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Treball final de grau

DYNAMICS OF FINITE BLASCHKE PRODUCTS

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Acknowledgements

Before venturing into complex dynamics, I would like to thank all of those who made this project possible. Hence, I want to thank my family and friends, who supported me throughout these last six years and helped to make every single day enjoyable, even days fully spent in the library, which were not few. I hope these words are enough to thank you for everything you have given me throughout these years.

Nevertheless, I am certain I would never be able to find enough words to thank both my advisers, Núria and Anna. I feel rewriting these fifty pages from scratch and filling them with acknowledgements would not be enough, since it would not do justice to the help you have given me, even during Christmas.

Abstract

The aim of this project is to characterise the dynamics of finite Blaschke products, which are precisely the proper maps of the unit disk. It is proven that, inside the unit disk, all points converge to a unique point, the Wolff-Denjoy point. We build a classification of finite Blaschke products according to the position of the Wolff-Denjoy point and the dynamics around it.

Finally, we study the restriction of finite Blaschke products to the unit circle and calculate explicitly a conjugacy to z^d. We end this work by showing a brief example of generalised Blaschke products, a nuanced variation of the previous family that presents rich dynamical phenomena, such as the emergence of Herman rings.

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Contents

Introduction

Complex dynamics comprises the study of the iterates of holomorphic maps on Riemann surfaces. In other words, given a Riemann surface U and a holomorphic map $f : U \longrightarrow U$, our focus lies on the sequence

$$z, f(z), f^2(z), \dots$$

for $z \in U$, where $f^n = f \circ \overset{(n)}{\cdots} \circ f$ denotes the n-th iterate of f at z. This string of values is known as the ***orbit*** of z, and our ultimate goal is to understand the orbit of all points in U.

The first remarkable results in this field are relatively recent, from the beginning of the 20th century. In fact, prior to the 1910s, very little was written on the topic, mainly centred in the local behaviour around fixed points, aligning more with the field of functional equations rather than complex dynamics. The initial spark can be traced back to Pierre Fatou in 1906, with the study of rational maps with only one fixed point. With additional hypotheses, he showed that orbits for these functions converge to this unique fixed point everywhere in the complex plane except for a closed, perfect set.

Figure 1: An example of the earlier results obtained by Fatou in 1906. All points in the plane tend to the fixed point ∞, except for a closed, perfect (and in this case, totally disconnected) set. Red regions indicate convergence to ∞, and the yellow gradient indicates rate of convergence.

The first, more general results establishing the foundations of holomorphic dynamics came a few years later, spurred by the *Great Prize Award in Mathematics* of 1918 from the Paris Academy of Sciences. Motivated by this event, P. Fatou and Gaston Julia independently laid the groundwork for complex dynamics, working in the iteration of rational maps in the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Their cornerstone was the concept of ***normality***, introduced by

Paul Montel in 1911. Making use of this powerful technique, they divided the Riemann sphere into two completely invariant sets, now bearing their names, the ***Fatou set*** and the ***Julia set***. In essence, the Fatou set contains the points with stable behaviour under iteration, in the sense that every point has a neighbourhood in which the asymptotic behaviour of the orbit is the same. On the other hand, the Julia set is the complement of the Fatou set, with chaotic dynamics, where nearby points may present different orbits.

Consequently, the study of rational maps involves characterising these two sets and the behaviour of the orbits within them. As Fatou already hinted in his earlier work, the Julia set is closed and perfect. Thus, the Fatou set is open and composed of different connected components, the ***Fatou components***. This exploration continued for over half a century, when Dennis Sullivan, in 1985, proved the ***No Wandering Domain Theorem*** (Theorem 2.28), asserting that the orbits of Fatou components of a rational map must be eventually periodic. The pinnacle of this discussion came with the ***Classification Theorem*** (Theorem 2.30), providing the ultimate classification of these Fatou components.

Fatou and Julia's pioneering work with rational maps in the Riemann sphere outlined the field of complex dynamics. Today, however, holomorphic dynamics reaches its true extent with the study of more general holomorphic maps in more varied Riemann surfaces. In fact, due to the well-known ***Uniformization Theorem*** (Theorem A.12), the study of the dynamics of any holomorphic function in a given Riemann surface is covered by the analysis of the iteration in the three following cases: in the Riemann sphere, in the complex plane, and in the unit disk. Thus, rational iteration represents only one of these three possibilities. We shall now move our attention to the case of iteration in the unit disk,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Due to earlier results by Riemann, Poincaré and Koebe, this last case encompasses iteration on all simply connected Riemann surfaces whose boundary contains at least three points (see Riemann Mapping Theorem A.10).

If the precursor of rational iteration is the study of functional equations, then the bedrock of iteration in the unit disk is ***Schwarz Lemma*** from 1880 (Lemma A.1), and its later generalisation by Carathéodory from 1912. In this case, the climax arrived in 1926, when Julius Wolff and Arnaud Denjoy independently formulated the ***Wolff-Denjoy Theorem*** (Theorem 2.31), a result strong enough to characterise every holomorphic self-map of $\mathbb{D}$. It asserts that if a holomorphic map defined in the unit disk is not an automorphism, then there exists a unique point in $\overline{\mathbb{D}}$, the ***Wolff-Denjoy point***, towards which all orbits converge. Notably, the function is only required to be holomorphic in $\mathbb{D}$, with no assumption of extension to $\partial\mathbb{D}$.

