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GRAU DE MATEMÀTIQUES

Treball final de grau

Public Exludable Goods and How to Finance Them: A Literature Review

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Realitzat a: Departamento Matemáticas-Informática

Barcelona, January 17, 2024

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Abstract

Since the transition of Ethereum from a *Proof-of-Work* to a *Proof-of-Stake* consensus protocol back in September 2022, the Proposer Buider Separation schema, or PBS, arose as the methodology for block producing. This protocol runs at a deficit for the relays, the trusted third party between the proposers and builders, fact that motivated our intrest in studying the properties of the protocol.

In this work we review the theoretical background needed to takle this analysis from a mechanism design perspective, we heavily relayand sumarise the the work of Roughgarden *et al.*, providing the tools we need for it.

Our main goal is to expose the basic notions of mechanism design theory and build a model capable of explaining the financing of *public excludable goods*, trying to provide a self sufficient and coherent thesis, even if disregarding problem that motivated this line of research.

To achieve our goal we start by introducing several basic concepts of different aspects of applied mathematics and economics, opening the way to the development of those ideas into more sophisticated concepts.

This theoretical approach peaks with the exposition of a selection of mechanisms and models and a review of their properties.

Chapter 1

Preliminary

Informal Greeting to the Topic

Never before have we seen a greater trend towards decentralization, which is gonna pose several technical and economical challenges that we must address and give answer to.
You might be aware of several of those challenges, some notorious challenges as up scaling blockchain, smart-contract security or legal framework discussion have never been more present in both the academic and the public debate, and are gonna occupy a spot in the academic discussion in the foreseeable future.
Let us join that discussion in this essay, and explore one of such challenges, where you are invited to reflect on the matter and participate in the debate. I am certainly down to discuss any ideas you might come with should you want to, trying to never assume any prior knowledge on the topic. This previous asseveration is gonna prove rather complicated due to the specifics on the topic that we have to run into, but I hope that you will lend me your patience and we can turn that around.
And last but not least I want to thank you, for taking the time to read this essay, desert from almost any examples, which makes it a rather dense reading, thank you.

1.1 Organization of the Work

The work will be organized as follows:

Our first chapter will have the objective of paving the road for the latter development of more complex ideas, it aims at establishing the prerequisites needed to later delve in mechanism design for the financing of public exludable goods. At the same time this introductory chapter will be divided into several sections, each of them aimed at putting the scope in a specific area. We start with an introduction on game theory, to be followed by another one discussing auction theory and further elaborating the concepts, we continue with a section on mechanism design only to conclude our introduction with a last section discussing the nature of public goods, taking our particular interest on the excludable ones.

In the first section of our second chapter we delve deeper in mechanism design and the financing of public exludable goods, in this case restraining ourselves to prior free and private valuations. We conclude the chapter with a second section where we also wonder on the economic efficiency on the mechanisms we propose.

In the third and last chapter we extend our model to consider interdependent valuations, putting the scope in the case of public signals.

We also have a conclusions section were we highlight the results of our examination.

And last, although the dissertations are fairly theoretical, we use the Annex to put the theory to work and apply our results to the problem that motivated our research, the relays financing on a PBS schema.

We hope that this gradual progression will help the adequate consolidation of concepts before elaborating on more complex material.

1.2 Introduction

In this chapter we aim to introduce the concepts from different branches of applied mathematics and economics we need as a prerequisite to explore financing of public goods.

These concepts might seem a little unrelated at the beginning, but the reader will corroborate them necessary as we advance in our dissertation.

When discussing financing of a public good it is often tricky to arrive at a mechanism that ensures that free-riding is not incentivised, we circumscribe our interest to the financing of the institution of relayer in a PBS schema. This relayer can be equiparated to the *auctioneer* in a traditional auction, and using the latter term will prove useful when discussing the concept since it is well established in the literature.

Let's put a pin on this for the moment, we will come back at this later.

Before we start let us begin with a non exhaustive notational cheatsheet:

Notation Cheatsheet

Symbol	Description
u_i	Player i's utility function
v_i	Player i's valuation function
s_i	Player i's strategy
A_i	Player i's set of actions $\{a_1, \ldots, a_k\}$
H_i	Player i's information set $\{h_1, \ldots, h_k\}$
N	A finite set $\{1, \ldots, n\}$
v_i	Player i's valuation function
t_i	Player i's threshold
w_i	Player i's signal
β_i	Player i's bidding strategy
b	Bid vector $(b_1, \ldots, b_n)$
b_i	Player i's bid
b_{-i}	Every bid other than player i's: $(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_k)$
ψ_i	Player i's payoff formula
x	Allocation map
p	Payment map
C	Cost map
S	Winner subset $(S \subseteq N)$
π	Social cost function
SW	Social Welfare function
$\mathcal{H}_n$	n-th harmonic number

Table 1.1: Notation

It is typical to find the notation "s" in the literature when referring to signals, but since we use it to denote strategy in our first chapter we will use "w" as an alternative

1.3 Introduction to Game Theory

Game theory is the branch of applied mathematics that aims to study strategic interactions among rational agents. It does so by modelling the behaviour of such agents and analysing the decision making process.

It can be argued that the problem at hand can be modelled as a game, therefore, in this section we will introduce concepts such as *utility*, *strategy*, *game* or *equilibrium*, all of them necessary to later discuss auctions and social welfare.

Without further ado, let us then start with one rhetorical question for the reader:

How would you give a magnitude to a preference?

1.3.1 Utility and Strategies

This idea is the *conditio sine qua non* for the study of rational decisionmaking, and perhaps the most prevalent concept in economics, we are talking about a concept that will allow us to establish a preference between the available alternatives, the *utility*, which we can define as:

Utility (see [MC95] pag. 9)

We call **Utility** to the function

$$\begin{aligned} u : D &\to \mathbb{R} \\ d &\mapsto u(d) \end{aligned}$$

that assigns a numerical value to an element $d \in D$ of a set of available alternatives in accordance to the individual's preferences.

For instance, if Alice prefers chocolate ice-cream to its mint flavoured equivalent, then:

$$u_A(\text{chocolate ice-cream}) > u_A(\text{mint ice-cream})$$

The reader must have noticed that we are not constraining any subjective preference. In fact, it will be most usual that different subjects have different utility functions, or even if two or more individuals had the same function, it could happen that their subjective situation would differ. So even if Alice and Bob like water the same (same function), if he is stranded in a desert while she's in a spa, he'll have a higher valuation of a glass of it.

Now that we know that individuals can order their preferences according to the utility they perceive from each object, we can move forward to the definition of strategy.

A **strategy** is a process of election of an action given the available information to a player at a given time, formally:

Strategy (see [MC95] pags. 228-230)

Given an information set H_i and the possible actions A of a game
A **Strategy** for player i is a function $s_i : H_i \to A_i$

Though this work we will restrict ourselves to the case where both H_i and A_i are finite.

An action can be any proactive or passive input the player makes that has an impact in the game (a bid in an auction, choosing heads or tails on a coin flip, doing nothing, etc.).

If we consider a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ we can choose our strategy probabilistically, this allows for the randomisation of strategies, formally:

Mixed Strategy (see [MC95] pag. 232)

Given a player's i finite pure strategy set S_i, a **mixed strategy** assigns a probability to each pure strategy:

$$\begin{aligned} \sigma_i : S_i &\to [0,1] \\ s_i &\mapsto \sigma_i(s_i) \end{aligned}$$

We will require that $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$

To illustrate this take the game Rock-Paper-Scissors (see [Rou13] chapter 1) where the player chooses each option with probability $\frac{1}{3}$. In such game neither player can increase their expected payoff via a unilateral deviation since every other action has the same expected payoff.

But, is there a better strategy?

It is only natural that an individual will prioritise those strategies yielding a higher expected utility, we say that an individual operating in such manner is a *rational individual*.

We will also say that a strategy is *weakly dominant* if no other strategy provides a higher utility.

1.3.2 Games

A **game** can be described as a model of strategic interactions between players, subject to constraints on the actions they can take. (see [OR94] pags. 2-11)

By that we mean that the individual's welfare depends not only on their actions but also on the actions of other individuals. (see [MC95] pags. 219-228)

Strategic Game [OR94]

A strategic game consists of:

- a finite set N (the set of players)
- for each player $i \in N$ a nonempty set A_i (the set of actions available to player i)
- for each player the utility induces preference among the available alternatives
- for every player we use an utility map

$$\begin{aligned} u_i : V &\to \mathbb{R} \\ v_i &\mapsto u_i(v_i) \end{aligned}$$

If the set A_i of actions of every player i is finite then the game is finite.

We can broadly categorise strategic games in:

- **Cooperative** or **Non-cooperative**: Depending on the type of actions available to the players. We say that a game is non cooperative if the sets of possible actions available to the individual players are primitives, on the other hand we say that a game is cooperative if the sets of possible actions of a group of players are primitives.
- **Strategic Games** or **Extensive Games**: We say a game is strategic if the players plan their actions independently form the other players and then take action simultaneously, whereas extensive game are more general models that contain strategic games, allowing players to consider their actions in scenarios where they know other players' decisions and actions or in scenarios where they do not have such information. Strategic games are also referred by as games in normal form in some parts of the literature.

- Games with **Perfect Information** and **Imperfect Information**: Depending on the information available to the players, a game is said to have perfect information if players have full information of each others past actions, in contrast we say that the game has imperfect information if that is not the case.

Strategic games can arrive at a steady state where no player has incentives to unilaterally alter their strategy. Once the player has formed a rational expectations of other players behaviour the optimal set of strategies is the one that maximises the players utility, we call this steady state an **equilibrium**.

1.3.3 Equilibrium

The prevalent solution concept in game theory is the **Nash equilibrium** (see [OR94] pags. 11-20), which characterizes a stable state within a strategic game. In this state, each player possesses a valid understanding of how others will behave and acts in a rational manner. However, it does not concern itself with investigating the path or methods leading to the establishment of this stable state.

Definition 1.1. *A Nash equilibrium of a strategic game is a profile of actions where the player doesn't have any better paying alternative actions, this is, the player has no incentives to deviate from its course of action since any other action will provide her with equal or lower utility. Formally:*

$$u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i)$$

Where $a^* \in A$ is a profile of actions for player $i \in N$

We can reformulate it as:

For any $a_{-i} \in A_{-i}$ we define $B_i(a_{-i})$ as the set of optimal actions of player i given a_{-i}

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \geq (a_{-i}, a_i^{'}) \quad \forall a_i^{'} \in A_i\} \tag{1.1}$$

We then say that B_i is the best response correspondence of player i where $a^*_i \in B_i(a^*_{-i}) \forall i \in N$ is an equilibrium action for player i, this is, she has no incentive to alter her actions because every other action at her disposal will yield an equal or lower utility.

Let's extend this concepts to a mixed strategy situation (see [MC95] pag. 250):

Definition 1.2. *Given a normal form game, the mixed strategies $s^*_1, \ldots, s^*_n$ constitute a mixed strategy Nash equilibrium if and only if $u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \quad \forall i \in N$.*

We use s_{-i} as the strategy profile of every player except i, in plain words the utility of the mixed strategy Nash equilibrium provides a higher or equal utility than any other strategy for all players.

