Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

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Treball final de grau

A geometric approach to wormhole theory

Autor: Juan Fernández Esteban

Director: **Dra. Joana Cirici**
Realitzat a: **Departament de Matemàtiques i Informàtica**

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and the universe said you are not separate from every other thing

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Someone important

Contents

Abstract

The main goal of this work is to give a rigorous definition of wormhole and show that some well known metrics given to wormholes - Ellis, Morris-Thorne and Schwarzschild - fit in such framework. We also introduce the necessary tools to understand the formalism.

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Introduction

As humans, we are captivated by the night sky. A simple gaze at the breathtakingly small stars leaves some of us only awed, others deeply in trance for hours. In this modern age, the truth about this mysterious shining dots has been unveiled, and they are, in essence, not different from our Sun. This revelation awakens not only many questions, but most importantly, it arouses a dream - a dream to catch the stars. Within the mind of the greatest dreamers, a world unfolds where humanity reigns over entire galaxies and navigating them at will.

But reality has its limitations: nothing can travel faster than light. This is not just a way of saying it. It does not make sense to travel faster than light, just as it does not make sense to have an object with imaginary mass. It's not that there is a magical wall that stops anything from moving faster than light. The structure of spacetime itself does not allow it.

Albert Einstein showed us in his theory of Special Relativity that space and time conspire against fast-moving objects: they change shape to stop things from accelerating further. Not content with that, he later postulated the beautiful theory of General Relativity (GR), which, based on the brilliant idea of H. Minkowski to join space and time into a single object, states that as a consequence of the presence of energy, space and time mix together and cease to be independent. This groundbreaking theory was proven more accurate than Newtonian mechanics, our previous conception of spacetime.

This theory comes with loads of incredible consequences, such as gravitational waves, black holes, or cosmic strings. It can feel like a mathematical playground. But this playground brings hope again to our interstellar dream: wormholes. A wormhole is a bridge that connects two completely separate regions of space, whether they are in the same universe or in different ones. Although this seems directly extracted from a science fiction book, they are actual solutions to the GR equations and they make physical sense. If one could indeed build a wormhole, any place in the universe could be reached in just the time spent going through the wormhole.

Wormholes are complex objects, and the construction of wormholes carries with it many unknowns and difficulties, such as topology change, causality violation, and negative energy. We will not be dealing with them in this paper, although the state-of-the-art knowledge is that none of these problems is for sure a deal-breaker.

The scope of this project is to introduce wormhole theory and give wormholes their deserved rigorous mathematical treatment, since there is a lack of a completely formal wormhole theory. We aim to provide a strict wormhole definition, since the existing definitions in the literature do not allow

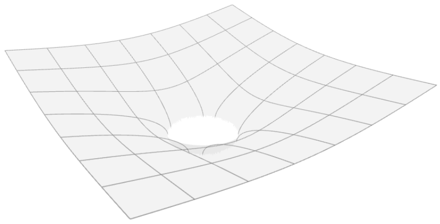

Figure 1: Schwarzschild metric visualization with some fixed parameters to embed it into $\mathbb{R}^3$. This is known as Flamm's paraboloid.

us to confidently classify spacetimes that do or do not have wormholes.

General Relativity has two postulates: the general principle of relativity, which states that the laws of physics are the same in all reference frames and leads us to tensors; and the mass equivalence principle, which states that the inertial mass and gravitational mass are the same and leads us to differential geometry. Therefore, we first review some basic definitions and results from differential geometry in chapter 1. Also, the invariance of the laws of electromagnetism and, in particular, the speed of light, leads to the study of Lorentzian geometry. Consequently, in chapter 2, we also review the main concepts of pseudo-Riemannian geometry and then, armed with this foundation, we are ready to briefly state the actual theory of General Relativity.

At the core of this theory are the Einstein field equations - a tensorial second-order differential equation. These equations, referred to in the plural since a tensorial equation can be reduced to equations for each component, establish the relationship between mass/energy distribution and spacetime structure (structure will be defined in detail). While obtaining the structure proves challenging when a mass/energy distribution is provided, obtaining the mass/energy distribution given the structure of spacetime is comparatively more tractable.

The most important solution of GR, after the flat, Minkowski spacetime of special relativity, is the Schwarzschild solution, which is a vacuum (without the presence of matter) spherically symmetric solution. In fact, by Birkhoff's theorem, it is the only vacuum spherically symmetric solution. It is known in general as the solution for a point mass located at the origin, or to hype things up, the basic black hole solution. A visualization has been provided in figure 1.

In this paper, we give a definition for wormhole as a region whose "space part" is homotopically equivalent to S^2. This involves the definition of a spe-

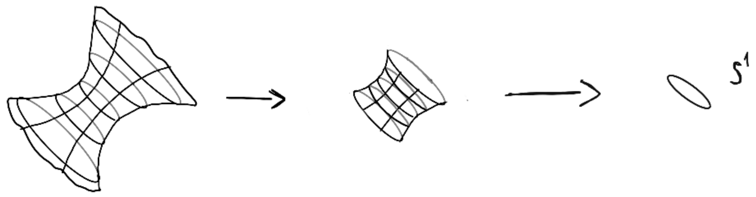

Figure 2: In one less dimension, the space part of a wormhole is homotopically equivalent to S^1 instead of S^2

cial type of vector representing our time direction, and defining orthogonal surfaces to it, which will represent space at any given moment in time. This surfaces, homotopically equivalent to S^2, can be thought as deformable into cylinders with spherical base instead of circular.

In chapter 3, we dive into wormhole theory. We construct wormholes with different metrics (which can be thought of as spacetime structures for the moment). These are well known metrics of wormholes, so we prove that they actually fit in the given definition.

The first one is the Ellis wormhole, discovered as part of a greater family of wormholes, the Ellis drainhole, in [Ell03]. This type of wormholes was supposed to be a non-singular substitute of the Schwarzschild metric (since the center of this metric has a singularity), but it was not very useful, since it is not a vacuum solution. The solution represents some fluid (named ether by Ellis) going through some kind of drainhole - hence the name. When setting the mass of ether going through to zero, the metric adopts its simplest form, the Ellis wormhole. This simplicity is also the reason why we decided to study the Ellis wormhole as a first dive into the theory. Despite its lack of complexity, it resulted to be a traversable wormhole, the first one, in fact! There are wormholes which can be traversed and wormholes which not, since traversing them would require an infinite amount of time.

The Ellis wormhole was actually the starting point for the construction of the wormhole seen in *Interstellar*. The construction of this wormhole was carried on in [JvTFT15]. For cinematic reasons, the Ellis wormhole alone was not enough to be visualized on the big screen, since some more freedom of parameters was needed. The wormhole used in *Interstellar* used 3 adjustable parameters, but the Ellis wormhole has only one, as the reader will be able to see.

For the Ellis wormhole, we can actually give an embedding, which makes it much easier to visualize what is happening. Since manifolds exist on their own, the Ellis wormhole exists without the need of anything else. However, for us, it is hard to think of curved spaces by themselves. It is much easier

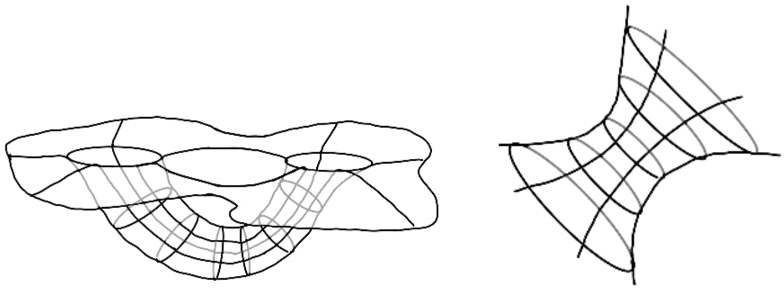

Figure 3: Diagrams of wormholes. The diagram on the left shows a spacetime with both mouths connected, and the one on the right, a spacetime containing just the wormhole. We will be working with the latter one.

to think of the space as a subspace - submanifold in this case - of $\mathbb{R}^n$. With that in mind, we can then visualize the manifolds properties in a much more intuitive way, since we are used to dealing with 2-spheres in $\mathbb{R}^3$ or even for objects embedded in higher dimensions (such as a Klein's bottle or a tesseract), with their projections. In fact, this idea has been a great tool to show General Relativity to non-scientist audiences. It is even the book cover of some GR text books! Wormholes can also be greatly benefited from this perspective, since the curvature is not as easy to visualize as the actual tunnel that everybody has seen. The reader who has not actually seen the tunnel can take a look at the diagram 3 so that the previous statement becomes true.

In the diagram, we are seeing just the equatorial section of the constant time sections, which allows us to lower the dimension from 4 to 2, allowing us to embed it into $\mathbb{R}^3$ (in this particular case). Obviously, when we say that we give an embedding, we are working with higher dimensions, so our embedding will need $\mathbb{R}^5$ or even $\mathbb{R}^6$.

We scale up in generality, and our next target is the Morris-Thorne wormhole - a family of very different wormholes, in fact. Morris and Thorne give in [MT88] a metric for spherically symmetric wormholes. From Birkhoff's theorem, we know that the only spherically symmetric vacuum solution must be the Schwarzschild metric. Therefore, the Morris-Thorne wormhole does contain matter in general - if it did not, it would be only one solution, the Schwarzschild one. It is a family of wormholes, determined by two functions: the redshift function, related to how much energy is pulled from photons - light particles - when they escape the wormhole, and the shape function, which is the one which gives the shape to the wormhole.

We show that the Ellis wormhole is a particular case of the Morris-

Thorne wormhole, which is not of much surprise since the Morris-Thorne wormhole aimed to be a generalization of the Ellis metric. Unexpectedly, we even show that a wormhole can be constructed with the same metric as the Schwarzschild black hole. This means that every Schwarzschild black hole could in fact be a wormhole (at least from just the perspective of General Relativity). This idea, ironically enough, solves the problem of the singularity at the center of the Schwarzschild solution, as Ellis aimed to patch with his solution. But this proposition is not new. It was discovered in [Fla15] in 1916 by Flamm, and much later, in 1935, revisited by Einstein and Rosen in [ER35], who also gave a new metric. This is, in fact, the first known wormhole. Unfortunately, the Einstein-Rosen bridge - the name given to the Schwarzschild wormhole - is not traversable, it is needed an infinite amount of time to do so.

Nevertheless, this idea raises an important aspect of GR. The theory does not deal with topology, which remains a free until chosen. Anyway, one can not tell whether black holes are wormholes or that, just black holes with a singularity, since anyone who sees it would not be able to tell us. However, this specific nuance stays at the core of wormhole mathematical falsifiability.

Knowing the metric of the Morris-Thorne wormhole, it is straightforward - although laborious - to find the matter distribution generating that particular structure. In this work, we calculated the energy distribution (represented by the energy-momentum tensor) of the Morris-Thorne wormhole. The analysis of this result reveals that the Schwarzschild solution is a vacuum solution - a predictable outcome given the assumption on the deduction of the metric. The surprise, however, lies on the Ellis wormhole. which, conversely, needs negative energy density. This means that there is no classical way of getting an Ellis wormhole, since we need negative mass. Matter like this is known as exotic matter, and there are certain effects in quantum mechanics that can generate exotic matter under some very specific circumstances. Thus, wormholes narrowly dodge another bullet, avoiding again another apparent flaw in their model.

