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# The Poisson process and some of its applications

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**To my brother Josep,**
**I wish you could see this.**

# Abstract

In this document we study the Poisson Process and some of its applications. This is a point process which is intended to model situations where one has a set of independent random points in a space. This situation appears naturally in many circumstances and therefore, the process has been widely known for decades. Throughout the memoir, a list of the abundant applications in which the Poisson process is used is given, and some are explained in detail. In addition, using the theory explained during the thesis, the Poisson process in one dimension is also applied to try to predict the number of traffic accidents in Catalonia in a given time interval, as well as the number of associated fatalities.



# Resum

En aquest document s'estudia el Procés de Poisson i algunes de les seves aplicacions. Aquest és un procés de punts que té com a objectiu modelar situacions en les quals hi ha un conjunt de punts aleatoris i independents en un espai. Aquesta situació apareix de forma natural en moltes circumstàncies, fent que el procés sigui àmpliament conegut des de fa dècades. Al llarg de la memòria es proporciona una llista de les nombroses aplicacions en les quals s'utilitza el Procés de Poisson i algunes d'aquestes s'expliquen en detall. A més, usant la teoria explicada durant aquest treball, també s'aplica el Procés de Poisson en una dimensió per intentar predir el nombre d'accidents de trànsit a Catalunya en un interval de temps determinat, així com el nombre de víctimes mortals associades.



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# Contents

# Introduction

In statistics and probability theory, a random collection of points in a given space (henceforth called $S$) is called a *point process* This essay studies the *Poisson process*, which is a random sequence of points in $S$ that satisfies the following two properties:

1. **Independence:** For any disjoint subsets $A, B \subseteq S$, the random variables $N(A)$ and $N(B)$ counting the number of points falling in each of these subsets, are independent.

2. **The Poisson distribution:** Given a subset $A \subseteq S$, the variable $N(A)$ follows a Poisson distribution. That is, there exists $\mu(A) \geq 0$, called the parameter of the Poisson variable, such that:

$$\mathbb{P}[N(A) = k] = e^{-\mu(A)} \cdot \frac{(\mu(A))^k}{k!}, \quad \text{for } k = 0, 1, \ldots$$

As a consequence of the Poisson distribution,

$$\mathbb{E}[N(A)] = \mathbb{V}[N(A)] = \mu(A).$$

It can be proven that $\mu(A)$ gives rise to a measure in the underlying space $S$ called the *intensity of the Poisson process.*
Reciprocally, and perhaps more relevantly, for every $\sigma$-finite and non-atomic measure $\mu$ in $S$, there exists a Poisson process with intensity $\mu$. This is going to be further developed in Chapter 1.7.

If $S = \mathbb{R}^d$, generally, the intensity $\mu$ of a Poisson process is given by the integral of a positive function on $S$ ($\mu$ is absolutely continuous with respect to the Lebesgue measure). The function $\lambda$ is called the *intensity function*, i.e.:

$$\mu(A) = \int_A \lambda(x)dx.$$

Although the two properties that define a Poisson process may seem as two separate characteristics, it is worth mentioning that they are related, since the

fundamental property of independence implies, at least in regular situations, that the counting function follows a Poisson distribution. This is seen in Chapter 1.

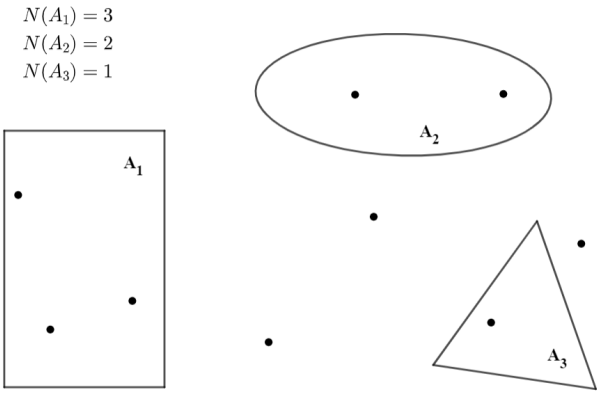


Figure 1: A Poisson process and some disjoint subsets in $S$.

The Poisson process is used to model a variety of situations, precisely because of the independence condition. It is used to study in astronomy, statistical mechanics, telecommunications, biology, economics, geology [14]...
For example, the distribution of chocolate chips in a biscuit can be viewed as a Poisson process in 3 dimensions [22] since the chocolate chips are randomly and independently distributed in the dough. In two dimensions, the location of trees in a forest, or traffic accidents, can be modelled with a Poisson process as well.
It is also used very often in queueing theory [14],[24]. For instance, in a supermarket queue, the arrival of the customers are the events and the average number of available supermarket checkouts per minute is the intensity $\mu$. This situation can be studied as a Poisson process, since the arrivals are in principle independent and at random times.

Even though the Poisson process is really helpful, obviously not everything can be modelled with it. There are many situations where there is interaction between the points. For example, the outbreaks of an infectious disease cannot be modeled by a Poisson process, it is necessary to take into account that there are positive correlations. In the opposite sense, if particles with the same charge are modelled, there will be negative correlations. In the above cases, the independence condition is not fulfilled.


Figure 2: A 1-D Poisson process.

In words of J. Kingman: *"There may be subtle effects which allow the presence of points in one region to influence those in another. There may be limitations which prevent two points from being too close together. There may be unexpected geometric patterns or regularities ("ley lines[2] ", for instance, in archaeology). A Poisson model is usually the simplest and, in a sense, the most random way in which to describe any particular phenomenon."*[3]

The Poisson process is not called so because it was invented or studied by the physicist and mathematician Siméon Denis Poisson (1781-1840) [14]. Actually, it takes this name because the Poisson distribution plays an important role in the process. Determining the origins of the process is not an easy task, since there are many notes with different notations from different authors in different countries that refer to the same process. However, as [14] explains, Norbert Wiener (1894-1964) is considered to be the first who provided a rigorous mathematical definition of the Poisson process. He referred to it as *"discrete chaos"* and *"Poisson chaos"*.

In this project we mostly work on spaces $S \subseteq \mathbb{R}^d$, for $d \in \mathbb{N}$. A Poisson process is called *homogeneous* (or *stationary* or *uniform*), when the parameter $\mu(A)$ of the counting variable $N(A)$, is precisely the Lebesgue measure of $A$, or a multiple of this. That is, $\mu(A) = \lambda|A|$, with $\lambda > 0$ independent of $A$. In this case, the likelihood of having a given number of points is the same in any subset of $S$ of equal measure. Therefore, this probability is invariant by rotations and translations. In any other case, the Poisson process is called *non-homogeneous* (or *non-stationary* or *non-uniform*). Then, the probability does depend on the location of the particular region and, in general, it will not be the same if the set is rotated or translated.

The memoir adheres to the following outline. In Chapter 1, based mainly on Kingman's book Chapters 1 and 2 [15], we first formalise the definition of Poisson process, constructing the probability space where it is defined.
Then, in Section 1.2 a heuristic argumentation is given to justify that the es-

[2] "*Ley lines*" are straight alignments drawn between various historic structures, prehistoric sites and prominent landmarks.

[3] p. 2 in Kingman's book [15].

sential property of the Poisson process, and the one that appears in so many applications, is the independence of different points. It is proven that, in a regular situation, this independence implies that the counting variable $N(A)$ follows a Poisson distribution.
From here, and bearing in mind that independence is the fundamental property, a couple of natural properties are seen. First, that the superposition of two Poisson processes is still a Poisson process (Theorem 1.11). Second, the mapping of a Poisson process is also a Poisson process (Theorem 1.14).
The next step is to see that, given a subset $B \subseteq S$, the Poisson process in $B$ conditioned on $N(B) = n$, has a multinomial distribution [1] of parameters $n$ and $p(A) = \frac{\mu(A)}{\mu(B)}$, for a subset $A \subseteq B$.
This indicates how to prove one of the most important results of this thesis: the existence theorem (Theorem 1.19). This states that, given a $\sigma$-finite and non-atomic measure $\mu$ in $S$, there exists a Poisson process with intensity $\mu$. This is seen in Section 1.7.
The existence theorem allows to construct Poisson processes having as average distribution almost any reasonable measure $\mu$.
To conclude the chapter we illustrate the versatility of the process by characterising the measures $\mu$ in $\mathbb{R}$ for which a Poisson process is almost surely a separate sequence.

Chapter 2 shows various applications of the Poisson process. The emphasis is placed on the study of road accidents with victims in Catalonia in the period from 2008 to 2020. This is done in Section 2.1. The above theory is applied to predict, by means of a non-homogeneous Poisson process, the expected number of accidents in a given period of 2021. *Weighted Poisson processes* are also introduced in order to find the expected number of fatalities in the analysed period. The usefulness of this could be crucial, as it could help governments to predict failures, and try to find solutions to minimise these numbers as much as possible. This application is based on two previous studies, one in Indonesia and another in Poland [12],[10],[9].
The following section shows a study carried out by R. D. Clarke in 1946, on the location of bombs dropped during World War II in London, by using a two-dimensional Poisson process. In the study of this application we follow the article by Shay and Shaw [24]. Along the same lines, and following [11], Section 2.2.2 explains the current usefulness of the Poisson process in finding the location of unexploded bombs.

In the last section of Chapter 2, right before the end of the project, we detail a list of fields to which the Poisson process has been applied.

# Chapter 1

# Formalization of the Poisson process

This chapter starts with the formalisation of the Poisson process. In the next section it is shown that the important property is that of independence. Subsequently, important theorems such as the Superposition Theorem and the Mapping Theorem are detailed. In the last section of the chapter, the Existence theorem is proved.
In this chapter we mainly rely on Kingman's book Chapters 1 and 2 [15].

## 1.1 Formal definition

To formally define a Poisson process it is necessary to recall some definitions.

**Definition 1.1.** *Let $S$ be a set and $\Sigma$ a $\sigma$-algebra over $S$. A positive measure is a map function $\mu$: $\Sigma \to [0, +\infty]$ such that*

*1)* $\mu(\emptyset) = 0$.

*2)* $\mu$ *is $\sigma$-additive, that is, for all countable collections $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in $\Sigma$,*

$$\mu(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=1}^{\infty} A_n.$$

*The pair $(S, \Sigma)$ is called a measurable space, and the members of the $\Sigma$ are called measurable sets.*

**Definition 1.2.** *A probability space is a triplet* $(\Omega, \mathscr{F}, \mathbb{P})$*, where* $\Omega$ *is the set of elementary outcomes (outcome space),* $\mathscr{F}$ *is a* $\sigma$*-field of subsets of* $\Omega$ *(events) and* $\mathbb{P}$ *is a probability measure, assigning a number in [0,1] to every event in* $\mathscr{F}$*.*

**Definition 1.3.** *A real-valued random variable is a function X:* $\Omega \to \mathbb{R}$ *that is measurable in* $(\Omega, \mathscr{F}, \mathbb{P})$*, which means that for all* $x \in \mathbb{R}$ *the event* $\{\omega \in \Omega: X(\omega) \leq x\}$ *belongs to* $\mathscr{F}$*.*

Observe that $X$ takes real values. A complex-valued random variables could be similarly defined as X: $\Omega \to \mathbb{C}$.
The random points, as in Figure 1, are either finite or countably infinite, subsets of the underlying space $S$, where $S$ is called the *state space*. Therefore, $S^\infty = S \times S \times \cdots \times S \times \cdots$ can be identified with the set of all (countable) sequences in $S$. A sequence $\{x_k\}_{k=1}^\infty$ is considered as an infinite vector in $S^\infty$, $\{x_k\}_{k=1}^\infty = (x_1, x_2, \ldots, x_n, \ldots)$. Our purpose is the construction of random sequences in $S$, also known as point processes, with certain properties.

**Definition 1.4.** *Let* $(S, \mathscr{F})$ *be a measurable space and* $\mu$ *a measure on it. The measure* $\mu$ *in* $S$ *is* $\sigma$*-finite if* $S$ *can be written as a countable union of measurable sets* $S_n$ *with* $\mu(S_n) < \infty$*, for all* $n \in \mathbb{N}$*.*

A measure with no atoms is called *non-atomic*, that is:

$$\mu(\{x\}) = 0, \quad \text{for any } x \in S.$$

If $\mu$ is non-atomic, the probability of having two coincident points is zero.

The most common case in applications is $S = \mathbb{R}^d$, $d \in \mathbb{N}$ and $\mu = m$, the Lebesgue measure. We will consider throughout the whole paper only state spaces $S \subseteq \mathbb{R}^d$, for some $d \in \mathbb{N}$, equipped with a $\sigma-$finite, non-atomic and positive measure $\mu$. However, a more generic theory could be studied with not many modifications.

