Modal Interval Probability: Application to Bonus-Malus Systems

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Classical intervals have been a very useful tool to analyze uncertain and imprecise mod-els, in spite of operative and interpretative shortcomings. The recent introduction of modal intervals helps to overcome those limitations. In this paper, we apply modal in-tervals to the field of probability, including properties and axioms that form a theoretical framework applied to the Markovian analysis of Bonus-Malus systems in car insurance. We assume that the number of claims is a Poisson distribution and in order to include uncertainty in the model, the claim frequency is defined as a modal interval; therefore, the transition probabilities are modal interval probabilities. Finally, the model is ex-emplified through application to two different types of Bonus-Malus systems, and the attainment of uncertain long-run premiums expressed as modal intervals.

Keywords: Modal intervals; interval probability; Markov chain; Bonus-Malus.

1. Introduction

Uncertainty and inaccuracy have been treated from different points of view. Thus, for instance, fuzzy sets and fuzzy numbers, introduced by Zadeh,[25] as well as rough sets[15,24,27] and intervals are widely used tools in this area. Since the beginning of the implementation of classical intervals,[9,11] they have been used in many fields. However, it was not until the recent introduction of modal intervals[5,16] that their use has taken an important step forward in two ways. On the one hand, their use has advanced in the operative sense, as modal intervals considerably extend the possibilities of solving problems whose resolution is not possible using classical intervals; and on the other hand, the advance has been in an interpretative sense, as the rigidity of the interpretation of interval calculus is overcome by the use of modal intervals.

One of the fields we consider in this paper is the application of modal intervals to probability.

Since the mid twentieth century, many theories of imprecise probabilities have been developed. Dempster[2] and Shafer[17] introduced the theory of evidence that characterizes uncertainties as discrete masses of probability associated with a set of power values. Another important contribution is the theory of possibility put forward by Zadeh[26] and Dubois and Prade[3] which represents uncertainties using necessity–possibility pairs. In 2000, Weichselberger[22] introduced an interval probability incorporating classical intervals into the probability value fulfilling the Kolmogorov properties. Wang[19,20] used generalized intervals, which conceptually lead to modal intervals, to study interval probability. Nowadays we can find some other studies about interval probability (see Refs. 7, 23).

Meanwhile, the Bonus-Malus system (BMS) has become the most common form of rating in car insurance. In this system, the bonus class of the policyholder is updated from one year to the next as a function of the current class and the number of claims made during the year: penalties are applied in the case of claims having been made, and premium discounts are achieved by claim-free policyholders.

The basic idea of this system is that the policyholder moves through different levels of premium according to the number of claims occurs. That is to say, bonuses are attained by not filing claims (a reward for careful drivers), and a malus is applied if many claims have been made. In a generic BMS, a basic premium is fixed depending on rating factors and the type of coverage. This basic premium is paid by drivers without a known claim history. Then, a Bonus-Malus scale is defined. This scale includes percentages of the basic premium to be paid after the occurrence of k claims. Each percentage of the basic premium is included in a state, and transitions between the states are fixed by the number of claims made.

Practically, penalties and discounts are included in the Bonus-Malus scale that is defined using a finite number of levels with their own premiums, and the transition rules between them that depend on the claims made. Assuming that the number of claims per year is independent, the process can be considered to be a Markov chain, and therefore, the transition probabilities must be obtained from the hypothesis used to model the claim experience. For more details concerning the BMS, see Refs. 1, 4, 6, 10, 13, 14.

In this paper, we propose an approach based on the uncertainty of the claim frequency. If the number of claims is a Poisson distribution with parameter λ, in order to introduce unpredictability, the claim frequency, λ, is not a certain value and can be considered as an interval. Niemiec[12] assumes a classical interval to describe the value of λ. Further, the transition probabilities become generalized imprecise probabilities.[20,21] We extend this point of view under the framework of modal intervals.

The paper is organized as follows: in Section 2, we present the main concepts involved in interval calculus; while Section 3 includes the required definition of modal interval probability. In Section 4, the BMS is modified through the introduction of uncertainty into the claim frequency, which means that we must use the concepts previously considered. Also in Section 4, we develop a numerical application.

2. Interval Calculus

Given two real numbers a and b such that $a \leq b$, classical interval theory defines the interval $[a, b]$ of the real line as:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

The set of classical intervals[11] is represented by $I(\mathbb{R})$.

Given a real continuous function, f, its extension on the set of classical intervals is represented by F and if $[a_1, b_1], \ldots, [a_n, b_n]$ are classical intervals, then F is defined as:

$$F([a_1, b_1], \ldots, [a_n, b_n]) = \{f(x_1, \ldots, x_n) \mid x_i \in [a_i, b_i]\}. \tag{1}$$

As f is a continuous function, the set:

$$Y = \{f(x_1, \ldots, x_n) \mid x_i \in [a_i, b_i]\},$$

is also a classical interval and thus:

$$Y = \left[\min_{x_i \in [a_i, b_i]} f(x_1, \ldots, x_n), \max_{x_i \in [a_i, b_i]} f(x_1, \ldots, x_n)\right].$$