To wrap things up, there are several aspects of wormhole theory which have been studied in a physics level, but no so much in a formal way. For different reasons - mainly the need for a broader understanding of some physical or mathematical concepts, which would add much more unneeded complexity to this work - the formalization of these concepts was discarded, but some hints at these topics were provided instead. These aspects include: time travel, which can be achieved by accelerating one of the mouths of a wormhole and joining them together, wormhole traversability and non orientable wormholes. This last one is really intriguing concept, since, travers-

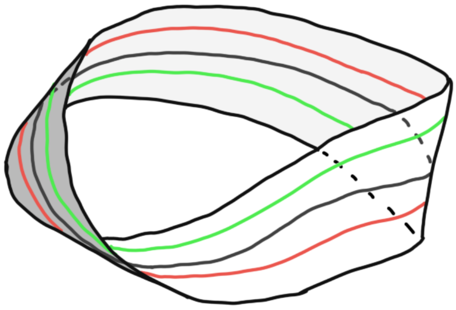

Figure 4: A Möbius strip representing the idea of a non orientable wormhole. Going around the strip turns left to right and right to left every time the same point has been reached. In a piece of paper, you would get one line on each (local) side of the paper, represented by red and green.

ing it would switch right and left. That is, by traversing it your heart would switch from left to right for the people observing you, and you would see people mirrored, so that freckle under her left eye would be under her right eye now!

Chapter 1

Differential geometry

Before diving into the world of General Relativity, we need some mathematical tools. The first and most basic one of them is differential geometry. GR craves for a new kind of mathematical space, something more general than $\mathbb{R}^n$; spaces that look locally like $\mathbb{R}^n$, but perhaps not globally. An example of this is the 2-sphere, S^2, but the main charm of these spaces is that they make sense on their own, without the need of being embedded into a higher dimensional space. These spaces are called manifolds.

In manifolds, the notion of straight lines, which represent trajectories of free particles (in inertial reference frames) in classical mechanics, tumbles. So what we are looking for in manifolds are spaces in which there are other types of trajectories for free particles. This is the interpretation we want for gravity, as a direct consequence of the first principle of General Relativity: the mass equivalence principle. By this principle, the resistance of a body to move by gravity and the amount a force exerted by gravity scales are the same. This means that mass can be ignored in gravity terms, and the path followed by particles on the space can be treated as a property of the space, and not the particle.

For this section, we will be mainly following [Lee13] and [CB03].

1.1 Differential manifolds

1.1.1 Definition

The first thing we want to mimic from the Euclidean space $\mathbb{R}^n$ is the topology. Therefore, our first step is to define sets with topological characteristics akin to $\mathbb{R}^n$.

Definition 1.1 (Topological manifold)**.** *Let $\mathcal{X}$ be a topological space. We say that $\mathcal{X}$ is a **topological manifold** if it satisfies:*

1. *$\mathcal{X}$ is locally Euclidean (locally homeomorphic to $\mathbb{R}^n$, with $n \in \mathbb{Z}_+$)*
2. *$\mathcal{X}$ is Hausdorff*
3. *$\mathcal{X}$ is second-countable (satisfies the second axiom of enumerability)*

In that case, n is called the ***dimension*** *of the topological manifold.*

This last property is not given by every author, but it will always be the case for us.

Now, we want a way of defining differentiation - we do not call it differential geometry for no reason. The idea is to cover the manifold with open subsets, in which we can work similarly to $\mathbb{R}^n$ regarding differentiation. With that purpose, the manifold is given local coordinates to each open subset, and forced that whenever this local coordinates are changed, this change of coordinates is differentiable.

Definition 1.2 (Chart)**.** *Let $\mathcal{M}$ be a topological manifold. A* ***chart*** *is a pair (U, ϕ) of an open subset $U \subset M$ and a homeomorphism $\phi : U \to \mathbb{R}^n$, where n is the dimension of the manifold.*

We also say that U is a ***coordinate open set*** *and that each component of ϕ is a* ***local coordinate****, so all of them will be a set of local coordinates.*

Definition 1.3 (Atlas)**.** *Let $\mathcal{M}$ be a topological manifold. A* ***differentiable atlas*** *(or, simply, atlas) of $\mathcal{M}$ is a set of charts of $\mathcal{M}$, $(U_i, \phi_i)_{i\in I}$ which satisfy:*

1. *$\mathcal{M} = \bigcup_{i\in I} U_i$*
2. *$\forall i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, we have $\phi_j \circ \phi_i^{-1}$ is $\mathcal{C}^\infty$ on its domain.*

Most of the time we will not be specifying which chart we choose, we will be talking in terms of some local coordinates. Notice that by definition, this is okay, since they must be defined all over the manifold. If we want some other coordinates, we can always change them.

Remark 1.4. The reunion of two atlases might not be another atlas, since two charts might not satisfy the second condition.

Definition 1.5 (Differential compatibility)**.** *Two atlases are* ***differentially compatible*** *if their reunion is another atlas.*

Proposition 1.6. *The differential compatibility relation between atlases is an equivalence relation.*

Proof. Let's check the properties of an equivalence relation.

- Reflexivity: every atlas is trivially compatible with itself.
- Symmetry: again, this is trivial, since we are asking for the same conditions to be true either way.
- Transitivity: we have $\mathcal{A} = (U_i, \phi_i)$, $\mathcal{B} = (V_j, \psi_j)$, $\mathcal{C} = (W_k, \delta_k)$. For any two i, k, we have that $\phi_i^{-1} \circ \delta_k = \phi_i^{-1} \circ \psi_j^{-1} \circ \psi_j \circ \delta_k$ for some j, which is $\mathcal{C}^\infty$. This can be done by picking the right open sets for each i, j, k, and covering all the domain of $\phi_i^{-1} \circ \delta_k$.

□

In our manifolds, we want to include every compatible coordinates, since they are valid coordinates. The previous proposition gives a clear hint on what the goal is: to use equivalence classes. We know that any two atlases are compatible with each other if and only if they are in the same class.

Definition 1.7 (Differentiable structure)**.** *Let $\mathcal{M}$ be a topological manifold, we define **differentiable structure** as a differential compatibility equivalence class (of atlases). A local chart will be any chart in any atlas of the equivalence class.*

Definition 1.8 (Smooth manifold)**.** *A **manifold** (or smooth manifold) is a topological manifold $\mathcal{M}$ together with a differentiable structure.*

So, informally, what we are saying is that a differential manifold is a set that resembles the Euclidean space, together with a way of differentiating. It is important to notice that once we get an atlas, we already have a manifold, since the equivalence class is completely determined.

Example 1.9. Here are some examples, which we will only be stating:

- $\mathbb{R}^n$ with the identity chart
- S^n, the n-sphere, with the stereographic projection with respect to two different points
- $\mathbb{RP}^n$, the real projective space, with the charts ϕ_i, $1 \leq i \leq n$, given by:

$$\phi_i([x_1 : \cdots : x_n]) = (\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$$

We obviously need something to differentiate: smooth maps between manifolds can be defined, and there is one intuitive way to do it: asking for the expression in local coordinates to be smooth. This is what the next definition is saying.

Definition 1.10 (Smooth map). *Let $\mathcal{M}, \mathcal{N}$ be two differential manifolds. Let $f : \mathcal{M} \to \mathcal{N}$ be a map between them. We say that f is **smooth** or differentiable if $\forall (U, \phi)$ local chart of $\mathcal{M}$ and (V, ψ) local chart of $\mathcal{N}$ we have $\psi \circ f \circ \phi^{-1}$ is differentiable (on the Euclidean space). We say that f is a **diffeomorphism** if it is smooth, has an inverse and it is also smooth.*

1.1.2 Tangent space

Now, since we are not working with subsets of the Euclidean space, it is not as easy to define the tangent space of a smooth manifold. We will be doing a bit of a trick; we will pick a vector space that behaves as expected: it should have the dimension of the manifold and only depend on a neighborhood of the point. With this in mind, the partial derivatives with respect to the local coordinates seem a good approach, since directional derivatives are one to one with vectors on the tangent space. We will be showing how we can build up this idea precisely next.

Definition 1.11 (Smooth function). *A **smooth function** on $\mathcal{M}$ is a smooth map $f : \mathcal{M} \to \mathbb{R}$, with the identity differentiable structure. The set of all smooth functions on $\mathcal{M}$ is denoted $\mathcal{F}(\mathcal{M})$.*

Remark 1.12. $\mathcal{F}(\mathcal{M})$ is a $\mathbb{R}$-commutative algebra with the sum, product with scalar and product of functions. In particular, it is also a ring.

Definition 1.13 (Derivation). *Given a smooth manifold $\mathcal{M}$ and a point $p \in \mathcal{M}$, an application $\delta_p : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$ is a **derivation** on $p \in \mathcal{M}$ if it satisfies:*

1. $\delta_p(f_1 + f_2) = \delta_p(f_1) + \delta_p(f_2)$
2. $\delta_p(\lambda f) = \lambda \delta_p(f)$
3. $\delta_p(f_1 \cdot f_2) = f_1(p) \cdot \delta_p(f_2) + \delta_p(f_1) \cdot f_2(p)$

That is, it is $\mathbb{R}$-linear (1 and 2) and satisfies the Leibniz rule (3).

Definition 1.14 (Tangent space). *Let $\mathcal{M}$ be a smooth manifold, and $p \in \mathcal{M}$. The set of all derivations on p is called the **tangent space** of $\mathcal{M}$ on p and we denote it $T_p\mathcal{M}$.*

Remark 1.15. The tangent space is a vector space. It can be easily checked from the fact that the arriving space is $\mathbb{R}$.

Now, if we have a smooth map between manifolds, the tangent space also has to be transformed accordingly. So there must be a relation between the map and the transformation of the tangent space. We can define another map, which we will call pushforward [1] and will go from one tangent space to the other linearly.

Definition 1.16 (Pushforward)**.** *Let $\mathcal{M}, \mathcal{N}$ be two smooth manifolds and $f : \mathcal{M} \to \mathcal{N}$, a smooth map between them. We define the **pushforward** (or differential) of f at a point $p \in M$ as the linear application $d_p f : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ such that:*

$$(d_p f(v_p))(g) = v_p(g \circ f) \qquad \forall g \in \mathcal{F}(\mathcal{N})$$

We have to see that this definition actually makes sense, and some properties that we are used to seeing in the standard $\mathbb{R}^n$ differential.

Proposition 1.17. *The pushforward is linear, and it is well-defined. Furthermore, if the smooth map is a diffeomorphism, then the pushforward is an isomorphism.*

Proof. Let's first see that it is well-defined: notice that $g \circ f \in \mathcal{F}(\mathcal{M})$. So $(d_p f(v_p))(g) = v_p(g \circ f)$ is well-defined.

Now let's see the linearity. We can see $d_p f(\lambda v_p + \mu w_p)(g) = (\lambda\, v_p + \mu\, w_p)(g \circ f) = \lambda\, v_p(g \circ f) + \mu\, w_p(g \circ f) = \lambda\, d_p f(v_p)(g) + \mu\, d_p f(w_p)(g)$.

Lastly, let's suppose $f : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism. Notice that by definition $d_p f^{-1}$ is the inverse of $d_p f$. Therefore, $d_p f$ is an isomorphism. □

Proposition 1.18 (Dimension of the tangent space)**.** *Let $\mathcal{M}$ be a smooth manifold of dimension n. Then, given any point $p \in \mathcal{M}$, it is true that $T_p\mathcal{M} \cong \mathbb{R}^n$.*

Proof. Let (U, ϕ) be a chart containing p. We will abuse notation, and also refer to U as a manifold. Notice that it constitutes one. We want to prove the following set of isomorphisms:

$$T_p\mathcal{M} \overset{(1)}{\cong} T_pU \overset{(2)}{\cong} T_{\phi(p)}\mathbb{R}^n \overset{(3)}{\cong} \mathbb{R}^n$$

(1), $T_p\mathcal{M} \cong T_pU$:

Can be seen at proposition 3.9 of [Lee13].

(2), $T_pU \cong T_{\phi(p)}\mathbb{R}^n$:

[1] Some authors call it the differential. We will choose the name pushforward since, as we will see, we will also define a pullback, and we want to make clear when we are pushing and when we are pulling -in the ways that we will define. Nevertheless, it must be clear the relation between the known differential on the Euclidean space and the pushforward on manifolds.

We have the diffeomorphism $\phi : U \to \phi(U)$. The pushforward $d_p\phi$ is an isomorphism.

