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The Gauss-Bonnet Theorem

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Abstract

The Gauss-Bonnet Theorem was first published by Gauss in 1827 for the case of a geodesic triangle on a surface. Since then, the theorem has progressively increased in generality. The purpose of this work is to prove it for the case of 2-dimensional Riemannian manifolds, while discussing the historic development of its other versions. For that, the necessary concepts of differential geometry are introduced, such as smooth manifolds, their tangent spaces, and the measurement of areas and angles via Riemannian metrics. The concepts of curves, lifts, orientability, and curvature are also adapted to the nature of manifolds. With that toolbox, the Rotation Index Theorem is proved, subsequently the Gauss-Bonnet Formula, and finally the Gauss-Bonnet theorem for orientable and non-orientable manifolds. The latter employs combinatorial arguments, combining local results to yield a global one. The most remarkable aspect of this theorem is precisely that it connects local properties of differential geometry, specifically the integral of the curvature, with a global topological invariant, the Euler characteristic.

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Chapter 1 Preliminary Theory

This chapter will present the majority of the necessary theory for establishing the Gauss-Bonnet Theorem. However, certain foundational results concerning manifolds will be presumed known, and in such instances, a source with the proof will be provided. Many of the definitions and derivations that appear here have interest on their own, and more details are available on [2], and [3]. Nevertheless, here only results relevant to the Gauss-Bonnet Theorem will be treated.

1.1 Smooth Manifolds

In this section, we delve into the realm of smooth manifolds: a framework essential for proving the Gauss-Bonnet theorem on 2-dimensional Riemannian spaces. They are the tool required to explore a geometric terrain where calculus extends beyond Euclidean spaces.

Definition 1.1. A topological space is *locally Euclidean of dimension n* if every point has a neighborhood homeomorphic to an open subset of \mathbb{R}^n .

Definition 1.2. An *n*-dimensional topological manifold or simply an *n*-manifold is a topological space that is second-countable Hausdorff and locally Euclidean of dimension *n*.

Definition 1.3. An *n*-dimensional topological manifold with boundary is a secondcountable Hausdorff space in which every point has a neighborhood homeomorphic to either an open subset of \mathbb{R}^n , or to an open subset of the closed half-space $\mathbb{H}^n = \{(x^1, ..., x^n) \in \mathbb{R}^n : x^n \ge 0\}$, whose boundary is $\partial \mathbb{H}^n =$ $\{(x^1, ..., x^n) \in \mathbb{R}^n : x^n = 0\}$.

Notice that the term manifold *with boundary* comprises in particular the manifolds *without boundary*. Therefore, one refers to manifolds *without boundary* simply as *manifolds*. Manifolds *with boundary* might be referred to as manifolds *with or without boundary*.

Definition 1.4. Let *M* be an *n*-dimensional manifold (without boundary). Then. a *coordinate chart for M* is a pair (U, φ) such that

- 1. $U \subseteq M$ is an open subset,
- 2. $\varphi: U \to \widehat{U}$ is a homeomorphism, where \widehat{U} is an open subset of \mathbb{R}^n or \mathbb{H}^n .
- (U, φ) is said to be a *chart containing* p if $p \in U \subseteq M$.

Definition 1.5. Let $U \subseteq \mathbb{R}^n$ be open, and let $F: U \to \mathbb{R}^k$ be a map. *F* is said to be *smooth* or *of class* C^{∞} if the partial derivatives of any order of its component functions are all continuous. A bijective smooth map whose inverse is also smooth is called a *diffeomorphism*.

Definition 1.6. Let *M* be a topological *n*-manifold with or without boundary, and let (U, φ) , and (V, ψ) be two coordinate charts for *M*. The maps $\psi \circ \varphi^{-1}$, and $\varphi \circ \psi^{-1}$ are called *transition maps*. Their domains are $\varphi(U \cap V)$, and $\psi(U \cap V)$, respectively.

Definition 1.7. The charts (U, φ) , and (V, ψ) are said to be *smoothly compatible* if their transition maps are smooth on their domain. Since they are the inverse of each other, they are diffeomorphisms.



Figure 1.1: The figure on the left represents *U* and *V* in *M*, while the figure on the right represents their image by φ and ψ , respectively, in \mathbb{R}^n .

Definition 1.8. A collection of coordinate charts whose domains cover M is called an *atlas for* M. Additionally, an atlas is *smooth* if any two charts in it are smoothly compatible. A smooth atlas A is *maximal* if it is not properly contained in any larger smooth atlas, which means that any chart that is smoothly compatible with every chart in A is already in A. **Definition 1.9.** A *smooth structure on M* is a smooth atlas that is maximal. A is a topological manifold with boundary equipped with a smooth structure is said to be a *smooth manifold with boundary*.

Let *M*, and *N* be *m*-, and *n*-dimensional smooth manifolds with or without boundary, respectively.

Definition 1.10. Let $F: M \to N$ be a map. It is said to be *smooth* if for any $p \in M$

- 1. there is a smooth chart (U, φ) for *M* containing *p*, and a smooth chart (V, ψ) for *N* containing F(p),
- 2. $F(U) \subseteq V$,
- 3. $\widehat{F} = \psi \circ F \circ \varphi^{-1} \colon \mathbb{R}^m \to \mathbb{R}^n$ is smooth. This function is called the *coordinate representation of* M.



Figure 1.2: Representation of two distinct smooth manifolds M, and N, with dimensions m, and n, respectively. Additionally, the charts outlined in Definition 1.10 are illustrated along with their resulting coordinate representations.

Definition 1.11. The set of smooth maps from *M* to *N* is denoted by $C^{\infty}(M, N)$, and the set of smooth maps from *M* to \mathbb{R} is simply notated as $C^{\infty}(M)$.

Now, manifolds with corners are introduced. For that, let *M* be a topological *n*-manifold with boundary, and let $\overline{\mathbb{R}}^n_+$ denote the set

$$\overline{\mathbb{R}}^{n}_{+} = \left\{ \left(x^{1}, \dots, x^{n} \right) \in \mathbb{R}^{n} \colon x^{1} \ge 0, \dots, x^{n} \ge 0 \right\}.$$
(1.1)

Definition 1.12. A *chart with corners for* M is a pair (U, φ) such that $U \subseteq M$ is open and φ is a homeomorphism from U to an open subset $\widehat{U} \subseteq \overline{\mathbb{R}}^n_+$. Two charts with corners are said to be *smoothly compatible* the same way as it was described in Definition 1.7.

Definition 1.13. A smooth structure with corners on M is a collection of

- 1. smoothly compatible charts (U, φ) such that $\varphi(U) \cap \partial \mathbb{H}^n = \emptyset$,
- charts (*U*, φ) with corners, which means that φ is a homeomorphism from *U* to an open subset of ℝⁿ₊,

that is maximal (in the sense of Definition 1.8), and whose domains cover M.

Now, the concept of tangent vectors is introduced. For that, now let M be a smooth manifold with or without boundary.

Definition 1.14. Let $v: C^{\infty}(M) \to \mathbb{R}$ be a map, $p \in M$, and $f, g \in C^{\infty}(M)$. Then, v is said to be a *derivation at* p if

$$v(fg) = f(p)vg + g(p)vf.$$
(1.2)

Definition 1.15. A *smooth vector bundle of rank k* is a pair of smooth manifolds *E* (*to-tal space*) and *M* (*base*) with or without boundary, and map $\pi: E \to M$ (*projection*) such that for every $p \in M$

- 1. $E_p = \pi^{-1}(p)$ is endowed with the structure of a *k*-dimensional real vector space,
- 2. there is a neighborhood *U* of *p*, and a diffeomorphism $\Phi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ (*smooth local trivialization*) such that
 - (a) $\pi_U \circ \Phi = \pi$, where π_U is the projection onto the first factor: $\pi_U \colon U \times \mathbb{R}^k \to U$,
 - (b) the restriction of Φ on $q \in U$, i.e. $\Phi(q, \cdot)$, is a linear isomorphism $E_q \to q \times \mathbb{R}^k \cong \mathbb{R}^k$.

Definition 1.16. Let $\pi: E \to M$ be a smooth vector bundle over M. Then, a *section* of E is a continuous map $\sigma: M \to E$ such that $\pi \circ \sigma = \text{Id}_M$. This is equivalent to imposing that $\sigma(p) \in E_p$, since by definition $E_p = \pi^{-1}(p)$. Let $U \subseteq M$ be an open subset, then a continuous map $\sigma: U \to E$ that satisfies $\pi \circ \sigma = \text{Id}_U$ is called a *local section of* E *over* M. For a smooth vector bundle $E \to M$, the set of smooth sections of E is denoted by $\Gamma(E)$.

Definition 1.17. A *local frame for* E *on* U is an ordered k-tuple $(\sigma_1, \ldots, \sigma_k)$ of local sections over an open subset $U \subseteq M$ such that at each $p \in U$, they form a base for E_p .

Definition 1.18. A *tangent vector at* p is a linear derivation at p. The set of all tangent vectors at p is called the *tangent space at* p and it is notated as T_pM .

Definition 1.19. Let *M* be an *n*-manifold, and $\varphi: U \to \widehat{U} \subseteq \mathbb{R}^n$ a smooth coordinate chart where $U \subseteq M$ is open. If the coordinate functions of φ are notated as (x^1, \ldots, x^n) then the *coordinate vectors* $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ are defined as the derivations that satisfy

$$\frac{\partial}{\partial x^{i}}\Big|_{p}f = \frac{\partial}{\partial x^{i}}\Big|_{\varphi(p)}\left(f \circ \varphi^{-1}\right).$$
(1.3)

Definition 1.20. Let $F: M \to N$ be a smooth map, and $p \in M$ a point. The *differential of F at p* is defined as the linear map

$$\begin{aligned} dF_p \colon T_p M &\to T_{F(p)N} \\ v &\mapsto dF_p(v), \end{aligned}$$
 (1.4)

given by $dF_p(v)f = v(f \circ F)$, where $f \in C^{\infty}(M)$.

Proposition 1.21. Let $F: M \to N$ be a smooth map, and $p \in M$ a point. Then dF_p is a well-defined linear map $T_pM \to T_{F(p)}N$.

Proof. Let $v_1, v_2 \in T_v M$ be arbitrary derivations, and $f \in C^{\infty}(M)$ a function.

$$((dF)_p(v_1+v_2))f = (v_1+v_2)(f \circ F) = v_1(f \circ F) + v_2(f \circ F).$$
(1.5)

Definition 1.22. The *tangent bundle of M* is the disjoint union of tangent spaces for all the points of *M*:

$$TM = \coprod_{p \in M} T_p M. \tag{1.6}$$

The set $\Gamma(TM)$ is notated as $\mathfrak{X}(M)$. A section of *TM* is called a vector field on *M*.

Remark 1.23. The tangent bundle is indeed a vector bundle, and that is a consequence of Lemma A.34 of [3], as exposed in page 384 of the same source.

Definition 1.24. Let $X, Y \in \mathfrak{X}(M)$. The Lie Bracket is defined as the map

$$[X, Y]: \quad C^{\infty}(M) \to C^{\infty}(M)$$
$$f \mapsto [X, Y]f = X(Yf) - Y(Xf).$$
(1.7)

Proposition 1.25. (Properties of Lie Brackets) Let M be a smooth manifold with or without boundary, and let $X, Y, Z \in \mathfrak{X}(M)$. Then the Lie Bracket

- 1. *is bilinear over* \mathbb{R} *as a function of the first or second argument. For example, for the first argument:* [aX + bY, Z] = a[X, Z] + b[X, Z], where $a, b \in \mathbb{R}$,
- 2. is antisymmetric: [X, Y] = -[Y, X],
- 3. satisfies [fX, gY] = fg[X, Y] + (fXg)Y (gYf)X, for $f, g \in C^{\infty}(M)$.

Proof. 1. and 2. are trivial. 3. is shown below:

$$[fX,gY] = fX(gY) - gY(fX) = f(X(g)Y + gXY) - g(Y(f)X - fYX)$$

= $fg[X,Y] + (fXg)Y - (gYf)X.$ (1.8)

Finally, there are some intuitive concepts in the Euclidean spaces that are to be translated to the realm of manifolds via the connections. While the notion of geodesics – the equivalent of straight lines in Euclidean spaces – may look as a safe path, characterizing them as shortest paths poses technical challenges. Instead, connections offer a powerful framework for differentiating vector fields along curves, defining geodesics, and enabling the concept of "parallel transport" of vectors along these curves.