The calculus $Y = F([a_1, b_1], \ldots, [a_n, b_n])$ can be semantically interpreted in two ways:

$$\begin{gathered}(\forall x_1 \in [a_1, b_1]) \ldots (\forall x_n \in [a_n, b_n]) (\exists y \in Y) \text{ such that } y = f(x_1, \ldots, x_n), \\ \text{or also:} \\ (\forall y \in Y)(\exists x_1 \in [a_1, b_1]) \ldots (\exists x_n \in [a_n, b_n]) \text{ such that } y = f(x_1, \ldots, x_n).\end{gathered} \tag{2}$$

Classical intervals present some shortcomings; one of which comes from the fact that the solution $[x, y]$ of the interval equation $[a, b] + [x, y] = [c, d]$ must satisfy $a + x = c$ and $b + y = d$. This solution exists on $I(\mathbb{R})$ only under the condition $b - a \leq d - c$, but even when the interval equation has a solution, this solution cannot be obtained by any syntactic interval computation on $I(\mathbb{R})$.

This problem is overcome by the use of modal intervals.

A modal interval is a pair consisting of a classical interval and a quantifier:

$$A = ([a, b], Q) \text{ where } [a, b] \in I(\mathbb{R}) \text{ and } Q \in \{\exists, \forall\}.$$

The set of modal intervals is represented by $I^*(\mathbb{R})$.

A modal interval A is said to be proper if $A = ([a, b], \exists)$; while A is said to be improper if $A = ([a, b], \forall)$.

The canonical notation of modal intervals expresses an improper interval $A = ([a, b], \forall)$ as $A = [b, a]$, and a proper interval $A = ([a, b], \exists)$ as $A = [a, b]$, identifying a proper interval with the associated classical interval.

Thus, using this canonical notation, the improper interval $([1,3], \forall)$ is represented by $[3,1]$, while the proper interval $([2,7], \exists)$ corresponds to the classical interval $[2,7]$.

We will say that the modality of an interval is proper if its quantifier is existential, while the modality will be improper if its quantifier is universal.

Within the set of modal intervals, we must emphasize the dual operator defined by

$$dual\,([a,b], Q) = \begin{cases} ([a,b], \forall) & \text{if } Q = \exists\,, \\ ([a,b], \exists) & \text{if } Q = \forall\,. \end{cases} \tag{3}$$

Thus, using the canonical notation of an interval, $dual\,([a,b]) = [b,a]$.

If f is a real continuous function, its extension on the modal intervals $X = ([a_1,b_1], \ldots, [a_n,b_n])$ is represented by $F(X)$ and it is defined similarly as the extension defined on the classical intervals (1) as

$$F(X) = \left[\min_{x_p \in X_P} \max_{x_i \in X_i} f(x_p, x_i)\,,\ \max_{x_p \in X_P} \min_{x_i \in X_i} f(x_p, x_i)\right]\,,$$

where X_p are the proper intervals that are components of X, and X_i are the improper intervals that are components of X.

Note that when the function f is reduced to one of the elementary arithmetic operations $\{+, -, \times, /\}$, the previous extension defines the calculation of these arithmetic operators on the set of the modal intervals. For more details, see Ref. 16.

The distributive property of the product with respect to the sum, which is not fulfilled in calculations with classical intervals, is still not fulfilled in the set of modal intervals and is reduced to a subdistributive property. In this, if A, B and C are modal intervals, then $A \times (B + C) \subseteq A \times B + A \times C$; although if B and C both belong to some zones[16,18] in the interval plane defined by the modal interval A, then the distributive law is fulfilled.

Using modal intervals, some of the limitations inherent to the set of classical intervals are overcome. Thus, the solution of the interval equation $[a,b] + [x,y] = [c,d]$ is $[x,y] = [c,d] - dual\,([a,b])$. In the same way, the solution of the interval equation $[a,b] \times [x,y] = [c,d]$ is $[x,y] = [c,d]\,/dual\,([a,b])$, bearing in mind that $0 \notin [a,b]$.

The interval equation $[3,6] + [x,y] = [4,8]$ has a solution in the set of classical intervals, but to evaluate it we need the dual operator of modal intervals described above (3). The solution of this equation is $[x,y] = [4,8] - dual\,([3,6])$ and hence $[x,y] = [1,2]$. Meanwhile, the equation $[3,6] + [x,y] = [5,7]$ has no solution in the set of classical intervals, but in contrast it does have a solution in the set of modal intervals. The solution is $[x,y] = [5,7] - dual\,([3,6])$, that is: $[x,y] = [2,1]$, which corresponds to an improper interval.

Another important contribution made by modal intervals is the improvement they offer of semantic interpretation in interval calculations [16, Theorem 3.3.1]. Thus, if f is a real continuous function and we consider the modal intervals

$X = ([a_1, b_1], \ldots, [a_n, b_n])$, the semantic interpretation of the calculus

$$Y = F(X) = \left[\min_{x_p \in X_P} \max_{x_i \in X_i} f(x_p, x_i), \max_{x_p \in X_P} \min_{x_i \in X_i} f(x_p, x_i) \right],$$

is:

$$(\forall x_p \in X_p)(\exists y \in Y)(\exists x_i \in X_i) \text{ such that } y = f(x_p, x_i) \quad \text{if } Y \text{ is proper,}$$
$$(\forall x_p \in X_p)(\forall y \in Y)(\exists x_i \in X_i) \text{ such that } y = f(x_p, x_i) \text{ if } Y \text{ is improper,}$$

where X_p are the proper interval components of X; and X_i are the improper interval components of X.

