest directed part from x to y, if it exists. Also, if $x \in \mathcal{D}$ a connected component, take the notation $z(\mathcal{D}) := z(x)$. Moreover, in the special connected component $\mathcal{C}$, taking a vertex such that its distance to the cycle coincides with the maximum possible, we call the vertex e and its only out-neighbour a; as they are our best candidates to fulfill these roles.

To make notation more simple, we also define the following term: given $x \in V(G)$ and $k > 0$ an integer, we define $x + k$ as the endpoint of the unique directed path of length k starting at x. It quickly follows from all the definitions that, for k, l positive integers, $x+k = x+l$ iff $d(x, x+k) = d(x, x+l)$

We now want to prove the following auxiliar result in order to complete the proof: *Claim*: given k a nonzero natural number and x a vertex,

$d(x, x+k) = d(x, x+d(e, e+k))$

Claim proof: one can quickly note that, for k a positive integer, it holds that

$$d(x,x+k) := \begin{cases} k & \text{if } k < l(x) \\ l(x) + ((k-l(x)) \ \ mod \ \ z(x)) & \text{otherwise} \end{cases}$$

where mod $z(x)$ is in reference to the residue of the division of $k - l(x)$ by $z(x)$, and this quantity defines the length of the cycle that is located in the same connected component as x. Let us split the proof in two cases: if $k < l(e)$, then $d(e, e+k) = k$ and the claim is obviously true. If $k \geq l(e)$, we can then use euclidean division in order to obtain an expression of the form $l(e) - k = q_k z(e) + r_k$, with $r_k < z(e)$, and, as $Z(\mathcal{D})|z(e)$, it is clear that $l(e) - k \equiv r_k(mod\ z_x)$, and so the equation $d(x, x + d(e, e + k)) = d(x, x + l(e) + r_k) = d(x, x + q_k z(e) + r_k) = d(x, x + k)$, which proves the claim. We now proceed with the proof of the general proposition: let us consider the element $\omega = e + (l(\mathcal{C}) + Z(\mathcal{C}) - 1)$. It is clear that $\omega \in Z$ (the cycle of the connected component), so, given what we have pointed out before, any vertex $v \in \mathcal{C}$ must reach the point eventually through a unique directed path. This means we can define the value $r(v) := d(e, \omega) - d(v, \omega)$ for any such v, as both quantities are defined and, as e is the element in $\mathcal{C}$ that is the furthest away from the cycle, the difference must be non-negative. Note that $d(v, \omega) = d(e + r(v), \omega)$, and so, $d(v + k, \omega) = d(e + r(v) + k, \omega)$ for any positive integer k, which means that $r(v + k) = r(e + r(v) + k) = d(e, e + r(v) + k)$ for any k as seen and any vertex $v \in \mathcal{C}$. With these properties in mind, we can now regard $V(G)$ as a magma (a set which is closed with respect to a binary operation) with the following operation:

$\forall y \in V(G)$, we define $ey = y$;

and, for every $x \neq e$ in $V(G)$, y in $V(G)$, we define it as

$$xy := \begin{cases} x + r(v) & \text{If } y \in \mathcal{C} \\ y & \text{otherwise} \end{cases}$$

By definition of the operation, e is already a left-neutral element, but we also have that $ye = y + r(e) = y$ for every y, and so it is a full-fledged neutral element. We only need to check that it is associative for $V(G)$ to be naturally interpreted as a monoid. Let us see it: let x, y, z be three different vertices in the graph, and let us divide the proof by cases. Suppose that $y \notin \mathcal{C}$, then yz will not be in $\mathcal{C}$ either, as it will be either equal to z if it is not an element of $\mathcal{C}$

or it will be equal to $y + r(z)$, remaining in the same connected component as y. Then, $x(yz) = yz = (xy)z$ by definition. Now suppose that $z \notin \mathcal{C}$. In this case, we will have that $(xy)z = z = yz = x(yz)$, using what we know about the operation. Then, only the case when both y, z are in $\mathcal{C}$ remains. Then, we have that $(xy)z = (x + r(y))z = x + (r(y) + r(z)) = x + d(x, x + r(y) + r(z))$ $x+d(e, e+r(y)+r(z)) = x+d(e, y+r(z)) = x+r(y+r(z)) = x+r(yz) = x(yz)$ thanks to the previously proven claim, and, therefore, the operation is associative and we obtain a monoid. Moreover, we have that $(x, y) \in E(G)$ iff $y = x + 1 \iff y = xa$, which clearly means that $G = \mathrm{Cay}(V(G), \{a\})$, as we wanted to show. $\square$

By relaxing a bit the specified conditions that we ask of this special connected component and the cycles of every one of them, we can also obtain a characterization for when a 1-outregular graph G is a semigroup digraph. This result was originally due to Zelinka, as presented in his paper referenced in [9]. The construction of the semigroup he claimed to represent the graph correctly, however, contained a mistake, making this proof (a corollary of thm.3.1) a fix to the original. This was first proved in [5]. Let us see this version:

Corollary 3.2 (Zelinka's theorem)**.** *Let G be a 1-outregular directed graph. Then, G is a semigroup graph if and only if G has a connected component C such that $Z(\mathcal{D})|Z(\mathcal{C})$ and $l(\mathcal{D}) \leq l(\mathcal{C}) + 1$ for every other connected component $\mathcal{D}$ of G.*

Proof. Let us begin by proving the left to right implication. Assume G is a semigroup digraph that is 1-outregular. Then, we have that $G = \mathrm{Cay}(S, \{a\})$ for some semigroup S, as we pointed out in the remark before. The strategy in simple: we want to use the preceding proposition as a lemma in order to obtain most of the properties immediately. The reference to do this is to consider the trivial monoid associated to any semigroup S, S^1, and we now see how the components of $\mathrm{Cay}(S^1, \{a\})$ relate to the original. If we take as $\mathcal{C}$ the connected component which contains a, in the Cayley graph for S^1, the corresponding connected component, call it $\mathcal{C}'$, is equal to $\mathcal{C}$ in everything but in the fact that we add to it the vertex 1 and the edge $(1, a)$, while the rest of connected components stay the same. In consequence, we have that

$Z(D) = Z(D')$ and $l(\mathcal{D}) = l(\mathcal{D}')$ for every connected component $\mathcal{D}'$ of the graph that is not $\mathcal{C}'$, $Z(\mathcal{C}') = Z(\mathcal{C})$ and $l(\mathcal{C}') \leq l(\mathcal{C}) + 1$ because of the only edge addition we have pointed out. Of course, as the Cayley graph of S^1 keeps being 1-outregular and is that of a monoid, we have that $Z(\mathcal{D}')|Z(\mathcal{C}')$ for any connected component and $l(\mathcal{D}') \leq l(\mathcal{C}')$. By the equalities we have seen just prior to this, we see that, clearly, $Z(\mathcal{D})|Z(\mathcal{C})$ and $l(\mathcal{D}) \leq l(\mathcal{C}) + 1$ for the corresponding connected components of G as well.
Conversely, suppose we have a graph G which satisfies these properties. If we take v the vertex in $\mathcal{C}$ such that $l(v) = l(\mathcal{C})$, we can create a new graph attaching a vertex to v (let us call it u) and the edge (u, v), and this new graph, call it H, will continue to be 1-outregular, as v must have indegree zero in order to satisfy the property of being the furthest away from the cycle. For this new graph and the corresponding connected component $\mathcal{C}_{\mathcal{H}}$ (the modified $\mathcal{C}$), C_H satisfies the hypotheses of the last proposition with respect to how it relates to other connected components $\mathcal{D}$, and so, it is a monoid graph taking the construction we have introduced in Thm.3.1, and u is the neutral element. Now, if we consider vertices of G with this operation, xy, we see that either $xy = y$ or $xy = x + r(y)$, so either xy is $y \neq u$ for $x, y \neq u$ or $xy = x + r(v)$, which is a vertex that must have indegree one, as it forms part of the directed path $x, x + 1, ...x + r(y)$, but, as we have just said before, a vertex such as u that has the property of being the furthest away from the cycle among all vertices in $\mathcal{C}$ must have indegree 0, and , thus, $u \neq x + r(y)$ either, making the operation closed over $V(G)$. As we have proved associativity before, we have that clearly $V(G)$ is a semigroup with it, and $G \cong \mathrm{Cay}(V(G), \{v\})$, as $r(v) = 1$ in the graph H where we originally defined the operation. □

In consequence, we have the immediate corollary, which has as a consequence that any tree (a connected, simple graph with no loops) is a monoid graph:

Corollary 3.3. *Let G be a connected, 1-outregular digraph. Then, G is a monoid digraph.*

In a sense, this is the best we can get in terms of characterizing families of outregular digraphs as either monoid or semigroup digraphs, as, for any other $k \geq 2$ that is a positive integer, it is possible to construct k-outregular digraphs such that they cannot be semigroup digraphs. An in-depth look at these graphs can be found in [5].

