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Calderón-Zygmund estimates for the Laplacian

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Abstract

Regularity theory for Partial Differential Equations might be one of the most important topics in the field. With many applications, some of them in areas further away like Mathematical Physics, learning the basic regularity estimates for the Laplacian seems a crucial step into understanding more general results and solutions. This project intends to provide the tools and proofs of the *Calderón-Zygmund estimates* for the Laplacian equation $\Delta u = f$, with $f \in L^p$. We will separate in three distinct cases: $p = 2$, $p \in (2, \infty)$ and $p = \infty$, each with a different proof. Further, using blow-up techniques introduced in [1] a new proof for the limiting case $p = \infty$ will be provided. Finally, we intend to remark a few points that could potentially lead towards a blow-up proof for the general L^p case.

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Introduction

After a long and interesting journey studying both Mathematics and Physics, I have come to realize that there are many things given as granted. Some of them not so obvious, like the recurrent example that any equation has a smooth solution, specially in the field of Physics. Of course, although not explained in the courses, these are checked, but the general feeling is that they are omitted. From a mathematical point of view it has always stricken me this kind of reasoning, since it is a little bit against the logical and rational thinking that mathematics have taught me.

For this reason, when speaking with Dr. Xavier Ros-Oton at the start of the project, it became clear that the study of regularity for Partial Differential Equations (PDE) would put my mind at ease, or at least try to. This field in PDE has been one of the most important, since it is not only a unifying problem in the area, but also has many important implications in other fields of study, such as Harmonic Analysis or Mathematical Physics.

Of course, there is no way of checking it for every single equation or case, and for simplicity, we decided that the best would be to just consider the Laplacian:

$$\begin{aligned} \Delta u &= f \;\; in \;\; \Omega, \\ u &= g \;\; in \;\; \partial\Omega. \end{aligned} \tag{1.1}$$

The idea, is that the function f is usually known, which means we know what type of regularity it follows, and we want to find out what happens with our solution u. Before continuing, we will usually consider that $g \equiv 0$, which considerably simplifies some arguments. The reason why we can do so, is that if it were not the case, then we could always re-define our problem such that a harmonic function v is the solution of:

$$\begin{aligned} \Delta v &= 0 \;\; in \;\; \Omega, \\ v &= g \;\; in \;\; \partial\Omega. \end{aligned} \tag{1.2}$$

Then the function $\omega = u - v$ would have the form of (1.1) with $\omega = 0$ in $\partial\Omega$, and studying it's regularity would allow us to learn much from u as well. What is implicit here, are the known results on harmonic functions, which will later be revised in the following chapter. Therefore, form now on, we will always consider problems of the type (1.1) with $g = 0$, except otherwise stated.

The next step, was to decide where would f belong to. The classical results on the matter consider the following two cases:

$$\begin{aligned} f \in C^{\alpha}(\Omega) \quad &\text{for} \quad \alpha \in (0,1). \\ f \in L^{p}(\Omega) \quad &\text{for} \quad p \in (1,\infty). \end{aligned} \tag{1.3}$$

In the first one, the results are called *Schauder estimates* and consider functions that can be treated point-wisely. In the second, called *Calderón-Zygmund estimates*, the functions have less information since they are only integrable. These results state that our solution u is "two-times" better than f.

The first goal of the project was to understand the tools and proofs required for these estimates to be true, specially focusing on the L^p case. After this deep comprehension, the original idea was to propose a new way of proving the *Calderón-Zygmund estimates* using a different type of approach, with a blow-up method, as in some of the *Schauder estimates* proofs. Up to the moment, this has not been possible for the general L^p case. Nonetheless a positive proof has been found for the special case $p = \infty$.

In the following sections, we will first introduce the basic tools needed for the L^p estimates, with a short mentioning on the C^α case. Then, we will show the proofs of the estimates for three different cases: $p = 2$, $p \in (2,\infty)$ and $p = \infty$, each following a different line of action. Finally, some of the problems encountered during the work will be remarked.

Preliminaries

The aim of this section is to introduce the basic tools needed for the following chapters, as well as some basic notations and results.

Before jumping into the regularity of our solution of (1.1), one might ask himself if this problem has, indeed, a solution. And even if existence is proved, that still does not mean it is unique. Thankfully, there are many methods to prove existence or uniqueness for the Laplacian, like the energy method, which are thoroughly explained in the references [1, 2, 3].

Nonetheless, it is still important to understand what it really means to be a solution of the *Dirichlet problem* (1.1). There are two basic ways of defining them, in the *weak* or in the *viscosity* sense, and although both have different and interesting properties, in our case we are only motivated to define the first one:

Definition 2.1. *We say that u is a weak solution to* (1.1)*, whenever $u \in H^1(\Omega)$, $u_{\partial\Omega} = g$ and:*

$$\int_\Omega \nabla u \nabla v = \int_\Omega fv \quad \text{for all } v \in H^1(\Omega) \text{ such that } v = 0 \text{ in } \partial\Omega. \tag{2.1}$$

Notice that the concept of H^1 has not yet been introduced, but will become clear in the *Sobolev spaces* section. At this point, we could start to think about the regularity, but first we introduce the main spaces we will be dealing with, and some of their properties.

2.1 Hölder spaces

Even if this project is not centered around the *Schauder estimates*, before defining the more complex Sobolev space, a short mentioning of Hölder spaces will be helpful.

Definition 2.2. *Consider $k \in \mathbb{N}$ and $\alpha \in (0,1)$, the space $C^{k,\alpha}(\overline{\Omega})$ is the set of functions $u \in C^k(\overline{\Omega})$ such that the following norm is finite:*

$$||u||_{C^{k,\alpha}(\overline{\Omega})} = ||u||_{C^k(\overline{\Omega})} + [D^k u]_{C^{0,\alpha}(\overline{\Omega})}, \tag{2.2}$$

where the norm:

$$||u||_{C^k(\overline{\Omega})} = \sum_{j=1}^{k} ||D^j u||_{L^\infty(\overline{\Omega})}, \tag{2.3}$$

and the semi-norm:

$$[u]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}. \tag{2.4}$$

Usually we will refer the $C^{0,\alpha}$ as C^α. As we will later see, there are some similarities with the *BMO* semi-norm. Despite that there are many interesting properties associated to this space, like the *Arzelà-Ascoli theorem* or some *interpolation inequalities*, these will not be covered. Again, one can find a very good recollection in [1].

2.2 L^p and Sobolev spaces

2.2.1 L^p spaces

First we will introduce the L^p spaces.

Definition 2.3. *Given $\Omega \in \mathbb{R}^n$ and $1 \leq p < \infty$ we define the $L^p(\Omega)$ space as:*

$$L^p(\Omega) = \{u \text{ measurable in } \Omega : \int_\Omega |u|^p dx < \infty\}. \tag{2.5}$$

It is a Banach space with the following norm:

$$||u||_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{1/p}. \tag{2.6}$$

In the case when $p = \infty$, $L^\infty(\Omega)$ is the set of bounded functions in Ω (up to a subset of measure 0) with $||u||_{L^\infty(\Omega)} = esssup_\Omega|u| = inf\{a \in \mathbb{R},\ \mu(u(x) < a) \neq 0\}$, where μ is the measure associated to the space.

There are three theorems worth mentioning at this point. The proofs will not be provided, but can easily be found in chapters 2 and 4 from Ref. [4]. These will be used constantly throughout the project:

Theorem 2.4. *(Hölder's inequality) Let (Ω, Σ, μ) be a measure space and let $p,\ q \in [1, \infty]$ with $1/p + 1/q = 1$. Then for all measurable real valued functions f and g on Ω:*

$$||fg||_{L^1(\Omega)} = ||f||_{L^p(\Omega)} \cdot ||g||_{L^q(\Omega)}. \tag{2.7}$$

Theorem 2.5. *(Minkowski's inequality) Let* (Ω, Σ, μ) *be a measure space and let* $p,\ q \in [1, \infty)$ *with* $1/p + 1/q = 1$*. Then for* $f,\ g \in L^p(\Omega)$ *we have that* $f + g \in L^p(\Omega)$ *and the triangle inequality holds:*

$$||f+g||_{L^p(\Omega)} \leq ||f||_{L^p(\Omega)} + ||g||_{L^p(\Omega)}. \tag{2.8}$$

Theorem 2.6. *(Young's convolution inequality) Let* (Ω, Σ, μ) *be a measure space and let* $p,\ q,\ r \in [1, \infty]$ *with* $1/p + 1/q = 1/r + 1$*. Then for* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$ *we have:*

$$||f * g||_{L^r(\mathbb{R}^n)} = ||f||_{L^p(\mathbb{R}^n)} \cdot ||g||_{L^q(\mathbb{R}^n)}. \tag{2.9}$$

where $*$ *denotes the integral convolution:*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \tag{2.10}$$

The idea, is that these inequalities allow us to compare $p - norms$ for different values of p, as well as using the triangular inequality. For example, with *Hölder's inequality* we can see that the L^2 norm is comparable to the L^1 norm, with a constant that only depends on the domain Ω:

$$||f||_{L^1(\Omega)} = ||1||_{L^2(\Omega)} \cdot ||f||_{L^2(\Omega)} = |\Omega|^{1/2} \cdot ||f||_{L^2(\Omega)}, \tag{2.11}$$

where again $|\Omega| = \mu(\Omega)$ refers to the measure of Ω. *Young's convolution inequality*, allows us to bound the norm of a convolution by the norms of our functions, which will be an important step in some of the proofs. Lastly, a very well known formula:

Theorem 2.7. *(Integration by parts). Assume* $\Omega \subset \mathbb{R}^n$ *is any bounded* C^1 *domain. Then, for any* $u, v \in C^1(\overline{\Omega})$ *we have*

$$\int_{\Omega} \partial_i u \cdot v \, dx = -\int_{\Omega} u \cdot \partial_i v \, dx + \int_{\partial\Omega} uv \cdot \nu_i \, dS, \tag{2.12}$$

where ν *is the unit (outward) normal vector to* $\partial\Omega$*, and* $i = 1, 2, \ldots, n$*.*

This theorem holds as well for functions $u, v \in H^1$, as will be later used in the L^2 chapter and has an implicit definition of weak derivative.

2.2.2 Sobolev spaces

Now, it is time to introduce the the Sobolev spaces $W^{k,p}(\Omega)$:

Definition 2.8. *Given $\Omega \in \mathbb{R}^n$ and $1 \leq p \leq \infty$ we define the Sobolev space $W^{k,p}(\Omega)$ as:*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega),\ \partial_j^m u \in L^p(\Omega) \text{ for } m = 1, ..., k, j = 1, ..., n\}. \tag{2.13}$$

This space is basically defined such that the weak derivatives of u up to order k are found in L^p. We can also see that it is a Banach space [3] when we endow it with the following norm:

$$||u||_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \right)^{1/p} \quad \text{for } 1 \leq p < \infty. \tag{2.14}$$

When $p = \infty$ we have to consider $||u||_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq k} ess \sup_\Omega |D^\alpha u|$. As with the L^p spaces, it is interesting to quote two properties which will be used along the project.