3. Interval Probability

Definition 1. Let Ω be a sample-space and let $\mathcal{A}$ be a σ-algebra of random events in Ω. A modal interval probability is a function $P : \mathcal{A} \to I^*(\mathbb{R})$ that satisfies the following axioms

- $\forall A \in \mathcal{A} \;\; P(A) \in I^*(\mathbb{R})$,
- $\forall A \in \mathcal{A} \;\; P(A) \geq [0, 0]$,
- $P(\Omega) = [1, 1]$,
- For any countable mutually disjoint events, $A_i \cap A_j = \varnothing$ for all $i \neq j$, then $P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$.

Proposition 1. *The calculus established in Axiom 4 of Definition 1 is semantically interpreted in the following way.*

If $\forall i \in \{1, \ldots, k\}$ $P(A_i)$ *are proper intervals and* $\forall j \in \{k+1, \ldots, n\}$ $P(A_j)$ *are improper intervals, then:*

- *If* $P\left(\bigcup_{i=1}^n A_i\right)$ *is a proper interval, the interpretation of the calculus* $P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$ *is:*
  $\{\forall p_i \in P(A_i)\}_{i=1,\ldots,k}$ $\exists p \in P\left(\bigcup_{i=1}^n A_i\right)$ $\{\exists p_j \in P(A_j)\}_{j=k+1,\ldots,n}$ *such that* $p = \sum_{i=1}^k p_i + \sum_{j=k+1}^n p_j$.
- *If* $P\left(\bigcup_{i=1}^n A_i\right)$ *is an improper interval, the interpretation of the calculus* $P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$ *is:*
  $\{\forall p_i \in P(A_i)\}_{i=1,\ldots,k}$ $\forall p \in P\left(\bigcup_{i=1}^n A_i\right)$ $\{\exists p_j \in P(A_j)\}_{j=k+1,\ldots,n}$ *such that* $p = \sum_{i=1}^k p_i + \sum_{j=k+1}^n p_j$.

Proof. As a consequence of the application of the $*$-semantic interval theorem [16, Theorem 3.3.1] to the calculus $P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i)$. □

Example 1. Let Ω be a sample-space, $\mathcal{A}$ a σ-algebra of random events in Ω, and P a modal interval probability function: $P : \mathcal{A} \to I^* (\mathbb{R})$.

If $A_1, A_2, A_3, A_4 \in \mathcal{A}$ are mutually disjoint events with probability values $P(A_1) = [0.1, 0.15]$, $P(A_2) = [0.5, 0.1]$, $P(A_3) = [0.1, 0.3]$ and $P(A_4) = [0.2, 0.1]$ then, as $P\left(\cup_{i=1}^{4} A_i\right) = \sum_{i=1}^{4} P(A_i)$ it will be $P\left(\cup_{i=1}^{4} A_i\right) = [0.9, 0.65]$.

As $P(A_1)$ and $P(A_3)$ are proper intervals, $P(A_2), P(A_4)$ and $P\left(\cup_{i=1}^{4} A_i\right)$ are improper intervals, it follows the semantic interpretation:

$(\forall p_1 \in [0.1, 0.15]) (\forall p_3 \in [0.1, 0.3]) (\forall p \in [0.65, 0.9]) (\exists p_2 \in [0.1, 0.5]) (\exists p_4 \in [0.1, 0.2])$ such that $p = p_1 + p_2 + p_3 + p_4$.

The following properties can be deduced from the axioms established in the above Definition 1, as their proofs are simple deductions from those axioms.

(1) $P(\varnothing) = [0, 0]$,

(2) If $A \subseteq B$ then $P(A) \leq P(B)$,

(3) $\forall A \in \mathcal{A} \;\; P(A) \leq [1, 1]$.

One of the most important properties in the calculation of probabilities relates the probability of an event A with the probability of its complementary event, A^c. When working with modal interval probabilities, this relationship is established using the dual operator in the following way:

(4) $\forall A \in \mathcal{A} \;\; P(A^c) = [1, 1] - dual\,(P(A))$.

Note that from the last Property 4, it is clear that the probability of A and the probability of A^c have distinct modalities. That is, $P(A)$ is a proper interval if and only if $P(A^c)$ is an improper interval.

This transcends classical interval probability theory, since using classical intervals it is impossible to obtain this result coherently.

The above Property 4, which uses the dual operator, is equivalent to:

$$\forall A \in \mathcal{A} \;\; P(A) + P(A^c) = [1, 1].$$

However, the equality $P(A^c) = [1, 1] - P(A)$ is obviously false, as explained in Section 2 above.

Example 2. Let Ω be a sample-space, $\mathcal{A}$ a σ-algebra of random events in Ω, and $P : \mathcal{A} \to I^* (\mathbb{R})$ a modal interval probability function. Let $A_1, A_2, A_3 \in \mathcal{A}$ be mutually disjoint events such that $A_1 \cup A_2 \cup A_3 = \Omega$.

If the interval values of the probabilities $P(A_1)$ and $P(A_2)$ are known: $P(A_1) = [0.1, 0.3]$ and $P(A_2) = [0.5, 0.6]$ then, as $P(A_1) + P(A_2) + P(A_3) = [1, 1]$, it will be $P(A_3) = 1 - dual\,(P(A_1) + P(A_2))$, that is $P(A_3) = [0.4, 0.1]$, which is semantically interpreted as:

$(\forall p_1 \in [0.1, 0.3]) (\forall p_2 \in [0.5, 0.6]) (\exists p_3 \in [0.1, 0.4])$ such that $p_1 + p_2 + p_3 = 1$.