Now that we have some theoretical background let us illustrate this concepts with a couple of examples:

Example 1.3. Prisoner's Dilemma

Consider two crime suspects put in different rooms for interrogation. There is no sufficient evidence so the sentence duration will depend on the confession of the crime.

The condestable offers each of them a deal where three scenarios can happen:

- If one of them makes a confession implicating their partner he will be used as a witness against the other suspect and receive immunity. On the other hand the prisoner who didn't confess would be sentenced to four years.
- If both confess they would both be sentenced to three years each.
- If non of them confesses they each would receive a sentence of one year.

We have two players in this game, and we represent each players available actions and **utilities** in different axis.

We say a player is rational if she aims to maximize her **utility**.

	Don't Confess	Confess
Don't Confess	-1, -1	-4, 0
Confess	0, -4	-3, -3

In this game the best social outcome is to cooperate (don't confess) since there are obvious advantages in the time served.

Still, each prisoner has incentives to behave opportunistically. From a selfish point of view, serving zero years is better than serving one, even if that causes the other prisoner to serve four, thus the **dominant strategy** for each prisoner is to confess, irrelevant of whatever the other prisoner does, leading to the worst social possible outcome. This serves as a perfect example of social-efficiency loss.

Both players act rationally, taking the action that aims to minimise the (in this case) disutility, the action "Confess" provides the prisoner with either 0 or 3 years to serve, and the action "Don't Confess" will lead to 1 or 4 years imprisonment.

This leads to the **Nash equilibrium** where both suspects serve 3 years.

Example 1.4. Matching Pennies

Two friends want to make a bet, they will flip a coin and reveal it at the same time. If both coins are equal (heads or tails) player 1 will pay one dollar to player 2. The opposite will happen if the coins do not match.

	Head	Tail
Head	-1, 1	1, -1
Tail	1, -1	-1, 1

There is no pure strategy Nash equilibrium in this case, choosing matching pennies has the same expected payment as choosing different pennies for both players, yet if we consider mixed strategies, choosing matching pennies and different pennies with equal probability leads us to a mixed strategy Nash equilibrium.

This is a game where players have diametrically opposed preferences, we will also call this type of games **strictly competitive games**, and in particular this is an example of **zero-sum game**, since whatever one player is winning, the other is loosing.

Proposition 1.5. *[Nas51] If we allow for mixed strategies, every strategic game* $(N, (A_i), u_i)$ *where the number of players and their action profiles are finite has a Nash equilibrium.*

The previous proposition can be easily proved with Kakatuni's fixed point theorem or with Brower's fixed point theorem, we will prescind of the proof ourselves but the avid reader can find it in several publications, among them the following: [OR94] (pag. 20) and [Gli52; Nas51].

1.4 Introduction to Auction Theory

Auctions have a long history dating back to ancient times, with Herodotus documenting their use in Babylon as far back as 500 BC (see [Kri09] pags. 1-8). However, we don't need to delve so deeply into history to find numerous contemporary examples of auctions. Nowadays, a variety of goods and services is regularly traded through auction forms, spanning wholesale fish, treasury bonds, art, and much more, examples go as far as the readers imagination.

We can say that an auction is the procurement of a good or service though competitive bidding.

1.4.1 Some Common Auction Forms

The open ascending price, or *English Auction*, is one of the oldest and perhaps the most prevalent auction form.

In an English auction, an auctioneer begins by calling a base price, an then proceeds to increment it as long as there are two or more interested bidders, the auction ends when only one bidder remains, and the last called price is the price at which the object is sold, this is, the price at which the second-last bidder dropped out (this can also be seen as the reservation price of the second-last bidder).

A *Dutch Auction* is an open descending price counterpart for its English cousin, in this modality the auctioneer begins by calling a high price, with presumably no buyers interested, and the proceeds to lower it until a bidder indicates interest, the object is then sold at that price.

The *Sealed-Bid First-Price Auction*, in which bidders indicate their bids in a sealed envelope and the highest bid wins and that is the price of the sell, the *Sealed-Bid Second-Price Auction* also exists, where the winner is the highest bidder as well, but the selling price is that of the second highest bid (see [Kri09] pags. 1-8).

1.4.2 Valuation

Auctions have such a prevalence in the modern world because the seller is unsure about the values that the bidders attribute to the object, as such they are an excellent tool for *price discovery*.
Bidders typically know the value of the object to themselves at the time of the bidding, although not necessarily the ones of the other bidders, we call that situation one with private values.

Private values can typically be assumed if the objects valuation of each bidder is independent from one another (see [Kri09] pags. 1-8), this is not always the case since some objects value can be correlated to others valuations of them, for instance, a tradeable stocks value won't necessarily be the price at which the holder attributes to it, rather than the value at which it can be sold at the marketplace. On top of it we can introduce expectations on the price development, which further complicate the valuation of some objects in which the value is drawn not only from a joint distribution, but where each agent has a different expectation for it's future value. We will debate this in due time, but let's keep it simple for the time being.

1.4.3 Private Auctions

Let us consider a single object for sale, where n bidders are interested and concur to the auction.

Each bidder i assigns the value v_i to the good , we will also call this valuation the agent's *type*. At the price v_i, bidder i would be indifferent between keeping the money or buying the object. Let us further impose that this value is only known by the bidder and that valuations are ***prior free***, this is, no bidder has any information on the valuations of other bidders, nor can they bound it in an interval.

Let us also assume that bidders are risk-neutral and seek to maximize utility.

Finally, let us impose that bidders are not subject to budget constraints, and that they can pay the required price should they win the auction.

In order to transform the value an agent perceives from an object to a bid, we use a bidding strategy $\beta_i : V \to \mathbb{R}_+$ (see [Kri09] pags. 11-13), which is the function according to which bidder i will determine their bid.

1.4.4 Second Price Private Auctions

If we circumscribe to private valuations, second-price private auctions are equivalent to open ascending price auctions (or English auctions as we referred them before) (see [Kri09] pags. 11-13). In a second price auction each bidder submits a bid b_i, the payoff for bidder i is then:

$$\psi_i(b_i) = \begin{cases} v_i - \max_{j \neq i} b_j & \text{if} \quad b_i > \max_{j \neq i} b_j \\ 0 & \text{if} \quad b_i \leq \max_{j \neq i} b_j \end{cases}$$

Proposition 1.6. *In a second-price sealed bid auction, bidding according to $\beta_i(v_i) = v_i$ is a weakly dominant strategy*

Proof. If we consider bidder i then let $p_i = \max_{j \neq i} b_j$ be the highest bid. By bidding $b_i = v_i$ the bidder will win the auction if $v_i > p_i$ and loose if $v_i < p_i$. At the price $v_i = p_i$ the bidder is indifferent between winning and loosing. Remember that p_i equals the second highest bid in this auction modality.

Now suppose that the bidder bids z_i instead of v_i and $z_i < v_i$, then if $v_i > z_i > p_i$ he still wins but his profit remains the same: $v_i - p_i$.

On the other hand if $v_i > p_i > z_i$ he would leave the same profit of $v_i - p_i$ unmaterialised.

The homologous argument can be done to disqualify bidding $b_i > v_i$. □

Having considered proposition 1.6 we wonder. How much is each bidder expected to pay in equilibrium? Fix bidder i let the variable $Y_i \equiv Y_i^{n-1}$ denote the highest bid among the $n-1$ remaining bidders. Then provided that we rearrange $V = v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_n$ we can say that Y_i is the highest order statistic of V. Now if we denote G as the distribution function of Y_i, then $\forall y \quad G(y) = F(y)^{n-1}$ so the expected payment for a bidder with valuation v in a second price auction (see [Kri09] pags. 11-13) can be written as:

$$\begin{aligned} m(x) &= P[\text{Win}] \times \mathbb{E}[\text{2nd highest bid} \mid \text{v is the highes bid}] \\ &= G(x) \times \mathbb{E}[Y_i | Y_i < v] \end{aligned}$$

1.4.5 Revenue and Efficiency

When comparing two auctions we will pay special attention to the concepts of *revenue*, or the expected selling price, and *efficiency*, this is, that the object ends up in the hands of the person that values it the most.

Efficiency is typically measured *ex post* and pays attention not to an individual but to the society as a whole, we can rephrase the latter asseveration as the following. An auction is efficient (see [Kri09] pags. 1-8) if it maximises the aggregated utility of its participants.

But, can't we trust the market to allocate objects efficiently?

The naive neoclassical answer would be yes, efficiency is relatively easy to achieve in oversimplistic models, but reality tends to be more complex.

Market failures often arise if we relax hypothesis, and not in vain, efficiency is one of the most studied properties in auction theory, just because when auctions become sufficiently complex, efficiency cannot be assumed.

This brings us to the next section, where we wonder on the best way to structure an auction so it has the best properties, among them we will aim to make our auctions efficient.

1.5 Introduction to Mechanism Design

So far we have introduced the concepts of games and auctions, yet we have not mentioned the governing institutions where they are circumscribed.

Both concepts deeply rely in the interaction with other individuals, and will follow certain governing procedures, either legally binding rulings or socially accepted courses of action. We need a protocol that rules over this institutions, and that partakers accept and abide by when taking part on them.

This rules need not be complex, for instance, when voting for the major of a town we can establish that the person with the most votes will get elected, when taking part on an auction we can accept that the highest bidder will get the object, etc.

1.5.1 Mechanisms

Since we don't intend to delve in social choice theory, we will define *allocation rule* and *payment rule* for the specific propose of studying bids, although the reader should consider that a more general definition is possible.

Allocation Rule (see [Rou13] chapter 3)

An **allocation rule** is a function

$$\begin{aligned} x : \mathbb{R}^n &\to \mathbb{R}^n \\ b &\mapsto x(b) \end{aligned}$$

that provides an allocation of the good(s) as a function of the bids b.

The allocation rule will serve us to identify the agent(s) to whom the good(s) is assigned. But for our considerations we will not simply be interested in any allocation rule, our attention will be centered in demanding that the allocation rule is efficient.

Efficient Allocation Rule (see [Kri09] pag. 75)

An allocation rule is said to be efficient if it maximizes social welfare, that is, for all $b \in B$,

$$x^*(b) \in \arg\max_{x \in X} \sum_{j \in N} x_j b_j$$

For instance if we have two ping pong paddles to gift, and Bob and Alice love ping-pong and Iosif hates it, then providing a paddle to Bob and Alice would carry more social welfare than gifting one to Iosif. We also need to introduce a concept to see what are the payments imposed after our auctions:

Payment Rule (see [Rou13] chapter 3)

A **payment rule** is a function

$$\begin{aligned} p : \mathbb{R}^n &\to \mathbb{R}^n \\ b &\mapsto p(b) \end{aligned}$$

that provides a price for the good(s) as a function of the bids.

For instance, in a second-price sealed bid auction the winner is the highest bidder and pays the second highest bid as a price, if we consider the bidder i as the winner and b_j as the second highest bid:

$$x(b) = (0, \dots, 0, \underbrace{1}_{i\text{-entry}}, 0, \dots, 0)$$

$$p(b) = (0, \dots, 0, \underbrace{b_j}_{i\text{-entry}}, 0, \dots, 0)$$

But allocation and payment rules are of little use for us by themselves, we still have to establish a connection between them, as such we can say that a *mechanism* is the set of rules that serve of guidance to assign a good under certain conditions. A good mechanism should provide incentives for rational agents to achieve the best aggregated outcome. This is often a decision in hands of the *policy maker*, without getting into detail the policy maker has an utility function too, which will typically depend on the aggregated utilities of the agents governed.