**Definition 1.5.** *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space. A Poisson process in* $S$ *is a map* $\Pi : \Omega \to S^\infty$ *satisfying two specific conditions.*

Let $S$ be a state space with a measure $\mu$, and $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space. In order to count the random points in subsets of $S$, it is fundamental to make a distinction among all the subsets. A reasonable family of them will be chosen and these will be called the *test sets*, as in Figure 1. Thus, if

$\Pi(w) = \{x_k(w)\}_{k=1}^{\infty}$ and $A$ is a test set, which means in particular that $A$ is $\mu$-measurable, then the random variable

$$N(A) = \#\{\Pi(w) \cap A\} = \#\{k | x_k(w) \in A\}$$

is defined as the number of points of $\Pi$ in $A$, which is a number in $\mathbb{N}$. Observe that the random variable

$$N(A) : \Omega \to \mathbb{N} = \{0, 1, 2, \ldots, \infty\}$$

is a function because the right-hand side depends on $\omega$. Moreover, this function must be measurable to become a random variable. Observe that for all $\mu$-measurable set $A \subseteq S$ and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$,

$$\{\omega \in \Omega | N(A) = n\} \in \mathscr{F}. \tag{1.1}$$

Then, $\Pi$ is a Poisson process if:

1) For any $A, B \subseteq S$ $\mu$-measurable sets with $A \cap B = \emptyset$, the random variables $N(A)$ and $N(B)$ are independent.

2) For any $A \subseteq S$ $\mu$-measurable set, the random variable $N(A)$ follows a Poisson distribution with parameter $\mu(A)$, that is:

$$\mathbb{P}[N(A) = k] = e^{-\mu(A)} \cdot \frac{(\mu(A))^k}{k!}, \quad \text{for } k \geq 0.$$

In most applications $S \subseteq \mathbb{R}^d$ and the considered test sets $A$ are Lebesgue measurable in $S$, and the $\sigma$-field is the one generated by the open subsets of $\mathbb{R}^d$.

**Remark 1.6.** The Poisson process has been defined as a random sequence of points in $S^{\infty}$. It is important to point out that it could be equivalently defined as a random atomic measure in $S$.
Recall that for $x \in S$ the *Dirac measure* $\delta_x$ on $S$, is defined on any set $A \subseteq S$ by:

$$\delta_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

The Poisson process $\Pi(\omega) = \{x_k(\omega)\}_k$ can be equivalently defined as the random atomic measure $\nu(\omega) = \sum_{k=1}^{\infty} \delta_{x_k(\omega)}$. Then, for a $\mu-$measurable $A \subseteq S$,

$$\nu_\omega(A) = \sum_{k=1}^{\infty} \delta_{x_k}(A) = \sum_{x_k \in A} 1 = \#(\{\lambda_k(\omega)\}_k \cap A) = N(A).$$

More formally, denoting by $T$ the set of measures of the form $\sum_{k=1}^{\infty} \delta_{x_k}$, where $x_k \in S$, the Poisson process can be equivalently defined as the map

$$\Pi : \Omega \to T, \qquad \omega \mapsto \nu_\omega = \sum_{k=1}^{\infty} \delta_{x_k(\omega)}.$$

Clearly each sequence $\{x_k\}_{k=1}^{\infty}$ gives rise to a measure $\nu_\omega = \sum_{k=1}^{\infty} \delta_{x_k(\omega)}$. Correspondingly, a random measure $\nu_\omega$ in $T$ will be of the form $\nu_\omega = \sum_{k=1}^{\infty} \delta_{x_k}$, hence it brings associated a random sequence $\{x_k\}_{k=1}^{\infty}$, which is a subset in $S$.

## 1.2 Why independence is the strong condition?

As mentioned in the introduction, we show here that, under mild regularity conditions, the independence hypothesis in definition 1.5 somehow implies that the random counting variables $N(A)$ must have Poisson distribution. Therefore, the most determining property of a Poisson process is the independence of the counting functions of disjoint sets.
We show this with a heuristic argument, in the simple setting $\mathbb{R}^d$.
Let $A_t$ be one parameter family of test sets, like a ball of radius $t \geq 0$, a cube of side $t$, or any increasing family of sets with regular boundary in $\mathbb{R}^d$. For $n \geq 0$, define the probabilities

$$p_n(t) = \mathbb{P}\{N(A_t) = n\},$$

$$q_n(t) = \mathbb{P}\{N(A_t) \leq n\}.$$

Observe that $q_n$ and $p_n$ have only jump discontinuities, because $N(A_t)$ increases with $t$ and $q_n$ decreases with $t$, so they are both differentiable almost everywhere.
The random variable $N(A_t)$ jumps from $n$ to $n+1$ when its enlarging boundary crosses one of the random points. The probability that this occurs between $t$ and $t+h$, with $h \in \mathbb{R}^+$, is the asymptotically probability that there is a point in the region between $A_t$ and $A_{t+h}$, which it is supposed small when $h$ is small. This is shown below.
More specifically, assume that

$$\mathbb{E}[N(A_{t+h}) - N(A_t) \geq 2] = \sum_{k=2}^{\infty} k \cdot \{\mathbb{P}[N(A_{t+h}) - N(A_t) = k]\} = o(h).$$

Under this mild assumption, let us estimate the probability of having points in $A_{t+h} \backslash A_t$.

**Claim:**

$$\begin{aligned}\mathbb{P}[N(A_{t+h}) - N(A_t) \geq 1] &= \mathbb{E}[N(A_{t+h})] - \mathbb{E}[N(A_t)] + o(h) \\ &= \mathbb{E}[N(A_{t+h} \backslash A_t)] + o(h).\end{aligned}$$

*Proof.* By definition and the assumption,

$$\begin{aligned}\mathbb{E}[N(A_{t+h})] - \mathbb{E}[N(A_t)] &= \mathbb{E}[N(A_{t+h}) - N(A_t)] \\ &= \sum_{k=0}^{\infty} k \cdot \{\mathbb{P}[N(A_{t+h}) - N(A_t) = k]\} = \mathbb{P}[N(A_{t+h}) - N(A_t) = 1] + \\ &+ \sum_{k=2}^{\infty} k \cdot \{\mathbb{P}[N(A_{t+h}) - N(A_t) = k]\} = \mathbb{P}[N(A_{t+h}) - N(A_t) = 1] + o(h).\end{aligned}$$

□

Define now

$$\mu(t) = \mathbb{E}\{N(A_t)\},$$

so that the Claim reads as

$$\begin{aligned}\mathbb{P}[N(A_{t+h}) - N(A_t) = 1] &= \mathbb{E}[N(A_{t+h})] - \mathbb{E}[N(A_t)] + o(h) \\ &= \mu(t+h) - \mu(t) + o(h).\end{aligned}$$

From here on, the goal is to demonstrate that $N(A_t)$ follows a Poisson distribution of parameter $\mu(t)$.
Fixed $n$, consider the following event $E = \{N(A_t) \leq n\}$, and let's see what happens when the time increases to $t + h$. There are two options:

1. $N(A_{t+h}) \leq n$. Since it is assumed that the probability of having two or more points in $A_{t+h} \backslash A_t$ is $o(h)$, thus, this essentially happens in two different cases:
   (a) $N(A_t) < n$.
   (b) $N(A_t) = n$, and there is no point in $A_{t+h} \backslash A_t$.
2. $N(A_t) = n$, and there is a point in $A_{t+h} \backslash A_t$. So that, $N(A_{t+h}) = n + 1$.

Hence, if $E_1 = \{N(A_{t+h}) \leq n\}$ and $E_2 = \{N(A_t) = n\} \cap \{N(A_{t+h} \backslash A_t) = 1\}$, then $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$. Therefore, $\mathbb{P}(E) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$.
Because $A_t$ and $A_{t+h} \backslash A_t$ are disjoint, by the independence assumption, $N(A_t)$ and $N(A_{t+h} \backslash A_t)$ are independent. Then, by the Claim,

$$\mathbb{P}(E_2) = \mathbb{P}[N(A_t) = n] \cdot \mathbb{P}[N(A_{t+h} \backslash A_t) = 1] = p_n(t) \cdot [\mu(t+h) - \mu(t) + o(h)].$$

Also,

$$q_n(t) = \mathbb{P}(E) = \mathbb{P}(E_1) + \mathbb{P}(E_2) = q_n(t+h) + p_n(t) \cdot [\mu(t+h) - \mu(t) + o(h)].$$

Hence,

$$q_n(t) - q_n(t+h) = p_n(t) \cdot [\mu(t+h) - \mu(t) + o(h)].$$

Dividing by $h$, and letting $h \to 0$,

$$-\frac{dq_n}{dt} = p_n \cdot \frac{d\mu}{dt}.$$

From this we will deduce that $p_n = \mathbb{P}[N(A_t) = n]$ has the values of a Poisson distribution of parameter $\mu(t)$.
Since $p_n = q_n - q_{n-1}$ and $p_0 = q_0 = \mathbb{P}[N(A_t) = 0]$, then $p_1 = q_1 - q_0 = q_1 - p_0$, so $q_1 = p_1 + p_0$.

For $n = 0$,

$$-\frac{dp_0}{dt} = p_0 \cdot \frac{d\mu}{dt}. \tag{1.2}$$

Also,

$$-\frac{dq_1}{dt} = -\frac{d(p_1 + p_0)}{dt} = p_1 \cdot \frac{d\mu}{dt},$$

hence

$$\frac{dp_1}{dt} = -\frac{dp_0}{dt} - p_1 \cdot \frac{d\mu}{dt} = (p_0 - p_1) \cdot \frac{d\mu}{dt}.$$

Let's prove by induction that for $n \geq 1$:

$$\frac{dp_n}{dt} = (p_{n-1} - p_n) \cdot \frac{d\mu}{dt}. \tag{1.3}$$

So the case $n = 1$ has been proven in (1.2) above.

Assuming this for $n - 1$, then, for $n$:

$$\begin{aligned}\frac{dp_n}{dt} &= \frac{d(q_n - q_{n-1})}{dt} = \frac{dq_n}{dt} - \frac{dq_{n-1}}{dt} \\ &= -p_n \cdot \frac{d\mu}{dt} + p_{n-1} \cdot \frac{d\mu}{dt} = (p_{n-1} - p_n) \cdot \frac{d\mu}{dt},\end{aligned}$$

as desired.
Now, integrating (1.2) on both sides and using that $p_0(0) = 1$ and $\mu(0) = 0 = \mathbb{E}[N(A_0)]$, the result is

$$\log p_0 + \mu = 0,$$

and consequently,

$$p_0(t) = e^{-\mu(t)}, \quad t \geq 0. \tag{1.4}$$

From (1.3), also

$$\frac{d}{dt}(p_n \cdot e^{\mu}) = \frac{dp_n}{dt} \cdot e^{\mu} + p_n \cdot e^{\mu} \cdot \frac{d\mu}{dt}$$
$$= e^{\mu} \cdot (p_{n-1} - p_n) \cdot \frac{d\mu}{dt} + p_n \cdot e^{\mu} \cdot \frac{d\mu}{dt} = p_{n-1} \cdot e^{\mu} \cdot \frac{d\mu}{dt}.$$

As $p_n(0) = 0$ for $n \geq 1$:

$$p_n(t) = e^{-\mu(t)} \int_0^t p_{n-1}(s) \cdot e^{\mu(s)} \frac{d\mu}{ds} ds.$$

Now, starting from (1.4) and by induction on $n$, we see that:

$$\mathbb{P}[N(A_t) = n] = p_n(t) = e^{-\mu(t)} \cdot \frac{\mu(t)^n}{n!}, \tag{1.5}$$

which means $N(A_t)$ follows a Poisson distribution of parameter $\mu(t) = \mathbb{E}\{N(A_t)\}$.

To see this for $n = 1$, observe that by (1.4), an integration yields

$$p_1(t) = e^{-\mu(t)} \int_0^t p_0(s) \cdot e^{\mu(s)} \frac{d\mu}{ds} ds = e^{-\mu(t)} \int_0^t \frac{d\mu}{ds} ds$$
$$= e^{-\mu(t)} \cdot [\mu(t) - \mu(0)] = e^{-\mu(t)} \cdot \frac{\mu(t)^1}{1!}.$$

Supposing (1.5) true until $n-1$, and substituting the variable $x = \mu(s)$, then, for $n$:

$$p_n(t) = e^{-\mu(t)} \int_0^t \frac{(\mu(s))^{n-1}}{(n-1)!} \cdot e^{-\mu(s)} \cdot e^{\mu(s)} \cdot \frac{d\mu}{ds} ds = \frac{e^{-\mu(t)}}{(n-1)!} \int_0^t (\mu(s))^{n-1} \cdot \frac{d\mu}{ds} ds$$
$$= \frac{e^{-\mu(t)}}{(n-1)!} \int_0^{\mu(t)} x^{n-1} dx = \frac{e^{-\mu(t)}}{(n-1)!} \cdot [\frac{x^n}{n}]_0^{\mu(t)} = e^{-\mu(t)} \cdot \frac{\mu(t)^n}{n!}.$$

## 1.3 The Superposition Theorem

The Poisson process has a number of special properties which make its use and the calculation of associated probabilities often surprisingly simple. One of them is the superposition property. Roughly stated, the superposition of Poisson processes $\Pi_i$ of intensity $\mu_i$ is again a Poisson process with intensity $\sum_i \mu_i$.