In the next result, we take a look at a smallest (in terms of vertices) possible outregular graph (which is 2-outregular) and provide an original proof of why it cannot be a semigroup digraph:

Proposition 3.5. *Let G be the following directed graph: $V(G) = \{x, y, z\}$, $E(G) = \{(x, x), (x, y), (y, x), (y, z), (z, y), (z, z)\}$. Then, G is not a semigroup digraph.*

Proof. Let us begin by remarking that G is clearly 2-outregular, as loops add one to both the indegree and the outdegree of a vertex. As a second observation, an important one, we see that G is strongly-connected, as any pair of vertices is joined by a directed path. In account of this, and, of course, $V(G)$ being finite, by Prop.2.12, we know that, if $G = \mathrm{Cay}(S, C)$ for a semigroup S and its corresponding connection set, S must be left-cancellative, which essentially means that $uS = S$ for any $u \in V(G)$. Let us now proceed to prove the result by reduction to the absurd. Suppose that $G = \mathrm{Cay}(S, C)$ in these conditions. Our goal is to find a contradiction for every case possible. In order to make observations simpler, we begin by noting that $C \neq S$, the connection set cannot be the whole semigroup. If that were the case, as every vertex has outdegree 2, there would have to be two different values, say $a, b \in S$ such that they satisfy $xa = xb$, for example. As S must be left-cancellative, we would have that $a = b$, generating a contradiction. On account of this very same observation about the outdegree, the connection set C cannot be a singleton, either, so it must be some two-element subset of S, of which there are just three. Let us find inconsistencies for every possible choice of C:

$C = \{x, y\}$: The trick here is simple. By the fact that S must be left-cancellative, we can deduce that the only element of S not in C must send every element to the only other element of $V(G)$ that is not edge-adjacent to them. In this case, this means that $xz = z$, $zz = x$, $yz = y$. Now, imposing the associativity we suppose for S, we should have that $(zx)z = z(xz) = zz = x$, so zx must be an element that is sent to x by z, and this cannot be either x or y, because $xz = z$, and $yz = y$ already. Therefore, it should be $zx = z$, which implies that $zy = y$. We can then consider the equation $(xz)y = x(zy)$, but $(xz)y = zy = y$, and $x(zy) = xy$, so it must be $xy = y$ and $xx = x$. Finally, if we consider the equation $y(xz) = (yx)z$, we obtain that $yz = y$ must be equal to $(yx)z$. Therefore, the only thing that can happen is that $yx = y$ and, in consequence, $yy = x$. This causes a contradiction, as $yz = yx$ would mean that $z = x$.

$C = \{z, y\}$: We use the same argument as before. In this case, we know that $xx = z$, $yx = y$ and $zx = x$. We can now proceed analogously: $(xz)x = x(zx) = xx = z$ mean that $xz = x$ and $xy = y$, and from $(zx)y = z(xy)$, we deduce that $zy = y$ and $zz = z$ as well. Finally, once again, we can take the equation $y(zx) = (yz)x$ to derive that $yx = y = (yz)x$, and so it must be $yz = y$ and $yy = x$. Because of the same reasons as before, $yx = yz$ and we have a contradiction.

This case can be summed up by saying that $f : S \longmapsto S$ defined as $f(x) = z$, $f(y) = y$, $f(z) = x$ is a graph isomorphism for G preserving the operations in S.

$C = \{x, z\}$: As in this case y is the odd one out, we know that $xy = z$, $zy = x$ and $y^2 = y$ by the left-cancellative property. But then, by associativity of the operation in S, we would need that $(xy)y = x(yy)$ is satisfied, when we already know that $(xy)y = zy = x$ and $x(yy) = xy = z$, making this case impossible as well and completing the proof. □

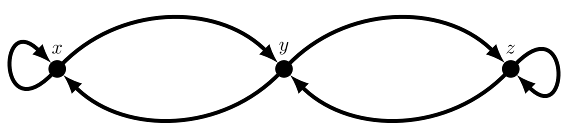

FIGURE 3. A simple drawing of a smallest outregular digraph G that is not a semigroup digraph.

The reason why we say it is a smallest non semigroup digraph is that there are no such graphs with less than three vertices, and it is not the unique graph satisfying this property having three vertices. The reason why there are no such graphs with less than two vertices is because of the results we have seen: any graph that is 0-outregular is a monoid digraph (represented by a monoid with as many elements as vertices of the graph and with $C = \{e\}$, the corresponding neutral element) and any 1-outregular digraph of two vertices must be a monoid graph if it is connected. That only leaves 2-outregular digraphs and 1-outregular digraphs that are disconnected. But these are easily identifiable, as there are only two: the digraph consisting of two vertices with a loop for each and the complete graph with loops of two vertices. The first is isomorphic to $\mathrm{Cay}(\mathbb{Z}/2\mathbb{Z}, \{0\})$, and the second to $\mathrm{Cay}(\mathbb{Z}/2\mathbb{Z}, \{0, 1\})$.

3.2. **Monoid and semigroup graphs.** So far, we were specifically concerned about digraphs admitting some monoid or semigroup representation via the Cayley construction. In this section, we tackle more general questions, as we set our sights in the aforementioned concept of monoid and semigroup graphs. Therefore, we use this section to talk about some special families of simple graphs that can or cannot be represented in this way, with an special focus on monoid graphs. As, in this new context, we do not care about the specific direction that edges take, we can afford some supositions to make things simpler. First, when S is a monoid, the underlying graph is not affected by whether $e \in C$ or not, so, in general, we will assume that $e \notin C$. The most important simplication, though, is a part of the content of the following result:

Proposition 3.6. *Let G a simple graph, and suppose that it is a monoid graph, so there are M a monoid, C a connection set of M such that $G = \underline{\mathrm{Cay}}(M, C)$. Then, if we define $N(e)$ to be the set of neighbours of e (the neutral element of M, we have that $G = \underline{\mathrm{Cay}}(M, N(e))$ as well.*

Proof. As it is clear that, of course, the set of vertices remains the same and all edges of the form (x, xc) for $c \in C$ are in $E_2 := E(\underline{\mathrm{Cay}}(M, N(e)))$ by definition of the Cayley graph ($C \subseteq N(e)$), we only have to see that every edge of E_2 is an edge of $E_1 := \underline{\mathrm{Cay}}(M, C)$. So, suppose that (w, v) is an edge in E_2 such that $wx = v$ for $x \in N(e)$ but not in C. Then, by definition of $N(e)$, there is some $u \in C$ such that $xu = e$, which means that $wu = vxu = v(xu) = v$, so (v, w) is in E_1. Of course, this completes the proof only because we are working with simple graphs and we consider the two directed edges to be the same. $\square$

Another thing we can do almost immediately is to use the results we showed in the preceding section, obtaining relevant examples of monoid graphs. If we take any tree T (a connected, simple graph without loops), we can give to them a 1-outregular representation in a way which allows us to apply Thm.3.1 the following way: we can take any vertex of T and add a loop to it, which will represent the cycle $\mathcal{C}$ that we know every connected, 1-outregular digraph has. Then, we pick the leaf that is the furthest away from our "cycle" and make it play the part of e, laying out directed paths that go from every leaf in the tree directly to the cycle via the shortest directed path. Of, course, this gives us as a consequence that:

Corollary 3.4. *Every finite tree T is a monoid graph.*

In fact, doing this for every corresponding connected component makes this true as well for *forests* (graphs where every connected component is a tree, graphs with no cycles) by Thm.3.1 and, by the same result, even some families of *pseudoforests*, which are graphs with at most one cycle per connected component. (exactly the ones satisfying the conditions exposed in the theorem). Another family of graphs of interest are *threshold* graphs, which can be defined as graphs that can be constructed from a starting vertex the following way: if the final graph has $n \in \mathbb{N} \setminus \{0\}$ vertices, for each of the $n-1$ corresponding steps, we add a new vertex which either remains disconnected or has edges to all the ones already present at the beginning of the step. This rather simple pattern of construction allows to quickly prove them as monoid graphs via this simple result.

Proposition 3.7. *Let G be a monoid graph. Then, the graphs $G_1 := (V(G) \cup \{x\}, E(G))$ and $G_2 := (V(G) \cup \{x\}, E(G) \cup \{(x, u)| \ u \in V(G)\})$ are also monoid graphs.*

Proof. So, suppose that $G = \underline{\text{Cay}}(M, C)$ for some monoid and some connection set. Let us now consider the new monoid $M' = M \cup \{x\}$, with the following operation for any pair a, b of its elements: $a * b := ab$ if both a, b are already elements of M and $x * a = a * x = x$ for any element in M'. Essentially, what we are doing is adding a zero element to M. Let us see that this preserves associativity, making M' a semigroup: consider $a, b, c \in M'$. If the three of them are elements of M, we know that $(ab)c = a(bc)$ because M is a monoid and, if any of the three equals x, then $a(bc) = x = (ab)c$ because it absorbs every element. Now, if we consider the digraph given by $\mathrm{Cay}(M', C)$, it is clear that there are no edges in it connecting to x, as $xC = \{x\}$ and the rest of products are confined to M by closure preserving the original graph. Then, we already have that $G_1 = \underline{\mathrm{Cay}}(M', C)$. As for G_2, we just have to take $C' = C \cup \{x\}$. The Cayley digraph corresponding to M', C' will be equal to the one that corresponds to M, C, except for the new vertex x which has edges connecting every element to it and only loops caused by left-multiplication. Therefore, $G_2 = \underline{\mathrm{Cay}}(M', C')$. $\square$

Graph powers of paths are also an interesting example of monoid graphs. By *path* of n vertices, we mean the graph P_n defined by $V(P_n) = \{1, 2, ..., n\}$ and $E(P_n) = \{(i, i+1) |\ i \in \{1, ..., n-1\}\}$, and by *kth power* of a graph G, we mean G^k s.t its vertices are equal to those of G and
$E(G^k) = E(G) \cup \{(x, y) |$ there is a path of length at most k joining $x, y\}$. We show it before finishing the section.