Proposition 2.9. *The space $H^1(\Omega) := W^{1,2}(\Omega)$ is a Hilbert space with the following scalar product:*

$$(u, v)_{H^1(\Omega)} = \int_\Omega uv + \int_\Omega \nabla u \nabla v. \tag{2.15}$$

Any bounded sequence $\{u_k\}_k \in H^1(\Omega)$ contains a weakly convergent subsequence $\{u_{k_j}\}_j$, meaning that there exists $u \in H^1(\Omega)$ such that:

$$(u_{k_j}, v)_{H^1(\Omega)} \longrightarrow (u, v)_{H^1(\Omega)} \quad \text{for any } v \in H^1(\Omega). \tag{2.16}$$

Moreover, such u will satisfy the lower semi continuity of the norm:

$$||u||_{H^1(\Omega)} \leq \liminf_{j\to\infty} ||u_{k_j}||_{H^1(\Omega)}, \tag{2.17}$$

and since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$:

$$||u||_{L^2(\Omega)} = \lim_{j\to\infty} ||u_{k_j}||_{L^2(\Omega)}. \tag{2.18}$$

Although it might look simple, this proposition will have important consequences later on. The main reason is that it will allow us to pass the limit for sequences found in $H^1(\Omega)$, as will be needed in the $p = 2$ and $p = \infty$ proofs. It also relates to the fact that only for $p = 2$ we have a Hilbert space, meaning that this space is the dual of itself at the same time. It is worth mentioning, that the second part of the proposition actually is a special case of the *Rellich-Kondrachov Compactness Theorem* for $p = 2$, as well as with the *Sobolev Embedding Theorem*. These ones can be found in [5, 3]. The second property is the following inequality [1, 3]:

Theorem 2.10. *(Poincaré inequality). Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $p \in [1, \infty)$. Then, for any $u \in W^{1,p}(\Omega)$ we have*

$$\int_\Omega |u - \overline{u}_\Omega|^p \, dx \leq C_{\Omega,p} \int_\Omega |\nabla u|^p dx. \tag{2.19}$$

where $\overline{u}_\Omega := ⨍_\Omega u = \frac{1}{|\Omega|} \int_\Omega u$, and

$$\int_\Omega |u|^p dx \leq C'_{\Omega,p} \left(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} |u|_{\partial\Omega}|^p \, d\sigma \right). \tag{2.20}$$

The constants $C_{\Omega,p}$ and $C'_{\Omega,p}$ depend only on n, p, and Ω.

In this case, the inequality allows us to bound both the u and ∇u norms by the norm of $D^2 u$. This is the main reason why most of the proofs of the estimates will not consider lower derivatives. Basically showing that $\nabla u \in L^p$ will already be true if we can see that $D^2 u$ belongs there.

2.2.3 BMO spaces

Finally it is of interest to explain the *BMO space*. it shares some similarities with the Sobolev and Hölder spaces. In the first case is because of the definition of the $W^{k,BMO}$ space, while on the second because of the semi-norm we will see. It will become important later on, when considering the $p = \infty$ case for the estimates.

Definition 2.11. *Given $\Omega \in \mathbb{R}^n$ and $1 \leq p < \infty$ we define the BMO space over Ω as the functions $u \in L^p(\Omega)$ for all $p \in [1, \infty)$ such that the seminorm:*

$$|u|_{BMO(\Omega),p} = \sup_{B \subset \Omega} \left(⨍_B |u - \overline{u}_B|^p \right)^{1/p} < \infty. \tag{2.21}$$

As well as with the Sobolev spaces, we can endow this space with a norm:

$$||u||_{BMO(\Omega)} = |u|_{BMO(\Omega),p} + ||u||_{L^p(\Omega)}. \tag{2.22}$$

This norm is defined by any p, and by Jensen's inequality [4] it can be seen that they are all equivalent [5]. This allows us to use the norm that better suits our problems. Later on we will be using it as:

$$||u||_{BMO(\Omega)} = |u|_{BMO(\Omega),1} + ||u||_{L^2(\Omega)}. \tag{2.23}$$

From now on, $|u|_{BMO(\Omega)}$ will denote $|u|_{BMO(\Omega),1}$ to simplify notation. Considering the already defined norm, as with the Sobolev spaces, we can define:

Definition 2.12. *Given $\Omega \in \mathbb{R}^n$ we define the $W^{k,BMO}(\Omega)$ space as:*

$$W^{k,BMO}(\Omega) = \{u \in BMO(\Omega),\ \partial_j^m u \in BMO(\Omega)\ for\ m = 1, ..., k, j = 1, ..., n\}. \quad (2.24)$$

2.3 Harmonic functions

Now that we already have a little idea on the spaces we will be dealing with, it is time to understand some of the basic properties related to harmonic functions. Just to remember, a harmonic function is the unique weak solution to the Dirichlet problem (1.1) when $f = 0$ [1]:

$$\Delta u = 0 \ \ in \ \ \Omega, \quad (2.25)$$

As has already been mentioned, we are already omitting the boundary conditions, since $g = 0$ unless otherwise stated. Using energy methods to proof the existence of solutions to the harmonic problem [1], one can see that these minimize the following integral:

$$\int_\Omega |\nabla u|^2. \quad (2.26)$$

Therefore, any other function v that has the same value as u on $\partial\Omega$ will have a larger $\int_\Omega |\nabla v|^2$. There are some very interesting properties which can only be defined for harmonic functions. For example, one of them is the Poisson kernel representation:

$$u(x) = \frac{c_n}{r} \int_{\partial B_r} \frac{\left(r^2 - |x|^2\right) u(y)}{|x - y|^n} dy, \quad (2.27)$$

defined in a domain Ω such that $B_r \subset \Omega$. This one has an immediate consequence [1]:

Corollary 2.13. *Let $\Omega \subset \mathbb{R}^n$ be any open set, and $u \in H^1(\Omega)$ be any harmonic function satisfying* (2.25) *in the weak sense. Then, u is C^∞ inside Ω. Furthermore, if u is bounded and harmonic in B_1, then the estimate:*

$$\|u\|_{C^k(B_{1/2})} \leq C_k \|u\|_{L^\infty(B_1)}, \quad (2.28)$$

holds for all $k \in \mathbb{N}$, and for some constant C_k depending only on k and n.

Proof. For any ball $B_r(x_0) \subset \Omega$, we will have it's Poisson kernel representation. and thanks to it, it is immediate to see that $u \in C^\infty(B_{r/2}(x_0))$ and the estimate above holds. Since this can be done for any ball $B_r(x_\circ) \subset \Omega$, we deduce that u is C^∞ inside Ω. □

This corollary also implies that if u is harmonic in Ω, then all of it's derivatives will be harmonic in Ω as well. This is because u is C^∞ inside Ω and therefore the partial derivatives can be interchanged. The same equation (2.25) then, is valid for all of it's derivatives.

A second and last property, similar to the Poisson Kernel representation, is the following:

Proposition 2.14. *(Mean value property) If u is harmonic in Ω, then:*

$$u(x) = \fint_{B_r(x)} u(y)dy \quad \textit{for any ball } B_r(x) \subset \Omega. \tag{2.29}$$

It allows us to determine the value of a harmonic function at a given point x just by considering one of the averages around that point. Finally, a well known result:

Theorem 2.15. *(Liouville's theorem with growth). Assume that u is a solution of $\Delta u = 0$ in $\mathbb{R}^n$ satisfying $|u(x)| \leq C(1 + |x|^\gamma)$ for all $x \in \mathbb{R}^n$, with $\gamma > 0$. Then, u is a polynomial of degree at most $\lfloor \gamma \rfloor$, where $\lfloor \gamma \rfloor$ denotes the floor function.*

Proof. Following the proof we can find in [1], let us define $u_R(x) := u(Rx)$, and notice that it is still harmonic: $\Delta u_R = 0$ in $\mathbb{R}^n$. From the above mentioned corollary 2.13 and the growth assumption we get:

$$R^k||D^k u||_{L^\infty(B_{R/2})} = ||D^k u_R||_{L^\infty(B_{1/2})} \leq C_k||u_R||_{L^\infty(B_1)} = C_k||u||_{L^\infty(B_R)} \leq C_k R^\gamma. \tag{2.30}$$

In particular, if $k = \lfloor \gamma \rfloor + 1$, then:

$$||D^k u||_{L^\infty(B_{R/2})} \leq C_k R^{\gamma - k} \to 0 \quad \text{as} \quad R \to \infty. \tag{2.31}$$

That is, $D^k u \equiv 0$ in $\mathbb{R}^n$, and u is a polynomial of degree $k - 1 = \lfloor \gamma \rfloor$. □

The theorem allows us to proof that a given function is a polynomial of a certain degree. As we can see, the growth condition does no necessarily be on the $|u(x)|$, since we find that the result holds for a bounded L^1 norm. Using the mean value property:

$$|u(x)| \leq \frac{1}{|B_r|} \int_{B_r(x)} |u(x)| = \frac{1}{r^n} ||u||_{L^1(B_r(x))}, \tag{2.32}$$

we find a bound on $|u(x)|$ depending on the L^1 norm in compact subsets. This shows that the L^∞ norm is comparable to the L^1 norm for harmonic functions. Choosing the subsets $B_r(x)$ accordingly, we will be able to use the above result for bounded L^1 functions.

Regularity estimates

Eventually, we have most of the tools needed for the study of the regularity estimates associated to the *Laplacian*. For simplicity, we will consider that the domain $\Omega = B_1$. After translation and re-scaling it would hold for a general Ω domain.

As mentioned before, the regularity of our solution u will depend on the regularity of f. Hence, the two most basic options are when:

$$\begin{aligned} f \in C^{\alpha}(\Omega) \quad &\text{for} \quad \alpha \in (0,1), \\ f \in L^{p}(\Omega) \quad &\text{for} \quad p \in (1,\infty). \end{aligned} \tag{3.1}$$

3.1 Schauder estimates

When $f \in C^{\alpha}(B_1)$, the theorems regarding the regularity of u are called *Schauder estimates*, and indeed prove that:

If u is a weak solution to the problem 1.1, with $f \in C^{\alpha}$, then $D^2u \in C^{\alpha}$ for $\alpha \in (0,1)$.

Following [1], there are different ways to prove these estimates. Usually they require to prove some propositions for C^{∞} functions, which then can be applied to treat the estimates. These can follow different ways such as constructing sequences which require some convergence theorems like the *Arzelà-Ascoli theorem*; the use of polynomial sequences to approximate the solution or through a blow-up method where we compare the solution to a harmonic problem in a smaller neighbourhood. All of them require some basic properties from *Hölder spaces*.

We could ask ourselves: *what happens when $\alpha = 0$ or $\alpha = 1$, namely that f is continuous or Lipschitz continuous?*. The short answer to it is that we can not assure the regularity of u, as can be seen with some counter-examples in [1].

3.2 Calderón-Zygmund estimates

In the case when $f \in L^p(B_1)$, the theorems regarding the regularity of u are called *Calderón-Zygmund estimates*, and prove that:

If u is a weak solution to the problem 1.1, with $f \in L^p$, then $D^2u \in L^p$ for $p \in (1, \infty)$.