**Definition 1.7.** *Let* $\Pi_1$ *and* $\Pi_2$ *be two general point processes in* $S$*. The superposition of both point processes is the union of all points in* $\Pi_1 \cup \Pi_2$*. So, if* $A \subseteq S$*, the new counting function for* $A$ *is*

$$N_{\Pi_1 \cup \Pi_2}(A) = N_{\Pi_1}(A) + N_{\Pi_2}(A).$$

Notice that superposition could as well be defined as the sum of two random atomic measures, as explained in Remark 1.6.

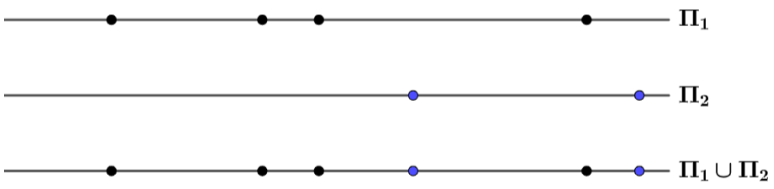


Figure 1.1: Superposition of two point processes.

To introduce the Superposition theorem, which has a huge importance, the following lemma needs to be stated. Its proof is very technical, so we skip it; the interested reader can find it in [15] p. 15.

**Lemma 1.8. (Disjointness Lemma)** *Let* $\Pi_1$ *and* $\Pi_2$ *be independent Poisson processes on* $S$ *with* $\mu_1$ *and* $\mu_2$ *as their respective intensities. Let* $A$ *be a measurable set with* $\mu_1(A) < \infty$ *and* $\mu_2(A) < \infty$*. Then* $\Pi_1$ *and* $\Pi_2$ *are disjoint with probability 1 on* $A$*, this is:*

$$\mathbb{P}\{\Pi_1 \cap \Pi_2 \cap A = \emptyset\} = 1. \tag{1.6}$$

This lemma actually extends to sets $A$ which are countable unions of sets of finite measure (for both $\mu_1$ and $\mu_2$).
The statement of Lemma 1.8 is not true without a finiteness assumption, as shown in the example below.

**Example 1.9.** Let $\Pi$ be the homogeneous Poisson process in $\mathbb{R}^2$, with intensity the Lebesgue measure. Let $\phi$ be the projection $\phi(x, y) = x$ onto the first coordinate. Then,

$$\phi(\Pi) = \{x | (x, y) \in \Pi\}$$

is a random countable subset of $\mathbb{R}$. Its counts have highly degenerate distributions: if $A \subseteq \mathbb{R}$ is a Borel set, then

$$N_{\Phi(\Pi)}(A) = \#\{\phi(\Pi) \cap A\} = \#\{\Pi \cap (A \times \mathbb{R})\}$$

has a Poisson distribution with parameter

$$\mu(A) = |A \times \mathbb{R}| = \iint_{A\times\mathbb{R}} dxdy = \begin{cases} 0, & \text{if } |A| = 0 \\ +\infty, & \text{if } |A| > 0. \end{cases}$$

Then the variables $N_{\Phi(\Pi)}(A)$ have degenerated distributions. For $k \in \mathbb{N}$:

- If $|A| = 0$,

$$\mathbb{P}[N_{\phi(\Pi)}(A) = k] = \mathbb{P}[N_\Pi(A \times \mathbb{R}) = k] = \begin{cases} 0, & k > 1 \\ 1, & k = 0. \end{cases}$$

- If $|A| > 0$,

$$\mathbb{P}[N_{\phi(\Pi)}(A) = k] = \mathbb{P}[N_\Pi(A \times \mathbb{R}) = k] = \begin{cases} 0, & k < \infty \\ 1, & k = \infty. \end{cases}$$

Now, $\Pi_1 = \Pi_2 = \phi(\Pi)$, with respective counting functions $N_1$ and $N_2$, are all degenerate, and therefore independent of one another.
It is immediate to check, case by case, that $\Pi_1$ and $\Pi_2$ are independent: for $n_1, n_2 \in \mathbb{N}$ and $A_1, A_2 \subseteq \mathbb{R}$,

$$\mathbb{P}[\{N_1(A_1) = n_1\} \cap \{N_2(A_2) = n_2\}] = \mathbb{P}[N_1(A_1) = n_1] \cdot \mathbb{P}[N_2(A_2) = n_2].$$

In summary, it has been proven that $\Pi_1$ and $\Pi_2$ are independent Poisson processes but (1.6) does not hold when $|A| > 0$.

A random variable that follows a Poisson distribution with parameter $\lambda \geq 0$, is noted as $\mathscr{P}(\lambda)$.
The following theorem is a clear example of the high relevance of the independence property. The result does not hold when the point processes have correlation, that is, when there is repulsion or attraction between points. Before, two results need to be stated.

**Lemma 1.10. (Countable Additivity)** *Let $X_j$, $j = 1, 2, \ldots$ be independent random variables, and assume that $X_j$ has the distribution $\mathscr{P}(\lambda_j)$ for each $j$. If,*

$$\sigma = \sum_{j=1}^{\infty} \lambda_j \tag{1.7}$$

*converges, then*

$$S = \sum_{j=1}^{\infty} X_j$$

*converges with probability 1, and S has distribution* $\mathscr{P}(\sigma)$*. If on the other hand, (1.7) diverges, then S diverges with probability 1.*

In particular, this lemma explains why $\mathscr{P}(0)$ and $\mathscr{P}(\infty)$ is well defined. The proof is the same as the familiar proof that the sum of two independent Poisson variables is again a Poisson variable, so we skip it.

**Theorem 1.11. (Superposition)** *Let* $\Pi_1$*,* $\Pi_2$*,... be a countable collection of independent Poisson processes on S and let* $\Pi_n$ *have intensity* $\mu_n$ *for each n. Then, their superposition*

$$\Pi = \bigcup_{n=1}^{\infty} \Pi_n$$

*is a Poisson process with intensity*

$$\mu = \sum_{n=1}^{\infty} \mu_n. \tag{1.8}$$

*Proof.* Let $N_n(A)$ denote the number of points of $\Pi_n$ in a $\mu_n$-measurable set A, for $n \in \mathbb{N}$. If $\mu_n(A) < \infty$ for all $n$, Lemma 1.8 shows that the random sets $\Pi_n$ are disjoint on $A$ almost surely, so that the number of points of $\Pi$ in $A$ is

$$N(A) = \sum_{n=1}^{\infty} N_n(A). \tag{1.9}$$

By the Countable Additivity Lemma, $N(A)$ has distribution $\mathscr{P}(\mu(A))$, where $\mu(A)$ is given by (1.8).
On the other hand, if $\mu_n(A) = \infty$ for some $n$, then $N_n(A) = N(A) = \infty$ and (1.9) holds trivially.
To prove the theorem it is enough to demonstrate that the random variables $N(A_1)$, $N(A_2)$, ... , $N(A_k)$ are independent if the sets $A_j$ are disjoint. But, this is clear because the double array of variables

$$N_n(A_j), \qquad j = 1, 2, \ldots, k \quad \text{and} \quad n = 1, 2, \ldots$$

are all independent, and $N(A_j)$ is defined in terms of a subset of these variables disjoint from those for other $j$.
This completes the proof, which for the sake of generality has been stated for countably infinite superpositions; it contains as an obvious corollary the corresponding result for finite superpositions.

□

Before finishing this section, we show another general frequently used consequence. Its proof is routine, so we skip it.

**Theorem 1.12. (Restriction)** *Let $\Pi$ be a Poisson process with intensity $\mu$ on $S$, and let $S_1$ be a measurable subset of $S$. Then, the random countable set*

$$\Pi_1 = \Pi \cap S_1$$

*can be regarded either as a Poisson process on $S$ with intensity*

$$\mu_1(A) = \mu(A \cap S_1)$$

*or as a Poisson process on $S_1$ whose intensity is the restriction of $\mu$ to $S_1$.*

## 1.4 The Mapping Theorem

Another important property of the Poisson process is that, under weak conditions ensuring that all the random points are mapped to distinct points, the mapping of a Poisson process into another space is also a Poisson process. To properly state the Mapping Theorem we recall the following definition.

**Definition 1.13.** *Given two measurable spaces $(S, \Sigma_1)$, $(T, \Sigma_2)$, a measurable mapping $f : S \to T$ and a measure $\mu : \Sigma_1 \to [0, +\infty]$, the push-forward of $\mu$ is the measure $f^*_\mu : \Sigma_2 \to [0, +\infty]$ defined by*

$$f^*_\mu(B) = \mu(f^{-1}(B)), \quad B \in \Sigma_2.$$

The push-forward measure is the induced measure from $\mu$ by the function $f$. If the function $f$ is clear from the context, it is noted as $f^*_\mu(B) = \mu^*(B)$.

**Theorem 1.14. (Mapping):** *Let $\Pi$ be a Poisson process with intensity $\mu$ on the state space $S$, and let $f : S \to T$ be a measurable function such that the push-forward measure $\mu^*$ has no atoms. Then $f(\Pi)$ is a Poisson process on $T$ having the induced measure $\mu^*$ as its intensity.*

*Proof.* The proof will be done only for injective functions, because the general case is very technical and outside the scope of this work. See [15] p. 19.
Assume thus that $f$ is a measurable and injective function. Then,

$$f^{-1}(B) = \{x \in S | f(x) \in B\}$$

is measurable for all measurable subset $B$ in $T$.
Hence, $f(\Pi) = \{f(x) | x \in \Pi\}$ is a random countable set in $T$, and the aim is

to prove that this is a Poisson process in $T$.
For $B$ measurable in $T$, the function

$$N^*(B) = \#\{f(\Pi) \cap B\}$$

counts the number of points of $f(\Pi)$ falling in $B$. As $f$ is injective, the new points $f(x)$, $x \in \Pi$ are distinct, so

$$N^*(B) = \#\{x \in \Pi | f(x) \in B\} = N(f^{-1}(B)),$$

and its distribution is a Poisson with parameter $\mu^*(B)$.
Furthermore, if $B_1, B_2, ..., B_k$ are disjoint, their inverse images are disjoint too and, consequently, $N^*(B_j)$ are independent.
In summary, it is been proven that if the points $f(x)$, $x \in \Pi$ are distinct, then $f(\Pi)$ is a Poisson process in $T$.

□

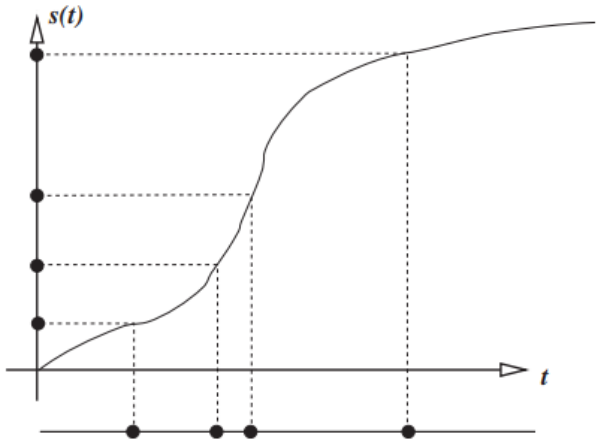


Figure 1.2: Application of a transformation $s$ to each individual point in a point process [3] p.17.

**Remark 1.15.** A careful reading of the proof shows that the result follows as soon as $f$ and $\Pi$ are such that, almost surely, for every $x, y \in \Pi$ with $x \neq y$ one has $f(x) \neq f(y)$. This is the same as requiring that $\{f(x) = f(y) | x, y \in \Pi, x \neq y\}$ is a subset with null measure.
The Mapping Theorem does not work with every function $f$, as the constant function shows, because of the fact that $f$ maps the whole of $\Pi$ onto a single point of $T$, which means that there are two distinct points with the same image. Note that if $f$ is a constant function, $\mu^*$ has a unique atom.

**Example 1.16.** The projection of a higher-dimensional Poisson process is a Poisson process.
To prove this, suppose that $\Pi \subseteq \mathbb{R}^D$ is a Poisson process with intensity $d\mu(x_1, x_2, ..., x_D) = \lambda(x_1, x_2, ..., x_D)dx_1 \cdots dx_D$, and consider the projection $f : \mathbb{R}^D \to \mathbb{R}^d$, for $d \leq D$,

$$f(x_1, x_2, ..., x_D) = (x_1, x_2, ..., x_d).$$

Then, for $B \subseteq \mathbb{R}^d$,

$$\begin{aligned}\mu^*(B) &= \int_{B\times\mathbb{R}^{D-d}} \lambda(x_1, x_2, \ldots, x_D)dx_1 \cdot dx_2 \cdots dx_D \\ &= \int_B \lambda^*(x_1, x_2, \ldots, x_d)dx_1 \cdot dx_2 \cdots dx_d\end{aligned}$$

where

$$\lambda^*(x_1, x_2, ..., x_d) = \int_{\mathbb{R}^{D-d}} \lambda(x_1, x_2, ..., x_D)dx_{d+1} \cdot dx_{d+2} \cdots dx_D. \tag{1.10}$$

As $\mu$ is $\sigma$-finite and $\mu^*$ is non-atomic, the Mapping Theorem can be applied and we can conclude that if (1.10) converges, then $f(\Pi)$ is a Poisson process with intensity $\lambda^*$ on $\mathbb{R}^d$.

