Advances in Mathematics 430 (2023) 109218

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Breakdown of homoclinic orbits to L_3 in the RPC3BP (II). An asymptotic formula

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ARTICLE INFO

Article history:
Received 17 September 2021
Received in revised form 31 January 2023
Accepted 7 July 2023
Available online 27 July 2023
Communicated by Vadim Kaloshin

Keywords:
Hamiltonian systems
Exponentially small phenomena
Splitting of separatrices
Celestial mechanics
Coorbital motions
L3 Lagrange point

ABSTRACT

The Restricted 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies called the primaries. If one assumes that the primaries perform circular motions and that all three bodies are coplanar, one has the Restricted Planar Circular 3-Body Problem (RPC3BP). In rotating coordinates, it can be modeled by a two degrees of freedom Hamiltonian, which has five critical points called the Lagrange points $L_1, \ldots, L_5$. The Lagrange point L_3 is a saddle-center critical point which is collinear with the primaries and beyond the largest of the two. In this paper, we obtain an asymptotic formula for the distance between the stable and unstable manifolds of L_3 for small values of the mass ratio $0 < \mu \ll 1$. In particular we show that L_3 cannot have (one round) homoclinic orbits.

If the ratio between the masses of the primaries μ is small, the hyperbolic eigenvalues of L_3 are weaker, by a factor of order $\sqrt{\mu}$, than the elliptic ones. This rapidly rotating dynamics makes the distance between manifolds exponentially small with respect to $\sqrt{\mu}$. Thus, classical perturbative methods (i.e. the Melnikov-Poincaré method) can not be applied.

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https://doi.org/10.1016/j.aim.2023.109218

The obtention of this asymptotic formula relies on the results obtained in the prequel paper [10] on the complex singularities of the homoclinic of a certain averaged equation and on the associated inner equation.
In this second paper, we relate the solutions of the inner equation to the analytic continuation of the parameterizations of the invariant manifolds of L_3 via complex matching techniques. We complete the proof of the asymptotic formula for their distance showing that its dominant term is the one given by the analysis of the inner equation.

Contents

1. Introduction

The Restricted Circular 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries, which perform a circular motion. If one also assumes that the massless body moves on the same plane as the primaries one has the Restricted Planar Circular 3-Body Problem (RPC3BP).

Let us name the two primaries S (star) and P (planet) and normalize their masses so that $m_S = 1-\mu$ and $m_P = \mu$, with $\mu \in \left(0, \frac{1}{2}\right]$. Choosing a suitable rotating coordinate system, the positions of the primaries can be fixed at $q_S = (\mu, 0)$ and $q_P = (\mu - 1, 0)$. Then, the position and momenta of the third body, $(q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, are governed by the Hamiltonian system associated to the Hamiltonian

$$h(q,p;\mu) = \frac{||p||^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{(1-\mu)}{||q-(\mu,0)||} - \frac{\mu}{||q-(\mu-1,0)||}. \tag{1.1}$$

Note that this Hamiltonian is autonomous. The conservation of h corresponds to the preservation of the classical Jacobi constant.

For $\mu > 0$, it is a well known fact that (1.1) has five critical points, usually called Lagrange points (see Fig. 1(a)). On an inertial (non-rotating) system of coordinates, the Lagrange points correspond to periodic dynamics with the same period as the two primaries, i.e. on a 1:1 mean motion resonance. The three collinear Lagrange points, L_1, L_2 and L_3, are of center-saddle type whereas, for small μ, the triangular ones, L_4 and L_5, are of center-center type (see, for instance, [55]).

Due to its interest in astrodynamics, a lot of attention has been paid to the study of the invariant manifolds associated to the points L_1 and L_2 (see [40,32,21]). The dynamics around the points L_4 and L_5 has also been heavily studied since, due to its stability, it is common to find objects orbiting around these points (for instance the Trojan and Greek Asteroids associated to the pair Sun-Jupiter, see [29,20,51]). Since the point L_3 is located "at the other side" of the massive primary, it has received somewhat less attention. However, the associated invariant manifolds (more precisely its center-stable and center-unstable invariant manifolds) play an important role in the dynamics of the RPC3BP since they act as boundaries of *effective stability* of the stability domains around L_4 and L_5 (see [31,54]). The invariant manifolds of L_3 play also a fundamental role in creating transfer orbits from the small primary to L_3 in the RPC3BP (see [37,56]) or between primaries in the Bicircular 4-Body Problem (see [38,39]).

Moreover, being far from collision, the dynamics close to the Lagrange point L_3 and its invariant manifolds for small μ are rather similar to that of other mean motion resonances which play an important role in creating instabilities in the Solar system, see [28]. On the contrary, since the points L_1 and L_2 are close to collision for small μ, the analysis of the associated dynamics is quite different.

Over the past years, one of the main focuses of study of the dynamics "close" to L_3 and its invariant manifolds has been the so called "horseshoe-shaped orbits", first

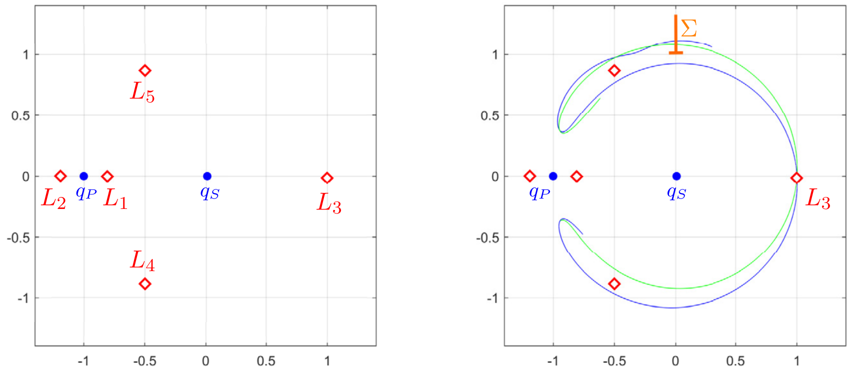

Fig. 1. (a) Projection onto the q-plane of the Lagrange points (red) for the RPC3BP on rotating coordinates. (b) Plot of the stable (green) and unstable (blue) manifolds of L_3, for $\mu = 0.0028$, on the q-plane. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

considered in [17], which are quasi-periodic orbits that encompass the critical points L_4, L_3 and L_5. The interest on these types of orbits arises when modeling the motion of co-orbital satellites, the most famous being Saturn's satellites Janus and Epimetheus, and near Earth asteroids. Recently, in [49], the authors have proved the existence of 2-dimensional elliptic invariant tori on which the trajectories mimic the motions followed by Janus and Epimetheus (see also [24,25,44,22,12,15,9,23]).

Rather than looking at stable motions "close to" L_3 as [49], the goal of this paper (and its prequel [10]) is rather different: its objective is to prove the breakdown of homoclinic connections to L_3. Indeed, since L_3 is a center-saddle critical point, it possesses 1-dimensional unstable and stable manifolds, which we denote by $W^{\mathrm{u}}(\mu)$ and $W^{\mathrm{s}}(\mu)$, respectively, and a 2-dimensional center manifold. Theorem 1.1 below gives an asymptotic formula for the distance between the stable and unstable invariant manifolds (at a suitable transverse section) for mass ratio $\mu > 0$ small enough.

1.1. The distance between the invariant manifolds of L_3

The one dimensional unstable and stable invariant manifolds of L_3 have two branches each (see Fig. 1(b)). One pair circumvents L_5, which we denote by $W^{\mathrm{u},+}(\mu)$ and $W^{\mathrm{s},+}(\mu)$, and the other, $W^{\mathrm{u},-}(\mu)$ and $W^{\mathrm{s},-}(\mu)$, circumvents L_4. Since the Hamiltonian system associated to the Hamiltonian h is reversible with respect to the involution

$$\Phi(q, p; t) = (q_1, -q_2, -p_1, p_2),$$

the $+$ branches of the invariant manifolds are symmetric with respect to the $-$ branches. Thus, we restrict our analysis to the positive branches.

To measure the distance between $W^{\mathrm{u/s},+}(\mu)$, we consider the symplectic polar change of coordinates

$$q = r\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \qquad p = R\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} - \frac{G}{r}\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}, \tag{1.2}$$

where R is the radial linear momentum and G is the angular momentum.

We consider the 3-dimensional section

$$\Sigma = \left\{(r,\theta,R,G) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}^2 \,:\, r>1,\, \theta = \frac{\pi}{2}\right\}$$

and denote by $(r_*^{\rm u}, \frac{\pi}{2}, R_*^{\rm u}, G_*^{\rm u})$ and $(r_*^{\rm s}, \frac{\pi}{2}, R_*^{\rm s}, G_*^{\rm s})$ the first crossing of the invariant manifolds with this section.

The next theorem measures the distance between these points for $0 < \mu \ll 1$.

Theorem 1.1. *There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$,*

$$\|(r_*^{\rm u}, R_*^{\rm u}, G_*^{\rm u}) - (r_*^{\rm s}, R_*^{\rm s}, G_*^{\rm s})\| = \sqrt[3]{4}\,\mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}}\left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log\mu|}\right)\right],$$

where:

- *The constant $A > 0$ is the real-valued integral*

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\,dx \approx 0.177744. \tag{1.3}$$

- *The constant $\Theta \in \mathbb{C}$ is the Stokes constant associated to the inner equation analyzed in [10] and in Theorem 3.13 below.*

Remark 1.2. We can prove the same result for any section

$$\Sigma(\theta_*) = \left\{(r,\theta,R,G) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}^2 \,:\, r>1,\, \theta = \theta_*\right\},$$

with $\theta_* \in (0, \theta_0)$ and $\theta_0 = \arccos\left(\frac{1}{2} - \sqrt{2}\right)$ (the value of μ_0 depends on how close to the endpoints of the interval θ_* is). The section $\theta = \theta_0$ is close to the "turning point" of the invariant manifolds (see Fig. 1(b)).

The constant A in (1.3) is derived from the values of the complex singularities of the separatrix of certain integrable averaged system, which is studied in the prequel paper [10]. The results obtained in [10] about this separatrix are summarized in Theorem 3.1 below.

The origin of the constant Θ appearing in Theorem 1.1 is explained in Theorem 3.13, which analyzes the so-called inner equation. This theorem is also proven in [10]. Moreover, in that paper it is seen, by a numerical computation,[1] that $|\Theta| \approx 1.63$. We expect that one should be able to prove that $|\Theta| \neq 0$ by means of rigorous computer computations (see [6]). Note that $|\Theta| \neq 0$ implies that there are not primary (i.e. one round) homoclinic orbits to L_3.

A fundamental problem in dynamical systems is to prove whether a given model has chaotic dynamics (for instance a Smale horseshoe). For many physically relevant models this is usually remarkably difficult. This is the case of many Celestial Mechanics models, where most of the known chaotic motions have been found in nearly integrable regimes where there is an unperturbed problem which already presents some form of "hyperbolicity". This is the case in the vicinity of collision orbits (see for example [46,13,16,47]) or close to parabolic orbits (which allows to construct chaotic/oscillatory motions), see [53,1,43,48,33,35,34]. There are also several results in regimes far from integrable which rely on computer assisted proofs [2,58,19,36]. The problem tackled in this paper and [10] is radically different. Indeed, if one takes the limit $\mu \to 0$ in (1.1) one obtains the classical integrable Kepler problem in the elliptic regime, where no hyperbolicity is present. Instead, the (weak) hyperbolicity is created by the $\mathcal{O}(\mu)$ perturbation, which can be captured considering an integrable averaged Hamiltonian along the $1:1$ mean motion resonance.[2]

One of the classical methods to construct chaotic dynamics is the Smale-Birkhoff homoclinic theorem by proving the existence of transverse homoclinic orbits to invariant objects, most commonly, periodic orbits. Certainly the breakdown of homoclinic orbits to the critical point L_3 given by Theorem 1.1 does not lead to the existence of chaotic orbits. However, one should expect that Theorem 1.1 implies that there exist Lyapunov periodic orbits exponentially close to L_3 whose stable and unstable invariant manifolds intersect transversally. This would create chaotic motions "exponentially close" to L_3 and its invariant manifolds (see [11]).

As already mentioned, Theorem 1.1 rules out the existence of primary homoclinic connections to L_3 in the RPC3BP for $0 < \mu \ll 1$. However, it does not prevent the existence of multiround homoclinic orbits, that is homoclinic orbits which pass close to L_3 multiple times. It has been conjectured (see for instance [14], where the authors analyze this problem numerically) that multi-round homoclinic connections to L_3 should exist for a sequence of values $\{\mu_k\}_{k\in\mathbb{N}}$ satisfying $\mu_k \to 0$ as $k \to \infty$.

A first step towards proving Arnold diffusion along the $1:1$ mean motion resonance in the 3-body problem? Consider the 3-Body Problem in the planetary regime, that is one massive body (the Sun) and two small bodies (the planets) performing approximate ellipses (including the "Restricted limit" when one of planets has mass zero). A fundamental problem is to assert whether such configuration is stable (i.e. is the Solar system

[1] One can find in the webpage https://github.com/margiralt/stokesConstantL3 the code for this numerical computation.

[2] The $1:1$ averaged Hamiltonian has been also studied to obtain "good" approximations for the global dynamics in the $1:1$ resonant zone, see for example [52,50] and the references therein.

stable?). Thanks to Arnold-Herman-Féjoz KAM Theorem, many of such configurations are stable, see [3,26]. However, it is widely expected that there should be strong instabilities created by Arnold diffusion mechanisms (as conjectured by Arnold in [4]). In particular, it is widely believed that one of the main sources of such instabilities dynamics are the mean motion resonances, where the period of the two planets is resonant (i.e. rationally dependent) [28].

The RPC3BP has too low dimension (2 degrees of freedom) to possess Arnold diffusion. However, since it can be seen as a first order for higher dimensional models, the analysis performed in this paper can be seen as a humble first step towards constructing Arnold diffusion in the $1:1$ mean motion resonance. In this resonance, the RPC3BP has a normally hyperbolic invariant manifold given by the center manifold of the Lagrange point L_3. This normally hyperbolic invariant manifold is foliated by the classical Lyapunov periodic orbits. One should expect that the techniques developed in the present paper would allow to prove that the invariant manifolds of these periodic orbits intersect transversally within the corresponding energy level of (1.1). Still, this is a much harder problem than the one considered in this paper and the technicalities involved would be considerable.

This transversality would not lead to Arnold diffusion due to the low dimension of the RPC3BP. However, if one considers either the Restricted Spatial Circular 3-Body Problem with small $\mu > 0$ which has three degrees of freedom, the Restricted Planar Elliptic 3-Body Problem with small $\mu > 0$ and eccentricity of the primaries $e_0 > 0$, which has two and a half degrees of freedom, or the "full" planar 3-Body Problem (i.e. all three masses positive, two small) which has three degrees of freedom (after the symplectic reduction by the classical first integrals) one should be able to construct orbits with a drastic change in angular momentum (or inclination in the spatial setting).

In the Restricted Planar Elliptic 3-Body Problem the change of angular momentum would imply the transition of the zero mass body orbit from a close to circular ellipse to a more eccentric one. In the full 3BP, due to total angular momentum conservation, the angular momentum would be transferred from one body to the other changing both osculating ellipses. This behavior would be analogous to that of [28] for the $3:1$ and $1:7$ resonances. In that paper, the transversality between the invariant manifolds of the normally hyperbolic invariant manifold was checked numerically for the realistic Sun-Jupiter mass ratio $\mu = 10^{-3}$. Arnold diffusion instabilities have been analyzed numerically for the Restricted Spatial Circular 3-Body Problem in [57].

1.2. The strategy to prove Theorem 1.1

The main difficulty in proving Theorem 1.1 is that the distance between the stable and unstable manifolds of L_3 is exponentially small with respect to $\sqrt{\mu}$ (this is also usually known as a *beyond all orders* phenomenon). This implies that the classical Melnikov Method [30] to detect the breakdown of homoclinics cannot be applied.

To prove Theorem 1.1, we follow the strategy of exponentially small splitting of separatrices (already outlined in [10]) which goes back to the seminal work by Lazutkin [41,42]. See [10] for a list of references on the recent developments in the field of exponentially small splitting of separatrices. In particular, we follow similar strategies of those in [8,7].

In the present work the first order of the difference between manifolds is not given by the Melnikov function. Instead, we must derive and analyze an inner equation which provides the dominant term of this distance. As a consequence, we need to “match” (i.e. compare) certain solutions of the inner equation with the parameterizations of the perturbed invariant manifolds.

The first part of the proof, that was completed in the prequel [10], dealt with the following steps:

A. We perform a change of coordinates to capture the slow-fast dynamics of the system. The first order of the new Hamiltonian has a saddle point with an homoclinic connection (also known as separatrix) and a fast harmonic oscillator. The change of coordinates is introduced in Section 2 and the properties of the new Hamiltonian are stated in Proposition 2.1, which corresponds to Theorem 2.1 in [10].
B. We study the analytical continuation of the time-parametrization of the separatrix of this first order. In particular, we obtain its maximal strip of analyticity and the singularities at the boundary of this strip. This is explained in Theorem 3.1 in Section 3.1, which corresponds to Theorem 2.2 and Proposition 2.3 in [10].
C. We derive the inner equation. This step is contained in Proposition 3.12, which corresponds to Proposition 2.5 of [10].
D. We study two special solutions which will be “good approximations” of the perturbed invariant manifolds near the singularities of the unperturbed separatrix (see Step F below). Such solutions, and their difference, are provided by Theorem 3.13, which corresponds to Theorem 2.7 of [10].

The remaining steps necessary to complete the proof of Theorem 1.1 are the following:

E. We prove the existence of the analytic continuation of the parametrizations of the invariant manifolds of L_3, $W^{\mathrm{u},+}(\delta)$ and $W^{\mathrm{s},+}(\delta)$, in an appropriate complex domain called boomerang domain. This domain contains a segment of the real line and intersects a sufficiently small neighborhood of the singularities of the unperturbed separatrix.
F. By using complex matching techniques, we show that, close to the singularities of the unperturbed separatrix, the solutions of the inner equation obtained in Step D are “good approximations” of the parameterizations of the perturbed invariant manifolds obtained in Step E.

G. We obtain an asymptotic formula for the difference between the perturbed invariant manifolds by proving that the dominant term comes from the difference between the solutions of the inner equation.

The structure of this paper goes as follows. In Section 2 we perform the change of coordinates introduced in Step A and state Theorem 2.2, which is a reformulation of Theorem 1.1 in this new set of variables. Then, in Section 3, we state the results concerning Steps B, C and D above (which are proven in [10]) and we carry out Steps E, F and G. These steps lead to the proof of Theorem 2.2. Sections 4 and 5 are devoted to proving the results in Section 3 which concern Steps E and F.

2. A singular formulation of the problem

The Lagrange point L_3 is a center-saddle equilibrium point, of the form $(d_\mu, 0, 0, d_\mu)$ with $d_\mu = 1 + \frac{5}{12}\mu + \mathcal{O}(\mu^3)$, of the Hamiltonian h in (1.1) whose eigenvalues, as $\mu \to 0$, satisfy (see [55])

$$\text{Spec} = \left\{\pm\sqrt{\mu}\,\rho(\mu), \pm i\,\omega(\mu)\right\}, \quad \text{with} \quad \begin{cases} \rho(\mu) = \sqrt{\frac{21}{8}} + \mathcal{O}(\mu), \\ \omega(\mu) = 1 + \frac{7}{8}\mu + \mathcal{O}(\mu^2). \end{cases}$$

The center and saddle eigenvalues are found at different time-scales. Moreover, when $\mu = 0$, the unstable and stable manifolds of L_3 "collapse" to a circle of critical points. Applying a suitable singular change of coordinates, which is based on the classical Poincaré variables (see [27,45]) and singular scalings, the Hamiltonian h can be written as a perturbation of a pendulum-like Hamiltonian weakly coupled with a fast oscillator.

The construction of this change of variables is presented in detail in Section 2.1 of [10]. To make the paper self-contained, in Appendix A, we give some details on the definition and properties of the Poincaré variables. In the present section we just describe the properties of the Hamiltonian (1.1) in these coordinates.

The Hamiltonian h expressed in the classical (rotating) Poincaré coordinates, $\phi_{\text{Poi}} : (\lambda, L, \eta, \xi) \to (q, p)$, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge dL + i\, d\eta \wedge d\xi$ and the Hamiltonian

$$H^{\text{Poi}} = H_0^{\text{Poi}} + \mu H_1^{\text{Poi}}, \tag{2.1}$$

with

$$H_0^{\text{Poi}}(L, \eta, \xi) = -\frac{1}{2L^2} - L + \eta\xi \qquad \text{and} \qquad H_1^{\text{Poi}} = h_1 \circ \phi_{\text{Poi}}. \tag{2.2}$$

Moreover, the critical point L_3 satisfies

$$\lambda = 0, \qquad (L, \eta, \xi) = (1, 0, 0) + \mathcal{O}(\mu) \tag{2.3}$$

and the linearization of the vector field at this point has, at first order, an uncoupled nilpotent and center blocks,

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + \mathcal{O}(\mu). \tag{2.4}$$

Since ϕ_{Poi} is an implicit change of coordinates, there is no explicit expression for H_1^{Poi}. However, it is possible to obtain series expansion in powers of $(L-1, \eta, \xi)$ (see Appendix B). These expansions have already been used in [10] (see Lemma 4.1).

To capture the slow-fast dynamics of the system, renaming

$$\delta = \mu^{\frac{1}{4}},$$

we perform the singular symplectic scaling

$$\phi_{\mathrm{sc}} : (\lambda, \Lambda, x, y) \mapsto (\lambda, L, \eta, \xi), \qquad L = 1 + \delta^2 \Lambda, \quad \eta = \delta x, \quad \xi = \delta y \tag{2.5}$$

and the time reparametrization $t = \delta^{-2}\tau$. Defining the potential

$$V(\lambda) = H_1^{\mathrm{Poi}}(\lambda, 1, 0, 0; 0) = 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}, \tag{2.6}$$

the Hamiltonian system associated to H^{Poi}, expressed in scaled coordinates, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge d\Lambda + i dx \wedge dy$ and the Hamiltonian

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + H_1, \tag{2.7}$$

where

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \qquad H_{\mathrm{osc}}(x, y; \delta) = \frac{xy}{\delta^2}, \tag{2.8}$$

$$H_1(\lambda, \Lambda, x, y; \delta) = H_1^{\mathrm{Poi}}(\lambda, 1 + \delta^2\Lambda, \delta x, \delta y; \delta^4) - V(\lambda) + \frac{1}{\delta^4} F_{\mathrm{p}}(\delta^2 \Lambda) \tag{2.9}$$

and

$$F_{\mathrm{p}}(z) = \left(-\frac{1}{2(1+z)^2} - (1+z) \right) + \frac{3}{2} + \frac{3}{2} z^2 = \mathcal{O}(z^3). \tag{2.10}$$

Therefore, we can define the “new” first order

$$H_0 = H_{\mathrm{p}} + H_{\mathrm{osc}}. \tag{2.11}$$

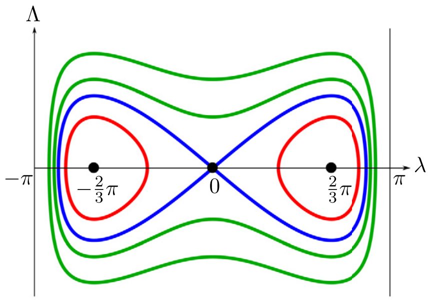

Fig. 2. Phase portrait of the system given by Hamiltonian $H_{\rm p}(\lambda, \Lambda)$ on (2.8). On blue the two separatrices.

From now on, we refer to H_0 as the unperturbed Hamiltonian and we identify H_1 as the perturbation.

The next proposition, proven in [10, Theorem 2.1], gives some properties of the Hamiltonian H.

Proposition 2.1. *The Hamiltonian H, away from collision with the primaries, is real-analytic in the sense of $\overline{H(\lambda, \Lambda, x, y; \delta)} = H(\overline{\lambda}, \overline{\Lambda}, y, x; \overline{\delta})$.*

Moreover, for $\delta > 0$ small enough,

- *The critical point L_3 expressed in coordinates (λ, Λ, x, y) is given by*

$$\mathfrak{L}(\delta) = \left(0, \delta^2 \mathfrak{L}_\Lambda(\delta), \delta^3 \mathfrak{L}_x(\delta), \delta^3 \mathfrak{L}_y(\delta)\right), \tag{2.12}$$

  with $|\mathfrak{L}_\Lambda(\delta)|$, $|\mathfrak{L}_x(\delta)|$, $|\mathfrak{L}_y(\delta)| \leq C$, for some constant $C > 0$ independent of δ.
- *The point $\mathfrak{L}(\delta)$ is a saddle-center equilibrium point and its linearization is*

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix} + \mathcal{O}(\delta).$$

  Therefore, it possesses a one-dimensional unstable and stable manifolds, $\mathcal{W}^{\rm u}(\delta)$ and $\mathcal{W}^{\rm s}(\delta)$.

The unperturbed system given by H_0 in (2.11) has two homoclinic connections in the (λ, Λ)-plane associated to the saddle point $(0, 0)$ and described by the energy level $H_{\rm p}(\lambda, \Lambda) = -\frac{1}{2}$ (see Fig. 2). We define

$$\lambda_0 = \arccos\left(\frac{1}{2} - \sqrt{2}\right), \tag{2.13}$$

which satisfies $H_{\rm p}(\lambda_0, 0) = -\frac{1}{2}$ so that, for the unperturbed system, λ_0 is the "turning point" in the (λ, Λ) variables. We will see that, in our regime, $\theta \approx \lambda$ and thus the value of θ_0 introduced in Remark 1.2 is indeed close to the "turning point" of the invariant manifolds (see Fig. 1(b)).

We rewrite Theorem 1.1, in fact the more general result in Remark 1.2, in the set of coordinates (λ, Λ, x, y). For $\lambda_* \in (0, \lambda_0)$, we consider the 3-dimensional section

$$\mathcal{S}(\lambda_*) = \left\{(\lambda, \Lambda, x, y) \in \mathbb{R}^2 \times \mathbb{C}^2 \ : \ \lambda = \lambda_*, \ \Lambda > 0, \ x = \overline{y}\right\},$$

which is transverse to the flow of H, and we define the first crossings of the invariant manifolds $\mathcal{W}^{\mathrm{u,s}}(\delta)$ with this section as $(\lambda_*, \Lambda_*^{\mathrm{u}}, x_*^{\mathrm{u}}, y_*^{\mathrm{u}})$ and $(\lambda_*, \Lambda_*^{\mathrm{s}}, x_*^{\mathrm{s}}, y_*^{\mathrm{s}})$.

Theorem 2.2. *Fix an interval $[\lambda_1, \lambda_2] \subset (0, \lambda_0)$ with λ_0 as given in (2.13). Then, there exists $\delta_0 > 0$ and $b_0 > 0$ such that, for $\delta \in (0, \delta_0)$ and $\lambda_* \in [\lambda_1, \lambda_2]$, the first crossings are analytic with respect to λ^* and*

$$|\Lambda_*^{\diamond}| \leq b_0, \qquad |x_*^{\diamond}|, |y_*^{\diamond}| \leq b_0 \delta^3, \qquad \diamond = \mathrm{u,s}. \tag{2.14}$$

Moreover,

$$|x_*^{\mathrm{u}} - x_*^{\mathrm{s}}| = |y_*^{\mathrm{u}} - y_*^{\mathrm{s}}| = \sqrt[6]{2}\, \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right],$$
$$|\Lambda_*^{\mathrm{u}} - \Lambda_*^{\mathrm{s}}| = \mathcal{O}(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}),$$

where A and Θ are the constants introduced in Theorem 1.1.

2.1. Proof of Theorem 1.1

To prove Theorem 1.1 (and Remark 1.2) from Theorem 2.2 we need to "undo" the changes of coordinates ϕ_{Poi} and ϕ_{sc} and adjust the section from λ = constant to θ = constant.

First, we consider the change ϕ_{sc} given by $(\lambda, L, \eta, \xi) = (\lambda, 1+\delta^2\Lambda, \delta x, \delta y)$, (see (2.5)). For $\lambda_* \in [\lambda_1, \lambda_2]$ we define

$$L^{\diamond}(\lambda_*; \delta) = 1 + \delta^2 \Lambda_*^{\diamond}, \quad \eta^{\diamond}(\lambda_*; \delta) = \delta x_*^{\diamond}, \quad \xi^{\diamond}(\lambda_*; \delta) = \delta y_*^{\diamond}, \quad \text{for } \diamond = \mathrm{u,s}. \tag{2.15}$$

Then, by Theorem 2.2, one has

$$\begin{aligned} |\Delta L(\lambda_*; \delta)| &= |L^{\mathrm{u}}(\lambda_*; \delta) - L^{\mathrm{s}}(\lambda_*; \delta)| = \mathcal{O}\left(\delta^{\frac{10}{3}} e^{-\frac{A}{\delta^2}}\right), \\ |\Delta \eta(\lambda_*; \delta)| &= |\eta^{\mathrm{u}}(\lambda_*; \delta) - \eta^{\mathrm{s}}(\lambda_*; \delta)| = \sqrt[6]{2}\, \delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right], \\ \Delta \xi(\lambda_*; \delta) &= \overline{\Delta \eta(\lambda_*; \delta)}. \end{aligned} \tag{2.16}$$

Next, we study the change ϕ_{Poi}. In the following result, we give a series expression of the polar coordinates with respect to the Poincaré elements. Even though its proof is a consequence of the definition of the Poincaré variables (see Section 4.1 in [10]), we provide it in Appendix A.

Lemma 2.3. *Fix $\varrho > 0$. Then, for $|(L-1,\eta,\xi)| \ll 1$ and $|\mathrm{Im}\,\lambda| \le \varrho$, the polar coordinates (r,θ,R,G) introduced in (1.2) satisfy*

$$
\begin{aligned}
r &= 1 + 2(L-1) - \frac{e^{-i\lambda}}{\sqrt{2}}\eta - \frac{e^{i\lambda}}{\sqrt{2}}\xi + \mathcal{O}(L-1,\eta,\xi)^2,\\
\theta &= \lambda + i\sqrt{2}e^{-i\lambda}\eta - i\sqrt{2}e^{i\lambda}\xi + \mathcal{O}(L-1,\eta,\xi)^2,\\
R &= \frac{ie^{-i\lambda}}{\sqrt{2}}\eta - \frac{ie^{i\lambda}}{\sqrt{2}}\xi + \mathcal{O}(L-1,\eta,\xi)^2, \qquad G = L - \eta\xi.
\end{aligned}
$$

Since in Theorem 2.2 the distance is measured in the section $\lambda = \lambda_*$ whereas the Theorem 1.1, and more generally Remark 1.2, measures it in the section $\theta = \theta^*$, we must "translate" the estimates in (2.16) to the new section. By Lemma 2.3, let g_θ be the function such that $\theta = \lambda + g_\theta(\lambda, L, \eta, \xi)$. Then, for $\diamond = \mathrm{u}, \mathrm{s}$, we consider

$$
F^\diamond(\lambda,\theta,\delta) = \theta - \lambda + g_\theta\left(\lambda, L^\diamond(\lambda;\delta), \eta^\diamond(\lambda;\delta), \xi^\diamond(\lambda;\delta)\right).
$$

Applying the Implicit Function Theorem, Lemma 2.3 and that, by (2.15), $L^\diamond(\lambda;0) = 1$ and $\eta^\diamond(\lambda;0) = \xi^\diamond(\lambda;0) = 0$, then there exist function $\widehat{\lambda}^\diamond(\theta;\delta)$ such that $F^\diamond(\widehat{\lambda}^\diamond(\theta;\delta),\theta,\delta) = 0$ and

$$
\begin{aligned}
\widehat{\lambda}^\diamond(\theta;\delta) = {} & \theta - i\sqrt{2}e^{-i\theta}\widehat{\eta}^\diamond(\theta;\delta) + i\sqrt{2}e^{i\theta}\widehat{\xi}^\diamond(\theta;\delta)\\
& + \mathcal{O}\left(\widehat{L}^\diamond(\theta;\delta) - 1, \widehat{\eta}^\diamond(\theta;\delta), \widehat{\xi}^\diamond(\theta;\delta)\right)^2,
\end{aligned}
\tag{2.17}
$$

with $\widehat{\eta}^\diamond(\theta;\delta) = \eta^\diamond(\widehat{\lambda}^\diamond(\theta;\delta);\delta)$, $\widehat{\xi}^\diamond(\theta;\delta) = \xi^\diamond(\widehat{\lambda}^\diamond(\theta;\delta);\delta)$ and $\widehat{L}^\diamond(\theta;\delta) = L^\diamond(\widehat{\lambda}^\diamond(\theta;\delta);\delta)$. Notice that, by (2.14) (plus Cauchy estimates for their derivatives) and (2.15),

$$
\widehat{\lambda}^\diamond(\theta;\delta) = \theta + \mathcal{O}(\delta^4).
$$

Thus, for any $[\theta_1,\theta_2] \subset (0,\lambda_0)$ and δ small enough, there exists $[\lambda_1,\lambda_2] \subset (0,\lambda_0)$ such that, for $\theta \in [\theta_1,\theta_2]$ one has $\widehat{\lambda}^{\mathrm{u,s}}(\theta;\delta) \in [\lambda_1,\lambda_2]$. In addition,

$$
\begin{aligned}
\widehat{L}^\diamond(\theta;\delta) &= L^\diamond(\theta;\delta) + \mathcal{O}(\delta^6) = 1 + \mathcal{O}(\delta^2),\\
\widehat{\eta}^\diamond(\theta;\delta) &= \eta^\diamond(\theta;\delta) + \mathcal{O}(\delta^8) = \mathcal{O}(\delta^4),\\
\widehat{\xi}^\diamond(\theta;\delta) &= \xi^\diamond(\theta;\delta) + \mathcal{O}(\delta^8) = \mathcal{O}(\delta^4).
\end{aligned}
\tag{2.18}
$$

Then, since $\Lambda_*^{\mathrm{u,s}} > 0$, by (2.15) one has that $\widehat{L}^{\mathrm{u,s}}(\theta;\delta) > 1$ for $\theta \in [\theta_1,\theta_2]$. Moreover, by Lemma 2.3 and taking δ small enough, one has $r^{\mathrm{u,s}}(\theta) - 1 > 0$.