Without further ado let's define mechanism formally:

Mechanism

A **Mechanism** is a pair $M = (x, p)$ where x and p are the aforementioned allocation and payment rules.

After this general definition we will typically refer to:

General Mechanism

We will say that a mechanism is a **General Mechanism** if:

- n strategic participants, or "agents;"
- a finite set Ω of outcomes;
- each agent i has a private valuation $v_i = u_i(o)$ for each outcome $o \in \Omega$.

After this couple of definitions it is a good time to revisit our definition of utility. The reader might have already deduced that given the bids $b = (b_1, ..., b_n)$ and a mechanism we could easily compute the utility of our n bidders.

Indeed, we can establish that bidder i with valuation v_i will use strategy $\beta(v_i) = b_i$ to obtain a bid, if we do this for all the bidders we would get b. Having obtained all the bids we can use $M(x(b), p(b))$ to obtain a map of bids and payments of the n bidders, where we can calculate $u'_i(b_i, p_i) = u_i(M(x(b), p(b)) = v_i x(b_i) - p(b_i) \in \mathbb{R}$

Moving forward we will use u' as our utility function by default, assuming it is quasi-linear, or in other words, that agents want to maximise $u'_i(b,p) = x_i(b)v_i - p_i(b)$.

Let us follow by introducing some desirable properties of a mechanism.

1.5.2 Various Properties for our Mechanisms

One of such properties we should aim at having when designing a mechanism is *ex post efficiency*. We have briefly introduced the concept in the previous section, but let us formally define it now:

Ex Post Efficiency

In the context of an auction, an allocation rule is said to be ***ex-post efficient*** if it assigns the good to the player that values it the most.

In an auction we typically have a non divisible good to assign, giving it to the bidder that has a highest valuation of it maximizes the aggregated utility.
Note that our definition refers to the allocation rule, despite of it we will (very) often abuse the concept and refer to *ex post efficient* mechanisms.

Individual Rationality (IR) (see [Kri09] pag. 66)

A mechanism is ***individually rational*** if $\forall i$ and b_i the equilibrium expected payoff $u_i(b_i) \geq 0$.

This definition follows the assumption that by not participating, an agent can "lock in" to a payoff of zero, thus an agent only has incentives for participation if the expected payoff is equal or better than in a non participation situation.

Truthfullness [Dob+18]

A mechanism is ***truthful*** if no player can strictly increase her utility by misreporting or lying about her valuation.

Pretty self explanatory, agents have incentives to report their true valuation if this property exists.

Incentive Compatibility (IC) (see [MC95] pag. 868)

An allocation rule $x(\cdot)$ is ***incentive compatible*** if the mechanism is individually rational and $M = (x, p)$ has an equilibrium $(s_1^*, ..., s_n^*)$ where $s_i^*(v_i) = v_i \quad \forall v_i \in V_i$.

The reader can note the subtle difference between truthfulness and incentive compatibility (if only because we require individual rationality for the latter), despite of this fact it is common to refer them indistinctly in the literature[2].

We say a mechanism is dominant-strategy incentive-compatible (DSIC) if truth telling is also the dominant strategy for agent i, regardless of the actions of the rest of agents.

This is, truth telling by every agent constitutes an equilibrium for

$$M = (x, p)$$

Implementable Allocation Rule (see [Rou13] chapter 3)

We say that a single parameter allocation rule is **implementable** if there is a payment rule p such that the mechanism $M = (x, p)$ is dominant-strategy incentive-compatible.

1.5.3 Theory in Action: Applying What we Learned

It is a good time to connect the concepts we have introduced this far, in fact we have already seen an example of a *ex post efficient*, truthful and incentive compatible mechanism: The second price sealed bid auction, where we proved that bidders have incentives to bid the exact amount at which they value the good, and which also ensures that the good goes to the bidder which values it the most.

Let's go ahead and prove it. As we can recall from 1.6, we have already seen truthfulness, lets see *ex post efficiency* and incentive compatibility.

Proposition 1.7. *A second-price sealed bid auction, is incentive compatible*

Proof. The intuition behind this is pretty straight forward, in plain words an equilibrium strategy for every bidder should be telling the truth.

[2] We will keep up appearances in this chapter but relax our diction in the following sections

It is no coincidence that we made a generalisation for every player when we proved 1.6, we can again take advantage of the proof to see that every bidder's optimal strategy is to bid according to its type, we will then arrive at an equilibrium where $s_i^*(v_i) = v_i$.

Individual rationality is also easy to see: Only the highest bidder has to make a payment, and as we can see $p_1 = v_2$ and the utility is then $u_1 = v_1 - v_2 \geq 0$ directly from the fact that $u_i = x_i(b)v_i - p_i(b)$, for the rest $x_j = 0$ and $p_j = 0$ so $u_j = 0$, therefore $u_i(b_i) \geq 0$. □

This serves us as a perfect example to illustrate how a mechanism can provide incentives to rational utility maximising agents to behave in a socially optimal manner, avoiding equilibria as the one we can recall from the prisoners dilemma.

Having this result let us move forward to prove *ex-post* efficiency.

Proposition 1.8. *A second-price sealed bid auction, with utility function:*

$$u_i = x_i(b)v_i - p_i(b)$$

allocation rule :

$$x_i(b) = \begin{cases} 1 & \text{if} \quad b_i > b_j \quad \forall j \neq i \\ 0 & \text{otherwise} \end{cases}$$

and payment rule:

$$p_i = \begin{cases} \max_{j \neq i} b_j & \text{if} \quad b_i > \max_{j \neq i} b_j \\ 0 & \text{if} \quad b_i \leq \max_{j \neq i} b_j \end{cases}$$

is efficient ex post.
In the unlikely event of a tie we will choose the winner randomly between the winner subset.

Proof. The auction proposed in the proposition assigns the good to the highest bidder and compels the second highest bid as payment, without loss of generality consider that $v_1 > v_2 \geq \ldots \geq v_n$. Using the property of truthfullness we know that $v_i = b_i$, then it is easy to see that $x = (1, 0, \ldots, 0)$ and $p = (v_2, 0, \ldots, 0)$, utility then is null for every bidder other than the first, who has $u_1 = v_1 - v_2 > 0$.

Due to the property of truthfulness we just mentioned we can asseverate that $v_1 > v_j \quad \forall j \in N \setminus \{1\}$ and the *ex post* efficiency is trivial.

In the event of a winning bid tie we have established that each winner has an equal probability of getting the good, the proof follows the same. Consider $v_1 = v_2 > v_3 \geq \ldots \geq v_n$ (for a tie with more than two bidders the proof is homologous).

Now $x = (1,0,\ldots,0)$ and $p = (v_3,0,\ldots,0)$ with probability $\frac{1}{2}$, and $x = (0,1,0,\ldots,0)$ and $p = (0,v_3,0,\ldots,0)$ with equal probability. Efficiency wise we are indifferent on which one of the two gets the good, since both $v_1 = v_2$ and $u_1 = u_2$ if they are assigned the good, to discard any other assignation we can refer to the sole winner proof in the above paragraph. □

1.5.4 Meyerson's Lemma

Let us end this section by stating a couple of theorems that will come in handy later:

Theorem 1.9. ***Myerson's Lemma (see [Rou13] chapter 3)***

(a) An allocation rule x is implementable if and only if it is monotone [3]*.*

(b) If x is monotone, then there is a unique payment rule such that the sealed-bid mechanism (x, p) is DSIC (assuming the normalization that $b_i = 0$ implies $p_i(b) = 0$).

(c) The payment rule in (b) is given by an explicit formula.

Proof. Let us have the DSIC mechanism $M(x, p)$, let us then establish that bidder i's bid is b_i and let b_{-i} be the set of bids of every other player. The notion of bidding anything other than $b_i \neq v_i$ can be disqualified in a similar manner as what we did in 1.6:
Let us have the bids $0 \leq y < z$, and let us assume that $z = v_i$ by the DSIC property the utility of bidding y would then be:

$$z \cdot x(z, b_{-i}) - p(z, b_{-i}) \quad \geq \quad z \cdot x(y, b_{-i}) - p(y, b_{-i})$$

Similarly, the bidder i might have the valuation y submit z as a false bid

$$y \cdot x(y, b_{-i}) - p(y, b_{-i}) \quad \geq \quad y \cdot x(z, b_{-i}) - p(z, b_{-i})$$

And we can rearrange the previous equations in:

$$z \cdot [x(y, b_{-i}) - x(z, b_{-i})] \leq p(y, b_{-i}) - p(z, b_{-i}) \leq y \cdot [x(y, b_{-i}) - x(z, b_{-i})]$$

We have established the monotony of x, now if we fix z and let y tend to z from above, then, x is flat except for a finite number of "jumps." If we then take the limit $y \to z$ in the previus equation, both sides of the equation become 0 if there is no

[3] a function f is monotonous if and only if $x \leq y$ implies that $f(x) \leq f(y)$

jump in x at z. If there is a jump of magnitude h at z, then the left- and right-hand sides both tend to $z \cdot h$, which implies the following constraint on p, for every z:

$$\text{jump in } p \text{ at } z = z \cdot \text{jump in } x \text{ at } z$$

Thus, assuming the normalization $p(0) = 0$, we've derived the following payment formula, for every bidder i, bids b_{-i} by other bidders, and bid b_i by i:

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{\ell} z_j \cdot \text{jump in } x_i(\cdot, b_{-i}) \text{ at } z_j \tag{1.2}$$

where $z_1, \ldots, z_\ell$ are the breakpoints of the allocation function $x_i(\cdot, b_{-i})$ in the range $[0, b_i]$. In the event of x being monotone but not piecewise constant we can provide another approach, for instance, suppose that x is differentiable. Dividing the payment difference sandwich by $y - z$ and taking the limit as $y \to z$ yields the constraint

$$p'(z) = z \cdot x'(z)$$

and, assuming $p(0) = 0$, the payment formula

$$p_i(b_i, b_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z, b_i)\, dz \tag{1.3}$$

for every bidder i, bid b_i, and bids b_{-i} by the others. When x is monotone and piecewise constant and p is defined by (1.2), then (x, p) is a DSIC mechanism. The same argument works more generally for monotone allocation rules that are not piecewise constant such as (1.3). □

Now that we have covered the mechanism design concepts we needed let us move forward to the next section, where we will briefly cover the concept of public goods and their peculiarities.

1.6 Introduction to Public Goods

In this section we intend to introduce a special type of good which can be consumed by several agents at the same time, this feature of *non-rival* consumption will also induce some differences on the way that these goods are traded respect to the goods we have studied this far. Another feature that typically characterises public goods is *non-excludability*, this is, the inability to limit its consumption. This features often lead to the free-rider problem.

In order to illustrate this think of a fireworks spectacle and two citizens, Alice and Bob as the consumers. Fireworks are non-rival, since its consumption by Alice doesn't erase them from the firmament, so Bob can enjoy the show too simultaneously, without diminishing the capacity of any other viewer to enjoy them.