## 1.5 Transforming a point process

A convenient feature of point processes is that new point process can be generated by some modifications. Two examples of such transformations have been already seen: superposition and mapping. Without going into too much detail, the purpose now is the discussion of two other possible modifications: thinning and cluster processes [3].

### 1.5.1 Thinning

Let $\Pi_0$ be a Poisson process in $S$. Then, *thinning* consists in deleting some points from $\Pi_0$. Suppose that every point $x$ in $\Pi_0$ is labelled with an indicator random variable $I_x$ taking the value 0 if the point $x$ has to be deleted or 1 if it has to be retained. Then the thinned process $\Pi_f$ is formed by those points $x$ from $\Pi_0$ such that $I_x = 1$.
There are two different ways of thinning: *dependent thinning*, when the random variables $I_x$ are dependent, and *independent thinning*, when $I_x$ are independent. If an homogeneous Poisson process is subjected to independent thinning, the

resulting thinned process is also Poisson. Just to show an example in $\mathbb{R}$, see Figure 1.3.

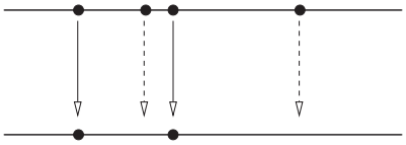

Figure 1.3: Thinning a point process [3] p.18.

### 1.5.2 Cluster Formation

We will now look at a point process, for instance a Poisson process, to which a transformation is applied; the result is, in general, no longer another Poisson process, precisely because of its fundamental characteristic of independence between the points.

Let $\Pi_0$ be a point process in $S$. *Clustering* is a modification which consists in replacing each point in $\Pi_0$ by a random finite set of points $\pi_x$, called the *cluster associated to* $x$. Then, the superposition of all clusters forms the new point process $\Pi_f$. It is usually assumed that the clusters $\pi_x$ for different points $x$ are independent. For example, see Figure 1.4.

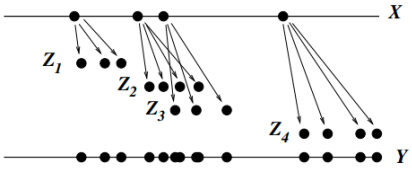


Figure 1.4: Cluster formation $(Z_i)$ to a point process $X$. The resulting point process is $Y$, [3] p. 19.

## 1.6 The Bernoulli process

We are interested in understanding, given a Poisson process $\Pi$ in a state space $S$, the distribution of the points in a region $A \subseteq S$ after conditioning on the number of points in $A$, which is $N(A)$. As we shall see soon, we get a multinomial distribution [1].

**Definition 1.17.** *Let $n \in \mathbb{N}$ be a number of independent and identically distributed trials where, each of which can result in one of $k$ classes. For each $i = 1, 2, ..., k$, let the chance of getting class $i$ on a single trial be $p_i$, so that*

$\sum_{i=1}^{k} p_i = 1$.
*For each* $i = 1, 2, ..., k$, *let* $N_i$ *be the number of trials that result in class* $i$, *so that* $\sum_{i=1}^{k} N_i = n$.
*Then the joint distribution of* $N_1, N_2, ..., N_k$ *is given by*

$$\mathbb{P}(N_1 = n_1, N_2 = n_2, ..., N_k = n_k) = \frac{n!}{n_1! n_2! ... n_k!} \cdot p_1^{n_1} \cdot p_2^{n_2} \cdot ... \cdot p_k^{n_k}. \quad (1.11)$$

*where* $n_i \geq 0$, *for* $1 \leq i \leq k$ *and* $\sum_{i=1}^{k} n_i = n$.
*This is called the* ***multinomial distribution*** *with parameters* $n$ *and* $p_1, p_2, ..., p_k$.
*Denoted as* $N \sim Multinomial(n, p)$.
*When there are only two classes, i. e.* $k = 2$, *the formula reduces to the familiar* ***binomial*** *formula.*

Let $B$ be a subset in $S$, and let $\Pi$ be a Poisson process on $B$ with intensity $\mu$ and satisfying $0 < \mu(B) < \infty$. Then, $\Pi$ is a finite subset of $B$ with probability one.

The goal now is to understand what happens if $\Pi$ is conditioned on the value of $N(B)$, which has a Poisson distribution $\mathscr{P}(\mu(B))$. To this end, define the conditional probability

$$\mathbb{P}_n\{\cdot\} = \mathbb{P}\{\cdot | N(B) = n\}.$$

Let $A_1, A_2, ..., A_k$ be disjoint subsets of $B$ and let $n_1, n_2, \ldots, n_k$ be integers such that $n_1 + n_2 + \ldots + n_k \leq n$. The aim is to determine

$$\begin{aligned}\mathbb{P}\{N(A_1) = n_1, \ldots, N(A_k) = n_k | N(B) = n\} \\ = \mathbb{P}_n\{N(A_1) = n_1, \ldots, N(A_k) = n_k\}.\end{aligned}$$

Consider $A_0 = (A_1 \cup \ldots \cup A_k)^c$ and $n_0 = n - \sum_{j=1}^{k} n_j$. Then, $B = \bigcup_{j=0}^{k} A_j$. By the independence,

$$\begin{aligned}&\mathbb{P}_n\{N(A_1) = n_1, N(A_2) = n_2, ..., N(A_k) = n_k\} = \\ &\quad = \mathbb{P}\{N(A_1) = n_1, N(A_2) = n_2, ..., N(A_k) = n_k | N(B) = n\} \\ &\quad = \frac{\mathbb{P}\{[N(A_1) = n_1, N(A_2) = n_2, ..., N(A_k) = n_k] \cap [N(B) = n]\}}{\mathbb{P}\{N(B) = n\}} \\ &\quad = \frac{\mathbb{P}\{N(A_1) = n_1, N(A_2) = n_2, ..., N(A_k) = n_k, N(A_0) = n_0\}}{\mathbb{P}\{N(B) = n\}} \\ &\quad = \frac{\prod_{j=0}^{k} \frac{e^{-\mu(A_j)} \mu(A_j)^{n_j}}{n_j!}}{\frac{e^{-\mu(B)} \mu(B)^n}{n!}} = \frac{n!}{n_0! n_1! \cdots n_k!} (\frac{\mu(A_0)}{\mu(B)})^{n_0} (\frac{\mu(A_1)}{\mu(B)})^{n_1} \cdots (\frac{\mu(A_k)}{\mu(B)})^{n_k}.\end{aligned}$$

Letting

$$p_j = p(A_j) = \frac{\mu(A_j)}{\mu(B)} \tag{1.12}$$

we see thus that $\overrightarrow{N(A)} = (N(A_0), N(A_1), \ldots, N(A_k))$ follows a multinomial distribution of parameters $n$ and $p = (p_0, p_1, \ldots, p_k)$.

**Remark 1.18.** Another way of seeing this is the following. If $X_1, X_2, ..., X_n$ are independent random variables, distributed over the space S according to the probability distribution $p = (p_0, p_1, \ldots, p_k)$, then $X_r$ are distinct with probability one (as a consequence of the Disjointness Lemma), so that $\{X_1, X_2, ..., X_n\}$ is a random set with $n$ elements. It is an easy multinomial calculation that

$$N(A) = \#\{r | X_r \in A\}$$

satisfies (1.11) with (1.12).

Hence, given $N(B)$, the points of a Poisson process look exactly like $N(B)$ independent random variables, with common distribution (1.12). In the following section a fundamental consequence of this result is going to be presented.

Here we exhibit another simple consequence. Let $Z = Z(\Pi)$ be a random variable and $\Pi$ a Poisson process with a finite intensity $\mu$ (so $Z$ is defined in terms of the Poisson process $\Pi$). Suppose that $\mathbb{E}(Z)$ could be computed in terms of its intensity $\mu$. Establishing $\mu(\cdot) = \mu p(\cdot)$, where $p$ is a fixed probability measure and $\mu = \mu(S)$ can vary, then, there exists a function $\phi$ such that $\mathbb{E}(Z) = \phi(\mu)$. If $\mathbb{E}_n(Z) = \mathbb{E}(Z|N(S) = n)$ is the corresponding expectation for the multinomial Process (1.11), then

$$\mathbb{E}(Z) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} e^{-\mu} \cdot \mathbb{E}_n(Z).$$

As $\mathbb{E}_n(Z)$ is independent of $\mu$:

$$\phi(\mu)e^{\mu} = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} e^{-\mu} \cdot \mathbb{E}_n(Z) \cdot e^{\mu} = \sum_{n=0}^{\infty} \frac{\mathbb{E}_n(Z)}{n!} \mu^n.$$

Thus, if $c_n = \dfrac{\mathbb{E}_n(Z)}{n!}$,

$$\phi(\mu)e^{\mu} = \sum_{n=0}^{\infty} c_n \mu^n.$$

Expanding $\phi(\mu)e^{\mu}$ as a power series in $\mu$, $\mathbb{E}_n(Z)$ can be acquired from the successive coefficients.

## 1.7 The Existence Theorem

Given a non-atomic and $\sigma$-finite measure $\mu$ on $S$, one may want to know if there exists a Poisson process with intensity $\mu$, a process for which $N(A)$ has distribution $\mathscr{P}(\mu(A))$ for all $A$ subset of $S$, and $N(A)$ is independent of $N(B)$ if $A \cap B = \emptyset$. The answer is positive assuming only mild conditions on $\mu$. Here it is shown how to construct such a process. In this way, a theory of great generality and wide applicability is obtained.

**Theorem 1.19. (Existence):** *Let $\mu$ be a $\sigma$-finite and non-atomic measure on $S$, then, there exists a Poisson process on $S$ having $\mu$ as its intensity.*

*Proof.* Let $\mu = \sum_{n=1}^{\infty} \mu_n,$, with $\mu_n(S) < \infty$. Without loss of generality, suppose $\mu_n(S) > 0$, for all $n \in \mathbb{N}$. On a suitable probability space construct independent random variables

$$N_n,\ X_{nr}\ (n, r = 1, 2, 3, ...)$$

such that the distribution of $N_n$ is $\mathscr{P}(\mu_n(S))$ and, for all $r$, the distribution of $X_{nr}$ is

$$p_n(\cdot) = \frac{\mu_n(\cdot)}{\mu_n(S)}, \tag{1.13}$$

where $N_n = N_n(S)$ is the total number of points in $S$. Note that in the previous section, the multinomial distribution shown how to choose $p_n(\cdot)$. Write

$$\Pi_n = \{X_{n1}, X_{n2}, ..., X_{nN_n}\}$$

and

$$\Pi = \bigcup_{n=1}^{\infty} \Pi_n. \tag{1.14}$$

Write

$$N_n(A) = \#\{\Pi_n \cap A\}.$$

Take now $A_1, A_2, \ldots, A_k$ mutually disjoint and fix $m \in \mathbb{N}$. Letting $A_0 = (A_1 \cup A_2 \cup ... \cup A_k)^c$ and $m_0 = m - (m_1 + m_2 + \ldots + m_k)$, then

$$\begin{aligned} \mathbb{P}\{N(A_1) = m_1, N(A_2) = m_2, \ldots, N(A_k) = m_k | N_n(S) = m\} = \\ = \frac{m!}{m_0! m_1! \cdots m_k!} p_n(A_0)^{m_0} p_n(A_1)^{m_1} \cdots p_n(A_k)^{m_k} \end{aligned} \tag{1.15}$$

Hence, the joint distribution of $N_n(A_1), \ldots, N_n(A_k)$ is given by

$$\begin{aligned}
&\mathbb{P}\{N_n(A_1) = m_1, N_n(A_2) = m_2, \ldots, N_n(A_k) = m_k\} = \\
&= \sum_{m=\sum_{j=1}^k m_j}^{\infty} \mathbb{P}\{N_n(A_1) = m_1, N_n(A_2) = m_2, \ldots, N_n(A_k) = m_k | N_n(S) = m\} \cdot \\
&\cdot \mathbb{P}(N_n(S) = m).
\end{aligned}$$

By (1.13) and (1.15), since $\mu_n(S) = \sum_{j=0}^k \mu_n(A_j)$,

$$\begin{aligned}
&\mathbb{P}\{N_n(A_1) = m_1, N_n(A_2) = m_2, \ldots, N_n(A_k) = m_k\} = \\
&= \sum_{m=\sum_{j=1}^k m_j}^{\infty} \frac{e^{-\mu_n(S)}(\mu_n(S))^m}{m!} \frac{m!}{m_0! m_1! \ldots m_k!} \prod_{j=0}^k p_n(A_j)^{m_j} \\
&= \sum_{m=\sum_{j=1}^k m_j}^{\infty} (\mu_n(S))^m \prod_{j=0}^k \frac{e^{-\mu_n(A_j)}}{m_j!} \frac{\mu_n(A_j)^{m_j}}{\mu_n(S)^{m_j}}.
\end{aligned}$$

