The community structure of the geometric soft configuration model

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Network models serve as an approach to explain the properties of real networks. The geometric soft configuration model, also known as the $\mathbb{S}^1/\mathbb{H}^2$ model, can be used to generate synthetic networks that replicate many features of real complex networks —sparsity, a heterogeneous degree distribution, the small world property, a high level of clustering, and more— while randomizing others. In this work, a range of parameters of the $\mathbb{S}^1/\mathbb{H}^2$ model has been explored, satisfactorily manipulating the level of heterogeneity of the degree distribution with the parameter γ and the level of clustering with the parameter β, in order to probe the level of control that is possible to attain in the generation of random networks. Recent theoretical evidence supports that hyperbolic networks like this one possess topological community structure, up to being maximally modular in the thermodynamic limit, even if the model is not purposefully equipped with geometric communities. The community structure of the $\mathbb{S}^1/\mathbb{H}^2$ model was put under scrutiny using computational simulations, revealing that synthetic networks generated according to it could be consistently partitioned with a high modularity. The modularity of equally sized angular partitions of the generated random networks was evaluated, confirming that this model tends to maximal modularity in the limit of large network size and in a regime of high clustering. The Louvain method for community detection in the topology of complex networks using modularity maximization was employed as well, giving rise to no significantly better results in comparison with the initial approach. With the $\mathbb{S}^1/\mathbb{H}^2$ model, it was also explored how much of the community structure of real networks can be attributed to the effect of clustering in combination with their heterogeneous degree distribution —networks with these two features are called hierarchical—. The results suggest that the communities detected in some real networks are, in part or totally, a byproduct of their hierarchicity.

I. INTRODUCTION

Complex systems are constituted by a large number of interacting entities. There is no agreement on a precise definition for them, however they may share several features, among which is the numerosity of their components and interactions, the fact that they can be heterogeneous, the presence of long range interactions or feedback, the system being open and out of equilibrium, and the emergence of more properties like nestedness, non-linearity, self-organization or robustness.

This kind of systems is present in nature under many guises, such as in human societies, computer networks, climate systems, neuronal systems, metabolic pathways, transportation networks, etc., and those of them that are constituted by discrete units can be represented in the form of networks where nodes represent the elements and links the interactions between them. Some relevant ubiquitous features of real complex networks are their sparsity, their heterogeneous degree distribution, the small-world property and an elevated level of clustering. Moreover, they are often found to be naturally divisible into densely connected groups.

Back in the 1990s, the search for community structure drew the attention of researchers in the context of social networks. The topic soon gained importance since communities oftentimes have been found to correspond to functional units in real networks. During the past twenty years, following the work of Girvan and Newman [1], there has been an increase in the research for community detection. It has been tackled most commonly as an optimization problem, consisting on choosing partitions of the network and devising a quantity to evaluate the quality of such partition: the modularity, Q, which measures the strength of a partition of a network and is computed as a subtraction of the fraction of connections that fall within communities minus the expected value of that same fraction in the case of a randomized version of the network. This quantity lies in the interval between -1 and 1, and it has been historically accepted that a modularity score above $Q = 0.3$ is a sign of community structure in a network [2].

In order to explain the aforementioned network properties, a selection of synthetic models has sprung up over the years. Hyperbolic models have been successful in reproducing many of their features, for instance sparsity, the scale-free property of the degree distribution, the small-world property, or a high level of clustering.

Turning to the aspect of their community structure, it is a notable thing when partitions with high values of

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modularity are reported in random networks with no explicit affinity between nodes. The modularity of random graph models has not yet been studied theoretically save for a limited selection of examples.

On the one hand, it must be noted that a high modularity alone does not imply the presence of community structure; this has been shown for networks in the Erdös–Rényi (ER) model, where high modularity has been detected while lacking meaningful community structure but as a consequence of finite size effects and fluctuations in the establishment of links [3], and for constant average degree d connected graphs, that exhibit an asymptotic modularity with a low bound of $\frac{2}{d}$ [4].

On the other hand, high values of the modularity function have been reported for hyperbolic network models in the thermodynamic limit, in this case, linked to the presence of communities that may be a residue of their hierarchicity. In fact, the Popularity-Similarity Optimization (PSO) model, which is a growing hyperbolic model, has been shown to be maximally modular in the thermodynamic limit [5].

In general, a high value of the modularity of a network must be supported with further evidence to declare the presence of meaningful community structure, since other sources may be the reason for the community structure being detected. Them being modular was not the initial intention when hyperbolic models where designed, however the finding of topological community structure in them meant that, if it were possible to understand and control it, some of the mechanisms for the emergence of community structure in real networks would be closer to be understood.

This work constitutes a study of the community structure of the geometric soft configuration model, also called the $\mathbb{S}^1/\mathbb{H}^2$ model, from which the PSO model is a growing variant. First will be laid the theoretical foundation around complex networks and the basic features that they ubiquitously exhibit in nature; community structure will be covered in more depth, as it is the central subject of this study. Then geometry will be introduced to present the $\mathbb{S}^1$ model and afterwards its isomorphic $\mathbb{H}^2$ model. The previous results about the modularity of hyperbolic models will then be reviewed. The practical work of this study will consist of generating synthetic networks by the $\mathbb{S}^1/\mathbb{H}^2$ model with control over their characteristics so that they can replicate the features of real networks. The modularity —along with the community structure— of random networks generated by this model will be evaluated as a function of the features that can be controlled: their degree distribution, their level of clustering and their size.

II. THEORETICAL FRAMEWORK

A. Basic topological features of real complex networks

Complex systems composed of discrete units are modeled as topological objects using graph representations where their elements are portrayed as nodes and the interactions between them as links connecting pairs of nodes. In graph theory, nodes are called vertices and links are called edges.

A set of useful global and local attributes will be presented hereunder to facilitate a quantitative study of complex networks.

The degree, k_i, of a node i is the number of interactions that it establishes with other nodes of the network.

The adjacency matrix of a network, A, is composed of:

$$a_{ij} = \begin{cases} 1 & i \ \& \ j \text{ connected} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In general, these interactions may have different intensities or weights, and there may be nodes with self-interactions, but in the scope of this work merely simple networks will be taken into consideration. Links are therefore unweighted and undirected, that is, all links have the same intensity and are bidirectional, and there are no self-interactions or self-loops.

A global quantity of networks that can be rapidly evaluated is the average degree, which is a ratio between the number of links, E, and the number of nodes, N:

$$\langle k \rangle = \frac{2E}{N} \tag{2}$$

It gives an idea of the level of connectivity of a network in a first global approximation.

1. *Sparsity*

Networks found in nature are typically sparse. This property refers to the density of links within a network, a network is said to be sparse when it has "much fewer" links than the maximum number of possible links in it, thus its average degree is "low".

The definition is formalized by establishing that a network is sparse if its average degree does not scale with its size, N.

2. *A scale-free degree distribution*

The degree distribution of many real networks follows a power law:

$$P(k) \sim k^{-\gamma} \tag{3}$$

Networks are said to be scale-free when the exponent of such power law lies in the interval $2 < \gamma < 3$. They are more heterogeneous the smaller the value of γ and more homogeneous when $\gamma > 3$.

In a broader sense, the actual ubiquitous feature in real networks is that they possess a fat-tailed distribution of degrees, being not as important whether the distribution is rigorously a power law or merely asymptotically.

3. *The small-world property*

As topological objects, distances are measured by the minimum number of edges that constitute the shortest path between two nodes, which is the smallest number of links that connects two nodes in the network.

Nodes may be connected in many ways, but a common property of real networks is that they are small-world, that is, that the distribution of shortest paths between all pairs of nodes is a homogeneous distribution with a well defined average value that is typically low. This property inherits its name from the context of social networks, where it manifests itself in strangers being linked by a short chain of acquaintances.

For network models, the definition is formalised by establishing that the number of nodes should grow exponentially with the network diameter, which is the longest of all the calculated shortest paths in the network.

4. *A high level of clustering*

Clustering or network transitivity is the property that states that two nodes that are connected to a third node have increased probability of being connected to one another, therefore forming a triangle. Thus, nodes in the same triangle are more similar, where similarity measures the affinity to form connections.