The difference between the invariant manifolds in a section of fixed $\theta \in [\theta_1, \theta_2]$ is given by

$$\Delta\widehat{\lambda}(\theta;\delta) = \widehat{\lambda}^{\mathrm{u}}(\theta;\delta) - \widehat{\lambda}^{\mathrm{s}}(\theta;\delta), \qquad \Delta\widehat{L}(\theta;\delta) = \widehat{L}^{\mathrm{u}}(\theta;\delta) - \widehat{L}^{\mathrm{s}}(\theta;\delta),$$
$$\Delta\widehat{\eta}(\theta;\delta) = \widehat{\eta}^{\mathrm{u}}(\theta;\delta) - \widehat{\eta}^{\mathrm{s}}(\theta;\delta), \qquad \Delta\widehat{\xi}(\theta;\delta) = \widehat{\xi}^{\mathrm{u}}(\theta;\delta) - \widehat{\xi}^{\mathrm{s}}(\theta;\delta).$$

Then, by (2.17) and (2.18), one has that

$$\Delta\widehat{\lambda}(\theta;\delta) = -i\sqrt{2}e^{-i\theta}\Delta\widehat{\eta}(\theta;\delta) + i\sqrt{2}e^{i\theta}\Delta\widehat{\xi}(\theta;\delta) + \mathcal{O}\left(\delta^2\Delta\widehat{L}(\theta;\delta), \delta^4\Delta\widehat{\eta}(\theta;\delta), \delta^4\Delta\widehat{\xi}(\theta;\delta)\right).$$

Moreover, by the mean value theorem, (2.16) and (2.18),

$$\begin{aligned}\Delta\widehat{L}(\theta;\delta) &= \Delta L(\widehat{\lambda}^{\mathrm{u}}(\theta;\delta);\delta) + \widehat{L}^{\mathrm{s}}(\widehat{\lambda}^{\mathrm{u}}(\theta;\delta);\delta) - \widehat{L}^{\mathrm{s}}(\widehat{\lambda}^{\mathrm{s}}(\theta;\delta);\delta) \\ &= \mathcal{O}(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}}) + \delta^2\mathcal{O}\left(\Delta\widehat{\lambda}(\theta;\delta)\right).\end{aligned}$$

Analogously,

$$\begin{aligned}\Delta\widehat{\eta}(\theta;\delta) &= \Delta\eta(\widehat{\lambda}^{\mathrm{u}}(\theta;\delta);\delta) + \delta^4\mathcal{O}\left(\Delta\widehat{\lambda}(\theta;\delta)\right), \\ \Delta\widehat{\xi}(\theta;\delta) &= \overline{\Delta\eta(\lambda^{\mathrm{u}}(\theta;\delta);\delta)} + \delta^4\mathcal{O}\left(\Delta\widehat{\lambda}(\theta;\delta)\right).\end{aligned}$$

Therefore, using (2.16), one can conclude that

$$|\Delta\widehat{\lambda}(\theta;\delta)| = \mathcal{O}\left(\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}\right), \qquad |\Delta\widehat{\eta}(\theta;\delta)| = \sqrt[6]{2}\,\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}\left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right],$$
$$|\Delta\widehat{L}(\theta;\delta)| = \mathcal{O}\left(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}}\right), \qquad \Delta\widehat{\xi}(\theta;\delta) = \overline{\Delta\widehat{\eta}(\theta;\delta)}.$$

Once we have adjusted the transverse section, it only remains to apply Lemma 2.3 to translate these differences to polar coordinates. That is,

$$\begin{aligned} r^{\mathrm{u}} - r^{\mathrm{s}} &= -\sqrt{2}\cos\theta\,\mathrm{Re}\,\Delta\widehat{\eta}(\theta;\delta) - \sqrt{2}\sin\theta\,\mathrm{Im}\,\Delta\widehat{\eta}(\theta;\delta) + \mathcal{O}(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}}), \\ R^{\mathrm{u}} - R^{\mathrm{s}} &= -\sqrt{2}\cos\theta\,\mathrm{Im}\,\Delta\widehat{\eta}(\theta;\delta) + \sqrt{2}\sin\theta\,\mathrm{Re}\,\Delta\widehat{\eta}(\theta;\delta) + \mathcal{O}(\delta^{\frac{16}{3}}e^{-\frac{A}{\delta^2}}), \\ G^{\mathrm{u}} - G^{\mathrm{s}} &= \mathcal{O}(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}}), \end{aligned}$$

which implies

$$\begin{aligned}\|(r^{\mathrm{u}}, R^{\mathrm{u}}, G^{\mathrm{u}}) - (r^{\mathrm{s}}, R^{\mathrm{s}}, G^{\mathrm{s}})\| &= \sqrt{2}\,|\Delta\widehat{\eta}(\theta;\delta)| + \mathcal{O}(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}}) \\ &= \sqrt[3]{4}\,\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}\left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right].\end{aligned}$$

To conclude the proof of Theorem 1.1, it is enough to recall that $\delta = \mu^{\frac{1}{4}}$.

3. Proof of Theorem 2.2

In this section, we present the main steps necessary to prove Theorem 2.2 (see the list in Section 1) and complete its proof. In Section 3.1 we summarize the results concerning the analysis of the separatrix of the unperturbed Hamiltonian H_{p} (see (2.8)) done in [10] (Step B). In Section 3.2, we prove the existence of parametrizations of the perturbed invariant manifolds in suitable complexes domains (Step E). In Section 3.3, we study the difference between the perturbed manifolds near the singularities of the perturbed separatrix. In particular, in Section 3.3.1, we summarize the results concerning the derivation (Step C) and analysis (Step D) of the inner equation obtained in [10] and, in Section 3.3.2, we compare certain solutions of the inner equation with the parametrizations of the perturbed manifolds by means of complex matching techniques (Step F). Finally, in Section 3.4, we combine all the previous results to obtain the dominant term of the difference between the invariant manifolds and prove Theorem 2.2 (Step G).

3.1. Analytical continuation of the unperturbed separatrix

The unperturbed Hamiltonian

$$H_0(\lambda, \Lambda, x, y) = H_{\mathrm{p}}(\lambda, \Lambda) + H_{\mathrm{osc}}(x, y)$$

(see (2.11)) possesses a saddle with two separatrices in the (λ, Λ)-plane (see Fig. 2). Let us consider the real-analytic time parametrization of the separatrix with $\lambda \in (0, \pi)$,

$$\begin{aligned} \sigma : \mathbb{R} &\to \mathbb{T} \times \mathbb{R} \\ t &\mapsto \sigma(t) = (\lambda_h(t), \Lambda_h(t)), \end{aligned} \tag{3.1}$$

with initial condition $\sigma(0) = (\lambda_0, 0)$ where $\lambda_0 = \arccos\left(\frac{1}{2} - \sqrt{2}\right) \in \left(\frac{2}{3}\pi, \pi\right)$.

The following result (which encompass Theorem 2.2, Proposition 2.3 and Corollary 2.4 in [10]) gives the properties of the analytic extension of $\sigma(t)$ to the domain

$$\begin{aligned} \Pi^{\mathrm{ext}}_{A,\beta} = &\{t \in \mathbb{C} \;:\; |\operatorname{Im} t| < \tan\beta \operatorname{Re} t + A\} \cup \\ &\{t \in \mathbb{C} \;:\; |\operatorname{Im} t| < -\tan\beta \operatorname{Re} t + A\}, \end{aligned} \tag{3.2}$$

with A as given in (1.3) (see Fig. 3).

Theorem 3.1. *The real-analytic time parametrization σ defined in (3.1) satisfies:*

- *There exists $0 < \beta_0 < \frac{\pi}{2}$ such that $\sigma(t)$ extends analytically to Π_{A,β_0}.*
- *$\sigma(t)$ has only two singularities on $\partial\Pi^{\mathrm{ext}}_{A,\beta_0}$ at $t = \pm iA$.*

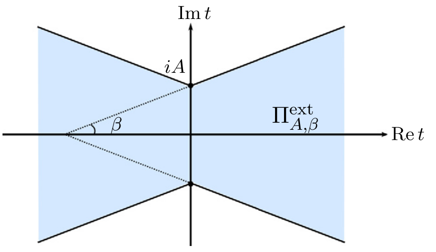

Fig. 3. Representation of the domain $\Pi^{\mathrm{ext}}_{A,\beta}$ in (3.2).

- *There exists* $\upsilon > 0$ *such that, for* $t \in \mathbb{C}$ *with* $|t - iA| < \upsilon$ *and* $\arg(t - iA) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$,

$$\lambda_h(t) = \pi + 3\alpha_+(t - iA)^{\frac{2}{3}} + \mathcal{O}(t - iA)^{\frac{4}{3}},$$
$$\Lambda_h(t) = -\frac{2\alpha_+}{3}\frac{1}{(t - iA)^{\frac{1}{3}}} + \mathcal{O}(t - iA)^{\frac{1}{3}},$$

with $\alpha_+ \in \mathbb{C}$ *such that* $\alpha_+^3 = \frac{1}{2}$.

An analogous result holds for $|t + iA| < \upsilon$, $\arg(t + iA) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ *and* $\alpha_- = \overline{\alpha_+}$.

- $\Lambda_h(t)$ *has only one zero in* $\overline{\Pi^{\mathrm{ext}}_{A,\beta_0}}$ *at* $t = 0$.

3.2. The perturbed invariant manifolds

In this section, following the approach described in [8,7,33], we study the analytic continuation of the parametrizations of the perturbed one-dimensional stable and unstable manifolds, $\mathcal{W}^{\mathrm{u}}(\delta)$ and $\mathcal{W}^{\mathrm{s}}(\delta)$.

Since we measure the distance between the invariant manifolds in the section $\lambda = \lambda_*$ (see Theorem 2.2), we parameterize them as graphs with respect to λ (whenever is possible) or, more conveniently, with respect to the independent variable u defined by $\lambda = \lambda_h(u)$.

To define these suitable parameterizations we first translate the equilibrium point $\mathfrak{L}(\delta)$ to $\mathbf{0}$ by the change of coordinates

$$\phi_{\mathrm{eq}} : (\lambda, \Lambda, x, y) \mapsto (\lambda, \Lambda, x, y) + \mathfrak{L}(\delta). \tag{3.3}$$

Second, we consider the symplectic change of coordinates

$$\phi_{\mathrm{sep}} : (u, w, x, y) \to (\lambda, \Lambda, x, y), \quad \lambda = \lambda_h(u), \quad \Lambda = \Lambda_h(u) - \frac{w}{3\Lambda_h(u)}. \tag{3.4}$$

We refer to (u, w, x, y) as the *separatrix coordinates*.

Let us remark that ϕ_{sep} is not defined for $u = 0$ since $\Lambda_h(0) = 0$ (see Theorem 3.1). We deal with this fact later when considering the domain of definition for u.

After these changes of variables, we look for the perturbed invariant manifolds as a graph with respect to u. In other words, we look for functions

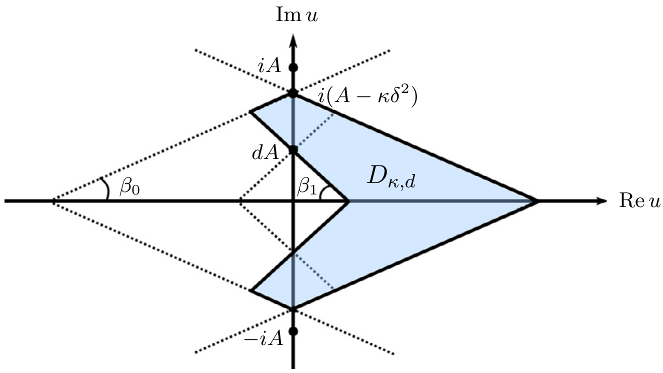

Fig. 4. The boomerang domain $D_{\kappa,d}$ defined in (3.7).

$$z^\diamond(u) = (w^\diamond(u), x^\diamond(u), y^\diamond(u))^T, \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s},$$

such that the invariant manifolds given in Proposition 2.1 can be expressed as

$$\mathcal{W}^\diamond(\delta) = \left\{ \left(\lambda_h(u), \Lambda_h(u) - \frac{w^\diamond(u)}{3\Lambda_h(u)}, x^\diamond(u), y^\diamond(u) \right) + \mathfrak{L}(\delta) \right\}, \quad \text{for } \diamond = \mathrm{u}, \mathrm{s}, \tag{3.5}$$

with u belonging to an appropriate domain contained in $\Pi^{\mathrm{ext}}_{A,\beta_0}$ (see (3.2)). The graphs z^{u} and z^{s} must satisfy the asymptotic conditions

$$\lim_{\mathrm{Re}\, u \to -\infty} \left(\frac{w^{\mathrm{u}}(u)}{\Lambda_h(u)}, x^{\mathrm{u}}(u), y^{\mathrm{u}}(u) \right) = \lim_{\mathrm{Re}\, u \to +\infty} \left(\frac{w^{\mathrm{s}}(u)}{\Lambda_h(u)}, x^{\mathrm{s}}(u), y^{\mathrm{s}}(u) \right) = 0. \tag{3.6}$$

Remark 3.2. Since the Hamiltonian H is real-analytic in the sense of $\overline{H(\lambda, \Lambda, x, y; \delta)} = H(\overline{\lambda}, \overline{\Lambda}, y, x; \overline{\delta})$ (see Proposition 2.1), then we say that $z(u) = (w(u), x(u), y(u))^T$ is real-analytic if it satisfies

$$w(\overline{u}) = \overline{w(u)}, \qquad x(\overline{u}) = y(u), \qquad y(\overline{u}) = x(u).$$

The classical way to study exponentially small splitting of separatrices, in this setting, is to look for solutions z^{u} and z^{s} in a certain complex common domain containing a segment of the real line and intersecting a $\mathcal{O}(\delta^2)$ neighborhood of the singularities $u = \pm iA$ of the separatrix.

Recall that the invariant manifolds can not be expressed as a graph in a neighborhood of $u = 0$. To overcome this technical problem, we find solutions z^{u} and z^{s} defined in a complex domain, which we call *boomerang domain* due to its shape (see Fig. 4). Namely,

$$\begin{aligned} D_{\kappa,d} = \{u \in \mathbb{C} \ : \ & |\mathrm{Im}\, u| < A - \kappa\delta^2 + \tan\beta_0 \mathrm{Re}\, u, |\mathrm{Im}\, u| < A - \kappa\delta^2 - \tan\beta_0 \mathrm{Re}\, u, \\ & |\mathrm{Im}\, u| > dA - \tan\beta_1 \mathrm{Re}\, u\}, \end{aligned} \tag{3.7}$$

where $\kappa > 0$ is such that $A - \kappa\delta^2 > 0$, β_0 is the constant given in Theorem 3.1 and $\beta_1 \in [\beta_0, \frac{\pi}{2})$ and $d \in (\frac{1}{4}, \frac{1}{2})$ are independent of δ.

Theorem 3.3. *Fix a constant $d \in (\frac{1}{4}, \frac{1}{2})$. Then, there exists $\delta_0, \kappa_0 > 0$ such that, for $\delta \in (0, \delta_0)$, $\kappa \geq \kappa_0$, the graph parameterizations z^{u} and z^{s} introduced in (3.5) can be extended real-analytically to the domain $D_{\kappa,d}$.*

Moreover, there exists a real constant $b_1 > 0$ independent of δ and κ such that, for $u \in D_{\kappa,d}$ we have that

$$|w^\diamond(u)| \leq \frac{b_1\delta^2}{|u^2 + A^2|} + \frac{b_1\delta^4}{|u^2 + A^2|^{\frac{8}{3}}}, \quad |x^\diamond(u)| \leq \frac{b_1\delta^3}{|u^2 + A^2|^{\frac{4}{3}}}, \quad |y^\diamond(u)| \leq \frac{b_1\delta^3}{|u^2 + A^2|^{\frac{4}{3}}}.$$

Notice that the asymptotic conditions (3.6) do not have any meaning in the domain $D_{\kappa,d}$ since it is bounded. Therefore, to prove the existence of z^{u} and z^{s} in $D_{\kappa,d}$ one has to start with different domains where these asymptotic conditions make sense and then find a way to extend them real-analytically to $D_{\kappa,d}$. We describe the details of these processes in the following Sections 3.2.1 and 3.2.2.

3.2.1. Analytic extension of the stable and unstable manifolds

The Hamiltonian H written in separatrix coordinates (see (3.3) and (3.4)) becomes

$$H^{\mathrm{sep}} = H_0^{\mathrm{sep}} + H_1^{\mathrm{sep}}, \tag{3.8}$$

with

$$H_0^{\mathrm{sep}} = w + \frac{xy}{\delta^2}, \qquad H_1^{\mathrm{sep}} = H \circ (\phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}}) - H_0^{\mathrm{sep}}. \tag{3.9}$$

Introducing the notation $z = (w, x, y)^T$ and defining

$$\mathcal{A}^{\mathrm{sep}} = \frac{i}{\delta^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.10}$$

the equations associated to the Hamiltonian H^{sep} can be written as

$$\begin{cases} \dot{u} = 1 + g^{\mathrm{sep}}(u, z), \\ \dot{z} = \mathcal{A}^{\mathrm{sep}} z + f^{\mathrm{sep}}(u, z), \end{cases} \tag{3.11}$$

where $g^{\mathrm{sep}} = \partial_w H_1^{\mathrm{sep}}$ and $f^{\mathrm{sep}} = (-\partial_u H_1^{\mathrm{sep}}, i\partial_y H_1^{\mathrm{sep}}, -i\partial_x H_1^{\mathrm{sep}})^T$. Consequently, the parameterizations $z^{\mathrm{u}}(u)$ and $z^{\mathrm{s}}(u)$ given in (3.5) satisfy the invariance equation

$$\partial_u z^\diamond = \mathcal{A}^{\mathrm{sep}} z^\diamond + \mathcal{R}^{\mathrm{sep}}[z^\diamond], \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s}, \tag{3.12}$$

with

$$\mathcal{R}^{\text{sep}}[\varphi](u)=\frac{f^{\text{sep}}(u,\varphi)-g^{\text{sep}}(u,\varphi)\mathcal{A}^{\text{sep}}\varphi}{1+g^{\text{sep}}(u,\varphi)}. \tag{3.13}$$

Remark 3.4. Note that one can use this invariance equation whenever

$$1+g^{\text{sep}}(u,\varphi)=1+\partial_w H_1^{\text{sep}}(u,\varphi)\neq 0.$$

This condition is satisfied in the different domains that are considered in this section and in the forthcoming ones and it is checked in Appendix B (see (B.16) and (B.32)). This fact is also used later in Section 3.3.

The first step is to look for solutions of this equation in the domains

$$D^{\mathrm{u},\infty}_{\rho_1}=\{u\in\mathbb{C}\;:\;\operatorname{Re}u<-\rho_1\}\,,\qquad D^{\mathrm{s},\infty}_{\rho_1}=\{u\in\mathbb{C}\;:\;\operatorname{Re}u>\rho_1\}\,, \tag{3.14}$$

for some $\rho_1>0$, which allows us to take into account the asymptotic conditions (3.6).

Proposition 3.5. *Fix $\rho_1>0$. Then, there exists $\delta_0>0$ such that, for $\delta\in(0,\delta_0)$, the equation (3.12) has a unique real-analytic solution $z^\diamond=(w^\diamond,x^\diamond,y^\diamond)^T$ in $D^{\diamond,\infty}_{\rho_1}$ (for $\diamond=\mathrm{u},\mathrm{s}$) satisfying the corresponding asymptotic condition (3.6).*

Moreover, there exists $b_2>0$ independent of δ such that, for $u\in D^{\diamond,\infty}_{\rho_1}$,

$$|w^\diamond(u)e^{-2\nu u}|\leq b_2\delta^2,\qquad |x^\diamond(u)e^{-\nu u}|\leq b_2\delta^3,\qquad |y^\diamond(u)e^{-\nu u}|\leq b_2\delta^3,$$

with $\nu=\sqrt{\frac{21}{8}}$ for $\diamond=\mathrm{u}$ and $\nu=-\sqrt{\frac{21}{8}}$ for $\diamond=\mathrm{s}$.

This proposition is proved in Section 4.1.

To extend analytically the invariant manifolds to reach the boomerang domain $D_{\kappa,d}$ we have to face the problem that these parameterizations become undefined at $u=0$. To overcome it, first we extend the solutions z^{u} and z^{s} of Proposition 3.5 to the *outer domains* (see Fig. 5)

$$\begin{aligned}D^{\mathrm{u},\mathrm{out}}_{\kappa,d_1,\rho_2}=\{u\in\mathbb{C}\;:\;&|\operatorname{Im}u|<A-\kappa\delta^2-\tan\beta_0\operatorname{Re}u,\\ &|\operatorname{Im}u|>d_1A+\tan\beta_1\operatorname{Re}u,\ \operatorname{Re}u>-\rho_2\}\,,\\ D^{\mathrm{s},\mathrm{out}}_{\kappa,d_1,\rho_2}=\big\{u\in\mathbb{C}\;:\;&-u\in D^{\mathrm{u},\mathrm{out}}_{\kappa,d_1,\rho_2}\big\},\end{aligned} \tag{3.15}$$

where $d_1\in(\frac14,\frac12)$ and $\rho_2>\rho_1$ are fixed independent of δ, and $\kappa>0$ is such that $A-\kappa\delta^2>0$.

Proposition 3.6. *Consider the functions z^{u}, z^{s} and the constant $\rho_1>0$ obtained in Proposition 3.5. Fix constants $\rho_2>\rho_1$ and $d_1\in(\frac14,\frac12)$. Then, there exist $\delta_0,\kappa_1>0$ such that, for $\delta\in(0,\delta_0)$, $\kappa\geq\kappa_1$, the functions $z^\diamond=(w^\diamond,x^\diamond,y^\diamond)^T$, $\diamond=\mathrm{u},\mathrm{s}$, can be extended analytically to the domain $D^{\diamond,\mathrm{out}}_{\kappa,d_1,\rho_2}$.*

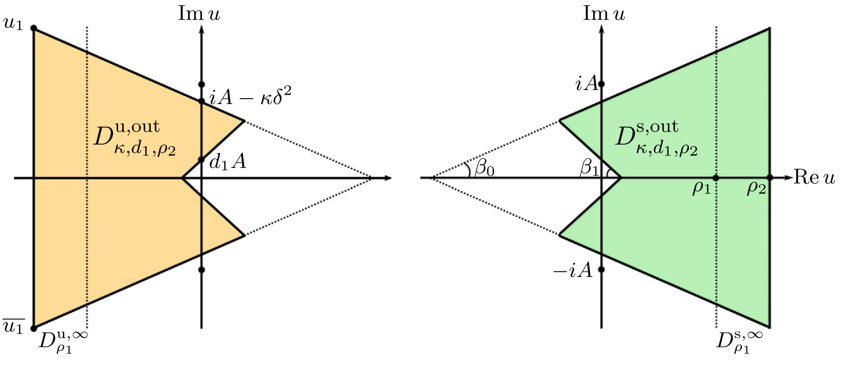

Fig. 5. The outer domains $D^{\mathrm{u,out}}_{\kappa,d_1,\rho_2}$ and $D^{\mathrm{s,out}}_{\kappa,d_1,\rho_2}$ defined in (3.15).

Moreover, there exists $b_3 > 0$ independent of δ and κ such that, for $u \in D^{\diamond,\mathrm{out}}_{\kappa,d_1,\rho_2}$,

$$|w^\diamond(u)| \le \frac{b_3\delta^2}{|u^2+A^2|} + \frac{b_3\delta^4}{|u^2+A^2|^{\frac{8}{3}}}, \quad |x^\diamond(u)| \le \frac{b_3\delta^3}{|u^2+A^2|^{\frac{4}{3}}}, \quad |y^\diamond(u)| \le \frac{b_3\delta^3}{|u^2+A^2|^{\frac{4}{3}}}.$$

This proposition is proved in Section 4.2.

Notice that taking ρ_2 big enough, $d_1 \le d$ and $\kappa_1 \le \kappa_0$ we have $D_{\kappa_0,d} \subset D^{\mathrm{s,out}}_{\kappa_1,d_1,\rho_2}$. Therefore, for the stable manifold z^{s}, Proposition 3.6 implies Theorem 3.3. However, we still need to extend further z^{u} in order to reach $D_{\kappa_0,d}$.

3.2.2. Further analytic extension of the unstable manifold

Since by Proposition 3.6 the unstable solution z^{u} is defined in $D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2}$. To prove Theorem 3.3 it only remains to extend it to the points in the boomerang domain $D_{\kappa_0,d}$ which do not belong to the outer unstable domain. Namely, we extend z^{u} to

$$\begin{aligned}\widetilde{D}_{\kappa,d} = \{u \in \mathbb{C} \ : \ &|\mathrm{Im}\, u| < A - \kappa\delta^2 - \tan\beta_0 \mathrm{Re}\, u, \\ &|\mathrm{Im}\, u| < dA + \tan\beta_1 \mathrm{Re}\, u, \ |\mathrm{Im}\, u| > dA - \tan\beta_1 \mathrm{Re}\, u\},\end{aligned} \tag{3.16}$$

for suitable κ and d (see Fig. 6). Notice that $\widetilde{D}_{\kappa,d} \subset D_{\kappa,d}$ and that $\widetilde{D}_{\kappa,d}$ only contains points at distance of $u = \pm iA$ of order 1 with respect to δ.

As we have mentioned, to measure the difference between the invariant manifolds $\mathcal{W}^{\mathrm{u}}(\delta)$ and $\mathcal{W}^{\mathrm{s}}(\delta)$ it is convenient to parameterize them as graphs (see (3.5)). However, these graph parametrizations are not defined at $u = 0$. Moreover, since all the fixed point arguments that we apply to obtain the graph parameterizations rely on complex path integration, we are not able to extend them to domains which are not simply connected. Therefore, to reach $\widetilde{D}_{\kappa,d}$ from $D^{\mathrm{u,out}}_{\kappa,d_1,\rho_2}$, we need to switch to a different parametrization that is well defined at $u = 0$.

The auxiliary parametrization we consider is the classical time-parametrization which is associated to the Hamiltonian H in (2.7). (Recall that the graph parametrization z^{u} was associated to the Hamiltonian $H^{\mathrm{sep}} = H \circ \phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}}$).

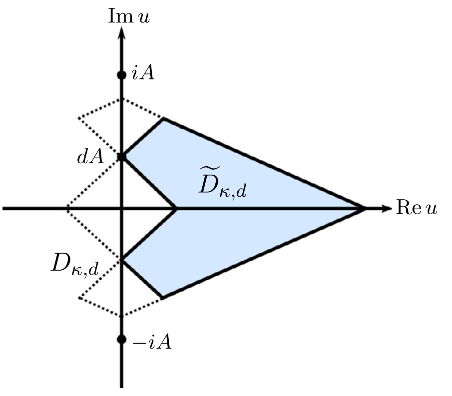

Fig. 6. The domain $\widetilde{D}_{\kappa,d}$ defined in (3.16).

This analytic extension procedure has three steps:

1. We consider the *outer transition domain* (see Fig. 7)

$$\begin{aligned}\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3} = \{v \in \mathbb{C} \;:\; & |\mathrm{Im}\, v| < A - \kappa_2\delta^2 - \tan\beta_0 \mathrm{Re}\, v, \\ & |\mathrm{Im}\, v| > d_2 A + \tan\beta_1 \mathrm{Re}\, v, \\ & |\mathrm{Im}\, v| < d_3 A + \tan\beta_1 \mathrm{Re}\, v\}\,,\end{aligned} \tag{3.17}$$

where $d_1 < d_2 < d_3 < \frac{1}{2}$ are independent of δ and $\kappa_2 > \kappa_1$ is such that $A - \kappa_2\delta^2 > 0$. Notice that $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3} \subset D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2}$.

Since $\dot{u} = 1 + o(1)$ (see (3.11)), we look for a real-analytic and close to the identity change of coordinates $u = v + \mathcal{U}(v)$ defined in $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$ such that the time-parametrization

$$\Gamma^{\mathrm{u}}(v) = \phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}}(v + \mathcal{U}(v), z^{\mathrm{u}}(v + \mathcal{U}(v))) \tag{3.18}$$

is a solution of the Hamiltonian H in (2.7). That is, $\dot{v} = 1$ and $\Gamma^{\mathrm{u}}(v) \in \mathcal{W}^{\mathrm{u}}(\delta)$ for $v \in \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$. See the details in Proposition 3.7 and Corollary 3.8 below.

2. We extend analytically the time-parametrization $\Gamma^{\mathrm{u}}(v)$ to reach the domain $\widetilde{D}_{\kappa,d}$. In particular, we extend Γ^{u} to the *flow domain*

$$\begin{aligned}D^{\mathrm{fl}}_{\kappa_3,d_4} = \{v \in \mathbb{C} \;:\; & |\mathrm{Im}\, v| < A - \kappa\delta^2 - \tan\beta_0 \mathrm{Re}\, v, \\ & |\mathrm{Im}\, v| < d_4 A + \tan\beta_1 \mathrm{Re}\, v\}\,,\end{aligned} \tag{3.19}$$

where $d_4 \in (d_2, d_3)$ is independent of δ and $\kappa_3 > \kappa_2$ is such that $A - \kappa_3\delta^2 > 0$. Notice that,

$$\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3} \cap D^{\mathrm{fl}}_{\kappa_3,d_4} \neq \emptyset \qquad \text{and} \qquad \widetilde{D}_{\kappa_4,d_5} \subset D^{\mathrm{fl}}_{\kappa_3,d_4},$$

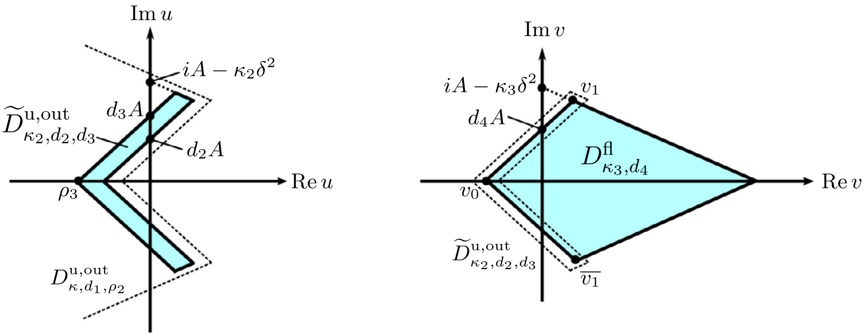

Fig. 7. The domain $\widetilde{D}^{\rm u,out}_{\kappa_2,d_2,d_3}$ given in (3.17) (left) and $D^{\rm fl}_{\kappa_3,d_4}$ in (3.19) (right).

for $d_5 \in (d_1, d_4)$ and $\kappa_4 > \kappa_3$. See the details in Proposition 3.9.

3. We prove that there exists a real-analytic close to the identity change of variables of the form $v = u + \mathcal{V}(u)$, $u \in \widetilde{D}_{\kappa_4,d_5}$, such that the function $z^{\rm u}(u)$ defined by

$$(u, z^{\rm u}(u)) = (\phi_{\rm eq} \circ \phi_{\rm sep})^{-1}\Big(\Gamma^{\rm u}(u + \mathcal{V}(u))\Big) \tag{3.20}$$

gives an invariant graph of $H^{\rm sep}$ in (3.8). See the details in Proposition 3.10 and Corollary 3.11 below.

As a consequence, we have extended analytically $z^{\rm u}$ to $\widetilde{D}_{\kappa_4,d_5}$.

For the first step, we look for a function $\mathcal{U}$ such that $(v + \mathcal{U}(v), z^{\rm u}(v + \mathcal{U}(v)))$ is a solution of the differential equations given by the Hamiltonian $H^{\rm sep}$ in (3.8). Therefore, $\mathcal{U}$ satisfies

$$\partial_v \mathcal{U}(v) = \partial_w H_1^{\rm sep}\left(v + \mathcal{U}(v), z^{\rm u}(v + \mathcal{U}(v))\right). \tag{3.21}$$

The next proposition ensures that $\mathcal{U}$ exists and it is well defined for $v \in \widetilde{D}^{\rm u,out}_{\kappa_2,d_2,d_3}$.