For a given event $A \in \mathcal{A}$, we define a conditional probability measure $P(\cdot|A)$ such that $P(B|A)$ is the conditional probability of B given A for any event $B \subseteq \Omega$.

Definition 2. If P is a modal interval probability, the modal interval conditional probability measure $P(\cdot|A)$ for an event $A \subseteq \Omega$ with $P(A) > [0,0]$ is defined by:

$$P(B|A) = \frac{P(B \cap A)}{dual\,(P\,(A))},$$

for any event $B \subseteq \Omega$.

Using the interval modal operation, we can show that the modal interval conditional probability $P(\cdot|A)$ is a probability measure, as

- $\forall B \in \mathcal{A} \quad P(B|A) \in I^*\,(\mathbb{R})$,
- $\forall B \in \mathcal{A} \quad P(B|A) \geq [0,0]$,
- $P\,(\Omega|A) = \frac{P(\Omega \cap A)}{dual(P(A))} = \frac{P(A)}{dual(P(A))} = [1,1]$,
- For any countable mutually disjoint events, $B_i \cap B_j = \varnothing$ for all $i \neq j$, applying Definition 2:

$$P\left(\bigcup_{i=1}^{n} B_i|A\right) = \frac{P\,((\bigcup_{i=1}^{n} B_i) \cap A)}{dual\,(P\,(A))}$$

and using the laws of the algebra of sets, it follows that $(\bigcup_{i=1}^{n} B_i) \cap A = \bigcup_{i=1}^{n} (B_i \cap A)$ and consequently:

$$P\left(\bigcup_{i=1}^{n} B_i|A\right) = \frac{P\,(\bigcup_{i=1}^{n} (B_i \cap A))}{dual\,(P\,(A))}\,.$$

Thus, applying Axiom 4 in Definition 1, we have:

$$P\left(\bigcup_{i=1}^{n} B_i|A\right) = \frac{\sum_{i=1}^{n} P\,(B_i \cap A)}{dual\,(P\,(A))}$$

which is

$$P\left(\bigcup_{i=1}^{n} B_i|A\right) = \frac{1}{dual\,(P\,(A))} \cdot \left(\sum_{i=1}^{n} P\,(B_i \cap A)\right)\,.$$

Finally, we can apply the distributive law in modal intervals as all the modal intervals are positive and hence they belong to the same distributive zone.[16,18] Thus, $P\,(\bigcup_{i=1}^{n} B_i|A) = \sum_{i=1}^{n} \frac{P(B_i \cap A)}{dual(P(A))}$ and it will be $P\,(\bigcup_{i=1}^{n} B_i|A) = \sum_{i=1}^{n} P\,(B_i|A)\,.$

From Definition 2, the equality $P(B \cap A) = P(B|A) \cdot P\,(A)$ is obviously fulfilled.

Given P a modal interval probability, a finite sequence of random variables $\{X_n\}_{n \in \{1,\ldots,k\}}$ verifies the Markov property if, for all $n \in \{1,\ldots,k\}$:

$$P\,(X_{n+1} = i_{n+1}|X_n = i_n, \ldots, X_1 = i_1) = P\,(X_{n+1} = i_{n+1}|X_n = i_n) = \mathbf{p}_{i_n i_{n+1}}$$

where $\mathbf{p}_{i_n i_{n+1}} = \left[\underline{p_{i_n i_{n+1}}}, \overline{p_{i_n i_{n+1}}}\right] \in I^*\,(\mathbb{R})\,.$

We can group the elements $\left[\underline{p_{i_n i_{n+1}}}, \overline{p_{i_n i_{n+1}}}\right]$ in a modal interval matrix $\mathbf{P}$ called the modal transition matrix:

$$\mathbf{P} = \begin{pmatrix} \left[\underline{p_{11}}, \overline{p_{11}}\right] & \cdots & \left[\underline{p_{1r}}, \overline{p_{1r}}\right] \\ \vdots & \ddots & \vdots \\ \left[\underline{p_{r1}}, \overline{p_{r1}}\right] & \cdots & \left[\underline{p_{rr}}, \overline{p_{rr}}\right] \end{pmatrix},$$

fulfilling for all $i \in \{1, \ldots, r\}$:

$$\sum_{j=1}^{r} \left[\underline{p_{ij}}, \overline{p_{ij}}\right] = [1, 1] \ .$$

A modal interval Markov chain is a pair $(\mathbf{P}, \mathbf{L}(0))$, where $\mathbf{P}$ is the transition modal matrix and $\mathbf{L}(0)$ the initial distribution vector.

4. Bonus-Malus System using Modal Intervals

In this section, we apply the foregoing concepts and definitions concerning modal interval probabilities to a BMS. Following Lemaire[10] , policyholders are divided into a finite number of classes, such that each policy stays in one class through each period, usually a year. The premium depends only on the class the policyholder belongs to and that class for a given period is fixed by the class in the previous period and the number of claims made during that period.