Definition 1.26. Let *M* be a smooth manifold with or without boundary, a *connection in the tangent bundle TM* is defined as any map

$$\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$(X, Y) \mapsto \nabla_X Y \tag{1.9}$$

such that ∇

1. is linear over $C^{\infty}(M)$ in the first argument:

$$\nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y, \qquad (1.10)$$

2. is linear over \mathbb{R} in the second argument:

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2, \tag{1.11}$$

3. satisfies this product rule:

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y, \qquad (1.12)$$

where $f, g \in C^{\infty}(M)$, $X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(M)$, and $a, b \in \mathbb{R}$.

The vector field $\nabla_X Y$ is called the *covariant derivative of* Y *in the direction* X.

1.2 Tensors

In the context of this document, it is understood that only real vector spaces will be discussed, and this will not be reiterated further.

Definition 1.27. Let *V* be an *n*-dimensional vector space, and let V^* be its dual space, whose elements are called covectors. A *mixed tensor of type* (j,k) is a multi-linear map

$$F: \overbrace{V^* \times \cdots \times V^*}^{j} \times \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}, \qquad (1.13)$$

where, for simplicity of notation, it has been assumed that the arguments are grouped in vectors and covectors, but this is not mandatory. The *set of mixed* (j,k)-*tensors on* V is denoted as $T^{j,k}(V)$.

Definition 1.28. A *covariant j-tensor* F *on* V is a mixed tensor of type (0, j),

$$F: \overbrace{V \times \cdots \times V}^{j} \to \mathbb{R}.$$
(1.14)

The set of covariant *j*-tensors on V is denoted as $T^{j}(V^{*})$.

Definition 1.29. A *contravariant j-tensor* F *on* V is a mixed tensor of type (j, 0),

$$F: \overbrace{V^* \times \cdots \times V^*}^{j} \to \mathbb{R}.$$
(1.15)

The set of contravariant *j*-tensors on V is denoted as $T^{j}(V)$.

There is a particularly interesting subset of $T^k(V^*)$:

Definition 1.30. Let $F \in T^k(V^*)$ be a covariant tensor. It is said to be *symmetric* if it is invariable under an interchange of any two of its arguments:

$$F(..., u_i, ..., u_j, ...) = F(..., u_j, ..., u_i, ...), \quad u_i, u_j \in V, \quad 1 \le i < j \le k.$$
 (1.16)

Then, the set of symmetric *k*-tensors of *V*, denoted by $\Sigma^k(V^*)$, is clearly a linear subset of $T^k(V^*)$.

Conversely, if the sign changes under an interchange of any two of its arguments, *F* is said to be *alternating*. The set of alternating *k*-tensors of *V* is denoted by $\Lambda^k(V^*)$, and it is a linear subset of $T^k(V^*)$.

In the study of manifolds, the vector spaces under consideration are predominantly T_pM , for $p \in M$. Consequently, it could be interesting to explore the disjoint union of tangent spaces at every point in M.

Definition 1.31. The bundle of (j, k)-tensors on M is defined as

$$T^{(j,k)}TM = \prod_{p \in M} T^{(j,k)} (T_p M)$$
(1.17)

The bundle of covariant *j*-tensors is denoted as $T^jT^*M = T^{(0,j)}TM$, and the bundle of contravariant *j*-tensors is denoted as $T^jTM = T^{(j,0)}TM$.

The *bundle of symmetric j-tensors* is

$$\Sigma^{k}T^{*}M = \coprod_{p \in M} \Sigma^{k} \left(T_{p}^{*}M\right).$$
(1.18)

Definition 1.32. A *tensor field on* M is a section of a tensor bundle over M. If it is a section of $T^{(0,1)}TM$ (a covariant 1-tensor field) then it is called a *covector field*. The space of all smooth covariant k-tensor fields is denoted as $\mathcal{T}^k(T^kT^*M)$. Similarly, the subbundle of T^kT^*M formed by the alternating tensor tensors is denoted by Λ^kT^*M , and an alternating tensor field on M is known as a k-form.

Definition 1.33. Let $(E_1, ..., E_n)$ be a smooth local frame for *TM*. Its associated *dual coframe*, $(\varepsilon_1, ..., \varepsilon_n)$, is a smooth covector field for which $\varepsilon^i(E_i) = \delta_i^i$.

Definition 1.34. Let V be an n-dimensional vector space, then a *density on* V is a function

$$\mu: \overbrace{V^* \times \cdots \times V^*}^n \to \mathbb{R}.$$
(1.19)

such that for every linear map $T: V \to V$, it holds

$$\mu(Tv_1, \dots, Tv_n) = |\det T| \, \mu(v_1, \dots, v_n).$$
(1.20)

Let (v_1, \ldots, v_n) be a base for *V*. If $\mu(v_1, \ldots, v_n) > 0$, then μ is said to be *positive*.

Proposition 1.35. *The previous definition is independent of the base of V.*

Proof. Let (v_1, \ldots, v_n) , and (u_1, \ldots, u_n) be two bases of V, $\mu(v_1, \ldots, v_n) > 0$, and let $A = (v_1 \ldots v_n)$, and $B = (u_1 \ldots u_n)$ be the matrices whose columns are the vectors of these bases, respectively. Via a series of elementary transformations, T_1, \ldots, T_k , one can obtain matrix B from matrix A: if $T = T_n \ldots T_1$, then B = TA. In that case, T is a linear map and thus, $\mu(u_1, \ldots, u_n) = \mu(Tv_1, \ldots, Tv_n) = |\det T| \mu(v_1, \ldots, v_n) > 0$.

Proposition 1.36. Let V be a vector space of dimension $n \ge 1$, and let μ_1 and μ_2 be densities on V. If $\mu_1(E_1, \ldots, E_n) = \mu_2(E_1, \ldots, E_n)$ for some basis $(E_i)_{i=1,\ldots,n}$ of V, then $\mu_1 = \mu_2$.

Proof. Let u_1, \ldots, u_n be arbitrary vectors. Define $T: V \to V$ as the unique linear map such that $T(E_i) = u_1$, for $i = 1, \ldots, n$. Then,

$$\mu_{1}(u_{1},...,u_{n}) = \mu_{1}(TE_{1},...,TE_{n}) = |\det T|\mu_{1}(E_{1},...,E_{n})$$

= $|\det T|\mu_{2}(E_{1},...,E_{n}) = \mu_{2}(TE_{1},...,TE_{n})$ (1.21)
= $\mu_{2}(u_{1},...,u_{n})$

Definition 1.37. Let (M, g) be a Riemannian manifold (with or without boundary). The *Riemannian density* is defined as the unique density such that for every local orthonormal frame $(E_i)_{i=1,...,n}$

$$\mu(E_1, \dots, E_n) = 1. \tag{1.22}$$

Additionally, if (M, g) is oriented, the Riemannian volume form dV_g is defined in the same way by means of a positively oriented orthonormal local frame:

$$dV_g(E_1,\ldots,E_n) = 1.$$
 (1.23)

Proposition 1.38. *The Riemannian density is well defined: it exists and is unique. Additionally, it is smooth and positive.* *Proof.* Let $(E_i)_{i=1,...,n}$ be a local orthonormal frame for *TM* on an open set $U \subseteq M$, and let $(\varepsilon^1, \ldots, \varepsilon^n)$ be its corresponding dual coframe. Define μ as $\mu = |\varepsilon_1 \land \ldots \land \varepsilon_n|$. For this definition, the wedge product is used (its exact definition can be promptly found at page 400 of [3]). Here, its explicit expression is given:

$$\varepsilon_1 \wedge \ldots \wedge \varepsilon_n(v_1, \ldots, v_n) = \Sigma_{\sigma \in S_n} \operatorname{sgn}(\sigma) (\varepsilon_1 \otimes \ldots \otimes \varepsilon_n) (v_{\sigma_{(1)}}, \ldots, v_{\sigma_{(n)}}),$$
 (1.24)

where S_n is the set of all *n*-permutations. This clearly shows that $\mu(E_1, \ldots, E_n) = 1$, and thus is positive.

To prove it is an smooth function it suffices to see that each ε_i is smooth. Therefore, the goal is to show that given a smooth frame (E_i) , its associated dual coframe (ε_i) is smooth. First, express each E_i , and ε^j in terms of the coordinate frame $(\partial/\partial x^i)$, and coordinate coframe (λ^i) , respectively:

$$E_i = a_i^k \frac{\partial}{\partial x^k}, \quad \varepsilon^j = b_l^j \lambda^l. \tag{1.25}$$

Notice that from the condition $\varepsilon^{j}(E_{i}) = \delta_{i}^{j}$ it follows that the matrices (a_{i}^{k}) and (b_{l}^{j}) are inverses of each other. Since the map of the matrix inversion is an smooth endomorphism of $GL(n, \mathbb{R})$, if either (a_{i}^{k}) or (b_{l}^{j}) are smooth, the other matrix will be smooth as well. In order to see that the coefficients a_{i}^{k} of E_{i} are continuous, notice that $E_{i}: U \to TM$ is smooth. For a $p \in U \subseteq M$, there is a coordinate chart $(W, \varphi), W \subseteq M$ open, and so its coordinate functions are notated as (x^{1}, \ldots, x^{n}) , where $x_{i} = \pi_{i} \circ \varphi$, and where $\pi_{i}: \mathbb{R}^{n} \to \mathbb{R}$ is a projection. Finally, notice that $a_{i} = E_{i}(x_{i}), \varphi, \pi_{i}$ and E_{i} are smooth, and it can be concluded that μ is smooth.

To see that μ is a density, observe that $(\varepsilon_1 \otimes \ldots \otimes \varepsilon_n)$ is an *n*-form, and then

$$(\varepsilon_1 \otimes \ldots \otimes \varepsilon_n) (Tv_1, \ldots, Tv_n) = T (\varepsilon_1 \otimes \ldots \otimes \varepsilon_n) (v_1, \ldots, v_n), \qquad (1.26)$$

where *T* is a linear map. Then, if one applies the modulus to the expression at Eq. (1.24), it yields that $\mu(Tv_1, \ldots, Tv_n) = |\det(T)|\mu(v_1, \ldots, v_n)$. Thus, uniqueness stems from Proposition 1.36. If two smooth coordinates, (U, φ) and (V, ψ) , overlap, $U \cap V \neq \emptyset$, then the two associated definitions of μ coincide, by the same uniqueness of Proposition 1.36, which means that the definition of μ is global. \Box

Corollary 1.39. The *Riemannian volume form* is well defined and unique. Let (E_1, \ldots, E_n) an oriented orthonormal frame (E_1, \ldots, E_n) , and $(\varepsilon_1, \ldots, \varepsilon_n)$ its corresponding dual coframe. In this base, dV_g is given by

$$dV_g = \varepsilon_1 \wedge \ldots \wedge \varepsilon_n. \tag{1.27}$$

Definition 1.40. Let $F: M \to N$ be a smooth map, $p \in M$ a point, and α a *k*-tensor $\alpha \in T^k(T^*_{F(p)}N)$. Let $v_1, \ldots, v_k \in T_pM$ be arbitrary derivations. Then the *pointwise pullback of* α *by* F *at* p is a tensor $dF^*_p(\alpha) \in T^k(T^*_pM)$ given by

$$dF_{p}^{*}(\alpha)\left(v_{1},\ldots,v_{k}\right)=\alpha\left(dF_{p}\left(v_{1}\right),\ldots,dF_{p}\left(v_{k}\right)\right).$$
(1.28)

If *A* is a covariant *k*-tensor field on *N*, the *pullback of A* is a *k*-tensor field F^*A on *M* such that

$$(F^*A)_p = dF_p^* \left(A_{F(p)} \right).$$
(1.29)

Lastly, two results necessary for the proof of the Gauss-Bonnet Formula (Theorem 3.29) are presented:

Proposition 1.41. Let ω be a smooth 1-form, and X, Y smooth field vectors. It holds that

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$
(1.30)

Proof. This result is very specific, and would require the introduction of more theory. Therefore, if necessary, consult Theorem 14.24 of [2]. \Box

1.3 Riemannian Metrics

Definition 1.42. Let *V* be a vector space. An *inner product on V* is a map

$$\begin{array}{c} \langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{R} \\ (u, v) \mapsto \langle u, v \rangle \end{array}$$
(1.31)

such that for any u, v, $w \in V$, and a, $b \in \mathbb{R}$, the following properties hold:

- 1. Symmetry: $\langle u, v \rangle = \langle v, u \rangle$,
- 2. Linearity on the first argument: $\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle$ (which implies bilinearity when combined with the previous property),
- 3. Positive definiteness: $\langle u, u \rangle \ge 0$, and $\langle u, u \rangle = 0$ if and only if u = 0.