The clustering coefficient is a local measure of the number of triangles in a network. It quantifies the tendency of nodes to form transitive relations. For each node, it is defined as the number of triangles such node is part of normalized by the maximum number of triangles it could form:

$$c_i = \frac{T_i}{\frac{k_i(k_i-1)}{2}} = \frac{2T_i}{k_i(k_i-1)} \tag{4}$$

where T_i is the number of triangles through node i. Note that Equation 4 is not defined for nodes with $k_i \leq 1$ since no triangles can be formed in such a case and therefore their clustering coefficient is trivially 0.

From the clustering coefficient of the nodes in a network, a degree dependent clustering spectrum can be computed by averaging the local clustering over each degree class, $G(k)$, which is the set of all the nodes with degree k, whose count for each k will be referred to as N_k.

$$\bar{c}_k = \frac{1}{N_k} \sum_{i \in G(k)} c_i \tag{5}$$

In many real networks this clustering spectrum shows a power-law behaviour, showing the hierarchical nature of these networks, in which higher degree nodes are expected to have a lower clustering coefficient. In general, a network is called hierarchical if its degree distribution is heterogeneous and it possesses a high clustering.

In order to have a global quantity to evaluate the level of clustering, it is common to measure the average level of clustering. For a network of size N:

$$\bar{c} = \frac{1}{N} \sum_{i=1}^{N} c_i \tag{6}$$

The inmediate way to do it is to take into account all the nodes of the network in this average, but there are variations that exclude nodes of degree 0 or even nodes of degree 1, the reasoning behind these choices being that these nodes possess a trivial clustering coefficient anyways. In this work, the average level of clustering has been computed excluding nodes of degree 0, nodes that are disconnected from the rest.

A high average level of clustering is essentially the fruit of local features that add up in a global scope as a large fraction of triangles.

It has been found that the level of clustering of real networks tends to be greater than the expected for the same networks if they were randomly linked. This is best exemplified in social networks, where it is common to observe that the relationships between people are usually clustered in families and friend groups.

B. Community structure

In a closely related matter, a complex network has community structure if its nodes can be grouped into sets so that two nodes are more likely to be connected if they belong to the same set or community and less likely if they belong to different communities. Thus, communities of nodes are more densely connected internally than between them. They declare groups of nodes that are strongly connected or that are similar. In general, communitites can be overlapping or non-overlapping. In the simple networks under consideration, only non-overlapping ones will be considered.

Real networks commonly exhibit compartmentalization. Communities in social networks may be groups of interest, shared place of origin, location or occupation; in citation networks, they may represent fields of research; in metabolic networks, communities may correspond to metabolic paths or cycles; in neural networks, they may be functional or structural units.

The community structure of complex networks provides information relating network function and topology. These two concepts are entangled with one another since the topology of a network plays a very important role in the transference of information through it. The ability to detect communities may shed light on the underlying relation between them.

With the goal of detecting meaningful communities in mind, a series of algorithms have been designed in the last twenty years to overcome the computational problem it presents. They share that they consist of optimizing some quality function, the evaluation of which method obtains the best results is still an open question. The most common approaches are based on the optimization of the modularity, others make use of statistical inference to maximize a likelihood function and a third class uses dynamical processes, particularly, they maximize the entropy of random walks, based on the idea that a random walk would spend more time lingering inside communities than jumping from one community to another.

This work is mainly focused on the first class, the modularity-based methods.

Given a network and a partition of said network, the modularity is defined to quantify how well this partition adjusts to a community structure of the network, if any. In other words, it evaluates the strength of a division of a network into communities, giving a measure of whether there are more links within the communities declared by the partition while fewer links remain between them.

Let a partition of a network into communities such that a node i belongs to the community C_i. In order to approach the measurement of the "goodness" of such a partition, it is a good place to start to look at the fraction of edges that fall within communities compared to the total number of edges:

$$\frac{\sum_{ij} a_{ij}\delta(C_i, C_j)}{\sum_{ij} a_{ij}} = \frac{1}{2E}\sum_{ij} a_{ij}\delta(C_i, C_j) \tag{7}$$

This quantity is a scalar between 0 and 1 and it is larger when there are many intracommunity edges. However, the trivial scenario when there is only one community gives rise to the maximum value. To avoid choosing this partition always as the best one, the current definition of modularity has been devised with the subtraction of the expected value of the fraction from Equation 7 in the case of a randomized version of the network. It is common to choose for the construction of the randomized version the configuration model.

Given a network, the configuration model [6] maintains the degrees of the nodes. Each node is assigned a number of edge stubs equal to its degree and the random network is built by choosing edge stubs uniformly at random and linking them together until there are no more left. Under this construction, the probability that a node i is connected to j is proportional to the product of their degrees:

$$p_{ij} = \frac{k_i k_j}{2E} \tag{8}$$

The resulting expression for the modularity is the following:

$$Q = \frac{1}{2E}\sum_{ij}(a_{ij} - \frac{k_i k_j}{2E})\delta(C_i, C_j) \tag{9}$$

The range of the modularity is then $[-1, 1]$, it is larger the more intracommunity links there are for the partition that it is evaluating. Being defined as such, the trivial partition now leads to a value of 0. The modularity of a partition is positive if there are more links between nodes in the same set than expected at random, and it is negative if there are more links between sets. An example of positive modularity in social networks arises when looking at the social relationships between individuals partitioned by ethnic class, while the modularity is negative for partnership relations partitioned by gender identity.

The maximum value that the modularity can reach signals the "best" partition of a network and it's called its modularity score. Therefore, the modularity score of a network —usually referred to as just the modularity of the network— lies in the range $[0, 1]$, being close to 0 for random networks.

The partitioning of a network into communities grew into an optimization problem, the main obstacle being the computational challenge this kind of problems pose as the size of the network increases. This is why a heuristic approach was adopted, giving rise to a selection of methods to tackle it. The modularity was devised as a

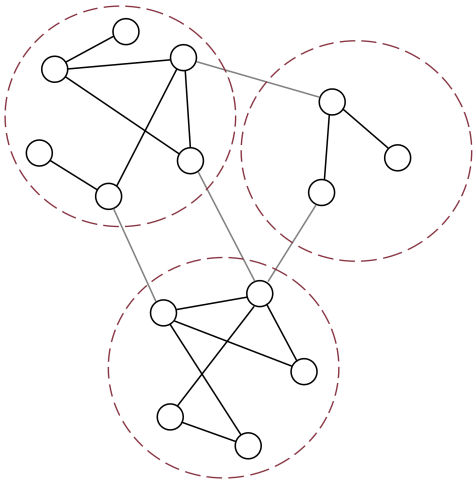


FIG. 1: A small network with community structure. Three communities are represented with dashed lines, with the intracommunity links in solid black and the intercommunity links represented in grey.

way to measure the community structure of a network, it is used as a quality function to be maximized by the algorithms that aim to detect community structure.

Some of the most famous classes of algorithms that have been proposed for this task are the following.

1. The Girvan-Newman method

One of the first methods to obtain a great success in community detection was devised by Girvan and Newman [1]. It is based on the betweenness of a node, which is the fraction of all the shortests paths between every pair of nodes in the network that pass through the node in question; it gives a sense of its centrality in the network. The method consists of two steps: first, iteratively removing the links with the highest betweenness, and second, a step for recalculation of the betweenness of all nodes affected by the removal of those links. These steps are repeated until no links remain. A dendrogram is extracted from this process and the step with the maximum value of the modularity is declared the best partition.

2. The greedy modularity maximization method

Differently from the previous approach, this is a local detection method, which makes it less computationally costly. The starting point is with each node in its own community. It then consists of joining the pair of communities that would lead to the maximum modularity until no possible increase is found, thus reaching a local optimum value.

3. The Louvain method

By Bondel et al. [7], this method starts similarly to the previous one with a community for each node, but in this case for each node it evaluates the change of modularity of removing it from its community and placing it on the neighbouring ones, and doing so when the value is maximum and positive. After iterating for the whole network, a local maximum is achieved, where no individual move can improve the modularity. Then, a new network is created rescaling the communities into new nodes and the process is repeated until the modularity increases no more. With this improvement, this method was shown to outperform its predecessors in terms of computation time and quality of partitions.