A BMS is analyzed considering the theory of Markov chains. It is necessary to define a scale system with r classes and a premium scale $\mathbf{b} = (b_1, \ldots, b_r)$, where b_i is the premium paid by policyholders in class i. The transition probabilities between classes are included in a transition matrix defined as $\mathbf{P} = [p_{ij}]$, where p_{ij} is the probability of moving for class i to class j, that is to say, the probability of a certain number of claims, defined in the Bonus-Malus scale system.

Let us define $\mathbf{L}(0) = (l_1^0, \ldots, l_r^0)$, $\sum_{i=1}^{r} l_i^0 = 1$ as the initial distribution of policyholders in each class. We can obtain the distribution of policyholders in each class in period $t = 1, 2, ...$, i.e. $\mathbf{L}(t) = (l_1^t, \ldots, l_r^t)$, as $\mathbf{L}(t) = \mathbf{L}(t-1) \cdot \mathbf{P}$ or, applying a recursive method, $\mathbf{L}(t) = \mathbf{L}(0) \cdot \mathbf{P}^t$. The steady-state distribution is the vector $\mathbf{L}(\infty) = (l_1^\infty, \ldots, l_r^\infty)$, which satisfies $\mathbf{L}(\infty) = \mathbf{L}(\infty) \cdot \mathbf{P}$, that is to say, the steady-state distribution does not change over time. Using the previous definitions, it is possible to obtain the average premium paid by a policy holder in the steady state, denoted as Π. The long-run premium is defined as $\Pi = \sum_{i=1}^{r} b_i l_i^\infty$.

In this paper, the claim frequency is uncertain. The origin of this uncertainty can be fluctuations, a lack of information that introduces errors in models, or numerical or measurement errors. To introduce this hypothesis into the model, we assume that the parameter that determines the number of claims is not a certain value, it is a modal interval, so we modify the analysis using modal interval probabilities.

Therefore, from now on, the transition probabilities are modal interval probabilities, $\mathbf{p_{ij}} = [\underline{p_{ij}}, \overline{p_{ij}}]$, where $\mathbf{P}$ is the modal transition matrix:

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_{11} = [\underline{p_{11}}, \overline{p_{11}}] & \cdots & \mathbf{p}_{1r} = [\underline{p_{1r}}, \overline{p_{1r}}] \\ \vdots & \ddots & \vdots \\ \mathbf{p}_{r1} = [\underline{p_{r1}}, \overline{p_{r1}}] & \cdots & \mathbf{p}_{rr} = [\underline{p_{rr}}, \overline{p_{rr}}] \end{pmatrix},$$

where for all $i \in \{1, \ldots, r\}$, $\sum_{j=1}^{r} \mathbf{p}_{ij} = [1, 1]$.

The distribution of policyholders in each class at time t is the vector of modal intervals $\mathbf{L}(t) = (\mathbf{L}_1^t, \ldots, \mathbf{L}_r^t)$, and $\sum_{j=1}^{r} \mathbf{L}_j^t = [1, 1]$. $\mathbf{L}(t)$ can be obtained as

$$\mathbf{L}(t) = \mathbf{L}(0) \cdot \mathbf{P} = (\mathbf{L}_1^0, \ldots, \mathbf{L}_r^0) \cdot \begin{pmatrix} \mathbf{p}_{11} & \cdots & \mathbf{p}_{1r} \\ \vdots & \ddots & \vdots \\ \mathbf{p}_{r1} & \cdots & \mathbf{p}_{rr} \end{pmatrix},$$

where the steady-state distribution $\mathbf{L}(\infty) = (\mathbf{L}_1^\infty, \ldots, \mathbf{L}_r^\infty)$ is

$$\mathbf{L}(\infty) = \mathbf{L}(\infty) \cdot \mathbf{P} = (\mathbf{L}_1^\infty, \ldots, \mathbf{L}_r^\infty) \cdot \begin{pmatrix} \mathbf{p}_{11} & \cdots & \mathbf{p}_{1r} \\ \vdots & \ddots & \vdots \\ \mathbf{p}_{r1} & \cdots & \mathbf{p}_{rr} \end{pmatrix},$$

and where $\mathbf{\Pi} = \sum_{i=1}^{r} b_i \mathbf{L}_i^\infty = [\underline{\Pi}, \overline{\Pi}]$ is the mean asymptotic premium: a modal interval.

Below, we present some applications to different forms of BMS.

From now on, we assume that the number of claims, N, a discrete random variable, follows a Poisson distribution, $N \sim Po(\lambda)$; and due to the uncertainty, we assume that the claim frequency, $\boldsymbol{\lambda}$, is a modal interval: $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]$. The density function required to obtain the probabilities of occurrences of k claims assuming $N \sim Po(\lambda)$ is

$$P(N = k) = \frac{\boldsymbol{\lambda}^k}{k!} e^{-dual(\boldsymbol{\lambda})},$$

where $P(N = k)$ is a modal interval probability. Assuming $N \sim Po(\boldsymbol{\lambda})$, we obtain

$$P(N = 0) = \frac{[\lambda_1, \lambda_2]^0}{0!} e^{-dual[\lambda_1, \lambda_2]} = [e^{-\lambda_1}, e^{-\lambda_2}],$$

$$P(N = 1) = \frac{[\lambda_1, \lambda_2]^1}{1!} e^{-dual[\lambda_1, \lambda_2]} = [\lambda_1 e^{-\lambda_1}, \lambda_2 e^{-\lambda_2}],$$