Proposition 3.8. *Let P be a finite path. Then, for every positive integer such that $1 \leq k \leq |V(P)|$, P^k is a monoid graph.*

Proof. For $k = 1$, the proof is immediate. Any path P is a type of tree, and we know those admit the 1-outregular representation we discussed in-depth, as a directed path starting at one leaf of the path and ending in a loop at the other end. For $k > 1$ in the defined range, if $P = \underline{\mathrm{Cay}}(M, \{a\})$ for some monoid M that gives this representation, we say that $P^k = \underline{\mathrm{Cay}}(M, \{a, a^2, ..., a^k\})$ The set of vertices are obviously the same, so we only have to see that they have the same edges. But, (x, y) is an edge of P^k if, and only if, there is a path $\lambda = xv_1(...)v_my$ of length at most k joining x and y, for $v_1, ..., v_m$ non repeating vertices. As there are m of them, the length of λ must be $m + 1 \leq k$. But this is equivalent to the equations $xa = v_1$, $v_ia = v_{i+1}$ for $1 \leq i \leq m-1$ and $v_ma = y$ in the chosen representation of P as a digraph. If we use those equations and substitute values from the last to the first one, we obtain that $xa^{m+1} = y$, and so (x, y) is an edge of our candidate underlying Cayley graph.

As for the converse implication, we have that $(x, y) \in \underline{\text{Cay}}(M, \{a, a^2, ..., a^k\})$ iff there is some $m \leq k$ such that $xa^m = y$. But this means they are connected by the path of length $m + 1$ $\{x, xa, xa^2, ..., xa^m = y\}$, and so it is an edge of P^k as well. $\square$

3.3. **Infinite families of outerplanar monoid graphs.** Having seen some interesting families of monoid graphs due to preliminary and already reviewed results, we dedicate this section to a little more focused study of families of monoid graphs that are outerplanar. Via a couple of results due to K.Knauer in collaboration with the author, we are able to provide an infinite family of such graphs. Let us see how. First, though, we introduce the definition of this concept.

Definition 3.9. *Let G be a graph. We say that it is* planar *if, colloquially, it can be drawn in the plane in such a way that no pair of edges intersect in any point that is not one of their endpoints. Formally, we define it as a graph that is embeddable in the plane, that is, if the graph admits a* graph embedding *that takes it to $\mathbb{R}^2$, a function $f : G \longmapsto \mathbb{R}^2$ which is injective, takes vertices of G to points in the plane and edges (x, y) to continuous paths ($f_{(x,y)} : [0, 1] \longmapsto \mathbb{R}^2$ s.t $f_{(x,y)}(0) = f(x)$ and $f_{(x,y)}(1) = f(y)$) in such a way that two edge representations only intersect if it is at one of their endpoints. When a graph is planar, its representation in the plane divides it in regions delimited by the edges of the graph, which we refer to as* faces. *We say that a planar graph is also* outerplanar *when all its vertices are located in the outermost face it defines in $\mathbb{R}^2$ (the face that is unbounded).*

A simple example of an outerplanar graph might be C_5, as the drawing we did in Figure 1 can easily show. The strategy to obtain an infinite family of outerplanar graphs such that are monoid graphs is composed of two simple parts: to obtain a way to construct an infinite number of outerplanar graphs easily, and to at least be able to represent with monoids an infinite subfamily of it. We begin by describing the process to obtain infinite outerplanar graphs that will make it possible: let us consider the family of graphs called 2-trees, and consider those that can be constructed in a finite number of steps following this process: take the smallest 2-tree, the triangle C_3, as the starting graph

and let us consider the following technique: from a given 2-tree, it is possible to obtain another one with exactly one more triangular face as follows: we add a new vertex u to the chosen $2-$tree, and we make it adjacent to a pair of vertices v, w such that (v, w) was already an edge of the 2-tree. This is called tackling the vertex u to the edge (v, w). Any 2-tree will be outerplanar as long as only one vertex is tackled to a given edge during the construction.
We will refer to the outerplanar graphs obtained in this manner as *triangular* graphs. It is worth noting that they are also maximal outerplanar graphs, in the sense that adding any other edge would cause the loss of the property.
This method is not without its faults, though, as this "assignment" is not a function from the set of trees satisfying these hypotheses to triangular graphs. Even for paths, if they have six vertices or more, there are more than one ways to follow the method in order to obtain essentially different graphs, depending of what common side we choose the triangles to share for every edge. Let us see a simple example for the path of four vertices P_6:

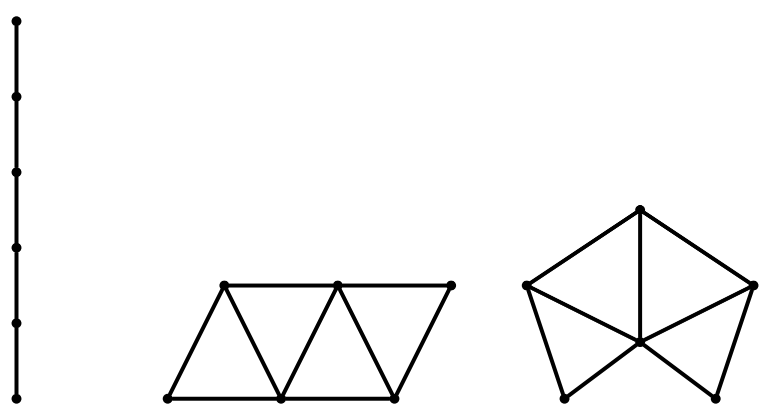

FIGURE 4. Different triangular graphs built using P_6. The first one depicts a triangular path with 4 triangular faces, which we will see is always of the form P^2 for a path P, and the second is a "bend", which we could describe as a path with an apex vertex P^+

The reason this begins to happen at exactly $n = 6$ for the number of vertices is because paths up to 5 vertices cannot represent triangular graphs with vertices of degree larger than 4. From 6 vertices on, if we join the triangles in a way that they "bend", we can obtain vertices of greater degree, while making paths with triangles as in the first example always leaves maximum degree equal to 4. This example features two different families of triangular graphs obtained this way. In the first case, the chosen sides where the triangles are joined are chosen in a way that the triangles form a path again, and, in the second one, they bend. *Triangular paths*, *bends* and *bifurcations* (which is when a triangle has other triangles glued to every one of its faces, coming from a bifurcation in the tree) are the main types of simple, regular (in the sense of shape, not referring to the graph notion) graphs that we can build this way, and they are the pieces we could use to build every possible triangular graph through the union set operation. The main problem, though, is that union is not that good of an operation when it comes to preserving semigroup structure, or not adding unwanted edges in Cayley constructions. Before returning to these difficulties, though, let us set our sights in the types of triangular graphs we *can* prove that are monoid graphs. Those are triangular paths and bends.

3.3.1. *Triangular paths.* As we have just mentioned before, triangular paths are the subfamily of triangular graphs where triangles are glued in a straight-line fashion, preserving the path structure somewhat. They are also called this way because they always come from paths when built using the vertex-to-triangle construction, and admit a fairly easy representation inherited from the results of the preceding section.

Proposition 3.9. *Let T_n be a triangular path, where $n \geq 1$ is its number of triangles. Then, $T_n = (P_{n+2})^2$, for P_{n+2} the path of $n+2$ vertices, making T_n always a monoid graph by Prop.3.8.*

Proof. Suppose we have a triangular path composed by n triangles, for some positive integer n. If $n = 1$, we can take P_3 and, taking the graph power, $P_3^2 \cong C_3$. This gives us an initial case we can use to fuel a proof by induction: suppose now $n > 1$ is such that $n - 1$ satisfies the property. The triangular path in $n - 1$ vertices (T_{n-1}) is now, by hypothesis, the square graph of a path, say P_{n+1}. We can then obtain the triangular path of n vertices (T_n) simply by adding a new vertex to P_{n+1} and adding an edge that joins it to an endpoint of it (a leaf), call it $P' = P_{n+2}$, and then taking the square power of it, $T_n = (P')^2$. The reason why this happens is that, for this P', taking the described graph power only adds the edge representing a length two path from P to the new vertex, and so it only adds a triangle to T_{n-1}. $\square$

Remark 7. *Let us note that this equation can be also read in the opposite direction: for* $n \geq 3$*,* $(P_n)^2 = T_{n-2}$*.*

Given that this case is pretty simple, we can do even more, we can find explicit monoid representations for the triangular paths T_n, for every natural n equal or greater than 1. It is worth noting that this is just a particular case of the Zelinka construction, representing the path as a directed path that ends in a loop.

