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Life settlement pricing with fuzzy parameters

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HIGHLIGHTS

- Key parameters to fit the price of life settlements are mortality multiplier and internal rate of return.
- These parameters are modelled as triangular fuzzy numbers.
- The fuzzy price of the life settlement is then obtained.
- This fuzzy price is approximated by three different approaches.
- Several properties of these triangular approximations are evaluated.

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ABSTRACT

Existing literature asserts that the growth of life settlement (LS) markets, where they exist, is hampered by limited policyholder participation and suggests that to foster this growth appropriate pricing of LS transactions is crucial. The pricing of LSs relies on quantifying two key variables: the insured's mortality multiplier and the internal rate of return (IRR). However, the available information on these parameters is often scarce and vague. To address this issue, this article proposes a novel framework that models these variables using triangular fuzzy numbers (TFNs). This modelling approach aligns with how mortality multiplier and IRR data are typically provided in insurance markets and has the advantage of offering a natural interpretation for practitioners. When both the mortality multiplier and the IRR are represented as TFNs, the resulting LS price becomes a FN that no longer retains the triangular shape. Therefore, the paper introduces three alternative triangular approximations to simplify computations and enhance interpretation of the price. Additionally, six criteria are proposed to evaluate the effectiveness of each approximation method. These criteria go beyond the typical approach of assessing the approximation quality to the FN itself. They also consider the usability and comprehensibility for financial analysts with no prior knowledge of FNs. In summary, the framework presented in this paper represents a significant advancement in LS pricing. By incorporating TFNs, offering several triangular approximations and proposing goodness criteria of them, it addresses the challenges posed by limited and vague data, while also considering the practical needs of industry practitioners.

1. Introduction

A life settlement (LS) refers to an arrangement wherein a policyholder sells their life insurance contract to an investor, rather than accepting the surrender value proposed by the insurer [60]. By entering into this agreement, the buyer obtains the right to receive the predetermined death benefit upon the insured's demise, as agreed upon between the insurer and policyholder, while also assuming the responsibility of paying the outstanding premiums. Several factors can influence a policyholder's decision to pursue an LS, including the perceived simplicity of the transaction, social influence, and perceived ethical considerations [14]. However, the primary motivation for opting for an LS, as opposed to claiming the surrender value, stems from the fact that the former generally offers a larger sum compared to the latter. This discrepancy arises from the pricing of LSs, which takes into account the actual mortality probabilities of the insured, exceeding the standard probabilities for their age [89], whereas the surrender value paid by the insurer is calculated using those standard probabilities.

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Fig. 1 illustrates the various parties and processes involved in an LS transaction. On the selling side, the policyholder, with the assistance of a broker, explores the secondary market to find the most favourable trade for their policy. Ideally, the policyholder's insurance broker would have informed them about the availability of LS agreements as an alternative to surrendering the policy during the initial purchase, i.e., in the primary market. On the buying side, the intermediaries, known as LS providers, purchase the insurance policies on behalf of institutional investors such as LS funds, specialized alternative investment funds, and insurance companies. These entities then engage in trading activities within the so-called tertiary markets [28]. The tertiary market involves the buying and selling of life insurance policies among investors. It is common for institutional investors that have acquired policies to securitize a portion of their portfolio. Therefore, negotiations in this segment encompass not only the policies themselves but also the trading of securities backed by the expected death benefits from the underlying policies.

The intricate nature of LSs implies the intervention of various parties that facilitate their development with the maximum guarantees for all parties. Among these parties, a crucial role is played by medical underwriters (MUs), also known as life expectancy (LE) providers, who offer a fundamental service. MUs issue LE certificates that include a mortality multiplier, which is used to determine the insured's LE (as discussed in Subsection 3.1). Both LS brokers and LS providers acquire these estimates to price the policies during the trading process. MUs derive mortality multipliers by combining information from various experts, including doctors, medical databases, life actuaries, and other relevant sources. Other parties include tracking agents who monitor insureds' health and report deaths, document depositaries for agreement-related paperwork, and independent legal and financial advisors who provide impartial guidance.

Recent research conducted by Andrés Sánchez and González Vila Puchades [13] highlights the significant development observed in the LS market, particularly in the United States, which is the most advanced market in this domain. Projections suggest that the market volume of LSs is expected to exceed $60 billion by 2025, with an estimated annual gross market potential of $212 billion between 2019 and 2028. Despite these promising figures, the growth of LSs is impeded by limited participation from policyholders. To address this issue, the authors conduct a study and conclude that performance expectancy plays a crucial role in fostering a positive attitude towards LSs. In other words, for LSs to be appealing to policyholders, they must offer significantly higher amounts than the alternative of surrendering the policy. Moreover, the perception of usefulness is enhanced by factors such as expected easiness, which relates to agile and efficient procedures, as well as social influence. The acceptance of LSs by policyholders relies on the endorsement of financial advisors and insurance brokers. It is imperative for these professionals to perceive LSs as a convenient option for their clients, as their support and positive perception can significantly influence policyholders' consideration of LS transactions.

Both performance expectancy and social influence emphasize the importance of proper pricing in LSs. By accurately pricing LSs, policyholders are more likely to receive attractive offers that surpass the value of policy surrender. This creates a financial incentive for policyholders to engage in LS transactions. Moreover, appropriate pricing contributes to the perception of usefulness. When LSs are priced accurately, policyholders develop confidence in the fairness and transparency of the process. They can trust that the offered amounts accurately reflect the true value of their policies. This perception of usefulness encourages policyholders to view LSs as a viable option for selling their life insurance policies. Additionally, adequate pricing is linked to social influence. The perception of financial advisors and insurance brokers plays a vital role in shaping policyholders' utilization of LSs. When these professionals view LSs as a convenient and advantageous option for their clients, they are more inclined to recommend and support such transactions. Accurate pricing ensures that financial advisors and insurance brokers can confidently present LSs as a valuable solution, thereby influencing policyholders to consider this type of agreement. In summary, it is evident that accurate valuation of LSs is pivotal in attracting policyholders, creating a perception of usefulness, and gaining support from financial advisors and insurance brokers. Through appropriate pricing, LSs can offer higher amounts, serve as a practical alternative to policy surrender, and generate positive social influence, all of which contribute to the growth of the LS market.

However, it is important to acknowledge that pricing LSs poses significant challenges. The pricing of LSs relies on the quantification of two essential variables: the insured's mortality multiplier (or its associated LE) and the internal rate of return (IRR). As mentioned earlier, the estimation of the mortality multiplier is carried out by MUs. On the other hand, the IRR is determined by a valuation agent. Unlike the mortality multiplier, which depends on factors such as the insured's physical and mental health, lifestyle, etc., the IRR is influenced by the inherent risk associated with LSs. This risk level is often higher than that of stocks [38]. The increased risk is primarily due to several uncertainty factors affecting LSs, such as longevity risk, inaccuracies in LE estimation, the potential for fraud within the underlying life insurance policy, limited liquidity in LS markets, insurer default risk, and other related variables [24,27]. In summary, the pricing of LSs relies on the quantification of the insured's LE, provided through mortality multipliers by LE providers, as well as the determination of the IRR, which reflects the heightened risk associated with LSs compared to other assets, stemming from various uncertainty sources impacting the market.

The actuarial literature presents various approaches to pricing LSs, including deterministic, probabilistic, stochastic, and fuzzy methods [1,30,57,72,89] The deterministic approach views the life insurance policy

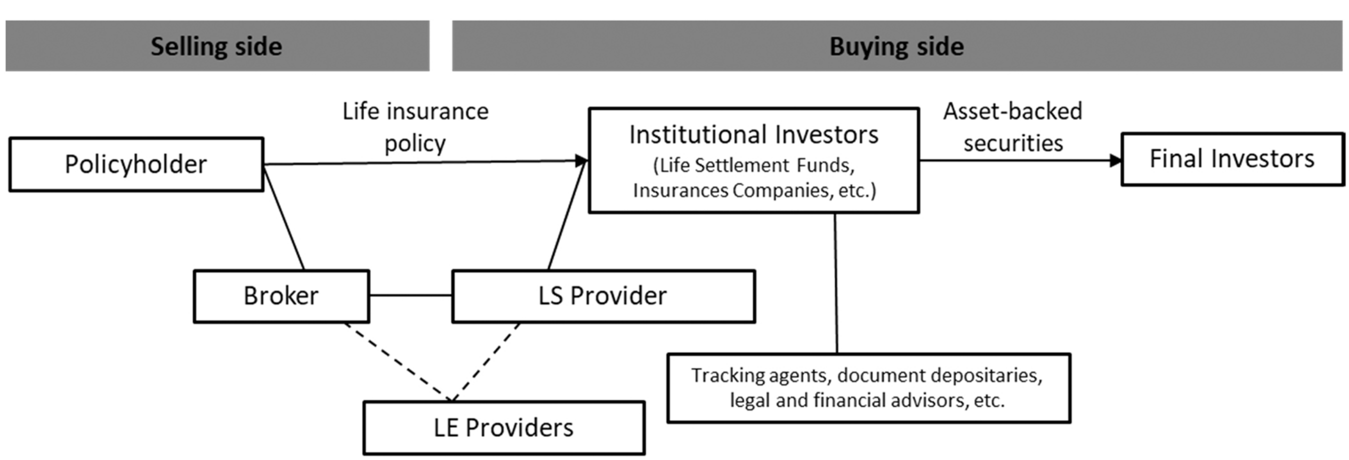


Fig. 1. Parties and processes in an LS transaction.
Source: Own elaboration from Braun et al. [28].

as a financial transaction with a term exactly equal to the insured's LE. It evaluates only the premiums and death benefit occurring over this term. Under the probabilistic approach, all premiums and the death benefit are considered, taking into account their probabilities of occurrence. In the stochastic approach, the insured's actual mortality probabilities obtained from their mortality multiplier are utilized. A random variable called "Whole years of life for the insured" is considered and a related random variable, "Stochastic price of the LS," is derived. While the mathematical expectation of this random variable is equal to the price obtained using the probabilistic approach, the stochastic approach provides a comprehensive understanding of all possible price values and their respective probabilities. The works by Zollars et al. [89], Lubovich et al. [57], and Brockett et al. [30] assume that both the mortality multiplier (if used) and the IRR are real (crisp) values. However, in the recent approach by Aalaei [1], it is argued that the intrinsic nature of the IRR cannot be modelled as a crisp value. Instead, the author proposes pricing LSs with the use of a fuzzy IRR.

LSs are traded on over-the-counter markets, which are characterized by lower levels of transparency compared to centralized markets. Moreover, each life insurance policy considered for trading in an LS transaction possesses unique characteristics that pose challenges when applying standard actuarial procedures for assessment. The available information for pricing a specific life policy often involves multiple sources of uncertainty [56,85,86]. This includes the presence of false or irrelevant information, the absence of crucial data, the difficulty of incorporating individual psychological profiles and resilience capacities, and uncertainties related to medical advances in medicines, treatments, and lifestyle choices. Following Viertl and Hareter [78], stochastic variability can be effectively addressed using probability theory, while fuzzy sets can account for other relevant kind of uncertainty, such as imprecision, as is the case in variables relevant to LSs. In the context of insurance, Shapiro [69] emphasizes the applicability of FNs, a specific type of fuzzy sets, in modelling parameters that require subjective actuarial judgment. FNs offer a means to represent and work with linguistic expressions, such as "approximately 6%" or "relatively high," which are often encountered in insurance settings.

Building upon the insights of Aalaei [1], this paper acknowledges the challenges associated with accurately estimating the IRR and recognizes the suitability of FNs for modelling the imprecision inherent in this variable. However, in contrast to the author's perspective, this study extends the application of FNs to also represent the insured's mortality multiplier, considering it as an imprecise value. For instance, if different MUs provide mortality multiplier estimates of 8 and 9, it can be inferred that the mortality multiplier for the insured lies in the range of "approximately 8.5". By representing both the mortality multiplier and the IRR as FNs, the pricing of LSs is performed using Fuzzy Financial Mathematics (FFM).

The foundation of FFM can be attributed to the pioneering works of Kaufmann [49], Buckley [31], and Li Calzi [54] in the late 1980 s and early 1990 s. Their contributions laid the groundwork for incorporating fuzzy sets into financial modelling and analysis. Since then, FFM has been further developed and applied in various research studies, including more recent papers like Voloshyn et al. [80], which build upon that foundation to explore a fuzzy mathematical model of financial risks. FFM has also been applied in actuarial settings, both in non-life and life insurance domains. In the context of non-life insurance, researchers such as Cummins and Derrig [35], Derrig and Ostaszewski [36], Andrés-Sánchez and Terceño [16], Andrés-Sánchez [6], Heberle and Thomas [43], and Villacorta et al., [79] utilize FFM to address uncertainty in non-life insurance pricing or reserving. Meanwhile, Mircea and Covrig [61] and Ungureanu and Vernic [77] develop cash-flow models with fuzzy parameters to assess the probability of ruin for insurers. On the other hand, Romaniuk [68] analyses the behaviour of a non-life insurer's portfolio, which consists of two layers: a classical risk process and a catastrophe bond. In the field of life insurance, Lemaire [53], Ostaszewski [66], Betzuen et al. [25], and Andrés-Sánchez and González-Vila [8–10] incorporate fuzzy interest rates within a classical actuarial mathematics framework to price endowments, annuities, and death benefits. Additionally, researchers like Anzilli [17], Anzilli and Facchinetti [18], and Anzilli et al. [19] adjust the value of life equity-linked contracts using the option-pricing framework of Brennan and Schwartz [29] and possibilistic parameters. Moreover, Andrés-Sánchez et al. [15] propose the incorporation of fuzzy information in pricing substandard annuities, while Omrani et al. [65] use fuzzy data envelopment analysis to evaluate the performance of Iranian insurance firms. Likewise, Andrade and Valencia [5] propose a fuzzy random survival forest to model lapse rates in a life insurance portfolio containing imprecise or incomplete data.

The body of literature outlined in above paragraph serves as the foundation for the present study, which aims to develop a novel framework for pricing LSs by modelling the uncertainty associated with information on mortality multipliers and IRRs using FNs. The proposed framework encompasses all the necessary steps for pricing an LS, including the calibration of uncertain parameters using limited available market data, calculation of a fuzzy price for the LS using FFM, and simplification of the fuzzy price to facilitate subsequent calculations and assessments. This approach may offer a valuable solution for pricing LSs, enabling parties involved in the LS industry to obtain comprehensive information regarding the value of LSs across various scenarios. Therefore, it facilitates the determination of appropriate LS prices and, consequently, contributes to fostering the desired growth of life insurance secondary markets, as previously discussed. By providing a robust methodology that accounts for uncertainty and imprecision, this paper intends to contribute to the advancement and development of the LS market.