They are also non-excludable, since once the firework is shining in the sky there is no way to preclude anybody from raising the head and observing them.

But then if Bob can enjoy a good and there is no credible way to exclude him from its consumption, what is the incentive for Bob to pay for a good he'll already be able to enjoy?

Imagine that both Bob and Alice have the wish to enjoy a firework spectacle which costs $100 to organise. And both Bob and Alice would pay $100 to do so. Now if Bob knows that Alice will pay anyways he can disclose a false willingness to pay in the interval [0, 100], and he could enjoy the show for free or pay only a fraction of it.

This summarises the free-rider problem, a typical challenge for the financing of public goods that our mechanisms have to take into consideration in order to avoid or, at least, mitigate such problems.

A ***rival good*** is a good whose consumption by one individual precludes the consumption by others.

Most of the goods are rival in its consumption, since its consumption typically depletes the good. For instance, if Alice purchases an ice cream and consumes it, nobody else can eat that ice cream.

A good is ***excludable*** if its consumption can be restricted to an individual or collective.

For instance, a plane ticket can only be used by the passenger that figures in the reservation.

Moving forward we will abuse the notation and consider public excludable goods unless stated otherwise, this not only is more aligned with the problem we want to tackle but also simplifies the elimination of the free-rider problem.

A ***public excludable good*** is a non-rival, excludable good.

As foreshadowed, the excludability feature allows for the selective provision of the good to certain individuals. Although free-riding is not eliminated by this feature alone it certainly simplifies things, since now we can have a traceability of who contributed to the financing of the good and who didn't, and exclude them from the consumption of the good (if that action of exclusion has credibility).

Imagine then we want to finance a public phone company, providing a phone to an individual costs $50 and the operating costs of the network is 1M regardless of the number of users.
In this example we have the excudability feature, nobody can use the network

without a phone and the permission of the network owner. We could provide the service only to those who paid for it. For instance, imagine that Alice an Bob pay each 500k to finance the fixed costs of the network and each of them has a phone. Now opening this service to other uses can only increase Alice and Bob's utility, since being able to call their friends and family doesn't increase their cost. If Rick buys a phone Bob and Alice have incentives to let him use the network they already paid for, and Rick has the incentive to use the network without contributing to its financing, what is more, if Alice and Bob's combined utility of having Rick use the network exceeds \$50 they would even provide the phone for him!

In this example we have inadvertently introduced secuentiality: Alice and Bob made the fist move in financing the company, only to discover that nobody else wants to pay after them. We have also introduced network economics, for the time-being let us only note that there are several goods that provide more utility the more people use them. A social network could be a perfect example of it, there is no gain on using a social network where you are the only user, who would see the pictures of your breakfast then?

1.6.1 Financing Concepts

In order to develop this freshly introduced concepts and dig deeper in the financing of this type of goods let us make some definitions:

Cost Function [Dob+18]

A ***cost function*** $C : 2^N \to \mathbb{R}^+$ is a function that specifies the cost of every possible allocation of goods and services.

We consider monotone cost functions throughout this dissertation unless stated otherwise.

We will also consider C(∅) = 0

In the previous example we assumed our cost function to be linear $C(Network) = 1M+$ \$50 $\times i$

So, if cost depends on the number of users and we face the financing of a public excludable good two questions rapidly arise. Who has to pay for the good? And how much?

We can now take advantage of what we learned on the auction theory and mechanism design sections and propose an auction-like mechanism to discover the value the bidders place in the public good, and then deliver the good to those willing to pay their share of it.

Let us provide another example where we can see this exclusion feature:

Consider that a group of friends is debating the purchase of a Ping-Pong table, the table costs \$1000 and every friend has its subjective valuation of the good v_i, in this heterogeneous group of friends there are some who like ping-pong and some who really don't care about it, which reflects in their valuation.

Now, let us take truthful bids from every friend in the group $\beta(v_i) = b_i$ for the $i \in N$ fiends in the group.

As in [Maz23], let us take the subset $S^\star \in \text{argmax}_{S \subseteq N}\{\sum_{i=1}^{n} b_i : b_i \geq \frac{C_r}{|S|}\}$

In equilibrium a friend will be included in $S^\star$ if $b_i \geq \frac{C_r}{|S^\star|}$

This is, if that friend's willingness to pay for the ping-pong table is greater or equal to its share of the table. For instance, if we have only 5 friends want to play pong-pong, and $b_1 = 500, b_2 = 450, b_3 = 400, b_4 = 300$ and $b_5 = 100$

Then player 5 (Iosif) doesn't like pong-pong enough to pay the share that would correspond to him if all 5 were to buy the table together, since $\frac{1000}{5} = 200 \geq b_5$, on the other hand $\frac{1000}{4} = 250 = p \leq b_i$ for $i \in [1, \ldots, 4]$, so Alice earns an utility of $v_1 - p = 500 - 250 = 250$, Bob earns an utility of $v_2 - p = 450 - 250 = 200$ and so on.

Since Iosif doesn't like ping-pong that much and forcing him to play would carry a disutility, we can exclude him from the purchase and the consumption of the good. This previous example presents us with a great opportunity to define a couple of more concepts:

A mechanism is said to balance the budget (see [Kri09] pag. 78) if for every realization of values, the net payments from agents sum to zero

$$\sum_{i \in N} p_i(b_i, b_{-i}) = 0$$

Where b_i is the bid of player i and b_{-i} is the bids of the players other than i.

This general definition could suit us finely, yet we have introduced the notion of cost (more precisely cost function), this cost needs not necessarily imply that an agent is bearing it/collecting such amount.
Instead we will fine tune our budget balance definition to the following:

Budget Balance

A mechanism is said to balance the budget if for every realization of values, the net payments from agents sum equals to the cost of the mechanism

$$\sum_{i \in N} p_i(b_i, b_{-i}) = C(S)$$

Where $S \subseteq N$ is the subset of agents being serviced with the good, we will later explore its financing, let it suffice from now that we want that the sum of payments of the agents enjoying the good equals its cost.

A lesser restrictive property, deeply related to the previous is ***no-deficit***, we say that a mechanism has such property if:

$$\sum_{i \in N} p_i(b_i, b_{-i}) \geq C(S)$$

Whereas budget balance is preferable, there are scenarios where we won't be able to attain it, having to satisfy ourselves with the no deficit property instead.

Chapter 2

Economic Efficiency

In the latter chapter we centered our efforts in providing the necessary tools for our analysis, we will now capitalise in our previous effort and put the theory to work, we will introduce a baseline cost-sharing scenario for a public excludable good, rich in axioms and hypothesis, and we will follow by relaxing some of those hypothesis, arriving at a more complex but also rich model.

In this chapter we aim to study efficient mechanisms for the financing of public excludable goods, we will use the concepts previously introduced and we will also specify some others to our particular topic. The reader might wonder, why such insistence in excludability?

We have previously mentioned the free-rider problem, let us provide now a more concise explanation on why we circumscribe our study in public excludable goods.

First of all we have implicitly assumed the absence of any institution with power to impose its sovereignty. Indeed, free-riding would not be a problem if an entity such a State could impose a tax, and then used that revenue to finance a public non excludable good.

If payment can't be compelled agents have an incentive to misreport their valuations and still enjoy the good.

For the avid reader a more detailed dissertation on private provision of public goods can be found in [MC95], section 11.C. Let us suffice with the notion that the following analysis cannot be extended to any public good, only to those which can be excluded at its consumption.

2.1 Some Well Known Mechanisms

In this section we heavily rely in the work of Roughgarden *et al.* ([Dob+18]), we will start with a collection of a couple of basic cost sharing mechanisms. We will put our focus in the advantages and disadvantages of each and expose the intricacies of the trade-off between the advantages and disadvantages of both.

2.1.1 The Vickrey Clarke Groves Mechanism

Let us then start with our definition of VCG mechanism for public excludable goods, we can provide the following algorithm:

VCG Mechanism (Public Excludable Good) [Dob+18]

1. Accept a bid b_i from each player i.
2. Choose the outcome $S := N$ if $\sum_{i \in N} b_i > 1$, and $S := \emptyset$ otherwise.
3. Charge each winner $i \in S$ the minimum bid for which she would still win (holding others' bids fixed), namely $\max\{0, 1 - \sum_{j \in N \setminus \{i\}} b_j\}$.

Where $S \subseteq N$ is the subset of "winners" of the auction. We also intend to maximize the total utility, and since the VCG mechanism is efficient by construction, $\sum_{i \in N} v_i - C(S)$ is maximal.

As explained in [Dob+18], the VCG mechanism is truthful and is also efficient, but among all the good properties of a VCG mechanisms there is one that we (dearly) miss.

Indeed, budget balancing is a desirable property that the VCG mechanism does not have by default, for instance if players have a valuation $v_i \geq \dfrac{1}{|N| - 1}$, then the VCG mechanism obtains a revenue of zero, yet the cost is still unitary, this means that, at least someone "in the game" ends up with a negative utility (typically the provider of the good, that can not recover its cost).

Example 2.1. Imagine that we want to finance a public transport route, let us unitarise the cost and consider a population of 5 subjects with values $v_1 = 0.5, v_2 = 0.4, v_3 = 0.35, v_4 = 0.2$ and $v_5 = 0.1$.

Since the VCG is truthful $b_i = v_i$ for all our players, so since $\sum_{i \in N} b_i = 1.55 > 1$ we are going to provide the good, and according to the VCG mechanism we just discussed the payments are:

$$
\begin{aligned}
p_5 &= \max\{0, 1 - (0.5 + 0.4 + 0.35 + 0.2)\} = 0 \\
p_4 &= \max\{0, 1 - (0.5 + 0.4 + 0.35 + 0.1)\} = 0 \\
p_3 &= \max\{0, 1 - (0.5 + 0.4 + 0.2 + 0.1)\} = 0 \\
p_2 &= \max\{0, 1 - (0.5 + 0.35 + 0.2 + 0.1)\} = 0 \\
p_1 &= \max\{0, 1 - (0.4 + 0.35 + 0.2 + 0.1)\} = 0
\end{aligned}
$$

So in this example nobody contributes to the financing of our public transport route yet the cost is still $C(S) = 1$.

But if VCG mechanism doesn't balance the budget, what other mechanisms can do this? Which trade-offs will we face? And why do we insist so much in balancing the budget?

Without budget balance there are little agents that would be inclined to provide a public good, it certainly complicates the private provision of it, let us then explore some alternatives.

2.1.2 The Shapley Value Mechanism

Since the VCG mechanism is not budget balanced and that is indeed a property we would like our mechanisms to have let us consider alternatives that do balance the budget, one of such is the following:

Shapley Value Mechanism (Public Excludable Good) [Dob+18]

1. Accept a bid b_i from each player i.
2. Initialize $S := N$.
3. If $b_i \geq 1/|S|$ for every $i \in S$, then halt with winners S, and charge each player $i \in S$ the price $p_i = 1/|S|$.
4. Let $i^* \in S$ be a player with $b_{i^*} < 1/|S|$.
5. Set $S := S \setminus \{i^*\}$ and return to Step 3.

Where we have unitarised the cost since we can normalise the valuations too without loss of generality.