C. Geometric models

Synthetic models constitute reference models from an statistical point of view, for features of interest in these networks can be selectively preserved while randomizing others. The most commonly used for this purpose due to its simplicity is the already described configuration model, which maintains the degree sequence of the original network while randomizing the connections. These objects are useful for comparison with their real counterparts and they may be enlightening on whether certain properties of reality are fruit of structural constraints, randomness, or something else entirely.

So far networks have been considered as purely topological objects where distances were measured as the shortest paths between nodes. However, many real networks are implicitly embedded in metric spaces. This way nodes may be equipped with coordinates on which the connections are dependent. There are networks for which the geometry is explicit, like a railway network, but even when this is not the case many networks can be equipped with latent spaces that induce an effective geometry.

The current state of the art follows the idea of equipping networks with underlying or hidden metric spaces, where distances are measured following the geodesic lines between two points. In this manner, geometric models are meant to aid in explaining some of the features of real networks. For example, in social networks, age stratification can be explained by a hidden coordinate marking the similarity between nodes, since people tend to make friends with people of similar age.

Apart from equipping real networks with an underlying geometry, models of geometric graphs on the hyperbolic plane can be designed to obtain random networks where the coordinates and connections between nodes are sampled at random but can replicate the global features of real networks through the laws imposed on their distribution. The geometric soft configuration model $\mathbb{S}^1$ for simple —undirected and unweighted— networks was introduced by Serrano et al. [8] and a purely geometric formulation of it, the $\mathbb{H}^2$ model, was subsequently formulated by Krioukov et al. [9].

1. The $\mathbb{S}^1$ model

A node i in a similarity space that takes the form of a one-dimensional sphere is determined by two variables: the hidden degree, κ_i, that quantifies its popularity, and the angular position where it is placed on the circle, θ_i.

The distance between two nodes i and j in a circle of radius $R_{\mathbb{S}^1}$ is:

$$d_{ij} = R_{\mathbb{S}^1} \Delta\theta_{ij} \tag{10}$$

where $\Delta\theta_{ij} = \pi - |\pi - |\theta_i - \theta_j||$. It is thus a measure of their similarity. For N nodes, the radius of the circle is adjusted to $R_{\mathbb{S}^1} = \frac{N}{2\pi}$ so that the density of nodes in the circle is 1.

It has been chosen a parametrization such that the angular positions and the hidden degrees of the nodes are uncorrelated. The nodes are sampled with angular positions so that they are distributed uniformly at random

along the circle. The distribution of hidden degrees of the nodes is defined so that it can display the emergence of patterns or features typical of real networks, with the following power law probability distribution:

$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma} \tag{11}$$

with a lower bound for the hidden degree, $\kappa_0 = \frac{\gamma-2}{\gamma-1}\langle k\rangle$, where $\langle k\rangle$ is the target average degree and γ a parameter that controls the steepness of the distribution function. The target average degree is usually small with respect to the size of the network, N, in this work it has been chosen that $\langle k\rangle = 10$.

This choice of distribution grants control over the expected degree of the nodes by the variation of the parameter γ. When γ is very close to 2, strong fluctuations occur in the model. For low values of γ in the region close to 2, the degree distribution of the network is heterogeneous, with most of the nodes with a lower degree and only a small fraction of them being highly connected. When γ reaches higher values, the homogeneity of the network generated increases. The range to be explored in this work is that of $\gamma \in [2.5, 3.5]$.

The connections between nodes depend on both how likely the nodes are to establish a connection, their popularities, and the distance between them in the similarity space, their similarities. To reproduce the properties of real networks in a simple manner, it has been chosen that two nodes, i and j, are connected with probability:

$$p_{ij} = \frac{1}{1 + (\frac{d_{ij}}{\mu\kappa_i\kappa_j})^{\beta}} \tag{12}$$

where the parameter β controls the level of clustering and $\mu = \frac{\beta}{2\pi<k>}\sin(\frac{\pi}{\beta})$ controls the average degree.

The synthetic networks in this model can reproduce the degree distribution of real networks via the manipulation of the parameter γ and their level of clustering with β. These networks can be considered as randomized versions of their real counterparts in the sense that they preserve the degree distribution and the level of clustering while the rest of features are maximally random. Thus, they can be used as synthetic models for the analysis of certain characteristics.

2. *The $\mathbb{H}^2$ model*

The $\mathbb{H}^2$ model is isomorphic to the $\mathbb{S}^1$ model. The angular coordinates remain the same and the hidden degrees are transformed into radial coordinates in the following manner:

$$r_i = R_{\mathbb{H}^2} - 2\log\frac{\kappa_i}{\kappa_0} \tag{13}$$

where the radius of the two-dimensional hyperbolic disc is fixed to $R_{\mathbb{H}^2} = 2\log\frac{N}{\pi\mu\kappa_0^2}$ so that $r_i = 2\log\frac{N}{\pi\mu\kappa_0\kappa_i}$. The number of nodes grows exponentially with the radius.

The choice of representation in the hyperbolic space is free, being the most common the Poincaré representation portrayed in Figure 2. In this work the representation used is the native representation, in which all distances have their true hyperbolic values.

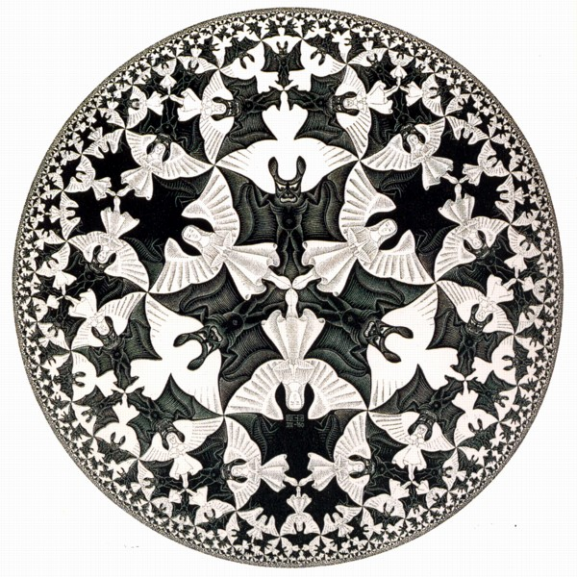


FIG. 2: M. C. Escher's "Circle Limit IV", (c) 1997. Cordon Art, Baarn, Holland. All rights reserved. The hyperbolic plane is usually represented with the Poincaré unit disc, $\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$, equipped with the hyperbolic metric, $ds^2 = \frac{du^2+dv^2}{(1-u^2-v^2)^2}$. This artistic representation shows how distances behave in the Poincaré disc and how objects are deformed in it.

After the transformation given by Equation 13, higher degree nodes are placed closer to the centre of the disc, while most of the nodes with lower degree lay on the boundary. This permits an improved visualization of the previously hidden degrees of nodes and it is the reason this transformation has been performed.

The probability of connection is transformed into a purely geometric function that now depends on the distances between nodes in the hyperbolic plane. This gives evidence of the fact that the effective geometry of the $\mathbb{S}^1/\mathbb{H}^2$ model is indeed hyperbolic. Equation 12 transforms into:

$$p_{ij} = \frac{1}{1 + e^{\frac{\beta}{2}(x_{ij} - R_{\mathbb{H}^2})}} \tag{14}$$

with $x_{ij} = r_i + r_j + 2\log\frac{\Delta\theta_{ij}}{2}$ the hyperbolic distance between two nodes i and j.

In order to obtain the desired synthetic networks, random networks were generated using the $\mathbb{S}^1$ model. To perform the transformation to the isomorphic $\mathbb{H}^2$ model it is only necessary the computation of the radial coordinate of each node.

3. The Geometric Critical Gap Method (G-CGM) for community detection

Apart from modularity-based methods, once networks are equipped with their latent geometry, other approaches to the detection of geometrical communities may arise making use of their coordinates. Based on the idea that angular communities are essentially sets of nodes that are similar, other criteria can be established to define partitions of the networks.