Definition 1.43. A vector space that possesses a specific inner product is called an *inner product space*.

Definition 1.44. The *length* or *norm of* $v \in V$ corresponds to

$$|u| = \langle u, u \rangle^{\frac{1}{2}}.\tag{1.32}$$

Definition 1.45. Let $u, v \in V$. The *angle between u and v* is defined uniquely as the angle θ comprised in $[0, \pi]$ given by

$$\cos \theta = \frac{\langle u, v \rangle}{|u| |v|}.$$
(1.33)

The vectors *u* and *v* are said to be *orthogonal* if $\langle u, v \rangle = 0$. If on top of that, each of their norm is 1, they are called *orthonormal*.

Definition 1.46. Let $S = \{u_1, ..., u_k\}$, $k \in a$, $u_i \in V$, be a set of vectors of V. Their *linear span* or *span of S*, notated as span(S), is the set of all the possible linear combinations of the vectors in S. A set of vectors is said to be *orthogonal* (*orthonormal*) if they are pairwise orthogonal (orthonormal).

Proposition 1.47. (The Gram-Schmidt Algorithm) Suppose V has dimension n and is endowed with an inner product. Let (u_1, \ldots, u_n) be an ordered basis for V. Then, a new orthonormal basis (b_1, \ldots, b_n) may be retrieved such that

$$\operatorname{span}(\{u_1, \dots, u_j\}) = \operatorname{span}(\{b_1, \dots, b_j\}), \quad 1 \le j \le n.$$
 (1.34)

This base is obtained by applying the following recursive algorithm:

$$b_{1} = \frac{u_{1}}{|u_{1}|}$$

$$b_{j} = \frac{u_{j} - \sum_{i=1}^{j-1} \langle u_{j}, b_{i} \rangle b_{i}}{|v_{j} - \sum_{i=1}^{j-1} \langle u_{j}, b_{i} \rangle b_{i}|}, \quad 2 \le j \le n.$$
(1.35)

Proof. Notice that b_1 is well defined because $u_1 \neq 0$. Similarly, b_j is well defined because since (u_1, \ldots, u_n) is a basis, then $u_j \notin \text{span}(u_1, \ldots, u_{j-1})$, for $j = 1, \ldots, n$. It is trivial to see that b_j is a linear combination of u_1, \ldots, u_j , which implies the condition in Eq. (1.34), and also the linear independence of the new base.

Definition 1.48. A *Riemannian metric on* M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M)$ such that when evaluated at any $p \in M$, notated as g_p , it corresponds to an inner product on T_pM . A *Riemannian manifold* is a pair (M, g), where M is a smooth manifold with or without boundary, and g is a Riemannian metric on M.

Proposition 1.49. Let M be a smooth manifold. Then, M admits a Riemannian metric.

Proof. Page 376 of [3] introduces the partitions of unity and smooth bump functions related to a manifold M, and shows their existance. Then, there is a partition of unity that covers M, and a collection of bump functions that are equal to 1 in some neighborhood of p, for every $p \in M$, and vanish outside a larger neighborhood. These functions, together with a Riemannian metric on \mathbb{R}^n (the Euclidean metric, for instance), can be used to pull back metrics to M from \mathbb{R}^n , locally.

1.4 Curves

Let (M, g) be a Riemannian manifold with or without boundary.

Definition 1.50. A *curve in* M is a continuous map $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an interval that may or may not include endpoints, or be bounded. It is said to be *smooth* if it is an smooth map from the manifold $I \subseteq \mathbb{R}$, which has boundary, to the manifold M. A curve is said to be a *curve segment* if its domain is a compact interval.

Definition 1.51. A *regular curve* is a smooth curve γ such that $\gamma' \neq 0$ for each $t \in I$.

Definition 1.52. Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval. A *partition of* [a, b] is a finite sequence $(a_0, ..., a_k)$ of real numbers such that $a = a_0 < a_1 < \cdots < a_k = b$. The intervals $[a_{i-1}, a_i]$ are called *subintervals of the partition*.

Definition 1.53. A curve is *piecewise regular* if there is a partition (a_0, \ldots, a_k) of [a, b] such that $\gamma|_{[a_{i-1},a_i]}$ is a regular curve segment for each $i = 1, \ldots, k$. An *admissible curve* is a piecewise regular curve segment. An *admissible partition for a curve* γ is a partition (a_0, \ldots, a_k) such that $\gamma|_{[a_{i-1},a_i]}$ is smooth for each $i = 1, \ldots, k$.

1.5 Covering maps

Let *M* and *M* be topological spaces that are both connected and locally pathconnected, and let $\pi: \widetilde{M} \to M$ be a surjective continuous map.

Definition 1.54. A set $A \subseteq M$ is *evenly covered* if π maps each connected component of $\pi^{-1}(A)$ homeomorphically onto A.

Definition 1.55. A map π is a *covering map* if every point of M has a connected neighborhood of U that is evenly covered. A covering map is *smooth* if \widetilde{M} and M are smooth manifolds with or without boundary and for each $p \in M$ there is a neighborhood U such that π maps each component of $\pi^{-1}(U)$ diffeomorphically onto U.

Definition 1.56. Let $U \subseteq M$ be an evenly covered open set, and π a covering map. The connected components of a of $\pi^{-1}(U)$ are called the *sheets of the covering over* U.

Definition 1.57. Let $\pi: \widetilde{M} \to M$ be a covering map, and let $F: B \to M$ be a map from a topological space B into M. A *lift of* F is a continuous map $\widetilde{F}: B \to \widetilde{M}$ such that $\pi \circ \widetilde{F} = F$.

Some relevant properties of these maps are displayed below:

Proposition 1.58. (Lifting Properties of Covering Maps)

Let $\pi: \widetilde{M} \to M$ be a covering map.

- (a) Unique lifting property: Let B be a connected topological space and $F : B \to M$ be a continuous map. Then, any two lifts of F are identical if they coincide at one point.
- (b) Path Lifting Property: Let $f : [0,1] \to M$ be a continuous path. For every $\tilde{p} \in \pi^{-1}(f(0))$, there is a unique lift $\tilde{f} : [0,1] \to \widetilde{M}$ of f such that $\tilde{f}(0) = \tilde{p}$.

Proof. See [2]; Theorem 11.12, Corollary 11.4, and Theorem 11.15, respectively.

Theorem 1.59. (Lifting Maps from Simply Connected Spaces) Let $\pi: \widetilde{M} \to M$ be a covering map, let B be a connected, locally path-connected, and simply connected map. Let $F: B \to M$ be a continuous map. Take $b \in B$, and $p \in \widetilde{M}$ such that $\pi(e) = F(b)$. Then, there is a lift $\widetilde{M}, \widetilde{F}: B \to \widetilde{M}$ of F such that $\widetilde{F}(b) = p$.

Proof. See Theorem 11.18 and Corollary 11.19 of [1].

1.6 Orientability

Definition 1.60. Let *V* be a finite-dimensional vector space. An *orientation of V* is an equivalence class of ordered bases for *V* such that two ordered bases are related if the determinant of the corresponding change-of-basis matrix is positive.

As a consequence, in every vector space there are two orientations.

Definition 1.61. Once an orientation is chosen, every basis is said to be *positively oriented* if it belongs to the chosen orientation, or *negatively oriented* if it does not. The orientation of the *standard basis* (e_1, \ldots, e_n) is referred to as the *standard orientation of* \mathbb{R}^n , where $e_i = (0, \ldots, 1, \ldots, 0)$, with the 1 occupying the *i*-th position.

Let *M* be a smooth manifold with or without boundary. With the concept of orientability of a vector space, one can define the orientability for *M*.

Definition 1.62. An *orientation for* M is a choice of orientation for each tangent space of M that is continuous. That means that for every point $p \in M$ there is a neighborhood in which there is a local frame that determines an orientation in the neighborhood of p.

Definition 1.63. Let *M* be *n*-dimensional. The manifold *M* is *orientable* if there exists an orientation for *M*. A smooth orientable manifold together with a choice of orientations forms an *oriented manifold*. A smooth coordinate chart $(U, (X^i))$ is an *oriented chart* if the coordinate frame $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$ is positively oriented for every point in *U*.

Proposition 1.64. (Orientation Determined by an *n*-Form) Let *M* be an *n*-manifold with or without boundary. If $\mu \in \Omega^n(M)$ is a nonvanishing *n*-form, then it determines a unique orientation of *M* by declaring a base (b_1, \ldots, b_n) for T_pM positively oriented if and only if $\mu_p(b_1, \ldots, b_n) > 0$. Conversely, if *M* is oriented, there is a smooth nonvanishing *n*-form that determines its orientation.

Proof. See Prop. 15.5 of [2].

This proposition inspires the following definitions.

Definition 1.65. A nonvanishing *n*-form on a smooth *n*-manifold is called an *ori*entation form. If *M* is an oriented smooth *n*-manifold and an orientation form μ determines its orientation, then μ is *positively oriented*.

Let *M*, and *N* be smooth *n*-manifolds with or without boundary, $F: M \rightarrow N$ a local diffeomorphism, and μ any positively oriented orientation form for *N*.

Definition 1.66. If *N* is oriented, $F^*\mu$ determines an orientation on *M* known as the *pullback orientation on M induced by F*, where μ is any positively oriented orientation form for *N*.

Definition 1.67. Let $\omega \in \Lambda^k(V^*)$ be an alternating *k*-vector, and let $v \in V$ be a vector. The operation named *interior multiplication by* v yields a (k-1)-tensor notated as $v \lrcorner \omega$ such that

$$(v \lrcorner \omega)(u_1, \dots, u_{k-1}) = \omega(v, w_1, \dots, w_{k-1}).$$
 (1.36)

Proposition 1.68. (Orientation of a Hypersurface) Let M be an oriented smooth nmanifold with or without boundary, and μ any positively oriented orientation form. Let $S \subseteq M$ be a smooth immersed hypersurface, $\iota: S \hookrightarrow M$ the inclusion, and $N: S \to TM$ a continuous vector field (this implies that N is a continuous map such that $N_p \in T_pM$, for every $p \in S$). If S is nowhere tangent to N, then S has a unique orientation determined by the (n - 1)-form $\iota^*(N \sqcup \mu)$.

An instance in which this proposition may be applied is when *M* has a boundary, since ∂M is a hypersurface.

Definition 1.69. Let *M* be a smooth manifold with boundary, and let *N* be a vector field along ∂M . *N* is said to be an *outward-pointing vector field* if for each $p \in \partial M$ there is a smooth curve $\gamma: (-\varepsilon, 0] \to M$ for which $\gamma(0) = p$, $\gamma'(0) = N_p$, and $N_p \notin T_p(\partial M)$.

Proposition 1.70. (Existence of a Global Smooth Outward-Pointing Vector Field) For every smooth manifold with boundary M there exists a global smooth outward-pointing vector field.

Proof. It suffices to take $-\partial/\partial x^n$ in boundary coordinates in a neighborhood of each $p \in \partial M$, and grouping them together with a partition of unity, whose definition is presented in page 376 of [3].

Theorem 1.71. (Stokes' Theorem of Manifolds with Corners) *Let* M *be an oriented smooth n-manifold with corners, and let* ω *be a compactly supported smooth* (n + 1)*-form on* M*. Then*

$$\int_{M} d\omega = \int_{\partial M} \omega. \tag{1.37}$$

Proof. See Theorem 16.11 of [2] first for the proof of the Stokes' Theorem of Manifolds without corners, and then see Theorem 16.25 of [2] which uses that previous result to prove the theorem for manifolds with corners. \Box

Since an outward-pointing vector field is nowhere tangent to ∂M , the next result follows from the last proposition.

Proposition 1.72. (Induced Orientation on a Boundary) Let M be an oriented smooth manifold with boundary, $\iota: \partial M \hookrightarrow M$ the inclusion map, N any outward-pointing vector field along ∂M , and μ a positively oriented orientation form for M. Then, ∂M is orientable, and there is a canonical orientation determined by $\iota^*(N \sqcup \mu)$. The canonical orientation is referred to as the induced orientation or Stokes orientation.

Proof. See [2] of Prop. 15.24,.