As the VCG mechanism, the shapley value mechanism is also truthful and it is budget balanced, yet it is not efficient.

Whereas VCG mechanism sacrifices budget balancing in pursuit of efficiency, the Shapley value mechanism dose the inverse trade-off.

Example 2.2. Imagine the same scenario as in example 2.1

Since the Shapley value mechanism is also truthful $b_i = v_i$ for all our players.

Then if we follow the algorithm, in the first iteration:

$S = N$, and $\frac{1}{|S|} = 0.2$, thus $p_i = 0.2$ and bidders 1,2,3 and 4 make the cut after the first iteration since $b_i \geq 0.2$ for all players in $i \in \{1, 2, 3, 4\}$

Now let us set $S = \{1, 2, 3, 4\}$ and proceed analogously, since now $\frac{1}{|S|} = 0.25$ now player 4 is left behind and $S = \{1, 2, 3\}$

The next iteration leaves $p_i = 0.33$ so now we can halt our algorithm, for $b_1, b_2, b_3 \geq 0.33$ and thus $S^* = \{1, 2, 3\}$.

Our mechanism is indeed budget balanced, since we are collecting $\sum p_{i \in S^*} = 3 \times 0.33 = 1$ (well, with our poetic license).

On the other hand, our mechanism is not efficient, but we have to wait a little longer until we discuss social welfare to see why.

2.1.3 Truthful Mechanisms Discussion

In order to ease the discussion regarding the winners of an action S we will use the following generalised proposition when referring to truthful mechanisms:

Proposition 2.3. ***[Dob+18]*** *Let M be a deterministic, truthful, and individually rational cost-sharing mechanism with player set N. Then, for every player $i \in N$ and bid vector b_{-i} for players other than i, there is a threshold $t_i(b_{-i}) \in \mathbb{R}^+ \cup \{+\infty\}$ such that:*

(i) If player i bids more than $t_i(b_{-i})$, then she is included in the output set S, at the price $t_i(b_{-i})$.

(ii) If player i bids less than $t_i(b_{-i})$, then she is excluded from S.

This previous proposition, which is actually a corollary of Meyersons lemma (1.9), will prove quite handy in several proofs later on, providing us with a powerful tool in the next section.

In the case of the VCG mechanism the threshold $t_i(b_{-i})$ would be the difference between the maximum reported welfare that can be achieved between the rest of the agents when agent i is not included in S and the welfare when i is indeed included in S.

On the other hand, when focusing on the Shapley value mechanism [Dob+18] the threshold is then:

$$t_i(b_{-i}) = \frac{1}{f_i(b_{-i})} + 1 \tag{2.1}$$

Where $f_i(b_{-i})$ is the size of the largest subset S of $N \setminus \{i\}$ such that $b_j \geq \dfrac{1}{|S|+1}$ $\forall \quad j \in S$. The following proposition will also help later on when studying efficiency in the next section.

Proposition 2.4. *[Dob+18] For every truthful cost-sharing mechanism M, there is a universally truthful cost-sharing mechanism M′ such that, for every bid vector b, the expected revenues of M and M′ are equal.*

We now have the means to select a subset of agents that have access to the good or service, and exclude those who don't value the good or service enough to pay for their share, thus avoiding the free-rider situation.

The mechanisms that we have studied this far are truthful and either budget balanced or efficient, yet we would like to attain all three properties at once.

A natural question emerges from this situation: Can we tailor a mechanism that, aside from being truthful, is both efficient and budget balanced?

The answer requires yet a bit more patience, generally speaking no truthful mechanism can be both efficient and budget balanced [Dob+18; GL79].

Still, everything is not lost (imagine reading though all this stuff only to end with: "Okay, this cite says impossible, that's all folks"), we will be able to tune the trade off between this two concepts to achieve a mechanism that suits the agents taking part in the game, some games will allow us to construct such mechanism, whereas we will find impossibility theorems in other scenarios.

2.2 Cost and Efficiency Comparison

So far we have stated what a public excludable good is, we have also introduced the notion of cost along with several other concepts related to the posterior analysis of a mechanism and we have wrapped up in an introductory example, yet the question we have hinted remains unasked: How can we finance a public excludable good?
And also, given several alternatives. How can we compare them and select the one with the best properties?

We will need a criteria to classify mechanisms and make such decisions, luckily, there are several ways to study the economic efficiency of a mechanism.

In this section we will measure the inefficiency of a cost sharing mechanism using the social cost, we will require a couple additional concepts to measure efficiency and we will also require some standard additional properties for our mechanism, that will facilitate our task (or make it possible in some cases).

2.2.1 Social Cost and Social Welfare

As exposed, we are in need of tools to compare mechanisms objectively, the most prevalent ones refer to the *ex post* efficiency of the society as a whole, we will first introduce social cost and social welfare as part of such tools.

Let us start by the notion of social cost, which we define as

$$\pi(S) = C(S) + \sum_{i \notin S} v_i \tag{2.2}$$

This is, the actual cost of the good and the valuations "unsatisfied" of the agents who do not value the good enough to pay their share.

Now that we have delved on the concept of social cost, it is a good moment to introduce social welfare as well.

We can define social welfare as the value provided by the mechanism minus its cost:

$$SW(S) = \sum_{i \in S} v_i - C(S)$$

We can now show that our Shapley value mechanism in example 2.2 is not efficient, since $\sum v_{i \in S^*} - C(S^*) = 0.5 + 0.4 + 0.35 - 1 = 0.25$, which is not the highest social welfare attainable.

Indeed, we could open the consumption of the good to players 3 and 4, for it doesn't increase the cost but the social welfare increases up to $\sum v_{i \in S} - C(S^*) = 0.5 + 0.4 + 0.35 + 0.2 + 0.1 - 1 = 0.55$.

The social welfare criteria is the standard when analysing the economic efficiency of a mechanism, yet Feigenbaum *et al.* [Fei+03] show that we cannot achieve a finite social cost approximation for a dominant strategy and budget balanced mechanism. The result holds even if we only require no deficit instead of budget balance, so we are forced to consider alternative criteria.

2.2.2 Alternative Criteria

Since we cannot solely rely on social cost and social welfare solely we are once again forced to explore alternative criteria for our efficiency analysis. It is to overcome this recently exposed limitations on the social cost and welfare criteria that Roughgarden and Sundararajan [RS09] propose the α-approximation criteria, which compares the mechanism with the optimally efficient one.

With that in mind, let us introduce a couple of new concepts:

Given a truthful mechanism, we say that a mechanism is *α-approximate* if the expected social cost is at most $\alpha \geq 1$ the expected social cost of the optimal outcome.

This is:

$$\pi(S) \leq \alpha\pi(S^{\star})$$

For instance, since the VCG mechanism is efficient (produces an outcome with minimal social cost), it is 1-approximate.

The other mechanism we have studied so far, the Shapley value mechanism, is $\mathcal{H}_n$-approximate, as proved by Roughgarden and Sundararajan [RS07]

Where $\mathcal{H}_n$ is the n_{th} harmonic number, this is:

$$\mathcal{H}_n = \sum_{i=1}^{n} \frac{1}{i}$$

We say that a mechanism is *β-budget-balanced*, if the

$$\sum_{i \in S} p_i \in \left[\frac{C(S)}{\beta}, C(S)\right]$$

where $\beta \geq 1$.

For instance, the Shapley value cost-sharing mechanism is 1-budget-balanced, since the price charged is $p_i = 1/|S|$, trivially the sum of $|S|$ times this amount equals to C(S), since we have made the cost unitary, if such were not the case the price would the be $p_i = C(S)/|S|$ and we would arrive at the same conclusion.

Before following on we will introduce a couple of more properties:

1. **Equal Treatment**: A mechanism satisfies equal treatment if and only if every two players i and j that submit the same bid receive the same allocation and price.

2. **Upper Semi-Continuity**: A mechanism satisfies upper semi-continuity if and only if the following condition holds for every player i and bids b_{-i} of the other players: if player i wins with every bid larger than b_i, then it also wins with the bid b_i.

3. **Consumer Sovereignty**: A mechanism satisfies consumer sovereignty if and only if, for all players i and bids b_{-i} of the other players, there exists a bid b_i such that player i wins when the bid profile is (b_i, b_{-i}).

We can now take advantage of this new definitions with the following characterization:

Proposition 2.5. ***[Dob+18]*** *A deterministic, truthful, and budget-balanced cost-sharing mechanism for public excludable good problems satisfies equal treatment, consumer sovereignty, and upper semi-continuity if and only if it is the Shapley value mechanism.*

The following proof (fully extracted from [Dob+18]) is of no particular interest for us, but it will help us later:

Proof. Fix such a mechanism M. We first note that all thresholds $t_i(b_{-i})$ induced by M must lie in $[0, 1]$: every threshold is finite by consumer sovereignty, and is at most 1 by the budget-balance condition. We proceed to show that for all players i and bids b_{-i} by the other players, the threshold function t_i has the same value as that for the Shapley value mechanism (2.1). We prove this by downward induction on the number of coordinates of b_{-i} that are equal to 1.

For the base case, fix i and suppose that b_{-i} is the all-ones vector. Suppose that $b_i = 1$. Since all thresholds are in $[0, 1]$ and M is upper semi-continuous, all players win. By equal treatment and budget-balance, all players pay $1/n$. Thus, $t_i(b_{-i}) = 1/n$ when b_{-i} is the all-ones vector, as for the Shapley value mechanism.

For the inductive step, fix a player i and a bid vector b_{-i} that is not the all-ones vector. Set $b_i = 1$ and consider the bid vector $b = (b_i, b_{-i})$. Let S denote the set of players j with $b_j = 1$. Let $R \geq S$ denote the output of the Shapley value mechanism for the bid vector b-the largest set of players such that $b_j \geq 1/|R|$ for all $j \in R$.

As in the base case, consumer sovereignty, budget-balance, and equal treatment imply that M allocates to all of the players of S at a common price p. For a player j outside S, b_{-j} has one more bid of 1 than b_{-i} (corresponding to player i), and the inductive hypothesis implies that its threshold is that of the Shapley value mechanism for the same bid vector b. For players of $R \setminus S$, this threshold is $1/|R|$. For a player outside R, this threshold is some value strictly greater than its bid. Since $b_j \geq 1/|R|$ for all $j \in R$ and M is upper semi-continuous, it chooses precisely the winner set R when the bid vector is b. This generates revenue $|S|p + (|R| - |S|)/|R|$. Budget-balance dictates that the common threshold p for all players of S, and in particular the value of $t_i(b_{-i})$, equals $1/|R|$. This agrees with player i's threshold for the bids b_{-i} in the Shapley value mechanism, and the proof is complete. □

2.2.3 Upper and Lower Efficiency Bounds

Now we move forward to discussing the upper and lower bounds of deterministic mechanisms, which will serve us later in our efficiency comparison. We will start with the symmetric lower bound case and then jump to discuss the randomization and its effects in efficiency.

Theorem 2.6. ***Lower Bound for Deterministic Symmetric Mechanisms [Dob+18]*** *No deterministic and budget-balanced cost-sharing mechanism for public excludable good problems that satisfies equal treatment is better than $\mathcal{H}_n$-approximate, where n is the number of players.*

Proof. Let us have a mechanism M as the one we describe. If M was to violate the consumer sovereignty rule, then we could find a player i and bids b_{-i} such that $t_i(b_{-i}) = +\infty$. If we were to let any valuation tend to infinity, then our mechanism would fail to achieve a finite social cost approximation factor.