The G-CGM method is inspired by the inhomogeneities in the angular distributions observed in real networks.

All angular gaps, $\Delta\theta$, between consecutive nodes are measured and those that exceed a critical value, $\Delta\theta_c$, separate adjacent angular communities. This critical gap is calculated assuming a Poisson point process on the $\mathbb{S}^1$ along which the angular gaps are exponentially distributed with rate $\delta = \frac{N}{2\pi}$.

$$\Delta\theta_c = 2\pi\frac{\log N + \gamma_E}{N} \tag{15}$$

where the Euler-Mascheroni constant has an approximate value of $\gamma_E \simeq 0.5772$.

4. Equally sized angular partitions

Following this idea of partitioning a geometric network by splitting the circle into a number of segments, q, where $2 \leq q \ll N$, when dealing with random networks where the nodes are uniformly distributed at random along their similarity space —such as the networks generated by the $\mathbb{S}^1/\mathbb{H}^2$ model—, the division can be made with no preference into equally sized segments of angular width $\frac{2\pi}{q}$.

These geometrical methods only take into consideration the angular position of the nodes and ignore their hyperbolic radial position when stablishing their partitions. Whilst the G-CGM aims to detect angular communities, the equally sized angular partitions will be used to find out if there is community structure independently of the angular coordinate.

5. Embeddings of real networks: Mercator

Regarding the fact that real networks have a hidden hyperbolic geometry, ways have been found to map them to these geometric models using statistical inference. This particular method consists on finding a hidden degree and an angular position for each node that maximize a likelihood function of the form:

$$\mathcal{L} = \prod_{i<j}(p_{ij})^{a_{ij}}(1-p_{ij})^{1-a_{ij}} \tag{16}$$

where p_{ij} in this case is Equation 12.

This is computationally expensive, but there have been devised methods that aim to guide the optimization, such as the tool Mercator [14], that combines this maximum likelihood approach with machine learning algorithms. Mercator embeds networks in the $\mathbb{S}^1/\mathbb{H}^2$ hyperbolic geometry, inferring the relevant coordinates as well as the global parameters.

Back to the example of age stratification, with an embedding like this one the ages of the individuals in a social network can be inferred based on the relationships between them.

D. Modularity in hyperbolic models

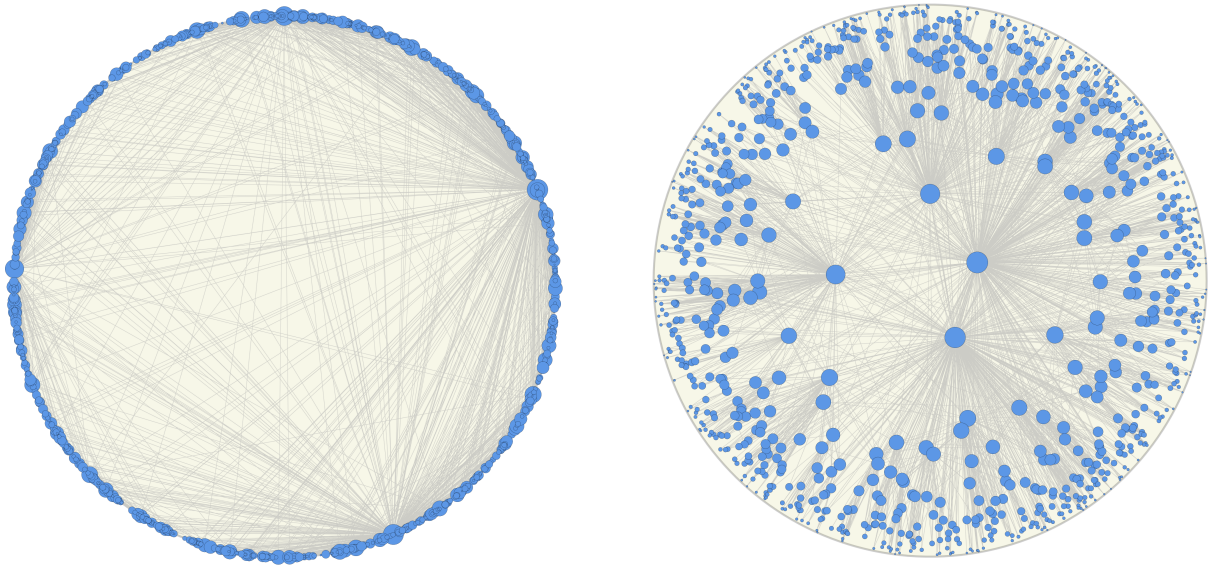
Hyperbolic models were not in principle designed with community structure in mind, it came as a byproduct of their hierarchicity —their heterogeneous degree distribution in combination with their high level of clustering—. Precise knowledge of the behaviour of the modularity in these models is key to understand the emergence of topological community structure and may shed light on the sources of community structure in real networks.

Chellig and Fountoulakis [11] proved theoretically that the modularity of the $\mathbb{H}^2$ model converges to one in probability for the limit of $N \to \infty$ and $T \to 0$ ($\beta \to \infty$). They attribute it to its hierarchical structure, which is a fruit of high clustering. They performed a partition of the Poincaré disc into a number of equal sectors and argue that since each sector tends to capture a large proportion of a clustered sub-set, this will make the modularity tend to one in probability.

Balogh et al. [5] studied both theoretically and experimentally the modularity of the Popularity-Similarity Optimization (PSO) model, which is a growing variant of the $\mathbb{S}^1$ model. They proved that the PSO model is maximally modular in the thermodynamic limit, when $N \to \infty$, and observed such behaviour performing equally sized angular partitions of networks by this model for an optimal number of communities that was a function of the size of the system.

The reason behind this is that a characteristic distance can be defined at which the connection probability between two nodes at such distance is close to zero, and thus, in the $N \to \infty$ limit, most links connect nodes that are close to each other in comparison with other parts of the hyperbolic disc, therefore the network becomes segmented with a maximal modularity. The angular width of the optimal partition is thus comparable to such characteristic distance.

They also suggest that being maximally modular in the thermodynamic limit is a universal feature of hyperbolic models, since the basis of the previous explanation applies for all without much tuning.



(a) Representation in the $\mathbb{S}^1$ geometry, where nodes are placed along the circle and their hidden degrees are represented by their size; the bigger the nodes the more connections they tend to make with the rest.

(b) Representation in the $\mathbb{H}^2$ geometry, where the hidden coordinate has been transformed into the hyperbolic radial position of the nodes; the bigger the nodes the closer they are to the centre and the more connections they tend to make with the rest.

FIG. 3: Example of a random network generated by the $\mathbb{S}^1/\mathbb{H}^2$ model with parameters $N = 1000$, $\gamma = 2.5$ and $\beta = 4$. Both representations are of the same network, and give weight to the decision of introducing the $\mathbb{H}^2$ for improved visualization.

III. RESULTS & DISCUSSION

The purpose of this work is to generate synthetic networks by the $\mathbb{S}^1/\mathbb{H}^2$ model, being able to control their degree distribution and their level of clustering, so that they can reproduce the heterogeneous degree distribution and the elevated level of clustering of real networks so that they are hierarchical. Then, the community structure of these synthetic networks will be evaluated with the quality function known as the modularity. In order to do so, the random networks generated will be partitioned into equally sized angular sectors, since there is no a priori preference for the angular coordinate as the nodes are distributed uniformly at random along the circle. This method of partitioning will be compared with the results obtained by modularity optimization methods, in particular the Louvain method.

The generation and the analysis of networks given their edgelists were performed with programs that were developed specially for this purpose in C++, the built-in functions of the PYTHON package NetworkX [12] and the tool Mercator.

A. Synthetic networks by the $\mathbb{S}^1/\mathbb{H}^2$ model

Following the depiction of the model, a program was designed to generate random networks by the $\mathbb{S}^1/\mathbb{H}^2$ model. From the way this model was defined it made possible to control the characteristics of the networks generated.