Proposition 3.7. *Let the function $z^{\rm u}$ and the constants ρ_2, d_1 and κ_1 be as obtained in Proposition 3.6 and consider constants $d_2, d_3 \in (d_1, \frac{1}{2})$ such that $d_2 < d_3$ and $\kappa_2 > \kappa_1$. Then, there exists δ_0 such that, for $\delta \in (0, \delta_0)$, the equation (3.21) has a real-analytic solution $\mathcal{U} : \widetilde{D}^{\rm u,out}_{\kappa_2,d_2,d_3} \to \mathbb{C}$.*

Moreover, for some constant $b_4 > 0$ independent of δ and for $v \in \widetilde{D}^{\rm u,out}_{\kappa_2,d_2,d_3}$, $\mathcal{U}$ satisfies

$$|\mathcal{U}(v)| \leq b_4 \delta^2 \qquad \textit{and} \qquad v + \mathcal{U}(v) \in D^{\rm u,out}_{\kappa_1,d_1,\rho_2}.$$

This proposition is proved in Section 4.3. Together with Proposition 3.7 implies the following corollary.

Corollary 3.8. *Under the hypothesis of Proposition 3.7, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, the function $\Gamma^{\rm u}$ in (3.18) is well defined and real-analytic in $\widetilde{D}^{\rm u,out}_{\kappa_2,d_2,d_3}$.*

On the following, we use without mention that $\Gamma^{\mathrm{u}}(v)$ can be split as

$$\Gamma^{\mathrm{u}}(v) = \Gamma_h(v) + \widehat{\Gamma}(v), \quad \text{with} \quad \begin{cases} \Gamma_h = (\lambda_h, \Lambda_h, 0, 0)^T, \\ \widehat{\Gamma} = (\hat{\lambda}, \hat{\Lambda}, \hat{x}, \hat{y})^T. \end{cases} \tag{3.22}$$

The next proposition extends the parametrization Γ^{u} to the domain $D^{\mathrm{fl}}_{\kappa_3,d_4}$ (see (3.19)).

Proposition 3.9. *Let the function Γ^{u} and the constants d_2, d_3 and κ_2 be as obtained in Corollary 3.8 and Proposition 3.7 and fix $d_4 \in (d_2, d_3)$ and $\kappa_3 > \kappa_2$. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, Γ^{u} can be real-analytically extended to $D^{\mathrm{fl}}_{\kappa_3,d_4}$.*

Moreover, there exists a constant $b_5 > 0$ independent of δ such that, for $v \in D^{\mathrm{fl}}_{\kappa_3,d_4}$,

$$|\hat{\lambda}(v)| \leq b_5 \delta^2, \qquad |\hat{\Lambda}(v)| \leq b_5 \delta^2, \qquad |\hat{x}(v)| \leq b_5 \delta^3, \qquad |\hat{y}(v)| \leq b_5 \delta^3.$$

This proposition is proved in Section 4.4.

For the third step, we "go back" to the graph parametrization $z^{\mathrm{u}}(u)$ by looking for a change $v = u + \mathcal{V}(u)$ for $u \in \widetilde{D}_{\kappa,d}$. Notice that, in order to satisfy equation (3.20) and recalling (2.12), $\mathcal{V}$ must be a solution of

$$\hat{\lambda}(u + \mathcal{V}(u)) = \lambda_h(u) - \lambda_h(u + \mathcal{V}(u)). \tag{3.23}$$

Then, one can easily recover the graph parametrization $(w^{\mathrm{u}}(u), x^{\mathrm{u}}(u), y^{\mathrm{u}}(u))$ using the equations

$$\begin{aligned} \Lambda_h(u) - \Lambda_h(u + \mathcal{V}(u)) - \frac{w^{\mathrm{u}}(u)}{3\Lambda_h(u)} + \delta^2 \mathfrak{L}_\Lambda(\delta) &= \hat{\Lambda}(u + \mathcal{V}(u)), \\ x^{\mathrm{u}}(u) + \delta^3 \mathfrak{L}_x(\delta) &= \hat{x}(u + \mathcal{V}(u)), \\ y^{\mathrm{u}}(u) + \delta^3 \mathfrak{L}_y(\delta) &= \hat{y}(u + \mathcal{V}(u)). \end{aligned} \tag{3.24}$$

The next proposition ensures that $\mathcal{V}$ exists and it is well defined in $\widetilde{D}_{\kappa,d}$ (see (3.16)).

Proposition 3.10. *Let the function Γ^{u} and the constants d_4 and κ_3 be as obtained in Proposition 3.9 and the constant d_1 as obtained in Proposition 3.6. Let us consider constants $d_5 \in (d_1, d_4)$ and $\kappa_4 > \kappa_3$. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, equation (3.23) has a real-analytic solution $\mathcal{V} : \widetilde{D}_{\kappa_4,d_5} \to \mathbb{C}$ satisfying*

$$|\mathcal{V}(u)| \leq b_6 \delta^2 \qquad \text{and} \qquad u + \mathcal{V}(u) \in D^{\mathrm{fl}}_{\kappa_3,d_4},$$

for some constant $b_6 > 0$ independent of δ and $u \in \widetilde{D}_{\kappa_4,d_5}$.

Proposition 3.10 is proved in Section 4.5. Summarizing all the previous results we obtain the following result.

Corollary 3.11. *Let the function $\mathcal{V}$ and the constants d_5 and κ_4 be as obtained in Proposition 3.10. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, equation (3.24) has a unique solution $z^{\rm u} = (w^{\rm u}, x^{\rm u}, y^{\rm u})^T : \widetilde{D}_{\kappa_4,d_5} \to \mathbb{C}^3$.*

Moreover, there exists a constant $b_7 > 0$ independent of δ such that, for $u \in \widetilde{D}_{\kappa_4,d_5}$,

$$|w^{\rm u}(u)| \le b_7\delta^2, \qquad |x^{\rm u}(u)| \le b_7\delta^3, \qquad |y^{\rm u}(u)| \le b_7\delta^3.$$

To finish this section, notice that, taking ρ_2 big enough, $d \ge d_5$ and $\kappa_0 \ge \kappa_4$ we have that

$$D_{\kappa_0,d} \subset D^{\rm u,out}_{\kappa_1,d_1,\rho_2} \cup \widetilde{D}_{\kappa_4,d_5}, \qquad \text{with} \qquad D^{\rm u,out}_{\kappa_1,d_1,\rho_2} \cap \widetilde{D}_{\kappa_4,d_5} \neq \emptyset,$$

and then, Corollary 3.11 and Proposition 3.6 imply the statements of Theorem 3.3 referring to the unstable manifold $z^{\rm u}$.

3.3. A first order of the invariant manifolds near the singularities

Let us consider the difference

$$\Delta z = (\Delta w, \Delta x, \Delta y)^T = z^{\rm u} - z^{\rm s},$$

where $z^{\rm u}$ and $z^{\rm s}$ are the perturbed invariant graphs given in Theorem 3.3. Since $z^{\rm u}$ and $z^{\rm s}$ satisfy the invariance equation (3.12), the difference Δz satisfies the linear equation

$$\partial_u \Delta z(u) = \mathcal{A}^{\rm sep}\Delta z(u) + \widetilde{\mathcal{B}}^{\rm spl}(u)\Delta z(u), \tag{3.25}$$

where $\mathcal{A}^{\rm sep}$ is as given in (3.10) and

$$\widetilde{\mathcal{B}}^{\rm spl}(u) = \int_0^1 D_z\mathcal{R}^{\rm sep}[\sigma z^{\rm u} + (1-\sigma)z^{\rm s}](u)d\sigma. \tag{3.26}$$

Since $z^{\rm u}$ and $z^{\rm s}$ are already defined in $D_{\kappa,d}$, $\widetilde{\mathcal{B}}^{\rm spl}(u)$ can be considered as a "known" function.

In addition, since the graphs of $z^{\rm u}$ and $z^{\rm s}$ belong to the same energy level of $H^{\rm sep}$ (see (3.8)), we have that

$$H^{\rm sep}(u, z^{\rm u}(u); \delta) - H^{\rm sep}(u, z^{\rm s}(u); \delta) = 0, \quad \text{for } u \in D_{\kappa,d}.$$

Therefore, we can reduce (3.25) to a two dimensional equation. Indeed, defining $\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_3)$ such that

$$\Upsilon(u) = \int_0^1 D_z H^{\rm sep}\left(u, \sigma z^{\rm u}(u) + (1-\sigma)z^{\rm s}(u)\right) d\sigma, \tag{3.27}$$

and applying the mean value theorem we have that

$$\Upsilon_1(u)\Delta w(u) + \Upsilon_2(u)\Delta x(u) + \Upsilon_3(u)\Delta y(u) = 0.$$

Notice that $\Upsilon_1(u) = 1+\int_0^1 \partial_w H_1^{\text{sep}}(u, \sigma z^{\text{u}}(u) + (1-\sigma)z^{\text{s}}(u))\, d\sigma$ and therefore $\Upsilon_1(u) \neq 0$ for $u \in D_{\kappa,d}$ (see Remark 3.4). Therefore, writing

$$\Delta w(u) = -\frac{\Upsilon_2(u)}{\Upsilon_1(u)}\Delta x(u) - \frac{\Upsilon_3(u)}{\Upsilon_1(u)}\Delta y(u) \tag{3.28}$$

and defining $\Delta\Phi = (\Delta x, \Delta y)^T$, the last two components of (3.25) are equivalent to

$$\partial_u \Delta\Phi(u) = \mathcal{A}^{\text{spl}}(u)\Delta\Phi(u) + \mathcal{B}^{\text{spl}}(u)\Delta\Phi(u), \tag{3.29}$$

where

$$\begin{aligned} \mathcal{A}^{\text{spl}} &= \begin{pmatrix} \frac{i}{\delta^2} + \widetilde{\mathcal{B}}^{\text{spl}}_{2,2} & 0 \\ 0 & -\frac{i}{\delta^2} + \widetilde{\mathcal{B}}^{\text{spl}}_{3,3} \end{pmatrix}, \\ \mathcal{B}^{\text{spl}} &= \begin{pmatrix} -\frac{\Upsilon_2}{\Upsilon_1}\widetilde{\mathcal{B}}^{\text{spl}}_{2,1} & \widetilde{\mathcal{B}}^{\text{spl}}_{2,3} - \frac{\Upsilon_3}{\Upsilon_1}\widetilde{\mathcal{B}}^{\text{spl}}_{2,1} \\ \widetilde{\mathcal{B}}^{\text{spl}}_{3,2} - \frac{\Upsilon_2}{\Upsilon_1}\widetilde{\mathcal{B}}^{\text{spl}}_{3,1} & -\frac{\Upsilon_3}{\Upsilon_1}\widetilde{\mathcal{B}}^{\text{spl}}_{3,1} \end{pmatrix}. \end{aligned} \tag{3.30}$$

Next, we give an heuristic idea of how to obtain an exponentially small bound for $\Delta y(u)$ for $u \in D_{\kappa,d}$. The case for Δx is analogous. If we omit the influence of $\widetilde{\mathcal{B}}^{\text{spl}}$, then there exists $c_y \in \mathbb{C}$ such that Δy is of the form

$$\Delta y(u) = c_y\, e^{-\frac{i}{\delta^2}u}.$$

Evaluating this function at the points

$$u_+ = i(A - \kappa\delta^2), \qquad u_- = -i(A - \kappa\delta^2),$$

one has $\Delta y(u_+) \sim c_y e^{\frac{A}{\delta^2}-\kappa}$. Then, since $\Delta y(u_+) \sim 1$, it implies that $c_y \sim e^{-\frac{A}{\delta^2}+\kappa}$ and, as a consequence, Δy is exponentially small for $u \in \mathbb{R}$. However, we are not interested in an upper bound of Δy but in an asymptotic formula. Thus we have to find the constant c_y, or more precisely a good approximation of it.

To this end, we need to give the main terms of Δy at $u = u_+$. Likewise we need to analyze $\Delta x(u) \sim c_x\, e^{\frac{i}{\delta^2}u}$ at $u = u_-$. To perform this analysis we proceed as follows:

1. We provide suitable solutions $Z_0^{\text{u,s}}(U)$ of the so-called inner equation. The inner equation, see [5,18], describes the dominant behavior of the functions z^{u} and z^{s} close to (one of) the singularities $u = \pm iA$. In particular, it involves the first order of the Hamiltonian H^{sep} close to a singularity and it is independent of the small parameter δ. See Section 3.3.1.

2. We check how well $z^{\mathrm{u,s}}(u)$ are approximated by $Z_0^{\mathrm{u,s}}(U)$ around the singularities $u = \pm iA$ by means of a complex matching procedure. See Section 3.3.2.

3.3.1. The inner equation

In this section we summarize the results on the derivation and study of the inner equation obtained in [10]. We focus on the inner equation around the singularity $u = iA$, but analogous results hold near $u = -iA$.

To derive the inner equation, we look for a new Hamiltonian which is a good approximation of H^{sep}, given in (3.8), in a suitable neighborhood of $u = iA$. First, we scale the variables (u, w, x, y) so that the graphs $z^{\mathrm{u,s}}(u)$ become $\mathcal{O}(1)$-functions when $u - iA = \mathcal{O}(\delta^2)$. Since, by Theorem 3.3, we have that

$$w^\diamond(u) = \mathcal{O}(\delta^{-\frac{4}{3}}), \qquad x^\diamond(u) = \mathcal{O}(\delta^{\frac{1}{3}}), \qquad y^\diamond(u) = \mathcal{O}(\delta^{\frac{1}{3}}), \qquad \text{for } \diamond = \mathrm{u,s},$$

we consider the symplectic scaling $\phi_{\mathrm{in}} : (U, W, X, Y) \to (u, w, x, y)$, given by

$$U = \frac{u - iA}{\delta^2}, \qquad W = \delta^{\frac{4}{3}} \frac{w}{2\alpha_+^2}, \qquad X = \frac{x}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+}, \qquad Y = \frac{y}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+}, \tag{3.31}$$

where $\alpha_+ \in \mathbb{C}$ is the constant given by in Theorem 3.1, which is added to avoid the dependence of the inner equation on it. Moreover, we also perform the time scaling $\tau = \delta^2 T$. We refer to (U, W, X, Y) as the *inner coordinates.*

Proposition 3.12. *The Hamiltonian system associated to* (3.8) *expressed in the inner coordinates is Hamiltonian with respect to the symplectic form* $dU \wedge dW + i dX \wedge dY$ *and*

$$H^{\mathrm{in}} = \mathcal{H} + H_1^{\mathrm{in}}, \tag{3.32}$$

where

$$\mathcal{H}(U, W, X, Y) = H^{\mathrm{in}}(U, W, X, Y; \delta)|_{\delta=0} = W + XY + \mathcal{K}(U, W, X, Y),$$

with

$$\begin{aligned}
\mathcal{K}(U, W, X, Y) &= -\frac{3}{4} U^{\frac{2}{3}} W^2 - \frac{1}{3U^{\frac{2}{3}}} \left(\frac{1}{\sqrt{1 + \mathcal{J}(U, W, X, Y)}} - 1 \right), \\
\mathcal{J}(U, W, X, Y) &= \frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X+Y)}{9U} \left(W - \frac{2}{3U^{\frac{2}{3}}} \right) \\
&\quad - \frac{4i(X-Y)}{3U^{\frac{2}{3}}} - \frac{X^2 + Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{3}}}.
\end{aligned}$$

Moreover, if $c_1^{-1} \leq |U| \leq c_1$ *and* $|(W, X, Y)| \leq c_2$ *for some* $c_1 > 1$ *and* $0 < c_2 < 1$*, there exist* $b_8, \gamma_1, \gamma_2 > 0$ *independent of* δ, c_1, c_2 *such that*

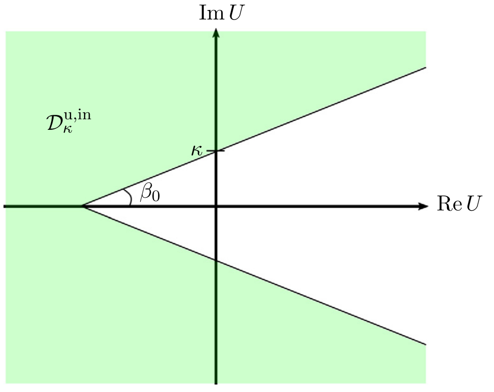

Fig. 8. The inner domain $\mathcal{D}_\kappa^{\mathrm{u,in}}$ for the unstable case.

$$|H_1^{\mathrm{in}}(U,W,X,Y;\delta)| \leq b_8 c_1^{\gamma_1} c_2^{\gamma_2} \delta^{\frac{4}{3}}. \tag{3.33}$$

This result is proven in [10] in Proposition 2.5.

Now, we present the study of the inner Hamiltonian $\mathcal{H}$. Denoting $Z = (W,X,Y)^T$, the equations associated to the Hamiltonian $\mathcal{H}$, can be written as

$$\begin{cases} \dot{U} = 1 + g^{\mathrm{in}}(U,Z), \\ \dot{Z} = \mathcal{A}^{\mathrm{in}} Z + f^{\mathrm{in}}(U,Z), \end{cases}$$

where

$$\mathcal{A}^{\mathrm{in}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \tag{3.34}$$

and $f^{\mathrm{in}} = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T$ and $g^{\mathrm{in}} = \partial_W \mathcal{K}$. We look for invariant graphs $Z = Z_0^{\mathrm{u}}(U)$ and $Z = Z_0^{\mathrm{s}}(U)$ of this equation, that satisfy the invariance equation also called *inner equation*,

$$\partial_U Z_0^\diamond(U) = \mathcal{A}^{\mathrm{in}} Z_0^\diamond + \mathcal{R}^{\mathrm{in}}[Z_0^\diamond](U), \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s}, \tag{3.35}$$

with

$$\mathcal{R}^{\mathrm{in}}[\varphi](U) = \frac{f^{\mathrm{in}}(U,\varphi) - g^{\mathrm{in}}(U,\varphi)\mathcal{A}^{\mathrm{in}}\varphi}{1 + g^{\mathrm{in}}(U,\varphi)}. \tag{3.36}$$

These functions Z_0^{u} and Z_0^{s} will be defined in the domains

$$\mathcal{D}_\kappa^{\mathrm{u,in}} = \{U \in \mathbb{C} : |\mathrm{Im}\, U| \geq \tan\beta_0 \mathrm{Re}\, U + \kappa\}, \qquad \mathcal{D}_\kappa^{\mathrm{s,in}} = -\mathcal{D}_\kappa^{\mathrm{u,in}},$$

respectively, for some $\kappa > 0$ and with β_0 as given in Theorem 3.3 (see Fig. 8). Moreover, we analyze the difference $\Delta Z_0 = Z_0^{\mathrm{u}} - Z_0^{\mathrm{s}}$ in the overlapping domain

$$\mathcal{E}_\kappa^{\mathrm{in}} = \mathcal{D}_\kappa^{\mathrm{u,in}} \cap \mathcal{D}_\kappa^{\mathrm{s,in}} \cap \{U \in \mathbb{C} \,:\, \mathrm{Im}\, U < 0\}\,.$$

Theorem 3.13. *There exist $\kappa_5, b_9 > 0$ such that for $\kappa \geq \kappa_5$, the equation (3.35) has analytic solutions $Z_0^\diamond(U) = (W_0^\diamond(U), X_0^\diamond(U), Y_0^\diamond(U))^T$, for $U \in \mathcal{D}_\kappa^{\diamond,\mathrm{in}}$, $\diamond = \mathrm{u}, \mathrm{s}$, satisfying*

$$|U^{\frac{8}{3}} W_0^\diamond(U)| \leq b_9, \qquad |U^{\frac{4}{3}} X_0^\diamond(U)| \leq b_9, \qquad |U^{\frac{4}{3}} Y_0^\diamond(U)| \leq b_9.$$

In addition, there exist $\Theta \in \mathbb{C}$, $b_{10} > 0$ independent of κ, and an analytic function $\chi = (\chi_1, \chi_2, \chi_3)^T$ such that, for $U \in \mathcal{E}_\kappa^{\mathrm{in}}$,

$$\Delta Z_0(U) = Z_0^{\mathrm{u}}(U) - Z_0^{\mathrm{s}}(U) = \Theta e^{-iU} \Big((0,0,1)^T + \chi(U) \Big),$$

with $|(U^{\frac{7}{3}} \chi_1(U), U^2 \chi_2(U), U \chi_3(U))| \leq b_{10}$.

This result is Theorem 2.7 of [10].

Remark 3.14. To obtain the analogous result to Theorem 3.13 near the singularity $u = -iA$, one must perform the change of coordinates

$$V = \frac{u + iA}{\delta^2}, \qquad \widehat{W} = \delta^{\frac{4}{3}} \frac{w}{2\alpha_-^2}, \qquad \widehat{X} = \frac{x}{\delta^{\frac{1}{3}} \sqrt{2} \alpha_-}, \qquad \widehat{Y} = \frac{y}{\delta^{\frac{1}{3}} \sqrt{2} \alpha_-},$$

where $\alpha_- \in \mathbb{C}$ is $\alpha_- = \overline{\alpha_+}$ (see Theorem 3.1). Then, for $V \in \overline{\mathcal{D}_\kappa^{\diamond,\mathrm{in}}}$, one can prove the existence of the corresponding solutions

$$\widehat{Z}_0^\diamond(V) = (\widehat{W}_0^\diamond(V), \widehat{X}_0^\diamond(V), \widehat{Y}_0^\diamond(V))^T, \qquad \text{where } \diamond = \mathrm{u}, \mathrm{s}.$$

Due to the real-analyticity of the problem (see Remark 3.2) we have that $\widehat{X}^\diamond(V) = \overline{Y^\diamond(U)}$. Therefore, the difference $\Delta \widehat{Z}_0 = \widehat{Z}_0^{\mathrm{u}} - \widehat{Z}_0^{\mathrm{s}}$, is given asymptotically for $U \in \overline{\mathcal{E}_\kappa^{\mathrm{in}}}$ by

$$\Delta \widehat{Z}_0(V) = \overline{\Theta} e^{iV} \Big((0,1,0)^T + \zeta(V) \Big),$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3)^T$ satisfies $|(V^{\frac{7}{3}} \zeta_1(V), V \zeta_2(V), V^2 \zeta_3(V)| \leq C$, for a constant C independent of κ.

3.3.2. Complex matching estimates

We now study how well the solutions of the inner equation approximate the solutions of the original system given by Proposition 3.6 in an appropriate domain. As in the previous section, we focus on the singularity $u = iA$, but analogous results can be proven for $u = -iA$ (see Remark 3.14). Let us recall that the functions $z^{\mathrm{u,s}}$ are expressed in the separatrix coordinates (see (3.4)) while the functions $Z_0^{\mathrm{u,s}}$ are expressed in inner coordinates (see (3.31)).

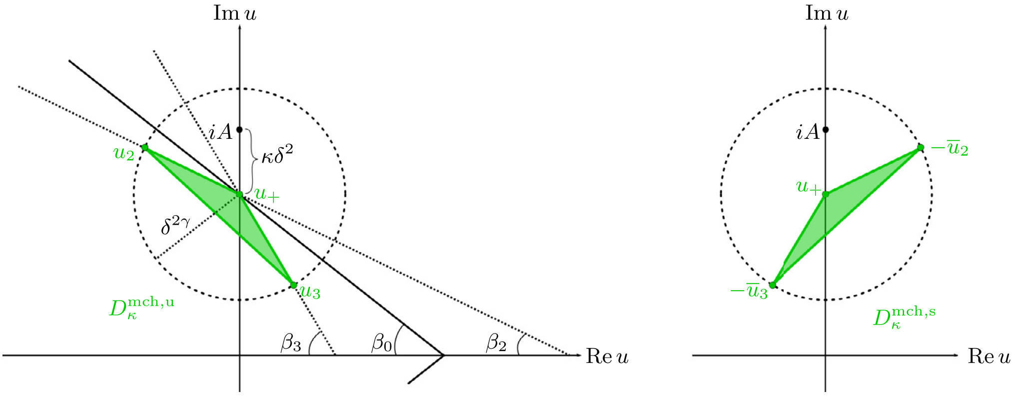

Fig. 9. The matching domains $D_\kappa^{\mathrm{mch,u}}$ and $D_\kappa^{\mathrm{mch,s}}$ in the outer variables.

We first define the matching domains in separatrix coordinates and, later, we translate them to the inner coordinates. Let us consider β_2, β_3, and γ independent of δ and κ, such that

$$0 < \beta_2 < \beta_0 < \beta_3 < \frac{\pi}{2}, \qquad \text{and} \qquad \gamma \in \left[\frac{3}{5}, 1\right),$$

with β_0 as given in Theorem 3.1. Then, we define $u_j \in \mathbb{C}$ $j = 2, 3$ (see Fig. 9), as the points satisfying:

- $\operatorname{Im} u_j = -\tan \beta_j \operatorname{Re} u_j + A - \kappa\delta^2$.
- $|u_j - u_+| = \delta^{2\gamma}$, where $u_+ = i(A - \kappa\delta^2)$.
- $\operatorname{Re} u_2 < 0$ and $\operatorname{Re} u_3 > 0$.

We define the matching domains in the separatrix coordinates as the triangular domains

$$D_\kappa^{\mathrm{mch,u}} = \widehat{u_+\, u_2\, u_3}, \qquad D_\kappa^{\mathrm{mch,s}} = \widehat{u_+\, (-\overline{u_2})\, (-\overline{u_3})}.$$

Let d_1, ρ_2 and κ_1 be as given in Proposition 3.6. Then, for $\kappa \geq \kappa_1$ and $\delta > 0$ small enough, the matching domains satisfy

$$D_\kappa^{\mathrm{mch,u}} \subset D_{\kappa,d_1,\rho_2}^{\mathrm{u,out}} \qquad \text{and} \qquad D_\kappa^{\mathrm{mch,s}} \subset D_{\kappa,d_1,\rho_2}^{\mathrm{s,out}}, \tag{3.37}$$

and, as a result, z^{u} and z^{s} are well defined in $D_\kappa^{\mathrm{mch,u}}$ and $D_\kappa^{\mathrm{mch,s}}$, respectively.

The matching domains in inner variables are defined by

$$\mathcal{D}_\kappa^{\mathrm{mch},\diamond} = \left\{ U \in \mathbb{C} \;:\; \delta^2 U + iA \in D_\kappa^{\mathrm{mch},\diamond} \right\}, \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s}, \tag{3.38}$$

with

$$U_j = \frac{u_j - iA}{\delta^2}, \qquad \text{for } j = 2, 3. \tag{3.39}$$

Therefore, for $U \in \mathcal{D}_\kappa^{\mathrm{mch},\diamond}$,

$$\kappa \cos \beta_2 \leq |U| \leq \frac{C}{\delta^{2(1-\gamma)}}.$$

By definition,

$$\mathcal{D}_\kappa^{\mathrm{mch,u}} \subset \mathcal{D}_\kappa^{\mathrm{u,in}} \qquad \text{and} \qquad \mathcal{D}_\kappa^{\mathrm{mch,s}} \subset \mathcal{D}_\kappa^{\mathrm{s,in}},$$

for $\kappa \geq \kappa_5$ (see Theorem 3.13). Thus, $Z_0^{\mathrm{u,s}}$ is well defined in $\mathcal{D}_\kappa^{\mathrm{mch,u,s}}$.

In order to compare $z^{\mathrm{u,s}}(u)$ and $Z_0^{\mathrm{u,s}}(U)$, we translate $z^{\mathrm{u,s}}$ to inner coordinates

$$Z^\diamond(U) = \left(W^\diamond, X^\diamond, Y^\diamond\right)^T(U) = \left(\delta^{\frac{4}{3}} \frac{w^\diamond}{2\alpha_+^2}, \frac{x^\diamond}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+}, \frac{y^\diamond}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+}\right)^T (\delta^2 U + iA), \tag{3.40}$$

with $\diamond = \mathrm{u,s}$ and $z^\diamond = (w^\diamond, x^\diamond, y^\diamond)^T$ are given in Proposition 3.6. Therefore, by (3.37), $Z^\diamond$ is well defined in the matching domain $\mathcal{D}_\kappa^{\mathrm{mch},\diamond}$ (which is expressed in inner variables).

Next theorem gives estimates for $Z^{\mathrm{u,s}} - Z_0^{\mathrm{u,s}}$.

Theorem 3.15. *Consider κ_1 and κ_5 as obtained in Proposition 3.6 and Theorem 3.13, respectively. Then, there exist $\gamma^* \in [\frac{3}{5}, 1)$, $\kappa_6 \geq \max\{\kappa_1, \kappa_5\}$ and $\delta_0 > 0$ such that, for $\gamma \in (\gamma^*, 1)$, there exists $b_{11} > 0$ satisfying that, for $U \in \mathcal{D}_\kappa^{\mathrm{mch},\diamond}$, $\kappa \geq \kappa_6$ and $\delta \in (0, \delta_0)$,*

$$|U^{\frac{4}{3}} W_1^\diamond(U)| \leq b_{11}\delta^{\frac{2}{3}(1-\gamma)}, \qquad |U\, X_1^\diamond(U)| \leq b_{11}\delta^{\frac{2}{3}(1-\gamma)}, \qquad |U\, Y_1^\diamond(U)| \leq b_{11}\delta^{\frac{2}{3}(1-\gamma)},$$

with $(W_1^\diamond, X_1^\diamond, Y_1^\diamond)^T = Z_1^\diamond = Z^\diamond - Z_0^\diamond$ and $\diamond = \mathrm{u,s}$.

This theorem is proven in Section 5.

3.4. The asymptotic formula for the difference

We look for an asymptotic expression for the difference

$$\Delta\Phi = (\Delta x, \Delta y)^T = (x^{\mathrm{u}} - x^{\mathrm{s}}, y^{\mathrm{u}} - y^{\mathrm{s}})^T,$$

where $(x^{\mathrm{u}}, y^{\mathrm{u}})$ and $(x^{\mathrm{s}}, y^{\mathrm{s}})$ are components of the perturbed invariant graphs given in Theorem 3.3. Recall that, by (3.29), $\Delta\Phi$ satisfies

$$\partial_u \Delta\Phi(u) = \mathcal{A}^{\mathrm{spl}}(u)\Delta\Phi(u) + \mathcal{B}^{\mathrm{spl}}(u)\Delta\Phi(u), \tag{3.41}$$

with $\mathcal{A}^{\mathrm{spl}}$ and $\mathcal{B}^{\mathrm{spl}}$ as given in (3.30). The equation is split as a dominant part, given by the matrix $\mathcal{A}^{\mathrm{spl}}$ and a small perturbation corresponding to the matrix $\mathcal{B}^{\mathrm{spl}}$. Therefore, it makes sense to look for $\Delta\Phi$ as $\Delta\Phi = \Delta\Phi_0 + h.o.t$ with a suitable dominant term $\Delta\Phi_0 = (\Delta x_0, \Delta y_0)^T$ satisfying

$$\partial_u \Delta\Phi_0(u) = \mathcal{A}^{\mathrm{spl}}(u)\Delta\Phi_0(u). \tag{3.42}$$

A fundamental matrix of (3.42), for $u \in D_{\kappa,d}$, is given by

$$\mathcal{M}(u) = \begin{pmatrix} m_x(u) & 0 \\ 0 & m_y(u) \end{pmatrix}, \tag{3.43}$$

with

$$\begin{aligned} m_x(u) &= e^{\frac{i}{\delta^2}u} B_x(u), \qquad B_x(u) = \exp\left(\int_{u_*}^{u} \widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,2}(s)ds \right), \\ m_y(u) &= e^{-\frac{i}{\delta^2}u} B_y(u), \quad B_y(u) = \exp\left(\int_{u_*}^{u} \widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}(s)ds \right), \end{aligned} \tag{3.44}$$

and a fixed $u_* \in D_{\kappa,d} \cap \mathbb{R}$. Then, $\Delta\Phi_0$ must be of form

$$\Delta\Phi_0(u) = \begin{pmatrix} \Delta x_0(u) \\ \Delta y_0(u) \end{pmatrix} = \begin{pmatrix} c_x^0 m_x(u) \\ c_y^0 m_y(u) \end{pmatrix}, \tag{3.45}$$

for suitable constants $c_x^0, c_y^0 \in \mathbb{C}$ which we now determine.

By Theorems 3.13 and 3.15 and using the inner change of coordinates in (3.31), we have a good approximation of $\Delta y(u)$ near the singularity $u = iA$ given by

$$\Delta y(u) \approx \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} \Delta Y_0 \left(\frac{u - iA}{\delta^2} \right).$$

Then, taking $u = u_+ = i(A - \kappa\delta^2)$, we have that

$$\Delta y(u_+) \approx \Delta y_0(u_+) \approx \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} \Delta Y_0 \left(\frac{u_+ - iA}{\delta^2} \right) = \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} e^{-\kappa} \Theta (1 + \chi_3(-i\kappa)).$$

Then, using that $\Delta y(u_+) \approx \Delta y_0(u_+) = c_y^0 m_y(u_+)$, and proceeding analogously for the component Δx at the point $u_- = -i(A - \kappa\delta^2)$ (see Remark 3.14), we take

$$c_x^0 = \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \overline{\Theta}\, \sqrt{2}\alpha_- B_x^{-1}(u_-) \qquad \text{and} \qquad c_y^0 = \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \Theta \sqrt{2}\alpha_+ B_y^{-1}(u_+). \tag{3.46}$$

To prove Theorem 2.2, we check that $\Delta\Phi_0(u)$ is the leading term of $\Delta\Phi(u)$, for $u \in \mathbb{R} \cap D_{\kappa,d}$, by estimating the remainder $\Delta\Phi_1 = \Delta\Phi - \Delta\Phi_0$.