The main contributions of this paper can be summarized in three points:

(1) By incorporating FNs to model the uncertainty surrounding both the mortality multiplier and the IRR, this study effectively addresses a literature gap that previously neglected to consider the mortality multiplier as a fuzzy variable. The inclusion of FNs for both variables allows for a more accurate representation of their imprecision, providing a comprehensive framework.
(2) A new methodology, inspired by Cheng [34], is presented. This methodology addresses the challenge of fitting fuzzy mortality multipliers and fuzzy IRRs using triangular FNs. This approach is aligned with the information available in the context of potential LS trades. The mortality multiplier is typically derived from a set of point estimates provided by two or three medical underwriters (MUs), as outlined by Xu [85]. On the other hand, the IRR is often obtained from crisp IRR values observed in recent comparable trades, as reported by AA Partners Ltd [2].
(3) The evaluation of LS prices using triangular parameters does not yield a triangular FN as the result. However, employing a triangular approximation can be highly valuable for fitting the final price of LSs. Triangular approximations strike a balance between simplifying computations and interpretations without oversimplifying the underlying complexity [39]. Therefore, we calculate three alternative approximations commonly used in fuzzy financial contexts. To evaluate the quality of these triangular approximations, we propose six novel criteria that focus on closeness, better adherence to the most reliable values, unbiasedness, preservation, interpretability, and ease of calculation. It is worth noting that the criteria we introduce go beyond the typical approach of assessing the approximation quality to the FN itself (see, e.g., [22,41,48]). Our criteria also consider the usability and comprehensibility for financial analysts and practitioners without prior knowledge of FNs. We believe this is a crucial aspect since, as mentioned, the LS-related insurance industry requires new approaches to improve pricing techniques and foster further development.

The remaining sections of this work are organized as follows. Section 2 provides an overview of some definitions of FNs used in this study. In Section 3, we present the concepts of conventional life actuarial mathematics necessary for LS pricing. The proposed methodology for pricing LSs is presented in Section 4. In Section 5, we show a practical application of the developed framework through a numerical example and provide a discussion of the obtained results. Finally, Section 6 summarizes the key conclusions of this paper and offers potential directions for future extensions.

2. Fuzzy numbers

2.1. Basics on fuzzy numbers

Fuzzy Sets, which were first introduced by Zadeh [87] in his seminal paper, form the basis of Fuzzy Set Theory (FST). A *fuzzy set* $\widetilde{A}$ can be defined as $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | x \in X\}$, where $\mu_{\widetilde{A}}$ is known as the membership function and is a mapping from the referential set X to the interval $[0,1]$, i.e. $\mu_{\widetilde{A}} : X \to [0,1]$. Therefore, 0 indicates non-membership in the fuzzy set $\widetilde{A}$ and 1 indicates absolute membership. Alternatively, a fuzzy set can be represented by its α-level sets or α-cuts. An α-cut is a crisp set A_α where $A_\alpha = \{x \in X | \mu_{\widetilde{A}}(x) \geq \alpha\}$, $\forall \alpha \in (0,1]$, with the convention that $A_{\alpha=0}$ is the closure of the support of $\widetilde{A}$, i.e. all $x \in X$ that $\mu_{\widetilde{A}}(x) \geq 0$. Within FST, a key concept for representing uncertain quantities is that of a FN. A *fuzzy number* (FN) is a fuzzy set $\widetilde{A}$ defined on the reference set $\mathbb{R}$. It is normal, i.e., its membership function $\mu_{\widetilde{A}}(x)$ attains the value 1 for some $x \in X$, convex, i.e., all its α-cuts are convex, and its α-cuts are closed and bounded intervals. Therefore, they can be represented as confidence intervals $A_\alpha = \left[\underline{A}(\alpha), \overline{A}(\alpha)\right]$, where $\underline{A}(\alpha)$ ($\overline{A}(\alpha)$) are continuously increasing (decreasing) functions of the membership level $\alpha \in [0,1]$. In this paper, to model fuzzy information, we use *triangular fuzzy numbers* (TFNs), which are denoted as $\widetilde{A} = (A, l_A, r_A)$. The membership function and its corresponding α-cuts are:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - A + l_A}{l_A} & A - l_A < x \leq A \\ \dfrac{A + r_A - x}{r_A} & A < x \leq A + r_A \\ 0 & \text{otherwise} \end{cases} \tag{1a}$$

$$A_\alpha = \left[\underline{A}(\alpha), \overline{A}(\alpha)\right] = [A - l_A(1-\alpha), A + r_A(1-\alpha)] \tag{1b}$$

The value A is the core (mode or centre) of the TFN $\widetilde{A}$ and can be understood as its most reliable value, i.e., $\mu_{\widetilde{A}}(A) = 1$. The parameters $l_A, r_A \geq 0$ are the spreads or radiuses and indicate the variability of $\widetilde{A}$ with respect to its core A. The interval $A_0 = \left[\underline{A}(0), \overline{A}(0)\right] = [A - l_A, A + r_A]$ is the support of $\widetilde{A}$. It contains all possible values of the variable $\widetilde{A}$ that are constrained by the lower and upper bounds $A - l_A$ and $A + r_A$, respectively.

TFNs are used in countless practical applications, including actuarial ones because they are easy to handle arithmetically and well-suited to quantify imprecise predictions. For example, an actuary's statement "I expect that over the next 2 years the yield on 10-year government bonds will be 1.5% and deviations no greater than 0.05%" can be quantified in a very natural way as (0.015, 0.005, 0.005). Furthermore, when the information about a variable is vague and imprecise, and no information about how to model this kind of uncertainty is available, the principle of parsimony leads to representing that information in the simplest way possible. The linear form of TFNs meets that requirement. On the other hand, this kind of fuzziness is very intuitive and can be interpreted naturally by practitioners.

We acknowledge that the use of TFNs may have limitations when dealing with imprecise information that involves multiple sources of vagueness. Various extensions of fuzzy sets have been proposed in the literature to address these limitations. For example, intuitionistic fuzzy sets introduced by Atanassov [21] allow for simultaneous consideration of the membership degree and non-membership degree of each element. Turksen [76] proposes interval-valued fuzzy sets, where the membership degree is represented by a closed subinterval of the unit interval. Hesitant fuzzy sets, suggested by Torra [75], are motivated by the difficulty that can arise when establishing the membership degree of an element and this difficulty is because there are some possible values that make to hesitate about which would be the right one. In summary, these modelling approaches provide a more nuanced and richer representation of variables subject to uncertainty compared to FNs. However, they require the estimation of a larger number of parameters. In contrast, the use of FNs, particularly linear ones, offers a simpler way to represent information [88]. This simplicity can be highly desirable in situations where information is scarce [48] or where human subjective judgment is necessary [46], such as in the case of the secondary market for life insurance, which operates as an over-the-counter market. Consequently, it is not surprising that a significant portion of the literature on fuzzy applications to actuarial and insurance topics incorporates the modelling of uncertainty using FNs with linear shapes.

The *expected interval* of the FN $\widetilde{A}$, $EI\left(\widetilde{A}\right)$, is a representative real interval of that FN:

$$EI\left(\widetilde{A}\right) = \left[\int_0^1 \underline{A}(\alpha)d\alpha, \int_0^1 \overline{A}(\alpha)d\alpha\right] \tag{2a}$$

and for $\widetilde{A} = (A, l_A, r_A)$:

$$EI\left(\widetilde{A}\right) = \left[A - \frac{l_A}{2}, A + \frac{r_A}{2}\right] \tag{2b}$$

Let $f(\cdot)$ be a continuous real valued function of n-real variables $x_j, j = 1, 2, \ldots, n$, and let $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n$ n FNs. The *extension principle* in Zadeh [87] allows us to define a FN $\widetilde{B}$ induced by the FNs $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n$ through $f(\cdot)$ as $\widetilde{B} = f\left(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\right)$. Although it is usually impossible to obtain the membership function of $\widetilde{B}$, it is often possible to obtain a closed expression for its α-cuts, B_α. If $f(\cdot)$ is increasing with respect to the first m variables, $m \leq n$, and decreases in the last $n - m$ variables, Buckley and Qu [33] demonstrate:

$$\begin{aligned} B_\alpha = \left[\underline{B}(\alpha), \overline{B}(\alpha)\right] = \Bigg[& f\left(\underline{A}_1(\alpha), \underline{A}_2(\alpha), \ldots, \underline{A}_m(\alpha), \overline{A}_{m+1}(\alpha), \overline{A}_{m+2}(\alpha), \ldots \overline{A}_n(\alpha)\right), \\ & f\left(\overline{A}_1(\alpha), \overline{A}_2(\alpha), \ldots \overline{A}_m(\alpha), \underline{A}_{m+1}(\alpha), \underline{A}_{m+2}(\alpha), \ldots, \underline{A}_n(\alpha)\right)\Bigg] \end{aligned} \tag{3a}$$

Eq. (3a) holds significant relevance in the present study because the functional relations that will be used to price LSs are continuous and exhibit an increasing or decreasing behaviour.

In the case in which $\widetilde{A}_j = (A_j, l_{A_j}, r_{A_j}), j = 1, 2, \ldots, n$, B_α in (3a) becomes:

$$\begin{aligned} \underline{B}(\alpha) = & f(A_1 - l_{A_1}(1-\alpha), \ldots, A_m - l_{A_m}(1-\alpha), A_{m+1} + r_{A_{m+1}}(1-\alpha), \ldots, \\ & A_n + r_{A_n}(1-\alpha)) \end{aligned} \tag{3b}$$

$$\begin{aligned} \overline{B}(\alpha) = & f(A_1 + r_{A_1}(1-\alpha), \ldots, A_m + r_{A_m}(1-\alpha), A_{m+1} - l_{A_{m+1}}(1-\alpha), \ldots, \\ & A_n - l_{A_n}(1-\alpha)) \end{aligned} \tag{3c}$$

and, generally, $\widetilde{B}$ does not maintain the triangular shape of $\widetilde{A}_j = (A_j, l_{A_j}, r_{A_j}), j = 1, 2, \ldots, n$..

2.2. Triangular approximations of nonlinear functions of triangular fuzzy numbers

FST has proven to be highly valuable in modelling imprecision, as evidenced by numerous scientific papers and practical applications. When evaluating nonlinear functions of TFNs, the result is not a TFN. However, once this result is obtained, the variable being modelled is described precisely, as its α-cuts yield exact numerical quantities. This observation gives rise to a dilemma concerning excessive precision in describing imprecise phenomena. While α-cuts provide detailed numeric information on membership grades, this level of detail can have both positive and negative implications. On one hand, analysts may benefit from abundant data, but on the other hand, an overly detailed description can overwhelm the general interpretation of the outcomes. Furthermore, complex shapes of membership functions can pose challenges in calculations. Simplified expressions, such as those related to linear expressions are often sufficient to capture vagueness and provide tractable tools for handling and interpretation (see, e.g., [51]). Consequently, various approximations of FNs have been developed. Among these, triangular approximations have garnered significant attention since they allow the resulted FN to be represented by only three parameters (i.e., the lower and upper bounds of the support, and the core). The objective of these approximations is to simultaneously reduce computational efforts and simplify interpretation, striking a balance between accuracy and practicality, and preserving the triangular shape of initial data. Grzegorzewski and Mrówka [39] highlight that a triangular approximation offers more richness compared to a simple transformation of an FN into a single crisp representative value. On the other hand, Grzegorzewski and Pasternak-Winiarska [41] argue that complex shapes of FNs can pose challenges in computations and intuitive result interpretation. Premature defuzzification can result in a significant loss of information, making it preferable to retain fuzzy information throughout the calculations as much as possible. The triangular approximation strikes a balance between computational simplification and interpretation, avoiding excessive simplification of fuzzy parameters. Furthermore, TFNs have an intuitive interpretation, making them potentially valuable in decision-making processes. This explains why there is a wide literature developing methods to approximate non-TFNs to triangular ones.

We present three possible triangular approximations to the FN $\widetilde{B}$ induced by f when evaluated with the TFNs $\widetilde{A}_j = (A_j, l_{A_j}, r_{A_j}), j = 1, 2, \ldots, n$. These approximations, that will be denoted as $\widetilde{B}^T = (B^T, l_{B^T}, r_{B^T})$, are the secant approach (SA), the gradient approach (GA) and the expected interval approach (EIA).

- Secant approach

This approach fits a triangular form to $\widetilde{B}$ by means of the secant lines linking the lower bounds of the 0-cut and the 1-cut (i.e., $\underline{B}(0)$ and $\underline{B}(1)$) and the upper bounds of these two α -cuts (i.e., $\overline{B}(0)$ and $\overline{B}(1)$). Therefore, it maintains the core and the support of the original FN $\widetilde{B}$. Despite being simple, this approach fits well not only several nonlinear arithmetical operations [49] but also financial functions [7,48,49] and actuarial calculations [15,43,79]. Thus:

$$B^T = \underline{B}(1) = \overline{B}(1) = f(A_1, \ldots, A_m, A_{m+1}, \ldots, A_n) \tag{4a}$$

$$l_{B^T} = B^T - f(A_1 - l_{A_1}, \ldots, A_m - l_{A_m}, A_{m+1} + r_{A_{m+1}}, \ldots, A_n + r_{A_n}) \tag{4b}$$

$$r_{B^T} = f(A_1 + r_{A_1}, \ldots, A_m + r_{A_m}, A_{m+1} - l_{A_{m+1}}, \ldots, A_n - l_{A_n}) - B^T \tag{4c}$$

- Gradient approach

The gradient approach is based on the approximation to non-linear operations with L-R FNs developed in Dubois and Prade [37]. It is built up from the first-order Taylor polynomial expansion from the 1-cut to any α-cut. So, let us approximate $\underline{B}(\alpha)$ in (3a) from $B(1)$ using the Taylor expansion to the first degree with $\underline{B}(\alpha) \approx \underline{B}(1) + \frac{d\underline{B}(1)}{d\alpha}(\alpha - 1)$, with analogous expression for $\overline{B}(\alpha)$. If we name the vector comprising the cores of $\widetilde{A}_j, j = 1, 2, \ldots, n, A_C = (A_1, A_2, \ldots, A_n)$, it is straightforward to see that $\widetilde{B}^T = (B^T, l_{B^T}, r_{B^T})$ where:

$$B^T = f(A_C) \tag{5a}$$

$$l_{B^T} = \sum_{j=1}^{m} \frac{\partial f(A_C)}{\partial x_j} l_{A_j} - \sum_{j=m+1}^{n} \frac{\partial f(A_C)}{\partial x_j} r_{A_j} \tag{5b}$$

$$r_{B^T} = \sum_{j=1}^{m} \frac{\partial f(A_C)}{\partial x_j} r_{A_j} - \sum_{j=m+1}^{n} \frac{\partial f(A_C)}{\partial x_j} l_{A_j} \tag{5c}$$

Andrés-Sánchez and Terceño [16] applied this triangular approximation to price life insurance and to calculate nonlife claim reserves. Andrés-Sánchez and González-Vila [10,11] also use this approach to simplify some calculations with fuzzy parameters in a life actuarial mathematics context. Often, this approximation allows obtaining the left and right spreads by using financial measures of volatility such as the interest rate duration in pricing life contingencies [10] or the so-called “the greeks” in option pricing [7].