Now, with this context in mind, we shall move into the matter of study of this project, a topic shaped by the two pinnacle results of complex dynamics, the Classification Theorem and the Wolff-Denjoy Theorem: ***finite Blaschke products***.

Finite Blaschke products

Finite Blaschke products of degree $k \geq 1$ are maps of the form

$$B(z) = e^{i\theta} \prod_{l=1}^{k} \frac{z - w_l}{1 - \overline{w_l} z}, \qquad \text{for } \theta \in \mathbb{R}, w_l \in \mathbb{D}, z \in \mathbb{D}.$$

They are named after ***Wilhelm Blaschke***, who introduced them in 1915 in the limit case where k approaches ∞, where they hold significant relevance in the context of Hardy spaces and inner functions. Nevertheless, our focus in this work lies in their finite form, considering a finite degree $k \geq 1$. They are not only rational maps of degree k but also self-maps of the unit disk, since, as one may notice from their definition, they are products of automorphisms of $\mathbb{D}$. From this duality stems the interesting dynamics of Blaschke products: they conform the intersection between iteration in the Riemann sphere and iteration in the unit disk. And here lies the goal of our work: to combine the results of Fatou and Julia with the results of Wolff and Denjoy, in order to give an accurate description of the dynamics of this family of maps.

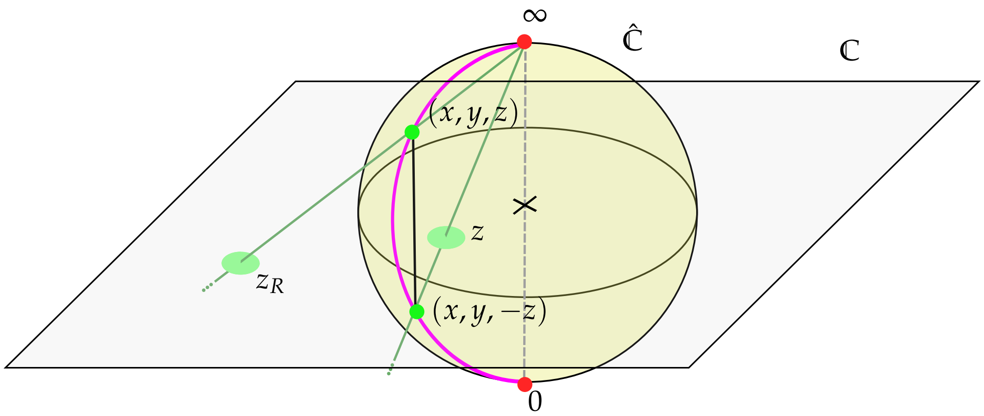


Figure 2: Sketch of the reflection property of finite Blaschke products. Each point $z \in \mathbb{D}$ has a reflection point $z_R = 1/\overline{z}$, and the extension of finite Blaschke products satisfies $B(z)\overline{B(z_R)} = 1$. Interestingly, considering the stereographic projection, in the Riemann sphere $\hat{\mathbb{C}}$ these points are reflections on the z-plane.

Before venturing into their dynamics, it should be noted that the definition of finite Blaschke products restricts them to the unit disk. However, as previously mentioned, they extend through the boundary as rational maps, holomorphic in all the Riemann sphere. The interesting aspect about this is that this extension to the whole Riemann sphere is not arbitrary: the behaviour of Blaschke products outside the unit disk is a mirror image of their behaviour within it. This phenomenon is known as the ***reflection property*** (Proposition 1.4) and it stems from an earlier result by Schwarz (Theorem A.4). Formally, it takes the form

$$B(z) = \frac{1}{\overline{B(1/\overline{z})}}, \text{ for all } z \in \hat{\mathbb{C}}.$$

Another important quality that can be hinted at first glance is that, for $k = 1$, finite Blaschke products conform the family of automorphisms of $\mathbb{D}$. Noteworthy, this can be generalised for higher degree. In particular, for $k \geq 2$, finite Blaschke products are precisely the family of ***holomorphic proper maps*** of the unit circle. In other words, they are the only holomorphic maps such that every point in $\mathbb{D}$ has exactly k preimages (counted with multiplicity). Another defining property of this family is their extension to the unit circle. Indeed, Fatou showed in 1923 (Theorem 1.6, [Fat23]) that any holomorphic self-map of $\mathbb{D}$ that can be extended continuously to $\partial\mathbb{D}$ leaving both $\mathbb{D}$ and $\partial\mathbb{D}$ invariant is a finite Blaschke product.

Thus, it is clear that finite Blaschke products (of degree $k \geq 2$) serve as an excellent common thread in the exploration of complex dynamics. Now, the main question remains: what can we say about the dynamics of this family?