In order to simplify the notation, throughout the rest of the document, we denote by C any positive constant independent of δ and κ to state estimates.

3.4.1. End of the proof of Theorem 2.2

We look for $\Delta\Phi_1$ as the unique solution of an integral equation. Since $\Delta\Phi$ satisfies (3.41), by the variations of constants formula

$$\Delta\Phi(u) = \begin{pmatrix} c_x m_x(u) \\ c_y m_y(u) \end{pmatrix} + \begin{pmatrix} m_x(u) \displaystyle\int_{u_-}^{u} m_x^{-1}(s)\, \pi_1 \left(\mathcal{B}^{\rm spl}(s)\Delta\Phi(s)\right) ds \\ m_y(u) \displaystyle\int_{u_+}^{u} m_y^{-1}(s)\, \pi_2 \left(\mathcal{B}^{\rm spl}(s)\Delta\Phi(s)\right) ds \end{pmatrix}, \tag{3.47}$$

where $\mathcal{M}(u)$ is the fundamental matrix (3.43), s belongs to some integration path in $D_{\kappa,d}$ and c_x and c_y are defined as

$$c_x = \Delta x(u_-) m_x^{-1}(u_-), \qquad c_y = \Delta y(u_+) m_y^{-1}(u_+). \tag{3.48}$$

For $k_1, k_2 \in \mathbb{C}$, we define

$$\mathcal{I}[k_1, k_2](u) = \left(k_1\, m_x(u), k_2\, m_y(u)\right)^T, \tag{3.49}$$

and the operator

$$\mathcal{E}[\varphi](u) = \begin{pmatrix} m_x(u) \displaystyle\int_{u_-}^{u} m_x^{-1}(s)\, \pi_1 \left(\mathcal{B}^{\rm spl}(s)\varphi(s)\right) ds \\ m_y(u) \displaystyle\int_{u_+}^{u} m_y^{-1}(s)\, \pi_2 \left(\mathcal{B}^{\rm spl}(s)\varphi(s)\right) ds \end{pmatrix}. \tag{3.50}$$

Then, with this notation, $\Delta\Phi_0 = \mathcal{I}[c_x^0, c_y^0]$ (see (3.46)) and equation (3.47) is equivalent to $\Delta\Phi = \mathcal{I}[c_x, c_y] + \mathcal{E}[\Delta\Phi]$. Since $\mathcal{E}$ is a linear operator, $\Delta\Phi_1 = \Delta\Phi - \Delta\Phi_0$ satisfies

$$\Delta\Phi_1(u) = \mathcal{I}[c_x - c_x^0, c_y - c_y^0](u) + \mathcal{E}[\Delta\Phi_0](u) + \mathcal{E}[\Delta\Phi_1](u). \tag{3.51}$$

To obtain estimates for $\Delta\Phi_1$, we first prove that $\mathrm{Id} - \mathcal{E}$ is invertible in the Banach space $\mathcal{X}_\times^{\rm spl} = \mathcal{X}^{\rm spl} \times \mathcal{X}^{\rm spl}$, with

$$\mathcal{X}^{\rm spl} = \left\{ \varphi : D_{\kappa,d} \to \mathbb{C} \;:\; \|\varphi\|^{\rm spl} = \sup_{u \in D_{\kappa,d}} \left| e^{\frac{A - |\operatorname{Im} u|}{\delta^2}} \varphi(u) \right| < +\infty \right\},$$

endowed with the norm

$$\|\varphi\|_\times^{\rm spl} = \|\varphi_1\|^{\rm spl} + \|\varphi_2\|^{\rm spl}, \tag{3.52}$$

for $\varphi = (\varphi_1, \varphi_2)$. Therefore, to prove Theorem 2.2 it is enough to see that $\Delta\Phi_1$ satisfies that $\|\Delta\Phi_1\|_\times^{\mathrm{spl}} \leq C\delta^{\frac{1}{3}} |\log\delta|^{-1}$.

First, we state a lemma whose proof is postponed to Appendix C.1.

Lemma 3.16. *Let κ_0, δ_0 be the constants given in Theorem 3.3. Then, there exists a constant $C > 0$ such that, for $\kappa \geq \kappa_0$, $\delta \in (0, \delta_0)$ and $u \in D_{\kappa,d}$, the function Υ in (3.27), the matrix $\mathcal{B}^{\mathrm{spl}}$ in (3.30) and the functions B_x, B_y in (3.44) satisfy for $\kappa \geq \kappa_0$, $\delta \in (0, \delta_0)$ and $u \in D_{\kappa,d}$,*

$$\begin{gathered} |\Upsilon_1(u) - 1| \leq \frac{C}{\kappa^2}, \quad |\Upsilon_2(u)| \leq \frac{C\delta}{|u^2 + A^2|^{\frac{4}{3}}}, \quad |\Upsilon_3(u)| \leq \frac{C\delta}{|u^2 + A^2|^{\frac{4}{3}}}, \\ C^{-1} \leq |B_*(u)| \leq C, \, * = x, y, \quad \text{and} \quad |\mathcal{B}^{\mathrm{spl}}_{i,j}(u)| \leq \frac{C\,\delta^2}{|u^2 + A^2|^2}, \, i, j = 1, 2. \end{gathered} \tag{3.53}$$

In the next lemma we obtain estimates for the linear operator $\mathcal{E}$ (see (3.50)).

Lemma 3.17. *Let κ_0, δ_0 be the constants as given in Theorem 3.3. There exists $b_{12} > 0$ such that for $\delta \in (0, \delta_0)$ and $\kappa \geq \kappa_0$, the operator $\mathcal{E} : \mathcal{X}_\times^{\mathrm{spl}} \to \mathcal{X}_\times^{\mathrm{spl}}$ in (3.50) is well defined and satisfies that, for $\varphi \in \mathcal{X}_\times^{\mathrm{spl}}$,*

$$\|\mathcal{E}[\varphi]\|_\times^{\mathrm{spl}} \leq \frac{b_{12}}{\kappa} \|\varphi\|_\times^{\mathrm{spl}}.$$

In particular, $\mathrm{Id} - \mathcal{E}$ *is invertible and*

$$\left\|(\mathrm{Id} - \mathcal{E})^{-1}[\varphi]\right\|_\times^{\mathrm{spl}} \leq 2 \|\varphi\|_\times^{\mathrm{spl}}.$$

Proof. Let us consider $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)^T$, $\varphi \in \mathcal{X}_\times^{\mathrm{spl}}$ and $u \in D_{\kappa,d}$. We only prove the estimate for $\mathcal{E}_2[\varphi](u)$. The corresponding one for $\mathcal{E}_1[\varphi](u)$ follows analogously.

By the definition of m_y in (3.44) and Lemma 3.16, we have that

$$\begin{aligned} |\mathcal{E}_2[\varphi](u)| &\leq C\delta^2 e^{\frac{\operatorname{Im} u}{\delta^2}} \left| \int_{u_+}^{u} e^{-\frac{\operatorname{Im} s}{\delta^2}} \frac{|\varphi_1(s)| + |\varphi_2(s)|}{|s^2 + A^2|^2} ds \right| \\ &\leq C\delta^2 e^{\frac{\operatorname{Im} u - A}{\delta^2}} \|\varphi\|_\times^{\mathrm{spl}} \left| \int_{u_+}^{u} e^{\frac{|\operatorname{Im} s| - \operatorname{Im} s}{\delta^2}} \frac{ds}{|s^2 + A^2|^2} \right|. \end{aligned}$$

Let us consider the case $\operatorname{Im} u < 0$. Then, for a fixed $u_0 \in \mathbb{R} \cap D_{\kappa,d}$, we define the integration path $\rho_t \subset D_{\kappa,d}$ as

$$\rho_t = \begin{cases} u_+ + 2t(u_0 - u_+) & \text{for } t \in (0, \frac{1}{2}), \\ u_0 + (2t - 1)(u - u_0) & \text{for } t \in [\frac{1}{2}, 1). \end{cases}$$

Then,

$$|\mathcal{E}_2[\varphi](u)| \leq C\delta^2 e^{-\frac{|\operatorname{Im} u|+A}{\delta^2}} \|\varphi\|_\times^{\mathrm{spl}} \left| \int_0^{\frac{1}{2}} \frac{dt}{|\rho_t - iA|^2} + \int_{\frac{1}{2}}^1 \frac{e^{\frac{2|\operatorname{Im}\rho_t|}{\delta^2}}}{|\rho_t + iA|^2} dt \right| \leq \frac{C}{\kappa} e^{\frac{|\operatorname{Im} u|-A}{\delta^2}} \|\varphi\|_\times^{\mathrm{spl}}.$$

If $\operatorname{Im} u \geq 0$, we consider the integration path $\rho_t = u_+ + t(u-u_+)$ for $t \in [0,1]$ and we obtain

$$|\mathcal{E}_2[\varphi](u)| \leq C\delta^2 e^{\frac{|\operatorname{Im} u|-A}{\delta^2}} \|\varphi\|_\times^{\mathrm{spl}} \left| \int_0^1 \frac{|u-u_+|}{|\rho_t - iA|^2} dt \right| \leq \frac{C}{\kappa} e^{\frac{|\operatorname{Im} u|-A}{\delta^2}} \|\varphi\|_\times^{\mathrm{spl}}.$$

Therefore, $\|\mathcal{E}_2[\varphi]\|^{\mathrm{spl}} \leq \frac{C}{\kappa} \|\varphi\|_\times^{\mathrm{spl}}$. □

Notice that, by (3.51), $\Delta\Phi_1$ satisfies

$$(\mathrm{Id} - \mathcal{E})\Delta\Phi_1(u) = \mathcal{I}[c_x - c_x^0, c_y - c_y^0](u) + \mathcal{E}[\Delta\Phi_0](u). \tag{3.54}$$

Since, by Lemma 3.17, $\mathrm{Id} - \mathcal{E}$ is invertible in $\mathcal{X}_\times^{\mathrm{spl}}$ we have an explicit formula for $\Delta\Phi_1$. Nevertheless, we still need good estimates for the right hand side with respect to the norm (3.52).

Lemma 3.18. *There exist $\kappa_*, \delta_0, b_{13} > 0$ such that, for $\kappa = \kappa_* |\log\delta|$ and $\delta \in (0, \delta_0)$,*

$$\left\|\mathcal{I}[c_x - c_x^0, c_y - c_y^0]\right\|_\times^{\mathrm{spl}} \leq \frac{b_{13}\,\delta^{\frac{1}{3}}}{|\log\delta|} \qquad \text{and} \qquad \|\mathcal{E}[\Delta\Phi_0](u)\|_\times^{\mathrm{spl}} \leq \frac{b_{13}\,\delta^{\frac{1}{3}}}{|\log\delta|},$$

with $\mathcal{I}$, (c_x^0, c_y^0), (c_x, c_y), $\mathcal{E}$ and $\Delta\Phi_0$ defined in (3.49), (3.46), (3.48), (3.50) and (3.45), respectively.

Proof. By the definition of the function $\mathcal{I}$,

$$\left\|\mathcal{I}[c_x - c_x^0, c_y - c_y^0]\right\|_\times^{\mathrm{spl}} = \left|c_x - c_x^0\right| \|m_x\|^{\mathrm{spl}} + \left|c_y - c_y^0\right| \|m_y\|^{\mathrm{spl}},$$

where m_x and m_y are given in (3.44). Then, by Lemma 3.16,

$$\|m_x\|^{\mathrm{spl}} = e^{\frac{A}{\delta^2}} \sup_{u \in D_{\kappa,d}} \left[e^{-\frac{\operatorname{Im} u + |\operatorname{Im} u|}{\delta^2}} |B_x(u)| \right] \leq C e^{\frac{A}{\delta^2}}, \qquad \|m_y\|^{\mathrm{spl}} \leq C e^{\frac{A}{\delta^2}},$$

and, as a result,

$$\left\|\mathcal{I}[c_x - c_x^0, c_y - c_y^0]\right\|_\times^{\mathrm{spl}} \leq C e^{\frac{A}{\delta^2}} \left(|c_x - c_x^0| + |c_y - c_y^0|\right). \tag{3.55}$$

We now obtain an estimate for $|c_y - c_y^0|$. The estimate for $|c_x - c_x^0|$ follows analogously.

By the definition of m_y (see (3.44)), one has

$$\left|c_y - c_y^0\right| = e^{-\frac{A}{\delta^2}+\kappa} \left|B_y^{-1}(u_+)\right| \left|\Delta y(u_+) - \Delta y_0(u_+)\right|. \tag{3.56}$$

Let us denote $\Delta Y = Y^{\mathrm{u}} - Y^{\mathrm{s}}$ where $Y^{\mathrm{u,s}}$ are given on (3.40). Recall that $Y^{\mathrm{u,s}} = Y_0^{\mathrm{u,s}} + Y_1^{\mathrm{u,s}}$ where $Y_0^{\mathrm{u,s}}$ is the third component of $Z_0^{\mathrm{u,s}}$, the solutions of the inner equation (see Theorems 3.13 and 3.15). We write,

$$\Delta y(u_+) = \sqrt{2}\alpha_+\delta^{\frac{1}{3}}\Delta Y\left(\frac{u_+ - iA}{\delta^2}\right) = \sqrt{2}\alpha_+\delta^{\frac{1}{3}}\left[\Delta Y_0\left(-i\kappa\right) + Y_1^{\mathrm{u}}\left(-i\kappa\right) - Y_1^{\mathrm{s}}\left(-i\kappa\right)\right].$$

By the definition of Δy_0 in (3.45) (see also (3.46)), we have $\Delta y_0(u_+) = \sqrt{2}\alpha_+\delta^{\frac{1}{3}}\Theta e^{-\kappa}$. Then, by (3.56) and Lemma 3.16,

$$\left|c_y - c_y^0\right| \le C\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}+\kappa}\Big[\left|\Delta Y_0\left(-i\kappa\right) - \Theta e^{-\kappa}\right| + \left|Y_1^{\mathrm{u}}\left(-i\kappa\right)\right| + \left|Y_1^{\mathrm{s}}\left(-i\kappa\right)\right|\Big],$$

and, applying Theorems 3.13 and 3.15, we obtain

$$\left|c_y - c_y^0\right| \le C\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}+\kappa}\left[\left|\chi_3(-i\kappa)e^{-\kappa}\right| + \frac{C}{\kappa}\delta^{\frac{2}{3}(1-\gamma)}\right] \le \frac{C}{\kappa}\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\left(1 + \delta^{\frac{2}{3}(1-\gamma)}e^{\kappa}\right),$$

where $\gamma \in (\gamma^*, 1)$ with $\gamma^* \in [\frac{3}{5}, 1)$ given in Theorem 3.15. Taking $\kappa = \kappa_* \left|\log\delta\right|$ with $0 < \kappa_* < \frac{2}{3}(1-\gamma)$, we obtain

$$\left|c_y - c_y^0\right| \le \frac{C\delta^{\frac{1}{3}}}{\left|\log\delta\right|}e^{-\frac{A}{\delta^2}}\left(1 + \delta^{\frac{2}{3}(1-\gamma)-\kappa_*}\right) \le \frac{C\delta^{\frac{1}{3}}}{\left|\log\delta\right|}e^{-\frac{A}{\delta^2}}.$$

This bound and (3.55) prove the first estimate of the lemma.

For the second estimate, it only remains to bound $\Delta\Phi_0$ and apply Lemma 3.17. Indeed, by the definition of $\Delta\Phi_0$ in (3.46), Lemma 3.16 and (3.55), we have that

$$\left\|\Delta\Phi_0\right\|_{\times}^{\mathrm{spl}} = \left\|\mathcal{I}[c_x^0, c_y^0]\right\|_{\times}^{\mathrm{spl}} \le Ce^{\frac{A}{\delta^2}}\left(\left|c_x^0\right| + \left|c_y^0\right|\right) \le C\delta^{\frac{1}{3}}.$$

Since $\kappa = \kappa_* \left|\log\delta\right|$ with $0 < \kappa_* < \frac{2}{3}(1-\gamma)$, Lemma 3.17 implies $\left\|\mathcal{E}[\Delta\Phi_0]\right\|_{\times}^{\mathrm{spl}} \le \frac{C\delta^{\frac{1}{3}}}{\left|\log\delta\right|}$. □

With this lemma, we can give sharp estimates for $\Delta\Phi_1$ by using equation (3.54). Indeed, since the right hand side of this equation belongs to $\mathcal{X}_{\times}^{\mathrm{spl}}$, by Lemma 3.17,

$$\Delta\Phi_1(u) = (\mathrm{Id} - \mathcal{E})^{-1}\left(\mathcal{I}[c_x - c_x^0, c_y - c_y^0](u) + \mathcal{E}[\Delta\Phi_0](u)\right).$$

Then, Lemmas 3.17 and 3.18 imply

$$\left\|\Delta\Phi_1\right\|_{\times}^{\mathrm{spl}} \le \frac{C\delta^{\frac{1}{3}}}{\left|\log\delta\right|}. \tag{3.57}$$

To prove Theorem 2.2, it only remains to analyze $B_x(u_-)$ and $B_y(u_+)$.

Lemma 3.19. *Let κ_* be as given in Lemma 3.18. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$ and $\kappa = \kappa_* |\log \delta|$, the functions B_x, B_y defined in (3.44) satisfy*

$$B_x^{-1}(u_-) = e^{-\frac{4i}{9}(\pi - \lambda_h(u_*))} \left(1 + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right),$$

$$B_y^{-1}(u_+) = e^{\frac{4i}{9}(\pi - \lambda_h(u_*))} \left(1 + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right),$$

where $u_\pm = \pm i(A - \kappa\delta^2)$.

This lemma is proven in Appendix C.2.

Let $u_* \in D_{\kappa,d} \cap \mathbb{R}$. We compute the first order of $\Delta\Phi_0(u_*) = (\Delta x_0(u_*), \Delta y_0(u_*))^T$. Since, by Theorem 3.1, $(\alpha_+)^3 = (\alpha_-)^3 = \frac{1}{2}$, and applying Lemma 3.19 and (3.46), we obtain

$$|\Delta x_0(u_*)| = |\Delta y_0(u_*)| = \sqrt[6]{2}\, |\Theta|\, \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left(1 + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right).$$

Moreover, by (3.57),

$$|\Delta x(u_*) - \Delta x_0(u_*)|, |\Delta y(u_*) - \Delta y_0(u_*)| \leq \frac{C\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}}{|\log \delta|}.$$

Finally, notice that the section $u = u_* \in D_{\kappa,d} \cap \mathbb{R}$ translates to $\lambda = \lambda^* := \lambda_h(u_*)$ (see (3.4)). Moreover, since $\dot{\lambda}_h = -3\Lambda_h$ (see (3.1)), one deduces that $\Lambda_h(u) > 0$ for $u > 0$. Therefore, by the change of coordinates (3.4), Theorem 3.3 and taking δ small enough,

$$\Lambda_*^\diamond = \Lambda_h(u_*) - \frac{w^\diamond(u_*)}{3\Lambda_h(u_*)} = \Lambda_h(u_*) + \mathcal{O}(\delta^2) > 0, \qquad \text{with } \diamond = \mathrm{u}, \mathrm{s},$$

and, therefore using formula (3.28) for Δw and Lemma 3.16, we obtain that

$$|\Lambda_*^{\mathrm{u}} - \Lambda_*^{\mathrm{s}}| \leq C\,|\Delta w(u_*)| \leq C\delta\,|\Delta x(u_*)| + C\delta\,|\Delta y(u_*)| \leq C\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}.$$

4. The perturbed invariant manifolds

In this section, we prove Theorem 3.3 by following the scheme detailed in Sections 3.2.1 and 3.2.2.

Throughout this section and the following ones, we denote the components of all the functions and operators by a numerical sub-index $f = (f_1, f_2, f_3)^T$, unless stated otherwise.

4.1. The invariant manifolds in the infinity domain

The first step is to prove Proposition 3.5, which deals with the proof of the existence of parameterizations z^{u} and z^{s} satisfying the invariance equation (3.12) and the asymptotic conditions (3.6). We only consider the −u− case, being the −s− case analogous.

Consider the invariance equation (3.12), $\partial_u z^{\mathrm{u}} = \mathcal{A}^{\mathrm{sep}} z^{\mathrm{u}} + \mathcal{R}^{\mathrm{sep}}[z^{\mathrm{u}}]$, with $\mathcal{A}^{\mathrm{sep}}$ and $\mathcal{R}^{\mathrm{sep}}$ defined in (3.10) and (3.13), respectively. This equation can be written as

$$\mathcal{L} z^{\mathrm{u}} = \mathcal{R}^{\mathrm{sep}}[z^{\mathrm{u}}], \qquad \text{with} \qquad \mathcal{L}\varphi = (\partial_u - \mathcal{A}^{\mathrm{sep}})\varphi. \tag{4.1}$$

In order to obtain a fixed point equation from (4.1), we look for a left inverse of $\mathcal{L}$ in a suitable Banach space. To this end, for a fixed $\rho_1 > 0$ and a given $\alpha \in \mathbb{R}$, we introduce

$$\mathcal{X}_{\alpha}^{\infty} = \left\{ \varphi : D_{\rho_1}^{\mathrm{u},\infty} \to \mathbb{C} \ : \ \varphi \text{ real-analytic}, \ \|\varphi\|_{\alpha}^{\infty} := \sup_{u \in D_{\rho_1}^{\mathrm{u},\infty}} |e^{-\alpha u}\varphi(u)| < \infty \right\},$$

and the product space $\mathcal{X}_{\times}^{\infty} = \mathcal{X}_{2\nu}^{\infty} \times \mathcal{X}_{\nu}^{\infty} \times \mathcal{X}_{\nu}^{\infty}$, with $\nu = \sqrt{\frac{21}{8}}$ endowed with the weighted product norm

$$\|\varphi\|_{\times}^{\infty} = \delta \|\varphi_1\|_{2\nu}^{\infty} + \|\varphi_2\|_{\nu}^{\infty} + \|\varphi_3\|_{\nu}^{\infty}.$$

Next lemmas, proven in [8], give some properties of these Banach spaces and provide a left inverse operator of $\mathcal{L}$.

Lemma 4.1. *Let $\alpha, \beta \in \mathbb{R}$. Then, the following statements hold:*

1. *If $\alpha > \beta \geq 0$, then $\mathcal{X}_{\alpha}^{\infty} \subset \mathcal{X}_{\beta}^{\infty}$. Moreover $\|\varphi\|_{\beta}^{\infty} \leq \|\varphi\|_{\alpha}^{\infty}$.*
2. *If $\varphi \in \mathcal{X}_{\alpha}^{\infty}$ and $\zeta \in \mathcal{X}_{\beta}^{\infty}$, then $\varphi\zeta \in \mathcal{X}_{\alpha+\beta}^{\infty}$ and $\|\varphi\zeta\|_{\alpha+\beta}^{\infty} \leq \|\varphi\|_{\alpha}^{\infty} \|\zeta\|_{\beta}^{\infty}$.*

Lemma 4.2. *The linear operator $\mathcal{G} : \mathcal{X}_{\times}^{\infty} \to \mathcal{X}_{\times}^{\infty}$ given by*

$$\mathcal{G}[\varphi](u) = \left(\int_{-\infty}^{u} \varphi_1(s)ds, \int_{-\infty}^{u} e^{-\frac{i}{\delta^2}(s-u)}\varphi_2(s)ds, \int_{-\infty}^{u} e^{\frac{i}{\delta^2}(s-u)}\varphi_3(s)ds \right)^T$$

is continuous, injective and is a left inverse of the operator $\mathcal{L}$.

Moreover, there exists a constant C independent of δ and ρ_1 such that, for $\varphi \in \mathcal{X}_{\times}^{\infty}$,

$$\|\mathcal{G}[\varphi]\|_{\times}^{\infty} \leq C \left(\|\varphi_1\|_{2\nu}^{\infty} + \delta^2 \|\varphi_2\|_{\nu}^{\infty} + \delta^2 \|\varphi_3\|_{\nu}^{\infty} \right).$$

Notice that the eigenvalues of the saddle point $(0,0)$ of $H_{\mathrm{p}}(\lambda, \Lambda)$ (see (2.8)) are $\pm\sqrt{\frac{21}{8}}$. Then, the parametrization of the separatrix $\sigma = (\lambda_h, \Lambda_h)$ (see (3.1)) satisfies

$$\lambda_h \in \mathcal{X}_\nu^\infty \quad \text{and} \quad \Lambda_h \in \mathcal{X}_\nu^\infty. \tag{4.2}$$

Therefore, z^{u} is a solution of (4.1) satisfying the asymptotic conditions (3.6) if and only if $z^{\mathrm{u}} \in \mathcal{X}_\times^\infty$ and satisfies the fixed point equation

$$\varphi = \mathcal{F}[\varphi] = \mathcal{G} \circ \mathcal{R}^{\mathrm{sep}}[\varphi].$$

Thus, Proposition 3.5 is a straightforward consequence of the following proposition.

Proposition 4.3. *There exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, equation $\varphi = \mathcal{F}[\varphi]$ has a solution $z^{\mathrm{u}} \in \mathcal{X}_\times^\infty$. Moreover, there exists a real constant $b_{14} > 0$ independent of δ such that $\|z^{\mathrm{u}}\|_\times^\infty \leq b_{14}\delta^3$.*

To see that $\mathcal{F}$ is a contractive operator, we have to pay attention to the nonlinear terms $\mathcal{R}^{\mathrm{sep}}$.

Lemma 4.4. *Fix $\varrho > 0$ and let $\mathcal{R}^{\mathrm{sep}}$ be the operator defined in (3.13). Then, for $\delta > 0$ small enough*[3] *and $\|\varphi\|_\times^\infty \leq \varrho\delta^3$, there exists a constant $C > 0$ such that*

$$\|\mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_{2\nu}^\infty \leq C\delta^2, \qquad \|\mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_\nu^\infty \leq C\delta, \qquad j = 2, 3,$$

and

$$\begin{aligned}
&\|\partial_w \mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_0^\infty \leq C\delta^2, && \|\partial_x \mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_\nu^\infty \leq C\delta, && \|\partial_y \mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_\nu^\infty \leq C\delta, \\
&\|\partial_w \mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_{-\nu}^\infty \leq C\delta, && \|\partial_x \mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_0^\infty \leq C, && \|\partial_y \mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_0^\infty \leq C, \quad j = 2, 3.
\end{aligned}$$

The proof of this lemma is postponed to Appendix B.1.

Proof of Proposition 4.3. Consider the closed ball

$$B(\varrho) = \left\{\varphi \in \mathcal{X}_\times^\infty \ : \ \|\varphi\|_\times^\infty \leq \varrho\right\}.$$

First, we obtain an estimate for $\mathcal{F}[0]$. By Lemmas 4.2 and 4.4, if δ is small enough,

$$\|\mathcal{F}[0]\|_\times^\infty \leq C\delta \,\|\mathcal{R}_1^{\mathrm{sep}}[0]\|_{2\nu}^\infty + C\delta^2 \,\|\mathcal{R}_2^{\mathrm{sep}}[0]\|_\nu^\infty + C\delta^2 \,\|\mathcal{R}_3^{\mathrm{sep}}[0]\|_\nu^\infty \leq \frac{1}{2} b_{14}\delta^3, \tag{4.3}$$

for some $b_{14} > 0$.

Then, it only remains to check that the operator $\mathcal{F}$ is contractive in $B(b_{14}\delta^3)$. Let $\varphi, \tilde{\varphi} \in B(b_{14}\delta^3)$. Then, by the mean value theorem,

[3] To simplify the exposition, in this lemma and in the technical lemmas from now on, we avoid referring to the existence of δ_0 and just mention that δ must be small enough. We follow the same convention for κ whenever is needed.

$$\mathcal{R}_j^{\mathrm{sep}}[\varphi] - \mathcal{R}_j^{\mathrm{sep}}[\widetilde{\varphi}] = \left[\int_0^1 D\mathcal{R}_j^{\mathrm{sep}}[s\varphi + (1-s)\widetilde{\varphi}]ds\right](\varphi - \widetilde{\varphi}), \qquad j = 1,2,3.$$

Applying Lemmas 4.1 and 4.4 and the above equality, we obtain

$$\begin{aligned} \|\mathcal{R}_1^{\mathrm{sep}}[\varphi] - \mathcal{R}_1^{\mathrm{sep}}[\widetilde{\varphi}]\|_{2\nu}^{\infty} \le \sup_{\zeta\in B(b_{14}\delta^3)} \Big[& \|\varphi_1 - \widetilde{\varphi}_1\|_{2\nu}^{\infty} \|\partial_w \mathcal{R}_1^{\mathrm{sep}}[\zeta]\|_0^{\infty} \\ & + \|\varphi_2 - \widetilde{\varphi}_2\|_{\nu}^{\infty} \|\partial_x \mathcal{R}_1^{\mathrm{sep}}[\zeta]\|_{\nu}^{\infty} + \|\varphi_3 - \widetilde{\varphi}_3\|_{\nu}^{\infty} \|\partial_y \mathcal{R}_1^{\mathrm{sep}}[\zeta]\|_{\nu}^{\infty} \Big] \le C\delta \|\varphi - \widetilde{\varphi}\|_{\times}^{\infty}, \\ \|\mathcal{R}_j^{\mathrm{sep}}[\varphi] - \mathcal{R}_j^{\mathrm{sep}}[\widetilde{\varphi}]\|_{\nu}^{\infty} \le \sup_{\zeta\in B(b_{14}\delta^3)} \Big[& \|\varphi_1 - \widetilde{\varphi}_1\|_{2\nu}^{\infty} \|\partial_w \mathcal{R}_j^{\mathrm{sep}}[\zeta]\|_{-\nu}^{\infty} \\ & + \|\varphi_2 - \widetilde{\varphi}_2\|_{\nu}^{\infty} \|\partial_x \mathcal{R}_j^{\mathrm{sep}}[\zeta]\|_0^{\infty} + \|\varphi_3 - \widetilde{\varphi}_3\|_{\nu}^{\infty} \|\partial_y \mathcal{R}_j^{\mathrm{sep}}[\zeta]\|_0^{\infty} \Big] \le C \|\varphi - \widetilde{\varphi}\|_{\times}^{\infty}, \end{aligned}$$

for $j = 2,3$. Then, by Lemma 4.2 and taking δ small enough,

$$\begin{aligned} \|\mathcal{F}[\varphi] - \mathcal{F}[\widetilde{\varphi}]\|_{\times}^{\infty} &\le C\delta \|\mathcal{R}_1^{\mathrm{sep}}[\varphi] - \mathcal{R}_1^{\mathrm{sep}}[\widetilde{\varphi}]\|_{2\nu}^{\infty} + C\delta^2 \sum_{j=2}^{3} \|\mathcal{R}_j^{\mathrm{sep}}[\varphi] - \mathcal{R}_j^{\mathrm{sep}}[\widetilde{\varphi}]\|_{\nu}^{\infty} \\ &\le C\delta^2 \|\varphi - \widetilde{\varphi}\|_{\times}^{\infty} \le \frac{1}{2} \|\varphi - \widetilde{\varphi}\|_{\times}^{\infty}. \end{aligned} \tag{4.4}$$

Then, by the definition of ϱ in (4.3) and (4.4), $\mathcal{F} : B(b_{14}\delta^3) \to B(b_{14}\delta^3)$ is well defined and contractive. Therefore, $\mathcal{F}$ has a fixed point $z^{\mathrm{u}} \in B(b_{14}\delta^3)$. □

4.2. The invariant manifolds in the outer domain

To prove Proposition 3.6, we must extend analytically the parameterizations z^{u} and z^{s} given in Proposition 3.5 to the outer domains, $D^{\mathrm{u,out}}_{\kappa,d_1,\rho_2}$ and $D^{\mathrm{s,out}}_{\kappa,d_1,\rho_2}$, respectively. Again, we only deal with the unstable -u- case, being the -s- case analogous. We prove the existence of z^{u} by means of a fixed point argument in a suitable Banach space.

Given $\alpha,\beta \in \mathbb{R}$, we consider the norm

$$\|\varphi\|_{\alpha,\beta}^{\mathrm{out}} = \sup_{u\in D^{\mathrm{u,out}}_{\kappa,d_1,\rho_2}} \left| g_\delta^{-\alpha}(u) \left(u^2 + A^2\right)^{\beta} \varphi(u)\right|, \quad g_\delta(u) = \frac{1}{|u^2 + A^2|} + \frac{\delta^2}{|u^2 + A^2|^{\frac{8}{3}}},$$

and the associated Banach space

$$\mathcal{X}_{\alpha,\beta}^{\mathrm{out}} = \left\{\varphi : D^{\mathrm{u,out}}_{\kappa,d_1,\rho_2} \to \mathbb{C} \ : \ \varphi \text{ real-analytic}, \ \|\varphi\|_{\alpha,\beta}^{\mathrm{out}} < \infty\right\}. \tag{4.5}$$

These Banach spaces have the following properties, which we use without mentioning along the section. Their proof follows the same lines as the proof of Lemma 7.1 in [8].