- Expected interval approach

The expected interval approach is built up by adapting the trapezoidal approximation method by Grzegorzewski and Mrówka [39, 40] and Ban [22] to the particular case of a triangular shape. Following this methodology, the approximation $\widetilde{B}^T$ must preserve $EI\left(\widetilde{B}\right)$ and minimize its distance to $\widetilde{B}$ with the distance measure $d\left(\widetilde{B}, \widetilde{B}^T\right)$:

$$d\left(\widetilde{B}, \widetilde{B}^T\right) = \sqrt{\int_0^1 \left(\underline{B}(\alpha) - \underline{B^T}(\alpha)\right)^2 d\alpha + \int_0^1 \left(\overline{B}(\alpha) - \overline{B^T}(\alpha)\right)^2 d\alpha}$$

Then, by solving:

$$\text{Minimize} \int_0^1 \left(\underline{B}(\alpha) - B^T + l_{B^T}(1-\alpha)\right)^2 d\alpha + \int_0^1 \left(\overline{B}(\alpha) - B^T - r_{B^T}(1-\alpha)\right)^2 d\alpha$$

subject to

$$B^T - \frac{l_{B^T}}{2} = \int_0^1 \underline{B}(\alpha) d\alpha$$

$$B^T + \frac{r_{B^T}}{2} = \int_0^1 \overline{B}(\alpha) d\alpha$$

$$l_A, r_A \geq 0,$$

we obtain:

$$\text{If} \quad 2\int_0^1 \underline{B}(\alpha) d\alpha + \int_0^1 \overline{B}(\alpha) d\alpha \leq 3\left(\int_0^1 \alpha \underline{B}(\alpha) d\alpha + \int_0^1 \alpha \overline{B}(\alpha) d\alpha\right) \leq \int_0^1 \underline{B}(\alpha) d\alpha + 2\int_0^1 \overline{B}(\alpha) d\alpha :$$

$$B^T = -\int_0^1 \underline{B}(\alpha) d\alpha - \int_0^1 \overline{B}(\alpha) d\alpha + 3\int_0^1 \alpha \underline{B}(\alpha) d\alpha + 3\int_0^1 \alpha \overline{B}(\alpha) d\alpha \tag{6a}$$

$$l_{B^T} = -4\int_0^1 \underline{B}(\alpha)d\alpha - 2\int_0^1 \overline{B}(\alpha)d\alpha + 6\int_0^1 \alpha\underline{B}(\alpha)d\alpha + 6\int_0^1 \alpha\overline{B}(\alpha)d\alpha \quad (6b)$$

$$r_{B^T} = 2\int_0^1 \underline{B}(\alpha)d\alpha + 4\int_0^1 \overline{B}(\alpha)d\alpha - 6\int_0^1 \alpha\underline{B}(\alpha)d\alpha - 6\int_0^1 \alpha\overline{B}(\alpha)d\alpha \quad (6c)$$

If $2\int_0^1 \underline{B}(\alpha)d\alpha + \int_0^1 \overline{B}(\alpha)d\alpha > 3\left(\int_0^1 \alpha\underline{B}(\alpha)d\alpha + \int_0^1 \alpha\overline{B}(\alpha)d\alpha\right)$:

$$B^T = \int_0^1 \underline{B}(\alpha)d\alpha \quad (6d)$$

$$l_{B^T} = 0 \quad (6e)$$

$$r_{B^T} = 2\int_0^1 \overline{B}(\alpha)d\alpha - 2\int_0^1 \underline{B}(\alpha)d\alpha \quad (6\ f)$$

Finally, if $3\left(\int_0^1 \alpha\underline{B}(\alpha)d\alpha + \int_0^1 \alpha\overline{B}(\alpha)d\alpha\right) > \int_0^1 \underline{B}(\alpha)d\alpha + 2\int_0^1 \overline{B}(\alpha)d\alpha$:

$$B^T = \int_0^1 \overline{B}(\alpha)d\alpha \quad (6\ g)$$

$$l_{B^T} = 2\int_0^1 \overline{B}(\alpha)d\alpha - 2\int_0^1 \underline{B}(\alpha)d\alpha \quad (6\ h)$$

$$r_{B^T} = 0 \quad (6i)$$

3. Pricing life settlements with crisp parameters

3.1. Pricing life contingencies with nonstandard probabilities

The present value of unitary whole life insurance to be paid at the end of the death year is:

$$A_x = \sum_{t=1}^{\omega-x} (1+i)^{-t}\ {}_{t-1/}q_x \quad (7a)$$

where ${}_{t-1/}q_x$ stands for the mortality probability between years $t-1$ and t for a person aged x years, ω is the maximum attainable age in the base mortality table and i is the interest rate used to price the life insurance.

Since ${}_{t-1/}q_x = {}_{t-1}p_x{\cdot}q_{x+t-1}$, being ${}_{t-1}p_x$ the probability that a person aged x survives $t-1$ years and q_{x+t-1} the one-year mortality probability of person aged $x+t-1$, i.e., the probability that this person dies within the next year, Eq. (7a) can be rewritten as:

$$A_x = \sum_{t=1}^{\omega-x} (1+i)^{-t}\ {}_{t-1}p_x{\cdot}q_{x+t-1} \quad (7b)$$

On the other hand, the present value of a unitary post payable whole life annuity is:

$$a_x = \sum_{t=1}^{\omega-x} (1+i)^{-t}\ {}_tp_x \quad (7c)$$

The LE of a person aged x is:

$$e_x = \sum_{t=1}^{\omega-x} {}_tp_x = \sum_{t=1}^{\omega-x}\prod_{k=0}^{t-1}(1-q_{x+k}) \quad (8)$$

Eqs. (7a)-(7c) and (8) implicitly suppose that present values are obtained with standard probabilities from a mortality table base. To price life contracts in the case of nonstandard LEs, such as LSs or enhanced annuities, these equations are also used, but standard probabilities are replaced by those that best suit the insured's health status. To obtain these nonstandard probabilities, the actual one-year mortality probability, q_x^*, is adjusted from a linear transformation of the standard one-year mortality probability q_x or, alternatively, from a linear transformation of the force of mortality [64,67]. The most usual way to fit q_x^* is by means of the so-called mortality multiplier model in such a way that:

$$q_x^* = \min\{1, \beta{\cdot}q_x\} \quad (9a)$$

where β is the mortality multiplier. Therefore, if $\beta > 1(\langle 1)$, the insured's LE is below (above) the average. It is easy to check in (9a) that q_x^* is increasing with respect to β if $\beta{\cdot}q_x < 1$. Otherwise, it is neither increasing nor decreasing.

From q_x^*, the following nonstandard probabilities can be easily obtained:

$${}_tp_x^* = \prod_{j=0}^{t-1}\left(1-q_{x+j}^*\right) = \prod_{j=0}^{t-1}\left(1-\min\{1,\beta q_{x+j}\}\right) \quad (9b)$$

$${}_{t|}q_x^* = {}_tp_x^*{\cdot}q_{x+t}^* = \min\{1,\beta q_{x+t}\}{\cdot}\prod_{j=0}^{t-1}\left(1-\min\{1,\beta q_{x+j}\}\right) \quad (9c)$$

Similarly, the adjusted LE e_x^* is:

$$e_x^* = \sum_{t=1}^{\omega-x} {}_tp_x^* = \sum_{t=1}^{\omega-x}\prod_{j=0}^{t-1}\left(1-\min\{1,\beta q_{x+j}\}\right) \quad (9d)$$

3.2. Determining life settlement prices

Let a life insurance be considered in (7a) for a person aged x_0 with premiums to be paid as in (7c). At the beginning of the contract, the insurer states the equilibrium between the benefit and premiums by equalling:

$$\sum_{t=1}^{\omega-x_0} C_{x_0+t}{\cdot}(1+i_0)^{-t}\ {}_{t-1/}q_{x_0} = \sum_{t=1}^{\omega-x_0} P_{x_0+t}{\cdot}(1+i_0)^{-t}\ {}_tp_{x_0}$$

where C_{x_0+t} and P_{x_0+t} are, respectively, the premium and the benefit payable at age x_0+t.

The interest rate at the initial stage, i_0, is commonly referred to as the technical interest rate. Its value is determined based on the insurer's ability to generate profits from investing premiums in various assets. Typically, these assets have low-risk profiles, such as government bonds, making i_0 a representation of the risk-free rate. Furthermore, it is assumed that the insured's LE is of standard nature. Once the life insurance contract is in effect, the policyholder possesses an asset in the form of the policy, which can be priced in the LS market. However, the probabilities and interest rate used to price the life insurance policy at this stage differ from those at the initial stage. The probabilities are adjusted to account not only for the insured's age but also for their individual circumstances, including health conditions and lifestyle, as assessed by the MUs. The interest rate employed to evaluate the contract, i.e., the expected IRR of the LS, encompasses various risks specific to this type of transaction. These risks include longevity risk, biases in LE evaluation, the credit risk associated with the insurer, the potential for policy rescission due to fraudulent claims, and liquidity risk, as identified by Braun and Xu [27]. Consequently, the interest rate used in LS pricing is typically significantly higher than the insurer's technical interest rate. Once the adjusted probabilities and the IRR have been determined by the involved parties, they can proceed to estimate the economic value of the LS.

As indicated in Section 1, there are several approaches to assess LS: deterministic, probabilistic, stochastic, and fuzzy methods ([30,57,72,89]; Aalaei, [1]). The probabilistic method, which is the most widespread method [2], values all premiums and benefits using conventional actuarial mathematics as a basis, e.g., (7a) and (7c). Therefore, the probabilistic price of the LS, $PLS_x \geq 0$, is:

$$PLS_x = \sum_{t=1}^{\omega-x} C_{x+t} \cdot (1+i)^{-t} {}_{t-1/}q_x^* - \sum_{t=1}^{\omega-x} P_{x+t} \cdot (1+i)^{-t} {}_t p_x^* \tag{10a}$$

To simplify our analysis, we will suppose that pending cash flows are constant, as in Brockett et al. [30]. Therefore, from (7a) and (7c), and symbolizing by A_x^* and a_x^* these expressions in the case of nonstandard probabilities, we can write:

$$PLS_x = C \cdot A_x^* - P \cdot a_x^* \tag{10b}$$

From (7b) and (9b), (10b) turns into:

$$PLS_x = C \sum_{t=1}^{\omega-x} (1+i)^{-t} \prod_{j=0}^{t-2} \left(1 - q_{x+j-1}^*\right) \cdot q_{x+t-1}^* - P \times \sum_{t=1}^{\omega-x} (1+i)^{-t} \prod_{j=0}^{t-1} \left(1 - q_{x+j-1}^*\right) \tag{10c}$$

The duration [59] is a concept widely used by practitioners to measure the sensitivity of assets and liabilities with respect to the interest rate. This measure was extended to life insurance pricing by Li and Panjer [55] (see Remarks 1 and 2 in Appendix A). Furthermore, Andrés-Sánchez and González-Vila [12] derive measures similar to interest rate duration for longevity risk, i.e., linked to the mortality multiplier β. To do so, they consider the probabilistic method (see Remark 4).

Let us build up similar measures of interest rate and mortality multiplier durations exposed in Remarks 1 and 4 for the price of the LS (10a). We can define the interest rate duration of PLS_x with respect to fluctuations of IRR, $D(PLS_x)$ in a similar way to the duration GAP in an asset liability setting in Bierwag and Kaufman [26]:

$$D(PLS_x) = \frac{-D\left(A_x^*\right) C A_x^* + D\left(a_x^*\right) P a_x^*}{PLS_x} \tag{11a}$$

Similarly, we define a mortality multiplier duration of the LS price as $DM(PLS_x)$ [12]:

$$DM(PLS_x) = \frac{DM\left(A_x^*\right) C A_x - DM\left(a_x^*\right) P a_x^*}{PLS_x} \tag{11b}$$

Consequently:

$$\Delta PLS_x \approx \left[\frac{1}{1+i} D(PLS_x) \Delta i + DM(PLS_x) \Delta \beta\right] PLS_x \tag{11c}$$

Example 1. Table 1 shows the values $PLS_x, D(PLS_x)$ and $DM(PLS_x)$ of a life insurance that is priced in the LS market at age $x = 35, 45, 55, 65, 75, 85$ with IRR $i = 12\%$ and $\beta = 8$. That policy was initially traded for $x_0 = 35$ years, $i_0 = 2.836\%$ and $C = 100000$ monetary units (m.u.) and standard probabilities from the mortality table of the Spanish female population in HMD [44]. Therefore, $P = 1070.02$ m.u.

4. A fuzzy framework for life settlement pricing

4.1. Pricing life contingencies with nonstandard probabilities

In the study by Aalaei [1], the author explores the deterministic, probabilistic, and stochastic approaches to LS pricing and extends them to incorporate fuzzy IRR. We acknowledge the inherent challenges in accurately estimating the IRR and recognize the suitability of FNs for modelling the imprecision associated with this variable. However, in contrast to Aalaei [1], our study goes a step further by also representing the insured's mortality multiplier using TFNs, treating it as an imprecise value. By incorporating TFNs for both the mortality multiplier and the IRR, our approach captures and accounts for the inherent imprecision and uncertainty in these variables within the LS pricing framework.

Table 1
Price, interest rate duration and mortality multiplier duration of an LS.

x	PLS_x	$D(PLS_x)$	$DM(PLS_x)$
35	613.64	-100.335	1.491
45	9391.86	-14.129	0.169
55	21069.30	-9.528	0.103
65	37975.05	-6.732	0.063
75	67715.03	-2.988	0.034
85	88615.86	-1.057	0.014

Note: $x_0 = 35, i_0 = 2.836\%, C = 100000, P = 1070.02$, IRR $i = 12\%$ and $\beta = 8$.