(3), $T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n$:

Let $x^1, \dots, x^n$ be coordinates on $\mathbb{R}^n$. Let $\frac{\partial}{\partial x^i}|_a$ be the partial derivative with respect to x^i at the point $a = \phi(p)$. We want to show that $\left(\frac{\partial}{\partial x^i}|_a\right)_{1\leq i\leq n}$ is a base of $T_a\mathbb{R}^n$.

Let $\delta_a \in T_a\mathbb{R}^n$, $f \in \mathcal{F}(\mathbb{R}^n)$. We can see that (with x^i the i-th coordinate function):

$$\begin{aligned} f(q) &= f(0) + \int_0^1 \frac{d}{dt}(f(tq))dt \\ &= f(0) + \int_0^1 \sum_{i=1}^n x^i(q)(\frac{\partial f}{\partial x^i}(tq))dt \\ &= f(0) + \sum_{i=1}^n x^i(q) \int_0^1 (\frac{\partial f}{\partial x^i}(tq))dt \end{aligned}$$

We can also prove that a derivation applied to a constant is zero, since $\delta_a(\lambda) = \lambda\delta_a(1) = \lambda(1\dot{\delta}_a(1)+\delta_a(1)\dot{1}) = 2\lambda\delta_a(1) = 2\delta_a(\lambda)$. Taking this into account and applying δ_a to f:

$$\begin{aligned} \delta_a(f) &= \sum_{i=1}^n \delta_a(x^i)\left(\int_0^1 \frac{\partial f}{\partial x^i}(a)dt\right) + \sum_{i=1}^n x^i(0)\delta_a\left(\int_0^1 \frac{\partial f}{\partial x^i}(tq)dt\right) \\ &= \sum_{i=1}^n \delta_a(x^i)\left.\frac{\partial}{\partial x^i}\right|_a (f) \end{aligned}$$

So $\left(\frac{\partial}{\partial x^i}|_a\right)_{1\leq i\leq n}$ generate the space, and the linear independence is trivial. □

We will sometimes abuse notation and write either $\partial_{x^i}|_p$ or $\frac{\partial}{\partial x^i}|_p$ to refer to $d_p i(d_p\phi^{-1}(\frac{\partial}{\partial x^i}|_{\phi(p)}))$, where $i : U \hookrightarrow \mathcal{M}$ is the inclusion map. Notice that the actual vector living in $T_p\mathcal{M}$ should be written as the last one, but we hope the reader will go easy on us when we prefer to write any of the first two.

Corollary 1.19 (Base of the tangent space). *Let $\mathcal{M}$ be a smooth manifold, and $p \in \mathcal{M}$, with $(x^i)_i$ some local coordinates. Then $\left(\frac{\partial}{\partial x^i}|_p\right)_i$ is a base of $T_p\mathcal{M}$.*

Now we can give a local form of the pushforward in terms of our new base. Notice that it is strongly related to the one given in the Euclidean space -and it makes sense, we are moving to the Euclidean space, differentiating and going back to our smooth manifold, so any counterintuitive result in this regard should not go unnoticed.

Proposition 1.20 (Local expression of the pushforward). *Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds, with (U, ϕ) and (V, ψ) respective local charts with coordinates $(x^i)_i$ and $(y^j)_j$ respectively. Let $f : \mathcal{M} \to \mathcal{N}$ be a smooth map. Then:*

$$d_p f \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \sum_j \frac{\partial f^j}{\partial x^i}(p) \cdot \left. \frac{\partial}{\partial y^j} \right|_{f(p)}$$

Proof. By applying the chain rule:

$$d_p f \left(\left. \frac{\partial}{\partial x^i} \right|_p \right)(g) = \left. \frac{\partial}{\partial x^i} \right|_p (g \circ f) = \sum_j \frac{\partial f^j}{\partial x^i}(p) \cdot \frac{\partial g}{\partial y^j}(f(p))$$

□

We can also have curves that lie in manifolds. For them, it also makes sense to define a tangent vector.

Definition 1.21 (Smooth curve). *Let $\mathcal{M}$ be a smooth manifold. A* ***smooth curve*** *on $\mathcal{M}$ is a smooth map $\gamma : I \to \mathcal{M}$, where I is an open interval.*

Definition 1.22 (Tangent vector of a curve). *Let $\mathcal{M}$ be a smooth manifold. The* ***tangent vector*** *of a curve γ at $\gamma(t_0)$ (with $t_0 \in I$) is $\dot{\gamma}(t_0) \in T_{\gamma(t_0)}\mathcal{M}$, defined by:*

$$\dot{\gamma}(t_0)(f) = \left. \frac{d(f \circ \gamma(t))}{dt} \right|_{t=t_0} \qquad \forall f \in \mathcal{F}(\mathcal{M})$$

The dual space of the tangent space (generally referred to as **the cotangent space**) will be of great importance as well. The surprising part here is that the pushforward of the coordinate functions is actually a base of the cotangent space:

Proposition 1.23 (Dual base of the tangent space). *Let $\mathcal{M}$ be a smooth manifold, $p \in \mathcal{M}$ and $(x_i)_i$ some local coordinates. The base of $T_p\mathcal{M}^*$, dual of $\left(\left. \frac{\partial}{\partial x^i} \right|_p \right)_i$ is $(d_p x^i)_i$*

Proof. Let's see:

$$d_p x^j \left(\left. \frac{\partial}{\partial x^i} \right|_p \right)(f) = \left. \frac{\partial}{\partial x^i} \right|_p (f \circ x^j) = \frac{\partial x^j}{\partial x^i} f'$$

Therefore $d_p x^j (\left. \frac{\partial}{\partial x^i} \right|_p)$ is 1 when $j = i$ and 0 when $j \neq i$. □

1.1.3 Submanifolds

As usual, we want to define a type of subset of a manifold which in some sense conserves its properties. To fulfill this objective, we introduce the concept of smooth submanifold. Unfortunately, it is not as straightforward as one might initially presume, since we need a way of relating the submanifold with its supermanifold, although this relationship is not too complex.

Definition 1.24 (Immersion)**.** *We say that a smooth map $f : \mathcal{M} \to \mathcal{N}$ is an* ***immersion*** *if $\forall p \in \mathcal{M}$, $d_p f$ is injective.*

Definition 1.25 (Embedding)**.** *We say that an immersion $\phi : \mathcal{M} \to \mathcal{N}$ is an* ***embedding*** *if ϕ is a homeomorphism between $\mathcal{M}$ and $\phi(\mathcal{M})$.*

Definition 1.26 (Submanifold)**.** *Let $\mathcal{M}$, $\mathcal{N}$ be smooth manifold, $\phi : \mathcal{N} \to \mathcal{M}$ an embedding. We say that (N, ϕ) is a* ***submanifold*** *of $\mathcal{M}$.*

There is a special type of submanifold which is going to be particularly important for us.

Definition 1.27 (Hypersurface)**.** *Let $\mathcal{M}$ be a smooth manifold of dimension n. A* ***hypersurface*** *of $\mathcal{M}$ is a submanifold of $\mathcal{M}$ of dimension $n-1$.*

Getting submanifolds, and even manifolds, can become really hard or laborious, and sometimes one might not be interested in local charts, but only on some special smooth manifold properties. To prove that a given set is a manifold, we want an easy way to get them. This way, even if it seems like magic, exists, and it relies on a beautiful theorem akin to the implicit function theorem. One may find the proof to this result in theorem (3.1.1) of [CB03] or corollary (5.14) of [Lee13].

Definition 1.28 (Regular value)**.** *Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map between manifolds. A point $q \in \mathcal{M}$ is a* ***regular value*** *of F if $\forall p \in F^{-1}(q)$, $d_p F$ is exhaustive. We say that $F^{-1}(q)$ is the* ***level set*** *of q.*

Theorem 1.29 (Regular value theorem)**.** *Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map between manifolds. Let $q \in \mathcal{M}$ be a regular value of F. Then the level set of q is a submanifold of $\mathcal{M}$, of dimension $dim\mathcal{M} - dim\mathcal{N}$.*

1.2 Vector and tensor fields

From here on, the summations will be getting worse, sometimes even with an open number of indexes. There is a notation which can help us lower the load on the notation: the Einstein summation convention. It establishes that whenever an index is repeated on a single term, there is a summation over all the values of that index.

Example 1.30. Let us give some examples, with the sum being over all the range of i or j:

- $a_i b_i = \sum_i a_i b_i$
- $a_{ij} b^i = \sum_i a_{ij} b^i$
- $a_{ij} b^i c_j = \sum_i \sum_j a_{ij} b^i c_j$

We will be using this notation from now on.

1.2.1 Vector fields and 1-forms

We can have some kind of object on manifolds assigning to each point of the manifold a vector in its tangent space, we will call that a vector field. In an intuitive way, we could have the vector field represent the flow of some liquid, or the direction of the hair of a "hairy" manifold in which all the hair has been combed in a way that no hair sticks out. Then we generally want the vector field to be smooth; we do not know what that means yet, but we will now define it.

Definition 1.31 (Vector field)**.** *Let $\mathcal{M}$ be a smooth manifold. We define a* ***vector field*** *over $\mathcal{M}$ as an assignment of every point $p \in \mathcal{M}$ to an element $X_p \in T_p\mathcal{M}$.*

Definition 1.32 (Smooth vector field)**.** *Let $\mathcal{M}$ be a smooth manifold of dimension n, X a vector field over $\mathcal{M}$. We say that X is a* ***smooth tensor field*** *if $\forall p \in \mathcal{M}$, in local coordinates, we can write:*

$$X = \lambda^i(x^j)\frac{\partial}{\partial x^i} = \lambda^i(x^1, \dots, x^n)\frac{\partial}{\partial x^i}$$

with λ^i differentiable. We will denote by $\Xi(\mathcal{M})$ the set of all smooth vector fields.

With the definition of vector fields comes a definition of an analogous in the cotangent space, so they, as vectors, also have their dual. These duals of the vector fields are called 1-forms.

Definition 1.33 (1-form)**.** *Let $\mathcal{M}$ be a smooth manifold of dimension n. We define a* ***1-form*** *as an assignment ω of every point $p \in \mathcal{M}$ to an element $\omega_p \in T_p\mathcal{M}$, such that in local coordinates we can write:*

$$\omega = \omega_i(x^j)dx^i$$

with ω_i differentiable. We will denote by $\Omega^1(\mathcal{M})$ the set of all 1-forms.

Remark 1.34. $\Omega^1(\mathcal{M})$ and $\Xi(\mathcal{M})$ are $\mathcal{F}(\mathcal{M})$-modules.

Remark 1.35. Notice that there is not a non-differentiable definition for 1-form. In fact, when we talk about vector fields, we will not be talking about non-smooth vector fields either, since they will not be of much utility for spacetime geometry.

In $\mathbb{R}^n$, it is easy to define any dimensional subspaces. Unfortunately, in a smooth manifold every point has its tangent space [2], which spices things up. Distributions appear to save the day as analogous to subspaces.

Definition 1.36 (Distribution)**.** *Let $\mathcal{M}$ be a smooth manifold. A **distribution** $\mathcal{D}$ of dimension r over $\mathcal{M}$ is an assignment of every point $p \in \mathcal{M}$ to a subspace $\mathcal{D}_p$, with dimension also r.*

*We say that $\mathcal{D}$ is **smooth** if $\forall p \in \mathcal{M}$ there is an open neighborhood U of p and r smooth vector fields $X_1, \ldots, X_r$ such that:*

$$\mathcal{D}_p =< X_1, \ldots, X_r > |_p$$

In some way, distributions are related to any dimensional subspaces as vector fields are related to vectors. The definition will be, nonetheless, of great importance, since distributions induce submanifolds, as k-subspaces induce k-linear manifolds. With that purpose we give the following definitions, which are going to be of key importance when we define a wormhole, since the submanifolds representing constant time sections at each point in time are generated by the space directions, and we want only this space sections to have these tunnels.