Figure 3a shows a graphical representation of an example of a random network in the $\mathbb{S}^1$ geometry, where nodes are placed on the circle uniformly at random equipped with hidden degrees that are distributed according to Equation 11, that influences the probability with which they are linked, specified by Equation 12. Figure 3b is a representation of the same network in the $\mathbb{H}^2$ disc, where nodes with higher hidden degree are displaced towards the centre of the disc via the transformation given by Equation 13. The latter is essentially the same network represented in a more visual manner, since it permits to see directly that nodes closer to the centre tend to establish more connections with the rest.

This example is constituted by $N = 1000$ nodes and $E = 4606$ edges. As expected by design, there is a notable heterogeneity in the node distribution, but still the density of nodes decreases rapidly towards the centre of the disc. Nodes closer to the centre possess a higher degree. The network is sparse, with a target average degree fixed to $\langle k \rangle = 10$, while the measured average degree was 9.2120, slightly below the target due to the random nature of the establishment of connections. Sometimes the average degree was observed to be below and sometimes above, oscillating around the target value. With an elevated value for the parameter β, the network also exhibits a relatively high level of clustering, with a total average of $\bar{c} = 0.7096$.

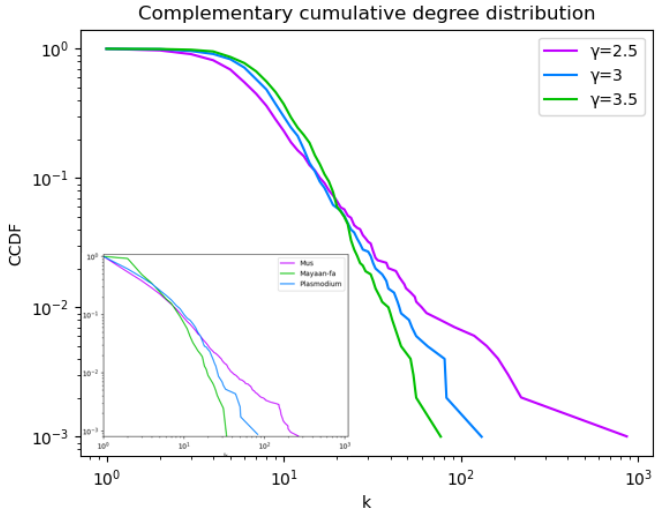


FIG. 4: CCDF of degrees of random synthetic networks of size $N = 1000$ and $\beta = 4$ for a few values of γ in the range for which the degree distribution is heterogeneous. The inset is the CCDF of real networks, they display a similar behaviour as that achieved by synthetic networks.

The degree distribution of the network can be controlled by the manipulation of the parameter γ.

Theoretically, it has been established that the distribution of hidden degrees becomes more homogeneous as γ increases. The distribution of hidden degrees is directly reflected in the degree distribution of the network, a similar behaviour is observed in the Complementary Cumulative Degree Distribution (CCDF) in Figure 4, reinforcing the idea that the network is heterogeneous when $2 < \gamma < 3$ and it becomes slightly more homogeneous as γ increases. Furthermore, in the range $10^1 - 10^2$ the plot shows a linear behaviour in the log-log scale, which corresponds to a power law behaviour, as expected for scale-free networks.

This model can display different degree distributions depending on the range of γ explored. The one chosen replicates the behaviour of real networks, for many of them are scale-free.

Turning to the aspect of clustering in these random networks, representing the clustering coefficient over degree classes with an exponential binning, as was done in Figure 5, it can be seen that its behaviour in all cases is that of a power law. The fact that the clustering coefficient decreases as the degree of the node grows shows that the networks generated by the $\mathbb{S}^1/\mathbb{H}^2$ model are hierarchical.

The level of clustering of the networks can be controlled by the variation of β.

Right from the definition of the connection probability in the $\mathbb{S}^1$ model (Equation 12), β established itself as an important parameter, it is now verified that the level of clustering grows with β. Each point in Figure 6 is an average of five simulations in a system of size $N = 2000$.

It gives an idea of the nature of this dependence, with a more pronounced growth at the beginning that slows down in the high regime near $\beta \lesssim 5$.

However, the size of the networks generated fixing the β parameter does not influence their level of clustering. For $\alpha = 2.5$ and $\beta = 4$, the level of clustering of networks ranging from five hundred to five thousand nodes lingered around the value $\bar{c} \simeq 0.68$.

These results were computed without taking into account nodes of degree 0, however it was confirmed that they were consistent with the quantity computed by the library NetworkX, which does account for all the nodes in the networks.

B. Community detection in the $\mathbb{S}^1/\mathbb{H}^2$ model

Independently of the characteristics specified for the synthetic networks, the angular coordinates of the nodes were always sampled uniformly at random along the circle. Trivially, no angular communities will be found under this construction.

However, partitions can be found for these synthetic networks that score an elevated value of the modularity. In this work, two methods of partitioning will be entertained: a method that divides the network into q equally sized angular sectors, where an optimal number of sectors can be found with maximum modularity for each network; and the Louvain method, an algorithm based on the optimization of the modularity to find the best partition of a given network. Both methods will be compared and the behaviour of the modularity with respect to the size of the network, the degree distribution and the level of clustering will be studied.

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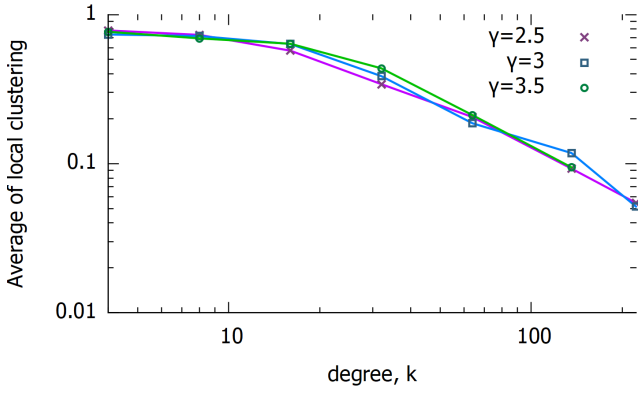


FIG. 5: Average of the local clustering coefficient over degree classes of random networks of size $N = 1000$ and $\beta = 4$ for several values of γ.

1. The absence of angular communities

A program that implemented the G-CGM was developed so that given the nodelist of a network —a list containing the coordinates of the embedding of all its nodes in its latent geometric space— it computed the number of angular communities that constitute it. These angular communities are therefore based merely on the similarity coordinate of the nodes and not on the connections between them. As expected for finite random networks in the $\mathbb{S}^1/\mathbb{H}^2$ ensemble, which are generated with angular positions distributed uniformly at random, the G-CGM does not find angular communities consistently, since the critical difference between angular positions is surpassed on very rare occasions and due to spurious fluctuations.

For the case of the random network that was represented in Figure 3 as an example, this method found merely two communities whose separation is a fruit of randomness and therefore not meaningful.

2. The modularity of equally-sized angular partitions

Being the networks generated angularly uniform, there is no a priori preference for choosing a partition based on the similarity between nodes. The choice of equally sized angular partitions is a simple approach that will prove more powerful than it seems.

The quantity that measures the quality of a partition, its modularity, defined by Equation 9, condensates into a subtraction of the fraction that evaluates the "goodness" of a partition minus that same fraction in the configuration model of the network. It will depend on q, the number of segments the network is partitioned into. The first thing to remark is that in general the values of the modularity score measured for random networks in the $\mathbb{S}^1/\mathbb{H}^2$ model were higher than expected, high enough to be considered as evidence for community structure in

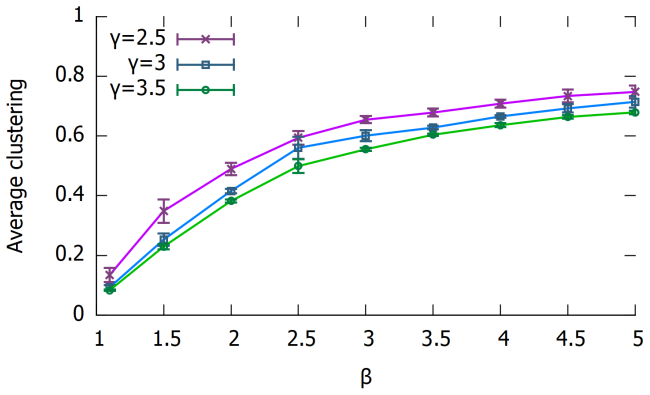


FIG. 6: Dependence of the average clustering on the parameter β in random networks of size $N = 2000$ for several values of γ. Each point is an average of five simulations.

these networks.