Let us suppose we have a triangular mortality multiplier $\widetilde{\beta} = (\beta, l_\beta, r_\beta)$ and a fuzzy IRR $\widetilde{i} = (i, l_i, r_i)$, whose α-cuts, $\forall \alpha \in [0,1]$, are, respectively:

$$\beta_\alpha = \left[\underline{\beta}(\alpha), \overline{\beta}(\alpha)\right] = \left[\beta - l_\beta(1-\alpha), \beta + r_\beta(1-\alpha)\right]$$

$$i_\alpha = \left[\underline{i}(\alpha), \overline{i}(\alpha)\right] = \left[i - l_i(1-\alpha), i + r_i(1-\alpha)\right]$$

By using $\widetilde{\beta}$, we first induce fuzzy one-year mortality probabilities $\widetilde{q}_x^*$, whose α-cuts, $q_x^*\alpha$, from (3b), (3c) and (9a) are:

$$q_x^*\alpha = \left[\underline{q_x^*}(\alpha), \overline{q_x^*}(\alpha)\right] = \left[\min\left\{1, \left[\beta - l_\beta(1-\alpha)\right] \cdot q_x\right\}, \min\left\{1, \left[\beta + r_\beta(1-\alpha)\right] \cdot q_x\right\}\right]$$

Therefore, $\widetilde{\beta}$ and $\widetilde{i}$ induce a fuzzy price of the LS, $\widetilde{PLS_x}$, whose α-cuts, $PLS_{x\alpha} = \left[\underline{PLS_x}(\alpha), \overline{PLS_x}(\alpha)\right]$ are obtained by using (3b)-(3c) and bearing in mind (see Remarks 7 and 8) that the price is decreasing (increasing) with respect to the IRR (mortality multiplier):

$$\underline{PLS_x}(\alpha) = C \sum_{t=1}^{\omega-x} \left(1 + i + r_i(1-\alpha)\right)^{-t} \prod_{k=0}^{t-2} \left(1 - \underline{q_{x+k-1}^*}(\alpha)\right) \cdot \underline{q_{x+t-1}^*}(\alpha) - P \times \sum_{t=1}^{\omega-x} \left(1 + i + r_i(1-\alpha)\right)^{-t} \prod_{k=0}^{t-1} \left(1 - \underline{q_{x+k-1}^*}(\alpha)\right) \tag{12a}$$

$$\overline{PLS_x}(\alpha) = C \sum_{t=1}^{\omega-x} \left(1 + i - l_i(1-\alpha)\right)^{-t} \prod_{k=0}^{t-2} \left(1 - \overline{q_{x+k-1}^*}(\alpha)\right) \cdot \overline{q_{x+t-1}^*}(\alpha) - P \times \sum_{t=1}^{\omega-x} \left(1 + i - l_i(1-\alpha)\right)^{-t} \prod_{k=0}^{t-1} \left(1 - \overline{q_{x+k-1}^*}(\alpha)\right) \tag{12b}$$

4.2. Steps of the proposed fuzzy framework

The proposed novel framework in this paper is structured around four distinct steps, as illustrated in Fig. 2. The first step involves fitting the TFNs for the mortality multiplier and the IRR, aligning them with the available information on LS prospects for a potential investor. The second step entails calculating the fuzzy price of the LS using Eqs. (12a)-(12b). Moving to the third step, we propose simplifying information by fitting a triangular price to the LS, which offers ease of interpretation and facilitates subsequent calculations. Finally, in the fourth step, we present six criteria that assist decision-makers in selecting their preferred approximation method to serve as a reference for the LS price. These criteria aim to guide the decision-making process and consider factors such as approximation quality, usability, and comprehensibility for individuals without prior knowledge of fuzzy numbers. In addition to Fig. 2, we have included pseudocode in Appendix B, which can serve as a useful reference for further development of the proposed framework into a formal programming language. However, it is worth noting that the calculations can also be run using conventional spreadsheet software such as Excel®.

Step 1: Fit the triangular fuzzy parameters from a set of crisp quantifications.

In practical scenarios, the insured's mortality multiplier and the IRR used for pricing LSs are often observed as a set of crisp values. The value of β (and the corresponding LE), is typically documented in LE certificates provided by at least two independent MUs. Xu [85] analyses the biases present in LE estimates from the four prominent American MUs, namely ITM 21st, AVS underwriting LLC, Fasano Associates Inc., and Longevity Services Inc. The study reveals that 67% of the LSs in the sample were evaluated by multiple MUs. Furthermore, the research also demonstrates statistically significant differences in valuations between two MUs.

As far as the IRR is concerned, AA Partners Ltd. [2] suggests adjusting it based on the IRR of recent comparable trades. This approach, known as the neighbourhood method, operates on the principle that investors tend to price similar assets similarly. In the context of LSs, this implies considering factors such as close face value, insured's LE and age, type of policy, among others. Fig. 3, taken from AA Partners Ltd. [2], presents a case study illustrating the process of determining an appropriate IRR for an 80-year-old individual.

The aggregation of crisp evaluations of the mortality multiplier and the IRR can be done using the method by Cheng [34], which proposes structuring the information of a variable that is observed as a set of real numbers, $\{a_1, a_2, \ldots, a_n\}$, by adjusting a TFN $\widetilde{A} = (A, l_A, r_A)$. In our problem, this set may be several mortality multipliers (or LEs) from different MUs or different IRRs registered for a group of LSs with homogeneous characteristics, as Fig. 3 displays. That method is developed sequentially as follows:

Step 1.1. Calculate the distance matrix $\boldsymbol{D} = \left[d_{ij}\right]_{n\times n}$, where the distance between the ith and the jth is $d_{ij} = \left|a_i - a_j\right|$. Of course, $d_{ii} = 0, d_{ij} = d_{ji}$.

Step 1.2. Calculate the mean distance of the ith observation to the other $n-1$ ones as $\overline{d}_i = \frac{\sum_{i=1}^{n} d_{ij}}{n-1}$. Therefore, $\overline{d}_i$ measures the distance of the ith opinion to the center of gravity of the pool. Of course, the weight of the value a_i to determine A is decreasing with respect to $\overline{d}_i$.

Step 1.3. Construct a matrix $\boldsymbol{P} = \left[p_{ij}\right]_{n\times n}$ to measure the importance of the ith observation over the jth one by doing $p_{ij} = \frac{\overline{d}_j}{\overline{d}_i}$. Therefore, $p_{ii} = 1$ and $p_{ij} = \frac{1}{p_{ji}}$.

Step 1.4. Fit the coefficients $w_i, i = 1, 2, \ldots, n$ to measure the importance degree of the ith observation in the group in such a way that $0 \leq w_i \leq 1$, $\forall i$.These coefficients must satisfy $p_{ij} = \frac{w_i}{w_j}$ and $\sum_{i=1}^{n} w_i = 1$. Therefore, $w_i = \frac{1}{\sum_{i=1}^{n} p_{ij}}$.

Step 1.5. The centre of $\widetilde{A}$ is $A = \sum_{i=1}^{n} w_i a_i$.

Step 1.6. Fit the parameters σ and η that are needed to adjust the spreads l_A and r_A:

$$\sigma = \sum_{i=1}^{n} w_i |A - a_i|$$

$$\eta = \frac{A - a^l}{a^r - A}$$

where $a^l = \frac{\sum_{i=1\; a_i<A}^{n} w_i a_i}{\sum_{i=1\; a_i<A}^{n} w_i}$ and $a^r = \frac{\sum_{i=1\; a_i>A}^{n} w_i a_i}{\sum_{i=1\; a_i>A}^{n} w_i}$.

Step 1.7. Calculate l_A and r_A by doing $l_A = \frac{3(1+\eta)\eta\sigma}{1+\eta^2}$ and $r_A = \frac{3(1+\eta)\sigma}{1+\eta^2}$.

It is easy to check that in the particular case of a set of two observations $\{a_1, a_2\}$, the TFN $\widetilde{A} = (A, l_A, r_A)$ is obtained as $A = \frac{a_1+a_2}{2}$ and $l_A = r_A = \frac{3}{2}|a_2 - a_1|$. In this regard, note that approximately 70% of LS transactions are evaluated by only two different LS providers [85].

Example 2. Let us show how to adapt this procedure to adjust a fuzzy IRR $\widetilde{i} = (i, l_i, r_i)$ to price the LS in Fig. 3, which is a real situation exposed in AA Partners Ltd. [2]. In this case, the set of observations on the IRR is {18.2%, 17.5%, 18.3%}. Therefore, the distance matrix of the IRR for the reference trades number 1, 2 and 3 are:

$$\boldsymbol{D} = \begin{bmatrix} 0 & 0.7 & 0.8 \\ 0.7 & 0 & 0.1 \\ 0.8 & 0.1 & 0 \end{bmatrix}$$

Then, the matrix $\boldsymbol{P}$, which shows the relative importance of each IRR in the pool, is:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0.53 & 0.6 \\ 0.53 & 1 & 0.1 \\ 0.6 & 1.125 & 1 \end{bmatrix}$$

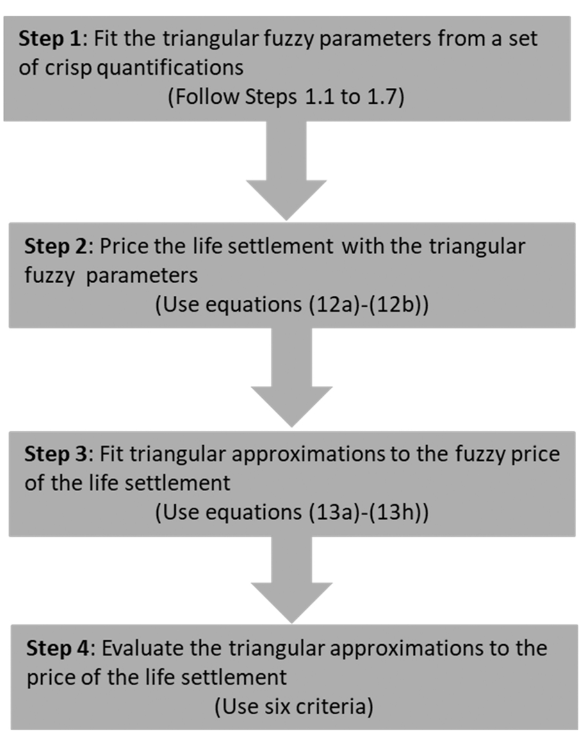
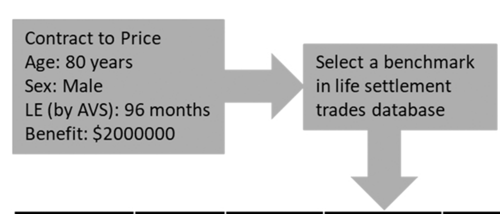


Fig. 2. Steps of the fuzzy framework to price LSs.
Source: Own elaboration.



Benchmark	Age	Sex	LE (by AVS in months)	Benefit	IRR (%)
Number 1	80.1	Male	97	$2000000	18.2
Number 2	79.6	Male	94	$2000000	17.5
Number 3	80	Male	95	$2000000	18.3

Fig. 3. Adjusting an IRR for an LS from a set of three reference trades.
Source: Own elaboration adapted from AA Partners Ltd. [2].

Furthermore, the weight matrix is $w = (0.2202, 0.4128, 0.3670)$ and, therefore, $i = 18.08\%$. To fit the spreads, l_R and r_R, we find that $\sigma = 0.2565$, $a^l = 17.5$ and $a^r = 18.25$. Therefore, $\eta = \frac{18.08-17.5}{18.25-18.08} = 3.54$, $l_A = \frac{3(1+3.54)\cdot 3.54\cdot 0.2565}{1+3.54^2} = 0.914$ and $r_A = \frac{3(1+3.54)\cdot 0.2565}{1+3.54^2} = 0.258$. As a result, the fuzzy IRR adjusted with the method in Cheng [34] is $\widetilde{i} = (18.08\%, 0.914\%, 0.258\%)$ and not an average crisp number, as it is proposed in AA Partners Ltd. [2].

In regards to the mortality multiplier, when having two different LE certificates (which is the common practice in the LS market as stated by [85]), a fuzzy triangular mortality multiplier can be obtained following the same process.

Step 2. Price the LS with triangular fuzzy parameters.

After performing Step 1, we have a triangular mortality multiplier $\widetilde{\beta} = (\beta, l_\beta, r_\beta)$ and a fuzzy IRR $\widetilde{i} = (i, l_i, r_i)$. To obtain the α-cuts of the fuzzy price of the LS, $\widetilde{PLS}_x$, Eqs. (12a) and (12b) have to be used. The resulting price is an FN that no longer retains the triangular shape.

Step 3: Fit triangular approximations to the fuzzy price of the LS.

By following the methods developed in Subsection 2.2, three alternative triangular approximations to $\widetilde{PLS}_x$ are obtained.

- Secant approach

This approach preserves the core and support of $\widetilde{PLS}_x$. Therefore, $\widetilde{PLS}^T_x = (PLS^T_x, l_{PLS^T_x}, r_{PLS^T_x})$ is fitted by using (4a)-(4c) and (12a)-(12b):

$$PLS^T_x = PLS_{x1} = C\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-2}(1-\min\{1,\beta\cdot q_{x+k-1}\})\cdot\min\{1,\beta\cdot q_{x+t-1}\} - P\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-1}(1-\min\{1,\beta\cdot q_{x+k-1}\}) \tag{13a}$$

$$\begin{aligned} l_{PLS^T_x} &= PLS^T_x - \underline{PLS_x}(0) \\ &= PLS^T_x - C\sum_{t=1}^{\omega-x}(1+i+r_i)^{-t}\prod_{k=0}^{t-2}\left[1-\min\{1,(\beta - l_\beta)\cdot q_{x+k-1}\}\right]\cdot\min\{1,(\beta-l_\beta)\cdot q_{x+t-1}\} + P \\ &\times\sum_{t=1}^{\omega-x}(1+i+r_i)^{-t}\prod_{k=0}^{t-1}\left[1-\min\{1,(\beta-l_\beta)\cdot q_{x+k-1}\}\right] \end{aligned} \tag{13b}$$

$$\begin{aligned} r_{PLS^T_x} &= \overline{PLS_x}(0) - PLS^T_x \\ &= C\sum_{t=1}^{\omega-x}(1+i-l_i)^{-t}\prod_{k=0}^{t-2}\left[1-\min\{1,(\beta+r_\beta)\cdot q_{x+k-1}\}\right]\cdot\min\{1,(\beta+r_\beta)\cdot q_{x+t-1}\} - P\sum_{t=1}^{\omega-x}(1+i-l_i)^{-t}\prod_{k=0}^{t-1}\left[1-\min\{1,(\beta+r_\beta)\cdot q_{x+k-1}\}\right] - PLS^T_x \end{aligned} \tag{13c}$$

- Gradient approach

To find PLS^T_x, the α-levels $PLS_{x\alpha}$ are approximated following (5a)-(5c) and using the durations for the price of the LS in (11a) and (11b). Therefore, $\widetilde{PLS}^T_x = \left(PLS^T_x, l_{PLS^T_x}, r_{PLS^T_x}\right)$ conserves the centre:

$$\begin{aligned} PLS^T_x &= PLS_{x1} \\ &= C\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-2}(1-\min\{1,\beta\cdot q_{x+k-1}\})\cdot\min\{1,\beta\cdot q_{x+t-1}\} - P \\ &\times\sum_{t=1}^{\omega-x}(1+i)^{-t}\prod_{k=0}^{t-1}(1-\min\{1,\beta\cdot q_{x+k-1}\}) \end{aligned} \tag{13d}$$