1.7 Curvature

For this section, notate the first, and second fundamental forms as I(X, Y), and II(X, Y), and the Weingarten map as $W_N(X)$, with $X, Y \in \mathfrak{X}$. It will be assumed that the reader is familiar with them, with the Gauss formula, and with the Gauss and Codazzi formula. However, these concepts are reviewed at pages 227-229 and 244 of [3].

Definition 1.73. Let (M, g) be a Riemannian manifold, and let *R* be a map defined as

$$R: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \quad (1.38)$$
$$(X, Y, Z) \quad \mapsto \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Proposition 1.74. The map R(X, Y)Z is multilinear over $C^{\infty}(M)$.

Proof. The linearity of R in X, and Y is shown on Prop. 7.3 of [3]. The linearity on Z is shown using the properties in Proposition 1.25 and in Definition 1.26.

$$R(X,Y)(fZ) = \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]} (fZ)$$

$$= \nabla_X (f \nabla_Y Z + (Yf) Z) - \nabla_Y (f \nabla_X Z - (Xf) Z) - f \nabla_{[X,Y]} Z - ([X,Y]f) Z$$

$$= f \nabla_X \nabla_Y Z + (Xf) \nabla_Y Z + (Yf) \nabla_X Z + X (Yf) Z$$

$$- f \nabla_Y \nabla_X Z - (Yf) \nabla_X Z - (Xf) \nabla_Y Z - Y (Xf) Z$$

$$- f \nabla_{[X,Y]} Z - ([X,Y]f) Z.$$

(1.39)

Finally, note that in the expression of the last identity, the first terms of each line correspond to fR(X, Y)Z. Also, the last term of lines 1, and 2 correspond to ([X, Y]f)Z, and then this and the rest of the terms cancel out. In conclusion, $R(X, Y)(fZ) = f\nabla_{[X,Y]}Z$, and *R* defines a (1,3)-tensor field on *M*.

Definition 1.75. The (*Riemann*) curvature tensor is a (0, 4)-tensor field defined as

$$Rm(X, Y, X, W) = \langle R(X, Y)Z, W \rangle_g.$$
(1.40)

Throughout the remaining part of this section, let (M, g) be an embedded n-dimensional Riemannian submanifold of an (n + 1)-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g}); (M, g) \hookrightarrow (\widetilde{M}, \widetilde{g})$. For every $p \in M$, local coordinartes x_1, \ldots, x_{n+1} may be chosen in a local neighborhood of p, U, such that locally $U \cap M = \{\widetilde{M}: x_{n+1} = 0\} \hookrightarrow U \subseteq \widetilde{M}$. This shows that there is a smooth unit normal vector field along $M, \partial/\partial x_{n+1}$ on a sufficiently small neighborhood of $p \in M$, and so it may be used when performing local calculations.

Definition 1.76. The *scalar second fundamental form of* M is the symmetric covariant 2-tensor field $h \in \Gamma(\Sigma^2 T^*M)$ given by $h = II_N$, so that

$$h(X,Y) = \langle N, II(X,Y) \rangle, \quad X,Y \in T^*M.$$
(1.41)

Equivalently, by virtue of the Gauss formula, $\widetilde{\nabla}_X Y = \nabla_X Y + II(X, Y)$; and the fact that *N* and $\nabla_X Y$ are orthogonal, an alternative definition is

$$II(X,Y) = h(X,Y)N.$$
(1.42)

Definition 1.77. Let N be a unit normal field. Then, the *shape operator of* M, s, is defined as

$$s = W_N \colon \mathfrak{X}(M) \to \mathfrak{X}(M),$$
 (1.43)

where W_N is the Weingarten map determined by N.

The relation between *h* and *s* is given by

$$\langle sX,Y\rangle = h(X,Y), \quad \forall X,Y \in \mathfrak{X}(M).$$
 (1.44)

Definition 1.78. The *Gaussian curvature* is given by K = det(s),

since $W_N : \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfies

$$\langle W_N(X), Y \rangle = II_N(X, Y) = \langle N, II(X, Y) \rangle.$$
 (1.45)

Since the ultimate goal of this document is to prove the Gauss-Bonnet theorem for a 2-dimensional Riemannian manifold, it will be advantageous to find a more comfortable expression for the Gaussian curvature locally, one that uses coordinates.

Proposition 1.79. Let (M, g) be a 2-dimensional Riemannian manifold, $p \in M$ arbitrary, and (a_1, a_2) an orthonormal basis for T_pM . Then

$$K(p) = Rm_p(a_1, a_2, a_2, a_1).$$
(1.46)

Proof. It suffices to see that $K(p) = \det(s_j^i) = \det(h_{ij}) = h_{11}^2 + h_{22}^2 - h_{12}h_{21}$ using the Gauss and Codazzi equations (Eq. 8.21 of [3]).

Chapter 2

Historical Development of the Gauss Bonnet Theorem

This chapter has primarily relied on sources such as [9] and [10], which provide an overview of the evolution of the Gauss-Bonnet theorem. They discuss the theorem's evolution in terms of its generality and also explore the various geometric insights offered by different proof methods.

The first proof of the Gauss-Bonnet Theorem was published on a a paper called *Disquisitiones Generales Circa Superficies Curvas* (1827) by Carl Friedrich Gauss (1777 – 1855) for the case of a geodesic triangle. A generalized version would later be published on 1848 by Pierre Ossian Bonnet (1819 – 1892), which was formulated as it follows:

Theorem 2.1. Let $M \subset \mathbb{R}^3$ be a surface whose domain \mathcal{D} is simply connected and with a boundary $\partial \mathcal{D}$, which is composed of finite number of smooth curves. If g is the Euclidean metric, let k_g be the geodesic curvature of the boundary. The interior angles at a vertex of the boundary are notated as α_j , K is the Gaussian curvature of M, and dA is the area element of M. Then

$$\int_{\partial \mathcal{D}} k_g ds + \sum_j \left(\pi - \alpha_j \right) + \int_{\mathcal{D}} K dA = 2\pi.$$
(2.1)

According to Bonnet's own memoir, *Mémoire sur la théorie générale des surfaces*, the mathematician Jacques Binet (1786–1856) similarly proved Gauss's Theorem in a note appended to a Memoir by Mr. Olinde Rodrigues that appeared in Volume III of the Correspondence of the École Polytechnique. However, Binet never officially published the theorem in question, which might explain why the theorem was eventually named the *Gauss-Bonnet* theorem.

What is modernly referred to as the *Gauss-Bonnet Theorem for compact orientable surfaces* is a version from 1888 attributed to Walther von Dyck (1856 – 1934):

Theorem 2.2. Let *M* be a compact orientable surface in \mathbb{R}^3 , and let its Euler characteristic (Definition 3.32) be $\chi(M)$. Then

$$\frac{1}{2\pi} \int_M K dA = \chi(M). \tag{2.2}$$

Notice that up until now, the term *surface* referred to the image of a map $F: U \to \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$ is open. In this work however, the Gauss-Bonnet theorem is proved on 2-dimensional manifolds, introducing a geometry that goes beyond Euclidean spaces. The history of this abstraction is parallel to the development of the theorem, when Bernhard Riemann (1826 – 1866) submitted his Ph.D. dissertation. His supervisor was Gauss, and the contents of his thesis were the introduction of what are now called Riemannian manifolds. From then onward, the theorem could be formulated in terms of smooth Riemannian 2-manifolds.

According to [11], Heinz Hopf (1894 – 1971) proposed in the late 1920s the generalization of the Gauss-Bonnet theorem to all manifolds with even dimension, and in 1925, he published the proof for the particular case of an embedded Riemannian hypersurface in an Euclidean space.

This problem was first solved in general by Allendoerfer and Weil in 1943. Nonetheless, the paper by Allendoerfer and Weil is rather complex. In contrast, Shiing-Shen Chern (1911 – 2004) offered a more straightforward, 6 pages long proof in 1944, which contributed to the popularization of the Gauss-Bonnet theorem. This result is known as the *Chern-Gauss-Bonnet T-heorem*.

Chapter 3

The Gauss-Bonnet Theorem

As previously mentioned, the Theorem of Gauss-Bonnet exists in multiple versions, and its generality varies based on the space in which it is formulated. This section begins by introducing essential notation and subsequently establishes the Rotation Index Theorem. Armed with these fundamental tools, combined with the concepts developed in Chapter 1, one embarks on proving the theorem specifically for smooth Riemannian 2-manifolds.

The reader who wants to explore more proofs can go to:

- Chapter 13 of [5] proves the Gauss-Bonnet theorem for compact parametrized surfaces in R³, and give some notions about the integration over such surfaces. It uses the same combinatorial arguments over the triangulations of a surface as the ones used in this document.
- 2. Chapter 9 of [3] proves the theorem the same as done way here: for the case of a 2-dimensional Riemannian surface orientable or non-orientable.
- 3. Chapter 6 of [4] offers a different method for the proof. The result is shown for a regular surface $S \subset \mathbb{R}^3$ endowed with an arbitrary Riemannian metric. First, it is shown that the integral $\int_S dA$ is independent of the Riemannian metric. Then, a convenient Riemannian metric is manufactured for the computation of the integral, which allows to analytically compute the result of the Gauss-Bonnet theorem.
- 4. Chapter 8 of [6] also proves the theorem with combinatorial arguments on a compact, regular surface of class C^3 , $S \subset \mathbb{R}^3$.

As stated in Corollary 5.4.5 of [5], every compact regular surface *S* in \mathbb{R}^3 is orientable, thus in that case the books just assume *S* to be oriented.

3.1 The Rotation Index Theorem

Let $\gamma : [a, b] \to \mathbb{R}^2$ be an admissible curve of the plane.

Definition 3.1. A curve γ is a *simple closed curve* when $\gamma(a) = \gamma(b)$, but it is injective on [a, b].

Definition 3.2. The *unit tangent vector field of* γ is a vector field defined along each smooth segment of γ . Its expression is

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} \tag{3.1}$$

Definition 3.3. A *tangent angle function for* γ is a continuous function θ : $[a, b] \rightarrow \mathbb{R}$ such that $T(t) = (\cos \theta(t), \sin \theta(t))$, for every $t \in [a, b]$.



Figure 3.1: Representation of the tangent angle function at certain points of γ .

Proposition 3.4. A tangent angle function θ : $[a, b] \to \mathbb{R}$ exists for γ : $[a, b] \to \mathbb{R}^2$.

Proof. Define the map $q: \mathbb{R} \to S^1$ as $q(s) = (\cos s, \sin s)$. Notice q is a smooth covering map. By the *Path Lifting Property* (Proposition 1.58 b)), there exists a lift $\theta: [a, b] \to \mathbb{R}$ of T(t):

$$q \circ \theta(t) = (\cos \theta(t), \sin \theta(t)) = T(t).$$
(3.2)

Furthermore, lifts are uniquely determined by the image of a single point, due to the *Unique Lifting Property* (Proposition 1.58 a)). Hence, any two of these lifts differ by a multiple of 2π .



Figure 3.2: Representation of the covering map *q* and of the composition $q \circ \theta$.

Definition 3.5. Let $\gamma: [a, b] \to \mathbb{R}^2$ be a continuously differentiable simple closed curve such that $\gamma'(a) = \gamma'(b)$, and let θ be any tangent angle function for γ . The *rotation index of* γ is

$$\rho(\gamma) = \frac{1}{2\pi} \left(\theta(b) - \theta(a)\right) \tag{3.3}$$

Proposition 3.6. $\rho(\gamma)$ is a multiple of 2π and it is independent of the choice of the tangent angle function.

Proof. Since $\gamma'(a) = \gamma'(b)$, the result arises from the identity

$$T(a) = (\cos \theta(a), \sin \theta(a)) = (\cos \theta(b), \sin \theta(b)) = T(b).$$
(3.4)

 $\rho(\gamma)$ is independent of the choice of the tangent angle function because the difference between two different angle functions is a multiple of 2π .

This definition is quite straightforward. However, it raises a question. What would happen if the curve was just piecewise regular instead of continuously differentiable? The next step is to define a certain type of piecewise regular curve that admits a tangent angle function.

Consider an admissible simple closed curve $\gamma : [a, b] \to \mathbb{R}^2$, and an admissible partition (a_0, \ldots, a_k) of [a, b].

Definition 3.7. The points $\gamma(a_i)$ are called the *vertices of* γ . The curve segments $\gamma|_{[a_{i-1},a_i]}$ are called the *edges* or *sides of* γ .