Definition 1.37 (Integral manifold)**.** *Let $(\mathcal{N}, i)$ be a submanifold of $\mathcal{M}$ a smooth manifold. We say that $\mathcal{N}$ is an **integral manifold** of a distribution $\mathcal{D}$ if it satisfies $di(T_p\mathcal{N}) = \mathcal{D}_{i(p)}$ $\forall p \in U$.*

Not all distributions create submanifolds. To exactly define which ones do induce new manifolds, we first need to define a mathematical artifact that can appear somewhat pathological at first glance.

Definition 1.38 (Lie bracket)**.** *Let $\mathcal{M}$ be a smooth manifold, we define its **Lie bracket** as the application:*

$$\begin{aligned} [\cdot\,,\cdot] : \Xi(\mathcal{M}) \times \Xi(\mathcal{M}) &\longrightarrow \Xi(\mathcal{M}) \\ (X, Y) \qquad &\longmapsto [X, Y] \end{aligned}$$

such that for every $f \in \mathcal{F}(\mathcal{M})$, it satisfies $[X, Y]f = X(Yf) - Y(Xf)$.

[2]Although they can be combined to form the tangent bundle

The Lie bracket is a particular case of a greater concept, the Lie derivative, which is defined for more mathematical objects than just vector fields. For now, we can stay with the idea that the Lie bracket is a kind of generalization to vector fields of the normal derivative.

Definition 1.39 (Involutive distribution)**.** *Let $\mathcal{M}$ be a smooth manifold, $\mathcal{D}$ a distribution over $\mathcal{M}$. We say that $\mathcal{D}$ is* ***involutive*** *if $\forall X, Y \in \mathcal{D}$, $[X, Y] \in \mathcal{D}$.*

The notions of involutive distribution and integral manifold are strongly related. The following results show that they are involutive distributions give arise to integral manifolds and vice versa. Their proofs can be found on proposition (19.3) and theorem (19.12), respectively, of [Lee13].

Proposition 1.40. *If on every $p \in \mathcal{M}$ there is an integral manifold of a distribution $\mathcal{D}$, then $\mathcal{D}$ is involutive.*

Theorem 1.41 (Frobenius theorem)**.** *If $\mathcal{D}$ is involutive, on every $p \in \mathcal{M}$ there is an integral manifold of $\mathcal{D}$.*

1.2.2 Tensor fields

Tensors are the core of General Relativity. They have a beautiful property; they do not depend on the local chart chosen. In physics, this translates to a non-dependence on the reference frame, which is one of the two principles of GR.

The intuitive idea of a tensor is a mathematical object which we feed any number of vectors, and we receive a number. Examples of this are a scalar product or linear forms. We ask the tensors to be linear on every input, which is a wonderful property, since, among other consequences, allows us to define a base for the tensor space. Notice that, defined this way, the previous property becomes obvious.

Definition 1.42 (Multilinear map)**.** *Let $E_1, \ldots, E_k, F$ be vector spaces. A map $f : E_1 \times \cdots \times E_k \to F$ is* ***multilinear*** *if it is linear on each variable when the others are fixed.*

Definition 1.43 (Tensor)**.** *Let E be a vector space. A (k, l)* ***tensor*** *over E is a multilinear application $T : \times^k E \times^l E^* \to \mathbb{R}$.*

By $\times^k E \times^l E^*$ we mean k copies of E and l copies of E^*. In fact, we see that we included the dual space in the tensor's input. In our framework, we will work with a scalar product, which as we know induces an isomorphism between the vector space and its dual space, so the choice of k and l will be merely a convention.

Definition 1.44 (Tensor product). *Let $E_1, \ldots, E_k, G_1, \ldots, G_l$ be vector spaces, F a field, $f : E_1 \times \cdots \times E_k \to F$ and $g : G_1 \times \cdots \times G_l \to F$ be multilinear maps. We define the tensor product of f and g as $f \otimes g : E_1 \times \cdots \times E_k \times G_1 \times \cdots \times G_l \to F$*

$$(f \otimes g)(u_1, \ldots, u_k, v_1, \ldots, v_l) = f(u_1, \ldots, u_k)g(v_1, \ldots, v_l)$$

Remark 1.45. In particular, the tensor product is defined for tensors, and it is straightforward to see that the result is another tensor.

Proposition 1.46 (Tensor vector space structure). *The set of all (k, l) tensors over a vector space E is also a vector space, and it is denoted $T_{(k,l)}(E)$.*

Proof. The fact that it is a vector space comes directly from the fact that the arrival set is $\mathbb{R}$. By using the standard properties of the sum and the product of $\mathbb{R}$ the eight axioms can be easily checked.

We can treat vectors $v \in E$ as linear applications $v : E^* \to \mathbb{R}$. Let $(e_i)_i$ be a base of E and $(\omega^i)_i$ its dual base. We propose as a base of $T_{(k,l)}(E)$ the set:

$$\{\omega^{j_1} \otimes \cdots \otimes \omega^{j_k} \otimes e_{i_1} \otimes \cdots \otimes e_{i_l} \in T_{(k,l)}(E) : 1 \leq j_1, \ldots, j_k, i_1, \ldots, i_l \leq n\}$$

The reader might be convinced on their own that the tensors in the set are linearly independent, by applying them to carefully selected elements of $\times^k E \times^l E^*$. To see that they generate $T_{(k,l)}(E)$, let $(v_1, \ldots, v_k, w_1, \ldots, w_k) \in \times^k E \times^l E^*$. This element can be decomposed into a linear combination of $k + l$-tuples of base elements of E and E^*. And since a tensor is multilinear, we end up with a linear combination of this tuples going through the tensor. But for this to be completely determined, it is enough with the given elements. $\square$

Now we want to do as we did with vectors and vector fields. To define tensor fields, we will be doing pretty much the same. Notice that we are interested in tensor fields mainly because of the notion of scalar products on manifolds, but we will see that in fact, differential geometry and general relativity have some other interesting tensor fields.

Definition 1.47 (Tensor field). *Let $\mathcal{M}$ be a smooth manifold of dimension n. We define a (k, l) **tensor field** over $\mathcal{M}$ as an assignment of every point $p \in \mathcal{M}$ to an element $K_p \in T_{(k,l)}(T_p\mathcal{M})$.*

Definition 1.48 (Smooth tensor field). *Let $\mathcal{M}$ be a smooth manifold of dimension n, K a (k, l) tensor field over $\mathcal{M}$. The local coordinates of K_p are:*

$$K_p = \lambda_{i_1,\ldots,i_k}^{j_1,\ldots,j_l} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_l}}$$

*We say that K is **smooth** if the functions $\lambda_{i_1,\ldots,i_k}^{j_1,\ldots,j_l}$ are.*

But now, we would like to define an extra object. When we have a smooth map between two manifolds, there is an induced way of transporting the tensor field from one manifold to the other. Note that we will be defining the pullback of tensor fields just for $(k, 0)$-smooth tensor field, since we are only interested in transporting our scalar products from one manifold to another.

Definition 1.49 (Pullback of tensor fields)**.** *Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds, K a $(k, 0)$-smooth tensor field. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map. The* ***pullback*** *of K, denoted $\phi^* K$ is defined as:*

$$(\phi^* K_p)(v_1, \dots, v_k) = K_p(d_p\phi(v_1), \dots, d_p\phi(v_k))$$

Lastly, we are interested in the Lie derivative of tensor fields. In particular, we are interested in the Lie derivative of $(2, 0)$-tensor fields. To avoid complicating things further, we will be only stating the local form of the Lie derivative of a $(2, 0)$-tensor field. If the tensor field is given by $g = g_{ij} dx^i \otimes dx^j$, then, its Lie derivative with respect to a vector field $X = X^k \partial_k$ is:

$$\mathcal{L}_X g = (X^k \partial_k g^{ij} + g^{kj} \partial_k X^i + g^{ik} \partial_k X^j) dx^i \otimes dx^j$$

At first, the interest in specifically $(2, 0)$-tensor fields might not be obvious, but the main point is to define the Lie derivative of metric tensors (our scalar product equivalent). There is a special kind of vector fields, called the **Killing vector fields**, which will be defined in more detail later. A Killing vector field X satisfies the Killing equation, $\mathcal{L}_X g = 0$. This means, in local coordinates:

$$X^k \partial_k g^{ij} + g^{kj} \partial_k X^i + g^{ik} \partial_k X^j = 0, \qquad \forall i, j$$

1.3 Connections

We will be defining a concept that will be core in our study of General Relativity. The idea of a connection is some kind of tool that connects - hence the name - local geometries. For a manifold, there are different ways of connecting its local geometries, so there are different connections available. We will see later, nevertheless, that introducing a metric locks the connection, which does in fact make sense.

Definition 1.50 (Connection)**.** *Let $\mathcal{M}$ be a smooth manifold. A* ***connection*** *is a map:*

$$\begin{aligned} \nabla : \Xi(\mathcal{M}) \times \Xi(\mathcal{M}) &\longrightarrow \Xi(\mathcal{M}) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

satisfying (for $f \in \mathcal{F}(\mathcal{M})$ and $X, X_1, X_2, Y, Y_1, Y_2 \in \Xi(\mathcal{M})$):

1. $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$
2. $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$
3. $\nabla_X(fY) = X(f)Y + f\nabla_X Y$
4. $\nabla_{fX} Y = f\nabla_X Y$

We need to prove now that the connection can be restricted to a chart, so we can find its expression on local coordinates. With that purpose, the next result is given. Its proof can be found in proposition (9.1.1) of [CB03].

Proposition 1.51 (Local connections). *Let ∇ be a connection on a manifold $\mathcal{M}$. For every open set $U \subset \mathcal{M}$, ∇ gives arise to a connection ∇^U on U, such that $\forall X, Y \in \Xi(\mathcal{M})$, $\nabla^U_{X_U} Y_U = \nabla_X Y|_U$.*

We will abuse notation and denote $\nabla^U = \nabla$.

Definition 1.52 (Christoffel symbols). *Let ∇ be a connection on a manifold $\mathcal{M}$. Let $(x_i)_i$ be some local coordinates (on a coordinate open set U). We define the* ***Christoffel symbols*** *of ∇ on U as Γ^k_{ij}, given by:*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

Proposition 1.53. *With the previous notation, the Christoffel symbols completely define ∇ on U.*

Proof. Let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$. We can see:

$$\nabla_X Y = X(Y^j)\frac{\partial}{\partial x^j} + X^i Y^j \Gamma^k_{ij} \frac{\partial}{\partial x^k} = (X(Y^k) + X^i Y^j \Gamma^k_{ij}) \frac{\partial}{\partial x^k}$$

which only depends on X, Y and the Christoffel symbols. $\square$

Another of the main benefits of defining a connection is the notion of geodesics. Geodesics are a special type of curve which satisfy a interesting relationship with the connection. We want to have them in mind since, in GR, free particles follow geodesics in the curved spacetime. We will see later that, once we have a metric, the connection is unique, so geodesics will be perfectly defined.

Definition 1.54 (Geodesic curve). *Let $\mathcal{M}$ be a smooth manifold with a connection ∇. A curve γ is a* ***geodesic*** *if it satisfies:*

$$\nabla_{\dot\gamma} \dot\gamma = 0$$

where $\dot\gamma$ is the tangent vector of γ.

We can finally define some interesting tensor fields, namely the torsion tensor and the curvature tensor. Their names are really well-chosen, since they represent exactly that. The torsion tensor is a tensor which in some way represents the torsion of geodesics, and the curvature tensor also represents in some way the curvature of these geodesics. Nevertheless, it is better to think of them as inherent properties of the manifold and not the geodesics.