The spreads are built up by evaluating (11a) and (11b) in the centres of the fuzzy IRR (i) and the fuzzy mortality multiplier (β). Therefore, by following (11c), we obtain:

$$l_{PLS^T_x} = \left[\frac{1}{1+i}D(PLS_x)r_i + DM(PLS_x)l_\beta\right]PLS_x \tag{13e}$$

$$l_{PLS^T_x} = \left[\frac{1}{1+i}D(PLS_x)r_i + DM(PLS_x)l_\beta\right]PLS_x \tag{13 f}$$

- Expected interval approach

In this case, $\widetilde{PLS}^T_x$ is the closest TFN to $\widetilde{PLS}_x$ that maintains its expected interval. To obtain it, we implement definite integrals in (6a)-(6i) by using Simpson's rule. As in Kaufmann [49], Jiménez and Rivas [48], Terceño et al. [74] and Andrés-Sánchez [7], we evaluate $\widetilde{PLS}_x$ on an eleven-membership level scale, i.e., for $\alpha = 0, 0.1, 0.2, \ldots, 1$, $\Delta\alpha = 0.1$. Therefore, to implement Simpson's rule, $\Delta\alpha = 0.1$ is taken:

$$\int_0^1 \underline{PLS_x}(\alpha)d\alpha \approx \frac{1}{30}\left[\underline{PLS_x}(0) + 2\sum_{j=1}^{4}\underline{PLS_x}\left(\frac{2j}{10}\right) + 4 \times\sum_{j=1}^{5}\underline{PLS_x}\left(\frac{2j-1}{10}\right) + \underline{PLS_x}(1)\right] \tag{13 g}$$

$$\int_0^1 \alpha\cdot\underline{PLS_x}(\alpha)d\alpha \approx \frac{1}{30}\left[2\sum_{j=1}^{4}\frac{2j}{10}\cdot\underline{PLS_x}\left(\frac{2j}{10}\right) + 4 \times\sum_{j=1}^{5}\frac{2j-1}{10}\underline{PLS_x}\left(\frac{2j-1}{10}\right) + \underline{PLS_x}(1)\right] \tag{13 h}$$

We proceed analogously to calculate $\int_0^1 \overline{PLS_x}(\alpha)d\alpha$ and $\int_0^1 \alpha\cdot\overline{PLS_x}(\alpha)d\alpha$.

Step 4: Evaluate the triangular approximations to the price of the LS

According to Grzegorzewski and Mrówka [39], when assessing the quality of an approximation, it is important to recognize that in an uncertain environment, allowing for some degree of deviation is reasonable. This is because the issues associated with vagueness are more qualitative in nature and less concerned with precision. Rather than searching for the best and universal approximation, the focus should be on evaluating the properties that are important for specific applications. The approximation should retain certain parameters of the original FN, be easy to implement, computationally efficient, and have a convincing interpretation [22,23,41,48]. The importance of these properties depends on the particular situation and the decision-maker involved. Building on these ideas, the criteria we introduce are not solely based on the quality of the approximation to the FN, but also on usability and ease of understanding for financial analysts and practitioners without prior knowledge of FNs. We consider this aspect to be crucial since, as described in Section 1, the LS-related insurance industry requires new approaches to improve pricing techniques and foster further development.

We propose 6 criteria to evaluate the quality of triangular approximations to $\widetilde{PLS}_x$. Some of them serve as guidelines for revised approximating methods, either explicitly or implicitly, and others are commonly accepted in computational mathematics. Whereas three of them are related to the errors by the triangular approximation, the last three take other issues into account. The error in the bounds of $PLS_{x\alpha}$ by the approximation $PLS^T_x\alpha$ is measured by means of the relative deviations in the bounds as:

$$\underline{e}(\alpha) = \frac{\left|\underline{PLS_x}(\alpha) - \underline{PLS_x^T}(\alpha)\right|}{\underline{PLS_x}(\alpha)} \text{ and } \overline{e}(\alpha) = \frac{\left|\overline{PLS_x}(\alpha) - \overline{PLS_x^T}(\alpha)\right|}{\overline{PLS_x}(\alpha)}$$

- Criterion 1: Closeness of the approximation
  This principle guides numerous methodologies for approximating FNs with TFNs, in which the parameters defining the approximation are adjusted by minimizing the distance between that approximation and the FN being approximated (e.g., [22,23,39,40]). Following Andrés-Sánchez [7], the average error is evaluated in $\alpha \in [\alpha^*, 1]$ at $\alpha^* = 0$ and $\alpha^* = 0.9$. If we consider $\alpha^* = 0$, we are assessing the closeness of $\widetilde{PLS}_x^T$ to $\widetilde{PLS}_x$ for the whole shape of the fuzzy variable, whereas by assessing $\alpha^* = 0.9$, we are interested only in the quality of the triangular approximation around the most reliable values. This last assessment is justified by the fact that in fuzzy financial pricing, prices whose membership levels are close to 1 could be of special interest ([42,62,63,81,83,84]). To measure average relative deviations, we use a weighted average measure of errors in lower and upper bounds since it is logical to consider more relevant those deviations in higher α-levels ([7]). Thus:

$$\underline{wae}(\alpha^*) = \frac{\int_{\alpha^*}^{1} \alpha \cdot \underline{e}(\alpha) d\alpha}{\int_{\alpha^*}^{1} \underline{e}(\alpha) d\alpha} \text{ and } \overline{wae}(\alpha^*) = \frac{\int_{\alpha^*}^{1} \alpha \cdot \overline{e}(\alpha) d\alpha}{\int_{\alpha^*}^{1} \overline{e}(\alpha) d\alpha}$$

  and we also evaluate the mean of weighted average errors in the bounds of the α-levels by doing the following:

$$wae(\alpha^*) = \frac{\underline{wae}(\alpha^*) + \overline{wae}(\alpha^*)}{2}$$

- Criterion 2: Better adherence to the most reliable values
  This criterion is related to the approximation of a parameterised FN through a function of the same nature by using the gradient of that function at the level $\alpha = 1$ [37]. It is also linked with the approximation of any FN through a trapezoidal FN that conserves the value and ambiguity of the original one, which is constructed by weighting the possible values the FN can take with its membership level [23]. It considers the sign of the relation between the errors $\underline{e}(\alpha)$ and $\overline{e}(\alpha)$ and the membership level α. A negative relation is preferable in such a way that the approximation capability decreases in values with lower α (and therefore with less reliability) than the opposite relation, that is, worse adherence to values with greater reliability. That relation is measured by means of the Spearman correlation of α with $\underline{e}(\alpha)$ and $\overline{e}(\alpha)$, $\rho\left(\alpha, \underline{e}(\alpha)\right)$ and $\rho(\alpha, \overline{e}(\alpha))$.
- Criterion 3. Unbiasedness
  This criterion considers the existence of a systematic bias in the approximation $\widetilde{PLS}_x^T$. It is preferable that the approximation $\widetilde{PLS}_x^T$ does not underestimate or overestimate $\widetilde{PLS}_x$ systematically in all α-levels. Although this criterion is not commonly considered explicitly in the literature on FNs approximation, it is a widely accepted principle in fields that deal with uncertain values, such as Statistics. Overestimated values are offset by the underestimated ones, minimizing the impact that approximation errors may have on subsequent calculations.
- Criterion 4. Preservation
  It is desirable that the triangular approximation preserves some of the representative features of the original FN, $\widetilde{PLS}_x$. This criterion is inherent in most approximation methods. The distinction among many of these methods lies in the characteristics that are retained. For instance, in Jiménez and Rivas [48], the approximated FN preserves the core and support of the original FN. In contrast, in Grzegorzewski and Mrówka [39] it maintains the expected interval and in Ban et al. [23] the value and ambiguity indicators are preserved.
- Criterion 5. Interpretability
  It embeds, of course, the fact that using a triangular shape allows an easy interpretation of the results (e.g., Grzegorzewski and Pasternak-Winiarska, [41]), but also an easy understanding of how the approximation method works. The interpretability of algorithms and their outcomes holds great significance in computational methods such as fuzzy systems [58] since it enhances the usability of the results. Hence, achieving an intuitively understandable interpretation of the approximation methodology is desirable for any financial analyst, irrespective of their proficiency in FST and FNs, as it enables its practical application within the industry.
- Criterion 6. Ease of calculation
  The desirability of closed-form solutions with respect to numerical solutions is an extensively acknowledged principle within the field of computational methods.

5. Results and discussion

To illustrate the steps depicted in Section 4, we consider a numerical application, which is the continuation of Example 1.

Example 1. *(continuation)*. Table 2 develops an empirical analysis of the same policy priced in Table 1. To implement the calculations, we use a fuzzy version[1] of the mortality multiplier and the IRR in that example, in such a way that $\widetilde{\beta} = \widetilde{8} = (8, 1, 1)$ and $\widetilde{i} = \widetilde{12\%} = (0.12, 0.01, 0.01)$, but we restrict the ages at which the policy is sold via an LS to $x = 55, 65, 75$. The results of this Table correspond to Step 2.

Note that although the mortality multiplier and the IRR are TFNs, the price of the LS does not maintain the triangular shape. In this regard, Table 2 shows that the first differences of the bounds of $PLS_{x\alpha}$ are not constant.

Before performing Step 3, it is interesting to compare the results in Table 2 with those obtained in the case of crisp values of mortality multiplier and IRR (Table 1). As can be seen, the crisp price of the LS is a particular case of the fuzzy one when $\alpha = 1$. That is, by using a fuzzy version of these two key parameters, it is possible to obtain not only a crisp value of the LS price but a wide range of the values this price can take with their associate level of membership. This result can be very useful in LS markets where, due to its nature, there is no precise information on the mortality multiplier and the IRR. Therefore, when faced with uncertainty, working with FNs instead of crisp ones offers several advantages. FNs explicitly capture uncertainty and imprecision, providing a more nuanced understanding of the range of possible values. Unlike crisp numbers that provide only a single precise value, FNs offer flexibility in representing uncertain information. They accommodate subjective assessments, expert opinions, and varying levels of confidence, allowing for more precise and nuanced modelling of uncertainty. Furthermore, FNs facilitate robust decision-making by enabling a comprehensive evaluation of potential scenarios (those associated with values of α). This broader perspective allows for more informed choices than that one in which only a crisp value is considered.

In Step 3, triangular approximations are obtained. Table 3 shows the triangular approximations of the prices displayed in Table 2. To implement the SA, it is sufficient to use the 1-cut and the 0-cut from Table 2. The GA can be obtained by using prices and duration measures

[1] For the sake of simplicity, we do not perform Step 1 in this example. However, fitting the fuzzy mortality multiplier and IRR could be done by following Steps 1.1–1.7.

Table 2
α-cuts of the price of the LS in Example 1 for $\widetilde{i} = (0.12, 0.01, 0.01)$ and $\widetilde{\beta} = (8, 1, 1)$.

α	$\underline{PLS_{55}}(\alpha)$	$\overline{PLS_{55}}(\alpha)$	$\Delta\underline{PLS_{55}}(\alpha)$	$\Delta\overline{PLS_{55}}(\alpha)$
$x = 55$				
1	21069.30	21069.30		
0.9	20675.01	21465.83	394.29	-396.53
0.8	20282.94	21864.62	392.07	-398.79
0.7	19893.08	22265.69	389.87	-401.07
0.6	19505.40	22669.07	387.68	-403.37
0.5	19119.88	23074.76	385.52	-405.69
0.4	18736.51	23482.79	383.37	-408.03
0.3	18355.26	23893.18	381.25	-410.39
0.2	17976.12	24305.95	379.14	-412.77
0.1	17599.07	24721.12	377.05	-415.17
0	17224.08	25138.71	374.98	-417.59
$x = 65$				
α	$\underline{PLS_{65}}(\alpha)$	$\overline{PLS_{65}}(\alpha)$	$\Delta\underline{PLS_{65}}(\alpha)$	$\Delta\overline{PLS_{65}}(\alpha)$
1	37975.05	37975.05		
0.9	37507.89	38441.96	467.16	-466.91
0.8	37040.45	38908.64	467.44	-466.68
0.7	36572.72	39375.10	467.73	-466.46
0.6	36104.68	39841.36	468.04	-466.26
0.5	35636.31	40307.45	468.37	-466.08
0.4	35167.58	40773.37	468.73	-465.92
0.3	34698.48	41239.14	469.10	-465.77
0.2	34228.98	41704.78	469.50	-465.64
0.1	33759.05	42170.30	469.92	-465.53
0	33288.68	42635.73	470.37	-465.43
$x = 75$				
α	$\underline{PLS_{75}}(\alpha)$	$\overline{PLS_{75}}(\alpha)$	$\Delta\underline{PLS_{75}}(\alpha)$	$\Delta\overline{PLS_{75}}(\alpha)$
1	67715.03	67715.03		
0.9	67305.01	68120.85	410.02	-405.82
0.8	66890.72	68522.57	414.29	-401.71
0.7	66472.07	68920.25	418.65	-397.68
0.6	66048.97	69313.98	423.10	-393.73
0.5	65621.32	69703.82	427.64	-389.85
0.4	65189.04	70089.85	432.28	-386.03
0.3	64752.01	70472.15	437.03	-382.29
0.2	64310.14	70850.77	441.87	-378.63
0.1	63863.32	71225.80	446.82	-375.03
0	63411.45	71597.29	451.87	-371.49

in Table 1 and the spreads of the IRR and the mortality multiplier. To obtain the EIA, it is necessary to use all the data in Table 2.

Step 4 needs some calculations that are collected in Table 4, C1, C2 (these last two in Appendix C) and Table 5, C3, C4, where C3 and C4 are also in Appendix C. These tables show the patterns described below.

- Criterion 1. Table 4, C1, C2 report that if the errors are evaluated at $\alpha^* = 0$, the EIA provides the closest approximation. Likewise, the SA and the GA generate similar errors. On the other hand, Table 5, C3, C4 display that at $\alpha^* = 0.9$, the lowest errors come from the GA and the largest from the EIA.
- Criterion 2. When the quality of the approximation for all possible values of α is assessed, the Spearman correlation in Table 4, C1, C2 shows a perfect negative relation between the errors and α for the GA. On the other hand, that correlation is small in the other methods, and its sign depends on the bound of the α-cut under assessment. If that analysis is done for $\alpha^* = 0.9$ (Table 5, C3, C4), it can be verified that the SA and the GA (EIA) provide a perfect negative (positive) Spearman correlation between α and the errors.