Definition 3.8. For a vertex $\gamma(a_i)$, its left- and right-hand velocity vectors are indicated by $\gamma'(a_i^-)$ and $\gamma'(a_i^+)$, respectively. Then, $T(a_i^-)$ and $T(a_i^+)$ are the *left-hand* and *right-hand unit vectors*, obtained by using the corresponding velocities.

Definition 3.9. Vertices are classified into three categories, for each one an associated *exterior angle* is defined. Let $(a = a_0, ..., a_k = b)$ be a partition of γ . A vertex $\gamma(a_i)$, 1 < i < k is

- (a) an *ordinary vertex* if T(a_i⁻) ≠ ±T(a_i⁺). Its exterior angle is defined as the oriented measure of the angle from T(a_i⁻) to T(a_i⁺) in the interval (−π, +π). It is defined with a positive sign if (T(a_i⁻), T(a_i⁺)) is a positively oriented basis for ℝ², and negative otherwise,
- (b) a *flat vertex* if $T(a_i^-) = T(a_i^+)$. Its exterior angle is zero.
- (c) a *cusp vertex* if $T(a_i^-) = -T(a_i^+)$. Its exterior angle is undefined, since it could be ambiguously π or $-\pi$.



Figure 3.3: Representation of a curve γ with partition $(a_0, \ldots, a_{i-1}, a_i, a_{i+1}, a_k)$, $k \in \mathbb{N}$. $\gamma(t)$ has a regular, a cusp, and a flat vertex at $t = a_{i-1}, a_i$, and a_{i+1} , respectively.

Definition 3.10. Let ε_i be the exterior angle of $\gamma(a_i)$. Then, the *interior angle at* $\gamma(a_i)$ is defined as $\theta_i = \pi - \varepsilon_i$.



Figure 3.4: Curve with an ordinary vertex $\gamma(a_i)$. The exterior angle ε_i and interior angle θ_i are indicated.

If i = 0 or i = k, then the vertex $\gamma(a) = \gamma(b)$ is being considered, and the definitions above need to be modified. In that case the vectors T(b) and T(a) must be used in the place of $T(a_i^-)$ and $T(a_i^+)$, respectively.

Definition 3.11. A *curved polygon* in the plane is an admissible simple closed curve without cusp vertices such that its image is the boundary of a precompact open set $\Omega \subseteq \mathbb{R}^2$, which is named the *interior of* γ .

Definition 3.12. A curved polygon with exactly three edges and three vertices in *M* is called a *curved triangle*.

Now, let $\gamma: [a, b] \to \mathbb{R}^2$ be a curved polygon with an admissible partition $(a_0, \ldots, a_k), k \in \mathbb{N}$.

Definition 3.13. γ is said to be *positively oriented* if it is parametrized so that at its smooth points γ' is positively oriented with respect to the induced or Stokes orientation on $\partial\Omega$, as defined at Proposition 1.72. Intuitively, this means that γ is parametrized in the counterclockwise direction, and therefore Ω is always to the left.

Definition 3.14. Let ε_i be the exterior angle at $\gamma(a_i)$, and ε_k at $\gamma(b)$. A *tangent angle function for a curved polygon* γ is a piecewise function θ : $[a, b] \rightarrow \mathbb{R}$ such that

- (a) $T(t) = (\cos \theta(t), \sin \theta(t))$ at each point where γ is smooth,
- (b) it is continuous from the right: $\theta(a_i) = \lim_{t \searrow a_i} \theta(t)$,
- (c) it satisfies $\theta(a_i) = \lim_{t \nearrow a_i} \theta(t) + \varepsilon_i$ for i > 0.

Proposition 3.15. (Existence of the Tangent Angle Function for a Curved Polygon) *There is a tangent angle function* θ : $[a, b] \rightarrow \mathbb{R}$ *for each curved polygon* γ : $[a, b] \rightarrow \mathbb{R}$ *.*

Proof. The first step is to define θ for $t \in [a_0, a_1]$ as any lift of T on this interval. Secondly, for i such that $1 \leq i < k$, θ is defined on $[a_i, a_{i+1}]$ as the unique lift that satisfies the property b) of Definition 3.14. For the uniqueness of the lift, Proposition 1.58 a) was used. Additionally, given any two tangent angle function, they differ by a constant multiple of 2π , which motivates the following definition.

Definition 3.16. The *rotation index of* γ , a curved polygon, is defined as

$$\rho(\gamma) = \frac{1}{2\pi} (\theta(b) - \theta(a)). \tag{3.5}$$

Definition 3.17. The *secant angle function of* γ , $\phi(t_1, t_2)$, measures the angle between the positive *x*-direction and the vector going from $\gamma(t_1)$ to $\gamma(t_2)$.

$$\rho(\gamma) = \frac{1}{2\pi} (\theta(b) - \theta(a)). \tag{3.6}$$

Theorem 3.18. (Rotation Index Theorem or Umlaufsatz, Heinz Hopf, 1935) The rotation index of a positively oriented curved polygon in the plane is +1.

Proof. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a positively oriented curved polygon. For simplicity, first it is assumed that all the vertices of γ are flat. Hence, γ' is continuous and $\gamma'(a) = \gamma'(b)$. Then, since additionally $\gamma(a) = \gamma(b)$, then the curved polygon γ can be extended to a continuous map from $\mathbb{R} \to \mathbb{R}^2$ just by making it periodic of period b - a. Consequently, the periodic γ has a continuous first derivative.

From now on, assume γ is periodic. If $T(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$, then let $\theta \colon \mathbb{R} \to \mathbb{R}$ be a lift of $T \colon \mathbb{R} \to \mathbb{S}^1$, as it was argued in Definition 3.3 using the mapping function q.

The restriction of θ to [a, b], $\theta |_{[a,b]}$, is a tangent angle function for γ that satisfies

$$\theta(b) = \theta(a) + 2\pi\rho(\gamma). \tag{3.7}$$

With that, define $\tilde{\theta}$ as $\tilde{\theta}(t) = \theta(t + b - a) - 2\pi\rho(\gamma)$, which is another lift of *T*:

$$(\cos\theta(t),\sin\theta(t)) = (\cos\theta(t+b-a),\sin\theta(t+b-a)) = T(t+b-a) = T(t), (3.8)$$

where the first equality is true because, as it had been argued, $\rho(\gamma)$ is an integer number, the second follows from the fact that θ is the lift of *T*, and the last is consequence of the periodicity of γ .

Observe that both $\hat{\theta}$ and $\hat{\theta}$ are lifts of *T* that coincide at one point ($\hat{\theta}(a) = \theta(a)$). The symbol \equiv will be used to denote that two maps are equal all over their domain. Then, by Proposition 1.58, $\tilde{\theta} \equiv \theta$, which entails that

$$\theta(t+b-a) \equiv \theta(t) = \theta(t) + 2\pi\rho(\gamma). \tag{3.9}$$

Now, let a_1 be any number $a_1 \in [a, b]$, and define $b_1 = a_1 + b - a$. Then the restriction $\gamma |_{[a_1,b_1]}$ is also a positively oriented curved polygon whose vertices are all flat, and whose tangent angle function is $\theta |_{[a_1,b_1]}$. Then, $\gamma |_{[a_1,b_1]}$ and $\gamma |_{[a,b]}$.

The rotation index of this curved polygon index $\gamma |_{[a_1,b_1]}$ coincides with that of γ :

$$\theta(b_1) - \theta(a_1) = \theta(a_1 + b - a) - \theta(a_1) = (\theta(a_1) + 2\pi\rho(\gamma)) - \theta(a_1) = 2\pi\rho(\gamma),$$
(3.10)

where it has been used that $\tilde{\theta} \equiv \theta$.

This result implies that the interval [a, b] can shifted arbitrarily. Thus, assume [a, b] ia an interval such that the *y*-coordinate of γ is minimum at the point *a*. Furthermore, γ' is independent of translations in the *xy*-plane so it can also be



Figure 3.5: On the left, an arbitrary curved polygon. On the right, the curved polygon with a shift in its domain and with translation described.

assumed that $\gamma(a)$ is the origin. Consequently, the image of γ resides in the closed upper half-plane, and T(a) = T(b) = (1,0). Also, θ will be chosen such that $\theta(a) = 0$ since different tangent angle functions differ by multiples of 2π .

Now, define t $\Delta \subseteq \mathbb{R}^2$ as the region $\Delta = (t_1, t_2)$: $a \leq t_1 \leq t_2 \leq b$. Let $V \colon \Delta \to \mathbb{S}^1$ be the map

$$V(t_1, t_2) = \begin{cases} \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|}, & t_1 < t_2 \text{ and } (t_1, t_2) \neq (a, b); \\ T(t_1), & t_1 = t_2; \\ -T(b), & (t_1, t_2) = (a, b). \end{cases}$$
(3.11)



Figure 3.6: Region $\Delta \subset \mathbb{R}^2$.

The next step is to show that *V* is a continuous function. This is readily seen for (t_1, t_2) such that $t_1 < t_2$ and $(t_1, t_2) \neq (a, b)$ because γ is continuous and injective. Next, the continuity at $(t, t) \in \Delta$ must be shown: let $(t_1, t_2) \in \Delta$ be such that $t_1 < t_2$ and apply the fundamental theorem of calculus:

$$\gamma(t_2) - \gamma(t_2) = \int_0^1 \frac{d}{ds} \gamma(t_1 + s(t_2 - t_1)) ds = \int_0^1 \gamma'(t_1 + s(t_2 - t_1))(t_2 - t_1) ds.$$
(3.12)

An inequality may be obtained from this equation, which in turn will be useful for computing the limits that verify the continuity of $V(t_1, t_2)$.

$$\frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1} - \gamma'(t) = \frac{1}{t_2 - t_1} \int_0^1 \gamma'(t_1 + s(t_2 - t_1))(t_2 - t_1)ds - \gamma'(t)$$

=
$$\int_0^1 \left[\gamma'(t_1 + s(t_2 - t_1)) - \gamma'(t)\right] ds.$$
 (3.13)

This implies that $\left|\frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1} - \gamma'(t)\right| \leq \int_0^1 |\gamma'(t_1 + s(t_2 - t_1)) - \gamma'(t)| ds$. Considering that [a, b] is a compact set in which γ' is continuous, by the Heine-

Considering that [a, b] is a compact set in which γ is continuous, by the Heine-Cantor theorem γ' is uniformly continuous. Then, the left-hand side of the previous equation can be made arbitrarily small by choosing (t_1, t_2) close to (t, t), resulting in

$$\lim_{\substack{(t_1,t_2)\to(t,t)\\t_1\le t_2}}\frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}=\gamma'(t),$$
(3.14)

At last, it is shown that the limit of $V(t_1, t_2)$, $t_1 < t_2$, coincides with V(t, t), $(t, t) \in \Delta$.

$$\lim_{\substack{(t_1,t_2)\to(t,t)\\t_1

$$= \frac{\gamma'(t)}{|\gamma'(t)|} = T(t) = V(t,t).$$
(3.15)$$

Similarly, the limit of $V(t_1, t_2)$, $t_1 = t_2$, coincides with V(t, t), $(t, t) \in \Delta$.

$$\lim_{\substack{(t_1,t_2)\to(t,t)\\t_1=t_2}} V(t_1,t_2) = \lim_{t_1\to t} T(t_1) = T(t) = V(t,t),$$
(3.16)

because T(t) is a continuous function.

With these two results, it is concluded that *V* is continuous at (t, t).

Lastly, the continuity at (a, b) is yet to be proved. For that, recall that γ has been extended by making it periodic with period b - a. Then

$$\lim_{\substack{(t_1,t_2)\to(a,b)\\t_1s_2}} \frac{\gamma(s_2)-\gamma(s_1)}{|\gamma(s_2)-\gamma(s_1)|} = -T(b) = V(a,b).$$
(3.17)

Finally, the continuity of *V* has been shown.

On the other hand, Theorem 1.59 implies that $V : \Delta \to S^1$ has a continuous lift $\varphi : \Delta \to \mathbb{R}$, which is uniquely determined if one sets $\varphi(a, a) = 0$. The theorem can be applied because Δ is connected, locally path-connected, and simply connected map.