Define $\pi_k(x) = x^k \cdot \dfrac{e^{-x}}{k!}$; then

$$\begin{aligned}
&\mathbb{P}\{N_n(A_1) = m_1, N_n(A_2) = m_2, \ldots, N_n(A_k) = m_k\} = \\
&= \sum_{m=\sum_{j=1}^k m_j}^{\infty} (\mu_n(S))^m \prod_{j=0}^k \pi_{m_j}(\mu_n(A_j)) \frac{1}{\mu_n(S)^{m_j}} \\
&= \sum_{m=\sum_{j=1}^k m_j}^{\infty} (\mu_n(S))^{m-\sum_{j=0}^k m_j} \prod_{j=0}^k \pi_{m_j}(\mu_n(A_j)) \\
&= \sum_{m=\sum_{j=1}^k m_j}^{\infty} 1 \cdot \prod_{j=0}^k \pi_{m_j}(\mu_n(A_j)).
\end{aligned}$$

Summing on all $m \geq \sum_{j=1}^k m_j$ is the same as summing on all

$$m_0 = m - \sum_{j=1}^k m_j \geq 0.$$

Thus, separating the product in the case $j = 0$ and the case $j \geq 1$, and

summing for all $m_0$:

$$\begin{aligned}\mathbb{P}\{N_n(A_1) = m_1, N_n(A_2) = m_2, \ldots, N_n(A_k) = m_k\} = \\ = \sum_{m=\sum_{j=1}^{k} m_j}^{\infty} \pi_{m_0}(\mu_n(A_0)) \prod_{j=1}^{k} \pi_{m_j}(\mu_n(A_j)) \\ = \sum_{m_0=0}^{\infty} \pi_{m_0}(\mu_n(A_0)) \prod_{j=1}^{k} \pi_{m_j}(\mu_n(A_j)) \\ = \prod_{j=1}^{k} \pi_{m_j}(\mu_n(A_j)) \sum_{m=m_0}^{\infty} \pi_{m_0}(\mu_n(A_0)).\end{aligned}$$

Since

$$\sum_{m_0=0}^{\infty} \pi_{m_0}(\mu_n(A_0)) = \sum_{m_0=0}^{\infty} e^{-\mu_n(A_0)} \cdot \frac{(\mu_n(A_0))^{m_0}}{m_0!} = 1,$$

it follows

$$\begin{aligned}\mathbb{P}\{N_n(A_1) = m_1, N_n(A_2) = m_2, \ldots, N_n(A_k) = m_k\} = \prod_{j=1}^{k} \pi_{m_j}(\mu_n(A_j)) \\ = \prod_{j=1}^{k} e^{-\mu_n(A_j)} \cdot \frac{(\mu_n(A_j))^{m_j}}{m_j!}.\end{aligned}$$

So the $N_n(A_j)$ are independent random variables with distributions $\mathscr{P}(\mu_n(A_j))$. Thus, $\Pi_n$ are independent Poisson processes with respective intensities $\mu_n$. Applying the Superposition Theorem it is concluded that (1.14) defines a Poisson process with intensity $\mu$.

□

Before finishing this section it is important to remark that the theorem is proven with the assumption of $\sigma$-finitude of the measure $\mu$, which is not strictly necessary.

### Separated Poisson process

The existence theorem allows the generation of Poisson processes with prefixed properties. To exemplify the importance of this theorem, we characterise the non-homogeneous Poisson processes on the line that are almost surely separated.

Let $\Pi$ be a Poisson process in $\mathbb{R}$ with intensity $\mu$. The objective is to find the measures for which the random points of the Poisson process are separated in $\mathbb{R}$.

**Definition 1.20.** *A sequence* $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ *is separated if*

$$\inf_{k \neq j} |x_k - x_j| > 0.$$

*It is said that* $\{x_k\}_{k=1}^{\infty}$ *is* $\delta$*-separated if* $|x_k - x_j| > \delta$*, for* $k \neq j$*.*

Separation is important in many problems, for example, interpolation in numerical analysis. In order to state the result, the real line is split in unit intervals $I_n = [n, n+1)$, $n \in \mathbb{Z}$.

**Theorem 1.21.** *Let* $\Pi$ *be a Poisson process in* $\mathbb{R}$ *with intensity* $\mu$*. Then,*

$$\mathbb{P}(\Pi \textit{ separated}) = \begin{cases} 1, & if \ \sum_{n\in\mathbb{Z}} \mu^2(I_n) < \infty \\ 0, & if \ \sum_{n\in\mathbb{Z}} \mu^2(I_n) = \infty. \end{cases}$$

Thus, for instance, in $S = \mathbb{R}$ with $d\mu(x) = \frac{1}{1+|x|}$, then:

$$\mu(I_n) = \int_n^{n+1} \frac{dx}{1+|x|} = \log(\frac{1+|n+1|}{1+|n|}) \backsim \frac{1}{1+|n|},$$

and

$$\sum_{n\in\mathbb{Z}} \mu^2(I_n) \backsim \sum_{n\in\mathbb{Z}} \frac{1}{(1+|n|^2)} < \infty.$$

Hence, $\Pi$ is almost surely a separated sequence.
On the other hand, considering the homogeneous Poisson process (i.e.: $d\mu = dx$), then $\sum_{n\in\mathbb{Z}} \mu^2(I_n) = \sum_{n\in\mathbb{Z}} 1 = \infty$, so $\Pi$ is not a separate sequence of random points with probability 1.

The main tool of the proof are the Borel-Cantelli Lemmas.

**Theorem 1.22. (Borel-Cantelli Lemma 1):** *Let* $\{E_n\}_{n=1}^{\infty}$ *be events such that*

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) < +\infty.$$

*Then,*

$$\mathbb{P}(\limsup_{n\to\infty} E_n) = 0.$$

**Theorem 1.23. (Borel-Cantelli Lemma 2):** *Let* $\{E_n\}_{n=1}^{\infty}$ *be independent events such that*

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty.$$

*Then*

$$\mathbb{P}(\limsup_{n\to\infty} E_n) = 1.$$

*Proof.* First, assume that $\sum_{n\in\mathbb{Z}} \mu^2(I_n) < \infty$. Consider the family of random variables

$$X_n = N_\Pi(I_n) = \#\{\Pi \cap I_n\}, n \in \mathbb{Z}.$$

These variables are independent with $\mathbb{E}(X_n) = \mu(I_n)$. Define the events

$$E_n = \{X_n > 1\}.$$

The aim is to see that $\mathbb{P}(X_n > 1)$ almost never happens. As $\sum_{n\in\mathbb{Z}} \mu^2(I_n) < \infty$, then $\mu(I_n) \to 0$. On the other hand,

$$\mathbb{P}(X_n > 1) = 1 - \mathbb{P}(X_n = 0) - \mathbb{P}(X_n = 1) = 1 - e^{-\mu(I_n)} - \mu(I_n)e^{-\mu(I_n)}$$

and, for $\mu(I_n)$ small enough, by Taylor's expansion around 0 of the exponential,

$$\begin{aligned} 1 - e^{-\mu(I_n)} - \mu(I_n)e^{-\mu(I_n)} &= 1 - (1 - \mu(I_n) + \frac{\mu^2(I_n)}{2} - \ldots) - \mu(I_n) \cdot (1 - \mu(I_n) \\ &+ \frac{\mu^2(I_n)}{2} - \ldots) = 1 - 1 + \mu(I_n) - \frac{\mu^2(I_n)}{2} + \ldots \\ &- \mu(I_n) + \mu^2(I_n) - \frac{\mu^3(I_n)}{2} + \ldots = \frac{\mu^2(I_n)}{2} + o(|\mu(I_n)|^3). \end{aligned}$$

Therefore,

$$\mathbb{P}(X_n > 1) = \frac{\mu^2(I_n)}{2} + o(|\mu(I_n)|^3) \approx \frac{\mu^2(I_n)}{2}.$$

Thus, $\sum_{n\in\mathbb{Z}} \mu^2(I_n) < \infty$ implies that $\sum_{n\in\mathbb{Z}} \mathbb{P}(X_n > 1) < \infty$. Consequently, by the first Borel-Cantelli Lemma, the probability of having an infinite number of intervals with $X_n > 1$ is zero. So there are almost surely only a finite number of intervals $I_n$ such that contain more than one point of the Poisson process. Strictly speaking, this is not enough to prove that $\Pi$ is a separated Poisson process, because there could be points arbitrarily approaching the boundaries of adjacent intervals $I_k$. To check that this case has null probability, consider intervals of the form $J_n = [n - \frac{1}{2}, n + \frac{1}{2})$ for all $n \in \mathbb{Z}$. As in the previous computations, only a finite number of $J_k$ has more than one point almost surely. Let

$$A_1 = \{\text{There are a finite number of } I_n \text{ with } X_n > 1\},$$

$$A_2 = \{\text{There are a finite number of } J_n \text{ with } N_\Pi(J_n) > 1\}.$$

As $\mathbb{P}(A_1) = \mathbb{P}(A_2) = 1$, it implies that $\mathbb{P}(A_1 \cap A_2) = 1$. Moreover, as $A_1 \cap A_2$ is an event for which $\Pi$ is separated, then, finally,

$$\mathbb{P}(\{\Pi \text{ is a separated sequence}\}) = 1.$$

Suppose now that $\sum_{n\in\mathbb{Z}} \mu^2(I_n) = \infty$. To prove that almost surely $\Pi$ is not separated, it is enough to see that for all $l_0 > 0$ in $\mathbb{N}$,

$$\mathbb{P}(\{\Pi \text{ is } 2^{-l_0}\text{- separated}\}) = 0.$$

Thus, fixing any $l_0$ in $\mathbb{N}$ and a separation of $2^{-l_0}$, split the interval $I_n$ into $2^{l_0}$ intervals with length $2^{-l_0}$, which will be called $I_{n,j}^{l_0}$ for $j = 1, 2, \ldots, 2^{l_0}$. Denoting $X_{n,j}^{l_0} = \#\{\Pi \cap I_{n,j}^{l_0}\}$, it is enough to check that

$$\mathbb{P}(\{X_{n,j}^{l_0} > 1 \text{ for an infinite number of } n \text{ and } j\}) = 1.$$

As the random variables $X_{n,j}^{l_0}$ are independent, applying the second Borel-Cantelli Lemma, it is only necessary to see that

$$\sum_{n\in\mathbb{Z}} \sum_{j=1}^{2^{l_0}} \mathbb{P}(X_{n,j}^{l_0} > 1) = +\infty.$$

But this is easy: naming $\mu_{n,j}^{l_0} = \mu(I_{n,j}^{l_0})$, then

$$\mathbb{P}(X_{n,j}^{l_0} > 1) = 1 - \mathbb{P}(X_{n,j}^{l_0} = 0) - \mathbb{P}(X_{n,j}^{l_0} = 1) = 1 - e^{-\mu_{n,j}^{l_0}} - \mu_{n,j}^{l_0} \cdot e^{-\mu_{n,j}^{l_0}}.$$

The larger the value of $\mu(I_n)$, the greater the probability of having more points in $I_n$. Therefore, the worst scenario is when $\mu(I_n) \to 0$, so that $\mu_{n,j}^{l_0}$ is very small. By the previous Taylor approximation,

$$\mathbb{P}(X_{n,j}^{l_0} > 1) = \frac{(\mu_{n,j}^{l_0})^2}{2} + o(|\mu_{n,j}^{l_0}|^3),$$

hence,

$$\sum_{n\in\mathbb{Z}} \sum_{j=1}^{2^{l_0}} \mathbb{P}(X_{n,j}^{l_0} > 1) \simeq \frac{1}{2} \sum_{n\in\mathbb{Z}} \sum_{j=1}^{2^{l_0}} \mu^2(I_{n,j}^{l_0}).$$

By the Cauchy-Schwarz inequality

$$\mu^2(I_n) \leq 2^{l_0} \sum_{j=1}^{2^{l_0}} \mu^2(I_{n,j}^{l_0}),$$

then, finally

$$\sum_{n\in\mathbb{Z}}\sum_{j=1}^{2^{l_0}}\mathbb{P}(X_{n,j}^{l_0}>1)\gtrsim\frac{1}{2}\cdot 2^{-l_0}\cdot\sum_{n\in\mathbb{Z}}\mu^2(I_n)=+\infty.$$

Applying the second Borel-Cantelli Lemma the proof is completed.

□

# Chapter 2

# Some applications of the Poisson process

## 2.1 Non-homogeneous compound Poisson process in one dimension applied to road traffic accidents with victims in Catalonia

We show here an example of the Poisson process applied to traffic accidents. We have mainly followed two studies, one from Indonesia [12], and the other, more detailed, from Poland [10],[9].
From now on, *road traffic accidents with victims* will be referred to as *traffic accidents*. The aim is to model traffic accidents to be able to predict the number of accidents in a period of time, as well as the number of fatalities or injured people. The use of a Poisson process in this situation seems reasonable, because in principle traffic accidents are independent; the fact that there was an accident yesterday does not affect the probability of having an accident today. Two accidents in the same section of the road can be related to other factors (road in bad condition, a sharp curve, etc.), but they are mutually independent. Thus, a non-homogeneous compound Poisson process is used to model traffic accidents with victims.