Lemma 4.5. *The following statements hold:*

1. *If $\varphi \in \mathcal{X}^{\text{out}}_{\alpha,\beta_1}$, then $\varphi \in \mathcal{X}^{\text{out}}_{\alpha,\beta_2}$ for any $\beta_2 \in \mathbb{R}$ and*

$$\begin{cases} \|\varphi\|^{\text{out}}_{\alpha,\beta_2} \le C\,\|\varphi\|^{\text{out}}_{\alpha,\beta_1}, & \text{for } \beta_2-\beta_1>0, \\ \|\varphi\|^{\text{out}}_{\alpha,\beta_2} \le C(\kappa\delta^2)^{\beta_2-\beta_1}\,\|\varphi\|^{\text{out}}_{\alpha,\beta_1}, & \text{for } \beta_2-\beta_1\le 0. \end{cases}$$

2. *If $\varphi \in \mathcal{X}^{\text{out}}_{\alpha,\beta_1}$, then $\varphi \in \mathcal{X}^{\text{out}}_{\alpha-1,\beta_2}$ for any $\beta_2 \in \mathbb{R}$ and*

$$\begin{cases} \|\varphi\|^{\text{out}}_{\alpha-1,\beta_2} \le C\,\|\varphi\|^{\text{out}}_{\alpha,\beta_1}, & \text{for } \beta_2-\beta_1>\frac{5}{3}, \\ \|\varphi\|^{\text{out}}_{\alpha-1,\beta_2} \le C\delta^2(\kappa\delta^2)^{(\beta_2-\beta_1)-\frac{8}{3}}\,\|\varphi\|^{\text{out}}_{\alpha,\beta_1}, & \text{for } \beta_2-\beta_1\le \frac{5}{3}. \end{cases}$$

3. *If $\varphi \in \mathcal{X}^{\text{out}}_{\alpha_1,\beta_1}$ and $\zeta \in \mathcal{X}^{\text{out}}_{\alpha_2,\beta_2}$, then $\varphi\zeta \in \mathcal{X}^{\text{out}}_{\alpha_1+\alpha_2,\beta_1+\beta_2}$ and*

$$\|\varphi\zeta\|^{\text{out}}_{\alpha_1+\alpha_2,\beta_1+\beta_2} \le \|\varphi\|^{\text{out}}_{\alpha_1,\beta_1}\,\|\zeta\|^{\text{out}}_{\alpha_2,\beta_2}.$$

4. *If $\varphi \in \mathcal{X}^{\text{out}}_{0,\beta+1}$ and $\zeta \in \mathcal{X}^{\text{out}}_{0,\beta+\frac{8}{3}}$, then $\varphi+\delta^2\zeta \in \mathcal{X}^{\text{out}}_{1,\beta}$ and*

$$\|\varphi+\delta^2\zeta\|^{\text{out}}_{1,\beta} \le \|\varphi\|^{\text{out}}_{0,\beta+1} + \|\zeta\|^{\text{out}}_{0,\beta+\frac{8}{3}}.$$

Let us recall that, by Proposition 3.5, the invariance equation (3.12) has a unique solution z^{u} in the domain $D^{\text{u},\infty}_{\rho_1}$ satisfying the asymptotic condition (3.6). Our objective is to extend analytically z^{u} to the outer domain $D^{\text{u,out}}_{\kappa,d_1,\rho_2}$. Notice that, since $\rho_1<\rho_2$, $D^{\text{u},\infty}_{\rho_1} \cap D^{\text{u,out}}_{\kappa,d_1,\rho_2} \neq \emptyset$ (see definitions (3.14) and (3.15) of $D^{\text{u},\infty}_{\rho_1}$ and $D^{\text{u,out}}_{\kappa,d_1,\rho_2}$).

As explained in Section 4.1, equation (3.12) is equivalent to $\mathcal{L}z^{\text{u}} = \mathcal{R}^{\text{sep}}[z^{\text{u}}]$ with $\mathcal{L}\varphi = (\partial_u - \mathcal{A}^{\text{sep}})\varphi$ and $\mathcal{R}^{\text{sep}}$ given in (3.13). In the following lemma we introduce a right-inverse operator of $\mathcal{L}$ defined on $\mathcal{X}^{\text{out}}_{\alpha,\beta}$.

Lemma 4.6. *Let us consider the operator $\mathcal{G}[\varphi] = (\mathcal{G}_1[\varphi_1], \mathcal{G}_2[\varphi_2], \mathcal{G}_3[\varphi_3])^T$, such that*

$$\mathcal{G}[\varphi](u) = \left(\int_{-\rho_2}^{u} \varphi_1(s)ds, \int_{\overline{u}_1}^{u} e^{-\frac{i}{\delta^2}(s-u)}\varphi_2(s)ds, \int_{u_1}^{u} e^{\frac{i}{\delta^2}(s-u)}\varphi_3(s)ds \right)^T,$$

where u_1 and $\overline{u}_1$ are the vertices of the domain $D^{\text{u,out}}_{\kappa,d_1,\rho_2}$ (see Fig. 5). Fix $\beta>0$. There exists a constant C such that:

1. *If $\varphi \in \mathcal{X}^{\text{out}}_{1,\beta}$, then $\mathcal{G}_1[\varphi] \in \mathcal{X}^{\text{out}}_{1,\beta-1}$ and $\|\mathcal{G}_1[\varphi]\|^{\text{out}}_{1,\beta-1} \le C\,\|\varphi\|^{\text{out}}_{1,\beta}$.*
2. *If $\varphi \in \mathcal{X}^{\text{out}}_{0,\beta}$, then $\mathcal{G}_j[\varphi] \in \mathcal{X}^{\text{out}}_{0,\beta}$, $j=2,3$, and $\|\mathcal{G}_j[\varphi]\|^{\text{out}}_{0,\beta} \le C\delta^2\,\|\varphi\|^{\text{out}}_{0,\beta}$.*

The proof of this lemma follows the same lines as the proof of Lemma 7.3 in [8].

Consider u_1 and $\overline{u_1}$ as in Fig. 5 and the function

$$F^0(u) = \left(w^{\mathrm{u}}(-\rho_2),\, x^{\mathrm{u}}(\overline{u}_1)e^{-\frac{i}{\delta^2}(\overline{u}_1-u)},\, y^{\mathrm{u}}(u_1)e^{\frac{i}{\delta^2}(u_1-u)}\right)^T.$$

Notice that, since $0<\rho_1<\rho_2$, we have $\{-\rho_2, u_1, \overline{u}_1\} \in D^{\mathrm{u},\infty}_{\rho_1}$. Therefore, by Proposition 3.5, z^{u} is already defined at these points. We define the fixed point operator

$$\mathcal{F}[\varphi] = F^0 + \mathcal{G} \circ \mathcal{R}^{\mathrm{sep}}[\varphi], \tag{4.6}$$

where the operator $\mathcal{R}^{\mathrm{sep}}$ is given in (3.13). Since $\mathcal{L}(F^0)=0$, by Lemma 4.6, a solution $z^{\mathrm{u}} = \mathcal{F}[z^{\mathrm{u}}]$ satisfies $\mathcal{L}z^{\mathrm{u}} = \mathcal{R}^{\mathrm{sep}}[z^{\mathrm{u}}]$ and by construction is the real-analytic continuation of the function z^{u} obtained in Proposition 3.5.

We rewrite Proposition 3.6 in terms of the operator $\mathcal{F}$ defined in the Banach space

$$\mathcal{X}^{\mathrm{out}}_{\times} = \mathcal{X}^{\mathrm{out}}_{1,0} \times \mathcal{X}^{\mathrm{out}}_{0,\frac{4}{3}} \times \mathcal{X}^{\mathrm{out}}_{0,\frac{4}{3}},$$

endowed with the norm

$$\|\varphi\|^{\mathrm{out}}_{\times} = \delta\, \|\varphi_1\|^{\mathrm{out}}_{1,0} + \|\varphi_2\|^{\mathrm{out}}_{0,\frac{4}{3}} + \|\varphi_3\|^{\mathrm{out}}_{0,\frac{4}{3}}.$$

Proposition 4.7. *There exist $\delta_0, \kappa_1 > 0$ such that, for $\delta \in (0,\delta_0)$ and $\kappa \geq \kappa_1$, the fixed point equation $z^{\mathrm{u}} = \mathcal{F}[z^{\mathrm{u}}]$ has a unique solution $z^{\mathrm{u}} \in \mathcal{X}^{\mathrm{out}}_{\times}$. Moreover, there exists a real constant $b_{15}>0$ independent of δ and κ such that $\|z^{\mathrm{u}}\|^{\mathrm{out}}_{\times} \leq b_{15}\delta^3$.*

We prove this proposition through a fixed point argument. First, we state a technical lemma, whose proof is postponed until Appendix B.2. Fix $\varrho>0$ and define

$$B(\varrho) = \left\{\varphi \in \mathcal{X}^{\mathrm{out}}_{\times} \,:\, \|\varphi\|^{\mathrm{out}}_{\times} \leq \varrho\right\}.$$

Lemma 4.8. *Fix $\varrho>0$ and let $\mathcal{R}^{\mathrm{sep}}$ be the operator defined in (3.13). For $\delta>0$ small enough and $\kappa>0$ big enough, there exists a constant $C>0$ such that, for $\varphi \in B(\varrho\delta^3)$,*

$$\|\mathcal{R}^{\mathrm{sep}}_1[\varphi]\|^{\mathrm{out}}_{1,1} \leq C\delta^2, \qquad \|\mathcal{R}^{\mathrm{sep}}_j[\varphi]\|^{\mathrm{out}}_{0,\frac{4}{3}} \leq C\delta, \qquad j=2,3,$$

and

$$\begin{aligned}
&\|\partial_w \mathcal{R}^{\mathrm{sep}}_1[\varphi]\|^{\mathrm{out}}_{1,\frac{1}{3}} \leq C\delta^2, && \|\partial_x \mathcal{R}^{\mathrm{sep}}_1[\varphi]\|^{\mathrm{out}}_{0,\frac{7}{3}} \leq C\delta, && \|\partial_y \mathcal{R}^{\mathrm{sep}}_1[\varphi]\|^{\mathrm{out}}_{0,\frac{7}{3}} \leq C\delta,\\
&\|\partial_w \mathcal{R}^{\mathrm{sep}}_2[\varphi]\|^{\mathrm{out}}_{0,\frac{2}{3}} \leq C\delta, && \|\partial_x \mathcal{R}^{\mathrm{sep}}_2[\varphi]\|^{\mathrm{out}}_{1,-\frac{2}{3}} \leq C, && \|\partial_y \mathcal{R}^{\mathrm{sep}}_2[\varphi]\|^{\mathrm{out}}_{0,2} \leq C\delta^2,\\
&\|\partial_w \mathcal{R}^{\mathrm{sep}}_3[\varphi]\|^{\mathrm{out}}_{0,\frac{2}{3}} \leq C\delta, && \|\partial_x \mathcal{R}^{\mathrm{sep}}_3[\varphi]\|^{\mathrm{out}}_{0,2} \leq C\delta^2, && \|\partial_y \mathcal{R}^{\mathrm{sep}}_3[\varphi]\|^{\mathrm{out}}_{1,-\frac{2}{3}} \leq C.
\end{aligned}$$

The next lemma gives properties of the operator $\mathcal{F}$.

Lemma 4.9. *Fix $\varrho > 0$ and let $\mathcal{F}$ be the operator defined in (4.6). Then, for $\delta > 0$ small enough and $\kappa > 0$ big enough, there exist constants $b_{16}, b_{17} > 0$ independent of δ and κ such that*

$$\|\mathcal{F}[0]\|_{\times}^{\text{out}} \leq b_{16}\delta^3.$$

Moreover, for $\varphi, \tilde{\varphi} \in B(\varrho\delta^3)$,

$$\delta \|\mathcal{F}_1[\varphi] - \mathcal{F}_1[\tilde{\varphi}]\|_{1,0}^{\text{out}} \leq b_{17}\left(\frac{\delta}{\kappa^2}\|\varphi_1 - \tilde{\varphi}_1\|_{1,0}^{\text{out}} + \|\varphi_2 - \tilde{\varphi}_2\|_{0,\frac{4}{3}}^{\text{out}} + \|\varphi_3 - \tilde{\varphi}_3\|_{0,\frac{4}{3}}^{\text{out}}\right),$$

$$\|\mathcal{F}_j[\varphi] - \mathcal{F}_j[\tilde{\varphi}]\|_{0,\frac{4}{3}}^{\text{out}} \leq \frac{b_{17}}{\kappa^2}\|\varphi - \tilde{\varphi}\|_{\times}^{\text{out}}, \qquad \textit{for} \quad j = 2, 3.$$

Proof. First, we obtain the estimates for $\mathcal{F}[0]$. By Proposition 3.5, we have that

$$|w^{\text{u}}(-\rho_2)| \leq C\delta^2, \quad |x^{\text{u}}(\overline{u_1})| \leq C\delta^3, \quad |y^{\text{u}}(u_1)| \leq C\delta^3$$

and, as a result, $\|F^0\|_{\times}^{\text{out}} \leq C\delta^3$. Then, applying Lemmas 4.6 and 4.8, we obtain

$$\|\mathcal{F}[0]\|_{\times}^{\text{out}} \leq \left\|F^0\right\|_{\times}^{\text{out}} + C\delta\left\|\mathcal{R}_1^{\text{sep}}[0]\right\|_{1,1}^{\text{out}} + C\delta^2 \textstyle\sum_{j=2}^{3}\|\mathcal{R}_j^{\text{sep}}[0]\|_{0,\frac{4}{3}}^{\text{out}} \leq C\delta^3.$$

For the second statement, since $\mathcal{F} = F^0 + \mathcal{G}\circ\mathcal{R}^{\text{sep}}$ and $\mathcal{G}$ is linear, we need to compute estimates for $\mathcal{R}^{\text{sep}}[\varphi] - \mathcal{R}^{\text{sep}}[\tilde{\varphi}]$. Then, by the mean value theorem,

$$\mathcal{R}_j^{\text{sep}}[\varphi] - \mathcal{R}_j^{\text{sep}}[\tilde{\varphi}] = \left[\int_0^1 D\mathcal{R}_j^{\text{sep}}[s\varphi + (1-s)\tilde{\varphi}]ds\right](\varphi - \tilde{\varphi}), \qquad j = 1, 2, 3.$$

In addition, by Lemmas 4.5 and 4.8, for $j = 2, 3$, we have the estimates

$$\|\partial_w\mathcal{R}_1^{\text{sep}}[\varphi]\|_{0,1}^{\text{out}} \leq \frac{C}{\kappa^2}, \quad \|\partial_x\mathcal{R}_1^{\text{sep}}[\varphi]\|_{1,-\frac{1}{3}}^{\text{out}} \leq \frac{C}{\delta}, \qquad \|\partial_y\mathcal{R}_1^{\text{sep}}[\varphi]\|_{1,-\frac{1}{3}}^{\text{out}} \leq \frac{C}{\delta},$$
$$\|\partial_w\mathcal{R}_j^{\text{sep}}[\varphi]\|_{-1,\frac{4}{3}}^{\text{out}} \leq \frac{C}{\kappa^2\delta}, \quad \|\partial_x\mathcal{R}_j^{\text{sep}}[\varphi]\|_{0,0}^{\text{out}} \leq \frac{C}{\kappa^2\delta^2}, \quad \|\partial_y\mathcal{R}_j^{\text{sep}}[\varphi]\|_{0,0}^{\text{out}} \leq \frac{C}{\kappa^2\delta^2}.$$

We estimate each component separately. For $j = 1$, we have that

$$\begin{aligned}
\delta\|\mathcal{R}_1^{\text{sep}}[\varphi] - \mathcal{R}_1^{\text{sep}}[\tilde{\varphi}]\|_{1,1}^{\text{out}} \leq \sup_{\zeta\in B(\varrho\delta^3)} \delta\Big[&\|\varphi_1 - \tilde{\varphi}_1\|_{1,0}^{\text{out}}\|\partial_w\mathcal{R}_1^{\text{sep}}[\zeta]\|_{0,1}^{\text{out}} \\
&+ \|\varphi_2 - \tilde{\varphi}_2\|_{0,\frac{4}{3}}^{\text{out}}\|\partial_x\mathcal{R}_1^{\text{sep}}[\zeta]\|_{1,-\frac{1}{3}}^{\text{out}} + \|\varphi_3 - \tilde{\varphi}_3\|_{0,\frac{4}{3}}^{\text{out}}\|\partial_y\mathcal{R}_1^{\text{sep}}[\zeta]\|_{1,-\frac{1}{3}}^{\text{out}}\Big] \\
\leq \frac{C\delta}{\kappa^2}\|\varphi_1 - \tilde{\varphi}_1\|_{1,0}^{\text{out}} &+ C\|\varphi_2 - \tilde{\varphi}_2\|_{0,\frac{4}{3}}^{\text{out}} + C\|\varphi_3 - \tilde{\varphi}_3\|_{0,\frac{4}{3}}^{\text{out}}.
\end{aligned}$$

Analogously, for $j = 2, 3$, we obtain

$$\begin{aligned}\|\mathcal{R}_j^{\mathrm{sep}}[\varphi]-\mathcal{R}_j^{\mathrm{sep}}[\tilde{\varphi}]\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq \sup_{\zeta\in B(\varrho\delta^3)} \Big[&\|\varphi_1-\tilde{\varphi}_1\|_{1,0}^{\mathrm{out}} \|\partial_w \mathcal{R}_j^{\mathrm{sep}}[\zeta]\|_{-1,\frac{4}{3}}^{\mathrm{out}} \\ &+ \|\varphi_2-\tilde{\varphi}_2\|_{0,\frac{4}{3}}^{\mathrm{out}} \|\partial_x \mathcal{R}_j^{\mathrm{sep}}[\zeta]\|_{0,0}^{\mathrm{out}} + \|\varphi_3-\tilde{\varphi}_3\|_{0,\frac{4}{3}}^{\mathrm{out}} \|\partial_y \mathcal{R}_j^{\mathrm{sep}}[\zeta]\|_{0,0}^{\mathrm{out}}\Big] \\ &\leq \frac{C}{\kappa^2\delta^2}\|\varphi-\tilde{\varphi}\|_{\times}^{\mathrm{out}}\end{aligned}$$

and, using Lemma 4.6, we obtain the estimates for the second statement. □

Lemma 4.9 shows that, by assuming κ big enough, operators $\mathcal{F}_2$ and $\mathcal{F}_3$ have Lipschitz constant less than 1 with the norm in $\mathcal{X}_{\times}^{\mathrm{out}}$. However, we are not able to control the Lipschitz constant of $\mathcal{F}_1$. To overcome this problem, we apply a Gauss-Seidel argument to define a new operator

$$\widetilde{\mathcal{F}}[z] = \widetilde{\mathcal{F}}[(w,x,y)] = \begin{pmatrix} \mathcal{F}_1[w,\mathcal{F}_2[z],\mathcal{F}_3[z]] \\ \mathcal{F}_2[z] \\ \mathcal{F}_3[z] \end{pmatrix},$$

which turns out to be contractive in a suitable ball and has the same fixed points as $\mathcal{F}$.

End of the proof of Proposition 4.7. We look for a fixed point of $\widetilde{\mathcal{F}}$. First, we obtain an estimate for $\|\widetilde{\mathcal{F}}[0]\|_{\times}^{\mathrm{out}}$. We rewrite it as

$$\widetilde{\mathcal{F}}[0] = \mathcal{F}[0] + \Big(\mathcal{F}_1[0,\mathcal{F}_2[0],\mathcal{F}_3[0]] - \mathcal{F}_1[0],0,0\Big)^T,$$

and we notice that, by Lemma 4.9, $\|(0,\mathcal{F}_2[0],\mathcal{F}_3[0])\|_{\times}^{\mathrm{out}} \leq \|\mathcal{F}[0]\|_{\times}^{\mathrm{out}} \leq C\delta^3$. Then, applying Lemma 4.9, there exists constant $b_{15}>0$ such that

$$\begin{aligned}\|\widetilde{\mathcal{F}}[0]\|_{\times}^{\mathrm{out}} &\leq \|\mathcal{F}[0]\|_{\times}^{\mathrm{out}} + \|\mathcal{F}_1[0,\mathcal{F}_2[0],\mathcal{F}_3[0]] - \mathcal{F}_1[0]\|_{1,0}^{\mathrm{out}} \\ &\leq \|\mathcal{F}[0]\|_{\times}^{\mathrm{out}} + C\|\mathcal{F}_2[0]\|_{0,\frac{4}{3}}^{\mathrm{out}} + C\|\mathcal{F}_3[0]\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq \frac{1}{2}b_{15}\delta^3.\end{aligned} \tag{4.7}$$

Now, we prove that the operator $\widetilde{\mathcal{F}}$ is contractive in $B(b_{15}\delta^3)$. Indeed, by Lemma 4.9, we have that, for $\varphi,\tilde{\varphi}\in B(b_{15}\delta^3)$,

$$\begin{aligned}\delta\|\widetilde{\mathcal{F}}_1[\varphi]-\widetilde{\mathcal{F}}_1[\tilde{\varphi}]\|_{1,0}^{\mathrm{out}} &\leq C\left(\frac{\delta}{\kappa^2}\|\varphi_1-\tilde{\varphi}_1\|_{1,0}^{\mathrm{out}} + \|\mathcal{F}_2[\varphi]-\mathcal{F}_2[\tilde{\varphi}]\|_{0,\frac{4}{3}}^{\mathrm{out}} + \|\mathcal{F}_3[\varphi]-\mathcal{F}_3[\tilde{\varphi}]\|_{0,\frac{4}{3}}^{\mathrm{out}}\right) \\ &\leq \frac{C\delta}{\kappa^2}\|\varphi_1-\tilde{\varphi}_1\|_{1,0}^{\mathrm{out}} + \frac{2C}{\kappa^2}\|\varphi-\tilde{\varphi}\|_{\times}^{\mathrm{out}} \leq \frac{C}{\kappa^2}\|\varphi-\tilde{\varphi}\|_{\times}^{\mathrm{out}}, \\ \|\widetilde{\mathcal{F}}_j[\varphi]-\widetilde{\mathcal{F}}_j[\tilde{\varphi}]\|_{0,\frac{4}{3}}^{\mathrm{out}} &= \|\mathcal{F}_j[\varphi]-\mathcal{F}_j[\tilde{\varphi}]\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq \frac{C}{\kappa^2}\|\varphi-\tilde{\varphi}\|_{\times}^{\mathrm{out}}, \qquad \text{for } j=2,3.\end{aligned}$$

Then, for $\kappa>0$ big enough, we have that $\|\widetilde{\mathcal{F}}[\varphi]-\widetilde{\mathcal{F}}[\tilde{\varphi}]\|_{\times}^{\mathrm{out}} \leq \frac{1}{2}\|\varphi-\tilde{\varphi}\|_{\times}^{\mathrm{out}}$. Together with (4.7), this implies that $\widetilde{\mathcal{F}}: B(b_{15}\delta^3)\to B(b_{15}\delta^3)$ is well defined and contractive. Therefore, $\widetilde{\mathcal{F}}$ has a fixed point $z^{\mathrm{u}}\in B(b_{15}\delta^3)$. □

4.3. Switching to the time-parametrization

In this section, by means of a fixed point argument, we prove Proposition 3.7. That is, we obtain a change of variables $\mathcal{U}$ satisfying (3.21), that is

$$\partial_v \mathcal{U} = R[\mathcal{U}] \quad \text{where} \quad R[\mathcal{U}] = \partial_w H_1^{\text{sep}}\left(v + \mathcal{U}(v), z^{\text{u}}(v + \mathcal{U}(v))\right). \tag{4.8}$$

To this end, we consider the Banach space

$$\mathcal{Y}^{\text{out}} = \left\{ \varphi : \widetilde{D}^{\text{u,out}}_{\kappa_2,d_2,d_3} \to \mathbb{C} \; : \; \varphi \text{ real-analytic}, \; \|\varphi\|_{\sup} := \sup_{v \in \widetilde{D}^{\text{u,out}}_{\kappa_2,d_2,d_3}} |\mathcal{U}(v)| < \infty \right\}. \tag{4.9}$$

First, we state a technical lemma. Its proof is a direct consequence of the proof of Lemma 4.8 (see also Remark B.12 in Appendix B.2).

Lemma 4.10. *Fix $\varrho > 0$. For $\delta > 0$ small enough and $\varphi \in \mathcal{Y}^{\text{out}}$ such that $\|\varphi\|_{\sup} \leq \varrho\delta^2$, there exists a constant $C > 0$ such that $\|R[\varphi]\|_{\sup} \leq C\delta^2$ and $\|DR[\varphi]\|_{\sup} \leq C\delta^2$.*

Let us define the operators

$$G[\varphi](v) = \int_{\rho_3}^{v} \varphi(s) ds \qquad \text{and} \qquad F = G \circ R, \tag{4.10}$$

where $\rho_3 \in \mathbb{R}$ is the rightmost vertex of the domain $\widetilde{D}^{\text{u,out}}_{\kappa_2,d_2,d_3}$ (see Fig. 7). Then, a solution $\mathcal{U} = F[\mathcal{U}]$ satisfies equation (4.8) and the initial condition $\mathcal{U}(\rho_3) = 0$.

Proof of Proposition 3.7. The operator G in (4.10) satisfies that, for $\varphi \in \mathcal{Y}^{\text{out}}$,

$$\|G[\varphi]\|_{\sup} \leq C\|\varphi\|_{\sup}. \tag{4.11}$$

Then, by Lemma 4.10, there exists $b_4 > 0$ independent of δ such that

$$\|F[0]\|_{\sup} \leq C\|R[0]\|_{\sup} \leq \frac{1}{2} b_4 \delta^2. \tag{4.12}$$

Moreover, for $\varphi, \tilde{\varphi} \in B(b_4\delta^2) = \left\{\varphi \in \mathcal{Y}^{\text{out}} \; : \; \|\varphi\|_{\sup} \leq b_4\delta^2\right\}$, by the mean value theorem and Lemma 4.10,

$$\|R[\varphi] - R[\tilde{\varphi}]\|_{\sup} = \left\| \int_0^1 DR[s\varphi + (1-s)\tilde{\varphi}] ds \right\|_{\sup} \|\varphi - \tilde{\varphi}\|_{\sup} \leq C\delta^2 \|\varphi - \tilde{\varphi}\|_{\sup}.$$

Then, by Lemma 4.10, (4.11), (4.12) and taking δ small enough, F is well defined and contractive in $B(b_4\delta^2)$ and, as a result, has a fixed point $\mathcal{U} \in B(b_4\delta^2)$.

It only remains to check that $v + \mathcal{U}(v) \in D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2}$ for $v \in \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$. Indeed, since $\|\mathcal{U}\|_{\sup} \leq b_4\delta^2$ and $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3} \subset D^{\mathrm{u,out}}_{\kappa,d_1,\rho_2}$, taking δ small enough the statement is proved. □

4.4. Extending the time-parametrization

In this section, we extend analytically the parametrization Γ^{u} given in Corollary 3.8 from the transition domain $D^{\mathrm{u,out}}_{\kappa,d_1,\rho_2}$ to the flow domain $D^{\mathrm{fl}}_{\kappa_3,d_4}$ (see (3.19)).

Since Γ^{u} satisfies the equations given by H in (2.7), $\widehat{\Gamma} = \Gamma^{\mathrm{u}} - \Gamma_h$ (see (3.22)) satisfies

$$\begin{cases} \partial_v \hat{\lambda} = -3\hat{\Lambda} + \partial_\Lambda H_1(\Gamma_h + \widehat{\Gamma}; \delta), \\ \partial_v \hat{\Lambda} = -V'(\lambda_h + \hat{\lambda}) + V'(\lambda_h) - \partial_\lambda H_1(\Gamma_h + \widehat{\Gamma}; \delta), \\ \partial_v \hat{x} = i\dfrac{\hat{x}}{\delta^2} + i\partial_y H_1(\Gamma_h + \widehat{\Gamma}; \delta), \\ \partial_v \hat{y} = -i\dfrac{\hat{y}}{\delta^2} - i\partial_x H_1(\Gamma_h + \widehat{\Gamma}; \delta), \end{cases}$$

which can be rewritten as $\mathcal{L}^{\mathrm{fl}}\,\widehat{\Gamma} = \mathcal{R}^{\mathrm{fl}}[\widehat{\Gamma}]$, where

$$\mathcal{L}^{\mathrm{fl}}\varphi = \left(\partial_v - \mathcal{A}^{\mathrm{fl}}(v)\right)\varphi, \qquad \mathcal{A}^{\mathrm{fl}}(v) = \begin{pmatrix} 0 & -3 & 0 & 0 \\ -V''(\lambda_h(v)) & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix}, \tag{4.13}$$

and

$$\mathcal{R}^{\mathrm{fl}}[\varphi](v) = \begin{pmatrix} \partial_\Lambda H_1(\Gamma_h(v) + \varphi(v); \delta) \\ T[\varphi_1](v) - \partial_\lambda H_1(\Gamma_h(v) + \varphi(v); \delta) \\ i\partial_y H_1(\Gamma_h(v) + \varphi(v); \delta) \\ -i\partial_x H_1(\Gamma_h(v) + \varphi(v); \delta) \end{pmatrix}, \tag{4.14}$$

with $T[\varphi_1] = -V'(\lambda_h + \varphi_1) + V'(\lambda_h) + V''(\lambda_h)\varphi_1$.

We look for $\widehat{\Gamma}$ through fixed point argument in the Banach space $\mathcal{X}^{\mathrm{fl}}_\times = \left(\mathcal{X}^{\mathrm{fl}}\right)^4$, where

$$\mathcal{X}^{\mathrm{fl}} = \left\{ \varphi : D^{\mathrm{fl}}_{\kappa_3,d_4} \to \mathbb{C} \; : \; \varphi \text{ real-analytic}, \, \|\varphi\|^{\mathrm{fl}} := \sup_{v \in D^{\mathrm{fl}}_{\kappa_3,d_4}} |\varphi(v)| < \infty \right\},$$

endowed with the norm

$$\|\varphi\|^{\mathrm{fl}}_\times = \delta\|\varphi_1\|^{\mathrm{fl}} + \delta\|\varphi_2\|^{\mathrm{fl}} + \|\varphi_3\|^{\mathrm{fl}} + \|\varphi_4\|^{\mathrm{fl}}.$$

A fundamental matrix of the linear equation $\dot{\xi} = \mathcal{A}^{\mathrm{fl}}(v)\xi$ is

$$\Phi(v) = \begin{pmatrix} 3\Lambda_h(v) & 3f_h(v) & 0 & 0 \\ -\dot{\Lambda}_h(v) & -\dot{f}_h(v) & 0 & 0 \\ 0 & 0 & e^{\frac{i}{\delta^2}v} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{\delta^2}v} \end{pmatrix} \quad \text{with} \quad f_h(v) = \Lambda_h(v)\int_{v_0}^{v} \frac{1}{\Lambda_h^2(s)} ds.$$

Note that $f_h(v)$ is analytic at $v = 0$.

To look for a right inverse of operator $\mathcal{L}^{\text{fl}}$ in (4.13), let us consider the linear operator

$$\mathcal{G}^{\text{fl}}[\varphi](v) = \left(\int_{v_0}^{v} \varphi_1(s)ds, \int_{v_0}^{v} \varphi_2(s)ds, \int_{\overline{v}_1}^{v} \varphi_3(s)ds, \int_{v_1}^{v} \varphi_4(s)ds \right)^T,$$

where v_0, v_1 and $\overline{v}_1$ are the vertexes of the domain $D^{\text{fl}}_{\kappa_3,d_4}$ (see Fig. 7). Then, the linear operator $\widehat{\mathcal{G}}[\varphi] = \Phi\mathcal{G}[\Phi^{-1}\varphi]$ is a right inverse of the operator $\mathcal{L}^{\text{fl}}$, and, for $\varphi \in \mathcal{X}^{\text{fl}}_{\times}$, satisfies

$$\begin{aligned} \|\widehat{\mathcal{G}}_1[\varphi]\|^{\text{fl}} + \|\widehat{\mathcal{G}}_2[\varphi]\|^{\text{fl}} &\le C\left(\|\varphi_1\|^{\text{fl}} + \|\varphi_2\|^{\text{fl}}\right), \\ \|\widehat{\mathcal{G}}_j[\varphi]\|^{\text{fl}} &\le C\delta^2\|\varphi_j\|^{\text{fl}} \text{ for } j = 3, 4. \end{aligned} \tag{4.15}$$

Next, we state a technical lemma providing estimates for $\mathcal{R}^{\text{fl}}$. Its proof is a direct consequence of the definition of the operator in (4.14) and Corollary B.7, which gives estimates for H_1^{Poi} in (2.1) (see also the change of coordinates (2.5) which relates H_1^{Poi} and H_1).