Additionally, the secant angle function φ defined at Definition 3.17 is related to the rotation index by

$$\rho(\gamma) = \frac{1}{2\pi} \left(\theta(b) - \theta(a) \right) = \frac{1}{2\pi} \left(\varphi(b, b) - \varphi(a, a) \right) = \frac{1}{2\pi} \varphi(b, b).$$
(3.18)

To see the second identity, define the map $\overline{\varphi}$: $[a, b] \to \mathbb{S}^1$ as $\overline{\varphi}(t) = \varphi(t, t)$. Notice that since $\varphi(s_1, s_2)$, with $(s_1, s_2) \in \Delta$, is a lift of $V(s_1, s_2)$, then $\overline{\varphi}(t)$ is a lift of V(t, t) = T(t), as well as $\theta(t)$. Hence, $\overline{\varphi}(t)$ and $\theta(t)$ differ only by a multiple of 2π , and the equality follows.

The only thing left to do is to compute the value of $\varphi(b, b)$. In order to do that, observe the points $(t_1, t_2) \in \Delta$ on the side of Δ for which $t_1 = a$, and $t_2 \in [a, b]$. On such a side, $V(a, t_2)$ is a vector whose tail is at the origin, and whose tail remains above the upper half-plane. Since $\varphi(a, a) = 0$ has been set, then it is necessary that $\varphi(a, t_2) \in [0, \pi]$. Additionally, seeing that $\varphi(a, b)$ represents the tangent angle of -T(b) = (-1, 0) shows that $\varphi(a, b) = \pi$ by continuity.

Analogously, on the side of Δ on which $t_2 = b$, and $t_1 \in [a, b]$ the vector $V(t_1, b)$ has its head at the origin and its tail in the upper half-plain, consequently $\varphi(t_1, b) \in [\pi, 2\pi]$ (taking into account that $\varphi(a, b) = \pi$). Since $\varphi(b, b)$ represents the tangent angle of T(b) = (1, 0), it can be concluded that $\varphi(b, b) = 2\pi$. As a result, $\rho(\gamma) = 1$.

The theorem has been proven now for a positively oriented curved polygon γ with only flat vertices. That first version of the theorem will be leveraged to show the result for a γ with ordinary vertices.



Figure 3.7: Visualization of V(a, t) and V(t, b), for $t \in [a, b]$.

Then let $\gamma(a_i)$ be an ordinary vertex of γ . Its exterior angle is notated as ε_i . Let α be a positive number smaller than $\frac{1}{2}(\pi - |\varepsilon_i|)$. By definition, θ is continuous from the right at a_i and $\lim_{t \nearrow a_i} \theta(t) = \theta(a_i) - \varepsilon_i$ (Definition 3.14). As a consequence, there is a δ small enough such that if $t \in (a_i, a_i + \delta)$ then $|\theta(t) - \theta(a_i)| < \alpha$, and if $t \in (a_i - \delta, a_1)$ then $|\theta(t) - (\theta(a_i) - \varepsilon_i)| < \alpha$.

The image of the compact set $[a, b] \setminus (a_i - \delta, a_i + \delta)$ is a compact set, because γ is a continuous map. Moreover, this image does not contain the point $\gamma(a_i)$, and so there exists an r small enough such that γ may only enter $\overline{B}_r(\gamma(a_i))$ when $t \in (a_i - \delta, a_i + \delta)$. The point for which $\gamma(t)$ enters $\overline{B}_r(\gamma(a_i))$ is noted $t_1 \in (a_i - \delta, a_i)$, whereas $t_2 \in (a_i, a_i + \delta)$ is the point for which $\gamma(t)$ leaves this ball (see Fig. 3.8). Because of how δ was chosen, the variance of $\theta(t)$ is smaller than α both in $t \in (t_1, a_i)$, and $t_2 \in (a_i, t_2)$. In the end, the total change of $\theta(t)$ when going through the interval [a, b], $\Delta\theta$, is bounded as $\varepsilon_i - 2\alpha < \Delta\theta < \varepsilon_i + 2\alpha$, which in turn means that $-\pi < \Delta\theta < \pi$, considering the definition of α .



Figure 3.8: On the left, a $\overline{B}_r(\gamma(a_i))$ where its *r* is too big. On the right, *r* is sufficiently small, and it fixes t_1 and t_2 .

As a final step, $\gamma|_{[t_1,t_2]}$ is replaced by a smooth curve segment δ whose velocity coincides with that of γ at $\gamma(t_1)$ and $\gamma(t_2)$, and whose tangent angle either increases or decreases monotonically from $\theta(t_1)$ to $\theta(t_2)$, just like an arc of a conic would do (Fig. 3.9). Since the change in tangent angle of δ is the angle between $T(t_1)$ and $T(t_2)$, comprised between $-\pi$ and π_0 , it must be exactly $\Delta\theta$.

In order to find this conic in an explicit way, it is convenient to transform the



Figure 3.9: Curve segment used to smooth the vertex $\gamma(a_i)$.

problem into a projective one. For that, notate $A = \gamma(t_1)$ and $B = \gamma(t_2)$, and let $r = \gamma(t_1) + \langle \gamma'(t_1) \rangle$ and $s = \gamma(t_1) + \langle \gamma'(t_1) \rangle$ be lines. The conic that is mentioned must have *r* and *s* as tangent lines at *A* and *B*, respectively. Additionally, let $C \in \mathbb{R}^2$ be a point that is not aligned with *A* and *B*. Then, homogeneous coordinates may be established: (0:0:1) is the point $r \cap s$, (1:0:0) is the point *A*, (0:1:0) is the point *B*. Then (a:b:1) is the point *C*, $a, b \in \mathbb{R}$, $a \neq 0$, $b \neq 0$.

The conic may be seen as conic section in a 3-dimensional projective space. Then, homogeneous coordinates (x : y : z : w) are used. Then, A = (1 : 0 : 0 : 0), B = (0 : 1 : 0 : 0), and C = (a : b : 1 : 1), and these points determine the plane

$$\alpha x - \beta y + \gamma z - \delta w = 0, \text{ where}$$

$$\alpha = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ b & 1 & 1 \end{vmatrix} = 0, \ \beta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ a & 1 & 1 \end{vmatrix} = 0, \ \gamma = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{vmatrix} = 1, \ \delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{vmatrix} = 1.$$

$$(3.19)$$

The equation of the cone is $x^2 + y^2 + z^2 = (x + y)^2$, or $xy = z^2$ if one cancels terms and applies the transformation $z \mapsto \sqrt{2}z$.

The final equation, z = w, yields the quadratic equation $xy = w^2$, using the equation of the cone. This is a homogeneous quadric equation in (x : y : w) determining a conic in the plane. Then, one could undo the projective transformation and obtain the desired conic in the original space.

3.2 The Gauss-Bonnet Formula

In this case, let (M, g) be an oriented 2-manifold.

Definition 3.19. A *curved polygon in* M is an admissible simple closed curve $\gamma \colon [a, b] \to M$ such that

- 1. it is the boundary of a precompact open set $\Omega \subseteq M$,
- 2. there is an oriented smooth coordinate disk (U, φ) such that $\overline{\Omega} \subseteq U$ and such $\varphi \circ \gamma$ is a curved polygon on the plane \mathbb{R}^2 . A coordinate disk is a coordinate chart (U, φ) such that $\varphi(U) = D$, where $D \subseteq \mathbb{R}^2$ is an open disk.

Definition 3.20. The *interior of* γ in this context corresponds to Ω .

The goal will be to follow an schema similar to the one used for the case of a plane curve, and as a result, the definitions used previously are now adapted.

Definition 3.21. Since (M, g) is an oriented manifold, by Proposition 1.72 ∂M has an induced orientation called the Stokes orientation. γ is said to be *positively oriented* if it is parametrized in the direction of the Stokes orientation.

Definition 3.22. The *unit tangent vector field* is defined on every smooth segment of γ as

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|_g}$$
(3.20)

Definition 3.23. The *exterior angle of* γ *at* $\gamma(a_i)$, ε_i , is the oriented measure of the angle $T(a_i^-)$ to $T(a_i^+)$ with respect to the *g*, the inner product of *M*, and the given orientation of *M*. It can be expressed as

$$\varepsilon_{i} = \frac{dV_{g}\left(T\left(a_{i}^{-}\right), T\left(a_{i}^{+}\right)\right)}{\left|dV_{g}\left(T\left(a_{i}^{-}\right), T\left(a_{i}^{+}\right)\right)\right|} \arccos\left\langle T\left(a_{i}^{-}\right), T\left(a_{i}^{+}\right)\right\rangle_{g}.$$
(3.21)

Definition 3.24. The *interior angle of* γ *at* $\gamma(a_i)$ is

$$\theta_i = \pi - \varepsilon_i. \tag{3.22}$$

Interior and exterior angles at $\gamma(a) = \gamma(b)$ are defined the same way they were at Definition 3.9.

With these definitions at hand, now the rotation index formulation can be adapted to the case of an oriented Riemannian 2-manifold. For that, let $\gamma : [a, b] \rightarrow M$ be a curved polygon whose interior is Ω , and let (U, φ) be an oriented smooth chart such that $\overline{\Omega} \subseteq U$. If γ , Ω , and g are transferred to the plane, then g may be assumed to be a metric on some open set $\widehat{U} \subseteq \mathbb{R}^2$, and γ is a curved polygon. The oriented orthonormal frame for g, (E_1, E_2) will be that obtained by applying the Gram-Schmidt algorithm to (∂_x, ∂_y) , which implies that E_1 is a positive scalar multiple of ∂_x everywhere in \widehat{U} . **Definition 3.25.** The *tangent angle function for* γ is a piecewise continuous function θ : $[a, b] \rightarrow \mathbb{R}$ such that it satisfies

$$T(t) = \cos(\theta(t)) E_1|_{\gamma(t)} + \sin(\theta(t)) E_2|_{\gamma(t)}, \qquad (3.23)$$

at each *t* in which $\gamma'(t)$ is continuous, and it is also continuous from the right and it satisfies $\theta(a_i) = \lim_{t \nearrow a_i} \theta(t) + \varepsilon_i$ and $\theta(b) = \lim_{t \nearrow a_i} \theta(t) + \varepsilon_k$. The existence of such a function, similarly to the planar case, arises from the fact that

$$T(t) = u_1(t)E_1|_{\gamma(t)} + u_2(t)E_2|_{\gamma(t)}, \qquad (3.24)$$

where $u_1, u_2: [a, b] \to \mathbb{R}$ are piecewise continuous functions that can be regarded as the coordinate functions of a map $(u_1, u_2): [a, b] \to \mathbb{S}^1$ since $|T(t)|_g = 1$, for every $t \in [a, b]$.

Definition 3.26. The *rotation index of* γ corresponds to

$$\rho(\gamma) = \frac{1}{2\pi} \left(\theta(b) - \theta(a) \right). \tag{3.25}$$

Note that the coordinate-independence of the rotation index is not evident since now the frame used is the varying (E_1, E_2) .

Lemma 3.27. Let M be an oriented Riemannian 2-manifold. The rotation index of every positively curved polygon in M is +1.

Proof. As described above, the definition of a curved polygon in M implies that there is an oriented coordinate chart that verifies the condition 2. of Definition 3.19. Such a chart can be used to regard γ as a curved polygon in the plane. Observe that $\theta(a)$ and $\theta(b)$ represent an angle between the same two vectors, and then $\rho(\gamma)$ must be an integer. However, notice that θ may be computed with different metrics such as the Euclidean, \overline{g} , or with g, hence $\rho(\gamma)$ could vary with the metric. Now, let g_s be an inner product defined as $g_s = sg + (1 - s)\overline{g}$, for s such that $0 \le s \le 1$. Then as it has been noted, the rotation index with respect to g_s , $\rho_{g_s}(\gamma)$, is an integer for each s, and therefore $f(s) = \rho_{g_s}(\gamma)$ is integer-valued.

Additionally, these facts show that f is a continuous function:

- 1. the g_s -orthonormal frame $(E_1^{(s)}, E_2^{(s)})$ described above depends continuously on s since they are computed via the Gram-Schmidt algorithm whose formulas are continuous on s,
- 2. on every interval $[a_{i-1}, a_i]$ in which γ is smooth, the functions u_1 and u_2 corresponding to the ones introduced at Eq. (3.24), but for g_s , can be expressed as

$$u_j(t,s) = \left\langle T_s(t), E_j^{(s)} \Big|_{\gamma(t)} \right\rangle_{g_s}, \qquad (3.26)$$

where $T_s = \frac{\gamma'(t)}{|\gamma'(t)|_{g_s}}$. As a result, u_1 and u_2 depend continuously on $(t,s) \in [a_{i-1}, a_i] \times [0, 1]$, and therefore the function $(u_1, u_2) \colon [a_{i-1}, a_i] \times [0, 1] \to \mathbb{S}^1$ has a continuous lift $\theta \colon [a_{i-1}, a_i] \times [0, 1] \to \mathbb{R}$, which is unique once the value at one point is determined (Proposition 1.58).