We construct such suiting monoids from the natural numbers in the following way: let $\mathbb{N}$ refer to the natural numbers as usual, zero included. For a fixed $n > 1$ natural as well, we define the following relation: given $x, y \in \mathbb{N}$, we have that $x \sim y \iff \begin{cases} x = y & \text{If } x, y \text{ satisfy that } x < n \text{ and } y < n \\ x, y \geq n \end{cases}$

It is almost immediate to see that this relation is an equivalence relation. It is obviously reflexive and symmetric, and transitivity is easily seen with a little thought: given a, b, c s.t $a \sim b$ and $b \sim c$, either it is because a, b are equal or because they are both greater than n. In the first case, $b \sim c$ only if c is equal to the other two, and in the second case, it must be that c is also greater or equal than n, giving the wanted property. We can now take $\mathbb{N}_{\infty,n} := \frac{\mathbb{N}}{\sim}$, and define the operation $[a] + [b] := [a + b]$, which is well-defined by construction of the relation (either $a, b < n$ and their classes have unique representatives or one of them is bigger than n or equal to it, and the sum will always be the same for any other representative of its class (due to how the order relation of the naturals behaves). So, we have a magma so far. We now see it possesses useful structure:

Proposition 3.10. *Let* n *be a natural number. Then,* $\mathbb{N}_{\infty,n}$ *is a monoid with the described operation.*

Proof. The class of 0 is obviously a neutral element, and $[n] := \infty$ is clearly a zero element in the sense of being absorbent. Moreover, the operation is always commutative. Therefore, it suffices to see that the operation is associative. Let $[x], [y], [z]$ be elements of our set. Then, if $x+y+z<n$, the operation behaves exactly as that of the normal naturals, and is clearly associative. If $x+y+z \geq n$, though, no matter in which order we do the operations, we will end up obtaining ∞ as our result, which means the operation is associative. □

Remark 8. *It is worthy to note that the associativity of the operation and the whole construction of the set does not use* 0 *at all, and so, if we did the same for* $\mathbb{N} \setminus \{0\}$*, we would still obtain a semigroup.*

Finally, we use the monoids we have built to represent triangular paths.

Proposition 3.11. *Let* $n > 1$ *be a natural number. Then,* $T_n = \underline{\mathrm{Cay}}(\mathbb{N}_{\infty,n+1}, \{\overline{1}, \overline{2}\})$

Proof. The reason why we need $3+(n-1)$ total vertices for n triangles is related to the proof of the preceding big result. 3 vertices are needed to represent the first triangle, and, from then on, every vertex we add makes it possible to represent another extra triangle. Therefore, making a total of $3+(n-1)$ vertices for n triangles. But, as $\mathbb{N}_{\infty,n}$ has $n+1$ elements, we have to choose the monoid for $3+(n-1)-1 = 3+(n-2) = n+1$. Now, for the Cayley representation, we have that, as $T_n = P^2$ for some path, the color $\overline{1}$ will give the 1-outregular representation of the path P, $\overline{2}$ will cover the rest of the edges and ∞ being the last vertex will just transform into loops any inputs from the connection set. For $\overline{n}$, both elements take it to ∞, making a multiple edge. □

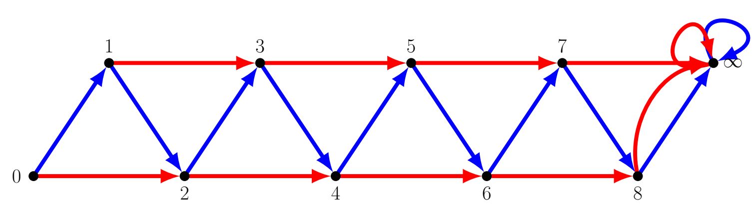

FIGURE 5. A drawing representing T_8 by $\mathbb{N}_{\infty,9}$, where color red represents the element $\overline{2}$ and blue represents element $\overline{1}$.

3.3.2. *Bends.* In the realm of possibilities given by the construction of triangular graphs from trees as we have established, bends are on the opposite end with respect to triangular paths: they are outerplanar graphs that are built by gluing together triangles in a way that increases the degree of a given vertex, say the center of the bend, as much as possible for any given number of triangles n. They can be represented in the plane as "almost-polygons", a decomposition of a regular polygon in equilateral triangles where there is one missing. In this case, given that we have already introduced the monoids $\mathbb{N}_{\infty,n}$, the proof to them being monoid graphs is much more straightforward. We first remind the result from Prop.3.7, which tells us in the first case that adding a zero element to a semigroup (an element such that the result of every operation that has it as one of the operands is equal to it) creates another semigroup. We can use this result and the preceding ones in order to see:

Proposition 3.12. *Let $n \geq 1$ denote the number of triangles of the graph. Let us refer to the bend of $n \geq 1$ triangles as B_n. Then,*
$B_n = \underline{\mathrm{Cay}}(\mathbb{N}_{\infty,n} \cup \{x\}, \{\overline{1}, x\})$*, for x a zero element added to $\mathbb{N}_{\infty,n}$*

Proof. By the aforementioned result, we know that $\mathbb{N}_{\infty,n} \cup \{x\}$ is a monoid, and the number of vertices $n = 3 + (n-3)$ is for the following reason. It will take a total of $3 + (n-1)$ total vertices to make the graph B_n, but in this representation, one spot is taken by x, so we need the corresponding "natural infitity" monoid for $3 + (n-2)$ elements, which is $\mathbb{N}_{\infty,3+(n-3)}$ on account of it having the element 0 as well. As for the Cayley representation this provides, we note that $\overline{1}$ will represent as an 1-outregular path digraph the perimeter of the "almost-polygon", ending in a loop at ∞. As for x, it just acts as the center of the bend, absorbing every other element. $\square$

Let us note that, as $B_n = T_n$ for $1 \leq n \leq 3$, they admit both Cayley representations. As we can easily see, examples described until now are clearly connected graphs. Our objective is now to see that there are also infinite examples of outerplanar monoid graphs that are not connected, see how to give explicit Cayley representations to some of them and prove an interesting characterization for families of outerplanar graphs that admit the $\underline{\mathrm{Cay}}(M, \{a, a^2\})$ representation.

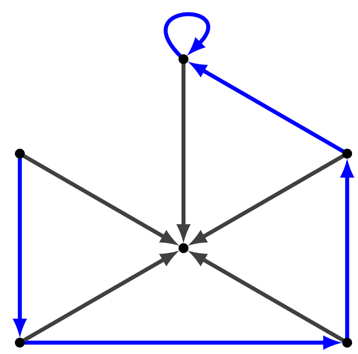

FIGURE 6. A drawing of the representation of B_4 by $\mathbb{N}_{\infty,4} \cup \{x\}$, where the color blue refers to $\overline{1}$ and the grey color to x. Clearly, the element with the loop is ∞.

In order to see this first point, we recover our discussion about unions of graphs. As we have briefly discussed above, in general, set theoretical union as usual is bad at preserving the semigroup structure of its components if we bestow it with some associative global operation, and the corresponding Cayley graphs tend to gain undesired edges. When it comes to disjoint union, though, things tend to be easier, as we have no intersections where two possibly very different operations have to agree. A very general approach to building semigroups as disjoint union of families of other semigroups can be seen in this result, found in [10]

Proposition 3.13. *Let $(Y, \geq)$ be a semigroup-partially-ordered set (any pair of elements α, β are such that $\alpha, \beta \leq \alpha\beta$), $\{S_\alpha\}_{\alpha \in Y}$ a family of semigroups. If, for any pair of elements $\alpha \geq \beta$ in Y there are semigroup morphisms $\phi_{\alpha,\beta} : S_\alpha \longmapsto S_\beta$ such that $\phi_{\alpha,\alpha} = Id_\alpha$ and $\phi_{\alpha,\lambda} = \phi_{\alpha,\beta} \circ \phi_{\beta,\lambda}$ for every triple of elements in Y such that $\alpha \geq \beta \geq \lambda$, then $S := \bigsqcup_{\alpha \in Y} S_\alpha$ is a semigroup with the following operation: $a * b = \phi_{\alpha,\alpha\beta}(a)\phi_{\beta,\alpha\beta}(b)$ for $a \in S_\alpha$ and $b \in S_\beta$.*

Proof. Let a, b, c be elements in S_A, S_B, S_C for $A, B, C \in Y$, respectively. Then, we have that $(a * b) * c = (\phi_{A,AB}(a)\phi_{B,AB}(b)) * c$
$= \phi_{AB,ABC}((\phi_{A,AB}(a)\phi_{B,AB}(b)))\phi_{C,ABC}(c) = \phi_{A,ABC}(a)\phi_{B,ABC}(b)\phi_{C,ABC}(c)$
$= a * (b * c)$ rearranging the second operation analogously. $\square$

This is a very general construction quite possibly based around the concept of *limit* in category theory which can be applied in a myriad of practical settings, specially for finite families, making it a very useful algebraic solution, but is clearly not suitable for our needs. Although we have a semigroup structure that will preserve the original structure of $S_\alpha \iff \alpha$ is an idempotent element of Y, it is clear that it will never provide Cayley graphs of the disjoint union that respect the original edges, as it takes operations from different semigroups in the family to one of them in particular, always creating cross edges between them. So, we need special ways of bestowing disjoint unions of semigroups with a semigroup structure that respects and keeps separated the respective Cayley graphs. With these needs in mind, K.Knauer proposed an idea that works for special cases of semigroups and choices of connected sets for the purposes of this work. Let us introduce it:

Proposition 3.14. *Let S, T be two semigroups, which we assume to be disjoint without loss of generality for our purposes. Then, if there exist special elements $a \in S$, $b \in T$ such that $as = a$ for all $s \in S$ and $bt = b$ for all $t \in T$ (they are left-zeros in their respective semigroups), and if we consider the function $\phi : S \sqcup T :\longmapsto \{a, b\}$ defined as $\phi(s) = b$ for every $s \in S$, $\phi(t) = a$ for every $t \in T$, $S \sqcup T$ is a semigroup with the operation defined as*

$$x * y := \begin{cases} xy & \text{if either } s, t \text{ are both in } S \text{ or in } T \\ x\phi(y) & \text{if they are in different semigroups of the union.} \end{cases}$$

If $a \in C$ and $b \in D$ for C, D the connection subsets of S and T, respectively, then we also have that $\mathrm{Cay}(S \sqcup T, C \sqcup D) = \mathrm{Cay}(S, T) \sqcup \mathrm{Cay}(T, D)$.