As well as with the *Schauder estimates*, there some different ways of proving these theorems, but these are more restricted since the L^p case is, up to my knowledge, not possible to treat point-wisely. Therefore, most of the proofs follow the methods introduced by *Calderón* and *Zygmund*, with the *Calderón-Zygmund decomposition argument* that can be found in [6, 5]. These usually use interpolation theorems, like the *Marcinkiewicz interpolation theorem*, the limiting case $p = 2$ as well as duality arguments. Nonetheless, there are some specific cases, as the aforementioned $p = 2$, when a much simpler approach can be taken.

Again, we could question ourselves what happens when $p = 1$ or $p = \infty$. The answer, again, is that we cannot assure that D^2u will be in L^p. When $p = 1$ we find an example in [5]:

Example 3.1. Consider the ball B_1 and the defined function over it as $u(r) = log(log(r^{-1}))$ written in polar coordinates. A straightforward computation leads to:

$$\Delta u = \frac{-1}{r^2 log^2(r)}, \tag{3.2}$$

which has a bounded $L^1(B_1)$ norm. On the other hand, one can compute the second derivative over r:

$$\partial_{rr}u = -\frac{log(r)+1}{r^2 log^2(r)} = -\frac{1}{r^2 log(r)}\left(\frac{log(r)+1}{log(r)}\right) \geq \frac{-1}{2r^2 log(r)}, \tag{3.3}$$

where in the last inequality we have considered a small enough r. Since $\int_0^\epsilon \frac{1}{r log(r)} dr$ is divergent for any $\epsilon \in (0, 1)$, we reach that $D^2u \notin L^1(B_1)$ as we wanted to see.

An alternative simple example for the $p = \infty$ case is given in [1]:

Example 3.2. Consider the function $u(x, y) = (x^2 - y^2)log(x^2 + y^2)$ in $\mathbb{R}^2$, then an easy computation shows that:

$$\partial_{xx}u = 2log(x^2 + y^2) + \frac{8x^2}{x^2 + y^2} - 2\left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2, \tag{3.4}$$

$$\partial_{yy}u = -2log(x^2 + y^2) - \frac{8y^2}{x^2 + y^2} + 2\left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2, \tag{3.5}$$

Clearly, one can see that both $\partial_{xx}u$ and $\partial_{yy}u$ are not in $L^\infty(\mathbb{R}^2)$, since there is no $M \in \mathbb{R}$ and set B with $|B| = 0$ such that $|\partial_{xx}u| < M$ and $|\partial_{yy}u| < M$ for any point in $\mathbb{R}^2 \setminus B$. This shows that $D^2u \notin L^\infty(\mathbb{R}^2)$. On the other hand, if we calculate:

$$\Delta u = \partial_{xx}u + \partial_{yy}u = 8\frac{x^2 - y^2}{x^2 + y^2}, \tag{3.6}$$

we clearly see that it is in $L^\infty(\mathbb{R}^2)$.

The main idea behind these examples is to combine a *harmonic* part with a function, such as the $log()$, that cause an issue at a some point/set of the domain.

Still, one can modify these estimates and find that in the $p = 1$ case, then $u \in \mathcal{H}^1$, which is the *Hardy space* and will not be explained in this project. While when $p = \infty$, then $D^2u \in BMO$ [7], as will be seen in future chapters. The relationship between these two spaces is given by *Fefferman* and *Stein*, stating that one is the dual of the other [5].

The main idea of this project was to understand these different L^p proofs, and try to apply the blow-up method normally used in the *Schauder estimates* for the L^p case. Up to this moment, only for the limiting case $p = \infty$ a positive proof has been found, due to the similarities between the BMO and *Hölder* semi-norms, since both consider supremums. Therefore, adapting the *Schuader estimate* proofs to the L^∞ case is much simpler.

L^2 case

When treating L^p spaces, the most simple case is when $p = 2$, since it is a Hilbert space and its dual is himself. This allows us to prove the *Calderón-Zygmund estimate* directly and without much complications, in a similar fashion as in the C^α cases. The proof is taken from [1], and then has a small follow up in order to end it.

Proposition 4.1. *Let $u, f \in C^\infty(B_1)$ be such that:*

$$\Delta u = f \ \ in \ \ B_1. \tag{4.1}$$

Then the following estimate holds:

$$||u||_{W^{2,2}(B_{1/2})} \leq C \left(||u||_{L^2(B_1)} + ||f||_{L^2(B_1)} \right), \tag{4.2}$$

where C is a constant that only depends on the dimension n.

Proof. Indeed, consider a fixed test function $\eta \in C_c^\infty(B_1)$ with $\eta \equiv 1$ in $B_{1/2}$, $\eta \equiv 0$ in $B_1 \setminus B_{3/4}$ and $\eta \geq 0$. Define $w := \eta u$ and integrating by parts twice leads to:

$$\begin{aligned} ||D^2 u||^2_{L^2(B_{1/2})} &= \sum_{i,j=1}^n \int_{B_{1/2}} |D_{ij} u|^2 \leq \sum_{i,j=1}^n \int_{B_1} |D_{ij} w|^2 \\ &= -\sum_{i,j=1}^n \int_{B_1} (D_{iij} w)(D_j w) = \sum_{i,j=1}^n \int_{B_1} (D_{ii} w)(D_{jj} w) \\ &= \int_{B_1} (\Delta w)^2 \leq C \int_{B_1} \left(u^2 + (\Delta u)^2 + |\nabla \eta|^2 |\nabla u|^2 \right). \end{aligned} \tag{4.3}$$

Since $\Delta w = \nabla \cdot \nabla(\eta u) = \nabla \cdot (\nabla \eta \cdot u + \eta \cdot \nabla u)$ and $C = C' \sup_{B_1}(\eta^2 + |\Delta \eta|^2)$ in the last inequality. Notice that C' is a dimensional constant. Again, integrating by parts twice:

$$\int_{B_1} |\nabla\eta|^2|\nabla u|^2 = -\int_{B_1} \nabla\left(|\nabla\eta|^2\nabla u\right) u = -\int_{B_1} |\nabla\eta|^2 u\Delta u \; - \int_{B_1} \nabla(|\nabla\eta|^2)u\nabla u$$
$$= -\int_{B_1} |\nabla\eta|^2 u\Delta u \; + \int_{B_1} \nabla\left[\nabla(|\nabla\eta|^2)u\right] u = -\int_{B_1} |\nabla\eta|^2 u\Delta u \; + \int_{B_1} \frac{1}{2}u^2\Delta|\nabla\eta|^2$$
$$\leq \tilde{C}\int_{B_1}\left(u^2 + (\Delta u)^2\right). \quad (4.4)$$

Where we have taken $\tilde{C} = \sup_{B_1}\left(|\nabla\eta|^2 + \Delta|\nabla\eta|^2\right)$. We have also been omitting the surface integral terms, because $u = 0$ at ∂B_1. Adding all up leads to:

$$||D^2u||^2_{L^2(B_{1/2})} \leq C\int_{B_1}\left(u^2 + f^2\right), \quad (4.5)$$

which directly leads to the desired result. □

This result, for C^∞ functions, bounds our $W^{2,2}$ norm, but we still need to prove the estimate. In order to do so, we will adapt *Corollary 2.16* found in [1]. We will use convolutions to approximate our function u by a smooth C^∞ function, allowing us to use the aforementioned result:

Corollary 4.2. *Let $u \in H^1(B_1)$ be a weak solution to:*

$$\Delta u = f \text{ in } B_1, \quad (4.6)$$

with $f \in L^2(B_1)$. Then u is $W^{2,2}$ inside B_1 and the estimate (4.1) holds.

Proof. Following the reference, let u be a solution in B_1 from $\Delta u = f$, with $f \in L^2(B_1)$. Let $\eta \in C_c^\infty(B_1)$ be any smooth function with $\eta \geq 0$ and $\int_{B_1}\eta = 1$. Let:

$$\eta_\epsilon = \epsilon^{-n}\eta(\frac{x}{\epsilon}). \quad (4.7)$$

It satisfies $\eta_\epsilon \in C_c^\infty(B_\epsilon)$ and $\int_{B_\epsilon}\eta_\epsilon = 1$. Consider the convolution:

$$u_\epsilon(x) = u * \eta_\epsilon(x) = \int_{B_\epsilon} u(x-y)\eta_\epsilon(y)dy, \quad (4.8)$$

which is C^∞ and satisfies:

$$\Delta u_\epsilon = f * \eta_\epsilon = f_\epsilon \quad in \quad B_{1-\epsilon}. \quad (4.9)$$

since u_ϵ is C^∞ we can use proposition 4.1 to get:

$$||u_\epsilon||_{W^{2,2}(B_{1/2})} \; \leq C\cdot(||u_\epsilon||_{L^2(B_1)} + ||f_\epsilon||_{L^2(B_1)}). \quad (4.10)$$

Now, considering *Young's convolution inequality* 2.6 for L^p norms, we get that:

$$||u_\epsilon||_{L^2(B_1)} \leq ||u||_{L^2(B_1)} * ||\eta_\epsilon||_{L^1(B_1)} = ||u||_{L^2(B_1)}, \tag{4.11}$$

and again:

$$||f_\epsilon||_{L^2(B_1)} \leq ||f||_{L^2(B_1)} * ||\eta_\epsilon||_{L^1(B_1)} = ||f||_{L^2(B_1)}. \tag{4.12}$$

Notice then, that the sequence u_ϵ is uniformly bounded in $B_{1/2}$. Using Proposition 2.9 there is a subsequence that converges to $u \in W^{2,2}(B_{1/2})$ with:

$$||u||_{W^{2,2}(B_{1/2})} \leq C \cdot (||u||_{L^2(B_1)} + ||f||_{L^2(B_1)}). \tag{4.13}$$

which completes the proof. □

This shows this simple case. As will be seen in the following chapters, we will constantly use this estimate to check some of the inequalities, as well as use properties which are only related to the L^2 space. Therefore, although simple, it is a very important step for the upcoming sections.

L^p case

Knowing the validity of the $p = 2$ case, it is time to understand and prove the *Calderón-Zygmund estimates* for $p \in (1, \infty)$:

Theorem 5.1. *Let $u \in H^1(B_1)$ be a weak solution to:*

$$\Delta u = f \text{ in } B_1, \tag{5.1}$$

with $f \in L^p(B_1)$. Then u is $W^{2,p}$ inside B_1 and the following estimate holds:

$$\int_{B_1} |D^2 u|^p \leq C \left(\int_{B_2} |u|^p + \int_{B_2} |f|^p \right). \tag{5.2}$$

As will become clear at the end, due to duality arguments, it is sufficient to only prove the estimate for $p > 2$. The arguments in this chapter will follow the notes on the matter [8].