Definition 1.55 (Torsion tensor). *Let $\mathcal{M}$ be a smooth manifold, ∇ a connection on $\mathcal{M}$. The* ***torsion tensor field*** *of ∇ is the tensor field $(2,1)$ given by:*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

Proposition 1.56. *The torsion field is actually a tensor field.*

Proof. We have to see that it is $\mathcal{F}(\mathcal{M})$-bilinear. Let's first see that it is linear on the first variable.

$$\begin{aligned} T(f_1X_1 + f_2X_2, Y) &= \nabla_{(f_1X_1+f_2X_2)}Y - \nabla_Y(f_1X_1 + f_2X_2) - [f_1X_1 + f_2X_2, Y] \\ &= f_1\nabla_{X_1}Y + f_2\nabla_{X_2)}Y - Y(f_1)X_1 - f_1\nabla_Y X_1 - Y(f_2)X_2 \\ &\quad - f_2\nabla_Y X_2 + Y(f_1)X_1 + Y(f_2)X_2 - [X_1,Y] - [X_2,Y] \\ &= f_1\,T(X_1,Y) + f_2\,T(X_2,Y) \end{aligned}$$

We used the properties of the Lie bracket of aditivity and $[fX,Y] = f[X,Y] - Y(f)X$, both of which can be easily checked.

And since $T(X,Y) = -T(Y,X)$, it must be bilinear. □

Definition 1.57 (Symmetric connection). *We say that a connection is* ***symmetric*** *if the torsion field is 0.*

Definition 1.58 (Curvature tensor). *Let $\mathcal{M}$ be a smooth manifold, ∇ a connection on $\mathcal{M}$. The* ***Riemann curvature tensor field*** *of ∇ is the tensor field $(3,1)$ given by:*

$$R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

Proposition 1.59. *The curvature field is actually a tensor field.*

Proof. As with the torsion field, it is enough to see that the given definition is $\mathcal{F}(\mathcal{M})$-multilinear. Since $R(X,Y,Z) = -R(Y,X,Z)$, we will check only X and Z. For X:

$$\begin{aligned}
R(f_1X_1 + f_2X_2, Y, Z) &= f_1\nabla_{X_1}\nabla_Y Z + f_2\nabla_{X_2}\nabla_Y Z - \nabla_Y f_1\nabla_{X_1} Z - \nabla_Y f_2\nabla_{X_2} Z \\
&\quad - \nabla_{[f_1X_1+f_2X_2,Y]} Z \\
&= f_1\nabla_{X_1}\nabla_Y Z + f_2\nabla_{X_2}\nabla_Y Z - f_1\nabla_Y\nabla_{X_1} Z - f_2\nabla_Y\nabla_{X_2} Z \\
&\quad - Y(f_1)\nabla_{X_1} Z - Y(f_2)\nabla_{X_2} Z - \nabla_{[X_1,Y]} Z - \nabla_{[X_2,Y]} Z \\
&\quad + Y(f_1)\nabla_{X_1} Z + Y(f_2)\nabla_{X_2} Z \\
&= f_1R(X_1,Y,Z) + f_2R(X_2,Y,Z)
\end{aligned}$$

And now, for Z:

$$\begin{aligned}
R(X,Y,f_1Z_1 + f_2Z_2) &= \nabla_X\nabla_Y(f_1Z_1 + f_2Z_2) - \nabla_Y\nabla_X(f_1Z_1 + f_2Z_2) \\
&\quad - \nabla_{[X,Y]}(f_1Z_1 + f_2Z_2) \\
&= \nabla_X(f_1\nabla_Y Z_1 + Y(f_1)Z_1) + \nabla_X(f_2\nabla_Y Z_2 + Y(f_2)Z_2) \\
&\quad - \nabla_Y(f_1\nabla_X Z_1 + X(f_1)Z_1) - \nabla_Y(f_2\nabla_X Z_2 + X(f_2)Z_2) \\
&\quad - f_1\nabla_{[X,Y]} Z_1 - [X,Y](f_1)Z_1 - f_2\nabla_{[X,Y]} Z_2 - [X,Y](f_2)Z_2 \\
&= f_1R(X,Y,Z_1) + f_2R(X,Y,Z_2)
\end{aligned}$$

□

Chapter 2

General relativity

Now we are ready to introduce the mathematical core of General Relativity. We join space and time together in a single manifold, the spacetime, and we introduce the concept of a metric. This works as one expects it to, we give every point in the manifold a metric for its tangent plane. Nevertheless, now the metric will not give us a real distance between points. In spacetimes, points are typically called **events**, because they represent, simply enough, a point in space and time. So the now "distance" measured will give us an idea of how much they are separated in causality (we are assuming there is a maximum speed, which in reality, there is: the speed of light).

For this, the sign of the scalar product of a vector with itself will be one if pointing into a causal direction, and other when pointing into a non-causal one. Don't worry if all of this sounds a bit too vague, since we are now formalizing all of these concepts.

2.1 Pseudo-Riemannian geometry

For this section, we will be following mainly [O'N83].

The first thing that must be determined is a way of distinguishing between space and time directions. For that purpose, the index of bilinear maps is used. The following result can be found in [O'N83]'s lemma (2.26).

Proposition 2.1 (Index invariance). *Every non-degenerate $(2, 0)$-tensor can be transformed by a change of basis to one whose matrix is diagonal with all entries being either $+1$ or -1, and the number of each is invariant.*

Definition 2.2 (Metric signature). *The pair (p, q), where p is the number of -1 and q the number of $+1$ is called the **metric signature**.*

Definition 2.3 (Metric tensor)**.** *Let $\mathcal{M}$ be a smooth manifold. A* ***metric tensor*** *(or, simply, metric) is a tensor field of type $(2,0)$ on $\mathcal{M}$ which satisfies:*

1. *g is symmetric*
2. *g is non-degenerate*
3. *Its metric signature is the same $\forall p \in \mathcal{M}$*

Definition 2.4 (Pseudo-Riemannian manifold)**.** *A* ***pseudo-Riemannian manifold*** *is a pair $(\mathcal{M}, g)$, where $\mathcal{M}$ is a smooth manifold and g is a metric tensor. The* ***metric signature*** *of $\mathcal{M}$ is the metric signature of g.*

Remark 2.5. In local coordinates $(x^i)_i$, since the metric is a $(2,0)$ tensor, it can be written $g = g_{ij} dx^i \otimes dx^j$, but since it is symmetric notation is typically abused, writing $g = g_{ij} dx^i dx^j$. In this new notation we can add together terms with a different order of the same indices (which will be the same by symmetry), we just need to keep this in mind if we want to switch back to the tensor notation.

Sometimes we also say $ds^2 = g = g_{ij} dx^i dx^j$, which will be our final notation. Also, we will place our indexes down if the metric is the usual, and for the inverse we will be placing them up as such g^{ij} [1].

Example 2.6. We can study the case of the spherical metric. We start from $\mathbb{R}^3$ and we subtract the center ($\mathbb{R}^3 \setminus \{0\}$). This is a manifold, since subtracting a point will not affect charts: open sets stay open sets and the local coordinates are one to one. We define the spherical coordinates, given by the inverse of the map: $\psi(r, \theta, \phi) = (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r \cos\theta)$. Notice that one chart is not enough to cover all $\mathbb{R}^3 \setminus \{0\}$ even if ψ will be the only application needed, so we will do a trick and define for each point (we can pick θ_0 and ϕ_0 as we want) a neighborhood: $U = \{\psi(r, \theta, \phi) : r \in (0, \infty),\ \theta \in (\theta_0 - \pi/2, \theta_0 + \pi/2),\ \phi \in (\phi_0 - \pi, \phi_0 + \pi)\}$.

Now the only difference between distinct charts is the open set, since with the same definition, the application ψ is a homeomorphism. We can compute the metric, and after some work, we get:

$$g = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

The metric does not depend on the chosen open set, since the coefficients will be the same (and, in fact, the tangent vectors will be the same as well).

[1] The convention states that for tensor coefficients we place indexes going $T_p\mathcal{M} \to \mathbb{R}$ down, and indexes going $T_p\mathcal{M}^* \to \mathbb{R}$ up, so repeated indexes should not be in the same position. It is possible to rise and lower indexes with the metric, which as we know induces an isomorphism $T_p\mathcal{M} \cong T_p\mathcal{M}^*$, so the meaning of the tensor does not vary. For example $T^i_j = g_{kj} T^{ik}$. We will not dive further into it, but formally, this is achieved by stating that we are working with equivalence classes of physically equivalent classes of tensors, instead of just tensors. More on the topic can be found in [SW77] section (1.0.1).

We need to consider that at times, our metric might become degenerate or singular in some local charts. This indicates the invalidity of the coordinates at that particular point. Consequently, we will need to either change coordinates or utilize a different chart for that specific point. Determining when we can execute such a change of coordinates or when the manifold cannot be extended is challenging. We will refer to these latter types of points as **singularities**.

In summary, zeros or singularities in the metric could either be artifacts or crucial missing points of the manifold. Differentiating between the two at a given point is a complex matter.

Proposition 2.7 (Levi-Civita connection uniqueness and existence)**.** *Let* $(\mathcal{M}, g)$ *be a pseudo-Riemannian manifold. Then there is a unique connection* ∇ *such that* $\forall X, Y, Z \in \Xi(\mathcal{M})$*:*

1. $T^{\nabla}(X, Y) = 0$ *(Symmetry)*
2. $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ *(Metric compatibility)*

Proof. We can express the metric compatibility property with 3 different permutations of the vector fields X, Y, Z.

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

We can use them as follows:

$$\begin{aligned} Xg(Y,Z) + Zg(X,Y) - Yg(Z,X) &= g(\nabla_Z Y - \nabla_Y Z, X) + g(\nabla_Z X + \nabla_X Z, Y) \\ &\quad + g(\nabla_X Y - \nabla_Y X, Z) \\ &= g([X,Y],Z) + g([Z,Y],X) + g([X,Z],Y) \\ &\quad + 2g(\nabla_Z X, Y) \end{aligned}$$

Therefore, moving terms around:

$$\begin{aligned} g(\nabla_Z X, Y) = &\frac{1}{2}\Big[Xg(Y,Z) + Zg(X,Y) - Yg(Z,X)\Big] \\ &- \frac{1}{2}\Big[g([X,Y],Z) + g([Z,Y],X) + g([X,Z],Y)\Big] \end{aligned}$$

Now, since at any given point $g(X, \dot{)}$ is an isomorphism whenever g is non-degenerate, it is clear that the vector field $\nabla_Z X$ exists and is unique. It is not very hard to check that this expression actually satisfies the defined properties of a connection. □

Definition 2.8 (Levi-Civita connection). *The previously stated connection is called the* ***Levi-Civita connection****.*

Proposition 2.9 (Christoffel symbols of the Levi-Civita connection). *The local form of the Christoffel symbols of the Levi-Civita connection is:*

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\lambda} (\partial_j \, g_{i\lambda} + \partial_i \, g_{\lambda j} - \partial_\lambda \, g_{ij})$$

where $g^{k\lambda}$ represents the inverse of the metric.

Proof. By using the formula deduced in the proof of proposition (2.7), we get:

$$g(\nabla_{\partial_i} \partial_j, \partial_k) = \frac{1}{2} \left[\partial_i g(\partial_k, \partial_j) + \partial_j g(\partial_i, \partial_k) - \partial_k g(\partial_j, \partial_i) \right]$$

since the Lie brackets vanish. By multiplying by the inverse of the metric, we get the formula. □

Notice that the symbols are always the same when we transpose the subindexes, since the metric tensor is, by definition, symmetric. This is key to save time when we have to compute $4^3 = 64$ symbols for a 4-dimensional pseudo-Riemannian manifold, which is generally a non-pleasant task.

Remark 2.10. The Christoffel symbols will be continuous functions as long as the metric is C^1, which will always be the case for us.