Of course, the answer to this question depends in the characteristics of the Wolff-Denjoy point, $z_0 \in \overline{\mathbb{D}}$. According to this point, we can build a classification of finite Blaschke products into four different types: ***elliptic, hyperbolic, simply parabolic and doubly parabolic***. In the elliptic case, z_0 is inside the unit disk and orbits inside $\mathbb{D}$ converge to it, while those in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ tend to its reflected point $1/\overline{z_0}$. In the other three cases, $z_0 \in \partial\mathbb{D}$, and hence all orbits in both $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ converge to $z_0 = 1/\overline{z_0}$. The difference between these three cases lies in nature of this convergence.

With this classification, we are able to fully characterise the dynamics of finite Blaschke products. In particular, we have the following result (Theorem 3.6).

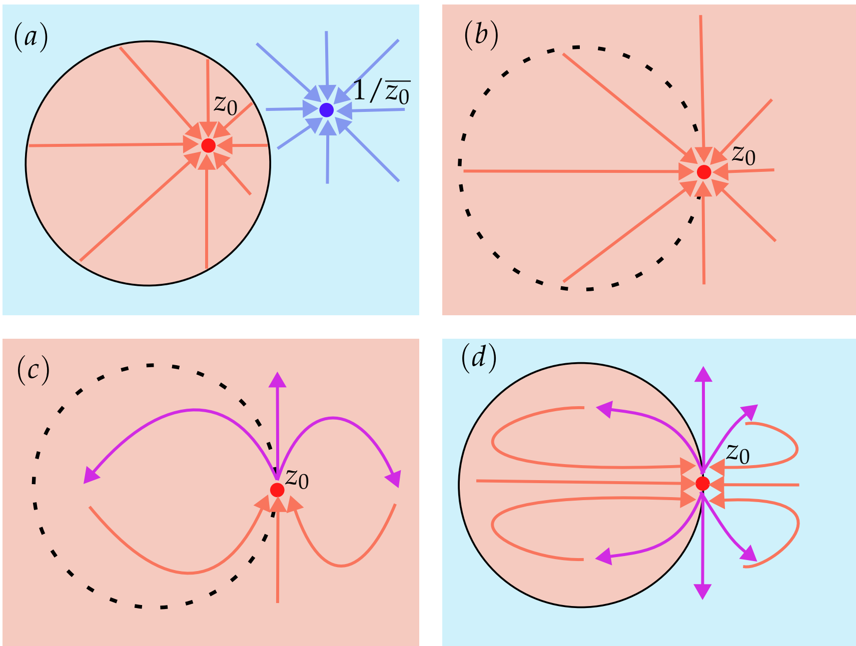


Figure 3: Characterisation of the dynamics of FBP. (a) shows the elliptic case, where the Fatou set has two connected components, $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. (b) and (c) show the hyperbolic and simply parabolic cases, respectively, where the Fatou set has only one connected component and the Julia set is a Cantor set of $\partial\mathbb{D}$. Finally, (d) shows the doubly parabolic case, where the Fatou set has again two connected components, $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Theorem (Characterisation of the dynamics of FBP). *Let B be a finite Blaschke product of degree $d \geq 2$. Then, the following holds.*

(a) If B is elliptic, then the Julia set of B is $\partial\mathbb{D}$ and the Fatou set of B consists of two invariant connected components, $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

(b) If B is hyperbolic, then the Julia set of B is a Cantor subset of $\partial\mathbb{D}$ and the Fatou set of B consists of one invariant connected component.

(c) If B is simply parabolic, the Julia set of B is a Cantor subset of $\partial\mathbb{D}$ and the Fatou set of B consists of one invariant connected component.

(d) If B is doubly parabolic, the Julia set of B is $\partial\mathbb{D}$ and the Fatou set of B consists of two invariant Fatou components, $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

See Fig. 3. This concise yet conclusive result entirely determines the dynamical partition of the Riemann sphere for Blaschke products. The Fatou set is composed of either one or two invariant Fatou components, where all orbits converge to a fixed point. In contrast, the Julia set is either connected, spanning the whole unit circle, or totally disconnected, being a Cantor subset of $\partial\mathbb{D}$. With this in mind, the natural next step towards characterising the dynamics of this family of maps involves investigating their behaviour on the unit circle.

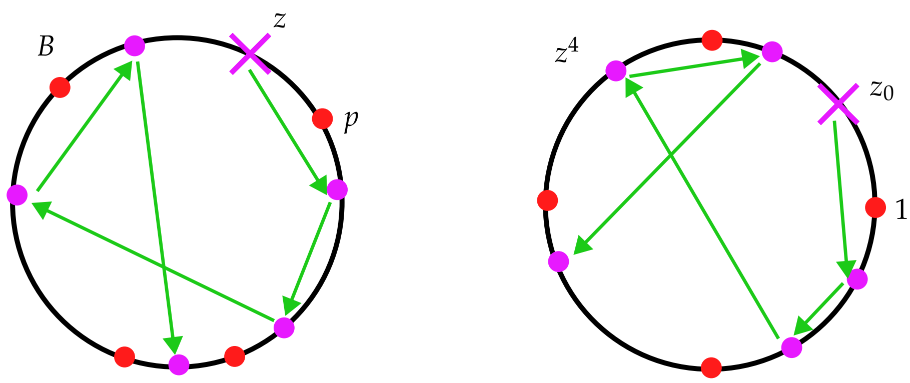
In this regard, previous work by Michael Shub with expanding maps from 1969 ([Shu69]) provides us with useful tools. Notably, Shub's Theorem (Theorem 4.3) establishes a unique ***semi-conjugacy*** between any finite Blaschke product of degree d, say B, and the map z^d in the unit circle. In other words, it guarantees the existence of a unique (not necessarily strictly) monotonous, surjective and continuous map $h : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ such that $h \circ B(z) = (h(z))^d$ for $z \in \partial\mathbb{D}$. Combining this result with the powerful properties of the Julia set, we conclude that, in the cases where the Julia set spans the entire unit circle, this semi-conjugacy is in fact a ***topological conjugacy***, i.e. it is a continuous, bijective map with continuous inverse. Specifically, we have the following (Theorem 4.4).