### 2.1.1 Model of the road accident number in Catalonia

A *non-homogeneous Poisson processes* (NPP) is considered due to the reason that, clearly, the probability of having an accident in a particular section of a road depends on time (season of the year, holidays, periods, etc.).
To formalise this, let $\{N(t) = N_{(0,t)} : t \geq 0\}$ be a non-homogeneous Poisson

process, where $N(t)$ is the Poisson variable that counts the number of accidents during the period $(0,t)$. Let $\lambda(x)$ be the intensity function, so that the intensity of a time interval $(s,t)$ is $\mu[(s,t)] = \int_s^t \lambda(x)dx$.

In this case, $\mu[(t,t+h)] = \int_t^{t+h} \lambda(x)dx$ and, by definition,

$$\begin{aligned}\mathbb{P}[N(t+h) - N(t) = k] = \mathbb{P}[N_{(t,t+h)} = k] &= \frac{\{\mu[(t,t+h)]\}^k}{k!} \cdot e^{-\mu[(t,t+h)]} \\ &= \frac{(\int_t^{t+h} \lambda(x)dx)^k}{k!} \cdot e^{-\int_t^{t+h} \lambda(x)dx}. \end{aligned} \tag{2.1}$$

The expectation and variance of NPP are given by

$$\Lambda(t) = \mathbb{E}[N(t)] = \mathbb{E}[N_{(0,t)}] = \int_0^t \lambda(x)dx. \tag{2.2}$$

$$\mathbb{V}(t) = \mathbb{V}[N(t)] = \mathbb{V}[N_{(0,t)}] = \int_0^t \lambda(x)dx.$$

The corresponding standard deviation is just $D(t) = \sqrt{\mathbb{V}[N(t)]}$.
Also, the expected value of an increment $N(t+h) - N(t)$ is

$$\mathbb{E}[N(t+h) - N(t)] = \mathbb{E}[N_{(t,t+h)}] = \int_t^{t+h} \lambda(x)dx. \tag{2.3}$$

The corresponding standard deviation is given by the square root of $\mathbb{V}[N_{(t,t+h)}]$.

Figure 2.1 details the number of accidents, including injuries and fatalities, for the period from January $1^{st}$ 2008 to December $31^{st}$ 2020. The goal is, placing ourselves temporarily in January $1^{st}$ 2021, to try to predict the number of traffic accidents and the number of fatalities for a particular period of time in 2021. An accurate model would provide very valuable information.
The following data comes from the Statisical yearbook of traffic accidents in Catalonia from Servei Català de trànsit de la Generalitat de Catalunya [23].

Figure 2.1: Accident data in Catalonia

| Number | Year | Interval | Centre | Accidents | Fatalities in 30 days | Injured | $\alpha$ | $\beta$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2008 | [0,1) | 0.5 | 24,590 | 452 | 32,659 | 0.018381 | 1.328142 |
| 2 | 2009 | [1,2) | 1.5 | 24,663 | 411 | 32,792 | 0.016665 | 1.329603 |
| 3 | 2010 | [2,3) | 2.5 | 24,132 | 381 | 32,329 | 0.015788 | 1.339673 |
| 4 | 2011 | [3,4) | 3.5 | 22,775 | 352 | 30,492 | 0.015456 | 1.338836 |
| 5 | 2012 | [4,5) | 4.5 | 23,368 | 336 | 31,630 | 0.014379 | 1.35356 |
| 6 | 2013 | [5,6) | 5.5 | 23,831 | 272 | 32,180 | 0.011414 | 1.350342 |
| 7 | 2014 | [6,7) | 6.5 | 23,828 | 272 | 32,087 | 0.011415 | 1.346609 |
| 8 | 2015 | [7,8) | 7.5 | 25,286 | 291 | 33,673 | 0.011508 | 1.331686 |
| 9 | 2016 | [8,9) | 8.5 | 26,995 | 282 | 35,999 | 0.010446 | 1.333543 |
| 10 | 2017 | [9,10) | 9.5 | 27,052 | 283 | 35,884 | 0.010461 | 1.326482 |
| 11 | 2018 | [10,11) | 10.5 | 26,907 | 326 | 35,426 | 0.012116 | 1.316609 |
| 12 | 2019 | [11,12) | 11.5 | 26,576 | 304 | 34,682 | 0.011439 | 1.305012 |
| 13 | 2020 | [12,13) | 12.5 | 17,779 | 204 | 22,530 | 0.011474 | 1.267225 |
| **Total** | | | 84.5 | 317,782 | 4166 | 422,363 | 0.170942 | 17.267324 |

In Figure 2.1 the parameter $\alpha$ is the rate of fatalities for accident and $\beta$ the rate of injured people for accident.

### Estimation of model parameters

The first step is to evaluate empirically the intensity $\Lambda(t)$. In both studies followed, it is assumed that the empirical hazard rate can be approximated by the linear function $\lambda(t) = at + b$. Of course, there are many other regression models to approximate this kind of data, such as polynomial of various degrees, exponential or logarithmic functions. The linear method is chosen to keep it simple.
Standard spreadsheets, such as Excel, yield the parameters $a, b$ of the linear regression directly from the table, using the least squares approximation and the standard formulas:

$$a = \frac{n\sum_{i=0}^{m} X_i Y_i - \sum_{i=0}^{m} X_i \sum_{i=0}^{m} Y_i}{n\sum_{i=0}^{m} X_i^2 - (\sum_{i=0}^{m} X_i)^2}, \quad b = \frac{\sum_{i=0}^{m} Y_i - a\sum_{i=0}^{m} X_i}{n}, \tag{2.4}$$

where $m$ is the number of studied years.

The computations have been rounded off to four decimals during the whole model, although they are done with more decimals in Excel.
Figure 2.2 shows the center of the interval of time in the horizontal axis and the number of accidents in the vertical axis. The blue line represents the empirical intensity of road accidents with victims in Catalonia, whereas the dotted line is the linear regression for the intensity $\lambda(t)$. Using (2.4), we get $a = 7.3571$ and $b = 24,396.9478$.

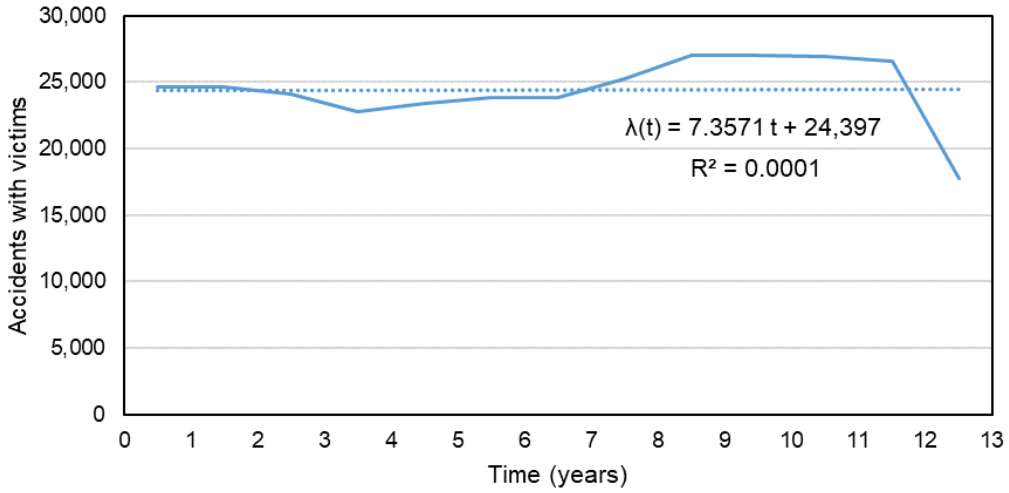


Figure 2.2: The empirical intensity of road accidents.

The value $R^2$ is a statistical measure that represents the proportion of the variance for a dependent variable that is explained by an independent variable in a regression model. Whereas correlation explains the strength of the relationship between an independent and a dependent variable, $R^2$ explains the extent to which the variance of one variable explains the variance of the second variable. So, if the $R^2$ of a model is 0.50, then approximately half of the observed variation can be explained by the inputs of the model. Thus, the larger $R^2$, the better the model. As it is explained in [8].

It is obvious that this graph is not the best approximation. Checking the $R^2$ value, it can be deduced that this approximation should be rejected, because $R^2 < 0.5$, as it is said in [17]: *"$R^2$ is too low for an empirical model in social science research. This range of $R^2$ is not acceptable".* After all, a forecast model is just a model. Also, looking at the last section of the graph, it is necessary to mention that the data from 2020 is quite exceptional, and can hardly match a regular model, since the traffic patterns during Covid-19 were completely altered.
Every model has its assumptions, and, if these are not met, it should not be relied upon. For example, just as a curiosity, if data from 2020 had not been contemplated, the regression would have changed, as well as the graph and the $R^2$ value. It would be Figure 2.3.

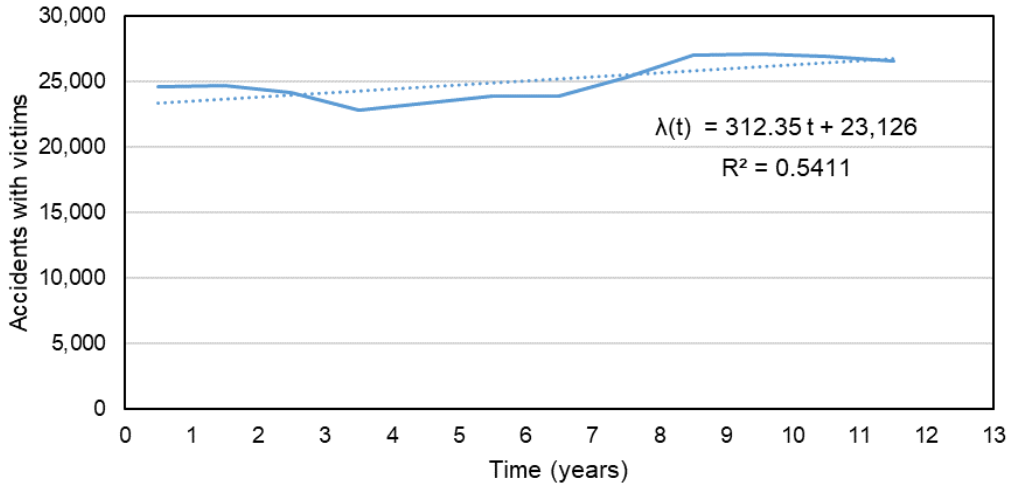


Figure 2.3: The empirical intensity of the road accidents until 2019.

In this instance, the $X$ and $Y$ axis are the same as for Figure 2.2. The blue function represents the number of accidents over time, but in this case omitting 2020. The dotted line is the regression showing the intensity of accidents but only until 2019. Then, the approximation improves substantially. So, the proposed Poisson model fits well in an ordinary situation. Nonetheless, this model does not predict correctly when extraordinary events occur, as it happens with most models. It seems that specialists, currently, favor a model based on approximation by a negative binomial over the Poisson model [21]. Keeping the data from 2020, the intensity function obtained is

$$\lambda(t) = 7.3571t + 24,396.9478, \quad \text{for } t \geq 0,$$

and from (2.2),

$$\Lambda(t) = \mathbb{E}[N(t)] = \int_0^t (7.3571x + 24,396.9478)dx = 3.6786t^2 + 24,396.9478t;$$

where $0 \leq t \leq 13$ [years].

### Prediction of the accident number

The aim now is to anticipate the number of traffic accidents, estimate the probability of a given number of these accidents and predict the fatalities in Catalonia from April $1^{st}$ to July $31^{st}$ 2021.

By (2.1) and (2.2), the present model gives, for any interval of time $(t, t+h)$:

$$\mathbb{P}[N_{(t,t+h)} = k] = \mathbb{P}[N(t+h) - N(t) = k] = \frac{[\Lambda(t+h) - \Lambda(t)]^k}{k!} e^{-[\Lambda(t+h)-\Lambda(t)]}.$$

As seen in (2.3), the expected number of accidents for the time interval $(t, t+h)$ is

$$\mathbb{E}[N(t+h) - N(t)] = \int_t^{t+h} \lambda(x)dx = \int_t^{t+h} (ax+b)dx = h(\frac{ah}{2} + b + at). \quad (2.5)$$

By (2.5), the standard deviation is then

$$D(t;h) = \sqrt{h(\frac{ah}{2} + b + at)}. \quad (2.6)$$

In this example, the period of time to study is in the interval $[13, 14)$. Since from January $1^{st}$ to April $31^{st}$ 2021 there are 90 days, the end of the interval corresponds to $t = 13 + \frac{90}{365} = \frac{967}{73} \approx 13.2466$ years. From April $1^{st}$ to July $31^{st}$ the number of years passed is $h = \frac{30+31+30+31}{365} = \frac{122}{365} \approx 0.3342$.
Using (2.5) and (2.6), the expected number of accidents is

$$\begin{aligned} E[N(t+h) - N(t)] &= \frac{122}{635}(\frac{7.3571 \cdot \frac{122}{635}}{2} + 24,396.9478 + 7.3571 \cdot \frac{967}{73}) \\ &\approx 8,187.5819. \end{aligned}$$

The corresponding standard deviation is

$$D(t;h) = \sqrt{h(\frac{ah}{2} + b + at)} = \sqrt{8,187.5819} \approx 90.4853.$$

In conclusion, the predicted number of accidents, from April $1^{st}$ to July $31^{st}$ 2021 is around 8,188 accidents, with a standard deviation of around 90.