Suppose now that our mechanism grants the consumer sovereignty property. The proof of proposition 2.5 shows that the outcome of the mechanism agrees with that of the Shapley value mechanism except on the measure-zero set of bid vectors for which there is at least one bid equal to $1/i$ for some $i \in \{1, \ldots, n\}$. As in Example 1.1, setting players' valuations to $v_i = \frac{1}{i} - \delta$ for each i, for arbitrarily small $\delta > 0$, shows that M is no better than $\mathcal{H}_n$-approximate. □

So far we have centered our efforts in the study of symmetric and deterministic mechanisms, but we are now in position to study upper and lower bounds for the non symmetric case, in particular we will take keen interest in our β-budget-balanced and α-approximated mechanisms.

But first, let us consider randomized mechanisms and their impact in our analysis before moving forward.

Randomization and its effects in the Upper and Lower Bounds

The theorem 2.6 is a crucial one in our discussion, we will now proceed to generalise its effects, we will address a 2-player scenario in order not to complicate the exposition, since the proceedings can be expanded to an arbitrary number of players with ease.

Proposition 2.7. ***Lower Bound for Deterministic Mechanisms [Dob+18]*** *A deterministic budget-balanced cost-sharing mechanism for a 2-player public excludable good problem is at least* 1.5*-approximate.*

Proof. Let us have a unitary bid for both players, so that our bid vector is $b_1 = b_2 = 1$. If a mechanism's outcome is a social cost approximation ratio better than 2, then it has to allocate the good or service to both players. Let us assume that we have such a mechanism, then player 1 pays p while player 2 pays $1 - p$. Without loss of generality, let $p \leq 0.5$. By proposition 2.3, the second player's threshold function satisfies $t_2(1) = 1 - p$.

Now let $b_1 = 1$ and $b_2 = 1 - p - \epsilon$ for $\epsilon > 0$ but sufficiently small. If both players win and our social cost is optimal, then our social cost equals 1. Since $t_2(1) = 1 - p$, then player 2cannot win in the presented mechanism, and the social cost is $1 + (1 - p - \epsilon) \geq 1.5 - \epsilon$ regardless of player 1 winning or loosing. □

We can also use a similar proof to asseverate that the best possible universally truthful and budget-balanced cost-sharing mechanism is 1.25-approximate, this is again in a in a two player scenario for public excludable goods.

Randomized Mechanisms

A randomized mechanism is a probability distribution over a deterministic mechanism. Randomization has several effects on efficiency, let us explore some of them related to our binding efforts:

Proposition 2.8. ***[Dob+18]*** *There is a universally truthful, budget-balanced, and* 1.25*-approximate mechanism for the two-player public excludable good problem.*

Proof. Let us select $\gamma \in [0, 1]$ uniformly at random for the first iteration. When players bids are at least γ and $1 - \gamma$, respectively for the 1^{st} and 2^{nd} player, then the mechanism halts with $S = \{1, 2\}$, $p_1 = \gamma$, and $p_2 = 1 - \gamma$. In the scenario where one of the player did not bid enough, such player is removed in the following iteration and the remaining player is asked to pay the full cost 1.

The mechanism is universally truthful and budget-balanced, since so far we're merely proposing that one player is expelled form S and the other bears the full cost. Regarding the expected social cost, leet us assume truthful bids so that $v_1 \geq v_2$ and let $x = v_1 + v_2 - 1$.

If $x < 0$, the optimal outcome is not providing the good, since players valuations are inferior to the cost, this is $S = \emptyset$ with probability 1. This outcome is also welfare-maximizing. If $v_2 \geq 1$, then $S = \{1, 2\}$ (since we are assuming that $v_1 \geq v_2$).

Now that we have considered the trivial scenarios, let $x, v_1, v_2 \in [0, 1]$, since the optimal social cost is 1 the mechanism will then select γ such that $v_1 \geq \gamma$ and $v_2 \geq 1 - \gamma$ with probability x. Both players win in this scenario, and the incurred social cost remains to be 1.

As opposed to this scenario consider now that, neither player wins, then the social cost would be $1+x$, and the expected approximation ratio we attain with the use of this algorithm and this valuation profile is $x \cdot 1 + (1-x) \cdot (1+x)$. Now, $x = 0.5$ maximizes this ratio, where the ratio is 1.25.

Last, if $v_1 \geq 1$ but $v_2 < 1$, both players win with probability v_2, and only player 1 wins with probability $1 - v_2$. The optimal social cost remains the same, 1, and the expected social cost is $v_2 \cdot 1 + (1-v_2)(1+v_2)$. Again, this is maximized when $v_2 = 0.5$, and the expected social cost is 1.25. □

Now that we have the upper bound for a randomized mechanism let us analyse the lower bound for it. Then we will proceed to the definition of a new mechanism.

Proposition 2.9. ***[Dob+18]*** *There is a constant $c > 0$ such that: no truthful-in-expectation and β-budget-balanced mechanism for public excludable good problems is better than $c \cdot \frac{\mathcal{H}_n}{\beta}$-approximate, where n is the number of players.*

Proof. Fix values for n and $\beta \geq 1$. No mechanism can be better than a 1-approximate, so we can assume that n is sufficiently large (otherwise let $c = \beta/\mathcal{H}_n$).

We will try to define a distribution over valuation profiles where the sum of the valuations is likely to be large but every mechanism is likely to produce an empty allocation. So let $a_1, \ldots, a_n$ be independent draws from the distribution with density $1/z^2$ on $[1, n]$ and remaining mass $(1/n)$ at zero. Set $v_i = a_i/(4n\beta)$ for each i and $V = \sum_{i=1}^{n} v_i$. We can note that V is likely to be at least a constant fraction of $(\ln n)/\beta$. To see why, we have

$$E[V] = nE[v_i] = \frac{\ln n}{4\beta}$$

and

$$Var[V] = nVar[v_i] \leq nE[v_i^2] = \frac{1}{16\beta^2},$$

thus

$$\sigma[V] \leq \frac{1}{4\beta}.$$

We can now use Chebyshev's inequality,and have

$$Pr[|X - E[X]| \geq \gamma \cdot \sigma[X]] \leq \frac{1}{\gamma^2}$$

for all $\gamma > 0$, we now can state that $V \geq (\ln n - 2)/(4\beta)$ with probability at least $3/4$. For sufficiently large n, $(\ln n - 2)/(4\beta)$ is at least $\mathcal{H}_n/(8\beta)$.

Let us now have a mechanism M, truthful in expectation and β-budget-balanced in expectation, this is, for every bid vector, the expected revenue of M is at least

a β fraction of its expected cost, the latter equals 1 minus the probability that no player wins.
We claim that the expected revenue of M, over both the random choice of the valuation profile and the internal coin flips of the mechanism, is at most $1/(4\beta)$. To see why the claim implies the theorem, note that this would imply that the expected cost of M is at most $1/4$, and so with probability at least $3/4$, M chooses the empty allocation.

Conditioned on the event that $\sum_{i \in U} v_i \geq \mathcal{H}_n/(8\beta)$, the probability that M chooses the empty allocation is at least $1/2$. Then, there exists a valuation profile v with $\sum_{i \in U} v_i \geq \mathcal{H}_n/(8\beta)$ such that, with probability at least $1/2$ over the internal randomness of M, M chooses the empty allocation. The expected social cost of M on this valuation profile is at least $\mathcal{H}_n/(16\beta)$, while the optimal social cost remains at most 1.

In order to prove the previous claim and upper bound the expected revenue of our mechanism with respect to this distribution over valuation profiles, we need first assume that M is a truthful deterministic mechanism. For every fixed threshold $t = t_i(b_{-i})$ that arises in the mechanism (2.3), the expected revenue extracted from player i is $t \cdot Pr[v_i \geq t] \leq 1/(4n\beta)$. Using now the linearity of expectation, the expected (over v) revenue of every deterministic truthful mechanism is at most $1/(4\beta)$. Now, we can observe that a universally truthful mechanism is just a distribution over a deterministic truthful mechanisms, then the expected revenue of every such mechanism is at most $1/(4\beta)$. Finally, thanks to 2.4 we have that for every truthful-in-expectation mechanism M, there is a universally truthful mechanism M' where M and M' have the same expected revenue on every bid profile. We then can asseverate that the expected revenue of every truthful-in-expectation mechanism is at most $1/(4\beta)$. □

We can now scale down the prices of a Shapley value mechanism by $\beta \geq 1$, which will provide us with a β-budgeted and $\left(\frac{\mathcal{H}_n}{\beta} + \beta\right)$-approximate mechanism that is truthful, thus a linear degradation in β of the lower bound in 2.9 is necessary at least up to $\beta \approx \sqrt{\mathcal{H}_n}$.

2.2.4 The Hybrid Mechanism

We have now paved the row introduce a new mechanism, with all this progression in mind we can now mix both the VCG and the Shapley mechanisms to attain a truthful and approximately efficient mechanism ($\mathcal{H}_n$-approximate), this **hybrid mechanism** is not budget-balanced, but it has the no deficit property.

Hybrid Mechanism [Dob+18]

1. Accept a bid b_i from each player i.
2. Let

$$S^* \in \arg\max_{S \subseteq N} \{\sum_{i \in S} b_i - C(S)\}$$

denote a welfare-maximizing outcome.

3. Initialize $S := S^*$.
4. If $b_i \geq \frac{C(S^*)}{|S|}$ for every $i \in S$, then halt with winners S.
5. Let $i^* \in S$ be a player with $b_{i^*} < \frac{C(S^*)}{|S|}$.
6. Set $S := S \setminus \{i^*\}$ and return to Step 4.
7. Charge each winner $i \in S$ a payment equal to the minimum bid at which i would continue to win (holding b_{-i} fixed).

Both the Shapley Value and the VCG mechanisms are truthful, the hybrid mechanisms inherits this property but we still consider necessary to prove that it is $\mathcal{H}_n$-approximate:

Proposition 2.10. *The proposed hybrid mechanism is $\mathcal{H}_n$-approximate*

Proof. Following [Dob+18] consider S^* the set of winners that optimises the social cost function (2.2), we can make use of the work by Roughgarden and Sundararajan ([RS09]) showing that the Shapley value mechanism causes an additive efficiency loss of $(\mathcal{H}_n - 1)$, scaling that by our cost $C(S^*)$, we get:

$$C(S) + \sum_{i \notin S} v_i \leq C(S^*) + \sum_{i \notin S^*} v_i + (\mathcal{H}_n - 1) \cdot C(S^*) \leq \mathcal{H}_n \cdot \left(C(S^*) + \sum_{i \notin S^*} v_i \right)$$

And we have the desired result. □

We can conclude the present chapter with this proof regarding the hybrid mechanism, we have provided two mechanisms, basic in the literature (Shapley value mechanism and VCG) and have followed the work of Roughgarden *et al.* in [Dob+18] to construct an hybrid mechanism, balancing the best of both words.

We would like to make an additional comment referring to the algorithms implementability before passing to the next chapter. As shown in [GS19], we can point out that the hybrid mechanism does not escalate well computationally on the number of players due to the second step of the algorithm.