Theorem (Shub's conjugacy)**.** *Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$. Then, there exists a conjugacy to z^d in $\partial\mathbb{D}$.*

However, the richness of this result extends further. Taking profit of the remarkable analytic properties of finite Blaschke products, we can explicitly determine this conjugacy in the elliptic and doubly parabolic case. Roughly speaking, we identify points whose orbit under B and z^d visit similar regions of the unit circle (see Fig. 4). This approach yields an explicit form of Shub's conjugacy through the ***itineraries*** of points, an instance of what is known as ***symbolic dynamics***.

At this point, one might wonder the relevance of this result within complex dynamics, and whether we have ventured too far from dynamics into analysis. Far from it, this result provides us with a ***conjugacy***, the fundamental tool in the study of dynamics. As a matter of fact, conjugacies preserve many essential dynamical properties, and hence, conjugating a finite Blaschke product B to a simpler map such as z^d is an excellent way of determining the dynamics of B. For instance, periodic points are preserved through conjugacies, and it is straight-forward to find that the periodic points of period $n \geq 1$ of z^d are precisely those of the form

$$p_{n,k} = e^{2\pi i r_{n,k}}, \quad \text{where } r_{n,k} = \frac{k}{d^n - 1}, \quad \text{for } k \in \{1, \ldots, d^n - 1\} \quad \text{and } k \not| d^n - 1.$$



$$S(z) = S_0(z_0) = 144213\ldots$$

Figure 4: Sketch of the conjugacy between an elliptic or doubly parabolic FBP B of degree 4 and z^4. We divided the unit circle into four numbered regions, and identified points whose orbits visited the same regions. The sequences $S(z)$ and $S_0(z_0)$ hold the information of which regions are visited.

Finally, once the behaviour of all orbits of finite Blaschke products is understood, one is tempted to seek a generalisation of these results to a family of maps exhibiting a similar structure. One possibility is to consider functions that still adhere to the reflection property, but not do not necessarily leave the unit disk invariant. This is accomplished by letting the zeros of finite Blaschke products to be ***outside the unit disk***. With this seemingly subtle modification, most of the previous results no longer apply and the dynamics of Blaschke products undergoes a significant transformation. The primary reason behind this drastic change lies in the existence of poles inside the unit disk, which invalidates the use of our main tool until now, the Wolff-Denjoy Theorem. We refer to this new family of functions as ***generalised finite Blaschke products***, and they, despite the change, remain rational maps. Consequently, the Classification Theorem still applies, affording us a glimpse into their emerging dynamics. See Fig. 5.

To delve into this exploration, we focus on a specific family of such generalised finite Blaschke products, taking the form

$$B_{\theta,a} = e^{i\theta} z^2 \frac{z-a}{1-az}, \text{ for } \theta \in \mathbb{R}, a > 1.$$

In this project, using the power of the Classification Theorem and drawing on intriguing results from number theory, we show that for certain pairs of parameters (θ, a), there exists an annular region spanning around the unit circle on which the dynamics is essentially that of an irrational rotation. This region is known as a ***Herman ring***, and, even though they were first conjectured to not exist for rational maps by Fatou and Julia themselves, their existence was proven in 1984 by Michael Herman in [Her84], using precisely generalised finite Blaschke products of odd degree as an example.

Structure of the project

The basic results from complex analysis and topology are assumed throughout the work. Nevertheless, we have included a short Appendix as a reminder, including some theorems and definitions that appear recurrently throughout the project.