$$P(N = 2) = \frac{[\lambda_1, \lambda_2]^2}{2!} e^{-dual[\lambda_1, \lambda_2]} = [\lambda_1^2 2e^{-\lambda_1}, \lambda_2^2 2e^{-\lambda_2}],$$

$$P(N \geq 1) = [1, 1] - dual(P(N = 0)) = [1, 1] - dual[e^{-\lambda_1}, e^{-\lambda_2}] = [1 - e^{-\lambda_1}, 1 - e^{-\lambda_2}],$$

$$P(N \leq 1) = P(N=0) + P(N=1) = [e^{-\lambda_1}(1+\lambda_1), e^{-\lambda_2}(1+\lambda_2)],$$

$$P(N \geq 2) = [1,1] - dual[P(N \leq 1)] = [1,1] - dual[e^{-\lambda_1}(1+\lambda_1), e^{-\lambda_2}(1+\lambda_2)]$$
$$= [1 - e^{-\lambda_1}(1+\lambda_1), 1 - e^{-\lambda_2}(1+\lambda_2)].$$

The previous probabilities are the modal transition probabilities included in the modal transition matrix $\mathbf{P}$. We will now focus our attention on two examples of BMS.

Example 3. [8, Cf. Example 2] We assume that the policyholder pays a premium c if one or more claims occurred in the preceding two-year period. In other situations the driver pays $a \leq c$. Three classes are defined, with the modal transition matrix being

$$\mathbf{P} = \begin{pmatrix} P(N \geq 1) & P(N=0) & 0 \\ P(N \geq 1) & 0 & P(N=0) \\ P(N \geq 1) & 0 & P(N=0) \end{pmatrix}.$$

If $N \sim Po([\lambda_1, \lambda_2])$,

$$\mathbf{P} = \begin{pmatrix} \left[1-e^{-\lambda_1}, 1-e^{-\lambda_2}\right] & \left[e^{-\lambda_1}, e^{-\lambda_2}\right] & 0 \\ \left[1-e^{-\lambda_1}, 1-e^{-\lambda_2}\right] & 0 & \left[e^{-\lambda_1}, e^{-\lambda_2}\right] \\ \left[1-e^{-\lambda_1}, 1-e^{-\lambda_2}\right] & 0 & \left[e^{-\lambda_1}, e^{-\lambda_2}\right] \end{pmatrix}.$$

We can obtain $\mathbf{L}(1)$, knowing that $\mathbf{L}(1) = \mathbf{L}(0) \cdot \mathbf{P}$. Assuming $\mathbf{L}(0) = (\mathbf{L}_1^0, \mathbf{L}_2^0, \mathbf{L}_3^0)$, $\sum_{i=1}^{3} \mathbf{L}_i^0 = [1,1]$,

$$\mathbf{L}(1) = \left(\left[1-e^{-\lambda_1}, 1-e^{-\lambda_2}\right], \left[\mathbf{L}_1^0 e^{-\lambda_1}, \mathbf{L}_1^0 e^{-\lambda_2}\right], \left[\left(\mathbf{L}_2^0 + \mathbf{L}_3^0\right) e^{-\lambda_1}, \left(\mathbf{L}_2^0 + \mathbf{L}_3^0\right) e^{-\lambda_2}\right]\right)$$

and considering $\mathbf{L}(t) = \mathbf{L}(t-1) \cdot \mathbf{P}$, we obtain

$$\mathbf{L}(2) = \mathbf{L}(3) = \ldots = \mathbf{L}(\infty) = \left(\mathbf{L}_1^\infty, \mathbf{L}_2^\infty, \mathbf{L}_3^\infty\right),$$

where:

$$\mathbf{L}_1^\infty = \left[1-e^{-\lambda_1}, 1-e^{-\lambda_2}\right],$$
$$\mathbf{L}_2^\infty = \left[e^{-\lambda_1}\left(1-e^{-\lambda_1}\right), e^{-\lambda_2}\left(1-e^{-\lambda_2}\right)\right],$$
$$\mathbf{L}_3^\infty = \left[e^{-2\lambda_1}, e^{-2\lambda_2}\right].$$

If $\lambda_1 < \lambda_2$, $\mathbf{L}_1^\infty$ is a proper interval and $\mathbf{L}_3^\infty$ is an improper interval. As for $\mathbf{L}_2^\infty$, it is a proper interval if $\lambda_1 < \lambda_2 < \ln(2)$, and an improper interval if $\ln(2) < \lambda_1 < \lambda_2$. If $\lambda_1 < \ln(2) < \lambda_2$, then $\mathbf{L}_2^\infty$ can be a proper or an improper interval.

Let us now calculate the long-run premium. Knowing that if the policyholder is in the first or second class the premium to be paid is c, and if the policyholder is in the third class, a is to be paid, $\mathbf{\Pi}$ is

$$\mathbf{\Pi} = c \cdot \mathbf{L}_1^\infty + c \cdot \mathbf{L}_2^\infty + a \cdot \mathbf{L}_3^\infty = [c + (a-c)e^{-2\lambda_1}, c + (a-c)e^{-2\lambda_2}].$$

Let us assume that $\lambda_1 < \lambda_2$, then $\mathbf{\Pi}$ and $\mathbf{L}_1^\infty$ are proper intervals and $\mathbf{L}_3^\infty$ is an improper interval. However, depending on the values of λ_1 and λ_2, $\mathbf{L}_2^\infty$ can be proper or improper.