Comparing with similar proofs like the C^α case, a small problem arises with L^p functions because there is no straightforward way to treat them point-wisely. Therefore, extra caution needs to be taken when treating inequalities and absolute values. The reason for this statement comes from a well know property of L^p functions (proposition 6.1 in [5]):

$$\int_\Omega |u|^p = p \int_0^\infty \lambda^{p-1} \left| \{x \in \Omega, |u(x)| > \lambda\} \right| d\lambda, \tag{5.3}$$

where $|\{\cdot\}|$ denotes the measure of the domain. If we want the norm to be finite, then for large λ the set $|\{x \in \Omega, |u(x)| > \lambda\}|$ has to have a small value. This allows us to treat the norm as the decay of this set knowing that the faster it decays the larger p can be.

Therefore, if we were able to see that the measure of one of these sets, for a fixed λ_0, is smaller than for a fixed value, we could end up proving that our function is in fact in L^p:

$$|\{|u(x)| > \lambda_0\}| \leq \epsilon |\{|u(x)| > 1\}|. \tag{5.4}$$

This type of inequality can be easily re-scaled with proper conditions on the data. Considering our estimate, we could try to prove something of the following sort: for a fixed value of λ_0, then for any ϵ there exists a δ_0 such that:

$$\left|\{|D^2u| > \lambda_0\}\right| \leq \epsilon \left(\left|\{|D^2u| > 1\}\right| + |\{|f| > \delta_0\}|\right). \tag{5.5}$$

It is straightforward to see that this estimate, after re-scaling:

$$\left|\{|D^2u| > \lambda_0\lambda\}\right| \leq \epsilon \left(\left|\{|D^2u| > \lambda\}\right| + |\{|f| > \delta_0\lambda\}|\right). \tag{5.6}$$

leads to equation 5.2 by substituting in Equation (5.3) we get:

$$\begin{aligned} ||D^2u||_{L^p}^p &= p\int_0^\infty \lambda^{p-1}\left|\{|D^2u(x)| > \lambda\}\right| d\lambda \\ &\leq \epsilon p \left(\int_0^\infty \lambda^{p-1}\left|\left\{|D^2u(x)| > \frac{\lambda}{\lambda_0}\right\}\right| d\lambda + \int_0^\infty \lambda^{p-1}\left|\left\{|f(x)| > \frac{\lambda\delta_0}{\lambda_0}\right\}\right| d\lambda\right) \\ &\leq \epsilon p \lambda_0^p ||D^2u||_{L^p}^p + C||f||_{L^p}^p. \end{aligned} \tag{5.7}$$

Choosing ϵ as small as necessary, we can make that $\epsilon p \lambda_0^p < 1$ and then the estimate follows. But again, careful consideration has to be taken into account at this point, since the condition $|D^2u(x)| < 1$ is unstable in the $W^{2,p}$ theory because of the nature of L^p functions. We cannot assure that this really happens for every point x we consider. Therefore this inequality is not well defined, and there is a need to find a much more stable function. The solution is using the *Hardy-Littlewood maximal function* defined as follows:

Definition 5.2. *For a locally integrable function v defined in $\mathbb{R}^n$ its maximal function at any point $x \in \mathbb{R}^n$ is:*

$$\mathcal{M}v(x) = \sup_{r>0} ⨍_{B_r(x)} v. \tag{5.8}$$

It can be defined as well for functions v over a domain $\Omega \in \mathbb{R}^n$, since we can always extend v by 0 outside of Ω. We will usually be omitting the domain where this function is defined, unless otherwise stated. The reason to introduce the maximal function is the relationship it has to the L^p norms:

Theorem 5.3. *Let $\Omega \in \mathbb{R}^n$, $v \in L^p(\Omega)$ and $\mathcal{M}v(x)$ its Hardy-Littlewood maximal function, then:*

$$||\mathcal{M}v||_{L^p(\Omega)} \leq C||v||_{L^p(\Omega)} \quad \textit{for any} \quad 1 < p < \infty. \tag{5.9}$$

$$|\{x \in \Omega, \mathcal{M}v(x) \geq \lambda\}| \leq \frac{C}{\lambda}||v||_{L^1(\Omega)}. \tag{5.10}$$

The first is called *strong p-p* estimate, while the second *weak 1-1* estimate. It shows us that the measures of $\{x \in \Omega, |v(x)| > \lambda\}$ and $\{x \in \Omega, \mathcal{M}v(x) > \lambda\}$ decay similarly. Still, the statement $\mathcal{M}u(x) \leq 1$ is more stable than $|D^2u(x)| \leq 1$, because the maximal function is invariant to scaling. This brings us to rewrite the inequality 5.5 in terms of the maximal function:

$$\left|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > \lambda_0^2\right\}\right| \leq \epsilon \left(\left|B_1 \cap \left\{\mathcal{M}|D^2u|^2 > 1\right\}\right| + \left|B_1 \cap \left\{\mathcal{M}|f|^2 > \delta_0^2\right\}\right|\right). \tag{5.11}$$

The value of δ_0 can be taken as small as possible since it is about f which we already know is in L^p. Again, in a similar fashion as the above case, we can see that this inequality implies our desired estimate.

Another important tool, called the *Vitali covering lemma* will allow us to cover a set with disjoint subsets which have a specific measure. This will partition our domain into smaller ones, where we control the measure of our desired function. During the proof of our theorem we will be using a similar result introduced and proved in the Ref. [7]:

Theorem 5.4. (**Modified Vitali**) *Let $0 < \epsilon < 1$ and $C \subset D \subset B_1$ be two measurable sets with $|C| < \epsilon|B_1|$ and satisfying that for every $x \in B_1$, with $|C \cap B_r(x)| \geq \epsilon|B_r|$, $B_r(x) \cap B_1 \subset D$. Then $|D| \geq \frac{1}{20^n \epsilon}|C|$.*

There are certain similarities between this theorem and the usual covering lemma used in the *Calderón-Zygmund estimates* proofs. This second is the *Calderón-Zygmund Decomposition* which uses cubes instead of balls, and can be found in the references [7, 5].

Now we move on to see the proof of a specific lemma, from which our estimate will follow, as happens in most cases when treating this type of inequalities.

Lemma 5.5. *Assume u is a solution of 1.1 in a domain Ω, with $B_4 \subset \Omega$. Then there exist a constant N_1 such that for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ so that if:*

$$\left\{\mathcal{M}|f|^2 \leq \delta^2\right\} \cap \left\{\mathcal{M}|D^2u|^2 \leq 1\right\} \cap B_1 \neq \emptyset, \tag{5.12}$$

then:

$$\left|\left\{\mathcal{M}|D^2u|^2 > N_1^2\right\} \cap B_1\right| < \epsilon|B_1|. \tag{5.13}$$

Proof. From 5.12 we see that there is a point $x_0 \in B_1$ such that:

$$\fint_{B_r(x_0)} |D^2u|^2 < 1, \qquad \fint_{B_r(x_0)} |f|^2 < \delta^2, \tag{5.14}$$

for any ball $B_r(x_0) \subset \Omega$, and in particular, considering a large enough r, for example $r = 8$ such that $B_4 \subset B_r(x_0)$:

$$\fint_{B_4} |D^2u|^2 \leq \frac{|B_r(x_0)|}{|B_4|} | \fint_{B_r(x_0)} |D^2u|^2 \leq \left(\frac{r}{4}\right)^n = 2^n, \tag{5.15}$$

and similarly:

$$\fint_{B_4} |f|^2 \leq 2^n\delta^2. \tag{5.16}$$

Using *Poincaré's inequality* (2.10) for ∇u leads to:

$$\fint_{B_4} \left|\nabla u - \overline{\nabla u}_{B_4}\right|^2 \leq C \fint_{B_4} |D^2u|^2 \leq C \fint_{B_4} |f|^2 \leq 2^n\delta^2 C = C_1, \tag{5.17}$$

where in the before-last inequality we used the L^2 - *estimate* (4.1). Let us define a new function v as the solution to:

$$\begin{cases} \Delta v = 0 \ \text{ in } \ B_4 \\ v = u - \overline{\nabla u}_{B_4}\mathbf{x} - \overline{u}_{B_4} \ \text{ in } \ \partial B_4 \end{cases} \tag{5.18}$$

By the properties of harmonic functions, since they minimize the gradient (2.26):

$$\int_{B_4} |\nabla v|^2 \leq \int_{B_4} |\nabla u - \overline{\nabla u}_{B_4}|^2 \leq C_1. \tag{5.19}$$

Therefore, using proposition 2.13 for the gradient as well as the *mean value property* and *Hölder's inequality* we reach:

$$||D^2v||^2_{L^\infty(B_3)} \leq C||\nabla v||^2_{L^\infty(B_{7/2})} \leq \tilde{C}||\nabla v||^2_{L^2(B_4)} \leq \tilde{C}C_1 \equiv N_0^2. \tag{5.20}$$

At the same time we can consider the function $u - v$ which thanks to the L^2 - *estimate* leads to:

$$\int_{B_3} |D^2(u-v)|^2 \leq C \int_{B_3} |f|^2 \leq C \int_{B_4} |f|^2 \leq C\delta^2, \tag{5.21}$$

since $\Delta(u - v) = f$ in B_3. Then using the *weak 1-1* estimate 5.10 we reach:

$$\lambda| \left\{x \in B_3,\ \mathcal{M}_{B_3}|D^2(u-v)|^2(x) > \lambda\right\} | \leq C \int_{B_3} |D^2(u-v)|^2 \leq C\delta^2, \tag{5.22}$$

where $\mathcal{M}_{B_3}g(x)$ denotes the maximal function over a domain $B_3 \subset \mathbb{R}^n$. After taking $\lambda = N_0^2$ and rewriting the constant:

$$\left|\left\{x \in B_1,\ \mathcal{M}_{B_3}|D^2(u-v)|^2(x) > N_0^2\right\}\right| \leq C\delta^2. \tag{5.23}$$

We have considered B_1 instead of B_3 so that we can use the constant N_0 defined above. The point now is to prove:

$$\left\{x \in B_1,\ \mathcal{M}|D^2u|^2(x) > N_1^2\right\} \subset \left\{x \in B_1,\ \mathcal{M}_{B_3}|D^2(u-v)|^2(x) > N_0^2\right\}. \quad (5.24)$$

with $N_1^2 = max\{4N_0^2, 2^n\}$. We will see it through contra-positive. Notice that if $y \in B_3$ then using 5.20 and that $a^2 + b^2 \geq 2ab$:

$$\begin{aligned}|D^2u(y)|^2 &= |D^2u(y)|^2 - 2|D^2v(y)|^2 + 2|D^2v(y)|^2 \\ &\leq 2|D^2u(y) - D^2v(y)|^2 + 2N_0^2. \quad (5.25)\end{aligned}$$

Let $x \in \left\{x \in B_1,\ \mathcal{M}_{B_3}|D^2(u-v)|^2(x) \leq N_0^2\right\}$, then, if $r \leq 2$ we have that $B_r(x) \subset B_3$ and therefore:

$$\begin{aligned}\sup_{r\leq 2} \fint_{B_r(x)} |D^2u|^2 &\leq \sup_{r\leq 2} \fint_{B_r(x)} 2|D^2(u-v)|^2 + 2N_0^2 \leq \\ &2\mathcal{M}_{B_3}|D^2(u-v)|^2(x) + 2N_0^2 \leq 4N_0^2, \quad (5.26)\end{aligned}$$

where in the last inequality we used the hypothesis over x. On the other hand, if $r > 2$, by the definition of x_0 given at the start, we have that $x_0 \in B_1 \subset B_r(x)$ and therefore $B_r(x) \subset B_{2r}(x_0)$, then by 5.14:

$$\fint_{B_r(x)} |D^2u|^2 \leq \frac{|B_{2r}(x_0)|}{|B_r(x)|} \fint_{B_{2r}(x)} |D^2u|^2 \leq 2^n, \quad (5.27)$$