Table 3
Triangular approximations to $\widetilde{PLS}_x$ in Table 2.

x	SA	GA	EIA
55	(21069.30, 3845.22, 4069.41)	(21069.30, 3953.99, 3953.99)	(21050.62, 3843.60, 4067.78)
65	(37975.05, 4686.37, 4660.68)	(37975.05, 4670.22, 4670.22)	(37977.20, 4684.73, 4659.13)
75	(67715.03, 4303.58, 3882.26)	(67715.03, 4078.89, 4078.89)	(67750.18, 4296.41, 3875.52)

- Criterion 3. In Table 4, C1, C2, it is possible to check that the SA and the GA produce systematic biases. Thus, SA underestimates (overestimates) the prices for ages $x = 65, 75$ ($x = 55$) $\forall\alpha$. In contrast, GA overestimates (underestimates) the price of the LSs for $x = 65, 75$ ($x = 55$). The EIA does not provide a systematic bias since the sign of the difference between the actual price and its approximation varies with α.
- Criterion 4. Whereas the SA preserves the core and the support of the original price, the GA preserves only the core. The EIA preserves the expected interval of the price and, consequently, any expected value but no visual features of the original shape.
- Criterion 5. The SA is based on the assessment of three scenarios: the most reliable and optimistic and pessimistic situations. In many fields, such as actuarial science, making assessments by evaluating several scenarios is very common. Therefore, this approximating method can be considered intuitive for a financial analyst. The GA is also understandable for any practitioner without knowledge of FNs. Its implementation requires only calculating the most reliable price and the maximum deviations from this price by using duration measures in the same manner that in conventional financial and actuarial analyses. EIA has deep roots in FST and, consequently, is more difficult to understand without knowledge of this theory.
- Criterion 6. While both the SA and the GA can be obtained with closed formulas and a low number of calculations, this does not follow for the EIA.

It is worth highlighting that the final choice of the triangular approximation to the value of the LS will depend on factors such as the intended use of the approximation (e.g., for interpretation purposes or for further calculations) or the relevance that the decision-maker deems each criterion should have. The application of multicriteria decision-making methods, as proposed by Irvanizam et al. [46], Irvanizam et al. [47], or Zhao [88], can be highly useful tools in this regard.

6. Conclusions and further research

In this study, we have introduced a novel framework for pricing LSs by employing FNs to account for the inherent uncertainty associated with two crucial parameters. Our approach builds upon the probabilistic pricing method, as established by previous works such as Zollars et al. [89], Lubovich et al. [57], and Brockett et al. [30]. However, we extend this method by utilizing TFNs to represent the IRR and the mortality multiplier. The adoption of TFNs is motivated by the inherent imprecision in the available information for determining these two variables, as highlighted in studies by Xu and Hoesch [86], Lim and Shyamalkumar [56], Xu [85], and Braun and Xu [27]. By incorporating FNs into the pricing framework, we are able to capture and model the inherent uncertainty more accurately, enhancing the reliability and robustness of LS pricing.

The pricing methodology presented in this study encompasses all the essential steps involved in pricing LSs using fuzzy parameters. Firstly, we focus on determining the fuzzy parameters by utilizing a fitting process that incorporates available information and expert knowledge. Next, the LS pricing formula is implemented, incorporating the fuzzy parameters into the calculation. Furthermore, to facilitate subsequent calculations and improve the interpretability of the results, we introduce triangular approximations to the fuzzy value of the LS. These approximations simplify the representation of the fuzzy value while preserving important characteristics, making it easier to handle and comprehend in practical applications. In addition to the triangular approximations, the last step incorporates their assessment using six different criteria. These criteria serve as guidelines for decision-makers in selecting their preferred approximation method and consider various factors such as the quality of the approximation, usability, and comprehensibility for individuals who may not have prior knowledge of FNs.

There is no optimal approximation method for the price of LSs. Each alternative has its own strengths and weaknesses. One of the advantages of the SA method is its intuitive interpretation and that it retains the core and support of the fuzzy actual price. SA is also straightforward to implement by applying the crisp pricing formula in three representative scenarios: the most feasible, optimistic, and pessimistic. On the other hand, the GA method excels in adhering closely to the original price at the values with the highest reliability ($\alpha^* \geq 0.9$). It also offers an intuitive interpretation without requiring prior knowledge of FNs and involves few computational calculations as well. The EIA also exhibits superiority in several criteria. This triangular approximation closely adheres to the entire shape of the original price, preserves its expected interval, and does not exhibit systematic biases in its α-cuts.

The present research has a specific focus on employing TFNs as a foundational element in the proposed framework, aiming to offer a comprehensive and detailed description of its implications in LS pricing. Despite potential limitations, TFNs strike a suitable balance by providing a parsimonious representation of vague data without oversimplifying its inherent structure, as highlighted in previous studies [39, 48]. Additionally, the suggested approach for adjusting TFNs to the key parameters draws inspiration from the methodology presented by Cheng [34]. Moreover, considering the feasibility of real-world applications in the industry, we believe that the proposed methodology utilizing FNs can be more easily implemented by professionals such as actuaries, insurance specialized lawyers, traders, and others, compared to more complex representations of uncertainty. The information provided by FNs is readily understandable by individuals without specialized knowledge of FST, which is not always the case for more intricate representations of uncertainty, as previously mentioned. This characteristic

Table 4
Evaluation of the triangular approximations to $\widetilde{PLS}_{55}$, $\alpha^* \geq 0$.

	(a)	(b)	(c)	(d)				
SA to $\widetilde{PLS}_{55}$								
α	$\underline{PLS_{55}}(\alpha)$	$\overline{PLS_{55}}(\alpha)$	$\underline{PLS^T_{55}}(\alpha)$	$\overline{PLS^T_{55}}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	21069.30	21069.30	21069.30	21069.30	0.00	0.00	0.000%	0.000%
0.9	20675.01	21465.83	20684.78	21476.24	9.77	10.41	0.047%	0.048%
0.8	20282.94	21864.62	20300.26	21883.18	17.31	18.56	0.085%	0.085%
0.7	19893.08	22265.69	19915.74	22290.13	22.66	24.43	0.114%	0.110%
0.6	19505.40	22669.07	19531.21	22697.07	25.82	28.00	0.132%	0.124%
0.5	19119.88	23074.76	19146.69	23104.01	26.81	29.25	0.140%	0.127%
0.4	18736.51	23482.79	18762.17	23510.95	25.66	28.16	0.137%	0.120%
0.3	18355.26	23893.18	18377.65	23917.89	22.39	24.71	0.122%	0.103%
0.2	17976.12	24305.95	17993.13	24324.83	17.01	18.88	0.095%	0.078%
0.1	17599.07	24721.12	17608.60	24731.77	9.54	10.65	0.054%	0.043%
0	17224.08	25138.71	17224.08	25138.71	0.00	0.00	0.000%	0.000%
	$EI\left(\widetilde{PLS}_{55}\right) = [19128.82, 23084.51]$					$\underline{wae}(0) = 0.083\%$		
	$EI\left(\widetilde{PLS}^T_{55}\right) = [19146.69, 23104.01]$					$\overline{wae}(0) = 0.077\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -0.09, \rho(\alpha, \overline{e}(\alpha)) = 0.09$					$wae(0) = 0.080\%$		
GA to $\widetilde{PLS}_{55}$								
α	$\underline{PLS_{55}}(\alpha)$	$\overline{PLS_{55}}(\alpha)$	$\underline{PLS^T_{55}}(\alpha)$	$\overline{PLS^T_{55}}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	21069.30	21069.30	21069.30	21069.30	0.00	0.00	0	0
0.9	20675.01	21465.83	20673.90	21464.70	-1.11	-1.13	0.005%	0.005%
0.8	20282.94	21864.62	20278.50	21860.10	-4.44	-4.52	0.022%	0.021%
0.7	19893.08	22265.69	19883.10	22255.50	-9.97	-10.20	0.050%	0.046%
0.6	19505.40	22669.07	19487.71	22650.90	-17.69	-18.17	0.091%	0.080%
0.5	19119.88	23074.76	19092.31	23046.30	-27.57	-28.46	0.144%	0.123%
0.4	18736.51	23482.79	18696.91	23441.69	-39.60	-41.09	0.211%	0.175%
0.3	18355.26	23893.18	18301.51	23837.09	-53.75	-56.09	0.293%	0.235%
0.2	17976.12	24305.95	17906.11	24232.49	-70.01	-73.46	0.389%	0.302%
0.1	17599.07	24721.12	17510.71	24627.89	-88.35	-93.23	0.502%	0.377%
0	17224.08	25138.71	17115.31	25023.29	-108.77	-115.42	0.631%	0.459%
	$EI\left(\widetilde{PLS}_{55}\right) = [19128.82, 23084.51]$					$\underline{wae}(0) = 0.088\%$		
	$EI\left(\widetilde{PLS}^T_{55}\right) = [19092.31, 23046.30]$					$\overline{wae}(0) = 0.073\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -1, \rho(\alpha, \overline{e}(\alpha)) = 1$					$wae(0) = 0.081\%$		
EIA to $\widetilde{PLS}_{55}$								
α	$\underline{PLS_{55}}(\alpha)$	$\overline{PLS_{55}}(\alpha)$	$\underline{PLS^T_{55}}(\alpha)$	$\overline{PLS^T_{55}}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	21069.30	21069.30	21050.62	21050.62	18.68	18.68	0.089%	0.089%
0.9	20675.01	21465.83	20666.26	21457.40	8.75	8.44	0.042%	0.039%
0.8	20282.94	21864.62	20281.90	21864.17	1.05	0.45	0.005%	0.002%
0.7	19893.08	22265.69	19897.54	22270.95	-4.46	-5.26	0.022%	0.024%
0.6	19505.40	22669.07	19513.18	22677.73	-7.78	-8.66	0.040%	0.038%
0.5	19119.88	23074.76	19128.82	23084.51	-8.94	-9.75	0.047%	0.042%
0.4	18736.51	23482.79	18744.46	23491.29	-7.95	-8.50	0.042%	0.036%
0.3	18355.26	23893.18	18360.10	23898.06	-4.84	-4.88	0.026%	0.020%
0.2	17976.12	24305.95	17975.74	24304.84	0.38	1.11	0.002%	0.005%
0.1	17599.07	24721.12	17591.38	24711.62	7.69	9.50	0.044%	0.038%
0	17224.08	25138.71	17207.02	25118.40	17.07	20.32	0.099%	0.081%
	$EI\left(\widetilde{PLS}_{55}\right) = [19128.82, 23084.51]$					$\underline{wae}(0) = 0.041\%$		
	$EI\left(\widetilde{PLS}_{55}\right) = [19128.82, 23084.51]$					$\overline{wae}(0) = 0.038\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -0.13\rho, (\alpha, (\alpha)) = 0.08$					$wae(0) = 0.040\%$		

Table 5
Analysis of errors by triangular approximations to $\widetilde{PLS}_{55}$ for $\alpha^* \geq 0.9$.

	SA		GA		EIA	
α	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	0.000%	0.000%	0.000%	0.000%	0.089%	0.089%
0.99	0.005%	0.005%	0.000%	0.000%	0.084%	0.083%
0.98	0.010%	0.011%	0.000%	0.000%	0.079%	0.078%
0.97	0.015%	0.016%	0.000%	0.000%	0.074%	0.073%
0.96	0.020%	0.021%	0.001%	0.001%	0.069%	0.067%
0.95	0.025%	0.026%	0.001%	0.001%	0.064%	0.062%
0.94	0.029%	0.031%	0.002%	0.002%	0.060%	0.058%
0.93	0.034%	0.035%	0.003%	0.003%	0.055%	0.053%
0.92	0.039%	0.040%	0.003%	0.003%	0.051%	0.048%
0.91	0.043%	0.044%	0.004%	0.004%	0.047%	0.044%
0.9	0.047%	0.049%	0.005%	0.005%	0.042%	0.039%
$\underline{wae}(0.9)$	0.024%		0.002%		0.065%	
$\overline{wae}(0.9)$	0.025%		0.002%		0.064%	
$wae(0.9)$	0.024%		0.002%		0.065%	
$\rho\left(\alpha, \underline{e}(\alpha)\right)$	-1		-1		1	
$\rho(\alpha, \overline{e}(\alpha))$	-1		-1		1	

is why FNs are commonly employed in multicriteria decision-making contexts [46,47,88]. Consequently, we emphasize the importance of obtaining triangular approximations to LSs prices and assuming initial parameters quantified with TFNs.

We believe that more research is needed on how to adjust the mortality multiplier and the IRR. This paper has proposed a method that takes the perspective of an external investor who only has LE certificates from MUs and IRRs of similar operations. However, other approaches can be taken and we reflect on how to approach such research below in this section.

The precise assessment of the mortality multiplier is challenging due to various circumstances. Firstly, the utilization of base mortality tables derived from the life insurance market introduces inaccuracies in mortality rates among the elderly population. This discrepancy arises from the scarcity of data for older individuals compared to those in younger or central age groups. Moreover, several other factors contribute to the biased and imprecise nature of the mortality multiplier estimation. These factors encompass potential misinformation provided by the insured parties, the absence of crucial data, and the inclusion of irrelevant or imprecise information [86]. Furthermore, MUs, as third-party entities involved in LS transactions, rely on publicly available sources to compile mortality data on the insureds. However, it is important to note that these acquired data often contain a significant proportion of unreported deaths. Accounting for the presence of unreported deaths is pivotal for ensuring a fair evaluation of the MUs methodology. Nonetheless, existing methods for addressing this issue either heavily rely on actuarial opinion or are based on certain assumptions [56]. Considering these factors, it is advisable to evaluate LS prices by introducing variability bands in the estimation of the mortality multiplier [86]. One effective approach for introducing such bands is by employing TFNs. In the empirical application conducted by Xu and Hoesch [86], the most reliable scenario for the mortality multiplier, denoted as β= 3, was used. Additionally, stress tests were performed by considering two other scenarios, one pessimistic and one optimistic, with deviations of 20%. This crisp mortality multiplier value can be quantified using a TFN as $\widetilde{\beta} = (3, 0.6, 0.6)$.