Case 1: $\ln(2) < \lambda_1 < \lambda_2$. $\mathbf{L}_1^\infty$ is a proper interval, and $\mathbf{L}_2^\infty$ and $\mathbf{L}_3^\infty$ are improper intervals, with $\mathbf{\Pi}$ also being a proper interval. The $*$-semantic interpretation is:

$$(\forall l_1^\infty \in \mathbf{L}_1^\infty)\,(\exists \pi \in \mathbf{\Pi})\,(\exists l_2^\infty \in \mathbf{L}_2^\infty)\,(\exists l_3^\infty \in \mathbf{L}_3^\infty) \text{ such that } \pi = c\cdot l_1^\infty + c\cdot l_2^\infty + a\cdot l_3^\infty\,.$$

Case 2: $\lambda_1 < \lambda_2 < \ln(2)$. $\mathbf{L}_1^\infty$ and $\mathbf{L}_2^\infty$ are proper intervals, $\mathbf{L}_3^\infty$ is an improper interval and the result $\mathbf{\Pi}$ is also a proper interval, the $*$-semantic interpretation is:

$$(\forall l_1^\infty \in \mathbf{L}_1^\infty)\,(\forall l_2^\infty \in \mathbf{L}_2^\infty)\,(\exists \pi \in \mathbf{\Pi})\,(\exists l_3^\infty \in \mathbf{L}_3^\infty) \text{ such that } \pi = c\cdot l_1^\infty + c\cdot l_2^\infty + a\cdot l_3^\infty.$$

If $\lambda_1 > \lambda_2$, $\mathbf{L}_1^\infty$ is an improper interval and $\mathbf{L}_3^\infty$ is a proper interval. As $\mathbf{L}_2^\infty$ is a proper interval if $\ln(2) < \lambda_2 < \lambda_1$, and an improper interval if $\lambda_2 < \lambda_1 < \ln(2)$. If $\lambda_2 < \ln(2) < \lambda_1$, then $\mathbf{L}_2^\infty$ can be a proper or an improper interval.

Let us assume $[\lambda_1, \lambda_2] = [0.038, 0.042]$, then the transition modal matrix is:

$$\mathbf{P} = \begin{pmatrix} [0.0372, 0.0411] & [0.9627, 0.9588] & 0 \\ [0.0372, 0.0411] & 0 & [0.9627, 0.9588] \\ [0.0372, 0.0411] & 0 & [0.9627, 0.9588] \end{pmatrix},$$

and assuming that the initial distribution is $\mathbf{L}(0) = (0.5, 0.3, 0.2)$ then, we can obtain $\mathbf{L}(1)$ and the steady-state distribution $\mathbf{L}(\infty)$

$$\mathbf{L}(1) = ([0.0372, 0.0411], [0.4813, 0.4794], [0.4813, 0.4794])\,,$$
$$\mathbf{L}(2) = \ldots = \mathbf{L}(\infty) = ([0.0372, 0.0411], [0.0358, 0.0394], [0.9268, 0.9194])\,.$$

If $c = 100$ and $a = 90$, the long-run premium is:

$$\mathbf{\Pi} = [90.7318, 90.8056].$$

As $\mathbf{L}_1^\infty = [0.0372, 0.0411]$, $\mathbf{L}_2^\infty = [0.0358, 0.0394]$ are proper intervals, $\mathbf{L}_3^\infty = [0.9268, 0.9149]$ is an improper interval, and the result $\Pi = [90.7318, 90.8056]$ is also a proper interval, the $*$-semantic interpretation is:

$$(\forall l_1^\infty \in [0.0372, 0.0411])\,(\forall l_2^\infty \in [0.0358, 0.0394])\,(\exists \pi \in [90.7318, 90.8056])$$
$$(\exists l_3^\infty \in [0.9149, 0.9268]) \text{ such that } \pi = 100\cdot l_1^\infty + 100\cdot l_2^\infty + 90\cdot l_3^\infty$$

That is, for any percentage of policies in the first class comprised between 3.72% and 4.11%, and for any percentage of policies in the second class comprised between 3.58% and 3.94%, there exists a value for the long-run premium comprised between 90.73 euros and 90.80 euros, which ensures the existence of a percentage of policies in the third class of between 91.49% and 92.68%.

Example 4. Irish Bonus-Malus system. This system is defined by the following table:

		Class after k claims		
i	b_i	0	1	2^+
6	100	5	6	6
5	90	4	6	6
4	80	3	6	6
3	70	2	5	6
2	60	1	4	6
1	50	1	3	6

The modal transition matrix is:

$$\mathbf{P} = \begin{pmatrix} P(N=0) & 0 & P(N=1) & 0 & 0 & P(N\geq 2) \\ P(N=0) & 0 & 0 & P(N=1) & 0 & P(N\geq 2) \\ 0 & P(N=0) & 0 & 0 & P(N=1) & P(N\geq 2) \\ 0 & 0 & P(N=0) & 0 & 0 & P(N\geq 1) \\ 0 & 0 & 0 & P(N=0) & 0 & P(N\geq 1) \\ 0 & 0 & 0 & 0 & P(N=0) & P(N\geq 1) \end{pmatrix},$$