Definition 2.11 (Induced metric). *Let $\mathcal{M}$ be a pseudo-Riemannian manifold, and $\mathcal{N}$ a smooth submanifold, with $i : \mathcal{M} \hookrightarrow \mathcal{N}$ the inclusion map. The pullback of the metric of $\mathcal{M}$ by i, that is $i^*(g)$ is called the* ***induced metric*** *whenever it is a metric.*

Notice that in matrix notation, if we have the pushforward di and the metric g, the matrix of the induced metric is $di \cdot g \cdot di^T$. Furthermore, since the pushforward has the local expression of the tangent vectors in each local coordinate, we can compute the local components of the induced metric through the scalar product of this tangent vectors

Definition 2.12 (Pseudo-Riemannian submanifold). *With the above notation, if there is an induced metric, $(\mathcal{N}, i^*(g))$ is a* ***pseudo-Riemannian subamnifold****.*

We will find later some cases on which a smooth submanifold might not be a pseudo-Riemannian submanifold because the pullback of the metric is not actually a metric.

Proposition 2.13 (Product manifold)**.** *Let* $(\mathcal{M}, g_{\mathcal{M}})$, $(\mathcal{N}, g_{\mathcal{N}})$ *be two pseudo-Riemannian manifolds, with* π, σ *the projections of* $\mathcal{M} \times \mathcal{N}$ *onto* $\mathcal{M}$ *and* $\mathcal{N}$ *respectively. Then:*

$$g = \pi^*(g_{\mathcal{M}}) + \sigma^*(g_{\mathcal{N}})$$

is a metric tensor on $\mathcal{M} \times \mathcal{N}$*, and therefore* $\mathcal{M} \times \mathcal{N}$ *it is a pseudo-Riemannian manifold.*

Proof. Let $v, w \in T_{(p,q)}(\mathcal{M} \times \mathcal{N})$. From the definition of pullback:

$$g(v, w) = g_{\mathcal{M}}(d\pi(v), d\pi(w)) + g_{\mathcal{N}}(d\sigma(v), d\sigma(w))$$

Then g must be symmetric. We want to see now non-degeneracy, so if there is any vector v such that $g(v, w) = 0,\ \forall w$. For a vector w with all of its nonzero coordinates on $\mathcal{M}$, it follows $d\sigma(w) = 0$, which means $g_{\mathcal{M}}(d\pi(v), d\pi(w)) = 0$. But then, $d\pi(w)$ can be any vector on $T_p(\mathcal{M})$. Therefore $d\pi(v) = 0$. Similarly, we get $d\sigma(v) = 0$, so it must be $v = 0$.

We have just left the constant index. But the orthonormal bases combine into a new orthonormal base. Hence, the index equals the sum of both indices, and it is constant. □

We want to define a special type of vector fields, related with the metric:

Definition 2.14 (Killing vector field)**.** *Let* $\mathcal{M}$ *be a smooth manifold. We say that a vector field* X *is a* ***Killing vector field*** *if* $\mathcal{L}_X g = 0$.

We are basically stating that the lie derivative of the metric tensor in the direction of the field X is zero. Notice that this must mean that, in some way, the metric does not change.

And now, an important special case:

Definition 2.15 (Lorentzian manifold)**.** *An n-dimensional* ***Lorentzian manifold*** *is a pseudo-Riemannian manifold with a metric of signature* $(1, n-1)$. *The same way, we also say that some metric is* ***Lorentzian*** *if its signature is* $(1, n-1)$.

This is the type of metric we will be dealing with from now on. The metric signature deserves its justification. The idea here is, as we said, the causality. By introducing one time dimension with an opposite sign as the spatial ones, we are making it so that the module of vectors pointing into the time direction and into the spatial ones have different sign.

This last definition will be of much use later on.

Definition 2.16 (Isometry)**.** *Let* $(\mathcal{M}, g_{\mathcal{M}})$, $(\mathcal{N}, g_{\mathcal{N}})$ *be pseudo-Riemannian manifolds. A diffeomorphism* $\phi : \mathcal{M} \to \mathcal{N}$ *is an* ***isometry*** *if it preserves the metric, that is,* $\phi^*(g_{\mathcal{N}}) = g_{\mathcal{M}}$. *We say, then, that* $\mathcal{M}$ *and* $\mathcal{N}$ *are* ***isometric***.

2.2 Spacetime geometry

For the geometry of spacetimes we will be following [HE75] and [SW77].

Definition 2.17 (Lorentzian spacetime)**.** *We define an n-dimensional* ***Lorentzian spacetime*** *as a connected n-dimensional Lorentzian manifold.*

Unless stated otherwise, a Lorentzian spacetime, or just spacetime, will be 4-dimensional.

Now we are going to give several definitions which seem pretty similar, but all of them will be necessary. The definitions have a physical relation of causality; if two different objects are given the same causal character, that will be because in some way they will mean similar things.

Definition 2.18 (Vector causal character)**.** *Let $\mathcal{M}$ be a Lorentzian spacetime, with g its metric. Let $p \in \mathcal{M}$, and $v \in T_pM$. Then:*

1. *If $g_p(v, v) < 0$, $\mathcal{N}$ is said to be* ***timelike****.*
2. *If $g_p(v, v) = 0$ and $v \neq 0$, $\mathcal{N}$ is said to be* ***lightlike****.*
3. *If $g_p(v, v) > 0$ or $v = 0$, $\mathcal{N}$ is said to be* ***spacelike****.*

We say that v is ***causal*** *if it is not spacelike.*

Definition 2.19 (Submanifold causal character)**.** *Let $\mathcal{M}$ be a Lorentzian spacetime, and let $\mathcal{N}$ be a submanifold. The pullback of the metric of $\mathcal{M}$ induces a new tensor on $\mathcal{N}$, let us denote it by g. Then:*

1. *If g is Lorentzian, $\mathcal{N}$ is said to be* ***timelike****.*
2. *If g is degenerate, $\mathcal{N}$ is said to be* ***lightlike****.*
3. *If g is definite positive, $\mathcal{N}$ is said to be* ***spacelike****.*

We say that $\mathcal{N}$ is ***causal*** *if it is not spacelike.*

Remark 2.20. A submanifold is only a pseudo-Riemannian submanifold if it is not lightlike, since in this case, g is not a metric tensor.

Definition 2.21 (Curve causal character)**.** *Let $\mathcal{M}$ be a Lorentzian spacetime. A* ***timelike/lightlike/spacelike/causal curve*** *is a curve whose tangent vector at each point is timelike/lightlike/spacelike/causal respectively.*

Definition 2.22 (Vector field causal character)**.** *Let $\mathcal{M}$ be a Lorentzian spacetime. A* ***timelike/lightlike/spacelike/causal vector field*** *is a vector field X which satisfies that $X_p \in T_p\mathcal{M}$ is timelike/lightlike/spacelike/causal respectively $\forall p \in \mathcal{M}$.*

Now a definition on the global character of spacetimes:

Definition 2.23 (Static spacetime). *Let $\mathbb{M}$ be a Lorentzian spacetime. We say that $\mathbb{M}$ is **static** if it admits a timelike Killing vector field whose orthogonal distribution is involutive.*

2.3 Einstein's field equations

Before giving the equation of General Relativity, we need to define some more curvature concepts. Notice that they might have stronger or weaker conditions for its definition, but it is better to have all of them clustered on here since their utility will be basically the field equations. Nevertheless, they will all obviously be defined in our Lorentzian spacetimes.

Definition 2.24 (Ricci tensor). *Let $\mathcal{M}$ be a smooth manifold, ∇ a connection on $\mathcal{M}$. The **Ricci curvature tensor field** of ∇ is the tensor field $(2,0)$ given at a point $p \in \mathcal{M}$ by the coefficients (in some local coordinates):*

$$Ric_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

Definition 2.25 (Scalar curvature). *Let $\mathcal{M}$ be a pseudo-Riemannian manifold. The **scalar curvature tensor field** of $\mathcal{M}$ is the tensor field $(0,0)$ given by (in some local coordinates):*

$$S = g^{\mu\nu} Ric_{\mu\nu}$$

The reader must remember that the repeated indices are the ones over which the summation is carried on. Implicit (not local) definitions can be given, but they become overly complicated, and they do not give anything in return.

Definition 2.26 (Einstein tensor). *Let $\mathcal{M}$ be a pseudo-Riemannian manifold. We define the **Einstein tensor field** of $\mathcal{M}$ as:*

$$G = Ric - \frac{1}{2} g S$$

Definition 2.27 (Einstein field equations). *Let $\mathbb{M}$ be a Lorentzian spacetime. An **Einstein field equation** is an equation $G = T$ for some tensor T. The **vacuum Einstein field equation** is an Einstein field equation with $T = 0$. If $\mathbb{M}$ satisfies the vacuum Einstein field equation, it is called an **Einstein manifold**.*

2.4 Solutions

The **Minkowski spacetime** is a really straightforward solution to the vacuum Einstein equation: that is, we choose $\mathbb{R}^4$ with the metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. Notice how the Riemann tensor vanishes, since the Christoffel symbols vanish, and consequently the Einstein tensor is exactly 0, condition for the Einstein vacuum equations.

Now we want something a bit more spicy, since, although Minkowski spacetime is the basis for special relativity, a theory which is interesting enough by itself, we want some curvature involve. We did not need to introduce all of these concepts to work on a flat spacetime.

The **Schwarzschild spacetime** is, by Birkhoff's theorem, the only spherically symmetric solution for the vacuum Einstein field equation. A beautiful, rigorous proof of this result can be seen at [vO19]. The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{1}{1 - \frac{r_s}{r}} dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$, all the spatial part (plus sign in the metric) in spherical coordinates.

We may notice two things: when $r_s = 0$, the metric is just Minkowski's, and the metric coefficients of dt^2 and dr^2 can go to zero and diverge at $r = 0$ and $r = r_s$.

Chapter 3

Wormholes and time travel

Here we get to the part that we were interested in: wormholes. For this section, we used the main text book on the topic: [Vis95]. Unfortunately, this book goes deeper into the physical meaning and stays a bit shallower than we want in the mathematical formalism. So we took the definition in there and rewrote it in a more rigorous way, to prove afterward that some old known wormholes are in fact wormholes with the definition.

We give then the following definition, based on definition (9.11) of [Vis95].

Definition 3.1 (Static wormhole)**.** *Let $\mathbb{M}$ be a static Lorentzian spacetime of dimension n. We say that an open set Ω contains a **static wormhole** if every integral submanifold of the distribution orthogonal to the timelike field is homotopy equivalent to S^{n-2}.*

We will be working only with static wormholes. Although the definition could be generalized, we did not find it necessary, since the wormholes that we will be studying will be static.

To understand a bit better the definition, what we are asking the wormhole to be, in an intuitive way, is that "it has a throat in the space dimensions".

Einstein field equations give us only a metric. We can play with this metric and find the curvature (either Riemann, Ricci or scalar). Nevertheless, our definition contains a purely topological term, homotopy. This seemingly simple problem is of greater magnitude than it first seems, and on here lies the actual debate on the existence of wormholes.

On a 2-dimensional Riemannian manifold, which has a positive definite metric, instead of non-degenerate, we have the beautiful Gauss-Bonnet theorem, which states that the Euler characteristic can be obtained from purely geometrical terms. Consequently, the possibility of completely characterizing topological surfaces from only a geometrical perspective appears.

But we are working (in general) with a higher dimensional pseudo - Riemannian manifold. Then, we do not have access to something as powerful as the Gauss-Bonnet theorem, but we do have access to an analogous, given by Chern in [Che63]. This article involves algebraic topology (including de Rham cohomology), completely out of the scope of the paper. Nevertheless, the theorem shows that the complete topological characterization is not generally possible, and therefore General Relativity has some freedom of topology.

For the case of wormholes, this implies that we might have some metric which can be the metric of a wormhole, but it might not be. Nevertheless, the following definition is important, because we need to know if a metric can be a wormhole metric.