The Poisson model allows further predictions. For example, the probability that the number of accidents in this period will be not less than $c = 7,991$ and not greater than $d = 8,340$ is

$$\begin{aligned} \mathbb{P}_{7,991 \le k \le 8,340} &= \mathbb{P}[7,991 \le N(t+h) - N(t) \le 8,340] \\ &= \sum_{k=7,991}^{8,340} \frac{8,187.5819^k}{k!} \cdot e^{-8,187.5819}. \end{aligned}$$

Counting this sum is not viable, so instead one uses the standard approximation of the Poisson with parameter $\lambda$ by the normal distribution with mean

$\lambda$ and standard deviation $\sqrt{\lambda}$. Then, normalizing to the standard normal $Z \sim N(0,1)$, and denoting $\phi(t) = \mathbb{P}(Z \leq t)$, the approximated probability is

$$\begin{aligned}\mathbb{P}_{7,991 \leq k \leq 8,340} &\approx \phi(\frac{8,340 - 8,187.5819}{90.4853}) - \phi(\frac{7,991 - 8,187.5819}{90.4853}) \\ &\approxeq 0.95 - 0.01 = 0.94.\end{aligned}$$

### 2.1.2 Non-homogeneous compound Poisson process

In order to anticipate the fatalities number, we use the non-homogeneous compound Poisson process.
A compound Poisson process is formed by adding random weights to each point of the previous Poisson process. It is reasonable to add weights to accidents, so that they account for the number of fatalities. For instance, an accident with three fatalities should have a bigger weight than another in which there are no deaths.
Formally, it is assumed that $\{N(t) : t \geq 0\}$ is a non-homogeneous Poisson process (NPP) determined by a function $\lambda(t) \geq 0$, for $t \geq 0$ as before, and $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables such that, for every $i \in \mathbb{N}$, $X_i$ takes value in $\mathbb{N}$ and is independent of $\{N(t) : t \geq 0\}$. A Poisson process of the form

$$X(t) = X_1 + X_2 + \cdots + X_{N(t)}, \quad t \geq 0$$

is said to be a *non-homogeneous compound Poisson process (NCPP).*

**Proposition 2.1.** *Let $\{X(t) : t \geq 0\}$ be an NCPP as above. If $\mathbb{E}[X_1^2] < \infty$, then:*

*(1)* $\mathbb{E}[X(t)] = \mathbb{E}[N(t)]\mathbb{E}(X_1) = \Lambda(t)\mathbb{E}(X_1)$.

*(2)* $\mathbb{V}[X(t)] = \Lambda(t)\mathbb{E}(X_1^2)$,

*where* $\Lambda(t) = \mathbb{E}[N(t)] = \int_0^t \lambda(x)dx$.

*Proof.* Applying the low of total expectation

$$\mathbb{E}[X(t)] = \mathbb{E}\{\mathbb{E}[X(t)|N(t)]\},$$

it is

$$\begin{aligned}\mathbb{E}[X(t)] &= \mathbb{E}\{\mathbb{E}[X(t)]|N(t)\} = \mathbb{E}\{\mathbb{E}[\sum_{i=1}^{N(t)} X_i]|N(t)]\} \\ &= \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=1}^{N(t)} X_i|N(t)=n]\mathbb{P}(N(t)=n) = \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=1}^{N(t)} X_i]\mathbb{P}(N(t)=n) \\ &= \sum_{n=0}^{\infty} E[X_1]n\mathbb{P}(N(t)=n) = \mathbb{E}(X_1)\mathbb{E}[N(t)].\end{aligned}$$

For (2), using the law of total variance

$$\mathbb{V}[X(t)] = \mathbb{E}\{\mathbb{V}[X(t)|N(t)]\} + \mathbb{V}\{\mathbb{E}[X(t)|N(t)]\}, \tag{2.7}$$

noting $(I)$ the first part of the sum and $(II)$ the second,

$(I)$

$$\begin{aligned}\mathbb{E}[X(t)|N(t)] &= \sum_{n=0}^{\infty} \mathbb{V}[\sum_{i=1}^{N(t)} X_i|N(t)=n]\mathbb{P}(N(t)=n) \\ &= \sum_{n=0}^{\infty} \mathbb{V}[\sum_{i=1}^{N(t)} X_i]\mathbb{P}(N(t)=n) = \sum_{n=0}^{\infty} \mathbb{V}(X_1)n\mathbb{P}(N(t)=n) \\ &= \mathbb{V}(X_1)\mathbb{E}[N(t)] = \mathbb{V}(X_1)\Lambda(t).\end{aligned}$$

$(II)$

$$\begin{aligned}\mathbb{V}\{\mathbb{E}[X(t)|N(t)]\} &= \mathbb{V}\{\mathbb{E}[\sum_{i=1}^{N(t)} X_i]|N(t)\} = \mathbb{V}[\mathbb{E}(X_1)N(t)] \\ &= \mathbb{V}[N(t)][\mathbb{E}(X_1)]^2 = [\mathbb{E}(X_1)]^2\Lambda(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{V}[X(t)] = \mathbb{V}(X_1)\Lambda(t) + [\mathbb{E}(X_1)]^2\Lambda(t) = \Lambda(t)\{\mathbb{E}(X_1^2) - [\mathbb{E}(X_1)]^2 + \\ + [\mathbb{E}(X_1)]^2\} = \Lambda(t)\mathbb{E}(X_1^2).\end{aligned}$$

□

**Corollary 2.2.** *Let* $\{X(t+h) - X(t) : t \geq 0\}$ *be an increment of the NCPP. If* $\mathbb{E}[X_1^2] < \infty$*, then:*

1. $\mathbb{E}[X(t+h) - X(t)] = \mathbb{E}[N(t+h) - N(t)] \cdot \mathbb{E}(X_1) = [\int_t^{t+h} \lambda(x)dx] \cdot \mathbb{E}(X_1)$.
2. $\mathbb{V}[X(t+h) - X(t)] = \mathbb{E}[N(t+h) - N(t)]\mathbb{E}(X_1^2)$,

*where* $\Lambda(t) = \mathbb{E}[N(t)] = \int_0^t \lambda(x)dx$.

### Prediction of accident consequences

As it is done in [9], [10] it is assumed that the random variables $X_i$, $i = 1, 2, \ldots$ have Poisson distribution with parameter $\mathbb{E}(X_i) = \mathbb{V}(X_i) = \eta$, $i = 1, 2, \ldots, N(t)$ and that the parameter $\eta$ also depends on time, that is $\eta = \eta(t)$.

In our opinion it is not obvious that the number of deaths per accident behaves as a Poisson, since, for example, the probability of 100 fatalities in a traffic accident is negligible. Perhaps it would be more reasonable to assume that $X_i$ follows a binomial with possible values $k = \{0, 1, 2, 3, 4\}$ and adding weights to it. But of course there may be a compelling reason for this assumption that we do not see. Anyway, we explain the study with this assumption.

The purpose at this point is to estimate $\eta(t)$, using the linear regression $\eta(t) = a_2 t + b_2$ described in (2.4).
To get the linear regression in this case, one considered the center of the time interval in the horizontal axis, as before, but now the frequency of fatalities with respect the road accident number is in the vertical axis, that is:

$$Y_i = \frac{\#\{\text{fatalities}\}}{\#\{\text{accidents with victims}\}}.$$

The result can be seen in Figure 2.4.

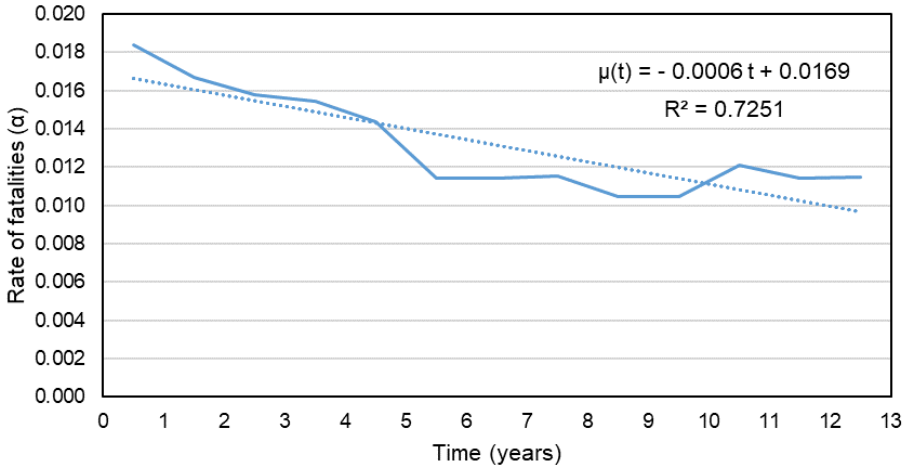


Figure 2.4: Frequency of fatalities against the road accident number (by a linear function).

Here the dotted line is $\eta(t)$, which is the new linear regression, and the blue function represents the frequency of fatalities with respect to the number of road accidents. As this graph shows, now $R^2 = 0.7251$, which is clearly better

than before.
One could still try to improve the value of $R^2$ by, for instance, approximating the data by a quadratic function instead of a linear one, in that case, we would get the Figure 2.5.

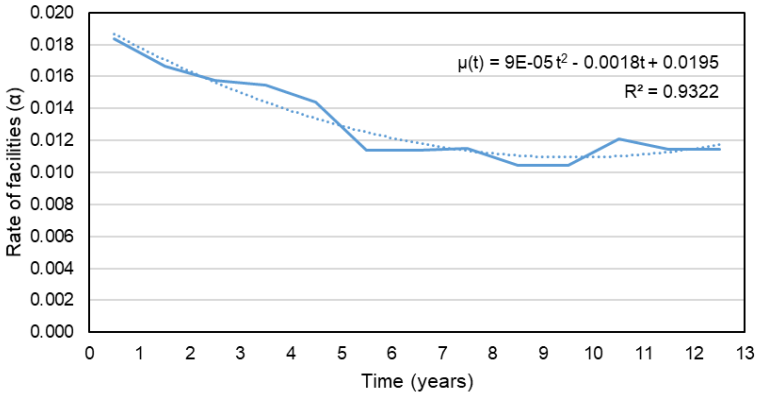


Figure 2.5: Frequency of fatalities against the road accident number (by a polynomial function).

Here $R^2 = 0.9322$, which is better than the previous value.

Continuing with the linear case, the regression line obtained is,

$$\eta(t) = -0.0006t + 0.0169, \quad t \geq 0. \tag{2.8}$$

In principle, in the interval of prediction $(t, t+h)$, for small $h$, the random variables $X_i$ are i.i.d. Nevertheless, the linear function above is almost constant in $(t, t+h)$ if the parameter $h$ is small. Thus, to simplify the model, it is supposed that

$$\mathbb{E}(X_i) = \mathbb{V}(X_i) = \hat{\eta} = \frac{\eta(t) + \eta(t+h)}{2}. \tag{2.9}$$

From (2.5) and Corollary (2.2) it is deduced that:

$$\begin{aligned} \mathbb{E}[X(t+h) - X(t)] &= \mathbb{E}[N(t+h) - N(t)] \cdot \hat{\eta} = \mathbb{E}[N(t+h) - N(t)] \\ &= h(\frac{ah}{2} + b + at) \cdot \hat{\eta}. \end{aligned} \tag{2.10}$$

$$\mathbb{V}[X(t+h) - X(t)] = \mathbb{E}[N(t+h) - N(t)] \cdot \mathbb{E}(X_1^2).$$

Now, recall that the expected number of fatalities at the time interval $(t, t+h)$

is given by the increment $X(t+h)-X(t)$. Thus, in order to predict the number of fatalities in road accidents in Catalonia from April $1^{st}$ to July $31^{st}$ 2021, the above identities are applied to the value $\eta(t)$ estimated in (2.8).
By (2.8) and (2.9),

$$\hat{\eta} = \frac{\eta(13.2466) + \eta(13.2466 + \frac{122}{365})}{2} \approxeq \frac{0.0093 + 0.0091}{2} = 0.0092.$$

Also

$$\mathbb{E}(X_i^2) = \mathbb{V}(X_i) + [\mathbb{E}(X_i)]^2 = \hat{\eta} + \hat{\eta}^2 = 0.0092.$$

Denoting by EFN to the expected number of fatalities and using (2.10), the predicted number of fatalities in the considered period of time is:

$$\begin{aligned}\mathbb{E}[X(t+h) - X(t)] &= \mathbb{E}[N(t+h) - N(t)] \cdot \hat{\eta} = 8187.5819 \cdot 0.0092 \\ &= \mathbf{EFN} = \mathbf{74.9998} \approx 75.\end{aligned}$$

Also by (2.10), the standard deviation of the fatality number (DFN) is:

$$\begin{aligned}D[X(t+h) - X(t)] &= \sqrt{\mathbb{E}[N(t+h) - N(t)] \cdot (\hat{\eta} + \hat{\eta}^2)} \\ &\approx \sqrt{8,187.5819 \cdot 0.0092} = \mathbf{DFN} = \mathbf{8.6998}.\end{aligned}$$

To conclude, the actual number of fatalities in this period is not known to us. Nonetheless, it is known that the total number for 2021 was 241. If distributed uniformly over the year, this would give approximately 80 fatalities for the period from April to July [23]. Note that this number is a bit off the 75 predicted by the model. This is not surprising since, in addition to the possible shortcomings of the model, it is observed in previous years that the majority of casualties happen in interurban zones and during the summer (i.e., they are not spread uniformly during the year) [23].
To compare the actual number of accidents with victims with the expected value found above, the same process could be applied.