This proves that $\mathcal{M}|D^2(u-v)|^2 \leq N_1^2$, as we wanted. Finally:

$$\begin{aligned}\left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2(x) > N_1^2\right\}\right| &\leq \left|\left\{x \in B_1,\ \mathcal{M}_{B_3}|D^2(u-v)|^2(x) > N_0^2\right\}\right| \\ &\leq \frac{C}{N_0^2} \int |f|^2 < C\delta^2 = \epsilon|B_1|, \quad (5.28)\end{aligned}$$

and taking δ satisfying the last equality leads to the desired result. □

The idea now, is to compare $\{\mathcal{M}|D^2u|^2 > N_1^2\}$ with $\{\mathcal{M}|D^2u|^2 > 1\}$ and see, in fact, that the second is much larger than the first. This is an immediate consequence from the Lemma 5.5 above:

Corollary 5.6. *Assume* u *is a solution of 1.1 in a domain* Ω *and* B *a ball so that* $4B \subset \Omega$. *If* $\left|\left\{\mathcal{M}|D^2u|^2(x) > N_1^2\right\} \cap B\right| \geq \epsilon|B|$ *then:*

$$B \subset \left\{\mathcal{M}|D^2u|^2(x) > 1\right\} \cup \left\{\mathcal{M}|f|^2 > \delta^2\right\}. \quad (5.29)$$

Proof. By doing the contra-positive argument of Lemma 5.5, we reach the desired result. □

Indeed, this Corollary is just a way of re-stating the anterior Lemma, but it is easier to understand, specially when comparing sets. Before seeing the proof of the theorem, we show the final piece:

Corollary 5.7. *Assume u is a solution of 1.1 in a domain Ω, with $B_4 \subset \Omega$, and that $\left|\{x \in B_1,\ \mathcal{M}|D^2u|^2 > N_1^2\}\right| \leq \epsilon|B_1|$. Then for $\epsilon_1 = 20^n\epsilon$:*

1. $$\left|\{x \in B_1,\ \mathcal{M}|D^2u|^2 > N_1^2\}\right| \leq \epsilon_1 \left(\left|\{x \in B_1,\ \mathcal{M}|D^2u|^2 > 1\}\right| + \left|\{x \in B_1,\ \mathcal{M}|f|^2 > \delta^2\}\right|\right). \tag{5.30}$$

2. $$\left|\{x \in B_1,\ \mathcal{M}|D^2u|^2 > N_1^2\lambda^2\}\right| \leq \epsilon_1 \left(\left|\{x \in B_1,\ \mathcal{M}|D^2u|^2 > \lambda^2\}\right| + \left|\{x \in B_1,\ \mathcal{M}|f|^2 > \delta^2\lambda^2\}\right|\right). \tag{5.31}$$

3. $$\left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2 > \left(N_1^2\right)^k\right\}\right| \leq \sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in B_1,\ \mathcal{M}|f|^2 > \delta^2\left(N_1^2\right)^{k-i}\right\}\right| + \epsilon_1^k \left|\{x \in B_1,\ \mathcal{M}|D^2u|^2 > 1\}\right|. \tag{5.32}$$

Where we have taken N_1 *and* ϵ *as in Lemma 5.5.*

Proof. Let us start from the beginning. For case number (1) let us consider the following 2 sets:

$$C = \{\mathcal{M}|D^2u|^2 > N_1^2\} \cap B_1, \tag{5.33}$$

$$D = \{\mathcal{M}|D^2u|^2 > 1\} \cup \{\mathcal{M}|f|^2 > \delta^2\} \cap B_1. \tag{5.34}$$

Since $N_1 > 1$, we have that $C \subset D \subset B_1$. Given a point $x \in B_1$, consider $B_r(x)$ with r small enough such that $B_{4r}(x) \subset B_1$, and that $|C \cap B_r(x)| \geq \epsilon|B_r|$. By Corollary 5.6, we have that $B_r(x) \subset D$. Finally, applying Theorem 5.4 we get $|C| \leq \epsilon_1|D|$, which basically implies (1).

For (2) let us consider $w = \lambda^{-1}u$, which obeys $\Delta w = \Delta(\lambda^{-1}u) = \lambda^{-1}f$. Then, applying (1) for w we end up with:

$$\left|\left\{x \in B_1,\ \mathcal{M}|D^2\lambda^{-1}u|^2 > N_1^2\right\}\right| \leq \epsilon_1 \left(\left|\left\{x \in B_1,\ \mathcal{M}|D^{2\lambda^{-1}}u|^2 > 1\right\}\right| + \left|\left\{x \in B_1,\ \mathcal{M}|\lambda^{-1}f|^2 > \delta^2\right\}\right|\right). \tag{5.35}$$

which is (2).

Lastly, for (3), notice that we can consider $\left(N_1^2\right)^k = \left(N_1^2\right)^{k-1} \cdot N_1^2$, and taking $\lambda_{k-1}^2 = \left(N_1^2\right)^{k-1}$ we can write:

$$\left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2 > \left(N_1^2\right)^k\right\}\right| = \left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2 > \lambda_{k-1}^2 N_1^2\right\}\right|. \tag{5.36}$$

Applying (2), we get:

$$\begin{aligned}&\left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2 > \left(N_1^2\right)^k\right\}\right| \\ &\quad \le \epsilon_1 \left(\left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2 > \lambda_{k-1}^2\right\}\right| + \left|\left\{x \in B_1,\ \mathcal{M}|f|^2 > \delta^2\lambda_{k-1}^2\right\}\right|\right).\end{aligned} \tag{5.37}$$

We can repeat the same argument for λ_{k-1}^2 till we reach $\lambda^2 = N_1^2$, which leads to:

$$\begin{aligned}&\left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2 > \left(N_1^2\right)^k\right\}\right| \\ &\quad \le \epsilon_1(\cdots \epsilon_1 \left(\left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2 > 1\right\}\right| + \left|\left\{x \in B_1,\ \mathcal{M}|f|^2 > \delta^2\right\}\right|\right) \\ &\qquad + \left|\left\{x \in B_1,\ \mathcal{M}|f|^2 > \delta^2\lambda_{k-1}^2\right\}\right|).\end{aligned} \tag{5.38}$$

Rearranging all the terms finally leads to (3). □

And finally we can prove the general result:

Proof. (Theorem 5.1) By multiplying, if necessary, u or f by a small constant, we can assume that $||f||_{L^p(B_4)} = \delta$ is small and that $\left|\left\{x \in B_1,\ \mathcal{M}|D^2u|^2 > N_1^2\right\}\right| \le \epsilon|B_1|$. If we can show that $\mathcal{M}|D^2u|^2 \in L^{p/2}(B_1)$, then we have that $D^2u \in L^p(B_1)$. By 5.9, if $f \in L^p$, then $\mathcal{M}|f|^2 \in L^{p/2}$, therefore considering the sums of squares to approximate the integral:

$$\begin{aligned}&\frac{N_1 - 1}{N_1} \sum_{i=1}^{+\infty} \left(\delta N_1^i\right)^p \left|\left\{\mathcal{M}|f|^2 > (\delta N_1^i)^2\right\}\right| \\ &\qquad = \sum_{i=1}^{+\infty} \left(\delta N_1^i - \delta N_1^{i-1}\right) (\delta N_1^i)^{p-1} \left|\left\{\mathcal{M}|f|^2 > (\delta N_1^i)^2\right\}\right| \\ &\qquad \le \int_0^\infty \lambda^{p-1} \left|\left\{\mathcal{M}|f|^2 > \lambda^2\right\}\right| \le \frac{1}{p}||\mathcal{M}|f|^2||_{L^{p/2}(B_1)} \le \frac{C}{p}||f||_{L^p(B_1)}.\end{aligned} \tag{5.39}$$

Rewriting leads to:

$$\sum_{i=1}^{+\infty} N_1^{ip} \left|\left\{\mathcal{M}|f|^2 > (\delta N_1^i)^2\right\}\right| \le \frac{CN_1}{p\delta^p(N_1 - 1)}||f||_{L^p(B_1)} \le C. \tag{5.40}$$

Hence using Corollary 5.7 and bounding the integral by rectangles, we can write that:

$$\begin{aligned}
&\fint_{B_1} \frac{|D^2u|^p}{p} \leq \sup_{B\subset B_1} \fint_B \frac{|D^2u|^p}{p} \leq \int_{B_1} \frac{\left(\mathcal{M}|D^2u|^2\right)^{p/2}}{p} \\
&= \int_0^\infty \lambda^{p-1} \left|B_1 \cap \{\mathcal{M}|D^2u|^2 \geq \lambda^2\}\right| \leq \sum_{k=0}^{+\infty} N_1^{kp} \left|B_1 \cap \left\{\mathcal{M}|D^2u|^2 \geq N_1^{2k}\right\}\right| \\
&\leq |B_1| + \sum_{k=1}^{+\infty} N_1^{kp} \left(\sum_{i=1}^{k} \epsilon_1^i \left|B_1 \cap \left\{\mathcal{M}|f|^2 > \delta^2 N_1^{2(k-i)}\right\}\right| + \epsilon_1^k \left|B_1 \cap \{\mathcal{M}|D^2u|^2 > 1\}\right| \right).
\end{aligned} \tag{5.41}$$

Rewriting the first term in the parentheses:

$$\begin{aligned}
&\sum_{k=1}^{+\infty} N_1^{kp} \sum_{i=1}^{k} \epsilon_1^i \left|B_1 \cap \left\{\mathcal{M}|f|^2 > \delta^2 N_1^{2(k-i)}\right\}\right| \\
&\qquad = \sum_{i=1}^{+\infty} N_1^{ip} \epsilon_1^i \sum_{k\geq i} N_1^{(k-i)p} \left|B_1 \cap \left\{\mathcal{M}|f|^2 > \delta^2 N_1^{2(k-i)}\right\}\right|.
\end{aligned} \tag{5.42}$$

Then the first inequality finally leads to :

$$\fint_{B_1} \frac{|D^2u|^p}{p} \leq |B_1| + 2C \sum_{i=1}^{+\infty} N_1^{ip} \epsilon_1^i, \tag{5.43}$$

where we have assumed that $\left|B_1 \cap \{\mathcal{M}|D^2u|^2 > 1\}\right|$ can be bounded with the L^2 - *estimate*. Taking ϵ_1 such that $\epsilon_1 N_1^p < 1$ finishes the proof.

□

What is interesting about this proof is that it shows a way of treating the L^p spaces. Since there is little information given by the functions, we have to find a way that is potentially "better". This is where the *Hardy-Littlewood maximal function* comes into play. This special function allows us to have a much more explicit treatment, since we can look at its absolute values at given points. The stability of it, as already mentioned, comes from the fact that its a *supremum* over the averaged integral around that point. Considering this, and the relationships with the L^p norms given by (5.9) and (5.10), it seems that it is natural to use this approach when trying to prove the estimate.