Indeed Roughgarden *et al.* [Dob+18] also concur in the fact that the hybrid mechanism is computationally expensive, they propose to use VCG and Shapley mechansim based alternatives instead.

Chapter 3

Extending the Model

So far we have assumed the players valuations to be private and that our mechanisms are prior free.

This assumptions imply that our previous mechanisms are sufficient for the analysis of the financing of several public excludable goods that fit this axioms, yet they still fall short to tackle other models.

Take for instance the financing of the auctioneer that conducts a bid, either an online portal or a person with a hammer and a baroque wig, this trusted third party is necessary to conduct an auction. We could view the cost of operating this third party as the public excludable good we want to finance, since it only renders a service to the parties concurring to the auction, but a new problem arises: The value of the auctioneer's services to each bidder might be correlated to the value that other bidders perceive from it, if only because the underlying good in auction might have this property too.

But, Why can't we use the previous models?

Since we have assumed private interdependent valuations there are several axioms that would deem our model inaccurate to the study of the proposed case.

Indeed, there are scenarios where the value of the good does not only depend on the private individual assessment of their value. We provided the stocks valuation as an example in section 1.4.2, but many more come to mind where the value of an object is not solely determined by the individual perception of it.

Art is another good example, for instance the price of a statue might not only depend on how much I enjoy contemplating it, but also on experts valuations, the amount of potential buyers (liquidity) of the piece, etc.

This particularity makes us wonder on the concept of interdependent valuations, which can affect the dynamics we have analysed so far and make it necessary for us to adapt our model.

In interdependent settings, it is not possible to design DSIC mechanisms because an agents valuation depends on all the signals elicited, thus precluding us from naively applying any of the previously discussed mechanisms.

The next strongest equilibrium notion we can consider is the *ex-post* efficiency, which we have already discussed up to some point, but it will require adaptations to tackle the signaling model.

As we just discussed, one of the mechanisms exposed this far can capture the fact that valuations might depend on each other, so in this chapter we are going to adapt the scope of our mechanism to consider ***public interdependent valuations*** in the scenario of *submodularity over signals*.

But why?

To answer this question we can use the following impossibility theorem [Ede+18], it will help us understand why we can't rely on the previously exposed mechanisms, and thus, we are forced to extend our models, we can summarize it as follows:

We cannot obtain a deterministic mechanism with any bounded ratio that is truthful, prior-free when the valuations do not satisfy single crossing. We refer to the same citation, where Eden *et al.* show that deterministic prior-free mechanisms do not satisfy the single crossing property, deeming the theorem applicable to our case.

3.1 Interdependent Values

In this line of research, the Interdependent Value Model (IDV) was first introduced by Milgrom and Weber [MW82] to consider this interdependent value interactions between agents.

In this model, valuation functions are public and each agent has a private signal about several outcomes.

3.1.1 Signals and Concept Adaptation

Since we are extending our model it is only natural that some of the concepts we have used this far in the prior free private independent valuation model require a fine tune to fit the public interdependent model, let us start this section by defining what a signal is:

Broadly speaking we can say that a signal is "incomplete information" about something. Perhaps the most prevalent example in the literature to illustrate this is the "lemons market" proposed by Akerloff [Geo+70].

Picture that you want to buy a used car, you can't possibly know the state of the vehicle at first glance, this happens due to the *information asymmetry*, where the seller knows the condition of the vehicle, but the buyer does not have the same information, and can not distinguish a car in good condition from a car in a bad shape (lemon). Even if you can't possibly know the precise condition of the car you can collect shreds of information, such as mileage, sensations on a test drive, the general overview, cosmetic damage, price, etc. that serve as a proxy of the state of the car. You can then aggregate those signals and form expectations about the state of the car and its value.

Formally a signal is a random variable $W \in [0, \omega]$, where ω can be ∞.

We will use $w_i \in [0, \omega_i]$ to denote player i's private signal. Since we are discussing interdependent values, signals are to be drafted from a joint distribution function F with density f over $\Omega = [0, \omega_i]^n$.

We will use the vector $w = (w_1, \cdots, w_n)$ to denote the collection of all signals.

Along this chapter we will consider that a players valuation can be expressed as a function of the signals, we will denote player i's valuation as $v_i(w)$.

As opposed to the previous chapter, where we treated v_i as a value, we will now define the map:

$$\begin{aligned} v_i : \Omega &\to \mathbb{R} \\ w &\mapsto v_i(w) \end{aligned}$$

To clarify it, if v'_i were the valuation as defined in the previous chapter, we shortened $v_i(good)$ to equal to the perceived value of the good for player i, v'_i.

In order not to complicate the model further we will introduce some standard assumptions in this line of research, specifically about the valuation functions v_i, we consider them:

- Public for every player
- Non-negative and normalized: $(v_i(\vec{0}) = 0)$
- Twice continuously differentiable
- Non-decreasing in all variables and strictly increasing in w_i
- Expectations are finite: $\mathbb{E}_w[v_i(w)] < \infty$

A mechanism is *ex-post* **Incentive Compatible (IC) [RTC16]** if, for every i, true signals w_i, signal profile w_{-i} and false signals $\tilde{w}_i$:

$$x_i(w)v_i(w) - p_i(w) \geq x_i(\tilde{w}_i, w_{-i})v_i(w) - p_i(\tilde{w}_i, w_{-i}), \tag{3.1}$$

The reader can recollect the incentive compatibility concept as previously exposed in section 1.5.2, now we have complemented the concept with the signals and a rough interpretation of the concept would be that, if the mechanism is **IC** the player prefers the payment of a truthful signal (the utility derived from the signal profile w_i is: $x_i(w)v_i(w) - p_i(w)$) over the payment of lying over the signal ($\tilde{w}_i$).

A mechanism is *ex-post* **Individualy Rantional (IR) [RTC16]** if, for every i and signal vector w:

$$x_i(w)v_i(w) - p_i(w) \geq 0. \tag{3.2}$$

Again, as we can recall from the definition in section 1.5.2 we require that the utility form participating in the mechanism is non negative, we again impose that principle, since the utility for player i is $x_i(w)v_i(w) - p_i(w)$.

If a mechanism is both *ex-post* IC and IR we will also call it **EPIC-IR** following the notation in Eden *et al.* ([EGZ22]).

A further step we need to extend our model is the definition of **conditional virtual values**

$$\varphi_i(w_i|w_{-i}) = -\frac{\frac{d}{dw_i}B_i(w_i|w_{-i})}{f_i(w_i|w_{-i})} = v_i(w) - \frac{1 - F_i(w_i|w_{-i})}{f_i(w_i|w_{-i})} \cdot \frac{d}{dw_i}v_i(w) \tag{3.3}$$

Where d denotes the derivative and the **conditional revenue curve** $B_i(\cdot|w_{-i})$ of bidder i is:

$$B_i(w_i|w_{-i}) = v_i(w) \int_{w_i}^{\omega_i} f_i(t|w_{-i})\, dt$$

In plain words the conditional revenue curve serves us to represent the expected revenue perceived by bidder i of the threshold price $v_i(w)$ given the signals w_{-i}.

Also the **conditional marginal density** $f_i(\cdot|w_{-i})$ for player i given the signals w_{-i} can be written as:

$$f_i(w_i|w_{-i}) = \frac{f(w)}{\int_0^{\omega_i} f(t, w_{-i})}\, dt$$

This three previous concepts, further developed in [RTC16] will provide a handy tool when adapting the Meyerson theorem we introduced in our first chapter (in section 1.9).

Proposition 3.1. *For every interdependent values setting, a mechanism is ex post IC and ex post IR if and only if for every i, w_{-i}, the allocation rule x_i is monotone non-decreasing* *in the signal w_i, and the following payment identity holds:*

$$p_i(w) = x_i(w)v_i(w) - \int_{v_i(0,w_{-i})}^{v_i(w_i,w_{-i})} x_i(v_i^{-1}(t|w_{-i}), w_{-i})\, dt - (x_i(0,w_{-i})v_i(0,w_{-i}) - p_i(0,w_{-i}))$$

$$p_i(0,w_{-i}) \leq x_i(0,w_{-i})v_i(0,w_{-i})$$

Myerson Mechanism for Interdependent Values [RTC16]

1. *Elicit signal reports w from the bidders*
2. *Maximize the conditional virtual surplus by allocating to the feasible set S with the highest non-negative conditional virtual value $\sum_{i\in S}\varphi_i(w_i|w_{-i})$, breaking ties lexicographically*
3. *Charge every winner i a payment $p_i(w) = v_i(w_i', w_{-i})$, where w_i' is the threshold signal such that, given the other signals w_{-i}, if i's signal were below the threshold, he would no longer win the auction*

Where $v_i^{-1}(t|w_{-i}, w_{-i})$ is the only element w' such that $v_i(w', w_{-i}) = t$

Proposition 3.2. *[RTC16] The expected revenue of an EPIC-IR equals its expected conditional virtual surplus, up to an additive factor:*

$$\mathbb{E}_w\left[\sum_i p_i(w)\right] = \mathbb{E}_w\left[\sum_i x_i(w)\varphi_i(w_i|w_{-i})\right] - \sum_i \mathbb{E}_{w_{-i}}\left[x_i(0,w_{-i})v_i(0,w_{-i}) - p_i(0,w_{-i})\right]$$

for every interdependent values setting.

The latter suggest that we need just maximize the conditional virtual surplus pointwise so we also maximise the expected revenue.

Whereas the previous characterization serves us generally in any interdependent valuation scenario, on the following pages we will restrain our efforts to the public interdependent valuation scenario.

3.1.2 The Potential Mechanism

Once again we have the tools to introduce a new mechanism at the twilight of our chapter, before we do this let us ask a rhetorical question to the reader:

Shouldn't we share our signals?

A classic example to illustrate the latter could prove convenient is the oil drilling rights auctions. First exposed by Wilson ([Wil66]), it serves us as as perfect example.

Imagine that Alice and Bob want to acquire the rights to drill oil in a tract of land, they are uncertain on the quantity they can extract from the well, and they each hire a specialist to aid with the measuring.

Alice's expert comes back with the signal w_1, whereas Bob's provides signal w_2.

Now both Alice and Bob can form expectations on the amount of oil they can extract. So, if they don't share the expert assessment:

$\mathbb{E}[v_1|w_1]$ represents Alice's valuation and $\mathbb{E}[v_2|w_2]$represents Bob's. But the amount of oil remains the same, so the value for Alice and Bob, although it can differ, is correlated.

Then, what happens if Bob's expert made an error, severely overestimating the quantity of oil? That would lead to Bob's valuation $\mathbb{E}[v_2|w_2]$ to be artificially high, and would cause Bob to overbid for the drilling rights.

When Bob realises that the signal he had used to form expectations was anomalous it's too late, the winners curse is complete.

Signals are positively correlated and having access to Alice's signal would affect Bob's expectations, and consequently, the value he attributes to the well. This is, if Bob had access to Alice's signal perhaps he would have been able to see that the value he was ready to pay for the rights was too high, avoiding (or at least moderating) his overbidding.