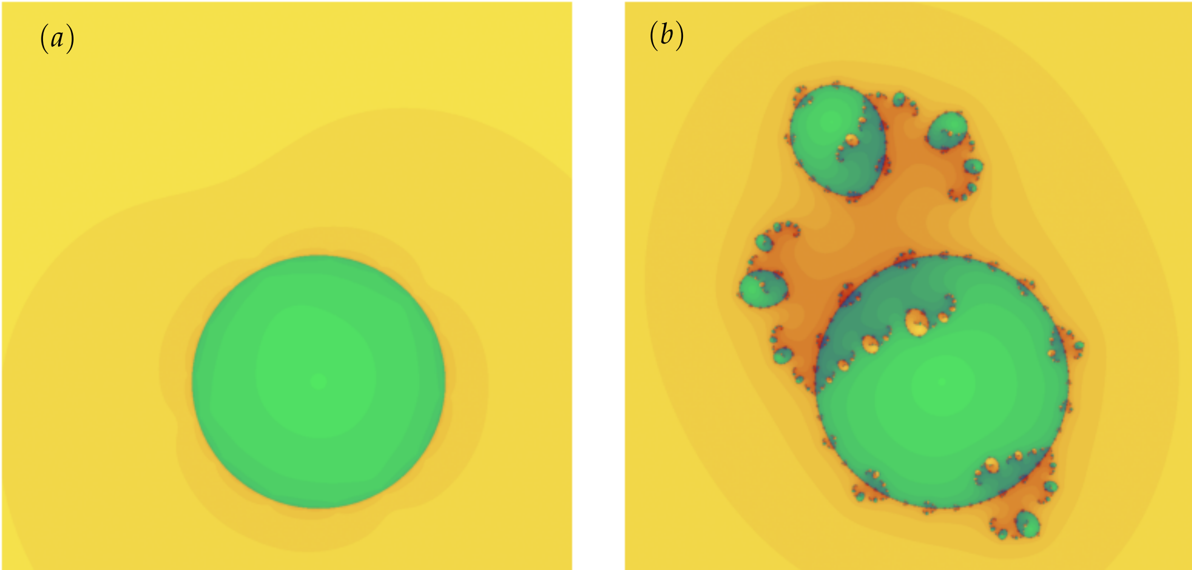


Figure 5: Example of the Fatou sets of the Blaschke product $B_w(z) = z^3 \frac{z-w}{1-\overline{w}z}$ for $w = a \in \mathbb{D}$ (elliptic Blaschke product, (a)), and for $w = 1/\overline{a} \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ (generalised Blaschke product, (b)), for $a = -0.1749 + 0.4129i$. Iterates in the green region converge to 0 and in the orange region to ∞, and the gradient represents the rate of convergence. Note that the elliptic case is extremely simple, and the only common property between both cases is the reflection property.

First, we devote Section 1 to the study of self-maps of the unit disk, and, in particular, to finite Blaschke products. We include the most relevant properties of this family of maps, which will be relevant when discussing their dynamics.

The second chapter is designed as a first introduction in complex dynamics. The Julia and Fatou sets, fixed points and the different types of Fatou components are formally defined, and the Classification Theorem is stated. We end the chapter with a review of the results characterising iteration in the unit disk. Some, but not all, proofs are included, since general theory is not the goal of the project.

In Chapter 3 we start discussing the dynamics of finite Blaschke products. We motivate their classification and prove the theorem characterising their Julia and Fatou sets.

Then, Chapter 4 ties the loose end of the third chapter, the behaviour in the unit disk and contains our main results. We introduce and prove Shub's Theorem, and, with a short introduction to symbolic dynamics, we explicitly find the conjugacy between elliptic and doubly parabolic finite Blaschke products and z^d in the unit circle.

Finally, the last chapter is to be taken as a short example that the dynamics of generalised finite Blaschke products presents essential differences with respect to the ordinary ones. We give some background in number theory and introduce a result determining the existence of Herman rings for certain maps. Afterwards, we prove that, for certain parameters, a certain family of degree 3 generalised finite Blaschke products satisfies the hypotheses of this result, and hence has a Herman ring.

Chapter 1

Holomorphic self-maps of the unit disk

We develop this chapter to the study of holomorphic self-maps of the ***unit disk***. Concerning notation, we denote the ***unit disk*** as:

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and refer to its boundary $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ as the ***unit circle***, and to its complement $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ as the ***exterior disk***.

First, we study the family of holomorphic automorphisms of $\mathbb{D}$, which turns out to reduce to two specific families of Möbius transformations. Subsequently, we introduce finite Blaschke products as the products of these Möbius maps. Then, we study some of their analytic properties, such as zeros, fixed points and their derivatives. Finally, we consider a generalisation of such maps, generalised finite Blaschke products.

For a deeper background in the concepts presented in this chapter, check [GMR18, Chapter 3], where most of the material in this chapter comes from.

1.1 Biholomorphisms of the unit disk

Our first step is to characterise holomorphic automorphisms of $\mathbb{D}$, i.e. biholomorphisms. We start by introducing two families of functions in $\mathbb{D}$.

Definition 1.1 (Rotations and Blaschke factors)**.** Let $\theta \in \mathbb{R}$ and $w \in \mathbb{D}$. We define the ***rotation of angle*** θ and the ***Blaschke factor of zero*** w as the maps:

$$\begin{aligned} \rho_\theta : \mathbb{D} &\longrightarrow \mathbb{D} \\ z &\longmapsto e^{i\theta} z \end{aligned} \qquad \text{and} \qquad \begin{aligned} \beta_w : \mathbb{D} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{z-w}{1-\overline{w}z}, \end{aligned}$$

respectively.

See Fig. 1.1 for a visual representation of these maps.

Theorem 1.2. *Let $Aut_{\mathcal{H}}(\mathbb{D})$ be the set of biholomorphisms of $\mathbb{D}$. Then,*

$$Aut_{\mathcal{H}}(\mathbb{D}) = \{\rho_\theta \circ \beta_\omega : \theta \in \mathbb{R}, \omega \in \mathbb{D}\},$$

where ρ_θ is the rotation of angle θ and β_w is the Blaschke factor of zero w.