Lemma 4.11. *Fix $\varrho > 0$ and consider $\varphi \in \mathcal{X}^{\text{fl}}_{\times}$ with $\|\varphi\|^{\text{fl}}_{\times} \le \varrho\delta^3$. Then, for $\delta > 0$ small enough, there exists a constant $C > 0$ such that the operator $\mathcal{R}^{\text{fl}}$ in (4.14) satisfies*

$$\begin{aligned} \|\mathcal{R}_1^{\text{fl}}[\varphi]\|^{\text{fl}}, \|\mathcal{R}_2^{\text{fl}}[\varphi]\|^{\text{fl}} \le C\delta^2, \qquad & \|\mathcal{R}_3^{\text{fl}}[\varphi]\|^{\text{fl}}, \|\mathcal{R}_4^{\text{fl}}[\varphi]\|^{\text{fl}} \le C\delta, \\ \|D_j\mathcal{R}_l^{\text{fl}}[\varphi]\|^{\text{fl}} \le C\delta, \qquad & j, l \in \{1, 2, 3, 4\}. \end{aligned}$$

Denote by $\mathbf{e}_j$, $j = 1, 2, 3, 4$, the canonical basis in $\mathbb{R}^4$. Noticing that, by Corollary 3.8, the function $\widehat{\Gamma} = (\hat{\lambda}, \hat{\Lambda}, \hat{x}, \hat{y})$ is already defined at $\{v_0, v_1, \overline{v}_1\} \in \widetilde{D}^{\text{u,out}}_{\kappa_2,d_2,d_3}$, we can consider the function

$$F^0(v) = \Phi(v)\left[\Phi^{-1}(v_0)\left(\hat{\lambda}(v_0)\mathbf{e}_1 + \hat{\Lambda}(v_0)\mathbf{e}_2\right) + \hat{x}(\overline{v}_1)\Phi^{-1}(\overline{v}_1)\mathbf{e}_3 + \hat{y}(v_1)\Phi^{-1}(v_1)\mathbf{e}_4\right].$$

Then, since $\widehat{\mathcal{G}}(F^0) = 0$, it only remains to check that $\mathcal{F} = F^0 + \widehat{\mathcal{G}} \circ \mathcal{R}^{\text{fl}}$ is contractive in a suitable ball of $\mathcal{X}^{\text{fl}}_{\times}$.

End of proof of Proposition 3.9. First, we obtain a suitable estimate for $\mathcal{F}[0]$. Applying Propositions 3.6 and 3.7 and using (3.18) we obtain that, for $v \in \widetilde{D}^{\text{u,out}}_{\kappa_2,d_2,d_3}$,

$$|\hat{\lambda}(v)| \le C\delta^2, \qquad |\hat{\Lambda}(v)| \le C\delta^2, \qquad |\hat{x}(v)| \le C\delta^3, \qquad |\hat{y}(v)| \le C\delta^3.$$

Therefore, since $\{v_0, v_1, \overline{v}_1\} \in \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$,

$$\|F^0\|^{\mathrm{fl}}_{\times} \le C\delta|\hat{\lambda}(v_0)| + C\delta|\hat{\Lambda}(v_0)| + C\,|\hat{x}(\overline{v}_1)| + C\,|\hat{y}(v_1)| \le C\delta^3$$

and, applying (4.15) and Lemma 4.11, there exists $b_5 > 0$ independent of δ such that

$$\|\mathcal{F}[0]\|^{\mathrm{fl}}_{\times} \le \|F^0\|^{\mathrm{fl}}_{\times} + \|\mathcal{G} \circ \mathcal{R}^{\mathrm{fl}}[0]\|^{\mathrm{fl}}_{\times} \le \frac{1}{2} b_5 \delta^3. \tag{4.16}$$

Let us define $B(b_5\delta^3) = \left\{\varphi \in \mathcal{X}^{\mathrm{fl}}_{\times} \, : \, \|\varphi\|^{\mathrm{fl}}_{\times} \le b_5\delta^3\right\}$. By the mean value theorem and Lemma 4.11, for $\varphi, \tilde{\varphi} \in B(b_5\delta^3)$ and $j = 1, .., 4$, we obtain

$$\|\mathcal{R}^{\mathrm{fl}}_j[\varphi] - \mathcal{R}^{\mathrm{fl}}_j[\tilde{\varphi}]\|^{\mathrm{fl}} \le \sum_{l=1}^{4} \left[\sup_{\zeta \in B(b_5\delta^3)} \left\{ \|D_l \mathcal{R}^{\mathrm{fl}}_j[\zeta]\|^{\mathrm{fl}} \right\} \|\varphi_l - \tilde{\varphi}_l\|^{\mathrm{fl}} \right] \le C\|\varphi - \tilde{\varphi}\|^{\mathrm{fl}}_{\times}.$$

Then, by (4.15) and taking δ small enough,

$$\begin{aligned} \|\mathcal{F}[\varphi] - \mathcal{F}[\tilde{\varphi}]\|^{\mathrm{fl}}_{\times} &\le C\delta \left[\sum_{j=1}^{2} \|\mathcal{R}^{\mathrm{fl}}_j[\varphi] - \mathcal{R}^{\mathrm{fl}}_j[\tilde{\varphi}]\|^{\mathrm{fl}} \right] + C\delta^2 \left[\sum_{l=3}^{4} \|\mathcal{R}^{\mathrm{fl}}_l[\varphi] - \mathcal{R}^{\mathrm{fl}}_l[\tilde{\varphi}]\|^{\mathrm{fl}} \right] \\ &\le C\delta \|\varphi - \tilde{\varphi}\|^{\mathrm{fl}}_{\times} \le \frac{1}{2} \|\varphi - \tilde{\varphi}\|^{\mathrm{fl}}_{\times}. \end{aligned} \tag{4.17}$$

Therefore, by (4.16) and (4.17), $\mathcal{F}$ is well defined and contractive in $B(b_5\delta^3)$ and, as a result, has a fixed point $\widehat{\Gamma} \in B(b_5\delta^3)$. □

4.5. Back to a graph parametrization

Now we prove Proposition 3.10 by obtaining the change of variables $\mathcal{V} : \widetilde{D}_{\kappa_4,d_5} \to \mathbb{C}$ as a solution of equation (3.23). This equation is equivalent to $\mathcal{V} = \mathcal{N}[\mathcal{V}]$ with

$$\mathcal{N}[\varphi](u) = \frac{1}{3\Lambda_h(u)} \left[\hat{\lambda}(u + \varphi(u)) + \lambda_h(u + \varphi(u)) - \lambda_h(u) + 3\Lambda_h(u)\varphi(u) \right].$$

We obtain $\mathcal{V}$ by means of a fixed point argument in the Banach space

$$\widetilde{\mathcal{Y}} = \left\{ \varphi : \widetilde{D}_{\kappa_4,d_5} \to \mathbb{C} \;:\; \varphi \text{ real-analytic}, \ \|\varphi\|_{\sup} := \sup_{u \in \widetilde{D}_{\kappa_4,d_5}} |\varphi(u)| < \infty \right\}.$$

Proof of Proposition 3.10. Let us first notice that, by Theorem 3.1,

$$C^{-1} \le \|\Lambda_h\|_{\sup} \le C. \tag{4.18}$$

Since $d_5 < d_4$ and $\kappa_4 > \kappa_3$, we have that $\widetilde{D}_{\kappa_4,d_5} \subset D^{\mathrm{fl}}_{\kappa_3,d_4}$, (see (3.16) and (3.19)). Then, applying Proposition 3.9, there exists $b_6 > 0$ independent of δ such that

$$\|\mathcal{N}[0]\|_{\sup} \leq \frac{1}{3}\|(\Lambda_h)^{-1}\|_{\sup}\|\hat{\lambda}\|_{\sup} \leq \frac{1}{2}b_6\delta^2.$$

Next, we compute the Lipschitz constant of $\mathcal{N}$ in $B(b_6\delta^2) = \{\varphi \in \widetilde{\mathcal{Y}} \,:\, \|\varphi\|_{\sup} \leq b_6\delta^2\}$. By the mean value theorem, for $\varphi, \tilde{\varphi} \in B(b_6\delta^2)$ and $\varphi_s = (1-s)\varphi + s\tilde{\varphi}$, we have that

$$\|\mathcal{N}[\varphi] - \mathcal{N}[\tilde{\varphi}]\|_{\sup} \leq \sup_{u\in \widetilde{D}_{\kappa_4,d_5}} \left| \int_0^1 D\mathcal{N}[\varphi_s](u)ds \right| \|\varphi - \tilde{\varphi}\|_{\sup}.$$

For $u \in \widetilde{D}_{\kappa_4,d_5}$ and δ small enough, we have that $u + \varphi_s(u) \in D^{\mathrm{fl}}_{\kappa_3,d_4}$. Therefore, by Proposition 3.9, (4.18) and recalling that $\dot{\lambda}_h = -3\Lambda_h$,

$$|D\mathcal{N}[\varphi_s](u)| \leq \frac{1}{3\,|\Lambda_h(u)|}\left\{|\partial_v\hat{\lambda}(u+\varphi_s(u))| + |\Lambda_h(u+\varphi_s(u)) - \Lambda_h(u)|\right\} \leq C\delta^2$$

and, taking δ small enough, $\|\mathcal{N}[\varphi]-\mathcal{N}[\tilde{\varphi}]\|_{\sup} \leq \frac{1}{2}\|\varphi-\tilde{\varphi}\|_{\sup}$. Therefore, the operator $\mathcal{N}$ is well defined and contractive in $B(b_6\delta^2)$ and, as a result, has a fixed point $\mathcal{V} \in B(b_6\delta^2)$.

Besides, since $\widetilde{D}_{\kappa_4,d_5} \subset D^{\mathrm{fl}}_{\kappa_3,d_4}$, we obtain that $u + \mathcal{V}(u) \in D^{\mathrm{fl}}_{\kappa_3,d_4}$ for $u \in \widetilde{D}_{\kappa_4,d_5}$ and δ small enough. □

5. Complex matching estimates

This section is devoted to prove Theorem 3.15 which provides estimates for $Z_1^{\mathrm{u,s}} = Z^{\mathrm{u,s}} - Z_0^{\mathrm{u,s}}$ in the matching domains $\mathcal{D}_\kappa^{\mathrm{mch,u}}$ and $\mathcal{D}_\kappa^{\mathrm{mch,s}}$, given in (3.38). We only prove the theorem for Z_1^{u}, being the proof for Z_1^{s} analogous.

5.1. Preliminaries and set up

Proposition (3.12) shows that the Hamiltonian H^{sep} expressed in inner coordinates, that is H^{in} as given in (3.32), is of the form $H^{\mathrm{in}} = W + XY + \mathcal{K} + H_1^{\mathrm{in}}$. Then, the equation associated to H^{in} can be written as

$$\begin{cases} \dot{U} = 1 + g^{\mathrm{in}}(U,Z) + g^{\mathrm{mch}}(U,Z), \\ \dot{Z} = \mathcal{A}^{\mathrm{in}} Z + f^{\mathrm{in}}(U,Z) + f^{\mathrm{mch}}(U,Z), \end{cases} \tag{5.1}$$

where $\mathcal{A}^{\mathrm{in}}$ is given in (3.34) and

$$\begin{aligned} f^{\mathrm{in}} &= \left(-\partial_U\mathcal{K}, i\partial_Y\mathcal{K}, -i\partial_X\mathcal{K}\right)^T, & g^{\mathrm{in}} &= \partial_W\mathcal{K}, \\ f^{\mathrm{mch}} &= \left(-\partial_U H_1^{\mathrm{in}}, i\partial_Y H_1^{\mathrm{in}}, -i\partial_X H_1^{\mathrm{in}}\right)^T, & g^{\mathrm{mch}} &= \partial_W H_1^{\mathrm{in}}. \end{aligned} \tag{5.2}$$

Notice that, since $(u, z^{\mathrm{u}}(u)) = \phi_{\mathrm{in}}(U, Z^{\mathrm{u}}(U))$ (see (3.40)), $(U, Z^{\mathrm{u}}(U))$ is an invariant graph of equation (5.1). Therefore, Z^{u} satisfies the invariance equation

$$\partial_U Z^{\mathrm{u}} = \mathcal{A}^{\mathrm{in}} Z^{\mathrm{u}} + \mathcal{R}^{\mathrm{in}}[Z^{\mathrm{u}}] + \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}],$$

with $\mathcal{R}^{\mathrm{in}}$ as defined in (3.36) and

$$\mathcal{R}^{\mathrm{mch}}[\varphi] = \frac{\mathcal{A}^{\mathrm{in}}\varphi + f^{\mathrm{in}}(U,\varphi) + f^{\mathrm{mch}}(U,\varphi)}{1 + g^{\mathrm{in}}(U,\varphi) + g^{\mathrm{mch}}(U,\varphi)} - \mathcal{A}^{\mathrm{in}}\varphi - \mathcal{R}^{\mathrm{in}}[\varphi]. \tag{5.3}$$

Similarly Z_0^{u} satisfies the invariance equation $\partial_U Z_0^{\mathrm{u}} = \mathcal{A}^{\mathrm{in}} Z_0^{\mathrm{u}} + \mathcal{R}^{\mathrm{in}}[Z_0^{\mathrm{u}}]$ (see Theorem 3.13) and, therefore, the difference $Z_1^{\mathrm{u}} = Z^{\mathrm{u}} - Z_0^{\mathrm{u}}$ must be a solution of

$$\partial_U Z_1^{\mathrm{u}} = \mathcal{A}^{\mathrm{in}} Z_1^{\mathrm{u}} + \mathcal{B}(U) Z_1^{\mathrm{u}} + \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}], \tag{5.4}$$

with

$$\mathcal{B}(U) = \int_0^1 D_Z \mathcal{R}^{\mathrm{in}}[(1-s)Z_0^{\mathrm{u}} + sZ^{\mathrm{u}}](U) ds. \tag{5.5}$$

The key point is that, since the existence of both Z_0^{u} and Z^{u} is already been proven, we can think of $\mathcal{B}(U)$ and $\mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}](U)$ as known functions. Therefore, equation (5.4) can be understood as a non homogeneous linear equation with independent term $\mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}](U)$. Moreover, defining the linear operator $\mathcal{L}^{\mathrm{in}}\varphi = (\partial_U - \mathcal{A}^{\mathrm{in}})\varphi$, equation (5.4) is equivalent to

$$\mathcal{L}^{\mathrm{in}} Z_1^{\mathrm{u}}(U) = \mathcal{B}(U) Z_1^{\mathrm{u}}(U) + \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}](U). \tag{5.6}$$

We prove Theorem 3.15 by solving this equation (with suitable initial conditions). To this end, we define the Banach space $\mathcal{X}_{\times}^{\mathrm{mch}} = \mathcal{X}_{\frac{4}{3}}^{\mathrm{mch}} \times \mathcal{X}_1^{\mathrm{mch}} \times \mathcal{X}_1^{\mathrm{mch}}$ with

$$\mathcal{X}_\alpha^{\mathrm{mch}} = \left\{ \varphi : \mathcal{D}_\kappa^{\mathrm{mch,u}} \to \mathbb{C} \ : \ \varphi \text{ real-analytic}, \ \|\varphi\|_\alpha^{\mathrm{mch}} = \sup_{U \in \mathcal{D}_\kappa^{\mathrm{mch,u}}} |U^\alpha \varphi(U)| < \infty \right\},$$

endowed with the product norm $\|\varphi\|_\times^{\mathrm{mch}} = \|\varphi_1\|_{\frac{4}{3}}^{\mathrm{mch}} + \|\varphi_2\|_1^{\mathrm{mch}} + \|\varphi_3\|_1^{\mathrm{mch}}$.

Next lemma gives some properties of these Banach spaces.

Lemma 5.1. *Let $\gamma \in [\frac{3}{5}, 1)$ and $\alpha, \beta \in \mathbb{R}$. The following statements hold:*

1. *If $\varphi \in \mathcal{X}_\alpha^{\mathrm{mch}}$, then $\varphi \in \mathcal{X}_\beta^{\mathrm{mch}}$ for any $\beta \in \mathbb{R}$. Moreover,*

$$\begin{cases} \|\varphi\|_\beta^{\mathrm{mch}} \leq C\kappa^{\beta-\alpha} \|\varphi\|_\alpha^{\mathrm{mch}}, & \text{for } \alpha > \beta, \\ \|\varphi\|_\beta^{\mathrm{mch}} \leq C\delta^{2(\alpha-\beta)(1-\gamma)} \|\varphi\|_\alpha^{\mathrm{mch}}, & \text{for } \alpha < \beta. \end{cases}$$

2. *If $\varphi \in \mathcal{X}_\alpha^{\mathrm{mch}}$ and $\zeta \in \mathcal{X}_\beta^{\mathrm{mch}}$, then $\varphi\zeta \in \mathcal{X}_{\alpha+\beta}^{\mathrm{mch}}$ and $\|\varphi\zeta\|_{\alpha+\beta}^{\mathrm{mch}} \leq \|\varphi\|_\alpha^{\mathrm{mch}} \|\zeta\|_\beta^{\mathrm{mch}}$.*

This lemma is a direct consequence of the fact that, as explained in Section 3.3.2, U satisfies

$$\kappa \cos \beta_2 \leq |U| \leq \frac{C}{\delta^{2(1-\gamma)}}. \tag{5.7}$$

Now, we present the main result of this section, which implies Theorem 3.15.

Proposition 5.2. *There exist $\gamma^* \in [\frac{3}{5}, 1)$, $\kappa_6 \geq \max\{\kappa_1, \kappa_5\}$, $\delta_0 > 0$ and $b_{18} > 0$ such that, for $\gamma \in (\gamma^*, 1)$, $\kappa \geq \kappa_6$ and $\delta \in (0, \delta_0)$, Z_1^{u} satisfies $\|Z_1^{\mathrm{u}}\|_{\times}^{\mathrm{mch}} \leq b_{18}\, \delta^{\frac{2}{3}(1-\gamma)}$.*

5.2. An integral equation formulation

To prove Proposition 5.2, we first introduce a right-inverse of $\mathcal{L}^{\mathrm{in}} = \partial_U - \mathcal{A}^{\mathrm{in}}$.

Lemma 5.3. *The operator $\mathcal{G}^{\mathrm{in}}[\varphi] = \left(\mathcal{G}_1^{\mathrm{in}}[\varphi_1], \mathcal{G}_2^{\mathrm{in}}[\varphi_2], \mathcal{G}_3^{\mathrm{in}}[\varphi_3]\right)^T$ defined as*

$$\mathcal{G}^{\mathrm{in}}[\varphi](U) = \left(\int_{U_3}^{U} \varphi_1(S)dS, \int_{U_3}^{U} e^{-i(S-U)}\varphi_2(S)dS, \int_{U_2}^{U} e^{i(S-U)}\varphi_3(S)dS \right)^T, \tag{5.8}$$

where U_2 and U_3 are introduced in (3.39), is a right inverse of $\mathcal{L}^{\mathrm{in}}$.

Moreover, there exists a constant $C > 0$ such that:

1. *Let $\alpha > 1$. If $\varphi \in \mathcal{X}_\alpha^{\mathrm{mch}}$, then $\mathcal{G}_1^{\mathrm{in}}[\varphi] \in \mathcal{X}_{\alpha-1}^{\mathrm{mch}}$ and $\left\|\mathcal{G}_1^{\mathrm{in}}[\varphi]\right\|_{\alpha-1}^{\mathrm{mch}} \leq C \|\varphi\|_\alpha^{\mathrm{mch}}$.*
2. *Let $\alpha > 0$, $j = 2, 3$. If $\varphi \in \mathcal{X}_\alpha^{\mathrm{mch}}$, then $\mathcal{G}_j^{\mathrm{in}}[\varphi] \in \mathcal{X}_\alpha^{\mathrm{mch}}$ and $\|\mathcal{G}_j^{\mathrm{in}}[\varphi]\|_\alpha^{\mathrm{mch}} \leq C \|\varphi\|_\alpha^{\mathrm{mch}}$.*

The proof of this lemma follows the same lines as the proof of Lemma 20 in [7]. Using the operator $\mathcal{G}^{\mathrm{in}}$, equation (5.6) is equivalent to

$$Z_1(U) = C^{\mathrm{mch}} e^{\mathcal{A}^{\mathrm{in}} U} + \mathcal{G}^{\mathrm{in}}\left[\mathcal{B} \cdot Z_1\right](U) + \left(\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z]\right)(U),$$

where $C^{\mathrm{mch}} = (C_W^{\mathrm{mch}}, C_X^{\mathrm{mch}}, C_Y^{\mathrm{mch}})^T$ is defined as

$$C_W^{\mathrm{mch}} = W_1(U_3), \qquad C_X^{\mathrm{mch}} = e^{-iU_3} X_1(U_3), \qquad C_Y^{\mathrm{mch}} = e^{iU_2} Y_1(U_2).$$

Then, defining the operator $\mathcal{T}[\varphi](U) = \mathcal{G}^{\mathrm{in}}\left[\mathcal{B} \cdot \varphi\right](U)$, this equation is equivalent to

$$(\mathrm{Id} - \mathcal{T}) Z_1^{\mathrm{u}} = C^{\mathrm{mch}} e^{\mathcal{A}^{\mathrm{in}} U} + \left(\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\right) \tag{5.9}$$

and therefore, to estimate Z_1^{u}, we need to prove that $\mathrm{Id} - \mathcal{T}$ is invertible in $\mathcal{X}_\times^{\mathrm{mch,u}}$.

Lemma 5.4. *Let us consider operators $\mathcal{B}$ and $\mathcal{G}^{\mathrm{in}}$ as given in (5.5) and (5.8). Then, for $\gamma \in [\frac{3}{5}, 1)$, $\kappa > 0$ big enough and $\delta > 0$ small enough, for $\varphi \in \mathcal{X}_\times^{\mathrm{mch}}$,*

$$\|\mathcal{T}[\varphi]\|_{\times}^{\mathrm{mch}} = \left\|\mathcal{G}^{\mathrm{in}}[\mathcal{B}\cdot\varphi]\right\|_{\times}^{\mathrm{mch}} \leq \frac{1}{2}\|\varphi\|_{\times}^{\mathrm{mch}}$$

and therefore

$$\left\|(\mathrm{Id}-\mathcal{T})^{-1}[\varphi]\right\|_{\times}^{\mathrm{mch}} \leq 2\|\varphi\|_{\times}^{\mathrm{mch}}.$$

To prove this lemma, we use the following estimates, whose proof is a direct result of Lemma 5.5 in [10].

Lemma 5.5. *Fix $\varrho > 0$ and take $\kappa > 0$ big enough. Then, there exists a constant C (depending on ϱ but independent of κ) such that, for $\varphi \in \mathcal{X}_{\times}^{\mathrm{mch}}$ with $\|\varphi\|_{\times}^{\mathrm{mch}} \leq \varrho$, the functions g^{in} and f^{in} in (3.36) and the operator $\mathcal{R}^{\mathrm{in}}$ in (5.2) satisfy*

$$\left\|g^{\mathrm{in}}(\cdot,\varphi)\right\|_{2}^{\mathrm{mch}} \leq C, \qquad \left\|f_1^{\mathrm{in}}(\cdot,\varphi)\right\|_{\frac{11}{3}}^{\mathrm{mch}} \leq C, \qquad \left\|f_j^{\mathrm{in}}(\cdot,\varphi)\right\|_{\frac{4}{3}}^{\mathrm{mch}} \leq C, \quad j=2,3$$

and

$$\begin{aligned}
&\left\|\partial_W \mathcal{R}_1^{\mathrm{in}}[\varphi]\right\|_{3}^{\mathrm{mch}} \leq C, \qquad \left\|\partial_X \mathcal{R}_1^{\mathrm{in}}[\varphi]\right\|_{\frac{7}{3}}^{\mathrm{mch}} \leq C, \qquad \left\|\partial_Y \mathcal{R}_1^{\mathrm{in}}[\varphi]\right\|_{\frac{7}{3}}^{\mathrm{mch}} \leq C,\\
&\left\|\partial_W \mathcal{R}_j^{\mathrm{in}}[\varphi]\right\|_{\frac{2}{3}}^{\mathrm{mch}} \leq C, \qquad \left\|\partial_X \mathcal{R}_j^{\mathrm{in}}[\varphi]\right\|_{2}^{\mathrm{mch}} \leq C, \qquad \left\|\partial_Y \mathcal{R}_j^{\mathrm{in}}[\varphi]\right\|_{2}^{\mathrm{mch}} \leq C, \quad j=2,3.
\end{aligned}$$

Proof of Lemma 5.4. Let Z^{u} be as given in (3.40). Then, by Proposition 3.6, estimates (5.7) and taking $\gamma \in [\frac{3}{5},1)$, we have that, for $U \in \mathcal{D}_{\kappa}^{\mathrm{mch,u}}$,

$$|W^{\mathrm{u}}(U)| \leq \frac{C}{|U|^{\frac{8}{3}}} + \frac{C\delta^{\frac{4}{3}}}{|U|} \leq \frac{C}{|U|^{\frac{8}{3}}}, \qquad \|X^{\mathrm{u}}\|_{\frac{4}{3}}^{\mathrm{mch}} \leq C, \qquad \|Y^{\mathrm{u}}\|_{\frac{4}{3}}^{\mathrm{mch}} \leq C. \tag{5.10}$$

Then, using also Theorem 3.13, we obtain that $(1-s)Z_0^{\mathrm{u}} + sZ^{\mathrm{u}} \in \mathcal{X}_{\times}^{\mathrm{mch}}$ for $s \in [0,1]$ and $\gamma \in [\frac{3}{5},1)$ and $\|(1-s)Z_0^{\mathrm{u}} + sZ^{\mathrm{u}}\|_{\times}^{\mathrm{mch}} \leq C$. As a result, using the definition of $\mathcal{B}$ in (5.5) and Lemma 5.5,

$$\begin{aligned}
&\|\mathcal{B}_{1,1}\|_{3}^{\mathrm{mch}} \leq C, \quad \|\mathcal{B}_{1,2}\|_{\frac{7}{3}}^{\mathrm{mch}} \leq C, \quad \|\mathcal{B}_{1,3}\|_{\frac{7}{3}}^{\mathrm{mch}} \leq C,\\
&\|\mathcal{B}_{j,1}\|_{\frac{2}{3}}^{\mathrm{mch}} \leq C, \quad \|\mathcal{B}_{j,2}\|_{2}^{\mathrm{mch}} \leq C, \quad \|\mathcal{B}_{j,3}\|_{2}^{\mathrm{mch}} \leq C, \quad \text{for } j=2,3.
\end{aligned} \tag{5.11}$$

Therefore, by Lemmas 5.3 and 5.1 and (5.11), we obtain

$$\begin{aligned}
\|\mathcal{T}_1[\varphi]\|_{\frac{4}{3}}^{\mathrm{mch}} &\leq C\left\|\pi_1(\mathcal{B}\varphi)\right\|_{\frac{7}{3}}^{\mathrm{mch}}\\
&\leq C\left[\|\mathcal{B}_{1,1}\|_{1}^{\mathrm{mch}}\|\varphi_1\|_{\frac{4}{3}}^{\mathrm{mch}} + \|\mathcal{B}_{1,2}\|_{\frac{4}{3}}^{\mathrm{mch}}\|\varphi_2\|_{1}^{\mathrm{mch}} + \|\mathcal{B}_{1,3}\|_{\frac{4}{3}}^{\mathrm{mch}}\|\varphi_3\|_{1}^{\mathrm{mch}}\right]\\
&\leq \frac{C}{\kappa^2}\|\varphi_1\|_{\frac{4}{3}}^{\mathrm{mch}} + \frac{C}{\kappa}\|\varphi_2\|_{1}^{\mathrm{mch}} + \frac{C}{\kappa}\|\varphi_3\|_{1}^{\mathrm{mch}} \leq \frac{C}{\kappa}\|\varphi\|_{\times}^{\mathrm{mch}}.
\end{aligned}$$

Proceeding analogously, for $j=2,3$, we have

$$\|\mathcal{T}_j[\varphi]\|_1^{\mathrm{mch}} \le C\left[\|\mathcal{B}_{j,1}\|_{-\frac{1}{3}}^{\mathrm{mch}}\|\varphi_1\|_{\frac{4}{3}}^{\mathrm{mch}} + \sum_{l=2}^{3}\|\mathcal{B}_{j,l}\|_0^{\mathrm{mch}}\|\varphi_l\|_1^{\mathrm{mch}}\right] \le \frac{C}{\kappa}\|\varphi\|_\times^{\mathrm{mch}}.$$

Taking $\kappa > 0$ big enough, we obtain the statement of the lemma. □

5.3. *End of the proof of Proposition 5.2*

To complete the proof of Proposition 5.2, we study the right-hand side of equation (5.9). First, we deal with the term $C^{\mathrm{mch}}e^{\mathcal{A}^{\mathrm{in}}U}$. Recall that U_2 and U_3 in (3.39) satisfy

$$\frac{C^{-1}}{\delta^{2(1-\gamma)}} \le |U_j| \le \frac{C}{\delta^{2(1-\gamma)}}, \qquad \text{for } j = 2, 3.$$

Then, taking into account that $W_1^{\mathrm{u}} = W^{\mathrm{u}} - W_0^{\mathrm{u}}$, (5.10) and Theorem 3.13 imply

$$|C_W^{\mathrm{mch}}| = |W_1^{\mathrm{u}}(U_3)| \le |W^{\mathrm{u}}(U_3)| + |W_0^{\mathrm{u}}(U_3)| \le \frac{C}{|U_3|^{\frac{8}{3}}} \le C\delta^{\frac{16}{3}(1-\gamma)}$$

and, as a result, by Lemma 5.1, $\|C_W^{\mathrm{mch}}\|_{\frac{4}{3}}^{\mathrm{mch}} \le C\delta^{\frac{8}{3}(1-\gamma)}$. Analogously, for $U \in \mathcal{D}_\kappa^{\mathrm{mch,u}}$,

$$|C_X^{\mathrm{mch}}e^{iU}| = |e^{i(U-U_3)}X_1^{\mathrm{u}}(U_3)| \le \frac{Ce^{-\mathrm{Im}\,(U-U_3)}}{|U_3|^{\frac{4}{3}}} \le C\delta^{\frac{8}{3}(1-\gamma)}$$

and then $\|C_X^{\mathrm{mch}}e^{iU}\|_1^{\mathrm{mch}} \le C\delta^{\frac{2}{3}(1-\gamma)}$. An analogous result holds for $C_Y^{\mathrm{mch}}e^{-iU}$. Therefore,

$$\|C^{\mathrm{mch}}e^{\mathcal{A}^{\mathrm{in}}U}\|_\times^{\mathrm{mch}} \le C\delta^{\frac{2}{3}(1-\gamma)}. \tag{5.12}$$

Now, we estimate the norm of $\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]$. The operator $\mathcal{R}^{\mathrm{mch}}$ in (5.3) can be rewritten as

$$\mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}] = \frac{f^{\mathrm{mch}}(1+g^{^{\mathrm{in}}}) - g^{\mathrm{mch}}(\mathcal{A}^{^{\mathrm{in}}}Z^{\mathrm{u}} + f^{^{\mathrm{in}}})}{(1+g^{^{\mathrm{in}}})(1+g^{^{\mathrm{in}}}+g^{\mathrm{mch}})}.$$

Then by (5.10), Lemmas 5.1 and 5.5 and taking κ big enough, we obtain

$$\begin{aligned} &\left\|g^{\mathrm{in}}(\cdot, Z^{\mathrm{u}})\right\|_0^{\mathrm{mch}} \le \frac{C}{\kappa^2} \le \frac{1}{2}, && \left\|iX^{\mathrm{u}} + f_2^{\mathrm{in}}(\cdot, Z^{\mathrm{u}})\right\|_0^{\mathrm{mch}} \le C, \\ &\left\|f_1^{\mathrm{in}}(\cdot, Z^{\mathrm{u}})\right\|_0^{\mathrm{mch}} \le C, && \left\|-iY^{\mathrm{u}} + f_3^{\mathrm{in}}(\cdot, Z^{\mathrm{u}})\right\|_0^{\mathrm{mch}} \le C. \end{aligned} \tag{5.13}$$

To analyze f^{mch} and g^{mch} (see (5.2)) we rely on the estimates for H_1^{in} in (3.33) and its derivatives, which can be easily obtained by Cauchy estimates. Indeed, they can be applied since $U \in \mathcal{D}_\kappa^{\mathrm{mch,u}}$ and, by (5.10),

$$|W^{\mathrm{u}}(U)|, |X^{\mathrm{u}}(U)|, |Y^{\mathrm{u}}(U)| \leq C.$$

Then, there exists $m > 0$ such that

$$|g^{\mathrm{mch}}(U, Z^{\mathrm{u}})| \leq C\delta^{\frac{4}{3}-2m(1-\gamma)}, \ |f_j^{\mathrm{mch}}(U, Z^{\mathrm{u}})| \leq C\delta^{\frac{4}{3}-2m(1-\gamma)}, \ \text{for } j = 1, 2, 3. \tag{5.14}$$

We note that, for $\gamma \in (\gamma_0^*, 1)$ with $\gamma_0^* = \max\{\frac{3}{5}, \frac{3m-2}{3m}\}$, we have that $\frac{4}{3} - 2m(1-\gamma) > 0$. Then, for $\gamma \in (\gamma_0^*, 1)$, δ small enough and κ big enough, using (5.13) and (5.14) we obtain

$$|\mathcal{R}_j^{\mathrm{mch}}[Z^{\mathrm{u}}](U)| \leq C\delta^{\frac{4}{3}-2m(1-\gamma)}, \qquad \text{for } j = 1, 2, 3.$$

Then, by Lemmas 5.1 and 5.3,

$$\begin{aligned} \|\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_{\times}^{\mathrm{mch}} &= \|\mathcal{G}_1^{\mathrm{in}} \circ \mathcal{R}_1^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_{\frac{4}{3}}^{\mathrm{mch}} + \textstyle\sum_{j=2}^{3} \|\mathcal{G}_j^{\mathrm{in}} \circ \mathcal{R}_j^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_1^{\mathrm{mch}} \\ &\leq C\|\mathcal{R}_1^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_{\frac{7}{3}}^{\mathrm{mch}} + \textstyle\sum_{j=2}^{3} C\|\mathcal{R}_j^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_1^{\mathrm{mch}} \leq C\delta^{\frac{4}{3}-2(m+\frac{7}{3})(1-\gamma)}. \end{aligned}$$

If we take $\gamma^* = \max\left\{\frac{3}{5}, \gamma_0^*, \gamma_1^*\right\}$ with $\gamma_1^* = \frac{3m+5}{3m+7}$, and $\gamma \in (\gamma^*, 1)$,

$$\|\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_{\times}^{\mathrm{mch}} \leq C\delta^{\frac{2}{3}(1-\gamma)}. \tag{5.15}$$

To complete the proof of Proposition 5.2, we consider equation (5.9). By Lemma 5.4, $(\mathrm{Id} - \mathcal{T})$ is invertible in $\mathcal{X}_{\times}^{\mathrm{mch}}$ and moreover

$$\begin{aligned} \|Z_1^{\mathrm{u}}\|_{\times}^{\mathrm{mch}} &= \left\|(\mathrm{Id} - \mathcal{T})^{-1}\left(C^{\mathrm{mch}} e^{\mathcal{A}^{\mathrm{in}} U} + \mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\right)\right\|_{\times}^{\mathrm{mch}} \\ &\leq 2\left\|C^{\mathrm{mch}} e^{\mathcal{A}^{\mathrm{in}} U} + \mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\right\|_{\times}^{\mathrm{mch}}. \end{aligned}$$

Then, it is enough to apply (5.12) and (5.15). □

Acknowledgments

M. Giralt and M. Guardia have received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 757802). This work is part of the grant PID-2021-122954NB-100 funded by the Spanish State Research Agency through the programs MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe". M. Guardia is also supported by the Catalan Institution for Research and Advanced Studies via an ICREA Academia Prize 2019. M. Giralt is also supported by the research project PRIN 2020XB3EFL "Hamiltonian and dispersive PDEs" funded by the the Italian Ministry of University and Research. This work is also supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).