After conducting some research, it seems that the Poisson process was applied some time ago (before 1995) to model traffic accidents in Catalonia, but it was later observed that this model has at least two drawbacks. On the one hand, the Poisson distribution assumes that the expected value and the variance are equal, which is generally not true in traffic accidents (the variance is usually greater [21]. On the other hand, by modelling traffic accidents by kilometres, there could be a huge number of kilometres with no accidents, which will produce a high frequency of zeros. This is not well suited for the Poisson distribution, even on stretches of 1 km in length, there are many intervals where no accidents occur, as [7] explains.

## 2.2 Non-homogeneous spatial Poisson process applied to the bombs location in London during the Second World War

### 2.2.1 South London

In this application we have mainly relied on a study from [24].

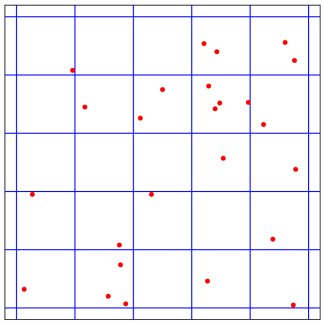

Figure 2.6: Computer simulation of part of the London grid and bombs location ( Freutci, via Wikimedia Commons).

An enthralling usage of the Poisson process in two dimensions is the modeling of the location of the V-1 bombs in south London during the WWII done by R. D. Clarke in 1946, a 31-year-old worker at the Prudential Assurance Company. He was interested in whether the bombs were dropped in clusters or randomly. He thought that humans are good in finding patterns and ascribing reasons to them, but statistical analysis could help to decide whether those patterns may be due to chance or to planing. Clarke decided to apply a statistical test to discover this allegation.

In his paper he chose an area of 144 $km^2$ situated in South London, where the mean number of V-1s seemed roughly constant. There were a total of 537 V-1 hits in this region. Then, the region was divided up into smaller squares measuring 0.25 $km^2$, and Clarke counted the number of V-1 hits in each individual square. After that he counted how many squares had no V-1s, how many squares had one V-1, and so on. With this notation, clusters would show up as squares with multiple V-1 hits. To test if V-1s tended to be clustered or not it was necessary to compare these results with what would arise from a random pattern. But there was a drawback. Nowadays, a computer can be used in order to simulate 537 random points. However, Clarke could not do this, so, deciding whether the bomb locations were random became essential for the study. At some point he believed that the Poisson process

could be applied to his project.
In this piece of work, an hypothetical uniform as well as random distribution of V-1s across South London was assumed, which means that the intensity $\lambda$ is assumed to be constant. In addition, he supposed that every V-1 was independent from the others. So, defining an event as *a V-1 landing in one of his 537 squares* and considering that the intensity was the *average density* of V-1s over the whole area, Clarke was able to obtain the expected results and compare them with the actual ones. Thus, according to the data, (see Figure 2.7)

$$\lambda = \frac{\#\{\text{V-1}\}}{\#\{\text{squares}\}} = \frac{537}{576} \approxeq 0.932.$$

He used the Poisson distribution to find the expected number of squares with $k$ V1-s hits for $k = 0, 1, \ldots, 4$ and $k \geq 5$. This is, for $X \sim \mathscr{P}(\lambda)$, then:

$$\begin{aligned}
&\mathbb{P}[X = 0] = e^{-\lambda} = e^{-\frac{537}{576}} \approx 0.394 \\
&\mathbb{P}[X = 1] = e^{-\lambda}\lambda = e^{-\frac{537}{576}} \frac{537}{576} \approx 0.367 \\
&\mathbb{P}[X = 2] = e^{-\lambda}\frac{\lambda^2}{2!} = e^{-\frac{537}{576}} \frac{(\frac{537}{576})^2}{2!} \approx 0.171 \\
&\ldots
\end{aligned}$$

Clarke obtained the following results:

| V1-s in square | Absolute frequency | Relative Frequency | Poisson Probability | Expected value |
|---|---|---|---|---|
| 0 | 229 | 0.398 | 0.394 | 226.743 |
| 1 | 211 | 0.366 | 0.367 | 211.39 |
| 2 | 93 | 0.161 | 0.171 | 98.539 |
| 3 | 35 | 0.061 | 0.053 | 30.622 |
| 4 | 7 | 0.012 | 0.012 | 7.137 |
| 5 and over | 1 | 0.002 | 0.003 | 1.569 |
| **Total** | 576 | 1 | 1 | 576 |

Figure 2.7: Results obtained by Clarke [6]. The expected number of squares is given by $\mathbb{P}(\lambda, k) \cdot 576$, where $\mathbb{P}(\lambda, k)$ is the mass function of observing $k$ events in a time interval for a Poisson variable of parameter $\lambda$.

Comparing the expected values with his observed values, Clarke concluded that it looked as if they adjusted almost perfectly. To confirm this, a chi-squared test was performed, where the null hypothesis was *V-1s did indeed follow the Poisson distribution* and a result of $p = 0.88$ was obtained, which

means that the null hypothesis was true with a 88% of probability.

During the following years, Clarke's publication became widely known. In 1950 Professor W. Feller of Cornell University said that *the above table indicates that there is perfect randomness and homogeneity, [...] but that to the untrained [eye], the randomness appears as regularity [...]* [26].
Nonetheless, Clarke only presented the summary table of his results [6], rather than the original. Maybe it was classified information at that time, but the trouble is that even now no one has been able to replicate the same results.

To confirm the veracity of this experiment, it has been repeated after 75 years. As Clarke published only a summary, the analysed area remains unknown. Therefore, an area with similar characteristics to that considered by Clarke has been chosen. In this case, however, it has been divided into 532 squares, so $\lambda = \frac{532}{576} \approxeq 0.924$. After doing the chi-square test, $p = 0.70$ is obtained. It is not as good as Clarke's, but it is really similar and also a positive result.

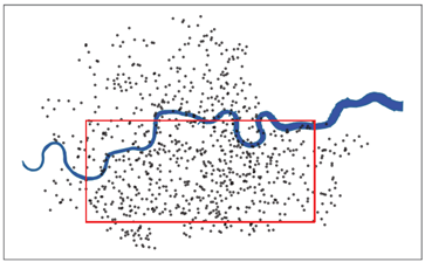

Figure 2.8: Black points indicate V-1 hits. Image from [24].

During these decades some studies have been presented with computational methods. However, no better results have been obtained. Statisticians think that the analysis performed by Clarke is correct but maybe only in this region; it is believed that Clarke was aware about this fact too.

In 1973, Thomas Pynchon wrote *Gravity's Rainbow*, a novel in which the Poisson distribution plays an important role. The *"flying bombs"* in this novel are not V-1s, but V-2s, and the statistician Roger Mexico realised that: *"More likely, Pynchon chose the V-2 because it suited his story better as a powerful image of random chance, falling without any warning whatsoever."*, [24].

The interested reader can read more about this in [24]. This article, written by Shay and Shaw, is the winner of Statistical Excellence Award for Early-Career Writing, awarded in partnership with the Young Statisticians Section of the Royal Statistical Society in 2019.

### 2.2.2 Germany

In this section we have mainly relied on a study from [11].
After these results, many other studies have been conducted. It is no longer just a question of curiosity, but of trying to find with reasonable certainty whether there were areas with greater risks, and if there were, which ones. This kind of studies are of great value to help finding unexploded bombs in old war zones, which can represent a threat for many years.

As an illustration, there is a whole paper conducting a study for WWII in Germany, where about 1.5 millions of tons of bombs were pounded and it is believed that around $10 - 15\%$ have not yet exploded [11]. Studying the spatial pattern of exploded bombs helps to determine *risk-zones*, regions with a likelihood of containing unexploded bombs. Assume that $X$ is a point process of the spatial locations of all bombs dropped in a particular window $W \subseteq \mathbb{R}^2$, and let $Y$ denote the observed point process of exploded bombs; then $Z = X \backslash Y$ is the point process of unexploded bombs. In this case, $Y$ is observed but $Z$ is not. If $q$ is the probability of a dropped bomb not exploding, assuming that this is homogeneous in $W$, and that $X$ is a non-homogeneous Poisson point process with intensity $\lambda_X(t)$ for $t \in W$, then, $\lambda_Y(t) = (1-q)\lambda_X(t)$ and $\lambda_Z(t) = q\lambda_X(t)$ are the respectively intensity functions for $Y$ and $Z$.
The entire statistical study to find high risk-zones can be read in [11].

Figure 2.9: NCAP aerial photo from 1944 showing the bombing of the V-2 rocket facility at Peenemünde, Germany. Image is available under a custom NCAP license - higher resolution images are available from NCAP, from [11].

## 2.3 Other examples

In this section we present briefly interesting applications of the Poisson process and give their references.

### 2.3.1 1D - Examples

Examples abound in one dimension because the random arrival of independent events in time is a very frequent occurrence in everyday life, like in the queueing theory mentioned previously [14],[24]. Examples of this are the forecasting of emergency calls or arrivals at an emergency service, goals scored in a football match [14], or email arrivals in an inbox.
In addition to those stated above, the Poisson process is to:

- **Economics.** As far back as 1903, Filip Lundberg studied in his doctoral thesis the probability of ruining an insurance company [26]. Some years later, Harald Cramér extended this study, which today is known as the *Cramér-Lundberg theory* (or *ruin theory*). Under the assumption that the times an insurer has to pay the costumers follows a Poisson process, the authors developed a theory of risk that is still in use [13].

- **Neural research.** Assuming that synapses are activated by independent Poisson processes [5].

- **Emission of alpha particles or heavy nuclei.** [18] It is found in the paper by Pere Puig and Joan-Francesc Barquinero, from the UAB [19].

### 2.3.2 2D - Examples

In two dimensions, the Poisson process is applied to the location of certain events, for instance, the distribution of salmon in the sea or the resolution of naval research problems [14].
Other examples are:

- **Molecules under a microscope.** The distribution density of these molecules is usually estimated to locate them under the microscope. In this study, the two-dimensional Poisson process is used to model the random activation of the molecules, in order to determine the location density in real time. In this way it is possible to optimise the performance of the microscope images without compromising the image quality [16].

- **Forest fires.** In [25] there is a study about forest fires in Portugal between 2009 and 2015, aiming to estimate the occurrence of these events

in time. Both the homogeneous and the non-homogeneous Poisson processes were used. A similar study was done in the Valencian Community [20].
In parallel and taking into account the danger of fires in Portugal, as it is explained in [4], "*fire stations [4] are typically non-uniformly distributed across space, and their service area is, in general, defined based on administrative boundaries*". But the location of FS is important when a fire starts. Obviously, the further away a FS is, the longer it will take before the fire fighters can start working. Therefore, studies are needed to find where FS would be best located. The study of FS using a spatial Poisson process is detailed in this article.

- **Wireless networks**. Around 1960, Edgar Gilbert developed a model of wireless networks based on the Poisson process. Today, this is considered the birth of the percolation theory. Mobile phone networks are an example of this (see [14]).

- **Distribution of plants**. The Swedish chemist Theodor Svedberg proposed, as early as 1922, to study the distribution of plants in plant communities by means of the Poisson point process. Years later more mathematicians joined the study and were able to make important contributions. Theodor Svedberg was awarded a Nobel prize in chemistry in 1926 [14].

### 2.3.3 3D - Examples

- **Astronomy**. The example in [22] shows how the Poisson process can be applied in 2 and 3 dimensions for modelling the stars in the sky.

- **Traffic accidents**. Considering the location and time of traffic accidents in an area, the 2D Poisson process becomes a 3-D model. This leads to a map with the location of the accidents and a third dimension for the time in which they occur [2].

To conclude, there are many applications of the Poisson process that are not going to be even mentioned in this essay due to lack of space. The intention was just to prove that the Poisson process is very versatile and has a huge range of applications. That a mathematical process is applied to so many scopes and for so long, is quite remarkable.

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