Proof. Let us see that the proposed operation works. As the result of the operation of any two elements falls in $S \sqcup T$ again, we only have to check associativity. Take, s, t, u elements in $S \sqcup T$. Then, if all of them are in S or in T, this follows directly from the respective semigroup structures. So, let us suppose that there is at least one element that is in a different component than the others, suppose $s, u \in S$, $t \in T$. Then, for the three possible positions of t, we have : $(s * u) * t = (su)\phi(t) = (su)a = s(ua) = s(u\phi(t)) = s * (u * t)$; $(s * t) * u = (sa)u = s(au) = sa = s * (tb) = s * (t * u)$; $t * (s * u) = t * (su) = tb = tbb = (tb)b = (tb) * u = (t * s) * u$. The case where two elements are in T and one is in S is solved the very same way, as the only argument used is the property of a, b being left-zeros in order for the operations to coincide when the element is in the middle, and so we have ourselves a semigroup. As for the last part, it is clear that if both $a \in C$, $b \in D$ for the corresponding connection sets of two Cayley graphs over S and T, the operation defined does not generate any cross edges by construction, but it does not even add new edges inside of any of the graphs separately, as $t * c = tb$ for every $c \in C$ and $t \in T$. Analogously, $s * d = sa$ for every $d \in D$ and $s \in S$, and we impose by hypothesis that these edges are already in the corresponding Cayley graphs. □

The first thing we can apply this result to is the classes of basic outerplanar graphs we know explicit monoid representations of: bends and triangular paths. Triangular paths do not have the unique left-zero of the monoid that represents them in the connection set, and, thus, are not suited to apply the result to them, except for the range of n where they are the same as bends. As for bends, a zero element is in the connection set for the representations of B_n for every $n \geq 1$, which gives us the following result directly as a corollary:

Corollary 3.5. *Let m, n be natural numbers bigger or equal than 1. Then, $B_n \sqcup B_m$ (the disjoint union of bends with n and m triangular faces, respectively) is a semigroup graph.*

Proof. Take the corresponding representations as monoid graphs for B_m and B_n as described in Prop.3.12 Then, as they satisfy the hypotheses of Prop.3.14, the disjoint union of both can be represented by the Cayley graph of the presented disjoint union semigroup. □

Of course, there are infinite examples of such disjoint unions, and all of them are examples of outerplanar, disconnected graphs that are semigroup. (the disjoint union of outerplanar graphs is outerplanar, as they do not share vertices nor form any new faces that are not already in one of the graphs by themselves). But this is not all that we can say in this respect; there are more ambitious goals we can use to test the tools we have introduced. In order to end this part of the work, we want to present an original research (although heavily reliant in Thm.3.1 by K.Knauer and Puig i Surroca) for a characterization of outerplanar graphs that admit the simplest Cayley representation that we know of, that which works for triangular paths: $\underline{\mathrm{Cay}}(M, \{a, a^2\}))$ for some monoid M and some element $a \in M$, which will provide infinite examples of outerplanar, disconnected *monoid* graphs. The strategy is the following: we want to see exactly how these graphs are related to the corresponding subgraphs $\mathrm{Cay}(M, \{a\})$, which are clearly monoid and 1-outregular, and, by Thm.3.1, directed pseudoforests as described. Then, we want to study how outerplanarity affects the options that are valid, and then recover which classes of outerplanar graphs admit the desired representation. We divide the process in different lemmas: the first one takes care of the relation between $\mathrm{Cay}(M, \{a, a^2\})$ and $\mathrm{Cay}(M, \{a\})$:

Lemma 3.1. *Let M be a semigroup, $a \in M$. Then, we have that* $\mathrm{Cay}(M, \{a, a^2\}) = (\mathrm{Cay}(M, \{a\}))^2$*, where by this exponent we mean the graph power in the* directed *sense, that is, $(x, y) \in E(G^2)$ for G a directed graph iff there is a* directed *path from x to y of length at most 2.*

Proof. Clearly the set of vertices of the two directed graphs is the same, so it suffices to see that they have the same (directed) edges. Let x, y be elements in M, then $(x, y) \in \mathrm{Cay}(M, \{a, a^2\})$ if, and only if, either $xa = y$ or $xa^2 = y$. In the first case, clearly (x, y) is already an edge of $\mathrm{Cay}(M, \{a\})$, and, in the second, the path $\{x, xa, xa^2 = y\}$ is a path of at most length 2 from x to y. (it could be less, for example if a is an idempotent).
Conversely, (x, y) is an edge of $(\mathrm{Cay}(M, \{a\}))^2$ if there is a directed path of at most length 2 from x to y. If it is of length 1, it is an edge of the Cayley graph and $xa = y$. If it is of length 2, it must be the unique directed path $\{x, xa, xa^2\}$ that can be achieved for this Cayley graph, with $xa \neq xa^2$, which excludes the previously considered case of a being an idempotent. In any case, it is clear that (x, y) is an edge of $\mathrm{Cay}(M, \{a, a^2\})$ □

This gives us a clear path to study representations of the form $\mathrm{Cay}(M, \{a, a^2\})$ from their subgraphs defined by just a, as we wanted. We now want to see how strong of a condition outerplanarity really is in this context. In order to be able to study this impact comfortably, we need to introduce a little machinery from *minor theory*, an important tool for almost any result in graph theory. We first introduce the notion of a *minor*.

Definition 3.10. *Let G be a simple, finite graph. Then, we say that H is a minor of G if it can obtained from G applying finitely many times one of more of these steps:*
(1) Erasing edges from G, that is, considering the same graph with a particular edge remove; (2) Removing isolated vertices, that is, those with degree 0.
(3) Contracting edges, that is, for an edge $e = (v, w)$, the graph resulting from contracting e is defined as a the graph where the vertices v, w are made into the same vertex, and this vertex inherits adjacency to all the other vertices that were adjacent to either v or w.; It is worth making the remark that is does not matter in which order this steps are applied in order to determine whether H is a minor of G or not.

The notion is specially potent since the work from N. Robertson and P. Seymour cited in [8] proved that some families of graphs satisfying a series of properties could be characterized in terms of excluded minors: that is, a graph G is in the family if and only if a given special graph cannot be obtained from G using the previously described process of obtaining minors.
For our case, we only need to know that there exists such an excluded minor characterization for outerplanar graphs, which we will cite as a lemma, but we will not prove.

Lemma 3.2. *Let G be a simple graph. Then, G is outerplanar iff neither K_4 or $K_{2,3}$ are minors of it.*

A complete proof of this result can be found in [2].

Lemma 3.3. *Let G be an outerplanar graph such that $G = \underline{\mathrm{Cay}}(M, \{a, a^2\})$ for a monoid M and an element a. Then, $G' = \underline{\mathrm{Cay}}(M, \{a\})$ does not contain any cycle C_n with $n \geq 4$*

Proof. Let $V(C_n) := \{v_0, ..., v_{n-1}\}$.Suppose any connected component of G' contained such a cycle. As a first observation, let us note that the only directed 1-outregular representation of a cycle is that of a directed path with

equal endpoints, that is, any directed representation where two edges "collide" (have opposite directions) is not 1-outregular. If that were the case, and v_k for $k \in \{0, 1, ..., n-1\}$ was the first vertex going out from v_0 in a directed path that encountered an edge in an opposite direction, say, the edge (v_{k+1}, v_k), by 1-outregularity and the need to represent every edge in the cycle, the following edge should have the same direction, so (v_{k+2}, v_{k+1}) is the edge in this directed representation. This argument can be iterated until the moment we arrive to v_0 again. Then, we have that (v_0, v_1) is an edge in the representation by hypothesis, and (v_0, v_{n-1}) as well by iterating the observation we just made. But that would make v_0 have outdegree 2, causing a contradiction. The same argument applies if we assume that the directed cycle follows the path $v_0, v_{n-1}, ...v_0$, and we can just rename the vertices without loss of generality in that case. Therefore, such a k cannot exist and the only possible representation is a directed n-cycle.
Having seen this, let us denote as $C = v_0, ..., v_{n-1}, v_0$ the directed cycle in question. As $n \geq 4$, we can take four different vertices $v_0, ..., v_3$, and then consider C^2. By applying the described steps for obtaining minors, we could clearly delete all edges belonging to length two paths in C^2 that are not contained in the subgraph induced by the selected four elements, and contract the rest of length one edges, obtaining a graph with K_4 as its underlying graph, which clearly makes the graph not outerplanar. $\square$

Therefore, if $G = \underline{\text{Cay}}(M\{a, a^2\})$ is to be outerplanar, we have that any connected component of $\text{Cay}(M, \{a\})$ can only have a 1-cycle (loop), a 2-cycle or a 3-cycle (triangle). Let us now see how outerplanarity restricts in which ways bifurcations in the connected components of the pseudotree $\text{Cay}(M, \{a\})$ can be found.