Appendix A. Poincaré variables

This appendix is devoted to define the (rotating) Poincaré coordinates for the RPC3BP through the Delaunay elements. We follow exactly the same scheme as the one in Section 2.1 of [10]. In addition we also provide the proof of Lemma 2.3.

Let $\phi_{\mathrm{pol}} : (r, \theta, R, G) \mapsto (q, p)$, be the symplectic polar change of coordinates defined in (1.2), where r is the radius, θ the argument of q, R the linear momentum in the r direction and G is the angular momentum. In these variables, the Hamiltonian (1.1), becomes

$$H^{\mathrm{pol}} = H_0^{\mathrm{pol}} + \mu H_1^{\mathrm{pol}}, \qquad H_0^{\mathrm{pol}}(r, R, G) = \frac{1}{2}\left(R^2 + \frac{G^2}{r^2}\right) - \frac{1}{r} - G. \tag{A.1}$$

The critical point L_3 (see [55] for the details) satisfies that, as $\mu \to 0$, $(r, \theta, R, G) = (d_\mu, 0, 0, d_\mu^2)$ being $d_\mu = 1 + \frac{5}{12}\mu + \mathcal{O}(\mu^3)$.

We introduce now the celebrated Delaunay elements, $(\ell, L, \hat{g}, G)$, where ℓ is the mean anomaly, $\hat{g}$ is the argument of the pericenter, L is the square root of the semi major axis and G is the angular momentum, (see [45]). It is well known that the action L is defined by

$$-\frac{1}{2L^2} = \frac{1}{2}\left(R^2 + \frac{G^2}{r^2}\right) - \frac{1}{r}$$

and the (osculating) eccentricity of the body is expressed as

$$e = \sqrt{1 - \frac{G^2}{L^2}} = \frac{\sqrt{(L-G)(L+G)}}{L}.$$

The so called "anomalies", the mean anomaly ℓ, the eccentric anomaly u, and the true anomaly f, satisfy the well known relations

$$\cos f = \frac{\cos u - e}{1 - e\cos u}, \qquad \sin f = \frac{\sqrt{1-e^2}\sin u}{1 - e\cos u}, \qquad u - e\sin u = \ell.$$

They are nothing but three angular parameters that define a position at the (osculating) ellipse. We have also the relations

$$r = L^2(1 - e\cos u) \qquad \text{and} \qquad \hat{\theta} = f + \hat{g}, \qquad \text{with} \qquad \hat{\theta} = \theta + t.$$

We consider now the rotating Delaunay coordinates (ℓ, L, g, G), where the new angle is defined as $g = \hat{g} - t$ (the argument of the pericenter with respect to the line defined by the primaries S and J). Then $\theta = f + g$ and the unperturbed Hamiltonian H_0^{pol} becomes $H_0^{\mathrm{pol}} = -\frac{1}{2L^2} - G$. Moreover, the critical point L_3 satisfies $\theta = \ell + g = 0$ and

$$L = \sqrt{\frac{d_\mu}{2-d_\mu^3}} = 1 + \mathcal{O}(\mu), \quad G = d_\mu^2 = 1 + \mathcal{O}(\mu), \quad L - G = \mathcal{O}(\mu^2).$$

Note that the Delaunay coordinates are not well defined for circular orbits ($e = 0$), since the pericenter, and as a consequence the angle g, are not well defined. To "remove" this singularity of the Delaunay coordinates, we use the classical Poincaré coordinates (λ, L, η, ξ) by means of

$$\lambda = \ell + g, \qquad \eta = \sqrt{L-G}e^{ig}, \qquad \xi = \sqrt{L-G}e^{-ig}.$$

Even though the Poincaré variables are defined through the Delaunay variables, they are analytic when the eccentricity tends to zero (i.e. at $L = G$), see [45,27]. The Hamiltonian equation associated to (A.1), expressed in Poincaré coordinates, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge dL + i\, d\eta \wedge d\xi$ and the Hamiltonian (2.1) in Section 2. We notice that in Poincaré coordinates, the critical point L_3 satisfies $(0, 1, 0, 0) + \mathcal{O}(\mu)$ (see (2.3)) and linearization (2.4).

Proof of Lemma 2.3 We use the formulae for the Poincaré elements and anomalies introduced previously. Fix $\varrho > 0$ and let $|(L - 1, \xi, \eta)| \ll 1$ and $|\mathrm{Im}\, \lambda| \leq \varrho$.

- The result for the angular momentum G is straightforward by the definition of the Poincaré elements.
- The radius r satisfies that $r = L^2(1 - e\cos u)$. In Section 4.1 of [10], it is seen that

$$\begin{aligned} e\cos u &= \frac{1}{\sqrt{2L}}\left(e^{-i\lambda}\eta + e^{i\lambda}\xi\right) + \mathcal{O}(e^{-i\lambda}\eta, e^{i\lambda}\xi)^2 \\ &= \frac{e^{-i\lambda}}{\sqrt{2}}\eta + \frac{e^{i\lambda}}{\sqrt{2}}\xi + \mathcal{O}(L-1,\eta,\xi)^2, \\ e\sin u &= \frac{i}{\sqrt{2L}}\left(e^{-i\lambda}\eta - e^{i\lambda}\xi\right) + \mathcal{O}(e^{-i\lambda}\eta, e^{i\lambda}\xi)^2 \\ &= \frac{ie^{-i\lambda}}{\sqrt{2}}\eta - \frac{ie^{i\lambda}}{\sqrt{2}}\xi + \mathcal{O}(L-1,\eta,\xi)^2. \end{aligned} \tag{A.2}$$

  As a result, we obtain the asymptotic expression for r.
- The angle θ satisfies that $\theta = \lambda + f - \ell$. Since the eccentric anomaly is given implicitly by $u - e\sin u = \ell$, we have that

$$\theta = \lambda + f - u + e\sin u. \tag{A.3}$$

  The true anomaly f and the eccentricity e satisfy that

$$\sin f = \frac{\sqrt{1-e^2}\sin u}{1 - e\cos u}, \qquad e^2 = 1 - \frac{G^2}{L^2} = \eta\xi\frac{(2L - \eta\xi)}{L}. \tag{A.4}$$

Then, by (A.2),

$$\sin f = (1 + e\cos u)\sin u + \mathcal{O}(L-1, \eta, \xi)^2 = \sin(u + e\sin u) + \mathcal{O}(L-1, \eta, \xi)^2.$$

By (A.3) and using that $|\mathrm{Im}\,\lambda| \leq \varrho$, one obtains $\theta = \lambda + 2e\sin u + \mathcal{O}(L-1, \eta, \xi)^2$, and applying (A.2) we obtain the corresponding asymptotic formula.

$$\theta = \lambda + 2e\sin u + \mathcal{O}(L-1, \eta, \xi)^2 = \lambda + i\sqrt{2}e^{-i\lambda}\eta - i\sqrt{2}e^{i\lambda} + \mathcal{O}(L-1, \eta, \xi)^2.$$

- The linear momentum R satisfies that

$$-\frac{1}{2L^2} = \frac{1}{2}\left(R^2 + \frac{G^2}{r^2}\right) - \frac{1}{r}.$$

Then, since $r = L^2(1 - e\cos u)$ and, by (A.4), $G^2 = L^2(1 - e^2)$, one obtains that

$$R^2 = \frac{(e\sin u)^2}{L^2(1 - e\cos u)^2}$$

and applying (A.2) we obtain the statement of the lemma.

Appendix B. Estimates for the invariant manifolds

In this appendix we prove the technical Lemmas 4.4 and 4.8. All these results involve, in some sense, estimates for the first and second derivatives of the Hamiltonian H_1^{sep} in (3.9). However, to obtain estimates for H_1^{sep}, we first obtain some properties of H_1^{Poi} (see (2.2)), which can be written as

$$H_1^{\mathrm{Poi}} = \frac{1}{\mu}\mathcal{P}[0] - \frac{1-\mu}{\mu}\mathcal{P}[\mu] - \mathcal{P}[\mu - 1], \tag{B.1}$$

where

$$\mathcal{P}[\zeta](\lambda, L, \eta, \xi) = \left(\|q - (\zeta, 0)\|^{-1}\right) \circ \phi_{\mathrm{Poi}}. \tag{B.2}$$

In [10] (see, in particular, Lemma 4.1), we computed the series expansion of $\mathcal{P}[\zeta]$ in powers of (η, ξ). In particular, $\mathcal{P}[\zeta]$ can be written as

$$\mathcal{P}[\zeta](\lambda, L, \eta, \xi) = \frac{1}{\sqrt{A[\zeta](\lambda) + B[\zeta](\lambda, L, \eta, \xi)}}, \tag{B.3}$$

where A and B are of the form

$$A[\zeta](\lambda) = 1 - 2\zeta\cos\lambda + \zeta^2, \tag{B.4}$$

$$\begin{aligned} B[\zeta](\lambda, L, \eta, \xi) = & 4(L-1)(1-\zeta\cos\lambda) + \frac{\eta}{\sqrt{2}}\left(3\zeta - 2e^{-i\lambda} - \zeta e^{-2i\lambda}\right) \\ & + \frac{\xi}{\sqrt{2}}\left(3\zeta - 2e^{i\lambda} - \zeta e^{2i\lambda}\right) + R[\zeta](\lambda, L, \eta, \xi) \end{aligned} \tag{B.5}$$

and, for fixed $\varrho > 0$, R is analytic and satisfies that

$$|R[\zeta](\lambda, L, \eta, \xi)| \le K(\varrho)\,|(L-1, \eta, \xi)|^2, \tag{B.6}$$

for $|\mathrm{Im}\,\lambda| \le \varrho$, $|(L-1, \eta, \xi)| \ll 1$ and $\zeta \in [-1, 1]$.

Then, wherever $|A[\zeta](\lambda)| > |B[\zeta](\lambda, L, \eta, \xi)|$, $\mathcal{P}[\zeta](\lambda, L, \eta, \xi)$ can be written as

$$\mathcal{P}[\zeta](\lambda, L, \eta, \xi) = \frac{1}{\sqrt{A[\zeta]}} + \sum_{n=1}^{+\infty} \binom{-\frac{1}{2}}{n} \frac{(B[\zeta])^n}{(A[\zeta])^{n+\frac{1}{2}}}. \tag{B.7}$$

Remark B.6. The Hamiltonian $H^{\mathrm{Poi}} = H_0^{\mathrm{Poi}} + \mu H_1^{\mathrm{Poi}}$ (see (2.1) and (B.1)) is analytic away from the collisions with the primaries, that is zeroes of the denominators of $\mathcal{P}[\mu]$ and $\mathcal{P}[\mu - 1]$. For $0 < \mu \ll 1$, one has

$$A[\mu] = 1 + \mathcal{O}(\mu), \qquad A[\mu - 1] = 2 + 2\cos\lambda + \mathcal{O}(\mu).$$

Therefore, in the regime that we consider, collisions with the primary S are not possible but collisions with P may take place at $\lambda \sim \pi$.

We now obtain estimates for H_1^{Poi} in domains "far" from $\lambda = \pi$.

Lemma B.7. *Fix $\lambda_0 \in (0, \pi)$ and $\mu_0 \in (0, \frac{1}{2})$ and consider the Hamiltonian H_1^{Poi} and the potential V introduced in (B.1) and (2.6), respectively. Then, for $|\lambda| < \lambda_0$, $|(L-1, \eta, \xi)| \ll 1$ and $\mu \in (0, \mu_0)$, the Hamiltonian H_1^{Poi} can be written as*

$$H_1^{\mathrm{Poi}}(\lambda, L, \eta, \xi; \mu) - V(\lambda) = D_0(\mu, \lambda) + D_1(\mu, \lambda)\big((L-1), \eta, \xi\big) + D_2(\lambda, L, \eta, \xi; \mu),$$

such that, for $j = 1, 2, 3$,

$$|D_0(\mu, \lambda)| \le K\mu, \qquad |(D_1(\mu, \lambda))_j| \le K, \qquad |D_2(\lambda, L, \eta, \xi; \mu)| \le K\,|(L-1, \eta, \xi)|^2,$$

with K a positive constant independent of λ and μ.

B.1. Estimates in the infinity domain

To prove Lemma 4.4, we need to obtain estimates for $\mathcal{R}^{\mathrm{sep}}$ and its derivatives. Let us recall that, by its definition in (3.13), for $z = (w, x, y)$ we have

$$\mathcal{R}^{\rm sep}[z] = \left(\frac{f_1^{\rm sep}(\cdot,z)}{1+g^{\rm sep}(\cdot,z)}, \frac{f_2{}^{\rm sep}(\cdot,z) - \frac{ix}{\delta^2}\, g^{\rm sep}(\cdot,z)}{1+g^{\rm sep}(\cdot,z)}, \frac{f_3{}^{\rm sep}(\cdot,z) + \frac{iy}{\delta^2}\, g^{\rm sep}}{1+g^{\rm sep}(\cdot,z)} \right)^T, \tag{B.8}$$

where $g^{\rm sep} = \partial_w H_1^{\rm sep}$ and $f^{\rm sep} = (-\partial_u H_1^{\rm sep}, i\partial_y H_1^{\rm sep}, -i\partial_x H_1^{\rm sep})^T$.

Therefore, we need to obtain first estimates for the first and second derivatives of $H_1^{\rm sep}$, introduced in (3.9), that is

$$H_1^{\rm sep} = H \circ (\phi_{\rm eq} \circ \phi_{\rm sep}) - \left(w + \frac{xy}{\delta^2} \right), \tag{B.9}$$

where $H = H_0 + H_1$ with $H_0 = H_{\rm p} + H_{\rm osc}$ (see (2.7), (2.11)).

Since (λ_h, Λ_h) is a solution of the Hamiltonian $H_{\rm p}$ and belongs to the energy level $H_{\rm p} = -\frac{1}{2}$,

$$H_0 \circ \phi_{\rm sep} = H_{\rm p}\left(\lambda_h(u), \Lambda_h(u) - \frac{w}{3\Lambda_h(u)}\right) + H_{\rm osc}(x,y;\delta) = -\frac{1}{2} + w - \frac{w^2}{6\Lambda_h^2(u)} + \frac{xy}{\delta^2}.$$

Therefore, by (B.9), the Hamiltonian $H_1^{\rm sep}$ can be expressed (up to a constant) as

$$H_1^{\rm sep} = M \circ \phi_{\rm sep} - \frac{w^2}{6\Lambda_h^2(u)}, \tag{B.10}$$

where

$$M(\lambda, \Lambda, x, y; \delta) = (H \circ \phi_{\rm eq})(\lambda, \Lambda, x, y; \delta) - H_0(\lambda, \Lambda, x, y).$$

In the following lemma we give properties of M.

Lemma B.8. *Fix constants $\varrho > 0$ and $\lambda_0 \in (0, \pi)$. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, $|\lambda| < \lambda_0$, $|\Lambda| < \varrho$ and $|(x,y)| < \varrho\delta$, the function M satisfies*

$$\begin{aligned}
&|\partial_\lambda M| \leq C\delta^2 \left|(\lambda,\Lambda)\right| + C\delta \left|(x,y)\right|, \qquad && |\partial_x M| \leq C\delta \left|(\lambda,\Lambda,x,y)\right|, \\
&|\partial_\Lambda M| \leq C\delta^2 \left|(\lambda,\Lambda)\right| + C\delta \left|(x,y)\right|, \qquad && |\partial_y M| \leq C\delta \left|(\lambda,\Lambda,x,y)\right|
\end{aligned}$$

and

$$\left|\partial_\lambda^2 M\right|, \left|\partial_{\lambda\Lambda} M\right|, \left|\partial_\Lambda^2 M\right| \leq C\delta^2, \qquad |\partial_{ij} M| \leq C\delta, \quad \text{for } i,j \in \{\lambda, \Lambda, x, y\}.$$

Proof. Applying $\phi_{\rm eq}$ (see (3.3)) to the Hamiltonian $H = H_0 + H_1$, we have that

$$\begin{aligned}
M &= (H_0 \circ \phi_{\rm eq} - H_0) + H_1 \circ \phi_{\rm eq} \\
&= \delta(x\mathfrak{L}_y + y\mathfrak{L}_x) + 3\delta^2 \Lambda \mathfrak{L}_\Lambda + \delta^4 \left(-\frac{3}{2}\mathfrak{L}_\Lambda^2 + \mathfrak{L}_x \mathfrak{L}_y \right) + H_1 \circ \phi_{\rm eq}.
\end{aligned} \tag{B.11}$$

Then,

$$|\partial_{ij}M| \leq \left|\partial_{ij}H_1(\lambda, \Lambda + \delta^2\mathfrak{L}_\Lambda, x + \delta\mathfrak{L}_x, y + \delta\mathfrak{L}_y; \delta)\right|, \tag{B.12}$$

for $i, j \in \{\lambda, \Lambda, x, y\}$. Since $|\Lambda| < \varrho$ and $|(x, y)| < \varrho\delta$, then $\left|\Lambda + \delta^2\mathfrak{L}_\Lambda\right| < 2\varrho$ and $\left|(x + \delta^3\mathfrak{L}_x, y + \delta^3\mathfrak{L}_y)\right| < 2\varrho\delta$, for δ small. By the definition of H_1 in (2.9) we have that,

$$H_1(\lambda, \Lambda, x, y; \delta) = H_1^{\mathrm{Poi}}\left(\lambda, 1 + \delta^2\Lambda, \delta x, \delta y; \delta^4\right) - V(\lambda) + \frac{1}{\delta^4}F_{\mathrm{p}}(\delta^2\Lambda),$$

where H_1^{Poi} is given in (2.2) (see also (2.5)), V is given (2.6) and F_{p} is given (2.10) and satisfies $F_{\mathrm{p}}(s) = \mathcal{O}(s^3)$. Since $\left|(\delta^2\Lambda, \delta x, \delta y)\right| < 2\varrho\delta^2 \ll 1$, we apply Lemma B.7 (recall that $\delta = \mu^{\frac{1}{4}}$) and Cauchy estimates to obtain

$$\left|\partial_\lambda^2 H_1\right| |\partial_{\lambda\Lambda}H_1|, \left|\partial_\Lambda^2 H_1\right| \leq C\delta^2, \quad |\partial_{ij}H_1| \leq C\delta, \text{ for } i, j \in \{\lambda, \Lambda, x, y\}. \tag{B.13}$$

Then, (B.12) and (B.13) give the estimates for the second derivatives of M.

For the first derivatives of M, let us take into account that, by Theorem 3.1, 0 is a critical point of both Hamiltonians $(H \circ \phi_{\mathrm{eq}})$ and H_0 and, therefore, also of $M = (H \circ \phi_{\mathrm{eq}}) - H_0$. This fact and the estimates of the second derivatives, together with the mean value theorem, gives the estimates for the first derivatives of M. □

End of the proof of Lemma 4.4. Let us consider $\varphi = (\varphi_w, \varphi_x, \varphi_y)^T \in \mathcal{X}_\times^\infty$ such that $\|\varphi\|_\times^\infty \leq \varrho\delta^3$. We estimate the first and second derivatives of H_1^{sep} evaluated at $(u, \varphi(u))$ (recall (B.8)), given by

$$\begin{aligned} H_1^{\mathrm{sep}}(u, \varphi(u); \delta) = M\left(\lambda_h(u), \Lambda_h(u) - \frac{\varphi_w(u)}{3\Lambda_h(u)}, \varphi_x(u), \varphi_y(u); \delta\right) \\ - \frac{\varphi_w^2(u)}{6\Lambda_h^2(u)}. \end{aligned} \tag{B.14}$$

First, let us define

$$\varphi_\lambda(u) = \lambda_h(u), \qquad \varphi_\Lambda(u) = \Lambda_h(u) - \frac{\varphi_w(u)}{3\Lambda_h(u)} \quad \text{and} \quad \Phi = (\varphi_\lambda, \varphi_\Lambda, \varphi_x, \varphi_y).$$

Since $\|\varphi\|_\times^\infty \leq \varrho\delta^3$ and $\lambda_h, \Lambda_h \in \mathcal{X}_\nu^\infty$ (see (4.2)),

$$\|\varphi_w\|_{2\nu}^\infty \leq C\delta^2, \qquad \|\varphi_x\|_\nu^\infty, \|\varphi_y\|_\nu^\infty \leq C\delta^3, \qquad \|\varphi_\lambda\|_\nu^\infty, \|\varphi_\Lambda\|_\nu^\infty \leq C. \tag{B.15}$$

Moreover since, by Theorem 3.1, $\lambda_h(u) \neq \pi$ for $u \in D_{\rho_1}^{\mathrm{u},\infty}$, we have that

$$|\varphi_\lambda(u)| = |\lambda_h(u)| < \pi, \quad |\varphi_\Lambda(u)| \leq Ce^{-\nu\rho_1} \leq C, \quad |(\varphi_x(u), \varphi_y(u))| \leq C\delta^3 e^{-\nu\rho_1} \leq C\delta^3$$

and, therefore, we can apply Lemma B.8 to (B.14). In the following computations, we use generously Lemma 4.1 without mentioning it.

1. First, we consider $g^{\rm sep} = \partial_w H_1^{\rm sep}$. By (B.14), we have that

$$g^{\rm sep}(u,\varphi(u)) = -\frac{\partial_\Lambda M \circ \Phi(u)}{3\Lambda_h(u)} - \frac{\varphi_w(u)}{3\Lambda_h^2(u)}.$$

Notice that, by Theorem 3.1, $|\Lambda_h(u)| \geq C$ for $u \in D^{\rm u,\infty}_{\rho_1}$. Then, $\|\Lambda_h^{-1}\|_{-\nu}^\infty \leq C$. Therefore, by Lemma B.8 and estimates (B.15), we have that

$$\begin{aligned} \|g^{\rm sep}(\cdot,\varphi)\|_0^\infty &\leq C\delta\left[\delta\|\varphi_\lambda\|_\nu^\infty + \delta\|\varphi_\Lambda\|_\nu^\infty + \|\varphi_x\|_\nu^\infty + \|\varphi_y\|_\nu^\infty\right] + C\|\varphi_w\|_{2\nu}^\infty \\ &\leq C\delta^2. \end{aligned} \tag{B.16}$$

To compute its derivative with respect to w, by (B.14), we have that

$$\partial_w g^{\rm sep}(u,\varphi(u)) = \frac{\partial_\Lambda^2 M \circ \Phi(u)}{9\Lambda_h^2(u)} - \frac{1}{3\Lambda_h^2(u)}$$

and, by Lemma B.8 and estimates (B.15), $\|\partial_w g^{\rm sep}(\cdot,\varphi)\|_{-2\nu}^\infty \leq C$. Following a similar procedure, we obtain $\|\partial_x g^{\rm sep}(\cdot,\varphi)\|_{-\nu}^\infty \leq C\delta$ and $\|\partial_y g^{\rm sep}(\cdot,\varphi)\|_{-\nu}^\infty \leq C\delta$.
2. Now, we obtain estimates for $f_1^{\rm sep} = -\partial_u H_1^{\rm sep}$. By (B.14), we have that

$$\begin{aligned} f_1^{\rm sep}(u,\varphi(u)) =& -\dot\lambda_h(u)\partial_\lambda M \circ \Phi(u) - \frac{\dot\Lambda_h(u)}{3\Lambda_h^3(u)}\varphi_w^2(u) \\ & - \left(\dot\Lambda_h(u) + \frac{\dot\Lambda_h(u)}{3\Lambda_h^2(u)}\varphi_w(u)\right)\partial_\Lambda M \circ \Phi(u). \end{aligned}$$

Then, since $\dot\lambda_h, \dot\Lambda_h \in \mathcal{X}_\nu^\infty$, by Lemma B.8 and estimates (B.15), we have that $\|f_1^{\rm sep}(\cdot,\varphi)\|_{2\nu}^\infty \leq C\delta^2$. To compute its derivative with respect to x, by (B.14),

$$\partial_x f_1^{\rm sep}(u,\varphi(u)) = -\dot\lambda_h(u)\partial_{x\lambda} M \circ \Phi(u) - \left(\dot\Lambda_h(u) + \frac{\dot\Lambda_h(u)}{3\Lambda_h^2(u)}\varphi_w(u)\right)\partial_{x\Lambda} M \circ \Phi(u)$$

and, therefore, $\|\partial_x f_1^{\rm sep}(\cdot,\varphi)\|_\nu^\infty \leq C\delta$. Similarly one can obtain $\|\partial_w f_1^{\rm sep}(\cdot,\varphi)\|_0^\infty \leq C\delta^2$ and $\|\partial_y f_1^{\rm sep}(\cdot,\varphi)\|_\nu^\infty \leq C\delta$.
3. Analogously to the previous estimates, we can obtain bounds for $f_2^{\rm sep} = i\partial_y H_1^{\rm sep}$ and $f_3^{\rm sep} = -i\partial_x H_1^{\rm sep}$. Then, for $j = 2, 3$, it can be seen that $\|f_j^{\rm sep}(\cdot,\varphi)\|_\nu^\infty \leq C\delta$, and differentiating we obtain $\|\partial_w f_j^{\rm sep}(\cdot,\varphi)\|_{-\nu}^\infty \leq C\delta$, $\|\partial_x f_j^{\rm sep}(\cdot,\varphi)\|_0^\infty \leq C\delta$ and $\|\partial_y f_j^{\rm sep}(\cdot,\varphi)\|_0^\infty \leq C\delta$.

Then, by the definition of $\mathcal{R}^{\rm sep}$ in (B.8) and the just obtained estimates, we complete the proof of the lemma. □

B.2. Estimates in the outer domain

To obtain estimates of $\mathcal{R}^{\mathrm{sep}}$, we write H_1^{sep} in (3.9) (up to a constant) as

$$H_1^{\mathrm{sep}} = H_1 \circ \phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}} - \frac{w^2}{6\Lambda_h^2(u)} + \delta(x\mathfrak{L}_y + y\mathfrak{L}_x) + 3\delta^2\mathfrak{L}_\Lambda\left(\Lambda_h(u) - \frac{w}{3\Lambda_h(u)}\right),$$

(see (B.10) and (B.11)). Then, by the definition of H_1 in (2.9), we obtain

$$\begin{aligned} H_1^{\mathrm{sep}} = & (H_1^{\mathrm{Poi}} - V) \circ \phi_{\mathrm{sc}} \circ \phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}} + \frac{1}{\delta^4} F_{\mathrm{p}}\left(\delta^2\Lambda_h(u) - \frac{\delta^2 w}{3\Lambda_h(u)} + \delta^4\mathfrak{L}_\Lambda\right) \\ & - \frac{w^2}{6\Lambda_h^2(u)} + \delta(x\mathfrak{L}_y + y\mathfrak{L}_x) + 3\delta^2\mathfrak{L}_\Lambda\left(\Lambda_h(u) - \frac{w}{3\Lambda_h(u)}\right), \end{aligned}$$

where H_1^{Poi} is given in (B.1), the potential V in (2.6) and F_{p} in (2.10). The changes of coordinates ϕ_{sc}, ϕ_{eq} and ϕ_{sep} are given in (2.5), (3.3) and (3.4), respectively.

Considering $z = (w, x, y)$, we denote the composition of change of coordinates as

$$(\lambda, L, \eta, \xi) = \Theta(u, z) = (\phi_{\mathrm{sc}} \circ \phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}})(u, z). \tag{B.17}$$

Then, since $\mu = \delta^4$, the Hamiltonian H_1^{sep} can be split (up to a constant) as

$$H_1^{\mathrm{sep}} = M_P + M_S + M_R, \tag{B.18}$$

where

$$M_P(u, z; \delta) = -\left(\mathcal{P}[\delta^4 - 1] - \frac{1}{\sqrt{2 + 2\cos\lambda}}\right) \circ \Theta(u, z), \tag{B.19}$$

$$M_S(u, z; \delta) = \left(\frac{1}{\delta^4}\mathcal{P}[0] - \frac{1 - \delta^4}{\delta^4}\mathcal{P}[\delta^4] - 1 + \cos\lambda\right) \circ \Theta(u, z), \tag{B.20}$$

$$\begin{aligned} M_R(u, z; \delta) = & -\frac{w^2}{6\Lambda_h^2(u)} + \delta^2\mathfrak{L}_\Lambda\left(3\Lambda_h(u) - \frac{w}{\Lambda_h(u)}\right) + \delta(x\mathfrak{L}_y + y\mathfrak{L}_x) \\ & + \frac{1}{\delta^4} F_{\mathrm{p}}\left(\delta^2\Lambda_h(u) - \frac{\delta^2 w}{3\Lambda_h(u)} + \delta^4\mathfrak{L}_\Lambda\right) \end{aligned} \tag{B.21}$$

and $\mathcal{P}$ is the function given in (B.2).

To obtain estimates for the derivatives of M_P, M_S and M_R, we first analyze the change of coordinates Θ in (B.17). It can be expressed as

$$\Theta(u, z) = \Big(\pi + \Theta_\lambda(u), 1 + \Theta_L(u, w), \Theta_\eta(x), \Theta_\xi(y)\Big), \tag{B.22}$$

where

$$\Theta_\lambda(u) = \lambda_h(u) - \pi, \qquad \Theta_\eta(x) = \delta x + \delta^4 \mathfrak{L}_x(\delta),$$
$$\Theta_L(u,w) = \delta^2 \Lambda_h(u) - \frac{\delta^2 w}{3\Lambda_h(u)} + \delta^4 \mathfrak{L}_\Lambda(\delta), \qquad \Theta_\xi(x) = \delta y + \delta^4 \mathfrak{L}_y(\delta).$$

Next lemma, which is a direct consequence of Theorem 3.1, gives estimates for this change of coordinates.