$\mathbf{L}^{\infty}$ case

The last case in this project is to see what happens with the *Calderón-Zygmund* estimate if $f \in L^{\infty}(B_1)$. As already mentioned, the estimates do not exactly hold for this particular case, but these can be rewritten. This is the reason why the *BMO* spaces have been introduced. We will see, in fact, that if $f \in L^{\infty}(B_1)$, then $u \in W^{2,BMO}(B_{1/2})$. What is interesting about the following proof, is that it follows a blow-up method, usually used for the C^{α} case, and up to our knowledge never applied for this type of problems. There are, of course, other ways to prove the same result, one of which is found in Ref. [7].

Proposition 6.1. *Let $u \in W^{2,BMO}(B_2)$ and $f \in L^{\infty}(B_2)$ such that $\Delta u = f$ in B_2. Then, for any δ there exists a constant C, depending only on δ and n, such that the following inequality holds:*

$$|D^2u|_{BMO(B_{1/2})} \;\leq\; \delta \cdot |D^2u|_{BMO(B_1)} \;+\; C \cdot (\; ||D^2u||_{L^1(B_1)} + ||f||_{L^{\infty}(B_1)}). \tag{6.1}$$

One can see, that this inequality is in fact very similar to the one found in the reference *Proposition 2.26*. Before seeing the proof of this proposition, we will first see a result for *BMO* functions when considering convergence in the L^2 norm. The proof follows a similar one from Lemma 3.4 in Ref. [1]:

Lemma 6.2. *Let Ω be a bounded domain and $(u_k)_{k\in\mathbb{N}}$, $u \in W^{1,2}(\Omega)$ a uniformly bounded sequence such that $u_k \longrightarrow u$ weakly in $W^{1,2}(\Omega)$. Then:*

$$|\nabla u|_{BMO(\Omega)} \;\leq\; \liminf_{k\to\infty} \; |\nabla u_k|_{BMO(\Omega)}. \tag{6.2}$$

Proof. Let $B \subset \Omega$ be any ball, we will first see that:

$$\fint_B |\nabla u - \overline{\nabla u}_B| \;\leq\; \liminf_{k\to\infty} \fint_B |\nabla u_k - \overline{\nabla u_k}_B|. \tag{6.3}$$

Consider $\eta \in C_c^\infty(B_1)$ smooth, with $\eta \geq 0$ and $\int_{B_1} \eta = 1$, and let $\eta_\epsilon = \epsilon^{-n}\eta(\frac{x}{\epsilon})$ so that the following convolutions are considered:

$$u_{k\,\epsilon}(x) = (u_k * \eta_\epsilon)(x) = \int_{B_\epsilon(x)} \eta(y)u_k(x-y)dy \quad and \quad u_\epsilon(x) = (u * \eta_\epsilon)(x). \tag{6.4}$$

Now, let $B' \subset B$ be such that for any $x \in B'$ we have that $B_\epsilon(x) \in B$. In particular, since $u_k \longrightarrow u$ in $W^{1,2}(\Omega)$ weakly, then the averages $\overline{u_k}_B \longrightarrow \overline{u}_B$ converge as well, as do the average over the gradients, and for every $x \in B'$ we have that $\nabla u_{k,\epsilon}(x) \longrightarrow \nabla u_\epsilon(x)$. This allows us to use *Fatou's Lemma* [4] over k:

$$\fint_{B'} |\nabla u_\epsilon - \overline{\nabla u_\epsilon}_B| \;\leq\; \liminf_{k\to\infty} \fint_{B'} |\nabla u_{k\,\epsilon} - \overline{\nabla u_{k\,\epsilon}}_B|. \tag{6.5}$$

We rewrite the right-hand side of the equation as:

$$\fint_{B'} |\nabla u_{k\,\epsilon} - \overline{\nabla u_{k\,\epsilon}}_B| \leq \fint_{B'} |\nabla u_{k\,\epsilon} - \overline{\nabla u_k}_B| \;+\; |\overline{\nabla u_k}_B - \overline{\nabla u_{k\,\epsilon}}_B|. \tag{6.6}$$

The last term of this equation is not in the integral, since it is constant with respect to it. Now we consider each term separately:

$$\begin{aligned}
\int_{B'} |\nabla u_{k\,\epsilon} - \overline{\nabla u_k}_B| \;&=\; \int_{B'} \Big| \int_{B_\epsilon(x)} \eta_\epsilon(x-y)(\nabla u_k(y) - \overline{\nabla u_k}_B)dy\Big|dx \\
&\leq \int_{B'} \int_{B_\epsilon(x)} \eta_\epsilon(x-y)|\nabla u_k(y) - \overline{\nabla u_k}_B|dydx \\
\leq \int_B |\nabla u_k(y) - \overline{\nabla u_k}_B| \int_{B_\epsilon(y)\cap B'} &\eta_\epsilon(x-y)dxdy \;\leq\; \int_B |\nabla u_k(y) - \overline{\nabla u_k}_B|dy.
\end{aligned} \tag{6.7}$$

In the last steps we have interchanged the order of the integrals, which leads to the redefinition of the sets over which we are integrating. On the second hand:

$$\begin{aligned}
\int_B \int_{B_\epsilon(x)} \nabla u_k(y)\eta_\epsilon(x-y)dydx \;&=\; \int_{\tilde{B}} \nabla u_k(y) \left(\int_{B_\epsilon(y)\cap B} \eta_\epsilon(x-y)dx \right) dy \\
&= \int_{B'} \nabla u_k + \int_{\tilde{B}\setminus B'} \nabla u_k(y) \left(\int_{B_\epsilon(y)\cap B} \eta(x-y)dx \right) dy.
\end{aligned} \tag{6.8}$$

Where $B \subset \tilde{B} \subset \Omega$ such that for every $x \in B$ we have that $B_\epsilon(x) \subset \tilde{B}$. This can be done since we can always choose ϵ at the start as small as desired for this condition to be true. Then by writing in terms of the average over B:

$$|\overline{\nabla u_{k\,\epsilon B}} - \overline{\nabla u_{kB}}| \;\leq \frac{1}{|B|}|\int_{\tilde{B}\setminus B'} \nabla u_k(y) \left(\int_{B_\epsilon(y)\cap B} \eta_\epsilon(x-y)dx\right) dy|$$
$$\leq \frac{1}{|B|}\int_{\tilde{B}\setminus B'} |\nabla u_k| \;\leq \frac{|\tilde{B}\setminus B'|^{1/2}}{|B|}||\nabla u_k||_{L^2(\tilde{B}\setminus B')} \;\leq \frac{\left((2\epsilon-\epsilon^2)|\tilde{B}|\right)^{1/2}}{|B|}||\nabla u_k||_{L^2(\Omega)}, \tag{6.9}$$

where we have used the fact that $\int_{B_\epsilon(x)} \eta_\epsilon(x)dx = 1$ and *Hölder's inequality* 2.4 for L^1 and L^2 norms. Putting all this together, we have that:

$$⨍_{B'} |\nabla u_{k\,\epsilon} - \overline{\nabla u_{k\,\epsilon B}}| \;\leq \frac{|B|}{|B'|}⨍_B |\nabla u_k - \overline{\nabla u_{kB}}| + \frac{\left((2\epsilon-\epsilon^2)|\tilde{B}|\right)^{1/2}}{|B|}||\nabla u_k||_{L^2(\Omega)}. \tag{6.10}$$

Since the sequence is uniformly bounded, then there exists $C \in \mathbb{R}$ such that $||\nabla u_k||_{L^2(\Omega)} \leq C^{1/2}$ and again by the weak convergence, we have that $\nabla u_\epsilon \longrightarrow \nabla u$ as $\epsilon \longrightarrow 0$ almost everywhere in B', again by *Fatou's Lemma* we can let $\epsilon \longrightarrow 0$ to deduce that:

$$⨍_{B'} |\nabla u - \overline{\nabla u_B}| \;\leq \liminf_{\epsilon\to 0} ⨍_B |\nabla u_\epsilon - \overline{\nabla u_{\epsilon B}}|$$
$$\leq\; \liminf_{\epsilon\to 0}\liminf_{k\to\infty} \frac{|B|}{|B'|}⨍_B |\nabla u_k - \overline{\nabla u_{kB}}| \;+ \frac{\left((2\epsilon-\epsilon^2)|\tilde{B}|C\right)^{1/2}}{|B|}. \tag{6.11}$$

If we now take into account that the measures of the sets and the second term of the right part do not depend on k, these limits can be rewritten as:

$$\liminf_{\epsilon\to 0}\frac{|B|}{|B'|}\liminf_{k\to\infty} ⨍_B |\nabla u_k - \overline{\nabla u_{kB}}| \;+ \liminf_{\epsilon\to 0}\frac{\left((2\epsilon-\epsilon^2)|\tilde{B}|C\right)^{1/2}}{|B|}$$
$$\longrightarrow\; \liminf_{k\to\infty} ⨍_B |\nabla u_k - \overline{\nabla u_{kB}}|, \tag{6.12}$$

which, after taking an increasing sequence of balls B' whose union is B eventually leads to our desired result:

$$⨍_B |\nabla u - \overline{\nabla u_B}| \;\leq\; \liminf_{k\to\infty} ⨍_B |\nabla u_k - \overline{\nabla u_{kB}}|. \tag{6.13}$$

The following remains true:

$$⨍_B |\nabla u - \overline{\nabla u_B}| \;\leq\; \liminf_{k\to\infty}\, \sup_{B\in\Omega} ⨍_B |\nabla u_k - \overline{\nabla u_{kB}}|. \tag{6.14}$$

Taking the supremum on the left-hand side, finally leads to:

$$|\nabla u|_{BMO(\Omega)} \leq \liminf_{k\to\infty} |\nabla u_k|_{BMO(\Omega)}. \tag{6.15}$$

□

At this point we have all the tools for the proof of proposition 6.1:

Proof. (Proposition 6.1) Following the steps in the reference, we will show it holds by contradiction. Let us suppose that there exists a $\delta \in \mathbb{R}$ such that for any constant $K \in \mathbb{R}$ there exist functions $u_k \in W^{2,BMO}(B_2)$ and $f_k \in L^\infty(B_2)$ satisfying $\Delta u_k = f_k$, and that (6.1) does not hold:

$$|D^2u_k|_{BMO(B_{1/2})} > \delta \cdot |D^2u_k|_{BMO(B_1)} + k \cdot (\, ||D^2u_k||_{L^1(B_1)} + ||f_k||_{L^\infty(B_1)}). \tag{6.16}$$

We now have to reach a contradiction. Choose a ball $\Omega_k \equiv B_{r_k}(x_k) \subset B_{1/2}$ and a $\beta \in \mathbb{R}$ such that:

$$\beta \cdot |D^2u_k|_{BMO(B_{1/2})} \leq \fint_{\Omega_k} |D^2u_k - \overline{D^2u_k}_{\Omega_k}|, \tag{6.17}$$