Lemma 3.4. *Let $G' = \text{Cay}(M, \{a\})$ be such that the underlying graph of $\text{Cay}(M, \{a, a^2\})$ is outerplanar. Then, if a connected component of G' contains a bifurcation (a directed representation of the simple graph S_3), the unique cycle of the connected component has all its vertices in this subgraph.*

Proof. The most direct way to translate the intuition of what a bifurcation is to graph theory is the graph $S_3 = K_{1,3}$, the star of three vertices. It is

clear that any bifurcation in a tree contains this simple graph as a subgraph. Let us take the following notations: $V(S_3) = \{0, 1, 2, 3\}$, where 0 refers to the degree 3 vertex. Out of all 8 ways to represent the edges $(0, i)$, $i \in \{1, 2, 3\}$ as directed edges, the only 1-outregular ones are isomorphic to one of these three cases: $\overrightarrow{S_{3,1}}$, with set of edges $\{(1, 0), (2, 0), (3, 0)\}$, where the vertex 0 has indegree 3. As the graph must be 1-outregular, another edge has to come out of 0. If it goes to 0, it will form a 1-cycle, and, if it goes to 1, 2, or 3, a 2-cycle. If we supposed that 0 goes to another vertex of G' v that is not in the bifurcation, when we take G'^2 we would have that the induced subgraph cannot be outerplanar. The situation is depicted in detail in Figure9, where we will prove this fact rigorously. In conclusion, there must be a 1 or 2-cycle in a 1-outerplanar version of this digraph.
$\overrightarrow{S_{3,2}}$ is the one defined by the set of edges $\{(2, 0), (3, 0), (0, 1)\}$ In this case, as this must be part of a digraph that is 1-outregular, 1 is the vertex that has en edge left to define. It it goes to itself or to other vertex of the triple $0, 2, 3$, it will generate a 1-cycle (loop), a 2-cycle or a 3-cycle. If it went to another vertex v of G' outside of the bifurcation, the length 2 paths $\lambda_i : i01$ for $i = 2, 3$ would generate edges $(i, 1)$ in G'^2 that would not allow for the edge $(0, v)$ to be represented in the plane in a way that G'^2 were outerplanar. Once again, the exact situation is depicted and shown after Figure10. Therefore, by our hypothesis, this case must also have a cycle in one of its vertices.
Finally, for $\overrightarrow{S_{3,3}}$ with set of edges $\{(0, 2), (3, 0), (1, 0)\}$, we have that the vertex that still needs to have another edge for because of the 1-outregularity of G' is 2. If it goes to any of the vertices already in S_3, once again, we have 1-cycle, a 2-cycle or a 3-cycle. And, if it went to another vertex v of G', once again, we would induce edges in G'^2 of the form $(1, 2)$, $(3, 2)$ and $(0, v)$. Once again, representing these as continuous paths in the plane, as well as the preexisting edges, will give no possibilities of an outerplanar representation. A completely rigorous way of seeing this is given after this statement when showing Figure11. In any case, this directed representation of S_3 must also have a cycle in its vertices, and all the rest are isomorphic to one of these by permuting the degree 1 vertices. $\square$

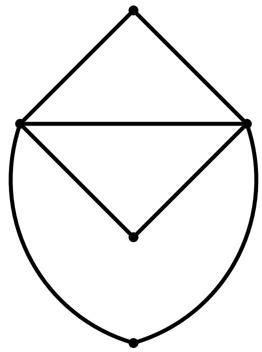

Figure 7. The three cases depicted in figures, 9, 10 and 11 have underlying simple graph isomorphic to the one depicted here, G.

We now repay what is due from the cases of the preceding lemma. We want to use 3.2 in order to prove that the squares of the digraphs considered in the figures do not represent outerplanar graphs. But G obviously has $K_{2,3}$ as a minor if we remove the diagonal from its square, so none of them can represent outerplanar graphs.

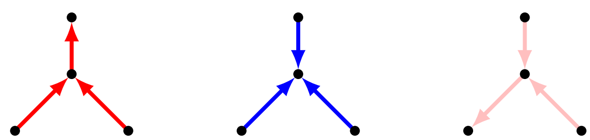

Figure 8. The three directed representations of S_3 that can be part of a 1-outregular graph, as seen in the last lemma.

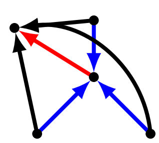

Figure 9. An example of a representation of $\overrightarrow{S_{3,1}}$ with an extra vertex v and edge $(0, v)$ which exemplifies that adding the edges for paths of length 2 (black in the picture) cannot result in an outerplanar graph.

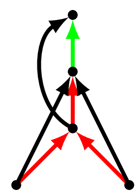

FIGURE 10. Drawing exemplifying the situation for $\overrightarrow{S_{3,2}}$ when we add to it a vertex v and take the square directed graph.

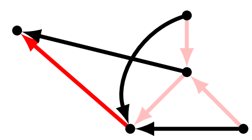

FIGURE 11. A drawing which depicts the situation for $\overrightarrow{S_{3,3}}$ and an extra vertex v

So far, we have seen that outerplanarity restricts in a great way the contexts in which "bifurcations" S_3 can present themselves: they only present themselves with the corresponding unique cycle of the connected component, which means, as cycles in every connected component are unique because of 1-outregularity, that they do not appear outside of them. Therefore, we have that connected components of $\mathrm{Cay}(M, \{a\})$ must be collections of paths converging to points of a 3-cycle, a 2-cycle or a 1-cycle.
We now study how a connected graph satisfying this properties can be, depending on the length of its unique cycle.

Proposition 3.15. *Let G be an outerplanar connected graph satisfying $G = \underline{\mathrm{Cay}}(M, \{a, a^2\})$ for a monoid M and a an element of it, and Z the length of the cycle in $\mathrm{Cay}(M, \{a\})$. If $Z = 3$, then up to three different directed paths go to Z, arriving in different vertices of the cycle. If $Z = 2$, then there are up to two directed paths going to Z. If $Z = 1$, G' is a finite collection of an unspecified amount of directed paths that intersect only at the 1-cycle, with also the possibility of bifurcations arriving at this point, having as a subgraph $\overrightarrow{S_{3,2}}$.*

Proof. We do the proof in reverse order as to how we have presented the facts: 1-cycles pose no problem to outerplanarity other than the subgraphs of the form $\overrightarrow{S_{3,2}}$, as squares of directed paths are known to be triangular paths, which we know to be outerplanar.

Bifurcations not containing the loop will not cause outerplanar graphs when taking the underlying graph of the graph square as seen in lemma 3.3, so, except for these bifurcations, there are none in these connected digraphs.

If $Z = 2$, then, having more than two paths arriving at any given point of the 2-cycle causes the directed power graph of exponent 2 to not be outerplanar, and, if two paths arrive at one point of the 2-cycle and one arrives to the other, the graph power of two is also never outerplanar. We check this facts in-depth when we depict them in Figures 12 and 13. In consequence, we are left with only the options presented in the statement of the proposition.

Finally, for $Z = 3$, we have that there can be no more than one directed path arriving to the same point of the 3-cycle. If there were at least two, then G' would have a subgraph isomorphic to $\overrightarrow{S_{3,2}}$ with the added vertex v as in the preceding proposition, and, as taking graph powers respects inclusions, we would have that G'^2 cannot be outerplanar. Therefore, the directed trees arriving at the 3-cycle contain no bifurcations, no cycles and, then, can only be directed paths. Because of what we have just argued, there can only be a path arriving per point, and so there can be up to three. □

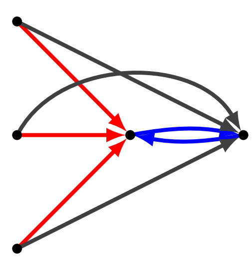

FIGURE 12. The subgraph induced by a 2-cycle with more than two directed paths meeting at one point and its corresponding graph square.

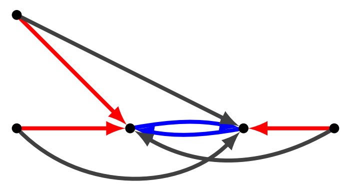

FIGURE 13. Drawing of the graph square for the second excluded case of the 2-cycle, when two paths arrive to one end and one to the other.

We can now note quickly that, in this case, the digraphs that appear in Figures 12 and 13 also define underlying graphs isomorphic to G as depicted in Figure 7, and so they cannot represent outerplanar graphs either, completing the proof.