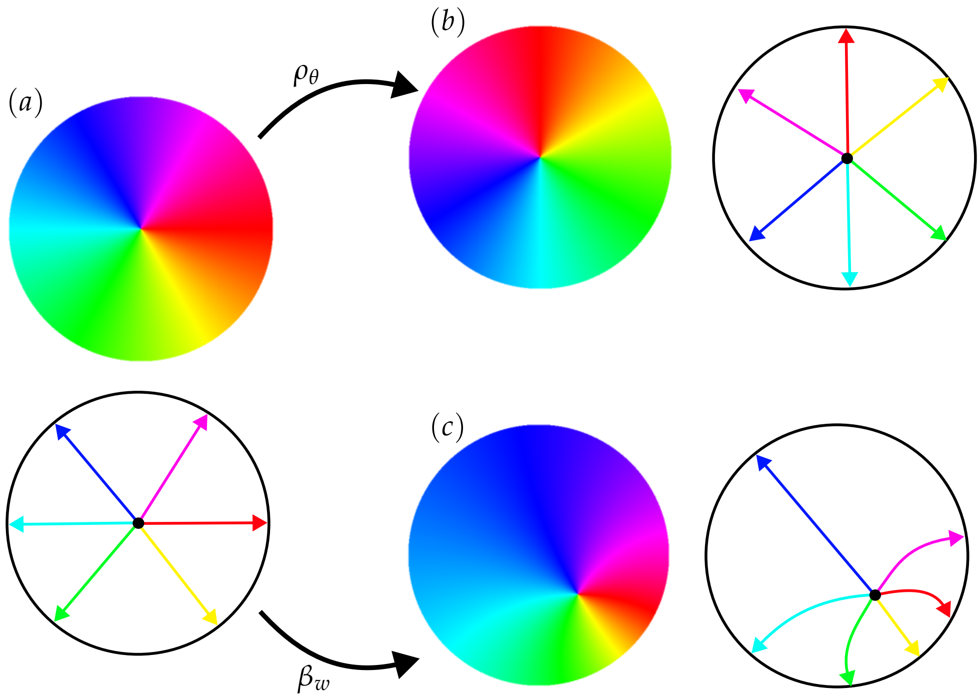


Figure 1.1: Action of ρ_θ and β_w over the unit disk. (a) represents the unit disk, (b) the image of the unit disk under ρ_θ for $\theta = \pi/2$ and (c) under β_w for $w = 3(1+i)/10$.

Proof. First, we shall see that maps of the form $\rho_\theta \circ \beta_w$ are biholomorphisms of $\mathbb{D}$. It is clear that for $\theta \in \mathbb{R}$, ρ_θ is a biholomorphism of the unit disk with inverse $\rho_{-\theta}$, since $|\rho_\theta(z)| = \left|e^{i\theta}z\right| = |z| < 1$ and $\rho_\theta(\rho_{-\theta}(z)) = z$ for $z \in \mathbb{D}$.

Next, for $w \in \mathbb{D}$, note that β_w is a Möbius transformation (see A.2). Hence, β_w is biholomorphic in $\hat{\mathbb{C}}$. We have, for $t \in \mathbb{R}$

$$\left|\beta_w(e^{it})\right| = \left|\frac{e^{it} - w}{1 - \overline{w}e^{it}}\right| = \frac{1}{|e^{it}|}\frac{|e^{it} - w|}{|e^{-it} - \overline{w}|} = 1, \tag{1.1}$$

and so, β_w sends $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Since $\beta_w(w) = 0$, we have that β_w must send $\mathbb{D}$ to $\mathbb{D}$. Thus, we have that β_w must be a biholomorphism of $\mathbb{D}$. It is direct to see that its inverse is β_{-w}.

It is left to see that any biholomorphism of $\mathbb{D}$ is of the form $\rho_\theta \circ \beta_w$, for some $\theta \in \mathbb{R}$ and $w \in \mathbb{D}$. Let f be a biholomorphism of $\mathbb{D}$. In particular, it must have a holomorphic inverse f^{-1}. Let $w = f^{-1}(0) \in \mathbb{D}$ and consider the Blaschke factor of zero $-w$ i.e. β_{-w}. Consider the self-map of $\mathbb{D}$ defined as $g = f \circ \beta_{-w}$.

Clearly, g is well-defined, holomorphic and satisfies $g(0) = f(\beta_{-w}(0)) = f(w) = 0$. Applying Schwarz Lemma A.1, we have $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$.

Moreover, since g is a composition of biholomorphisms, it must be, indeed, a biholomorphism. So, $g^{-1} = \beta_w \circ f^{-1}$ is well-defined and holomorphic. Furthermore, it satisfies $g^{-1}(0) = \beta_w(f^{-1}(0)) = \beta_w(w) = 0$. Thus, applying Schwarz Lemma A.1 again, we have $\left|g^{-1}(z)\right| \leq |z|$ for all $z \in \mathbb{D}$. Hence, we must have, for all $z \in \mathbb{D}$,

$$|z| = \left|g^{-1}(g(z))\right| \leq |g(z)| \leq |z|, \quad \text{and so} \quad |g(z)| = |z| \text{ for all } z \in \mathbb{D}.$$

Thus, again as a consequence of Schwarz Lemma A.1, we have that for all $z \in \mathbb{D}$ $g(z) = e^{i\theta}z = \rho_\theta(z)$, for some $\theta \in \mathbb{R}$. Hence, we have $f \circ \beta_{-w} = \rho_\theta$, and so $f = \rho_\theta \circ \beta_w$, as we wanted to see. □

1.2 Proper self-maps of the unit disk

Next, we move from bijective maps into higher degree self-maps of $\mathbb{D}$.