Now let us take a further step and define a new mechanism:

Potential Mechanism for Public Interdependent Values [Maz24]

Input: Signals $w_1, \ldots, w_n$.
Output: The set of agents S^* that are served and the payment vector $p = (p_1, \ldots, p_n)$.

1. Choose set $S^* \in \arg\max_{S \subseteq N} \left\{\sum_{i \in S} v_i(w) - \mathcal{H}_{|S|}\right\}$.
2. Set payments $p_i = \inf_{w \geq 0}\{v_i(w', w_{-i}) : x_i(t, w_{-i}) = 1\}$ if $i \in S^*$, and 0 otherwise.

The potential mechanism has some of our favourite properties:

Proposition 3.3. *The potential mechanism for interdependent public valuations is EPIC-IR, no-deficit and at most $\mathcal{H}_n$-approximate.*

Proof. Let us split the proof into parts:

- *EPIC-IR:* Since the allocation is determined in the first step of the algorithm, we can then refer to proposition 3.1 for the payment rule, let's break it down: The allocation rule is monotone and non decreasing in $w_1, \dots, w_n$, since $v_i(w)$ are monotone an non-decreasing in w.
  If $i \notin S^*$, then the payment of $p_i(w) = 0$ and $x_i(w) = 0$
  Now let $\tilde{w}_i = \text{argmin}_{w \geq 0}\{v_i(w', w_{-i}) : x_i(t, w_{-i}) = 1\}$, if $\tilde{w}_i = 0$, then $p_i(w) = v_i(0, s_{-i}) = v_i(0, s_{-i})x_i(0, s_{-i})$ when we hold the inequality of proposition 3.1. Else, if $i \notin S^*$ we find ourselves in the same situation, in either case $p_i(w) = v_i(0, s_{-i}) = v_i(0, s_{-i})x_i(0, s_{-i})$. Last if $i \in S^*$:
  $$x_i(w)v_i(w) - \int_{v_i(0,w_{-i})}^{v_i(w_i,w_{-i})} x_i(v_i^{-1}(t|w_{-i}, w_{-i}))\, dt = x_i(w)v_i(w) - \int_{0}^{v_i(w_i,w_{-i})} 1_{t \geq v_i(\tilde{w}, w_{-i})}\, dt$$
  $$= x_i(w)v_i(w) - [v_i(w) - v_i(\tilde{w}, w_{-i})]$$
  $$= v_i(\tilde{w}, w_{-i}) = p_i(w)$$
  Thus, the mechanism is EPIC-IR by the application of proposition 3.1.

- *$\mathcal{H}_n$-approximate:* Let S^* be the set of bidders served by our mechanism, then $\sum_{i \in S^*} v_i(w) - \mathcal{H}_{|S^*|} \geq \sum_{i \in N} v_i(w) - \mathcal{H}_n$, therefore:
  $$\mathcal{H}_n \geq \mathcal{H}_n - \mathcal{H}_{|S^*|} \geq \sum_{i \in N \setminus S^*} v_i(w) + C(S^*) = \pi(S^*)$$

- *No-deficit:* Let S^* be a non empty allocation, let $\tilde{w}_i$ be the signal that realizes p_i, also let S_i be the allocation with signals $(\tilde{w}_i, w_{-i})$, then by its definition $p_i = \dfrac{1}{|S_i|}$.
  Since the valuations v_i are monotone and $\tilde{w}_i \leq w_i$ then $|S_i| \leq |S^*|$ and thus $p_i \geq \dfrac{1}{|S^*|}$.
  Therefore adding all payments results in: $\sum_{i \in S^*} p_i \geq 1 = C(S^*)$

□

In this chapter we have seen the inability of our previous models, namely the VCG, the Shapley value and the hybrid mechanisms, to tackle a scenario where valuations are interdependent, we have also seen that when players share their signals the potential mechanism is a perfect candidate to come to rescue and that has several of the properties we like on a mechanism.

Chapter 4

Conclusions

Along this paper we have introduced the theoretic prerequisites needed for the comprehension and understanding of the financing of public exludable goods from an algorithmic mechanism design perspective.

We have provided the necessary background to understand the concept of mechanism, we have then introduced several mechanisms and then elaborated a review of their most significant properties. We have analysed the VCG mechanism, the Shapley value mechanism and the hybrid mechanism for the prior free and private valuations case. We have also provided the tools for the economic efficiency comparison of a mechanism and later introduced the hybrid mechanism, a blend of the aforementioned two that balances the strengths of both mechanisms. We have also analysed the public interdependent valuations scenario, where we have provided the potential mechanism algorithm as our reference tool.

After passing that milestone we feel that there are still open questions left behind:

- In our annex we propose a protocol that forces the bidder to share a boundary of their signals to participate in the mechanism, we sill wonder: Are there better ways to stimulate individual incentives on signal sharing?
- Can we further extend the model to consider Private Interdependent Valuations?

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Appendix A

Annex

A.1 Theory in Action

In this last section we will expose the problem that motivated our research, the relayer financing in a PBS schema. We will prescind of the definitions of the blockchain architectural concepts, which are not strictly necessary for our analysis, sufficient to say that we will trace parallelisms between the agents that participate in this schema, and our model. The reader can find more in the Proposer-Builder separation (PBS) in [But21; Hei+23], we will content ourselves with a basic approach to the concept as follows.

The PBS splits the block proposer and the builder in two different roles, the builder, aggregates the transactions in a blockchain and constructs the block, which then sends to the proposer. The proposer is tasked with the validation and proposition of the block to its peers, this mechanics require from a trusted third party mediator in order to ensure the adequate functioning of the protocol, the relayer.

As of the moment of writing this, the relayer infrastructure is financed at a deficit in the Ethereum blockchain, some agents pay for the cost of operating the relay upfront and then cry about it.

Since the existence of relayers is necessary for the PBS block building, the absence of this figure is not an option in this block building method blockchains.

We don't intend to delve into the details of this protocol, sufficient to say that this distribution ensures the specialization of agents in different aspects of the block production.

Typically the builders have better understanding on the miner extractable value (MEV) and the proposers have the infrastructure demanded by the protocol to build blocks.

Since proposers are "not that good" as builders in observing MEV they "outsource" the construction of the block to specialised agents (builders) and collect payments from them to accept their blocks and then send them to their peers.

On a nutshell, the relayer infrastructure is the public excludable good to be financed, the proposers sell their capacity to build blocks to builders, that get their revenue from the MEV extraction minus the cost of "buying" the block proposing rights.

The interactions between builders and proposers constitute a zero sum game, so we don't intend on paying attention to it, on the other hand, MEV is not observable, so the builders may have (and in fact is typical) different valuations regarding the proposition of a block.

We will now capitalize in our theoretical recompilation and apply the different models to the problem at hand.

A.2 First Attempt, a Naive Application of the Hybrid Mechanism

As we promised in the second chapter, we will first apply the hybrid mechanism (2.2.4) assuming private independent valuations.

We will leave out the application of the VCG and Shapley value mechanisms at this time, since they don't improve the current real live scenario (the VCG runs at a deficit which is the same scenario we see in the Ethereum chain and the Shapley value mechanism is not efficient and we are not willing to propose that trade-off)

Consider C_r to be the cost of operating the relay, let us also commit to a fee parameter $\gamma \in [0, 1]$ and consider an arbitrary number of bidders.

We can apply the algorithm we provided in 2.2.4 pretty straight forwardly, we will only modify the steps 2, 4 and 5, since we are considering the possibility of a fee (if the fee is 0 the reader can disregard this comment and apply the algorithm for the mechanism directly).

If $\gamma > 0$ then, as we said, we have to adapt the algorithm:

Hybrid Mechanism with fee

1. Accept a bid b_i from each player i.
2. Let
$$S^\star \in \text{argmax}_{S \subseteq N}\{\sum_{i=1}^{n} b_i : (1-\gamma)b_i \geq \frac{C_r}{\mid S \mid}\}$$
denote a welfare-maximizing outcome.
3. Initialize $S := S^*$.
4. If $(1-\gamma)b_i \geq \dfrac{C(S^*)}{|S|}$ for every $i \in S$, then halt with winners S.
5. Let $i^* \in S$ be a player with $(1-\gamma)b_{i^*} < \dfrac{C(S^*)}{|S|}$.
6. Set $S := S \setminus \{i^*\}$ and return to Step 4.
7. Charge each winner $i \in S$ a payment equal to the minimum bid at which i would continue to win (holding b_{-i} fixed).

This alternative is practically equal to the original hybrid mechanism, but we are considering that the protocol owner may charge a fee, as is typical in the blockchain ecosystem.

On a separate note on implementation, we should also consider that all transactions are public by default in a typical blockchain, and our mechanisms assume private bids for an adequate functioning.

We can easily bypass this limitation with the use of criptographical proofs to show that $(1-\gamma)b_i \geq \frac{C_r}{|S|}$ without the publication of the bid.

The application of the mechanism will then result in an $\mathcal{H}_n$-approximate and no deficit financing of the relay, we need also consider the fee γ in the efficiency loss, but since our proposed fee is a constant times the bid, the task is trivial.

A.3 Second Attempt, an Application of Potential Mechanism for Public Interdependent Valuations

In this scenario we need our protocol to publish the signals of the agents, (otherwise we would have to explore a private interdependent valuations model), remember that in this model the valuation functions are public, so any agent can calculate another ones valuation given that they now the signals.

But the fact remains, How can we make out participants share heir signals?

We can establish a commit and reveal phase through the use of smart contracts, on the first one players input their signal in the commit phase, when all players have done so the signal vector w is then published.

We can now start our potential mechanism by following our algorithm and as 3.3 dictates we have an EPIC-IR, no-deficit and at most $\mathcal{H}_n$-approximate mechanism.

A.4 Criticism

In the present annex we have considered two scenarios, one with private independent and prior free valuations and another one with public interdependent valuations both our proposed scenarios have good theoretical results, yet in the relays financing problem is a bit more complex than that. First we extended the model from the private independent valuation model to a public interdependent one. An argument could be made that the public knowledge of the valuation functions is quite a stretch, and indeed, quoting Bergemann and Morris in [BM05] *"The mechanism design literature assumes too much common knowledge of the environment among the players and planner"*. This one we just made (publicly known valuation functions) could arguably be one of such assumptions, on the other hand we could try interpolating every agents valuation given sufficient iterations of the game if not for one other limitation:

The blockchain environment is quite prone to enabling false name strategies, this feature comes with the privacy it provides, but any agent could easily set up a secondary (tertiary or even n-esimal) address to operate, for third parties those accounts could look and even operate as separate entities.

This feature makes it imperative that we demand *Sybil resistance* in our mechanisms, following the research of Mazorra [MDP23; Roi23] we can establish that the VCG mechanism, the Shapley value mechanism or the hybrid mechanism for the financing of public exludable goods we introduced in chapter 2.1 and 2.2.4 are not Sybil-proof. Our potential mechanism of chapter 3.1.2 for public interdependent valuations is no better (the proof Mazorra provides in [Roi23] is meant for a different scenario, luckily it applies to our case of public interdependent valuations potential mechanism just the same).

The options we provided clearly fall short to tackle a complex and intricate problem, further research is needed to adapt our mechanisms to be sybil-proof. As we also mentioned in 2.2.4, our algorithms should also be computationally implementable and should also be realistic in the assumptions they make.