Observe that this implies that $r_k \to 0$ when $k \to \infty$ as:

$$\begin{aligned}
&\fint_{\Omega_k} |D^2u_k - \overline{D^2u_k}_{\Omega_k}| \leq \fint_{\Omega_k} |D^2u_k| + \fint_{\Omega_k} |\overline{D^2u_k}_{\Omega_k}| \\
&= \frac{1}{|\Omega_k|}||D^2u_k||_{L^1(\Omega_k)} + |\overline{D^2u_k}_{\Omega_k}| \leq \frac{1}{|\Omega_k|}||D^2u_k||_{L^1(\Omega_k)} + \frac{1}{|\Omega_k|}\int_{\Omega_k} |D^2u_k| \\
&\leq \frac{2||D^2u_k||_{L^1(B_1)}}{|\Omega_k|},
\end{aligned} \tag{6.18}$$

using (6.16) and (6.17):

$$\beta \cdot |D^2u_k|_{BMO(B_{1/2})} \leq \frac{2 \cdot |D^2u_k|_{BMO(B_{1/2})}}{|\Omega_k| k} \implies |\Omega_k| \leq \frac{2}{\beta k}. \tag{6.19}$$

Since $|\Omega_k| = |B_{r_k}(x_k)| = r_k^n$ we reach the stated result. Let us now define the following functions:

$$\tilde{u}_k(x) = \frac{u_k(x_k + r_k x) - p_k(x)}{r_k^2 \cdot |D^2u_k|_{BMO(B_1)}}, \qquad \tilde{f}_k(x) = \frac{f_k(x_k + r_k x) - \overline{f_k}_{\Omega_k}}{|D^2u_k|_{BMO(B_1)}}, \tag{6.20}$$

where $x \in B_{1/2r_k}$ and $p_k(y)$ is a quadratic polynomial chosen such that the following conditions are fulfilled:

$$\overline{\tilde{u}_k}_{B_1} = \overline{\nabla \tilde{u}_k}_{B_1} = \overline{D^2\tilde{u}_k}_{B_1} = 0, \tag{6.21}$$

meaning that it is of the form:

$$p_k(x) = (\overline{u_k}_{\Omega_k} + M) + r_k(\overline{\nabla u_k}_{\Omega_k} + N) \cdot x + x^T \cdot r_k^2 \left(\frac{\overline{D^2 u_k}_{\Omega_k}}{2} \right) \cdot x, \qquad (6.22)$$

with $M, N \in \mathbb{R}$ constants to be determined from (6.21). It is clear from the definition of $\tilde{u}_k$ and f_k that:

$$\Delta \tilde{u}_k = \frac{\Delta u_k(x_k + r_k x) - r_k^2 \cdot \overline{\Delta u_k}_{\Omega_k}}{r_k^2 \cdot |D^2 u_k|_{BMO(B_1)}} = \frac{r_k^2 \cdot f_k(x_k + r_k x) - r_k^2 \cdot \overline{f_k}_{\Omega_k}}{r_k^2 \cdot |D^2 u_k|_{BMO(B_1)}} = \tilde{f}_k(x) \ in \ B_{1/2r_k}. \qquad (6.23)$$

Notice that this new function has a bounded BMO norm:

$$\begin{aligned} |D^2 \tilde{u}_k|_{BMO(B_{1/2r_k})} &= \sup_{B_r(x) \subset B_{1/2r_k}} \fint_{B_r(x)} \frac{|D^2 u_k(x_k + r_r x) - \overline{D^2 u_k}_{B_r(x)}|}{|D^2 u_k|_{BMO(B_1)}} \\ &\leq \sup_{B_r(y) \subset B_1} \fint_{B_r(y)} \frac{|D^2 u_k(y) - \overline{D^2 u_k}_{B_r(y)}|}{|D^2 u_k|_{BMO(B_1)}} = \frac{|D^2 u_k|_{BMO(B_1)}}{|D^2 u_k|_{BMO(B_1)}} = 1. \qquad (6.24) \end{aligned}$$

We can see as well that its L^1 norm, is strictly larger than 0:

$$\begin{aligned} ||D^2 \tilde{u}_k||_{L^1(B_1)} &= \int_{B_1} \frac{|D^2 u_k(x_k + r_r x) - \overline{D^2 u_k}_{\Omega_k}|}{|D^2 u_k|_{BMO(B_1)}} = \frac{1}{r_k^n} \int_{\Omega_k} \frac{|D^2 u_k(y) - \overline{D^2 u_k}_{\Omega_k}|}{|D^2 u_k|_{BMO(B_1)}} \\ &= \frac{\fint_{\Omega_k} |D^2 u_k(y) - \overline{D^2 u_k}_{\Omega_k}|}{|D^2 u_k|_{BMO(B_1)}} \geq \frac{\beta \cdot |D^2 u_k|_{BMO(B_{1/2})}}{|D^2 u_k|_{BMO(B_1)}} > \beta\delta, \qquad (6.25) \end{aligned}$$

where we have used equations (6.16) and (6.17). Using *Hölder's inequality* 2.4 between the L^1 and the L^2 norms, as well as the *Calderón-Zygmund estimate* for *p=2* we reach:

$$\beta\delta < ||D^2 \tilde{u}_k||_{L^1(B_1)} \leq C \cdot ||D^2 \tilde{u}_k||_{L^2(B_1)} \leq C \cdot (||\tilde{u}_k||_{L^2(B_2)} + ||f_k||_{L^2(B_2)}). \qquad (6.26)$$

The constant C only depends on n and B_1. On the other hand, for a given $R > 1$ with $B_R \subset B_{1/2r_k}$:

$$\begin{aligned} ||\tilde{f}_k||_{L^\infty(B_R)} &= \frac{||f_k(x_k + r_k x) - \overline{f_k}_{\Omega_k}||_{L^\infty(B_R)}}{|D^2 u_k|_{BMO(B_1)}} \\ &\leq \frac{||f_k(x_k + r_k x)||_{L^\infty(B_R)} + ||\overline{f_k}_{\Omega_k}||_{L^\infty(B_R)}}{|D^2 u_k|_{BMO(B_1)}} \leq \frac{||f_k||_{L^\infty(B_1)} + ||\overline{f_k}_{\Omega_k}||_{L^\infty(B_R)}}{|D^2 u_k|_{BMO(B_1)}} \\ &\leq \frac{2 \cdot ||f_k||_{L^\infty(B_1)}}{|D^2 u_k|_{BMO(B_1)}} < \frac{2 \cdot |D^2 u_k|_{BMO(B_{1/2})}}{K \cdot |D^2 u_k|_{BMO(B_1)}} \leq \frac{2}{k}, \qquad (6.27) \end{aligned}$$

where we considered that the *Minkowski's inequality* 2.5 still holds for L^∞ if we assume that the limits on f_k exist. We used as well the definition of $\overline{f_k}_{\Omega_k}$, that $f_k \in L^\infty(B_1)$ and again (6.16). This means, that as $k \to \infty$ then $||\tilde{f}_k||_{L^\infty(B_{1/2r_k})} \to 0$ and, therefore, that $\tilde{f}_k \to 0$ strongly in L^∞. Moreover, for a large enough k, $\beta\delta < C \cdot ||\tilde{u}_k||_{L^2(B_2)}$.

Using the proposition 2.9 from Sobolev spaces, in particular since $W^{2,BMO} \subset W^{2,2}$ and that the sequence $D^2\tilde{u}_k$ is uniformly bounded in compact subsets $B_R \subset B_{1/2r_k}$ for $R > 1$:

$$\begin{aligned} ||D^2\tilde{u}_k||_{L^1(B_R)} &= \int_{B_R} |D^2\tilde{u}_k| \leq \int_{B_R} |D^2\tilde{u}_k - \overline{D^2u_k}_{B_R}| + \int_{B_R} |\overline{D^2u_k}_{B_R}| \\ &\leq R^n + \fint_{B_R} |D^2\tilde{u}_k| = R^n + \frac{1}{R^n}||D^2\tilde{u}_k||_{L^1(B_R)}, \end{aligned} \tag{6.28}$$

meaning that $||D^2\tilde{u}_k||_{L^1(B_R)} \leq \frac{R^{2n}}{R^n-1}$. By *Poincaré's inequality* we have a bound on $\tilde{u}_k$ as well. Then, there exists a weakly convergent subsequence in $W^{1,2}$ such that $\tilde{u}_k \to u$ and $\nabla\tilde{u}_k \to \nabla u$ strongly in L^2 while $D^2\tilde{u}_k \to D^2u$ weakly in L^2. It follows that the next properties are satisfied:

$$\overline{u}_{B_1} = \overline{\nabla u}_{B_1} = 0 \quad C \cdot ||u||_{L^2(B_2)} \geq \beta\delta. \tag{6.29}$$

$$|D^2u|_{BMO(\mathbb{R}^n)} \leq \liminf_{k\to\infty} |D^2\tilde{u}_k|_{BMO(B_{1/2r_k})} \leq 1. \tag{6.30}$$

Lemma 6.2 has been used for the *BMO* lower semi-continuity. considering the definition of weakly convergent given in proposition 2.9, if we take a test function $\eta \equiv 1$ then:

$$0 = \overline{D^2\tilde{u}_k}_{B_1} = \fint_{B1} D^2\tilde{u}_k \cdot 1 \longrightarrow \fint_{B_1} D^2u \cdot 1 = \overline{D^2u}_{B_1}, \tag{6.31}$$

Since the left hand side of the above convergence is equally 0, then $\overline{D^2u}_{B_1} = 0$. And again by the definition of a weak solution of $\Delta\tilde{u}_k = f_k$, given $\eta \in W^{1,2}$:

$$\int_{B_{1/2r_k}} \nabla\tilde{u}_k\nabla\eta \longrightarrow \int_{\mathbb{R}^n} \nabla u\nabla\eta = 0 \longleftarrow \int_{B_{1/2r_k}} \tilde{f}_k\eta. \tag{6.32}$$

This implies that $\Delta u = 0$ in $\mathbb{R}^n$, and hence u is harmonic in $\mathbb{R}^n$ and $u \in C^\infty(B_R)$ for any ball B_R. Moreover, considering the *mean value property* for harmonic functions 2.14 the above conditions can be rewritten as:

$$\overline{u}_{B_1} = \overline{\nabla u}_{B_1} = \overline{D^2u}_{B_1} = 0 \Longrightarrow u(0) = \nabla u(0) = D^2u(0) = 0. \tag{6.33}$$

If we now consider a ball $B_R(x) \subset \mathbb{R}^n$ sufficiently large such that $B_1 \subset B_R(x)$:

$$\frac{1}{R^n}||D^2u||_{L^1(B_R(x))} = \fint_{B_R(x)} |D^2u| \leq \fint_{B_R(x)} |D^2u - \overline{D^2u}_{B_R(x)}| + \fint_{B_R(x)} |\overline{D^2u}_{B_R(x)}| \leq 1 + |\overline{D^2u}_{B_R(x)}|. \quad (6.34)$$