Definition 1.3 (Finite Blaschke product)**.** Let $\theta \in \mathbb{R}$ and $w_1, \dots, w_k \in \mathbb{D}$ for some integer $k \geq 1$. The ***finite Blaschke product*** (FBP) of rotation θ and zeros $w_1, \dots, w_k$ is the function $B : \mathbb{D} \longrightarrow \mathbb{D}$ defined as

$$B(z) = \rho_\theta \circ \left(\prod_{l=1}^{k} \beta_{w_l}(z) \right) = e^{i\theta} \prod_{l=1}^{k} \frac{z - w_l}{1 - \overline{w_l} z}. \tag{1.2}$$

We call $k \geq 1$ the ***degree*** of B.

It is clear from the last section 1.1 that $B(\mathbb{D}) = \mathbb{D}$ and hence B is a holomorphic self-map of $\mathbb{D}$. In particular, finite Blaschke products of degree $k = 1$ are biholomorphisms of the form $\rho_\theta \circ \beta_w$.

Note that the definition of FBP restricts their domain to $\mathbb{D}$. However, as these products are indeed rational functions, they can be extended meromorphically to $\mathbb{C}$, or, in fact, holomorphically to the Riemann sphere $\hat{\mathbb{C}}$. Furthermore, this extension is given as a reflection about the unit circle. Formally, we have the following result.

Proposition 1.4 (Reflection property)**.** *Let $B : \mathbb{D} \longrightarrow \mathbb{D}$ be a finite Blaschke product. Then, B can be extended holomorphically to a rational function $B : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, satisfying*

$$\overline{B(z)} = \frac{1}{B(1/\bar{z})}, \quad \forall z \in \hat{\mathbb{C}}.$$

Proof. First, note that B is indeed a rational function (it is clear from Def. 1.3). It is left to see that it satisfies the property of reflection. To do so, we shall find an appropriate map ϕ such that $\phi \circ B \circ \phi^{-1}$ satisfies the hypothesis of Schwarz Reflection Principle A.4.

Consider the Möbius transformation (see Appendix A.2):

$$\phi(z) = -i\frac{z-1}{z+1}, \quad \phi^{-1}(z) = \frac{i-z}{i+z}$$

Since Möbius transformation send circles to circles or straight lines, we have that $\phi(\partial\mathbb{D}) = \mathbb{R}$, as $\phi(1), \phi(i), \phi(-i) \in \mathbb{R}$. Since $\phi(0) = i$, we can conclude that $\phi(\mathbb{D}) = \mathbb{H}_+$ and $\phi(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}) = \mathbb{C} \setminus \mathbb{H}_+$, where $\mathbb{H}_+ := \{z \in \hat{\mathbb{C}} : \text{Im}(z) > 0\}$ is the ***upper half plane*** (see Fig. 1.2).

Now, consider a FBP $B : \mathbb{D} \longrightarrow \mathbb{D}$. We can define $f := \phi \circ B \circ \phi^{-1} : \mathbb{H}_+ \longrightarrow \mathbb{H}_+$. As a composition of holomorphic functions, f must be holomorphic in $\mathbb{H}_+$. Furthermore, since B extends continuously to $\partial\mathbb{D}$, f must do so for $\mathbb{R}$. Then, by Schwarz Reflection Principle A.4, f has a unique extension $F : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ defined as

$$F(z) = \overline{F(\bar{z})}, \quad \text{for } z \in \mathbb{C} \setminus \mathbb{H}_+$$

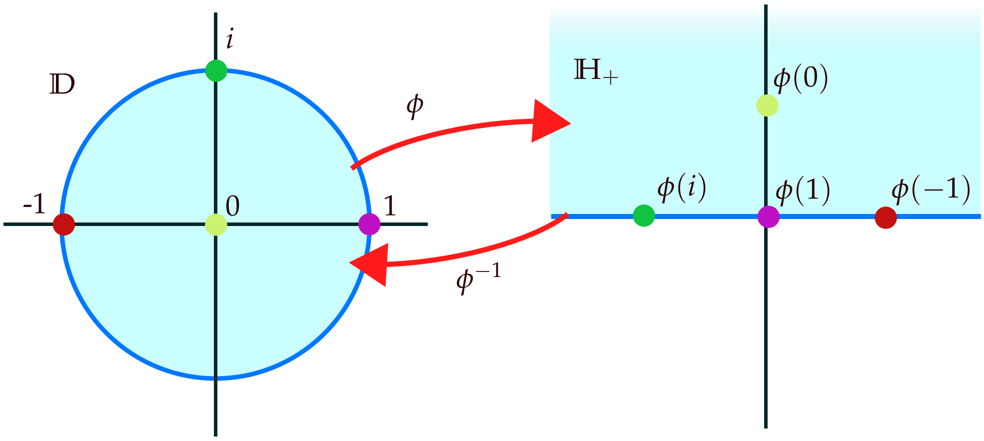


Figure 1.2: Sketch of the Möbius transformation $\phi : \mathbb{D} \longrightarrow \mathbb{H}_+$.