3. at every vertex, it follows from Eq. (3.21) that the exterior angle depends continuously on *s*. This is true because $\theta(b)$ is defined as $\theta(b) = \lim_{t \nearrow a_i} \theta(t) + \varepsilon_k$ (Definition 3.25).

Then, as it had been declared, f is continuous and integer valued, and so it follows that

$$\rho_{g}(\gamma) = f(1) = f(0) = \rho_{\overline{g}}(\gamma) = 1,$$
(3.27)

where the last identity is given by the Theorem 3.18.

From now on, for simplicity, the curved polygon γ will be given a unit-speed parametrization ($|\gamma'(t)| = 1$), and so $T(t) = \gamma'(t)$. Moreover, a normal vector field along the smooth parts of γ is uniquely determined if one imposes that $(\gamma'(t), N(t))$ is an oriented orthonormal basis for $T_{\gamma(t)}M$. This is equivalent to say that N points inward over $\partial\Omega$ if γ is positively oriented.

Definition 3.28. The *signed curvature of* γ at its smooth points of is given by

$$\kappa_N(t) = \left\langle D_t \gamma'(t), N(t) \right\rangle_g.$$
(3.28)

Notice that if the expression $|\gamma'(t)|_g^2 = \langle \gamma'(t), \gamma'(t) \rangle_g \equiv 1$ is differentiated, it yields $\langle D_t \gamma'(t), \gamma'(t) \rangle_g = 0$, thus $D_t \gamma'(t) = \kappa_N(t)N(t)$. $\kappa_N(t)$ is positive if γ is curving toward Ω , and negative if it is curving away.

Theorem 3.29. (The Gauss-Bonnet Formula) Let (M, g) be an oriented Riemannian 2-manifold, let $\gamma: [a, b] \rightarrow M$ be a positively oriented curved polygon in M with an admissible partition (a_1, \ldots, a_k) , and Ω its interior. It follows that

$$\int_{\Omega} K \, dA + \int_{\gamma} \kappa_N \, ds + \sum_{i=1}^k \varepsilon_i = 2\pi, \qquad (3.29)$$

with K being the Gaussian curvature of g, dA the Riemannian volume form, ε_i the exterior angle of γ at $\gamma(a_i)$, for i such that $1 \le i \le k$, and the second integral is taken with respect to arc length.

Proof. Let (x, y) be oriented smooth coordinates on an open set U containing Ω . As it has been already argued, the existence of such coordinates stems directly from the definition of a curved polygon in M. Let $\theta: [a, b] \to \mathbb{R}$ be a tangent angle function for γ . By means of the Rotation Index Theorem $(\theta(b) - \theta(a))$, and the Fundamental Theorem of Calculus $(\int_{a_{i-1}}^{a_i} \theta'(t) dt = a_i - a_{i-1})$, it follows that

$$2\pi = \theta(b) - \theta(a) = \sum_{i=1}^{k} \varepsilon_i + \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \theta'(t) \, dt.$$
(3.31)

Notice that when adding up the terms $a_i - a_{i-1}$, all but the first and last terms cancel out. Consequently, in order to obtain Eq. (3.29), a relation among θ' , κ_N , and *K* must be retrieved.

Again, let (E_1, E_2) be the oriented *g*-orthonormal frame obtained from $(\partial/\partial_x, \partial/\partial_y)$ via the Gram-Schmidt algorithm. Then, by the definitions of θ and *N*, these identities hold:

$$\gamma'(t) = \cos \theta(t) E_1|_{\gamma(t)} + \sin \theta(t) E_2|_{\gamma(t)},$$

$$N(t) = -\sin \theta(t) E_1|_{\gamma(t)} + \cos \theta(t) E_2|_{\gamma(t)}.$$
(3.32)

Henceforth, the dependence on t will not be explicitly noted to unburden the notation.

Subsequently, γ' is differentiated, which yields

$$D_{t}\gamma' = -(\sin\theta)\theta' E_{1}|_{\gamma} + (\cos\theta)\theta' E_{2}|_{\gamma} + (\cos\theta)\nabla_{\gamma'} E_{1}|_{\gamma} + (\sin\theta)\nabla_{\gamma'} E_{2}|_{\gamma} = \theta'N + (\cos\theta)\nabla_{\gamma'} E_{1}|_{\gamma} + (\sin\theta)\nabla_{\gamma'} E_{2}|_{\gamma}.$$
(3.33)

On the other hand, since (E_1, E_2) is an orthonormal frame, a few relations may be derived from the covariant derivatives of the different inner products, which are constant. Let $v \in T_{\gamma}M$ be any vector, then

$$\int_{\gamma} f \, ds = \int_{a}^{b} f(t) \left| \gamma'(t) \right|_{g} dt. \tag{3.30}$$

Let (M, g) be a Riemannian manifold, and γ a smooth curve segment. If $f: [a, b] \to \mathbb{R}$ is a continuous function, the *integral of f with respect to arc length* is

$$0 = \nabla_{v} |E_{1}|^{2} = \nabla_{v} \langle E_{1}, E_{1} \rangle \qquad = 2 \langle \nabla_{v} E_{1}, E_{1} \rangle, \qquad (3.34)$$

$$= \nabla_{v} |E_{2}|^{2} = \nabla_{v} \langle E_{2}, E_{2} \rangle \qquad = 2 \langle \nabla_{v} E_{2}, E_{2} \rangle, \qquad (3.35)$$

$$0 = \nabla_{v} \langle E_{1}, E_{2} \rangle \qquad \qquad = \langle \nabla_{v} E_{1}, E_{2} \rangle + \langle E_{1}, \nabla_{v} E_{2} \rangle. \tag{3.36}$$

From Eq. (3.34) and Eq. (3.35), it is deduced that $\nabla_v E_1$ is a multiple of E_2 , and that $\nabla_v E_2$ is a multiple of E_1 . Eq. (3.36) is used to define the 1-form ω such that

$$\omega(v) = \langle E_1, \nabla_v E_2 \rangle = - \langle \nabla_v E_1, E_2 \rangle.$$
(3.37)

If all these facts are combined, the covariant derivatives of the basis vectors may be expressed as

$$\nabla_{v}E_{1} = -\omega(v)E_{2},$$

$$\nabla_{v}E_{2} = \omega(v)E_{1}.$$
(3.38)

Hence, ω completely determines the connection in *U*. If one replaces Eq. (3.38) in Eq. (3.33), then the signed curvature of γ yields

$$\begin{aligned}
\kappa_{N} &= \langle D_{t}\gamma', N \rangle_{g} \\
&= \langle \theta'N, N \rangle_{g} + \cos \theta \left\langle \nabla_{\gamma'} E_{1} |_{\gamma}, N \right\rangle_{g} + \sin \theta \left\langle \nabla_{\gamma'} E_{2} |_{\gamma}, N \right\rangle_{g} \\
&= \langle \theta'N, N \rangle_{g} - \omega(\gamma') \cos \theta \left\langle E_{2}, N \right\rangle_{g} + \omega(\gamma') \sin \theta \left\langle E_{1}, N \right\rangle_{g} \\
&= \langle \theta'N, N \rangle_{g} - \omega(\gamma') \cos^{2} \theta - \omega(\gamma') \sin^{2} \theta \\
&= \theta' - \omega(\gamma').
\end{aligned}$$
(3.39)

Accordingly, Eq. (3.31) becomes

$$2\pi = \sum_{i=1}^{k} \varepsilon_i + \sum_{i=1}^{k} \int_{a_1-1}^{a_i} \kappa_N(t) dt + \sum_{i=1}^{k} \int_{a_1-1}^{a_i} \omega(\gamma'(t)) dt$$

$$= \sum_{i=1}^{k} \varepsilon_i + \int_{\gamma} \kappa_N ds + \int_{\gamma} \omega.$$
 (3.40)

The only remaining thing to show is that $\int_{\gamma} \omega = \int_{\Omega} K dA$. Observe that Ω is a smooth manifold with corners, and as a result, the Stokes' Theorem of Manifolds with Corners (Theorem 1.71) may be applied on $\int_{\gamma} \omega$, which yields

$$\int_{\gamma} \omega = \int_{\Omega} d\omega. \tag{3.41}$$

0

Hence, if one can prove that $d\omega = K dA$, the proof would be completed. For that, it is necessary to use Corollary 1.39, which shows that $dA(E_1, E_2) = 1$ since (E_1, E_2) is an oriented orthonormal frame. Then,

$$K dA(E_1, E_2) = K = Rm(E_1, E_2, E_2, E_1)$$

= $\langle R(E_1, E_2)E_2, E_1 \rangle_g$ (3.42)

Now, the expression $R(E_1, E_2)E_2$ may be expanded using the definition of the curvature tensor (Definition 1.75), Eq. (3.38), and the properties of the connections (Definition 1.26):

$$R(E_{1}, E_{2})E_{2} = \nabla_{E_{1}}\nabla_{E_{2}}E_{2} - \nabla_{E_{2}}\nabla_{E_{1}}E_{2} - \nabla_{[E_{1}, E_{2}]}E_{2}$$

$$= \nabla_{E_{1}} (\omega(E_{2})E_{1}) - \nabla_{E_{2}} (\omega(E_{2})E_{1}) - \omega ([E_{1}, E_{2}]) E_{1}$$

$$= (E_{1} \omega(E_{2})) E_{1} + \omega(E_{2})\nabla_{E_{1}} (E_{1})$$

$$- (E_{2} \omega(E_{2})) E_{1} - \omega(E_{2})\nabla_{E_{2}} (E_{1}) - \omega ([E_{1}, E_{2}]) E_{1}$$
(3.43)

Finally, the resulting expression can be replaced at Eq. (3.42):

$$K dA(E_{1}, E_{2}) = \langle (E_{1} \omega(E_{2})) E_{1} + \omega(E_{2}) \nabla_{E_{1}} (E_{1}) - (E_{2} \omega(E_{2})) E_{1} - \omega(E_{2}) \nabla_{E_{2}} (E_{1}) - \omega ([E_{1}, E_{2}]) E_{1}, E_{1} \rangle_{g}$$
(3.44)
$$= (E_{1} \omega(E_{2})) - (E_{2} \omega(E_{2})) - \omega ([E_{1}, E_{2}]) = d\omega (E_{1}, E_{2}),$$

where, again, Eq. (3.38) has been used, altogether with Proposition 1.41.

3.3 The Gauss-Bonnet Theorem

Definition 3.30. Let *M* be a compact 2-manifold. A *triangulation of M* is a finite collection of curved triangles such that, when any two of them intersect, they do so exactly at one of their vertices or along one of their edges. If *M* is smooth, the triangulation is said to be *smooth*.

Theorem 3.31. (Existence of the Triangulation of Compact 2-manifolds)

Any compact 2-manifold without boundary M has a triangulation.

Proof. A succinct proof can be found in [7]. However, this proof in turn employs the Jordan-Schoenflies Theorem. Alternatively, a self-contained proof can be found at Section 3 of [8] for regular surfaces in \mathbb{R}^3 .



Figure 3.10: a) Intersections allowed between triangles. b) Forbidden intersections.

Definition 3.32. *Let M be a* 2*-manifold with a given triangulation. The* Euler(–Pointcaré) characteristic of *M is defined as*

$$\chi(M) = T - E + V, \qquad (3.45)$$

where T is the number of triangles or faces of the triangulation, E is the number of edges, and V the number of vertices.



Figure 3.11: An example of the triangulation fo the sphere. In this case, $\chi(S^2) = T - E + V = 6 - 9 + 5 = 2$, respectively. On the right, the curved triangles have been replaced by triangles.

It cannot be presumptively assumed that the Euler characteristic of *M* does not depend on the chosen triangulation. Nevertheless, it will be demonstrated that, in fact, it does not.

Theorem 3.33. (The Gauss-Bonnet Theorem for orientable 2-manifolds) Let (M, g) be a smoothly triangulated compact Riemannian 2-manifold. dA denotes the Riemannian density. If K is the Gaussian curvature of g, and dA is its Riemannian density, then

$$\int_{M} K dA = 2\pi \chi(M). \tag{3.46}$$

Proof. M can be assumed to be connected because if it is not, the following deduction can be applied to each of its connected components and then these results can be added up.