These values of modularity would mean in practice a good confidence in the partition associated, but it is known that a high modularity score alone is not enough to indicate the presence of community structure in a network. Whether this elevated modularity score is a manifestation of the presence of actual community structure cannot be readily answered without more analytical sustenance.

In order to go deeper and quantify the tendencies of the modularity, random networks generated by the $\mathbb{S}^1/\mathbb{H}^2$ model were analyzed exploring a wide range system size, and how the modularity changed depending on the number of segments they were partitioned into, the heterogeneity of their degree distribution and their level of clustering.

In Figure 7a, for networks of smaller sizes, the error bars are bigger due to the effect of randomness in the generation of the networks. As the size of the synthetic networks generated was increased, the modularity of each partition grew towards a maximum in the limit of large network size for all the values of the parameters studied. This maximum of the modularity depends highly on the number of segments chosen. Depending on the size of the system, there is a value of q that is associated with the maximum modularity attainable, being between 9 and 15 parts for networks of sizes between 1000 and 5000. This showcases that there is an optimal q for which the modularity will be maximal in the limit of large network size. This result is supported by the theoretical and computational analysis by Balogh et al. [5] of the PSO model, proven to be maximally modular in the thermodynamic limit.

The method of partitioning of the network, even though regarded as simple in the beginning, permitted to reach a partition with maximal modularity, proving thus that there is no need for a more sophisticated partition, although this will be confirmed when applying the Louvain method hereinafter.

To study the effect of the degree distribution and the level of clustering on the modularity, the number of segments of the partition was fixed to $q = 5$.

Figure 7b shows the previously seen increment of the modularity with respect to the size of the system, now adding information about its behaviour depending on the shape of the degree distribution of the networks being generated. Still in the range in which networks are considered scale-free, the modularity is lower and more fluctuating for smaller values of γ, while around $\gamma \approx 3$ it is already close to its maximum for such partition.

Figure 7c shows the influence of the level of clustering on the growth of the modularity with the size of the system. The higher the level of clustering, the higher is the modularity. Although this increment is not very significant for values of β bigger than 3, as it can be seen that the curves for $\beta = 3$ and $\beta = 5$ are close to being overlapped.

It is thus concluded that the modularity of networks generated by the $\mathbb{S}^1/\mathbb{H}^2$ model with an heterogeneous degree distribution is maximal for an optimal number of communities in the limit of large network size and in the limit of high clustering.

This is supported by the work of Chellig et al. [11] proving that the modularity converges to one in probability as $N \to \infty$ in the limit of $T \to 0$ ($\beta \to \infty$) for the $\mathbb{H}^2$ model due to its hierarchical structure.

Putting together the consistently high modularity measured in this work and the previous theoretical results for hyperbolic models, it can be concluded that the $\mathbb{S}^1/\mathbb{H}^2$ model possesses topological community structure.

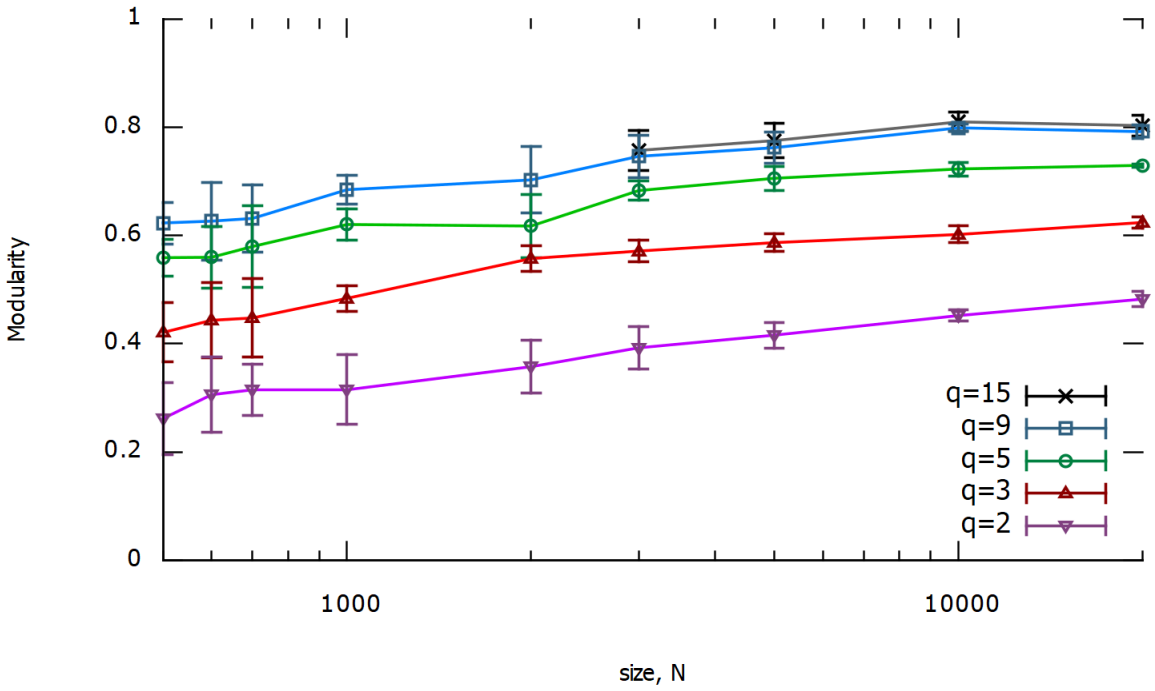
3. Modularity optimization methods

On a different note, methods that rely on modularity to determine the best partition of a network were also studied to see if there was an improvement in terms of the value of the modularity.

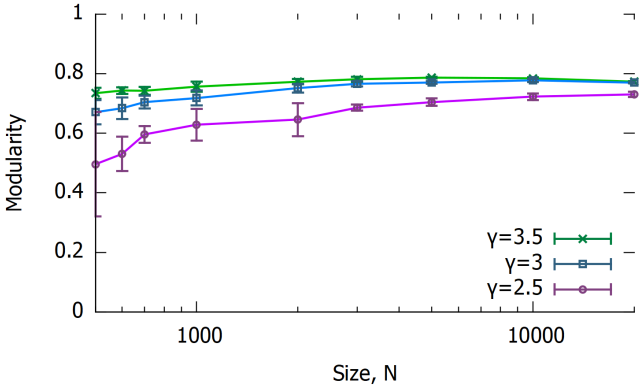
For the case of the random network used as an example, the greedy modularity maximization method found an optimal partition into 18 sets and it returned a value of the modularity of $Q = 0.641741$ associated with it. The Louvain method is however more computationally efficient and gives better results. It was implemented via the algorithm based on the article by Bondel et al. [7] in which the method was proposed. It was developed by two of its authors, Lefebvre and Guillaume, and it consistently found partitions of synthetic networks generated according to the $\mathbb{S}^1/\mathbb{H}^2$ model. Its "best" partition for the network used as an example with a value $Q = 0.676772$ of the modularity associated, a better result than the obtained by the greedy modularity maximization method for the same network.

Then, the behaviour of the modularity of synthetic networks as a function of their size was studied, this time finding the "best" partition with a modularity maximization method, choosing the Louvain method as the more effective.