Definition 3.2 (Wormhole metric)**.** *We say that a metric is a* ***static wormhole metric*** *if there is a pseudo-Riemannian manifold with that metric which is a static wormhole.*

Remark 3.3. If we have some metric of the type $ds^2 = -dt^2 + g(x, y, z)$, with g definite positive, we will have a static spacetime: it is easy to see that ∂_t is a killing vector field, and that $< \partial_x, \partial_y, \partial_z >$ is an orthogonal distribution. Furthermore, by setting t constant we get a manifold (from the regular value theorem, by setting $F(t, x, y, z) = t$ any value is a regular value, since $dF = (1, 0, 0, 0)$). It is obvious that these manifolds are integral manifolds of the distribution, so it must be involutive. Additionally, these manifolds are spacelike.

3.1 Ellis wormhole

The Ellis wormhole is a particular case of a much broader wormhole family, the Ellis drainhole, treated by Ellis in [Ell03]. It is the earliest known traversable wormhole, that meaning that it can be traversed in a finite amount of time (this term will be considered explained further in section (3.4)).

Definition 3.4 (Ellis wormhole metric)**.** *The Ellis wormhole metric is defined as:*

$$ds^2 = -dt^2 + d\rho^2 + (\rho^2 + n^2)d\Omega^2$$

We will define a 4-manifold embedded into a higher dimension Euclidean space in hopes that the induced metric arises a wormhole metric. This method will also provide us with a direct way of visualizing the wormhole.

Proposition 3.5. *The Ellis wormhole metric is a wormhole metric.*

Proof. We will be defining a subset of $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ with the metric $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) + dw^2$, which is a pseudo-Riemannian manifold, consequently of being a product of pseudo-Riemannian manifolds. In particular, we may define $r = \sqrt{\rho^2 + n^2}$, $w = n\,\text{arcsinh}(\rho/n)$. Let us write the subset explicitly:

$$\{(t, r, \theta, \phi, w) : r = \sqrt{\rho^2 + n^2},\ w = n\,\text{arcsinh}(\rho/n),\ \rho \in \mathbb{R}\}$$

As in spherical coordinates, we have to take into account that we need more than one chart to cover this, since we cannot have a single chart that "covers" all the domain of θ and ϕ. Nevertheless, we will be using another approach to show that this is in fact a manifold, the regular value theorem.

Notice that, since $\rho = n\sinh(w/n)$, it must be $r = n\cosh(w/n)$. If we define $F(t, r, \theta, \phi, w) = r - n\cosh(w/n)$, it is clear that the subset is the level set of 0, that is, $F^{-1}(0)$. We just need to see that d_pF is exhaustive $\forall p \in F^{-1}(0)$. If we compute it, $d_pF = (0, 1, 0, 0, \sinh(w/n))$, which cannot be zero, so it must have range 1, and therefore it must be exhaustive.

Then, we have a 4-manifold, our (yet to prove) spacetime:

$$F^{-1}(0) = \mathbb{R} \times \{(r, \theta, \phi, w) : r = n\cosh(w/n)\}$$

Now we are getting closer. For the next part, we will be finding the induced metric onto our 4-manifold (which will induce another trivially onto $\Sigma = \{t_0\} \times \{(r, \theta, \phi, w) : r = n\cosh(w/n)\}$). This way we will show that we have a spacetime.

We will use the charts given by $\psi^{-1}(t, \rho, \theta, \phi) = (t,\ \sqrt{(\rho^2 + n^2)},\ \theta,\ \phi,\ n\,\text{arcsinh}(\rho/n))$, and an open set $U = \{\psi(t, r, \theta, \phi) : t \in \mathbb{R},\ r \in (0, \infty),\ \theta \in (\theta_0 - \pi/2, \theta_0 + \pi/2),\ \phi \in (\phi_0 - \pi, \phi_0 + \pi)\}$, for any $\theta \in [-\pi/2, \pi/2)$, $\phi \in [0, 2\pi)$. It is not hard to check that these are in fact charts. We can compute the tangent vectors:

$$\begin{aligned}
\partial_t &= (1, 0, 0, 0, 0) \\
\partial_\rho &= (0, \frac{\rho}{\sqrt{\rho^2 + n^2}}, 0, 0, \frac{n}{\sqrt{\rho^2 + n^2}}) \\
\partial_\theta &= (0, 0, 1, 0, 0) \\
\partial_\phi &= (0, 0, 0, 1, 0)
\end{aligned}$$

We can now compute the induced metric from the metric given at the start. And, surprise, we get (taking into account that $r = \sqrt{\rho^2 + n^2}$):

$$ds^2 = -dt^2 + d\rho^2 + (\rho^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

We have an isometric 4-manifold to the Ellis (yet to prove) wormhole. It clearly has Lorentzian signature. Moreover, we know that the Ellis metric

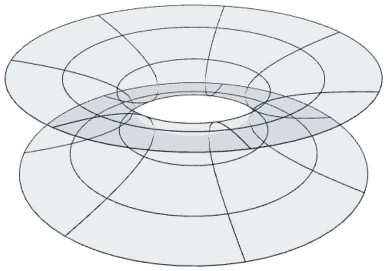

Figure 3.1: Ellis wormhole metric visualization with some fixed parameters to embed it into $\mathbb{R}^3$

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is static from the remark (3.3), and that ∂_t is the timelike Killing field, and Σ is the orthogonal distribution.

For the connection, we will see that it is arc connected: any two points (picking a chart on which both points lie) can be connected by the concatenation of the curves moving independently and continuously on each coordinate. Notice that this concatenation must be continuous, since the curve lies on the manifold always. So the manifold is a spacetime.

Finally, we just have left to prove that Σ is homotopy equivalent to S^2. For this we will find a deformation retract of Σ homeomorphic to S^2. Any sphere will be enough for this purpose, since the rest is a trivial homeomorphism. For that, we will choose the one with radius n.

Let us choose a point $(\rho_0, \theta_0, \phi_0)$. Our continuous map will be $r(s) = (\rho_0 s, \theta_0, \phi_0)$. It is really easy to check that this map "carries" the point at $s = 1$ given by $(\sqrt{(\rho_0^2 + n^2)},\ \theta_0,\ \phi_0,\ n \operatorname{arcsinh}(\rho_0/n))$ to the point $(n,\ \theta_0,\ \phi_0,\ 0)$ at $s = 0$, which is on the sphere of radius n. It is clear that $r(s) \in \Sigma$ and it is obviously continuous. So we just proved that Σ is homotopy equivalent to S^2.

So the Ellis wormhole metric is actually a wormhole metric. $\square$

3.2 Morris-Thorne wormhole

Morris and Thorne proposed a general, spherically symmetric wormhole metric in [MT88], and we would like to show that their general metric can be used to define a wormhole with our previous definition.

For this one, we will not be working with an embedding. Although it is a much clearer way of showing that some metric is in fact a wormhole metric, and there is in fact an embedding that we found for this metric, it requires the mathematical treatment of degenerate and/or singular metrics, which is outside the scope of this project.

Definition 3.6 (Traversable wormhole metric)**.** *The **Morris-Thorne wormhole metric** is defined as:*

$$ds^2 = -e^{2a(r)}dt^2 + \frac{1}{1 - \frac{b(r)}{r}}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

*where $a, b : \mathbb{R}_{>0} \to \mathbb{R}$ are smooth functions. The function a is the **redshift function** and b is the **shape function**. This last one must satisfy:*

1. *b has exactly one point $r_0 \in \mathbb{R}_{>0}$ such that $b(r_0) = r_0$.*
2. *$b(r) - b'(r)r > 0$, where b' is the derivative of b.*

To find an embedding, we could use a similar approach as [Fro59] (who gives the name to the embedding) and [PS12], who did it in a more formal way and even gave a general method to find embeddings. We could define a subset of $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ with the metric $ds^2 = -d\tau_0^2 + d\tau_1^2 + dw^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$, so the metric signature is $(-, +, +, +, +, +)$, which is a pseudo-Riemannian manifold, consequently of being a product of pseudo-Riemannian manifolds. As we said, we did not give enough tools to correctly work with the embedding, but it could be found (and it can be a nice exercise) with:

$$\begin{aligned}
\tau_0 &= e^{a(r)} \sinh t \\
\tau_1 &= e^{a(r)} \cosh t \\
w &= \pm \int_{r_0}^{r} \sqrt{\frac{1}{\frac{r}{b(v)} - 1} - (a'(v)e^{a(v)})} dv
\end{aligned}$$

The $\pm$ sign comes from the fact that this metric is thought to be use with two charts.

As we said, we will be doing a workaround. Before doing anything, we can see that our choice of coordinates is not the best: the coefficient of the dr^2 term diverges at $r = r_0$, so it is not well-defined. Furthermore, and as previously stated, this metric is thought to be used with two charts; one for each side of the wormhole. To deal with this, we can make a change of coordinates. Notice that we do not expect for the change of coordinate to be one to one, since we are joining two charts in one. We may define:

$$l = \pm \int_{r_0}^{r} \frac{dv}{\sqrt{1 - \frac{b(v)}{v}}}$$

Notice that $l = 0$ when $r = r_0$ in any case, and it can be checked using the given properties of b that this integral does in fact converge. What we do need is the $r(l)$ is defined. For that we define the function:

$$F(r,l) = l^2 - \left(\int \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \right)^2$$

And by the implicit function theorem we just need to see that $dF/dr \neq 0$. We may compute it:

$$\frac{dF}{dr} = \frac{-2}{\sqrt{1 - \frac{b(r)}{r}}} \int \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}} \neq 0 \quad \forall r \neq r_0$$

and

$$\frac{dF}{dr} \xrightarrow{r \to r_0} \frac{r_0^2}{b(r_0) - b'(r_0) r_0} \neq 0$$

So $r(l)$ is well-defined. The metric transforms in this local coordinates to:

$$ds^2 = -e^{2A(l)} dt^2 + dl^2 + r^2(l)(d\theta^2 + sin^2\theta d\phi^2)$$

With $A(l) = a(r(l))$.

Definition 3.7. *The* ***proper distance form*** *of the Morris-Thorne wormhole metric is:*

$$ds^2 = -e^{2A(l)} dt^2 + dl^2 + r^2(l)(d\theta^2 + sin^2\theta d\phi^2)$$

and the coordinate l is called the ***signed proper radial distance****.*

Now our metric seems more manageable.

Proposition 3.8. *The Morris-Thorne wormhole metric is a wormhole metric.*

Proof. We can define the smooth manifold $\mathbb{R} \times \mathbb{R} \times S^2$ and, with the proper distance form of the Morris-Thorne wormhole metric, we have a Lorentzian spacetime.

Notice how ∂_t is a timelike killing vector field, since $\mathcal{L}_{\partial_t} g = 0$, and $< \partial_l, \partial_\theta, \partial_\phi >$ is orthogonal to it. It is also clear that this distribution is involutive, and that the induced metric onto the integral manifolds Σ_t is:

$$ds^2 = dl^2 + r^2(l)(d\theta^2 + sin^2\theta d\phi^2)$$

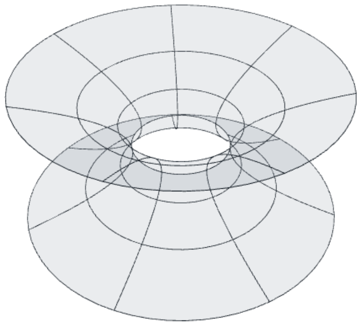

Figure 3.2: Einstein-Rosen metric visualization with some fixed parameters to embed it into $\mathbb{R}^3$

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These manifolds are spacelike. Furthermore, we can define a retraction such that $l = l_0 s$, which when $s = 0$ makes $l = 0$ and when $s = 1$, $l = l_0$. This is a continuous map, so the result is a deformation retract, and it is clearly S^2. □

Remark 3.9. The Ellis wormhole is a Morris-Thorne wormhole; if we define $\rho = l$, then $r(l) = l^2 + n^2$, $b(r) = \frac{n^2}{r}$ and $a(r) = 0$.