Lemma B.9. *Fix $\varrho > 0$ and $\delta > 0$ small enough. Then, for $\varphi \in B(\varrho\delta^3) \subset \mathcal{X}_\times^{\mathrm{out}}$,*

$$\|\Theta_\lambda\|_{0,-\frac{2}{3}}^{\mathrm{out}} \le C, \qquad \|\Theta_L(\cdot,\varphi)\|_{0,\frac{1}{3}}^{\mathrm{out}} \le C\delta^2, \qquad \|\Theta_\eta(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \le C\delta^4,$$
$$\left\|\Theta_\lambda^{-1}\right\|_{0,\frac{2}{3}}^{\mathrm{out}} \le C, \qquad \|1+\Theta_L(\cdot,\varphi)\|_{0,0}^{\mathrm{out}} \le C, \qquad \|\Theta_\xi(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \le C\delta^4.$$

Moreover, its derivatives satisfy

$$\|\partial_u\Theta_\lambda\|_{0,\frac{1}{3}}^{\mathrm{out}} \le C, \qquad \|\partial_u\Theta_L(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \le C\delta^2, \qquad \|\partial_w\Theta_L(\cdot,\varphi)\|_{0,-\frac{1}{3}}^{\mathrm{out}} \le C\delta^2,$$
$$\|\partial_{uw}\Theta_L(\cdot,\varphi)\|_{0,\frac{2}{3}}^{\mathrm{out}} \le C\delta^2, \qquad \partial_x\Theta_\eta, \partial_y\Theta_\xi \equiv \delta, \qquad \partial_w^2\Theta_L, \partial_x^2\Theta_\eta, \partial_y^2\Theta_\xi \equiv 0.$$

In the next lemma we obtain estimates for the derivatives of M_P.

Lemma B.10. *Fix $\varrho > 0$, $\delta > 0$ small enough and $\kappa > 0$ big enough. Then, for $\varphi \in B(\varrho\delta^3)$ and $* = x, y$,*

$$\|\partial_u M_P(\cdot,\varphi)\|_{1,1}^{\mathrm{out}} \le C\delta^2, \quad \|\partial_w M_P(\cdot,\varphi)\|_{1,-\frac{2}{3}}^{\mathrm{out}} \le C\delta^2, \quad \|\partial_* M_P(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \le C\delta,$$
$$\|\partial_{uw} M_P(\cdot,\varphi)\|_{1,\frac{1}{3}}^{\mathrm{out}} \le C\delta^2, \quad \|\partial_{u*} M_P(\cdot,\varphi)\|_{0,\frac{7}{3}}^{\mathrm{out}} \le C\delta, \quad \left\|\partial_w^2 M_P(\cdot,\varphi)\right\|_{0,\frac{4}{3}}^{\mathrm{out}} \le C\delta^4,$$
$$\|\partial_{w*} M_P(\cdot,\varphi)\|_{0,\frac{5}{3}}^{\mathrm{out}} \le C\delta^3, \quad \left\|\partial_*^2 M_P(\cdot,\varphi)\right\|_{0,2}^{\mathrm{out}} \le C\delta^2, \quad \|\partial_{xy} M_P(\cdot,\varphi)\|_{0,2}^{\mathrm{out}} \le C\delta^2.$$

Proof. We consider $\varphi \in B(\varrho\delta^3) \subset \mathcal{X}_\times^{\mathrm{out}}$ and we estimate the derivatives of $\mathcal{P}[\delta^4 - 1] \circ \Theta(u,\varphi(u))$. We first we obtain bounds for $A[\delta^4-1]$ and $B[\delta^4-1]$ (see (B.4) and (B.5)). To simplify the notation, we define

$$\widetilde{A}(u) = A[\delta^4-1](\pi + \Theta_\lambda(u)), \qquad \widetilde{B}(u,z) = B[\delta^4-1] \circ \Theta(u,z). \tag{B.23}$$

In the following computations we use extensively the results in Lemma 4.5 without mentioning them.

1. Estimates of $\widetilde{A}(u)$: Defining $\hat{\lambda} = \lambda - \pi$, by (B.4),

$$A[\delta^4-1](\hat{\lambda}+\pi) = 2(1-\cos\hat{\lambda}) - 2\delta^4(1-\cos\hat{\lambda}) + \delta^8.$$

   Then, applying Lemma B.9,

$$\|\sin\Theta_\lambda\|^{\text{out}}_{0,-\frac{2}{3}} \le C\,\|\Theta_\lambda\|^{\text{out}}_{0,-\frac{2}{3}} \le C, \qquad \left\|(1-\cos\Theta_\lambda)^{-1}\right\|^{\text{out}}_{0,\frac{4}{3}} \le C\,\left\|\Theta_\lambda^{-2}\right\|^{\text{out}}_{0,\frac{4}{3}} \le C$$

and, as a result,

$$\begin{aligned} \|\widetilde{A}^{-1}\|^{\text{out}}_{0,\frac{4}{3}} &\le C\,\left\|(1-\cos\Theta_\lambda)^{-1}\right\|^{\text{out}}_{0,\frac{4}{3}} \le C, \\ \|\partial_u\widetilde{A}\|^{\text{out}}_{0,-\frac{1}{3}} &\le C\,\|\sin\Theta_\lambda\|^{\text{out}}_{0,-\frac{2}{3}}\,\|\partial_u\Theta_\lambda\|^{\text{out}}_{0,\frac{1}{3}} \le C. \end{aligned} \tag{B.24}$$

2. Estimates of $\widetilde{B}(u,\varphi(u))$: Considering the auxiliary variables $(\hat{\lambda},\hat{L}) = (\lambda-\pi, L-1)$, we have that

$$\begin{aligned} B[\delta^4-1](\pi+\hat{\lambda},1+\hat{L},\eta,\xi) = {} & 4\hat{L}(1-\cos\hat{\lambda}+\delta^4\cos\hat{\lambda}) \\ & + \frac{\eta}{\sqrt{2}}(-3+2e^{-i\hat{\lambda}}+e^{-2i\hat{\lambda}}+\delta^4(3+e^{-2i\hat{\lambda}})) \\ & + \frac{\xi}{\sqrt{2}}(-3+2e^{i\hat{\lambda}}+e^{2i\hat{\lambda}}+\delta^4(3+e^{2i\hat{\lambda}})) \\ & + R[\delta^4-1](\pi+\hat{\lambda},1+\hat{L},\eta,\xi). \end{aligned} \tag{B.25}$$

Then, by the estimates in (B.6) and Lemma B.9,

$$\begin{aligned} \|\widetilde{B}(\cdot,\varphi)\|^{\text{out}}_{1,-2} \le {} & C\,\left\|\Theta_L(\cdot,\varphi)\Theta_\lambda^2\right\|^{\text{out}}_{0,-1} + \frac{C}{\delta^2}\,\|\Theta_\eta(\cdot,\varphi)\Theta_\lambda\|^{\text{out}}_{0,\frac{2}{3}} \\ & + \frac{C}{\delta^2}\,\|\Theta_\xi(\cdot,\varphi)\Theta_\lambda\|^{\text{out}}_{0,\frac{2}{3}} + \frac{C}{\delta^2}\,\left\|(\Theta_L,\Theta_\eta,\Theta_\xi)^2\right\|^{\text{out}}_{0,\frac{2}{3}} \le C\delta^2. \end{aligned} \tag{B.26}$$

Now, we look for estimates of the first derivatives of $\widetilde{B}(u,\varphi(u))$. By its definition in (B.23) and the expression of Θ in (B.22), we have that

$$\begin{aligned} \partial_u\widetilde{B} &= \left[\partial_\lambda B[\delta^4-1]\circ\Theta\right]\partial_u\Theta_\lambda + \left[\partial_L B[\delta^4-1]\circ\Theta\right]\partial_u\Theta_L, \\ \partial_w\widetilde{B} &= \left[\partial_L B[\delta^4-1]\circ\Theta\right]\partial_w\Theta_L, \\ \partial_x\widetilde{B} &= \left[\partial_\eta B[\delta^4-1]\circ\Theta\right]\partial_x\Theta_\eta, \qquad \partial_y\widetilde{B} = \left[\partial_\xi B[\delta^4-1]\circ\Theta\right]\partial_y\Theta_\xi. \end{aligned} \tag{B.27}$$

Differentiating (B.25) and applying Lemma B.9,

$$\begin{aligned} \left\|\partial_\lambda B[\delta^4-1]\circ\Theta(\cdot,\varphi)\right\|^{\text{out}}_{1,-\frac{4}{3}} &\le C\,\|\Theta_L(\cdot,\varphi)\Theta_\lambda\|^{\text{out}}_{-\frac{1}{3}} + \frac{C}{\delta^2}\,\|\Theta_\eta(\cdot,\varphi)\|^{\text{out}}_{0,\frac{4}{3}} \\ &\quad + \frac{C}{\delta^2}\,\|\Theta_\xi(\cdot,\varphi)\|^{\text{out}}_{0,\frac{4}{3}} + C\delta^2 \le C\delta^2, \\ \left\|\partial_L B[\delta^4-1]\circ\Theta(\cdot,\varphi)\right\|^{\text{out}}_{1,-\frac{7}{3}} &\le C\,\left\|\Theta_\lambda^2\right\|^{\text{out}}_{0,-\frac{4}{3}} + \frac{C}{\delta^2}\,\|\Theta_L(\cdot,\varphi)\|^{\text{out}}_{0,\frac{1}{3}} + \frac{C}{\kappa} \le C, \\ \left\|\partial_* B[\delta^4-1]\circ\Theta(\cdot,\varphi)\right\|^{\text{out}}_{0,-\frac{2}{3}} &\le C\,\|\Theta_\lambda\|^{\text{out}}_{0,-\frac{2}{3}} + \frac{C}{\kappa} \le C, \quad \text{for } * = \eta,\xi. \end{aligned}$$

Then, using also (B.27) and taking $*=x,y$,

$$\|\partial_u \widetilde{B}(\cdot,\varphi)\|_{1,-1}^{\mathrm{out}}, \|\partial_w \widetilde{B}(\cdot,\varphi)\|_{1,-\frac{8}{3}}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_* \widetilde{B}(\cdot,\varphi)\|_{0,-\frac{2}{3}}^{\mathrm{out}} \leq C\delta. \tag{B.28}$$

Analogously, for the second derivatives, one can obtain the estimates

$$\begin{aligned} &\|\partial_{uw} \widetilde{B}(\cdot,\varphi)\|_{1,-\frac{5}{3}}^{\mathrm{out}} \leq C\delta^2, \;\; \|\partial_w^2 \widetilde{B}(\cdot,\varphi)\|_{0,\frac{2}{3}}^{\mathrm{out}} \leq C\delta^4, \;\; \|\partial_{u*} \widetilde{B}(\cdot,\varphi)\|_{0,\frac{1}{3}}^{\mathrm{out}} \leq C\delta, \\ &\|\partial_{w*} \widetilde{B}(\cdot,\varphi)\|_{0,-\frac{1}{3}}^{\mathrm{out}} \leq C\delta^3, \;\; \|\partial_*^2 \widetilde{B}(\cdot,\varphi)\|_{0,0}^{\mathrm{out}} \leq C\delta^2, \;\; \|\partial_{xy} \widetilde{B}(\cdot,\varphi)\|_{0,0}^{\mathrm{out}} \leq C\delta^2. \end{aligned} \tag{B.29}$$

Now, we are ready to obtain estimates for $M_P(u,\varphi(u))$ by using the series expansion (B.7). First, we check that it is convergent. Indeed, by (B.24) and (B.26), for $u \in D_{\kappa,d_1,\rho_2}^{\mathrm{u,out}}$ and taking κ big enough we have that

$$\left| \frac{\widetilde{B}(u,\varphi(u))}{\widetilde{A}(u)} \right| \leq \|\widetilde{B}(\cdot,\varphi)\|_{0,-\frac{4}{3}}^{\mathrm{out}} \|\widetilde{A}^{-1}\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq \frac{C}{\kappa^2\delta^2} \|\widetilde{B}(\cdot,\varphi)\|_{1,-2}^{\mathrm{out}} \leq \frac{C}{\kappa^2} \ll 1.$$

Therefore, by (B.3) and (B.19),

$$\begin{aligned} |M_P(u,\varphi(u))| \leq & \left| \frac{1}{\sqrt{A[\delta^4-1](\lambda_h(u))}} - \frac{1}{\sqrt{2+2\cos\lambda_h(u)}} \right| \\ & + C\frac{|\widetilde{B}(u,\varphi(u))|}{|\widetilde{A}(u)|^{\frac{3}{2}}}. \end{aligned} \tag{B.30}$$

Then, to estimate M_P and its derivatives, it only remains to analyze the u-derivative of its first term. Indeed, by the definition of $A[\delta^4-1]$ in (B.4).

$$\left\| \partial_u \left(\frac{1}{\sqrt{A[\delta^4-1](\lambda_h(u))}} - \frac{1}{\sqrt{2+2\cos\lambda_h(u)}} \right) \right\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C\delta^4. \tag{B.31}$$

Therefore, applying estimates (B.24), (B.26), (B.28), (B.29) and (B.31), to the derivatives of M_P and using (B.30), we obtain the statement of the lemma. □

Analogously to Lemma B.10, we obtain estimates for the first and second derivatives of M_S and M_R (see (B.20) and (B.21)).

Lemma B.11. *Fix $\varrho > 0$, $\delta > 0$ small enough and $\kappa > 0$ big enough. Then, for $\varphi \in B(\varrho\delta^3)$ and $*=x,y$, we have*

$$\begin{aligned} &\|\partial_u M_S(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_w M_S(\cdot,\varphi)\|_{0,-\frac{1}{3}}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_* M_S(\cdot,\varphi)\|_{0,0}^{\mathrm{out}} \leq C\delta, \\ &\|\partial_{uw} M_S(\cdot,\varphi)\|_{0,\frac{2}{3}}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_{u*} M_S(\cdot,\varphi)\|_{0,\frac{1}{3}}^{\mathrm{out}} \leq C\delta, \quad \|\partial_w^2 M_S(\cdot,\varphi)\|_{0,-\frac{2}{3}}^{\mathrm{out}} \leq C\delta^4, \\ &\|\partial_{w*} M_S(\cdot,\varphi)\|_{0,-\frac{1}{3}}^{\mathrm{out}} \leq C\delta^3, \quad \|\partial_*^2 M_S(\cdot,\varphi)\|_{0,0}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_{xy} M_S(\cdot,\varphi)\|_{0,0}^{\mathrm{out}} \leq C\delta^2 \end{aligned}$$

and

$$\|\partial_u M_R(\cdot,\varphi)\|_{1,1}^{\text{out}} \le C\delta^2, \quad \|\partial_w M_R(\cdot,\varphi)\|_{1,-\frac{2}{3}}^{\text{out}} \le C\delta^2, \quad \|\partial_* M_R(\cdot,\varphi)\|_{0,0}^{\text{out}} \le C\delta,$$
$$\|\partial_{uw} M_R(\cdot,\varphi)\|_{1,\frac{1}{3}}^{\text{out}} \le C\delta^2, \quad \partial_{u*} M_R(\cdot,\varphi) \equiv 0, \quad \left\|\partial_w^2 M_R(\cdot,\varphi)\right\|_{0,-\frac{2}{3}}^{\text{out}} \le C,$$
$$\partial_{w*} M_R(\cdot,\varphi) \equiv 0, \quad \partial_*^2 M_R(\cdot,\varphi) \equiv 0, \quad \partial_{xy} M_R(\cdot,\varphi) \equiv 0.$$

End of the proof of Lemma 4.8. We start by estimating the first and second derivatives of $H_1^{\text{sep}}(u,\varphi(u);\delta)$ in suitable norms. Recall that by (B.18), $H_1^{\text{sep}} = M_P + M_S + M_R$. Therefore, taking $\varphi \in B(\varrho\delta^3) \subset \mathcal{X}_\times^{\text{out}}$ and applying Lemmas B.10 and B.11:

1. For $g^{\text{sep}} = \partial_w H_1^{\text{sep}}$ one has

$$\|g^{\text{sep}}(\cdot,\varphi)\|_{1,-\frac{2}{3}}^{\text{out}} \le \|\partial_w M_P(\cdot,\varphi)\|_{1,-\frac{2}{3}}^{\text{out}} + C\,\|\partial_w M_S(\cdot,\varphi)\|_{0,-\frac{1}{3}}^{\text{out}} + \|\partial_w M_R(\cdot,\varphi)\|_{1,-\frac{2}{3}}^{\text{out}}$$
$$\le C\delta^2$$

and, in particular, for κ big enough

$$\|g^{\text{sep}}(\cdot,\varphi)\|_{0,0}^{\text{out}} \le C\kappa^{-2} \ll 1. \tag{B.32}$$

Analogously, $\|\partial_w g^{\text{sep}}(\cdot,\varphi)\|_{0,-\frac{2}{3}}^{\text{out}} \le C$ and $\|\partial_* g^{\text{sep}}(\cdot,\varphi)\|_{0,\frac{5}{3}}^{\text{out}} \le C\delta^3$, for $* = x,y$.

2. For $f_1^{\text{sep}} = -\partial_u H_1^{\text{sep}}$, one has that

$$\|f_1^{\text{sep}}(\cdot,\varphi)\|_{1,1}^{\text{out}} \le \|\partial_u M_P(\cdot,\varphi)\|_{1,1}^{\text{out}} + C\,\|\partial_u M_S(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\text{out}} + \|\partial_u M_R(\cdot,\varphi)\|_{1,1}^{\text{out}} \le C\delta^2,$$

$\|\partial_w f_1^{\text{sep}}(\cdot,\varphi)\|_{1,\frac{1}{3}}^{\text{out}} \le C\delta^2$ and $\|\partial_* f_1^{\text{sep}}(\cdot,\varphi)\|_{0,\frac{7}{3}}^{\text{out}} \le C\delta$, for $* = x,y$.

3. For $f_2^{\text{sep}} = i\partial_y H_1^{\text{sep}}$ and $f_3^{\text{sep}} = -i\partial_x H_1^{\text{sep}}$, we can obtain the estimates

$$\begin{aligned}\|f_2(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\text{out}} &\le \|\partial_y M_P(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\text{out}} + C\,\|\partial_y M_S(\cdot,\varphi) + \partial_y M_R(\cdot,\varphi)\|_{0,0}^{\text{out}} \le C\delta,\\ \|f_3(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\text{out}} &\le \|\partial_x M_P(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\text{out}} + C\,\|\partial_x M_S(\cdot,\varphi) + \partial_x M_R(\cdot,\varphi)\|_{0,0}^{\text{out}} \le C\delta.\end{aligned} \tag{B.33}$$

Analogously, we have that $\|\partial_w f_j^{\text{sep}}(\cdot,\varphi)\|_{0,\frac{5}{3}}^{\text{out}} \le C\delta^3$ and $\|\partial_* f_j^{\text{sep}}(\cdot,\varphi)\|_{0,2}^{\text{out}} \le C\delta^2$, for $j = 2,3$ and $* = x,y$.

Joining these estimates and taking κ big enough, we complete the proof of the lemma. □

Remark B.12. Note that $\widetilde{D}_{\kappa_2,d_2,d_3}^{\text{u,out}} \subset D_{\kappa,d_1,\rho_2}^{\text{u,out}}$ and $\mathcal{Y}^{\text{out}} \subset \mathcal{X}_{0,0}^{\text{out}}$ (see (4.9) and (4.5)). Then, the proof of Lemma 4.10 is a direct consequence of the estimates for g^{sep} and its derivatives in Item 1 above and the fact that, by (3.11) and (4.8),

$$R[\mathcal{U}](v) = \partial_w H_1^{\text{sep}}\left(v+\mathcal{U}(v), z^{\text{u}}(v+\mathcal{U}(v))\right) = g^{\text{sep}}\left(v+\mathcal{U}(v), z^{\text{u}}(v+\mathcal{U}(v))\right).$$

Appendix C. Estimates for the difference

In this section we prove Lemmas 3.16 and 3.19.

C.1. Proof of Lemma 3.16

First, we prove the estimates for the operator Υ given in (3.27). For $\sigma \in [0,1]$, we define $z_\sigma = \sigma z^{\mathrm{u}} + (1-\sigma) z^{\mathrm{s}}$ with $z_\sigma = (w_\sigma, x_\sigma, y_\sigma)^T$. Then, by Theorem 3.3, for $u \in D_{\kappa,d}$, we have that

$$|w_\sigma(u)| \leq \frac{C\delta^2}{|u^2+A^2|} + \frac{C\delta^4}{|u^2+A^2|^{\frac{8}{3}}}, \qquad |x_\sigma(u)|, |y_\sigma(u)| \leq \frac{C\delta^3}{|u^2+A^2|^{\frac{4}{3}}}. \tag{C.1}$$

Recalling that $H^{\mathrm{sep}} = w + \frac{xy}{\delta^2} + H_1^{\mathrm{sep}}$ (see (3.8)), one has

$$\begin{aligned} |\Upsilon_1(u) - 1| &\leq \sup_{\sigma\in[0,1]} \left|\partial_w H_1^{\mathrm{sep}}(u, z_\sigma(u))\right|, \\ |\Upsilon_2(u)| &\leq \frac{|y_\sigma(u)|}{\delta^2} + \sup_{\sigma\in[0,1]} \left|\partial_x H_1^{\mathrm{sep}}(u, z_\sigma(u))\right|, \\ |\Upsilon_3(u)| &\leq \frac{|x_\sigma(u)|}{\delta^2} + \sup_{\sigma\in[0,1]} \left|\partial_y H_1^{\mathrm{sep}}(u, z_\sigma(u))\right|. \end{aligned}$$

Then, by (C.1) and applying (B.32) and (B.33) in the proof of Lemma 4.8 we obtain the estimates for Υ_1, Υ_2 and Υ_3.

We also need estimates for the matrix $\widetilde{\mathcal{B}}^{\mathrm{spl}}$ given in (3.26), which satisfies

$$|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{i,j}(u)| \leq \sup_{\sigma\in[0,1]} \left|\left(D_z \mathcal{R}^{\mathrm{sep}}[z_\sigma](u)\right)_{i,j}\right|,$$

for $z_\sigma = \sigma z^{\mathrm{u}} + (1-\sigma) z^{\mathrm{s}}$. Then, by (C.1) and applying Lemma 4.8, for $u \in D_{\kappa,d}$,

$$\begin{aligned} &\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,1}(u)\right| \leq \frac{C\delta}{|u^2+A^2|^{\frac{2}{3}}}, && \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,1}(u)\right| \leq \frac{C\delta}{|u^2+A^2|^{\frac{2}{3}}}, \\ &\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,2}(u)\right| \leq \frac{C}{|u^2+A^2|^{\frac{1}{3}}} + \frac{C\delta^2}{|u^2+A^2|^2}, && \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,2}(u)\right| \leq \frac{C\delta^2}{|u^2+A^2|^2}, \\ &\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,3}(u)\right| \leq \frac{C\delta^2}{|u^2+A^2|^2}, && \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}(u)\right| \leq \frac{C}{|u^2+A^2|^{\frac{1}{3}}} + \frac{C\delta^2}{|u^2+A^2|^2}. \end{aligned} \tag{C.2}$$

Then, by (3.53) and taking κ big enough,

$$\left|\mathcal{B}^{\mathrm{spl}}_{1,1}(u)\right| \leq \frac{|\Upsilon_2(u)|}{|\Upsilon_1(u)|} \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,1}(u)\right| \leq \frac{C\delta^2}{|u^2+A^2|^2},$$

$$\left|\mathcal{B}^{\mathrm{spl}}_{1,2}(u)\right| \leq \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,3}(u)\right| + \frac{|\Upsilon_3(u)|}{|\Upsilon_1(u)|}\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,1}(u)\right| \leq \frac{C\delta^2}{|u^2+A^2|^2}$$

and analogous estimates hold for $\mathcal{B}^{\mathrm{spl}}_{2,1}$ and $\mathcal{B}^{\mathrm{spl}}_{2,2}$.

Finally, we compute estimates for $B_y(u)$ (see (3.44)) and $u \in D_{\kappa,d}$. The estimates for $B_x(u)$ can be computed analogously. Let us consider the integration path $\rho_t = u_* + (u - u_*)t$, for $t \in [0,1]$. Then

$$B_y(u) = \exp\left(\int_0^1 \widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,2}(\rho_t)(u-u_*)dt\right).$$

Using the bounds in (C.2), we have that

$$|\log B_y(u)| \leq C\,|u-u_*| \left|\int_0^1 \frac{1}{|\rho_t^2+A^2|^{\frac{1}{3}}} + \frac{\delta^2}{|\rho_t^2+A^2|^2}dt\right| \leq C,$$

which implies $C^{-1} \leq |B_y(u)| \leq C$.

C.2. Proof of Lemma 3.19

We only give an expression for $B_y(u_+)$. The result for $B_x(u_-)$ is analogous. First, we analyze $\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}$.

Lemma B.13. *For $\delta > 0$ small enough, $\kappa > 0$ large enough and $u \in D_{\kappa,d}$, the function $\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}$ defined in (3.26) is of the form*

$$\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}(u) = -\frac{4i}{3}\Lambda_h(u) + \delta^2 m(u;\delta),$$

for some function m satisfying

$$|m(u;\delta)| \leq \frac{C}{|u^2+A^2|^2}.$$

Proof. Let us define $z_\tau = \tau z^{\mathrm{u}} + (1-\tau)z^{\mathrm{s}}$ and recall that, for $u \in D_{\kappa,d}$,

$$\widetilde{\mathcal{B}}_{3,3}(u) = \int_0^1 \partial_y \mathcal{R}^{\mathrm{sep}}_3[z_\tau](u)d\tau. \tag{C.3}$$

Then, by the expression of $\mathcal{R}^{\mathrm{sep}}_3$ in (B.8), the estimates in the proof of Lemma 4.8 (see Appendix B.2) and Theorem 3.3, we have that

$$\partial_y \mathcal{R}_3^{\mathrm{sep}}[z_\tau](u) = \frac{i}{\delta^2} g^{\mathrm{sep}}(u, z_\tau(u)) + \delta^2 \widetilde{m}(u;\delta),$$

where $|\widetilde{m}(u;\delta)| \leq \frac{C}{|u^2+A^2|^2}$. In the following, to simplify notation, we denote by $\widetilde{m}(u;\delta)$ any function satisfying the previous estimate. Since $g^{\mathrm{sep}} = \partial_w H_1^{\mathrm{sep}}$, by (B.18) one has

$$g^{\mathrm{sep}}(u, z_\tau(u)) = \partial_w M_P(u, z_\tau(u);\delta) + \partial_w M_S(u, z_\tau(u);\delta) + \partial_w M_R(u, z_\tau(u);\delta),$$

with M_P, M_S and M_R as given in (B.19), (B.20) and (B.21), respectively. Then, taking into account that $F_{\mathrm{p}}(s) = 2z^3 + \mathcal{O}(z^4)$ (see (2.10)) and following the proofs of Lemmas B.10 and B.11, it is a tedious but an easy computation to see that,

$$\begin{aligned} g^{\mathrm{sep}}(u, z_\tau(u)) =& \partial_w M_P(u,0,0,0;\delta) + \partial_w M_S(u,0,0,0;\delta) \\ &- \frac{w_\tau(u)}{3\Lambda_h^2(u)} - \frac{\delta^2 \mathfrak{L}_\Lambda(\delta)}{\Lambda_h(u)} - 2\delta^2\Lambda_h(u) + \delta^4\widetilde{m}(u;\delta) \end{aligned}$$

and, by (C.3),

$$\begin{aligned} \widetilde{\mathcal{B}}_{3,3}(u) =& \frac{i}{\delta^2}\left[\partial_w M_P(u,0,0,0;\delta) + \partial_w M_S(u,0,0,0;\delta)\right] \\ &- i\frac{w^{\mathrm{u}}(u) + w^{\mathrm{s}}(u)}{6\delta^2\Lambda_h^2(u)} - i\frac{\mathfrak{L}_\Lambda(\delta)}{\Lambda_h(u)} - 2i\Lambda_h(u) + \delta^2\widetilde{m}(u;\delta). \end{aligned} \tag{C.4}$$

Next, we study the terms $w^{\mathrm{u,s}}(u)$. Since $H^{\mathrm{sep}} = w + \frac{xy}{\delta^2} + M_P + M_S + M_R$ (see (3.8) and (B.18)), one can see that

$$H^{\mathrm{sep}}(u, z^{\mathrm{u}}(u);\delta) = H^{\mathrm{sep}}(u, z^{\mathrm{s}}(u);\delta) = \lim_{\mathrm{Re}\, u \to \pm\infty} H^{\mathrm{sep}}(u,0,0,0;\delta) = \delta^4 K(\delta),$$

with $|K(\delta)| \leq C$, for δ small enough. Then, by Theorem 3.3, for $\diamond = \mathrm{u},\mathrm{s}$,

$$|w^\diamond(u) + M_P(u, z^\diamond(u);\delta) + M_S(u, z^\diamond(u);\delta) + M_R(u, z^\diamond(u);\delta)| \leq \frac{C\delta^4}{|u^2+A^2|^{\frac{8}{3}}}.$$

Again, following the proofs of Lemmas B.10 and B.11, one obtains

$$\left|w^\diamond(u) + M_P(u,0,0,0;\delta) + M_S(u,0,0,0;\delta) + \delta^2\Lambda_h(u)(3\mathfrak{L}_\Lambda + 2\Lambda_h^2(u))\right| \leq \frac{C\delta^4}{|u^2+A^2|^{\frac{8}{3}}},$$

and, by (C.4),

$$\begin{aligned} \widetilde{\mathcal{B}}_{3,3}(u) =& -\frac{4i}{3}\Lambda_h(u) + \frac{i}{\delta^2}\left[\partial_w M_P(u,0,0,0;\delta) + \frac{M_P(u,0,0,0;\delta)}{3\Lambda_h^2(u)}\right] \\ &+ \frac{i}{\delta^2}\left[\partial_w M_S(u,0,0,0;\delta) + \frac{M_S(u,0,0,0;\delta)}{3\Lambda_h^2(u)}\right] + \delta^2\widetilde{m}(u;\delta). \end{aligned}$$

Therefore, it only remains to check that

$$\left|\partial_w M_{P,S}(u,0,0,0;\delta) + \frac{M_{P,S}(u,0,0,0;\delta)}{3\Lambda_h^2(u)}\right| \le \frac{C\delta^4}{|u^2+A^2|^2}.$$

Indeed, by (B.7) and the definition (B.19) of M_P, one has

$$M_P(u,w,0,0;\delta) = \mathcal{M}_P\left(u, \delta^2\Lambda_h(u) - \frac{\delta^2 w}{3\Lambda_h(u)} + \delta^4\mathfrak{L}_\Lambda(\delta)\right),$$

where $\mathcal{M}_P(u,\Lambda)$ is an analytic function for $u \in D_{\kappa,d}$ and $|\Lambda| \ll 1$. Moreover, following the proof of Lemma B.10, there exist a_0 and a_1 such that

$$|\mathcal{M}_P(u,\Lambda) - a_0(u;\delta) - a_1(u;\delta)\Lambda| \le \frac{C\Lambda^2}{|u^2+A^2|^2},$$

with

$$|a_0(u;\delta)| \le \frac{C\delta^4}{|u^2+A^2|^{\frac{2}{3}}}, \qquad |a_1(u;\delta)| \le \frac{C}{|u^2+A^2|^{\frac{2}{3}}}.$$

Therefore,

$$\begin{aligned}\left|\partial_w M_P(u,0,0,0;\delta) + \frac{M_P(u,0,0,0;\delta)}{3\Lambda_h^2(u)}\right| &\le \frac{|a_0(u)|}{3\Lambda_h^2(u)} + \frac{\delta^4\mathfrak{L}_\Lambda(\delta)\,|a_1(u)|}{3\Lambda_h^2(u)} + \frac{C\delta^4}{|u^2+A^2|^2}\\ &\le \frac{C\delta^4}{|u^2+A^2|^2}.\end{aligned}$$

An analogous estimate holds for M_S. □

End of the proof of Lemma 3.19. By Lemma B.13 and recalling that $u_+ = iA - \kappa\delta^2$,

$$\begin{aligned}\log B_y(u_+) = \int_{u_*}^{u_+} \widetilde{\mathcal{B}}_{3,3}^{\mathrm{spl}}(u)du = -\frac{4i}{3}\int_{u^*}^{iA}\Lambda_h(u)du\\ + \frac{4i}{3}\int_{u_+}^{iA}\Lambda_h(u)du + \delta^2\int_{u^*}^{u_+} m(u;\delta).\end{aligned} \tag{C.5}$$

Then, by Theorem 3.1 and taking into account that $\kappa = \kappa_*\,|\log\delta|$ (see Lemma 3.18), we obtain

$$\left|\log B_y(u_+) + \frac{4i}{3}\int_{u^*}^{iA}\Lambda_h(u)du\right| \le \frac{C}{\kappa} + C\kappa^{\frac{2}{3}}\delta^{\frac{4}{3}} + \frac{C\delta^2}{|u_* - iA|} \le \frac{C}{|\log\delta|}.$$

Finally, recalling that $\dot{\lambda}_h = -3\Lambda_h$, applying the change of coordinates $\lambda = \lambda_h(u)$ and using that $\lambda_h(iA) = \pi$, we have that

$$\frac{4i}{3}\int_{u^*}^{iA} \Lambda_h(u)du = -\frac{4i}{9}\int_{\lambda_h(u_*)}^{\pi} d\lambda = -\frac{4i}{9}\left(\pi - \lambda_h(u_*)\right).$$

Joining the last statements with (C.5), we obtain the statement of the lemma. □

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