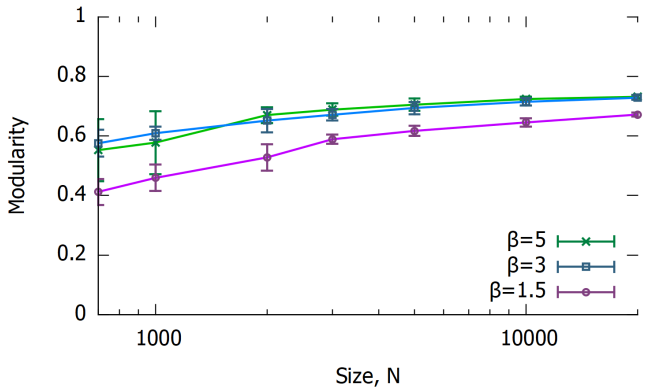
Figure 8a shows a slight growth of the modularity as the size of the systems increases. However, in comparison with Figure 7a, higher values of the modularity have been generally reached with the method of equally sized angular partitions. For the case of the Louvain method, the growth seemed to be less pronounced and the average values were usually below the average modularity for the best number of equally sized partitions. The fact that the results for the Louvain method fluctuate more can be attributed to the method itself because of the diversity of the paths it may choose to group the nodes into communities, rather than always choosing the same angular sectors. The number of communities given by the Louvain method naturally grows with the size of the system, and so it becomes more variable, this is an indicator that the community structure being detected in



(a) Modularity of several equally sized angular partitions with respect to the size of the networks with fixed $\gamma = 2.5$ and $\beta = 4$.



(b) Modularity of equally sized angular partitions into $q = 5$ sets with respect to the size of the networks for several values of the parameter γ and with fixed $\beta = 4$.



(c) Modularity of equally sized angular partitions into $q = 5$ sets with respect to the size of the networks with fixed $\gamma = 2.5$ and for several levels of clustering.

FIG. 7: Variation of the modularity of equally sized angular partitions of synthetic networks generated by the $\mathbb{S}^1/\mathbb{H}^2$ model with respect to size, number of communities, degree distribution and level of clustering. Each point corresponds to an average of ten simulations.

the $\mathbb{S}^1/\mathbb{H}^2$ model is indeed residual.

Having found no improvement over the equally sized angular partitions, it is to be concluded that the $\mathbb{S}^1/\mathbb{H}^2$ model has topological community structure despite its angular distribution.

C. Geometrical communities in real networks

Lastly, and in order to corroborate how the features discussed appear frequently in diverse real networks, a selection of examples was analysed.

The edgelists —lists containing the pairs of nodes that are connected, defining thus a network— of real complex networks were extracted from several sources: the genetic interactions in organisms [13], Mus and Drosophila; the protein-protein interactions in the Homo sapiens by the Reactome project [14]; a series of sociograms developed by J. L. Moreno [15]; the network of 500 busiest US commercial airports [16]; the passenger flights between US airports in december of 2010 [17]; and the flights between airports of the world extracted from Openflights.org [18].

Real networks are commonly sparse and their degree distribution is heterogeneous. The inset in Figure 4 represents the Complementary Cumulative Distribution Function (CCDF) of degrees of some of these networks. There is a similar tendency on the decreasing behaviour of the degree distribution, it is close to that of a power law. As stated in the pertinent section, it is not necessary to observe a strict power law to infer the heterogeneity of the network, an asymptotic behaviour such as the one observed suffices because the sufficient factor is that the distribution is heavy tailed.

Applying the tool Mercator to these real neworks, they were embedded in the hyperbolic geometry of the $\mathbb{S}^1/\mathbb{H}^2$ model. The coordinates of the nodes were inferred in the latent geometry along with a set of global parameters that are displayed for each case in Table I.

Networks with different levels of clustering were analysed. The tool Mercator excludes also nodes of degree 1 in the calculation, therefore these values for the average clustering are slightly higher than the measured excluding only disconnected nodes. Mercator infers the hidden coordinates and in particular the value of β by adjusting the average clustering to the one observed in the real network up to the desired precision. This is congruent with what was observed in the results in Table I, where these two quantities vary in conjunction.

It is also expected for real networks that they may possess some community structure based on the connectivity and similarity of the nodes and, as such, the Louvain method found partitions of them with sufficiently high values of the modularity associated to be accepted as communities at first glance. Some partitions inspire confidence, like the partition with $Q = 0.82$ for the Moreno propro network, while others, like the two networks from US airports, have a value of the modularity too low to promptly accept the partition as "good".

To better understand the sources of community structure in real networks, with the $\mathbb{S}^1/\mathbb{H}^2$ model there have been generated synthetic networks that replicate some global features of the real networks studied previously while randomizing the rest. They all had a heterogeneous degree distribution with fixed $\gamma = 2.5$ and they could reproduce the desired average degree. With the parameter β inferred by Mercator, it was possible to replicate the level of clustering of those real networks as well. The case of the 500 busiest US airports was represented as an example in Figure 9 since being the smallest network makes it good for visualization.

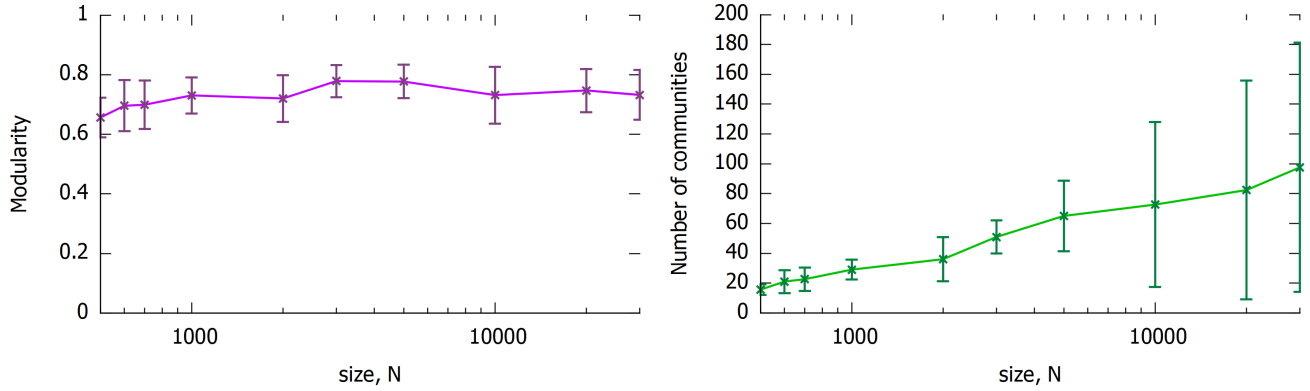
With knowledge about the topological community structure of synthetic networks by this model due to their hyperbolic geometry and hierarchical structure, it is time to raise the question of how much of the community structure of real networks can be explained by this source. Table I also shows the average values of the modularity for equally sized angular partitions into 15 communities of synthetic networks that emulate the real networks previously analysed as examples.

For the Drosophila and Mus networks, as well as for the Reactome and Moreno propro networks, the topological modularity of the synthetic networks was below the modularity measured by the Louvain method in their respective real networks. For the cases of the Moreno names and health networks, the topological community structure of the synthetic networks was tied to values of the modularity that already surpased the modularity measured by the Louvain method in the real networks. A similar tendency in a smaller measure was found for the US airport networks, while for the Openflights network the modularity of the synthetic version almost did not differ from the modularity of the real network.

It was not expected to find an exact match, since the topological community structure of the model may play a part, but there are more sources of community structure in real networks. In Figure 9, it can be seen the comparison between the real network embedded in the latent metric space by Mercator and a synthetic network generated by the $\mathbb{S}^1/\mathbb{H}^2$ model that replicates its global features. The differences are visible to the naked eye, but in terms of community structure, one aspect to remark is that since this model generates the nodes with angular distribution uniformly at random it does not have angular communities, which appear frequently in nature. To take them into account the model can be improved so it introduces a dependence between the two coordinates, and the impact of the possible upgrades on the topological community structure would need to be documented to get closer to reality.

IV. CONCLUSIONS

Real networks were embedded in hyperbolic geometry, which granted knowledge about their hidden coordinates and their global parameters so that synthetic hyperbolic



(a) The modularity shows a slight growth as the size of the random networks increases.

(b) The best partition is generally constituted by more communities as the size of the random networks is increased. However this number also varies a lot as the size increases.

FIG. 8: Modularity and number of communities by the Louvain algorithm of synthetic networks generated by the $\mathbb{S}^1/\mathbb{H}^2$ model as a function of the size of the networks with fixed $\gamma = 2.5$ and $\beta = 4$. Each point is an average of ten simulations.