Since *M* is a triangulated manifold, there is a triangulation $\{T_j\}_{j=1,...,T}$ of *M*. Then, *T* is the number of triangles, and let the number of edges, and vertices be *E*, and *V*, respectively. For each triangle T_j , their three edges, three vertices, and three interior angles are notated as E_{jk} , V_{jk} , and θ_{ij} for k = 1, 2, 3, respectively. Assume for the time being that *M* is orientable with a given orientation. Then, by the definition of triangulation, for each triangle T_j there is a positively oriented curved polygon $\tilde{E}: [a_{j1}, b_{j3}] \rightarrow M$ such that its regular curve segments are $\tilde{E}_{jk}: [a_{jk}, b_{jk}] \rightarrow M$. If θ_{ij} is an interior angle, then $2\pi - \theta_{ij}$ is its corresponding external angle. Using this notation, one can apply the Gauss-Bonnet formula, Eq. (3.29), on the curved triangle T_j :

$$\int_{T_j} K \, dA + \sum_{k=1}^3 \int_{\widetilde{E}_{jk}} \kappa_N \, ds + \sum_{i=1}^3 (\pi - \theta_{jk}) = 2\pi. \tag{3.47}$$

Thus, the addition of Eq. (3.47) for all the triangles T_j , j = 1, ..., T yields

$$\sum_{j=1}^{T} \int_{T_j} K \, dA + \sum_{j=1}^{T} \sum_{k=1}^{3} \int_{\widetilde{E}_{jk}} \kappa_N \, ds + \sum_{j=1}^{T} \sum_{k=1}^{3} (\pi - \theta_{jk}) = 2T\pi.$$
(3.48)

Before continuing, it will be useful to show that the sum of the interior angles of each vertex is 2π . This is obvious for the standard metric, but it needs to be shown for an arbitrary Riemannian metric. For a given vertex $e = V_{jk}$, its corresponding interior angles are those between the curves in contact with e that compose the triangles of the triangulation. In turn, the angle between two curves is the angle between their tangent vectors at the point where they coincide. Let $u_1, \ldots, u_k, u_{k+1} = u_1$ be these tangent vectors at e in T_pM . Let (e_1, e_2) be an orthonormal basis of T_pM with respect to g. Then, the unit tangent vectors may be expressed as $\hat{u} = \cos(\theta_i)e_1 + \sin(\theta_i)e_2$, with $\theta_1 = 0$, and the sum of the interior angles corresponds to $\theta(s_{k+1})$, which must be a multiple of 2π . To conclude that is 2π , use the same argument found in the proof of Theorem 3.18.

Now, several terms of Eq. (3.48) may be simplified:

- 1. $\sum_{j=1}^{T} \int_{T_j} K dA = \int_M K dA$, since the collection of triangles of the triangulation cover *M* and $T_i \cap T_j$ has measure 0, for every $i \neq j, i, j = 1, ..., T$,
- 2. $\sum_{j=1}^{T} \sum_{k=1}^{3} \int_{\widetilde{E}_{jk}} \kappa_N = \sum_{jk} \int_{\widetilde{E}_{jk}} \kappa_N = \sum_{jk,lm,E_{jk}=E_{lm}} \left(\int_{\widetilde{E}_{jk}} \kappa_N + \int_{\widetilde{E}_{lm}} \kappa_N \right) =$ $\sum_{jk,E_{jk}} \left(\int_{\widetilde{E}_{jk}} \kappa_N - \int_{\widetilde{E}_{jk}} \kappa_N \right) = 0.$ This identity comes from regrouping the terms and noticing that for each edge E_{jk} there are two curve segments that

go through the edge but have opposite direction (observe the edges of figure Fig. 3.12),



Figure 3.12: The triangulation of Fig. 3.11 has been cut out and "flattened" so that it is easier to be analyzed on the paper. Therefore, the vertices have been named with letters and the edges through which the cutting has been made are labeled with numbers. Once the cutting has been performed, the edges that appear twice should be identified. Notice each triangle has an orientation compatible with the original orientation of $M = S^2$.

3. $\sum_{j=1}^{T} \sum_{k=1}^{3} (\pi - \theta_{jk}) = 3T\pi - \sum_{j=1}^{T} \sum_{k=1}^{3} (\theta_{jk}) = \sum_{Vjk} \sum_{\theta_{lm} \text{ of } Vjk} = \sum_{Vjk} 2\pi = 2V\pi$, where, in the additions, all the interior angles pertaining to each vertex. The result arises then from realizing that the sum of all the interior angles associated with a each vertex add up to 2π ,



Figure 3.13: Depiction of the tangent space of the vertex p of the triangulation on Fig. 3.11. Some curve segments of the triangulation have been notated as \tilde{E}_i . $\tilde{E}_i(p)$ are tangent angle of M at p. It is shown that the sum of the interior angles of this vertex, i.e., the sum of the angles between the tangent vectors of $\tilde{E}_i(p)$ corresponds to $\theta_1 + \theta_2 + \theta_3 = 2\pi$.

4. Notice that if the sum of the number of edges of each separate triangle of

the triangulation (3*E*) is computed, then each edge has been counted twice, and therefore 2E = 3T, or equivalently, T = 2E - 2T.

All in all, it follows that

$$\int_{M} K \, dA = 2V\pi - T\pi = 2V\pi - (2E - 2T)\pi = 2\pi\chi(M). \tag{3.49}$$

Lemma 3.34. Let $(\widetilde{M}, \widetilde{g})$ and (M, g) be compact and connected Riemannian manifolds, and let $\pi \colon \widetilde{M} \to M$ be a k-sheeted Riemannian covering. Then

$$\int_{\widetilde{M}} dV_{\widetilde{g}} = k \int_{M} dV_{g}.$$
(3.50)

Proof.

$$\int_{\widetilde{M}} dV_{\widetilde{g}} = \int_{\widetilde{M}} \pi^* dV_g = k \int_M dV_g.$$
(3.51)

For more information on the integration on manifolds see Chapter 16 of [2].

Theorem 3.35. (The Gauss-Bonnet Theorem for nonorientable 2-manifolds) *The Eq.* (3.46) *of Theorem 3.33 holds if M is nonorientable as well. Now, dA denotes the Riemannian volume form.*

Proof. Just as in Theorem 3.33, *M* can be assumed to be connected. Also, observe that $\int_M K dA$ is the same regardless of whether dA is the Riemannian density or the Riemannian volume form.

The hypothesis of the theorem satisfy those of Proposition B.18 of [3]. Therefore, there is an oriented smooth manifold \widehat{M} and a two-sheeted smooth covering map $\widehat{\pi}: \widehat{M} \to M$. This result might indicate a possible path for proving the theorem: one could construct an orientable Riemannian manifold that meets the hypothesis of the orientable version of the Gauss-Bonnet Theorem (Theorem 3.33) in such a way that the computations on $(\widetilde{M}, \widetilde{g})$ are related to those on (M, g).

In that order of ideas, *M* is compact because it is the preimage of the compact *M* by π , a covering with finite fibers. On the other hand, \widetilde{M} can be endowed with the Riemannian metric corresponding to the pull back of $g: \widehat{g} = \widehat{\pi}^* g$. For that metric, the Gauss curvature is $\widehat{K} = \widehat{\pi}^* K$, and then $\widehat{\pi}^*(K dA) = \widehat{K dA}$

Assume *M* has a given triangulation. Now, the complex part is to fabricate a triangulation on \widehat{M} such that its Euler characteristic is related to that of *M*.

Let γ be a curved triangle of the triangulation of M whose interior is notated as Ω . By the definition of curved polygon (Definition 3.11), there is a smooth coordinate disk (U, φ) , such that $\overline{\Omega} \subseteq U$. Additionally, $\varphi \circ \gamma$ is a curved polygon on \mathbb{R}^2 whose interior is notated as Ω_0 . Therefore, $\varphi(\overline{\Omega}) = \overline{\Omega}_0$. Hence, $F = \varphi|_{\overline{\Omega}_0} : \overline{\Omega}_0 \to \overline{\Omega}$ is a diffeomorphism, by the definition of chart (see Fig. 3.14).



Figure 3.14: Let (U, φ) be a chart of a smooth manifold *M*. Since φ , and id are charts of *U*, and $\varphi(U)$, respectively, the diagram on the left shows that id is a diffeomorphism. Then, the diagram on the right shows that φ^{-1} is smooth.

Then, Corollary A.57 of [3] implies that φ^{-1} and F have a lift to \widehat{M} . If two lifts have a point in common in their image, they are identical (Proposition 1.58), and since \widehat{M} is 2-sheeted, then there are exactly two lifts $F_1, F_2: \overline{\Omega}_0 \to \widehat{M}$ such that $F = \widehat{\pi} \circ F_i$, with i = 1, 2. From this identity, and the fact that F is injective, it is shown that F_i is injective. Finally, the triangles derived from the triangulation of M can be lifted, which yields a triangulation of \widehat{M} with as many triangles, edges, and vertices as those of the original triangulation times the number of lifts: $\chi(\widehat{M}) = 2\chi(M)$.

Ultimately, the Gauss-Bonnet theorem for the orientable 2-manifold \widehat{M} (Theorem 3.33), together with the properties found for its constructed triangulation yield the desired result:

$$2\int_{M} K dA = \int_{\widehat{M}} \widehat{K} \widehat{dA} = 2\chi(\widehat{M}) = 4\pi\chi(M).$$
(3.52)

3.4 Application of the Gauss-Bonnet Theorem

First, a few terms and notation must be introduced:

If two Riemannian manifolds S_1 and S_2 are homeomorphic, one writes $S_1 \simeq S_2$. The connected sum operation is notated with #. The connected sum of *n* copies of a surface *S* is indicated as $nS = S\# \xrightarrow{n} \#S$. Denote the sphere as \mathbb{S}^2 , the torus as \mathbb{T} , and the projective plane \mathbb{P} . Then, remember this famous topological result: **Theorem 3.36. (Classification Theorem for Compact Topological 2-manifolds)** *Let M be a compact and connected topological 2-manifold.*

- 1. If M is orientable, then M is homeomorphic to \mathbb{S}^2 or to $n\mathbb{T}$,
- 2. If M is nonorientable, then M is homeomorphic to $n\mathbb{P}$.

Proof. See Theorems 6.15 and 10.22 of [2] for a proof for Topological manifolds. \Box

Additionally, for the connected sum of a pair of 2-manifolds, M_1 and M_2 , it holds $\chi(M_1 # M_2) = \chi(M_1) + \chi(M_2) - \chi(S^2)$, and then $\chi(n\mathbb{T}) = 2 - 2n$, and $\chi(n\mathbb{P}) = 2 - n$. As a result,

$$\chi(M) = \begin{cases} 2, \text{ if } M \simeq \mathbb{S}^2, \\ 2 - 2n, \text{ if } M \simeq n\mathbb{T}, \\ 2 - n, \text{ if } M \simeq n\mathbb{P}. \end{cases}$$
(3.53)

Then, this theorem combined with the Gauss-Bonnet theorem yields the following result.

Corollary 3.37. *Let* (M, g) *be a compact Riemannian* 2*-manifold, and let* K *be the corresponding Gaussian curvature. Then*

- 1. *if* $M \simeq \mathbb{S}^2$ *or* $M \simeq \mathbb{P}$ *, then* K > 0 *somewhere,*
- 2. if $M \simeq \mathbb{T}$ or $M \simeq 2\mathbb{P}$, then either $K \equiv 0$ or K takes both negative and positive values,
- 3. otherwise, K < 0 somewhere.

Proof. The sign of the characteristic of Euler may be computed for each scenario. The case 1. corresponds to $\chi(M) > 0$ ($\chi(\mathbb{S}^2) = 2$, $\chi(\mathbb{P}) = 1$). The case 2. corresponds to $\chi(M) = 0$. The case 3. corresponds to $\chi(M) < 0$.

By the Gauss-Bonnet Theorem, $\int_M K dA = 2\pi \chi(M)$, and then deductions about the sign of *K* can be made for each case. In 1., if $\chi(M) > 0$, then K > 0 somewhere. In 2., if $\chi(M) = 0$, then K = 0 everywhere, or *K* takes negative and positive values. In 3., $\chi(M) < 0$, and then K < 0 somewhere.

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