So, the only thing to check for it to be a characterization of connected outerplanar graphs admitting this representation is that the explained types of directed pseudotrees give underlying outerplanar graphs after taking the directed graph square. But this is clear for some cases: if Z is a 1-cycle and the connected component contains no bifurcations, only connected paths converging to the loop, taking graph square will give, for every directed path, a triangular path, all of them meeting at the vertex which is a loop in the directed representation. This, we know to be outerplanar. If Z is a two cycle and there is only one directed path arriving to one point of the cycle, then the underlying graph of the graph directed power of two will also be a triangular path, and the same in the case where there are two directed paths, each arriving to a different point of the two cycle. For $Z = 3$, if there are up to two different directed paths arriving each to a different point of the 3-cycle, the underlying graph of $\mathrm{Cay}(M, \{a, a^2\})$ will also be isomorphic to a triangular path.

Therefore, there are really only three new cases uncovered by our lemmas. Outerplanar graphs associated to bifurcations, either in a 1-cycle or a 2-cycle, and outerplanar graphs given by the case of a 3-cycle attached to exactly three directed paths. We now represent them in an outerplanar way, making effective that all of them are outerplanar. The reason we can affirm this is that, for the whole family of graphs to be outerplanar in every case, we only have to check what happens around the corresponding cycle. From that point on, the only changes in the graph are given by the length of the directed paths that arrive to it, but they always give triangular paths when taking their directed graph square and then taking the underlying simple graph, which is a process we know well.

In other words, these families of graphs follow predictable patterns we know can be made outerplanar, except around their corresponding cycles. Moreover, we have enough checking for every case individually, as when multiple directed paths and "bifurcations" of the form $\overrightarrow{S_{3,2}}$ coincide in the same 1-cycle, they do not add any directed paths of length up to 2 to each other. It is enough to study whether they are outerplanar or not separately.

In fact, bifurcations that end up in a 2-cycle give underlying graphs isomorphic to those given by bifurcations that end in a 1-cycle. Therefore, we only have to represent the case for the 3-cycle with three directed paths and one out of the two we have mentioned, and see that they are indeed outerplanar. We see it for a bifurcation $\overrightarrow{S_{3,2}}$ that ends in a 1-cycle:

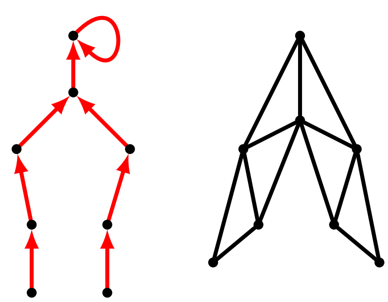

FIGURE 14. Underlying graph of the directed graph power of two of a bifurcation ending in a 1-cycle. We will call these graphs, clearly outerplanar, *arrowheads*. Note that it contains a subgraph isomorphic to B_4

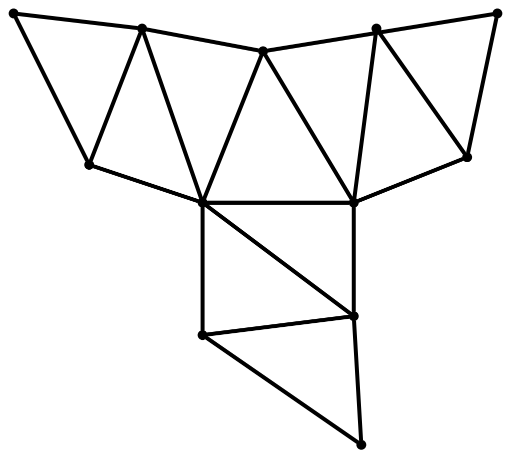

FIGURE 15. The underlying simple graph given by the described process by taking the 3-cycle with three directed paths of three vertices each. This is what we call a triangular bifurcation.

And, so, we can conclude that all graphs that are given by directed graph squares of pseudotrees as described in Prop.3.15 define underlying graphs that are outerplanar, and Prop.3.15 is, in fact, a characterization of the connected outerplanar graphs of this form, as all of them are monoid with a representation of the needed form, by lemma3.1 and the fact that we know that the pseudotrees that represent them are monoid.
We can finally state the complete characterization, as a corollary of all other results:

Corollary 3.6. *Let G be graph of the form $G = \underline{\mathrm{Cay}}(M, \{a, a^2\})$ for a monoid M and an element $a \in M$. Then, G is outerplanar if and only if the directed pseudoforest $\mathrm{Cay}(M, \{a\})$ satisfies that, given $\mathcal{C}$ its connected component with maximum length ($l(\mathcal{C}) \geq l(\mathcal{D})$ and $Z(\mathcal{D})|Z(\mathcal{C})$ for every other connected component $\mathcal{D}$), one of these three cases must hold:*
Either $Z(\mathcal{C}) = 3$ (and then $Z(\mathcal{D}) = 3$ or 1 for any other connected component), $Z(\mathcal{C}) = 2$ and every other component has cycle length 2 or 1 as well or $Z(\mathcal{C}) = 1$ and every connected component is of the same type, having only

1-cycles.

In all cases, the corresponding connected components are as described in 3.15.

Proof. The implication from left to right is a direct consequence of everything we have proved until now, 3.1, 3.3, 3.15.

The converse implication is also direct, as every of the forms allowed for a graph of the form $\underline{\text{Cay}}(M, \{a, a^2\})$ satisfying these restrictions for its connected components defines outerplanar underlying graphs. If every connected component is outerplanar, so is the whole of G. □

3.4. **Non-monoid graphs.** As we have briefly discussed before, being a monoid graph, or at least a semigroup graph, at least intuitively, should be more common that its opposite, as that would imply that the given graph admits *no* Cayley representation. This seems to be the case when checking the literature as well, as the only examples produced to date (to the knowledge of the author) are found in [5]. We now show the proof for the family of graphs $K_4 \sqcup C_l$, for $l > 1$ positive integer satisfying that $2, 3$ do not divide it found in the referenced work, and discuss some of its consequences.

Proposition 3.16. *Let us consider the family of graphs of the form $G_l := K_4 \sqcup C_l$, for $l > 1$ positive integers satisfying that $2, 3$ do not divide l. Then, G_l is not a monoid graph.*

Proof. We want a proof by contradiction. Assume that $G_l = \underline{\text{Cay}}(M, C)$, for some monoid M and some connection set C. Let us now denote as $\overrightarrow{K_4}$ and $\overrightarrow{C_l}$ the corresponding components of the directed graph $\text{Cay}(M, C)$, with possibly loops and some anti-parallel arcs. We know see that, as l is an odd number, there must be a pair of consecutive edges in $\overrightarrow{C_l}$, which, as it is a Cayley graph, must translate to a path of different vertices $\{u, uc, ucc'\}$, for $u \in M$, $c, c' \in C$, and u, ucc' not adjacent. If this were not the case, it would need to happen that every pair of consecutive edges representing the cycle are in opposite directions: but, since there are an odd number of vertices, this means there are an odd number of edges in the cycle as well, and so, this process means that the edge joining the last point in the directed cycle to the first must form a length two directed path with the directed edge going from the first to the second vertex, and so this cannot happen. Now, if we have our three consecutive vertices as described, and we supposed that the neutral element is in the $\overrightarrow{K_4}$ component of G_l, we would have that e, ec, ecc' must satisfy that either a pair of them are equal or that they are neighbours (adjacent by a directed edge), which would cause a contradiction with what we know about them by hypothesis, as any of these cases would imply that $\overrightarrow{C_l}$ would not describe an underlying graph that is a cycle. In any case, if our graph is to be represented by a monoid via the Cayley construction, the neutral element cannot be in the directed component corresponding to the complete graph. So, we have that $e \in \overrightarrow{C_l}$. Of course this means that $N(e) = \{c, c'\}$ for two different elements, and, by Prop.3.6, we can restrict ourselves to $C = N(e)$. The periods of c, c' can only be $1, 2$ or l, and, so, $\underline{\mathrm{Cay}}(M, \{c\})$ and $\underline{\mathrm{Cay}}(M, \{c'\})$ must be pseudoforests where the unique cycles in each of them cannot have length 3 or 4. Therefore, if we look at the Cayley graph generated by c and c' in the $\overrightarrow{K_4}$ directed component, they can generate at most three of the edges in K_4. This can be checked case by case, exhausting all possible directed representations, which we omit for the sake of brevity. (process would go as follows: starting in any vertex of the connected component, and checking every possible case for $c \in C$: it forms a loop or goes to another vertex, then goes to a third vertex or forms a 2-cycle, and so on, imposing that no length 3 or 4 cycles can be achieved). So, as K_4 has six edges, this implies that the only way it can be represented is if both directed graphs represent 3 non-loop edges each, and then there is no edge corresponding to both c and c', and, if we group the edges in K_4 in function of whether they come from c or c', we obtain two copies of the path P_4. We can now consider two cases: if there is some edge in $\overrightarrow{C_l}$ corresponding to both c, c', as we can consider an endomorphism taking e to some vertex in $\overrightarrow{K_4}$ (they must exist, as we know left-multiplication by any element defines an endomorphism of the Cayley graph), then we have that there must be an element in this connected component that has a loop.(this is because such a morphism will take the elements in C_l to K_4, and, as we have seen that the Cayley graphs there induced by c, c' are disjoint, the only way an edge is shared there if it is a loop). So, we can suppose that every edge in C_l is represented by either c or c', and then, as l is odd, there must be a loop in $\overrightarrow{C_l}$. Let us say this element causing the loop is c without loss of generality. By the same argument as before, this implies there is a loop in $\overrightarrow{K_4}$ once again. In order for the proof to be over, we show that this fact produces a contradiction. So, suppose x is the vertex with this loop, and we have that it will be in the copy of P_4 induced by c. This means it is the only cycle of this copy, as the corresponding Cayley graph is 1-outregular and, of course, monoid. Therefore, there must be v, w such that they are different and $vc = v$, $wc = x$, no matter which is the directed representation of the underlying path. This implies that the elements e, c, c^2 are all different. By this fact and the hypothesis that all edges correspond either to c or c' but not both, it is easy to see that $cc' \in \{e, c, c^2\}$. Then, $xc' = xcc' = x$, and so, there are no more loops in $\overrightarrow{K_4}$. Then, we have that $wc' = vcc' \in \{v, w, x\}$, which is a contradiction, since, with everything we know, it could only be $wc' = vcc' = x$, and then the c'-path could not possibly cover three different non-loop edges. $\square$