In the context of LS pricing, the IRR comprises a risk-free rate, similar to the technical interest rate, and a risk premium associated with various sources of risk. These risk sources include longevity risk, the volume of pending premiums at the agreement date, insurer default risks, rescission risk, and liquidity risk [24,27]. Estimating the IRR requires addressing the primary question of accurately determining the risk premium. To tackle this issue, Braun and Xu [27] propose adjusting the risk premium based on proxy variables representing each source of uncertainty using conventional regression methods. Thus, these authors consider several quantifiers as proxies for longevity risk, including LE, insured amount, differences between LEs from different sources, the number of available LEs for the policy, whether the policy is traded on the secondary or tertiary market, and the insured's age. The volume of outstanding premiums at the agreement date is captured by considering the sum of projected premiums as a rate of the death benefit. Default risk is evaluated based on the insurer's credit rating, while rescission risk is modelled over the policy tenure, representing the time elapsed from policy acquisition to the LS date. Furthermore, liquidity risk is assumed to be quantifiable through the intercept of the regression model. To make predictions expressed as FNs using the crisp output of conventional regression methods, two frameworks proposed by Buckley [32] and Al-Kandari et al. [4] can be employed. Buckley [32] suggests interpreting the statistical confidence intervals of the coefficients as TFNs. On the other hand, the approach presented by Al-Kandari et al. [4] differs slightly. Their method involves constructing fuzzy predictions based on estimating the residuals' confidence intervals, instead of using interval estimates of the coefficients.

In addition to conventional regression methods, Fuzzy Regression (FR) can be employed to establish the relationship between the risk premium and the input variables suggested by Braun and Xu [27]. However, it is important to note that FR accounts for errors stemming from the fuzziness inherent in the system, whereas conventional regression methods assume errors arise from random disturbances. FR has found applications in various actuarial analyses. For instance, Andrés-Sánchez and Terceño [16] and Shapiro and Koissi [70] use FR to fit the term structure of interest rates. Koissi and Shapiro [50], Andrés-Sánchez and González-Vila Puchades [11], and Szymański and Rossa [73] propose fuzzy versions of the Lee-Carter model. Additionally, Apaydin and Baser [20] and Woundjiagué et al. [82] apply FR to estimate claiming reserves. It is worth noting that FR allows for the consideration of input variables, such as the insured's LE or the relative sum of outstanding premiums, as fuzzy rather than crisp values.

We acknowledge that our proposed LS pricing approach utilizing tools from FST has certain limitations that warrant further research. To simplify the analysis, we have focused on scenarios involving constant premiums and death benefits. However, it is straightforward to extend the findings to encompass crisp variable cash flows. Additionally, we believe that applying fuzzy tools to model uncertainty regarding outstanding premiums and benefits could prove fruitful in certain types of life insurance. In any case, our approach could be a starting point for applying more complex representations of vagueness and uncertainty in life insurance pricing, such as intuitionistic, neutrosophic FNs or spherical fuzzy sets. These extensions of fuzzy sets offer valuable approaches for handling different aspects of imprecision (see, e.g., [3,45,52,71]) and could be a promising direction for future investigations on LSs pricing. Undoubtedly, such representations allow for more nuanced information modelling compared to what TFNs allow. However, it is important to point out that these instruments require the estimation of a larger number of parameters compared to FNs. This aspect is crucial given our problem setting, where operations are conducted in over-the-counter financial markets, each transaction can be considered a "tailored suit," and the reference information for setting the parameters of a specific transaction is very limited. In this regard, as indicated by Jiménez and Rivas [48], when the information on a variable is relatively scarce and vague, parsimonious and simple representations, such as linear forms, are desirable.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability

No data was used for the research described in the article.

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Appendix A

This appendix collects some important remarks related to Subsections 3.2 and 4.1.

Remark 1. The interest rate duration of an asset or liability is the weighted average of the maturity of its cash flows by their present value. We denote the duration of life insurance in (7a) as $D(A_x)$ being:

$$D(A_x) = \frac{\sum_{t=1}^{\omega-x} t\cdot(1+i)^{-t}\ {}_{t-1/}q_x}{A_x}$$

and, similarly, for the annuity in (7c), the duration $D(a_x)$ is:

$$D(a_x) = \frac{\sum_{t=1}^{\omega-x} t\cdot(1+i)^{-t}\ {}_{t}p_x}{a_x}$$

Note that whereas the price of the unitary whole life annuity is greater than that of whole life insurance, the duration of whole life insurance is above the duration of the annuity.

Remark 2. The duration is linked with elastic derivatives of present values:

$$D(A_x) = -\frac{1}{1+i}\cdot\frac{\frac{\partial A_x}{\partial i}}{A_x} \text{ and } D(a_x) = -\frac{1}{1+i}\cdot\frac{\frac{\partial a_x}{\partial i}}{a_x}$$

When analysing the sensitivity of values with respect to the interest rate, it is very common to use the linear approximation of the present value that comes from the first-order Taylor polynomial expansion:

$$\Delta A_x \approx -\frac{1}{1+i}D(A_x)\cdot A_x\cdot\Delta i \text{ and } \Delta a_x \approx -\frac{1}{1+i}D(a_x)\cdot a_x\cdot\Delta i$$

We symbolize by $D(A_x^*)$ the interest rate duration obtained with nonstandard probabilities in the case of the life insurance A_x^*. Similarly, for the life annuity a_x^* we use $D(a_x^*)$.

Remark 3. The first derivatives of p_x^* and ${}_{t-1/}q_x^*$with respect to β are [12]:

$$\frac{\partial_t p_x^*}{\partial\beta} = {}_tp_x^{*'} \approx -{}_tp_x^*\sum_{k=0}^{t-1}\frac{q_{x+k}}{p_{x+k}^*} \text{ and } \frac{\partial_{t-1/}q_x^*}{\partial\beta} = {}_{t-1/}q_x^{*'} \approx {}_{t-1/}q_x^*\left[\frac{1}{\beta} - \sum_{k=0}^{t-2}\frac{q_{x+k}}{p_{x+k}^*}\right]$$

Remark 4. By considering the probabilistic method, measures similar to the interest rate duration for the longevity risk can be defined [12]. These measures of sensitivity are denoted by $DM(A_x^*)$ and $DM(a_x^*)$ for the life insurance and the annuity, respectively:

$DM(A_x^*) = \frac{\frac{\partial A_x^*}{d\beta}}{A_x^*}$ and $DM(a_x^*) = \frac{\frac{\partial a_x^*}{d\beta}}{a_x^*}$.

By using (7a), (7c) and expressions in Remark 3, it can be demonstrated that:

$\frac{\partial A_x^*}{d\beta} = \sum_{t=1}^{\omega-x}(1+i)^{-t}{}_{t-1/}q_x^{*'}$ and $\frac{\partial a_x^*}{d\beta} = \sum_{t=1}^{\omega-x}(1+i)^{-t}{}_tp_x^{*'}$.

Therefore, mortality multiplier durations for the life insurance and annuity are, respectively:

$$DM(A_x^*) = \frac{\sum_{t=1}^{\omega-x}(1+i)^{-t}{}_{t-1/}q_x^{*'}}{A_x^*} \geq 0 \text{ and } DM(a_x^*) = \frac{\sum_{t=1}^{\omega-x}(1+i)^{-t}{}_tp_x^{*'}}{a_x^*} \leq 0$$

Remark 5. The fluctuations of life contingency prices when the mortality multiplier varies can be approximated linearly by using a first-order Taylor polynomial expansion as follows:

$$\Delta A_x^* \approx DM(A_x^*)\cdot A_x^*\cdot\Delta\beta \text{ and } \Delta a_x^* \approx DM(a_x^*)\cdot a_x^*\cdot\Delta\beta$$

Notice that whereas A_x^* is increasing with respect to β, since growth in β implies an earlier payment of the death benefit, a_x^* is decreasing with respect to the mortality multiplier because when the mortality multiplier increases, the expected number of annuity payments decreases.

Remark 6. Let Δi be a variation in the IRR and $\Delta\beta$ in the mortality multiplier. By using the results in Remarks 2 and 5, we can write ΔPLS_x as:

$$\Delta PLS_x \approx \left[-\frac{1}{1+i}D(A_x^*)\Delta i + DM(A_x^*)\Delta\beta\right]CA_x^* + \left[\frac{1}{1+i}D(a_x^*)\Delta i - DM(a_x^*)\Delta\beta\right]Pa_x^* =$$

$$\frac{1}{1+i}\left[-D(A_x^*)CA_x^* + D(a_x^*)Pa_x^*\right]\Delta i + \left[DM(A_x^*)CA_x - DM(a_x^*)Pa_x^*\right]\Delta\beta$$

Remark 7. The price of an LS is a decreasing function with respect to the interest rate. Let us suppose that $\Delta\beta = 0$ in expression in Remark 6. Then, for $\Delta i \geq 0$, $\Delta PLS_x \leq 0$ and if $\Delta i \leq 0$, $\Delta PLS_x \geq 0$ because $D(A_x^*) > D(a_x^*)$ (see Remark 1) and $C\,A_x^* \geq P\,a_x^*$ since $PLS_x \geq 0$.

Remark 8. . The price of an LS is an increasing function with respect to the mortality multiplier. Let us suppose that $\Delta i = 0$ in expression in Remark 6. Then, for $\Delta\beta \geq 0$, $\Delta PLS_x \geq 0$ and for $\Delta\beta \leq 0$, $\Delta PLS_x \leq 0$ because $DM(A_x^*) \geq 0$and $DM(a_x^*) \leq 0$ (see Remark 5).

Appendix B

Although the proposed framework can be implemented using spreadsheet software such as Excel®, we provide pseudocode. It serves as a schematic representation and as a basis for programming that framework in languages such as R or Python.

```
Step 1: Fit the triangular fuzzy parameters from a set of crisp quantifications
 1.1 Calculate the distance matrix D
    For i = 1 to n:
      For j = 1 to n:
        d_ij = |a_i - a_j |
        If i = j:
          d_ij = 0
 1.2 Calculate the mean distance ¯d_i for each observation
    For i = 1 to n:
      ¯d_i = (∑_(j=1)^(n) d_ij ) / (n - 1)
 1.3 Construct the matrix P to measure the importance of each observation
    For i = 1 to n:
      For j = 1 to n:
        p_ij = ¯d_j / ¯d_i
        If i = j:
          p_ij = 1
        Else:
          p_ij = 1 / p_ji
 1.4 Fit the coefficients w_i to measure the importance degree of each observation
    For i = 1 to n:
      w_i = 1 / (∑_(j=1)^(n) p_ij )
 1.5 Calculate the centre A of A˜
    A = ∑_(i=1)^(n) (w_i * a_i)
 1.6 Fit the parameters σ and η
    σ = ∑_(i=1)^(n) (w_i * |A - a_i|)
    a¯l = (∑_(i=1)^(n) (w_i * a_i) for a_i < A) / (∑_(i=1)^(n) (w_i) for a_i < A)
    a¯r = (∑_(i=1)^(n) (w_i * a_i) for a_i > A) / (∑_(i=1)^(n) (w_i) for a_i > A)
    η = (A - a^l) / (a^r - A)
 1.7 Calculate the spreads l_A and r_A
    l_A = (3 * (1 + η) * η * σ) / (1 + η^2 )
    r_A = (3 * (1 + η) * σ) / (1 + η^2 )

Step 2: Price the life settlement with the triangular fuzzy parameters
  PLS_x = calculateFuzzyPrice(β(α), i(α))
    LeftPLS_x(α)=calculatePresentValue(Leftβ(α),Righti(α))
    RightPLS_x(α)=calculatePresentValue(Rightβ(α),Lefti(α))
```

```
Step 3: Fit triangular approximations to the fuzzy price of the life settlement
  3.1. Secant approach
    PLS_x¯T  = CalculatePLS_x¯T(C, i, w, x, q, β, P)  // Eq. (13a)
    l_PLS_x¯T = CalculateLeftPLS_x¯T(PLS_x¯T, C, i, w, x, q, β, P, r_i, l_β)  // Eq. (13b)
    r_PLS_x¯T = CalculateRightPLS_x¯T(PLS_x¯T, C, i, w, x, q, β, P, l_i, r_β)  // Eq. (13c)
  3.2. Gradient approach
    3.2.1. Fit the center:
          PLS_x¯T = CalculatePLS_x¯T(C, i, w, x, q, β, P)  // Eq. (13d)
    3.2.2. Fit the spreads:
         l_PLS_x¯T = CalculateLeftSpreadPLS_x¯T(PLS_x^T, C, i, r_i, l_β, D, D_M)  // Eq. (13e)
         r_PLS_x¯T = CalculateRightSpreadPLS_x¯T(PLS_x^T, C, i, l_i, r_β, D, D_M)  // Eq. (13f)
  3.3. Expected interval approach
    3.3.1. Consider an eleven-membership level scale α=0, 0.1, 0.2, ..., 1, Δα=0.1.
    3.3.2. Implement Simpson's rule for Δα=0.1:
          IntegralPLS_x¯T = SimpsonRuleIntegrationPLS_x¯T(PLS_x¯T)  // Eq. (13g)
          IntegralAlphaPLS_x¯T = SimpsonRuleIntegrationAlphaPLS_x¯T(PLS_x¯T)  // Eq.
(13h)
    3.3.3. Proceed analogously to calculate the integral of upper α cuts

Step 4: Evaluate the triangular approximations to the price of the life settlement
  4.1 Calculate the error in the bounds of 〖PLS_x〗_α by the approximation 〖PLS_x¯T〗
_α:
     Lefte(α) = abs(LeftPLS_x(α) - LeftPLS_x¯T(α))/ LeftPLS_x(α)
     Righte(α) = abs(RightPLS_x(α) - RightPLS_x¯T(α)) / RightPLS_x(α)
  4.2 Put into work criterion to evaluate triangular approximations:
    4.2.1 Criterion 1: Closeness of the approximation.
         Leftwae(α) = integral(α* to 1) [α * Lefte(α)] / integral(α* to 1) [Lefte(α)]
          Rightwae(α) = integral(α* to 1) [α * Righte(α)] / integral(α* to 1) [Righte(α)]
           wae(α) = (Rightwae(α) + Rightwae(α)) / 2
    4.2.2 Criterion 2: Better adherence to the most reliable values.
           ρ(α, Lefte(α)) = calculateSpearmanCorrelation(α, Lefte(α))
           ρ(α,¯e(α)) = calculateSpearmanCorrelation(α, ¯e(α))
    4.2.3 Criterion 3: Unbiasedness.
            unbiasedness = checkUnbiasedness(PLS_x¯T)
    4.2.4 Criterion 4: Preservation.
            preservation = checkPreservation(PLS_x, PLS_x¯T)
   4.2.5 Criterion 5: Interpretability.
            interpretability = evaluateInterpretability(PLS_x¯T)
    4.2.6 Criterion 6: Ease of calculation.
           easeOfCalculation = evaluateEaseOfCalculation(PLS_x¯T)
  4.3 Based on the above criteria, choose the preferred approximation.
    preferredApproximation = choosePreferredApproximation(wae(α), ρ(α, Lefte(α)),
ρ(α,Righte(α)), unbiasedness, preservation, interpretability, easeOfCalculation)
```

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Appendix C

Table C1
Evaluation of the triangular approximations to $\widetilde{PLS}_{65}$, $\alpha^* \geq 0$.