Network	N	E	E^{synth}	$< k >$	$< k >^{synth}$	$\bar{c}$	β	μ	Number of communities	Q	Q^{synth}	ΔQ
Mus	7402	16858	12200 ± 400	4.56	3.30 ± 0.11	0.13	1.03	0.0031	378	0.68	0.26 ± 0.02	-0.42 ± 0.02
Drosophila	8114	38909	39900 ± 1100	9.59	9.8 ± 0.3	0.12	1.05	0.0026	128	0.52	0.272 ± 0.017	-0.248 ± 0.017
Reactome	5973	145778	148000 ± 7000	48.81	50 ± 2	0.65	2.09	0.0068	376	0.78	0.60 ± 0.03	-0.18 ± 0.03
Moreno propro	1458	1948	1600 ± 100	2.67	2.15 ± 0.14	0.14	1.29	0.0501	432	0.82	0.61 ± 0.02	-0.21 ± 0.02
Moreno names	1707	9059	9200 ± 800	10.61	10.8 ± 0.9	0.71	3.12	0.0396	85	0.51	0.64 ± 0.08	$+0.13 \pm 0.08$
Moreno health	2539	10455	9600 ± 1200	8.24	7.6 ± 0.9	0.15	1.30	0.0165	15	0.29	0.55 ± 0.03	$+0.26 \pm 0.03$
US airport 2010	1858	17214	18900 ± 1100	18.53	20.4 ± 1.2	0.63	1.36	0.0073	297	0.35	0.44 ± 0.03	$+0.09 \pm 0.03$
US airport 500	500	2980	2900 ± 200	11.92	11.6 ± 0.9	0.73	1.73	0.0225	13	0.35	0.44 ± 0.04	$+0.09 \pm 0.04$
Openflights	2905	15645	15200 ± 800	10.77	10.5 ± 0.6	0.59	1.85	0.0271	58	0.64	0.69 ± 0.03	$+0.05 \pm 0.03$

TABLE I: Parameters of real networks inferred by Mercator and their number of optimal partitions and modularity by the Louvain method. Synthetic networks were generated by the $\mathbb{S}^1/\mathbb{H}^2$ model based on the real networks, the modularity is the one associated to equally sized angular partitions into 15 communities. The values are an average of five simulations.

networks could be widely designed in imitation of them. Synthetic networks in the $\mathbb{S}^1/\mathbb{H}^2$ model were successfully generated with a heterogeneous degree distribution that was tuned by the parameter γ in the range between 2.5 and 3.5 and with a high level of clustering that was controlled by the parameter β over 1.

The community structure of the $\mathbb{S}^1/\mathbb{H}^2$ model was put under scrutiny, while expounding the literature about hyperbolic models.

Angular community structure is rooted in the similarity between nodes, not their importance in the network; this is why a method like the G-CGM that merely takes into account the angular positions of the nodes did not find angular communities in these random networks. However, equally sized angular partitions have been found to be associated with a high modularity in this model, indicating the presence of topological community structure despite the distribution uniformly at random of the similarity coordinate.

In general, a high modularity alone does not sufficiently declare that a network possesses community structure. An example of this are networks in the ER model, which are proven to have no community structure but still can be found partitions of them with a high modularity associated. However, for hyperbolic models there has been found evidence that supports their topological community structure. The PSO model is known to be maximally modular in the thermodynamic limit with equally sized angular partitions and the $\mathbb{H}^2$ model has been theoretically proven to converge to one in probability in the thermodynamic limit and high clustering regime.

This work is focused on the study of the $\mathbb{S}^1/\mathbb{H}^2$ model. In agreement with previous results, there are equally sized angular communities in these networks associated with high values of the modularity. The dependecies of the modularity were investigated and, as the level of clustering and the size of the networks were increased, the modularity of each partition converged to its maximum value. There is an optimal number of communities depending on the system size for which the modularity is maximal. To guarantee that these values of the modularity were indeed the maximum attainable for each case, the performance of the latter method was compared with modularity optimization methods, more concretely with the Louvain method, finding no improvement over the previous results. This further supports the fact that the $\mathbb{S}^1/\mathbb{H}^2$ model possesses topological community structure that is fruit of its hierarchical hyperbolic nature.

Comparing synthetic networks by this model with their real counterparts, it was corroborated that, even if this topological community structure is not always enough to replicate the observations in real networks, it constitutes is a starting point to understand the emergence of community structure in nature.

APPENDIX

A. Mathematical developments

1. *Sampling the hidden degrees of the $\mathbb{S}^1$ model*

Given by Equation 11 the desired distribution for the hidden degrees of the nodes in this model, any set of numbers distributed uniformly at random can be transformed into a set that obeys the desired distribution if the following process can be resolved.

First, from a normalized probability distribution function, it can be obtained the cumulative distribution function, which is the probability of a node having at most hidden degree κ:

$$F(\kappa) = 1 - (\frac{\kappa_0}{\kappa})^{\gamma-1} \tag{17}$$

Then the complementary cumulative degree distribution corresponds to the probability of a node having a hidden degree larger than κ.

$$1 - F(\kappa) = (\frac{\kappa_0}{\kappa})^{\gamma-1} \tag{18}$$

From the cumulative distribution function, it is to be obtained, if possible, its inverse. In this case:

$$F^{-1}(y) = \kappa_0(1-y)^{\frac{-1}{\gamma-1}}$$

which is well defined for $0 < y < 1$.

Applying this function to numbers distributed uniformly at random between 0 and 1 serves as a transformation of their distribution to the one desired.

B. Code availability

The programs developed for the generation and analysis of networks for this work are available on request to the author.

ACKNOWLEDGMENTS

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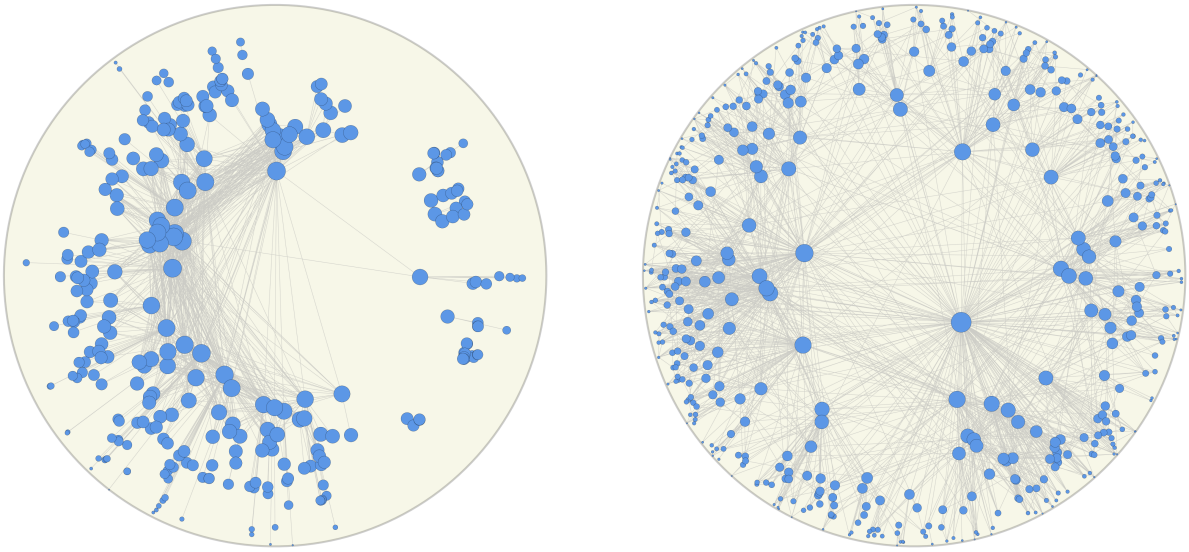

(a) Real network embedded in the geometric space by Mercator.

(b) Synthetic network generated by the $\mathbb{S}^1/\mathbb{H}^2$ model with $E = 2596$ and $\langle k \rangle = 10.38$.

FIG. 9: Network of the 500 busiest airports in the US in the $\mathbb{H}^2$ geometry. Comparison between the real network and the synthetic network that replicates its global features.

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