Now we might also consider the Scwharzschild metric again:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{1}{1 - \frac{r_s}{r}} dr^2 + r^2 d\Omega^2$$

It also has the form of a wormhole:

Remark 3.10. The Schwarzschild metric is a wormhole metric with $b(r) = r_s$ and $a(r) = \frac{1}{2} log(1 - r_s/r)$.

We just proved that Schwarzschild black holes are actually static wormholes! They are in fact called Einstein-Rosen bridges, discovered by L. Flamm in [Fla15], in 1916, and given the name by [ER35], who also gave the Einstein-Rosen metric. They used a similar approach as we did: they used $u^2 = r - r_s$, and they got the **Einstein-Rosen metric**:

$$ds^2 = -\frac{u^2}{u^2 + r_s} dt^2 + 4(u^2 + r_s) du^2 + (u^2 + r_s)(d\theta^2 + sin^2\theta d\phi^2)$$

We might now calculate the Christoffel symbols of the Morris-Thorne wormhole metric. We will prefer to compute them in the classical form, rather than in the proper distance form, just because it is the most commonly used chart. Nevertheless, if we were to work with geodesics traversing $l = 0$, the rigorous way would be to use the latter one. By using proposition 2.9, we can compute them:

$$
\begin{aligned}
\Gamma^t_{tr} &= a'(r) \\
\Gamma^r_{tt} &= a'(r)e^{2a(r)}\left(1 - \frac{b(r)}{r}\right) \\
\Gamma^r_{rr} &= \frac{1}{2}\frac{b'(r)r - b(r)}{r(r - b(r))} \\
\Gamma^r_{\theta\theta} &= b(r) - r \\
\Gamma^r_{\phi\phi} &= (b(r) - r)\sin^2\theta \\
\Gamma^\theta_{r\theta} &= \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} &= -\cos\theta\sin\theta \\
\Gamma^\phi_{r\phi} &= \frac{1}{r} \\
\Gamma^\phi_{\theta\phi} &= \frac{\cos\theta}{\sin\theta}
\end{aligned}
$$

With the remaining ones being either zero or a permutation of the subindexes which results on the symbols being equal. With this calculation done, we can now work with geodesics on Morris-Thorne wormholes! We can take it a step further, and compute the Einstein tensor, to find the needed energy-momentum tensor for our wormhole.

$$
\begin{aligned}
G_{tt} &= e^{-2a(r)}\frac{b'(r)}{r^2} \\
G_{rr} &= \left(1 - \frac{b(r)}{r}\right)\left[-\frac{b(r)}{r^3} + 2\left(1 - \frac{b(r)}{r}\right)\frac{a'(r)}{r}\right] \\
G_{\theta\theta} &= \frac{1}{r^2}\left(1 - \frac{b(r)}{r}\right)\left[a''(r) + \left(a'(r) + \frac{1}{r}\right)\left(a'(r) + \frac{b(r) - b'(r)r}{2r(r - b(r))}\right)\right] \\
G_{\phi\phi} &= \frac{G_{\theta\theta}}{\sin^2\theta}
\end{aligned}
$$

And the remaining coefficients are 0. This Einstein tensor is equal to the energy momentum tensor, so what we actually found is the energy distribution necessary to match the Morris-Thorne wormhole metric. Notice that, as we said, topology is not included in the equations, so we can only say that if a Morris-Thorne wormhole was actually found, it would have this energy-momentum tensor.

It can be checked that with $b(r) = r_s$ and $a(r) = \frac{1}{2}log(1 - r_s/r)$, that is, the Schwarzschild wormhole, the Einstein tensor vanishes, as we expected (the reader should remember that the Schwarzschild solution is a vacuum solution).

A more interesting result can be obtained by computing the Einstein tensor of the Ellis wormhole. By setting $a(r) = 0$ and $b(r) = \frac{n^2}{r}$, we get the following Einstein tensor:

$$
\begin{aligned}
G_{tt} &= -\frac{n^2}{r^4} \\
G_{rr} &= \frac{(n^2 - r^2)n^2}{r^6} \\
G_{\theta\theta} &= \frac{n^2}{r^6} \\
G_{\phi\phi} &= \frac{n^2}{r^6 \sin^2\theta}
\end{aligned}
$$

Now, the first component is negative. This component of the energy-momentum tensor, in physics, stands for the energy density[1], and it is negative [2]. Here is one of the complications of building wormholes, they need negative energy density. This state of matter is generally called *exotic matter*.

Just to let know the reader that not all is lost, it should be stated that there are way rounds to this problem. The most brought up solution to this problem lies in the domain of quantum field theory, the *Casimir effect* (see [Vis95] section (12.3.2)), which allows for negative energy density under certain conditions.

3.3 Time travel

Time travel is also something appearing with General Relativity. In fact, the theory alone, as with wormholes, does not forbid going back in time. Obviously, this implies a direct break in causality, as everybody has discussed in any interesting social situation. This section aims to give some

[1]In fact, it does stand for the energy density but only in an orthonormal reference frame. For the case of the Ellis wormhole, the component stays the same, so there is no need to worry further. The skeptical reader may check on their own, since it is not a hard exercise.

[2]The reader may have noticed that the energy density may change when changing coordinates. The problem lies, anyway, in the fact that there are some coordinates in which the density is negative.

insight into what role does time travel play in the scene, with a bit of a hint on its relation to wormholes.

For this small section, [Vis95] was used, in combination with [HE75].

Definition 3.11 (Time-orientable spacetime). *A Lorentzian spacetime $\mathbb{M}$ is* ***time-orientable*** *if it admits a timelike vector field.*

Remark 3.12. Notice that any static spacetime is trivially a time-orientable spacetime.

Definition 3.13 (Future and past directed curves). *Let $\mathbb{M}$ be a time-orientable Lorentzian spacetime. A* ***future directed curve*** *is a causal curve γ such that $g_p(\dot\gamma(p), T_p) < 0$ at any given point $p \in \mathbb{M}$. Here $\dot\gamma$ is the tangent vector and T is the killing timelike vector field defined by the time-orientability condition.*

We can analogously define ***past directed curve****, with $g_p(\dot\gamma(p), T_p) > 0$.*

Definition 3.14 (Causal future and past). *Let $\mathbb{M}$ be a time-orientable Lorentzian spacetime. We define the* ***causal future*** *of $p \in \mathcal{M}$ as the set of points $q \in \mathcal{M}$ such that there is a future-directed causal curve γ such that $\gamma(0) = p$ and $\gamma(1) = q$, and it is denoted $J^+(p)$.*

We can analogously define ***causal past****, denoted by $J^-(p)$.*

Definition 3.15 (Causality violation region). *We define the* ***causality violation region*** *of a point $p \in \mathbb{M}$ a time-orientable spacetime as $J^0(p) = J^-(p) \cap J^+(p)$. We define the causality violation region of the whole spacetime as $J^0(\mathbb{M}) = \bigcup_{p\in\mathbb{M}} J^0(p)$.*

In a physically intuitive way, time travel will be possible if the causality violation region of the spacetime is non-empty.

Remark 3.16. Notice that if we find any closed causal curves, we will have a non-empty causality violation region.

Although not with the mathematical rigor we aim for in this paper, [MTY88] shows that *a wormhole can be modified to produce closed causal curves*. We decided not to include a rigorous proof of this fact because it would require a deeper knowledge on topics such as special relativity, which would make this paper lose its main focus. Nevertheless, it is interesting to know that wormholes and time travel are strictly related.

3.4 Speculative extensions

This is where the rigor on this paper ends. Wormhole theory is a complex topic with not too many short term applications. This implies that the theory might not be as developed as other branches of General Relativity, not to talk about its formalization. Nonetheless, this section can give some insights about the next steps in wormhole theory formalization. We might be giving some new notions, but just to explain this possible next steps.

The Morris-Thorne wormhole metric, although a fantastic take on wormhole metrics, it is not the only one we can find, since it assumes spherical symmetry. Another interesting proposal on them is given by Visser on [Vis89], where a cubical wormhole is stated. The formalism applied for its construction, the junction construction formalism, makes use of the second fundamental form, treating the matter distribution as a thin shell.

Definition 3.17. *Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold, $\mathcal{H}$ a pseudo-Riemannian orientable hypersurface. The* ***second fundamental form*** *of $\mathcal{H}$ is the assignation defined at each point by $\mathrm{II}_p : T_p\mathcal{H} \times T_p\mathcal{H} \to T_p\mathcal{M}$ given at each point by:*

$$\mathrm{II}_p(u, v) = -g_p(d_pN(u), v)N_p\,, \qquad \forall u, v \in T_p\mathcal{H}$$

where N is the Gauss map of $\mathcal{H}$.

And the Gauss map is a smooth map which assigns each point of $\mathcal{H}$ a vector orthogonal to its tangent space. In some way, they represent the shape of the hypersurface. With this idea in mind, the second fundamental form makes for a great mathematical tool to deal with approximations of thin matter layers, which is exactly the case for the Visser wormhole.

Furthermore, observers can be introduced, and, with them, the notion of proper time. An observer is represented by a timelike curve. As particles, an observer will be free if it follows a geodesic. The proper time of an observer is defined as follows. Let $\gamma(\lambda)$ be the path that the observer is following. The proper time between λ_1 and λ_2 can be defined as:

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-g(\dot{\gamma(v)}, \dot{\gamma(v)})}dv$$

From this concept, the notion of **traversable wormhole** can be defined, as a wormhole for which a particle can traverse the wormhole in a finite amount of proper time for any observer. We should first define what does "traversing the wormhole" mean, although we could avoid this by just focusing on reasonably short timelike curves, which should not result in pathological divergences. Again there would be need of defining what "reasonably short" means, but that should be easier to define. Another option would be to try to define a traversing direction (or a set of them).

Until now, we have been working with wormholes as a structure. Notice how we could now place a wormhole in a universe, by using the connected sum. We should only be careful that the result is still a Lorentzian spacetime. Now this could result in a non orientable spacetime. If that was the case, we would call the wormhole a non orientable wormhole.

Definition 3.18. *Let $\mathcal{M}$ be a smooth manifold. We say that two charts (U, ϕ) and (V, ψ) with respective local coordinates $(x^i)_i$ and $(y^j)_j$* ***have the same orientation*** *if $\forall p \in U \cap V$:*

$$det(\left.\frac{\partial y^j}{\partial x^i}\right|_p) > 0$$

Definition 3.19. *A smooth manifold $\mathcal{M}$ is said to be* ***orientable*** *if there is an atlas such that all of its charts have the same orientation.*

The last important idea are the energy conditions. In General Relativity, there are some extra assumptions which are not a required part of the theory, but instead are extra conditions which a spacetime can satisfy so that it automatically satisfies other more interesting conditions. It can be seen as something similar to functions satisfying the continuity condition or the differentiability condition. From just knowing that, a ton of other conclusions can be extracted. The study of energy conditions in wormholes is also interesting.

Let us state some of the energy conditions:

Definition 3.20. *A Lorentzian spacetime $\mathbb{M}$ is said to satisfy the* ***null energy condition*** *if at every point $p \in \mathbb{M}$ every null vector K_p satisfies:*

$$T_p(K_p, K_p) \geq 0$$

where T is the energy-momentum tensor.

Definition 3.21. *A Lorentzian spacetime $\mathbb{M}$ is said to satisfy the* ***weak energy condition*** *if at every point $p \in \mathbb{M}$ every timelike vector V_p satisfies:*

$$T_p(V_p, V_p) \geq 0$$

where T is the energy-momentum tensor.

Definition 3.22. *A Lorentzian spacetime $\mathbb{M}$ is said to satisfy the* ***strong energy condition*** *if at every point $p \in \mathbb{M}$ every timelike vector V_p satisfies:*

$$(T - \frac{1}{2}Tr(T)g)_p(V_p, V_p) \geq 0$$

where T is the energy-momentum tensor and g is the metric.

Notice how the weak energy condition is not satisfied by the Ellis wormhole. The study of the energy conditions in different type of wormholes is not a difficult idea to formalize, and brings easily many general results to wormhole theory.

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