This, combined with the fact that:

$$\frac{1}{R^n}|\overline{D^2u}_{B_R(x)}| = \frac{1}{R^n}|\overline{D^2u}_{B_1} - \overline{D^2u}_{B_R(x)}| \leq \frac{1}{R^n}\fint_{B_1} |D^2u - \overline{D^2u}_{B_R(x)}| \leq \fint_{B_R(x)} |D^2u - \overline{D^2u}_{B_R(x)}| \leq 1, \quad (6.35)$$

leads to $||D^2u||_{L^1(B_R(x))} \leq R^n(R^n+1)$. Using *Liouville's theorem 2.15* for harmonic functions defined on $\mathbb{R}^n$ with growth, we reach the conclusion that D^2u has to be a polynomial of degree at most n, which means that u will be a polynomial of degree n+2. But, since the BMO semi-norm of D^2u is bounded, if we use the same argument as above for a polynomial with just one term of degree $q \leq n$ in a ball B_R with $R > 1$ large enough, then:

$$\frac{1}{|B_R|}\int_{B_R} |D^2u| = \frac{M \cdot R^{q+n}}{R^n} = M \cdot R^q \leq |D^2u|_{BMO(B_R)} \leq 1, \quad (6.36)$$

will hold if and only if $q = 0$. Remembering the above mentioned properties, the only possibility is that $u \equiv 0$ and since $C \cdot ||u||_{L^2(B_2)} > \beta\delta$ we reach a contradiction as desired. □

Remark 6.3. In this proof we used *Liouville's Theorem* for a function that has a bounded L^1 norm, as has previously been introduced in the preliminaries section. Recall, that our bound was $||D^2u||_{L^1(B_R(x))} \leq R^n(1+R^n)$. Since u is harmonic in $\mathbb{R}^n$, then D^2u is harmonic as well. Now, if R is large enough, namely, that $B_R(x)$ is such that $B_1 \subset B_R(x)$, then we have that $R \geq 1 + |x|$. This leads to $|D^2u(x)| \leq (1+R^n) \approx C(1+|x|^n)$, which is the desired bound for the theorem.

At this point we have just proved the estimate for functions that are already in $W^{2,BMO}$. Again, following the steps in the references, two more propositions are needed in order to reach the final result. First of all, a new estimate, in this case for the BMO norm:

Proposition 6.4. *Let $u \in W^{2,BMO}(B_1)$ and $f \in L^\infty(B_1)$ such that $\Delta u = f$ in B_1. Then:*

$$||u||_{W^{2,BMO}(B_{1/4})} \leq C \cdot (||u||_{L^2(B_1)} + ||f||_{L^\infty(B_1)}), \quad (6.37)$$

where C is a constant only depending on n.

Proof. Following the first proof of theorem 2.20 in Ref. [1], let us define:

$$[\,D^2u]^{*}_{B_{1/2}} = \sup_{B_\rho(x_0)\subset B_{1/2}} (2\rho)^2 |D^2u|_{BMO(B_{\rho/2}(x_0))}. \tag{6.38}$$

Given a ball $B_\rho(x_0) \subset B_{1/2}$, we can cover $B_{\rho/2}(x_0)$ with N smaller balls $B_{\rho/8}(z_j)_{1\le j\le N}$, with $z_j \in B_{\rho/2}(x_0)$. Considering that $B_{\rho/2}(z_j) \subset B_1$, then by taking $r = \rho/2$ we get:

$$\left(\frac{\rho}{2}\right)^2 |D^2u|_{BMO(B_{\rho/8}(z_j))} \le \sup_{B_r(x_0)\subset B_{1/2}} r^2 |D^2u|_{BMO(B_{r/4}(x_0))}, \tag{6.39}$$

which leads to:

$$\begin{aligned} \rho^2|D^2u|_{BMO(B_{\rho/2}(x_0))} &\le C\cdot\rho^2 \sum_{j=1}^{N} |D^2u|_{BMO(B_{\rho/8}(z_j))} \\ &\le C\cdot 2^2 N \sup_{B_r(x_0)\subset B_{1/2}} r^2|D^2u|_{BMO(B_{r/4}(x_0))}. \end{aligned} \tag{6.40}$$

Here we have used the *sub-additivity* of integrals and that C is an integration constant not depending on ρ. By taking the supremum on the left hand side:

$$[\,D^2u]^{*}_{B_{1/2}} \le C \sup_{B_r(x_0)\subset B_{1/2}} r^2|D^2u|_{BMO(B_{r/4}(x_0))}. \tag{6.41}$$

Now, using the proposition (6.1) for a ball $B_\rho(x_0) \subset B_{1/2}$:

$$\begin{aligned} \rho^2|D^2u|_{BMO(B_{\rho/4}(x_0))} &\le \delta\cdot\rho^2|D^2u|_{BMO(B_{\rho/2})} + C\cdot(\,||D^2u||_{L^1(B_{1/2})} + ||f||_{L^\infty(B_{1/2})}) \\ &\le \delta[\,D^2u]^{*}_{B_{1/2}} + C\cdot(\,||D^2u||_{L^1(B_{1/2})} + ||f||_{L^\infty(B_{1/2})}). \end{aligned} \tag{6.42}$$

And again, taking the supremum on the left side, and using equation (6.41) as well as $|D^2u|_{BMO(B_{1/4})} \le [\,D^2u]^{*}_{B_{1/2}}$ (which can be seen for $\rho = 1/2$):

$$|D^2u|_{BMO(B_{1/4})} \le C\cdot(\,||D^2u||_{L^1(B_{1/2})} + ||f||_{L^\infty(B_{1/2})}). \tag{6.43}$$

Considering again *Hölder's inequality* 2.4 and the *Calderón-Zygmund estimate* for $p = 2$, we reach:

$$\begin{aligned} ||D^2u||_{BMO(B_{1/4})} &= |D^2u|_{BMO(B_{1/4})} + ||D^2u||_{L^1(B_{1/4})} \\ &\le C\cdot(\,||D^2u||_{L^1(B_{1/2})} + ||f||_{L^\infty(B_{1/2})}) \\ &\le C\cdot(\,||u||_{L^2(B_1)} + ||f||_{L^2(B_1)} + ||f||_{L^\infty(B_1)}), \end{aligned} \tag{6.44}$$

which after a proper constant leads to the desired result. □

Finally, we are ready to prove the estimate for the general case, which is a consequence of the above results. This Corollary, follows the same idea as Corollary 2.16 found in Ref. [1].

Corollary 6.5. *(Corollary 2.16 Ros-Oton) Let u be a weak solution to:*

$$\Delta u = f \text{ in } B_1, \tag{6.45}$$

with $f \in L^\infty(B_1)$. Then u is $W^{2,BMO}$ inside B_1 and the estimate (6.37) holds.

Proof. Let u be a solution in B_1 from $\Delta u = f$, with $f \in L^\infty(B_1)$. Let $\eta \in C_c^\infty(B_1)$ be any smooth function with $\eta \geq 0$ and $\int_{B_1} \eta = 1$. Re-scale η as:

$$\eta_\epsilon = \epsilon^{-n}\eta(\frac{x}{\epsilon}), \tag{6.46}$$

which satisfies $\eta_\epsilon \in C_c^\infty(B_\epsilon)$ and $\int_{B_\epsilon} \eta_\epsilon = 1$. Consider the convolution:

$$u_\epsilon(x) = u * \eta_\epsilon(x) = \int_{B_\epsilon} u(x-y)\eta_\epsilon(y)dy. \tag{6.47}$$

Recall that any convolution with a C^∞ function is C^∞ as well. In particular we have that:

$$\Delta u_\epsilon = f * \eta_\epsilon = f_\epsilon \quad in \quad B_{1-\epsilon}. \tag{6.48}$$

Since u_ϵ is C^∞ we can use proposition 6.37 to get:

$$||u_\epsilon||_{W^{2,BMO}(B_{1/4})} \leq C \cdot (||u_\epsilon||_{L^2(B_1)} + ||f_\epsilon||_{L^\infty(B_1)}). \tag{6.49}$$

Now, considering *Young's convolution inequality* 2.6 for L^p norms, we get that:

$$||u_\epsilon||_{L^2(B_1)} \leq ||u||_{L^2(B_1)} * ||\eta_\epsilon||_{L^1(B_1)} = ||u||_{L^2(B_1)}, \tag{6.50}$$

and:

$$||f_\epsilon||_{L^\infty(B_1)} \leq ||f||_{L^\infty(B_1)} * ||\eta_\epsilon||_{L^1(B_1)} = ||f||_{L^\infty(B_1)}. \tag{6.51}$$

This proves that the sequence u_ϵ is uniformly bounded in $B_{1/4}$ and using proposition 2.9 there is a subsequence that converges to $u \in W^{2,BMO}(B_{1/4})$. Meaning that:

$$||u||_{W^{2,BMO}(B_{1/4})} \leq C \cdot (||u||_{L^2(B_1)} + ||f||_{L^\infty(B_1)}), \tag{6.52}$$

because the limits for the L^2, as well as the BMO norm, hold 6.2. After proper re-scaling, we reach our desired result. □

This completes the proof of the *Calderón-Zygmund estimates* for $p \in [2, \infty]$. The remaining cases, namely for $p \in [1, 2)$, can be easily seen through a duality argument, which introduces the relationship between different L^p spaces. This is why the proof is simplified to the mentioned cases.

If we were to consider interpolation arguments [5, 6] as well, we could indeed proof the *Calderón-Zygmund estimates* just for the easier cases $p = 2$ and $p = \infty$, and use both results to see the validity for the general L^p case.

Final remarks

Since the beginning, the main goal of the project was to adapt or find a blow-up argument that could prove the general L^p case. There have been many difficulties involved when trying this type of proof, as it usually requires a type of point-wise interpretation. Of course, when treating with L^p functions, this is rather difficult, and proving an estimate of the sort of (6.1) seems rather hard. The reason behind it is that L^p functions are only integrable, and could be non-convergent in a 0 measure set. This is a problem, since we cannot choose smaller balls within our domain, where, for example, the L^p norm has a larger density. This probably helped to motivate the search and finding of the covering lemmas introduced in the L^p section chapter 5.

Another aspect to take into account, could be not to directly work with L^p norms. One could try to reach an inequality that involved the measurable sets $\{x \in \Omega, |\mathcal{M}|D^2u| > \lambda\}$ seen in the provided L^p proof, or considering maximal functions as the *Hardy-Littlewood* $\mathcal{M}u(x)$ or the *sharp* $\mathcal{M}^{\#}u(x)$ one introduced in Ref. [5]. The idea here, could be to bound the value of the maximal function at a given point. Nonetheless, one has to be careful when doing so, because these type of functions could have a value of ∞ at some points.

Further, another approach could have been to find a different inequality, which would not require a specific point. One example of such is the one introduced in *Proposition 4.5* found in Ref. [9]. The idea behind this one, is that we define a sequence based on the supremum and it's characteristics, avoiding us the use of δ as in Equation 6.1.

This short points remark a little bit the problems I have encountered when trying to prove the L^p case, but I believe that further work in the matter could provide a positive argument. In spite of that, it has been an interesting journey, full of joy and frustration at given points and a first immersion into mathematical research.

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