SA to $\widetilde{PLS}_{65}$								
	(a)	(b)	(c)	(d)				
α	$\underline{PLS_{65}}(\alpha)$	$\overline{PLS_{65}}(\alpha)$	$\underline{PLS^T_{65}}(\alpha)$	$\overline{PLS^T_{65}}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	37975.05	37975.05	37975.05	37975.05	0.00	0.00	0.000%	0.000%
0.9	37507.89	38441.96	37506.41	38441.12	-1.47	-0.84	0.004%	0.002%
0.8	37040.45	38908.64	37037.78	38907.19	-2.67	-1.45	0.007%	0.004%
0.7	36572.72	39375.10	36569.14	39373.26	-3.58	-1.85	0.010%	0.005%
0.6	36104.68	39841.36	36100.50	39839.32	-4.18	-2.04	0.012%	0.005%
0.5	35636.31	40307.45	35631.87	40305.39	-4.44	-2.05	0.012%	0.005%
0.4	35167.58	40773.37	35163.23	40771.46	-4.35	-1.91	0.012%	0.005%
0.3	34698.48	41239.14	34694.59	41237.53	-3.89	-1.61	0.011%	0.004%
0.2	34228.98	41704.78	34225.96	41703.60	-3.02	-1.18	0.009%	0.003%
0.1	33759.05	42170.30	33757.32	42169.66	-1.73	-0.64	0.005%	0.002%
0	33288.68	42635.73	33288.68	42635.73	0.00	0.00	0.000%	0.000%
	$EI\left(\widetilde{PLS}_{65}\right) = [35634.83, 40306.76]$					$\underline{wae}(0) = 0.0073\%$		
	$EI\left(\widetilde{PLS}^T_{65}\right) = [35631.87, 40305.39]$					$\overline{wae}(0) = 0.0032\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -0.09, \rho(\alpha, \overline{e}(\alpha)) = 0.13$					$wae(0) = 0.0052\%$		
GA to $\widetilde{PLS}_{65}$								
	(a)	(b)	(c)	(d)				
α	$\underline{PLS_{65}}(\alpha)$	$\overline{PLS_{65}}(\alpha)$	$\underline{PLS^T_{65}}(\alpha)$	$\overline{PLS^T_{65}}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	37975.05	37975.05	37975.05	37975.05	0.00	0.00	0.000%	0.000%
0.9	37507.89	38441.96	37508.03	38442.07	0.14	0.11	0.000%	0.000%
0.8	37040.45	38908.64	37041.01	38909.10	0.56	0.46	0.002%	0.001%
0.7	36572.72	39375.10	36573.98	39376.12	1.26	1.02	0.003%	0.003%
0.6	36104.68	39841.36	36106.96	39843.14	2.28	1.78	0.006%	0.004%
0.5	35636.31	40307.45	35639.94	40310.16	3.63	2.72	0.010%	0.007%
0.4	35167.58	40773.37	35172.92	40777.18	5.33	3.82	0.015%	0.009%
0.3	34698.48	41239.14	34705.90	41244.21	7.41	5.07	0.021%	0.012%
0.2	34228.98	41704.78	34238.87	41711.23	9.90	6.45	0.029%	0.015%
0.1	33759.05	42170.30	33771.85	42178.25	12.80	7.95	0.038%	0.019%
0	33288.68	42635.73	33304.83	42645.27	16.15	9.54	0.049%	0.022%
	$EI\left(\widetilde{PLS}_{65}\right) = [35634.83, 40306.76]$					$\underline{wae}(0) = 0.006\%$		
	$EI\left(\widetilde{PLS}^T_{65}\right) = [35639.94, 40310.16]$					$\overline{wae}(0) = 0.004\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -1, \rho(\alpha, \overline{e}(\alpha)) = 1$					$wae(0) = 0.005\%$		
EIA to $\widetilde{PLS}_{65}$								
	(a)	(b)	(c)	(d)				
α	$\underline{PLS_{65}}(\alpha)$	$\overline{PLS_{65}}(\alpha)$	$\underline{PLS^T_{65}}(\alpha)$	$\overline{PLS^T_{65}}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	37975.05	37975.05	37977.20	37977.20	-2.15	-2.15	0.006%	0.006%
0.9	37507.89	38441.96	37508.72	38443.11	-0.84	-1.15	0.002%	0.003%
0.8	37040.45	38908.64	37040.25	38909.02	0.20	-0.38	0.001%	0.001%
0.7	36572.72	39375.10	36571.78	39374.94	0.94	0.16	0.003%	0.000%
0.6	36104.68	39841.36	36103.30	39840.85	1.38	0.51	0.004%	0.001%
0.5	35636.31	40307.45	35634.83	40306.76	1.48	0.68	0.004%	0.002%
0.4	35167.58	40773.37	35166.36	40772.68	1.23	0.69	0.003%	0.002%
0.3	34698.48	41239.14	34697.88	41238.59	0.60	0.55	0.002%	0.001%
0.2	34228.98	41704.78	34229.41	41704.50	-0.43	0.28	0.001%	0.001%
0.1	33759.05	42170.30	33760.94	42170.41	-1.88	-0.11	0.006%	0.000%
0	33288.68	42635.73	33292.46	42636.33	-3.78	-0.59	0.011%	0.001%
	$EI\left(\widetilde{PLS}_{65}\right) = [35634.83, 40306.76]$					$\underline{wae}(0) = 0.003\%$		
	$EI\left(\widetilde{PLS}^T_{65}\right) = [35634.83, 40306.76]$					$\overline{wae}(0) = 0.002\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -0.21, \rho(\alpha, \overline{e}(\alpha)) = 0.44$					$wae(0) = 0.003\%$		

Table C2
Evaluation of the triangular approximations to $\widetilde{PLS}_{75}$, $\alpha^* \geq 0$.

SA to $\widetilde{PLS}_{75}$								
	(a)	(b)	(c)	(d)				
α	$\underline{PLS_{75}}(\alpha)$	$\overline{PLS_{75}}(\alpha)$	$\underline{PLS^T_{75}}(\alpha)$	$\overline{PLS^T_{75}}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	67715.03	67715.03	67715.03	67715.03	0.00	0.00	0.000%	0.000%
0.9	67305.01	68120.85	67284.67	68103.25	-20.34	-17.60	0.030%	0.026%
0.8	66890.72	68522.57	66854.31	68491.48	-36.41	-31.09	0.054%	0.045%
0.7	66472.07	68920.25	66423.95	68879.71	-48.11	-40.54	0.072%	0.059%

(continued on next page)

Table C2 (*continued*)

SA to $\widetilde{PLS}_{75}$								
α	(a) $\underline{PLS}_{75}(\alpha)$	(b) $\overline{PLS}_{75}(\alpha)$	(c) $\underline{PLS}^T_{75}(\alpha)$	(d) $\overline{PLS}^T_{75}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
0.6	66048.97	69313.98	65993.60	69267.93	-55.37	-46.04	0.084%	0.066%
0.5	65621.32	69703.82	65563.24	69656.16	-58.08	-47.66	0.089%	0.068%
0.4	65189.04	70089.85	65132.88	70044.39	-56.16	-45.47	0.086%	0.065%
0.3	64752.01	70472.15	64702.52	70432.61	-49.48	-39.53	0.076%	0.056%
0.2	64310.14	70850.77	64272.17	70820.84	-37.97	-29.93	0.059%	0.042%
0.1	63863.32	71225.80	63841.81	71209.07	-21.51	-16.73	0.034%	0.023%
0	63411.45	71597.29	63411.45	71597.29	0.00	0.00	0.000%	0.000%
	$EI\left(\widetilde{PLS}_{75}\right) = [65601.97, 69687.94]$					$\underline{wae}(0) = 0.0525\%$		
	$EI\left(\widetilde{PLS}^T_{75}\right) = [65563.24, 69656.16]$					$\overline{wae}(0) = 0.0415\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -0.09, \rho(\alpha, \overline{e}(\alpha)) = 0.09$					$wae(0) = 0.0470\%$		
GA to $\widetilde{PLS}_{75}$								
α	(a) $\underline{PLS}_{75}(\alpha)$	(b) $\overline{PLS}_{75}(\alpha)$	(c) $\underline{PLS}^T_{75}(\alpha)$	(d) $\overline{PLS}^T_{75}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	67715.03	67715.03	67715.03	67715.03	0.00	0.00	0	0
0.9	67305.01	68120.85	67307.14	68122.92	2.13	2.06	0.003%	0.003%
0.8	66890.72	68522.57	66899.25	68530.81	8.53	8.24	0.013%	0.012%
0.7	66472.07	68920.25	66491.36	68938.69	19.29	18.45	0.029%	0.027%
0.6	66048.97	69313.98	66083.47	69346.58	34.51	32.61	0.052%	0.047%
0.5	65621.32	69703.82	65675.58	69754.47	54.26	50.65	0.083%	0.073%
0.4	65189.04	70089.85	65267.70	70162.36	78.66	72.51	0.121%	0.103%
0.3	64752.01	70472.15	64859.81	70570.25	107.80	98.10	0.166%	0.139%
0.2	64310.14	70850.77	64451.92	70978.14	141.78	127.37	0.220%	0.180%
0.1	63863.32	71225.80	64044.03	71386.03	180.71	160.23	0.283%	0.225%
0	63411.45	71597.29	63636.14	71793.92	224.69	196.62	0.354%	0.275%
	$EI\left(\widetilde{PLS}_{75}\right) = [65601.97, 69687.94]$					$\underline{wae}(0) = 0.050\%$		
	$EI\left(\widetilde{PLS}^T_{75}\right) = [65675.58, 69754.47]$					$\overline{wae}(0) = 0.043\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -1, \rho(\alpha, \overline{e}(\alpha)) = 1$					$wae(0) = 0.047\%$		
EIA to $\widetilde{PLS}_{75}$								
α	(a) $\underline{PLS}_{75}(\alpha)$	(b) $\overline{PLS}_{75}(\alpha)$	(c) $\underline{PLS}^T_{75}(\alpha)$	(d) $\overline{PLS}^T_{75}(\alpha)$	(a) − (b)	(c) − (d)	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	67715.03	67715.03	67750.18	67750.18	-35.15	-35.15	0.052%	0.052%
0.9	67305.01	68120.85	67320.54	68137.73	-15.52	-16.88	0.023%	0.025%
0.8	66890.72	68522.57	66890.89	68525.28	-0.17	-2.71	0.000%	0.004%
0.7	66472.07	68920.25	66461.25	68912.83	10.82	7.42	0.016%	0.011%
0.6	66048.97	69313.98	66031.61	69300.39	17.35	13.59	0.026%	0.020%
0.5	65621.32	69703.82	65601.97	69687.94	19.35	15.89	0.029%	0.023%
0.4	65189.04	70089.85	65172.33	70075.49	16.71	14.36	0.026%	0.020%
0.3	64752.01	70472.15	64742.69	70463.04	9.32	9.10	0.014%	0.013%
0.2	64310.14	70850.77	64313.05	70850.59	-2.91	0.18	0.005%	0.000%
0.1	63863.32	71225.80	63883.41	71238.15	-20.09	-12.35	0.031%	0.017%
0	63411.45	71597.29	63453.77	71625.70	-42.32	-28.41	0.067%	0.040%
	$EI\left(\widetilde{PLS}_{75}\right) = [65601.97, 69687.94]$					$\underline{wae}(0) = 0.024\%$		
	$EI\left(\widetilde{PLS}^T_{75}\right) = [65601.97, 69687.94]$					$\overline{wae}(0) = 0.022\%$		
	$\rho\left(\alpha, \underline{e}(\alpha)\right) = -0.19, \rho(\alpha, \overline{e}(\alpha)) = 0.19$					$wae(0) = 0.023\%$		

Table C3

Analysis of errors by triangular approximations to $\widetilde{PLS}_{65}$ for $\alpha^* \geq 0.9$.

	SA		GA		EIA	
	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	0.000%	0.000%	0.000%	0.000%	0.006%	0.006%
0.99	0.000%	0.000%	0.000%	0.000%	0.005%	0.005%
0.98	0.001%	0.000%	0.000%	0.000%	0.005%	0.005%
0.97	0.001%	0.001%	0.000%	0.000%	0.005%	0.005%
0.96	0.002%	0.001%	0.000%	0.000%	0.004%	0.005%
0.95	0.002%	0.001%	0.000%	0.000%	0.004%	0.004%
0.94	0.002%	0.001%	0.000%	0.000%	0.004%	0.004%
0.93	0.003%	0.002%	0.000%	0.000%	0.003%	0.004%
0.92	0.003%	0.002%	0.000%	0.000%	0.003%	0.003%
0.91	0.004%	0.002%	0.000%	0.000%	0.003%	0.003%
0.9	0.004%	0.002%	0.000%	0.000%	0.002%	0.003%
$\underline{wae}(0.9)$	0.002%		0.000%		0.004%	
$\overline{wae}(0.9)$	0.001%		0.000%		0.004%	

(*continued on next page*)

Table C3 (*continued*)

	SA		GA		EIA	
	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
$wae(0.9)$	0.002%		0.000%		0.004%	
$\rho\left(\alpha,\underline{e}(\alpha)\right)$	-1		-1		1	
$\rho(\alpha,\overline{e}(\alpha))$	-1		-1		1	

Table C4
Analysis of errors by triangular approximations to $\widetilde{PLS}_{75}$ for $\alpha^* \geq 0.9$.

	SA		GA		EIA	
α	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$	$\underline{e}(\alpha)$	$\overline{e}(\alpha)$
1	0.000%	0.000%	0.000%	0.000%	0.052%	0.052%
0.99	0.003%	0.003%	0.000%	0.000%	0.049%	0.049%
0.98	0.007%	0.006%	0.000%	0.000%	0.046%	0.046%
0.97	0.010%	0.008%	0.000%	0.000%	0.043%	0.043%
0.96	0.013%	0.011%	0.001%	0.000%	0.040%	0.040%
0.95	0.016%	0.014%	0.001%	0.001%	0.037%	0.038%
0.94	0.019%	0.016%	0.001%	0.001%	0.034%	0.035%
0.93	0.022%	0.019%	0.002%	0.001%	0.031%	0.032%
0.92	0.025%	0.021%	0.002%	0.002%	0.028%	0.030%
0.91	0.027%	0.024%	0.003%	0.002%	0.026%	0.027%
0.9	0.030%	0.026%	0.003%	0.003%	0.023%	0.025%
$\underline{wae}(0.9)$	0.015%		0.001%		0.037%	
$\overline{wae}(0.9)$	0.013%		0.001%		0.038%	
$wae(0.9)$	0.014%		0.001%		0.038%	
$\rho\left(\alpha,\underline{e}(\alpha)\right)$	-1		-1		1	
$\rho(\alpha,\overline{e}(\alpha))$	-1		-1		1	

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