If $N \sim Po([\lambda_1, \lambda_2])$, with $[\lambda_1, \lambda_2] = [0.038, 0.042]$, then:

$$P(N=0) = [0.9627, 0.9588] \ ,$$
$$P(N=1) = [0.0365, 0.04027] \ ,$$
$$P(N\geq 1) = [0.0372, 0.0411] \ ,$$
$$P(N\geq 2) = [0.0007, 0.0008] \ .$$

If $\mathbf{L}(0) = (0.1, 0.2, 0.3, 0.18, 0.12, 0.1)$:

$$\mathbf{L}(1) = ([0.2888, 0.2876], [0.2888, 0.2876], [0.1769, 0.1766], [0.1228, 0.1231], [0.1072, 0.1079], [0.0153, 0.0169]),$$

$$\mathbf{L}(2) = ([0.5560, 0.5516], [0.1703, 0.1693], [0.1288, 0.1296], [0.1138, 0.1151], [0.0212, 0.0233], [0.0096, 0.0108]),$$

$$\vdots$$

with the steady-state distribution being:

$$\mathbf{L}(27) = \mathbf{L}(28) = \ldots = \mathbf{L}(\infty) = (\mathbf{L}_1^\infty, \mathbf{L}_2^\infty, \mathbf{L}_3^\infty, \mathbf{L}_4^\infty, \mathbf{L}_5^\infty, \mathbf{L}_6^\infty),$$

where $\mathbf{L}_1^\infty = [0.9206, 0.9118]$, $\mathbf{L}_2^\infty = [0.0356, 0.0391]$, $\mathbf{L}_3^\infty = [0.0370, 0.0407]$, $\mathbf{L}_4^\infty = [0.0034, 0.0042]$, $\mathbf{L}_5^\infty = [0.0022, 0.0027]$ and $\mathbf{L}_6^\infty = [0.0009, 0.0011]$.

Then, the calculus

$$\mathbf{\Pi} = 50 \cdot \mathbf{L}_1^\infty + 60 \cdot \mathbf{L}_2^\infty + 70 \cdot \mathbf{L}_3^\infty + 80 \cdot \mathbf{L}_4^\infty + 90 \cdot \mathbf{L}_5^\infty + 100 \cdot \mathbf{L}_6^\infty ,$$

leads to the long-run premium, $\mathbf{\Pi}$:

$$\mathbf{\Pi} = [51.316, 51.474] .$$

As $\mathbf{L}_1^\infty$ is an improper interval, and the other values $\mathbf{L}_2^\infty, \mathbf{L}_3^\infty, \mathbf{L}_4^\infty, \mathbf{L}_5^\infty, \mathbf{L}_6^\infty$ and $\mathbf{\Pi}$ are proper intervals, the semantic interpretation for this result is:

$$(\forall l_2^\infty \in \mathbf{L}_2^\infty)\,(\forall l_3^\infty \in \mathbf{L}_3^\infty)\,(\forall l_4^\infty \in \mathbf{L}_4^\infty)\,(\forall l_5^\infty \in \mathbf{L}_5^\infty)\,(\forall l_6^\infty \in \mathbf{L}_6^\infty)\,(\exists \pi \in \mathbf{\Pi})\,(\exists l_1^\infty \in \mathbf{L}_1^\infty)$$
$$\text{such that } \pi = 50 \cdot l_1^\infty + 60 \cdot l_2^\infty + 70 \cdot l_3^\infty + 80 \cdot l_4^\infty + 90 \cdot l_5^\infty + 100 \cdot l_6^\infty .$$

That is, for any percentage of policies in the second class (l_2^∞) between 3.56 and 3.91%, for any percentage of policies in the third class (l_3^∞) between 3.7% and 4.07%, for any percentage of policies in the fourth class (l_4^∞) between 0.34% and 0.402%, for any percentage of policies in the fifth class (l_5^∞) between 0.22% and 0.27%, and for any percentage of policies in the sixth (l_6^∞) class between 0.09% and 0.11%, there exists a value of the long-term premium ($\mathbf{\Pi}$) between 51.316 and 51.474, that ensures the existence of policies in the first class between 91.18% and 92.06%.

Let us emphasize that in the last two examples we have used an improper interval, as in Example 3, $\mathbf{L}_3^\infty$ is an improper interval and in Example 4, $\mathbf{L}_1^\infty$ is also an improper interval.

5. Conclusions

In this paper, we have treated the concept of modal interval probability as an extension of classical interval probability. We study the probabilistic axioms under the point of view of modal interval analysis.

Using the modal interval probability, we can solve some problems inherent to the classical interval probability. These problems are solved both from the point of view of the calculation and from the interpretative point of view. We have taken advantage of these modal probabilities to include uncertainty inside the probabilities that define the Markovian analysis of Bonus-Malus systems.

In the examples provided in the text, it becomes clear not only the consistency of the modal probabilistic calculation but also the correct semantic interpretations that do not exist in the classic interval probabilistic calculation.

For future research on the structure here presented, it is interesting to consider the following two lines: The first one, deepen the use of modal intervals in the field of interval probability. The second one, consider different fields that have already been discussed in the classical interval probability theory. In this setting, we will be able to provide the correct semantic interpretation of the calculations.

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