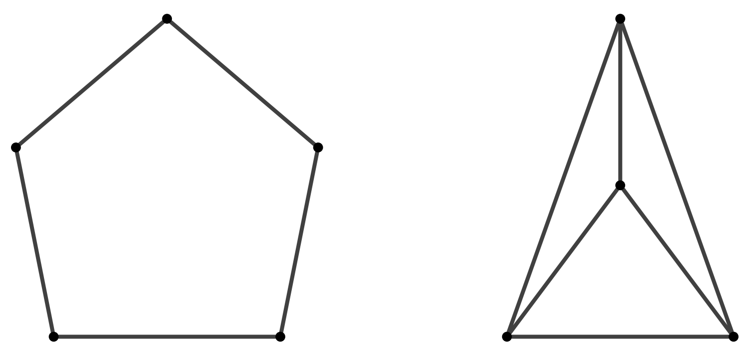

FIGURE 16. $K_4 \sqcup C_4$, the smallest example we know of a non nonoid graph.

Although we know that these graphs are not monoid, we do not know as of now whether or not they admit semigroup representations. Even though they are far from conclusive, we dedicate the rest of the section to provide a couple of results that show that $K_4 \sqcup C_5$, the smallest possible graph of the family, if it is a semigroup graph, does not admit some "naive" or simple representations. Let us see it.

Proposition 3.17. *Let $G = K_4 \sqcup C_5$. Then, G does not admit any semigroup Cayley representation such that the elements in K_4 form a 4 element semigroup S and C_5 is represented by $\mathbb{Z}/5\mathbb{Z}$ with $1 : a \in C$. (and thus the semigroup that represents G is equal to $S \sqcup \mathbb{Z}/5\mathbb{Z}$ with an associative operation that respects the corresponding restrictions):*

Proof. First, we notice that, as all edges in K_4 must be represented, there must be at least two elements of $S \in C$ for C the connection set, as any other element of $\mathbb{Z}/5\mathbb{Z}$ in C would cause extra edges to appear in the cycle or just add loops if it is the neutral element of the group. Let us first consider connection sets without the neutral element of the field of 5 elements. For any element $s \in S$ in the connection set, we must have that as stays in $\overrightarrow{C_5}$. It also is impossible that C contains the whole S, as ar_i can only have three values that respect the cycle if $r_i \in C$, and then there would be some different values, r_i, r_j with $ar_i = ar_j$. As a is cancellative because it has an inverse, we would have that both elements must be the same. Therefore, the connection set must be either equal to a and two elements of S or to a with three of them. For an element $r_i \in C$, it is clear that the only possible values the operation can take are $ar_i = a$, $ar_i = a + a$ or $ar_i = a - a = 0$. If the first case were true, then, by associativity, $a(r_i r_i) = (ar_i)r_i = 3a$, and, then, $r_i r_i$ is an element that brings a to $a + 2$, so $r_i r_i$ must not be in the connection set, say it is some other r_k with k different form i and j. But then, making the same assumption, we have that $r_i r_k$ must be an element that sends a to $a + 3$, so it has to be the only element remaining in S, call it s, given that it cannot be in the connection set. But then, $r_i s$ sends a to $a + 4$, which is different from all other results and cannot be in the connection set, and so we should have some value in S sending a to two different values, a clear contradiction. The same argument could be made from an element r_i that sends a to $a - a$ operating from the right, and so, every element of the connection set from S must fix a when operating from the right. This implies that, if our connection set has two elements of S, $C = \{a, r_i, r_j\}$, then $s(ar_i) = (sa)r_i = sa$ for any element $s \in S$ (and for r_j as well), and then, the vertex sa, clearly in $\overrightarrow{K_4}$ because $a \in C$, is only given loops by r_i, r_j. But a is also right-cancellative, so $Sa = S$, and, then, every element in S can be written as sa for some other element. In conclusion, this means that all edges of the connected component of K_4 should be covered just with a, 1-outregularly, which is clearly impossible. Therefore, only the case with three elements of S remains, but the same happens again. Any element from S in the connection set fixes a from the right, and then only adds loops to every vertex of S, leaving the six edges of the connected component to be covered just by the color a. Therefore, this representation is impossible. Finally, if the neutral element of $\mathbb{Z}/5\mathbb{Z}$ e were to be in the connection set as well, we would have that $r_i e = (r_j a)e = r_j(ae) = r_j a = r_i$ for every i from 1 to 4, because of $aS = S$ and e being a neutral element, and then it would just add loops, not contributing to the representation. □

So, if G_l admits semigroup representations of this form, they must work in unusual ways, with the cycle being represented by elements in the connection set that are in S.
The second idea that one would have in terms of intuition, while gradually increasing the complexity of the representation, would be to apply previously seen results to make semigroups from disjoint unions of them, such as the one given in Prop.3.15 K_4 poses no problem for such a representation, as we can just pick the group $\mathbb{Z}/3\mathbb{Z}$ with a zero element attached to it, let us call it x, and take the connection set $C = \{1, x\}$. As for C_5, however, we can see that it does not admit the simplest possible representation of this form:

Proposition 3.18. *Let $G = C_5$, $S = \{b, c_1, a_1, a_2, c_2\}$ such that b is a left-zero element. Then, G does not admit any Cayley representation such that $b \in C$ with $c_1 b = b$ and $c_2 b = b$. (we will refer to the c_i with i being 1 or 2 as extremes) and $a_i b = a_i$ for any $i \in \{1, 2\}$*

Proof. Let us prove it directly imposing associativity:
If we have that $a_1 b = a_1$, $a_2 b = a_2$, then we can deduce that:
$a_i a_j = (a_i b) a_j = a_i(b a_j) = a_j$, and then $A = \{a_1, a_2\}$ is the left-zero semigroup. Moreover, then $a_i c_j = (a_i b) c_j = a_i(b c_j) = a_i$, and so there is no way that the edge between a_1, a_2 needed in the representation of the cycle can be drawn, making this case impossible. □

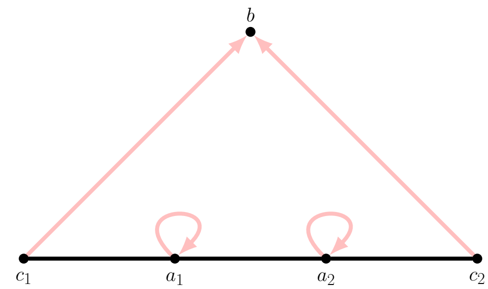

FIGURE 17. A diagram depicting how the type of representation of C_5 proved to not work in Prop.3.18 would go, with the action of b represented by color pink.

Other options have been found to be in contradiction with associativity under some extra assumptions, but not in all cases, and so, they could still be a way to obtain semigroup representations of the graph with this method. We have also checked some semigroup Cayley tables of 5 element semigroups, covering most of them that are *inverse*, but to no avail. Checking all other possibilities computationally could be a promising way of attacking the problem, as this seems to be as far as the pure deductive reasoning from the few properties we have imposed seems to go.

4. Conclusions

In this work, we have tackled mainly two questions which were originally posed in [5] by K.Knauer and Puig i Surroca, with diverse degrees of success. For question 6.3, asking whether every graph is a semigroup graph or not, we have studied the smallest candidate to be a negative example, the graph we already know that is not monoid, $K_4 \sqcup C_5$, with limited success. We have ruled out with original results representations that could be thought of as the most intuitive, but we recognize that our work has limited value to add in that direction and that computational methods should be applied. As for question 6.2, one of its parts asking whether or not there are outerplanar graphs that are monoid, we have found a caracterization of when graphs that admit a very particular Cayley representation, those of the form $\underline{\mathrm{Cay}}(M, \{a, a^2\})$, can be outerplanar, obtaining infinite families that are both monoid and outerplanar. For very simple cases of 2-trees, we have also been able to provide explicit monoid representations.

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