Next, consider the function $\phi^{-1} \circ F \circ \phi : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$. Note that for $z \in \mathbb{D}$, this function coincides with B, in other words, it is an extension of B. As a consequence of the Identity Principle A.3, it must be the only possible holomorphic extension. Furthermore, for $z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, we have

$$\phi^{-1} \circ F \circ \phi(z) = \phi^{-1} \circ \overline{f} \circ \overline{\phi(z)} = \phi^{-1} \circ \overline{\phi} \circ B \circ \phi^{-1} \circ \overline{\phi(z)}.$$

Finally, note that $\phi^{-1} \circ \overline{\phi(z)} = 1/\overline{z}$, and so:

$$\phi^{-1} \circ F \circ \phi(z) = \frac{1}{\overline{B(1/\overline{z})}}, \quad \text{for } z \in \hat{\mathbb{C}} \setminus \mathbb{D},$$

as we wanted to see. □

Another interesting property of finite Blaschke products is properness, i.e. every point in $\mathbb{D}$ has exactly k preimages, where k is the degree of the FBP. More details can be found in A.5.

Proposition 1.5. *Let B be a finite Blaschke product of degree k. Then, $B|_{\mathbb{D}}$ (resp. $B|_{\partial\mathbb{D}}$ and $B|_{\hat{\mathbb{C}}\setminus\overline{\mathbb{D}}}$) is a proper self-map of degree k of $\mathbb{D}$ (resp. of $\partial\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$).*

Proof. We want to see that every point in $\mathbb{D}$, $\partial\mathbb{D}$ or $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ has exactly k preimages in the same set, counted with multiplicity.

Hence, consider $a \in \mathbb{C}$. The preimages of a are exactly the solutions of the equation $B(z) = a$. Writing B in the form (1.2) for some $\theta \in \mathbb{R}, w_1, \dots, w_k \in \mathbb{D}$, the equation $B(z) = a$ is equivalent to

$$e^{i\theta} \prod_{l=1}^{k} (z - w_l) = a \prod_{l=1}^{k} (1 - \overline{w_l} z),$$

which is a polynomial equation of degree k. Then, a must have exactly k preimages in $\mathbb{C}$, counted with multiplicity.

Next, we have to see that these preimages are in the same set as a. Recall that $B(\mathbb{D}) = \mathbb{D}$. Hence, applying the Reflection Property 1.4, we know that:

$$B(z) = \frac{1}{\overline{B(1/\overline{z})}} \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}, \quad \text{for } z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}},$$

$$B(z) = \frac{1}{\overline{B(z)}} \text{ and so } |B(z)|^2 = 1, \quad \text{for } z = \frac{1}{\overline{z}} \in \partial\mathbb{D}.$$

In other words, $B(\mathbb{D}) = \mathbb{D}, B(\partial\mathbb{D}) = \partial\mathbb{D}$ and $B(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}) = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Then, if $a \in \mathbb{D}$ (or $\partial\mathbb{D}, \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$), all k preimages of a must also be in $\mathbb{D}$ (or, respectively, $\partial\mathbb{D}, \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$), as we wanted to see.

For the point $a = \infty \in \hat{\mathbb{C}}$, note that their preimages are exactly the poles of B, and these are the k values $1/\overline{w_l} \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. □

Therefore, finite Blaschke products of degree $k \geq 2$ are surjective self-maps of the unit disk $\mathbb{D}$, but they are not one-to-one. In fact, they cover $\mathbb{D}$ exactly k times. We will now see that, in the same way as FBP of degree 1 generated the biholomorphisms of $\mathbb{D}$, FBP are precisely the family of proper self-maps of $\mathbb{D}$.

Theorem 1.6 (Fatou's Theorem)**.** *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic self-map of the disk into itself such that*

$$\lim_{|z|\to 1^-} |f(z)| = 1$$

Then, f is a finite Blaschke product. Consequently, any holomorphic proper self-map of $\mathbb{D}$ is a FBP.

Proof. As a consequence of the Identity Theorem A.3, if f has infinitely many zeros in $\mathbb{D}$, they must accumulate in $\partial\mathbb{D}$. However, as $\lim_{|z|\to 1^-} |f(z)| = 1 \neq 0$, f must have a finite number of zeros in $\mathbb{D}$. Let us consider a finite Blaschke product B with the same zeros as f, counted with multiplicity, of arbitrary rotation. Then, we can consider the map $h : \mathbb{D} \longrightarrow \mathbb{C}$ defined as

$$h(z) = \begin{cases} \lim_{w\to z} \frac{f(w)}{B(w)}, & \text{if } B(z) = 0 \\ \frac{f(z)}{B(z)}, & \text{otherwise.} \end{cases}$$

Since f and g have the exact same zeros, the function h is well-defined, holomorphic and different from zero in all $\mathbb{D}$. This way, the function $1/h$ is also holomorphic in all $\mathbb{D}$. Furthermore,

$$\lim_{|z|\to 1^-} |h(z)| = \lim_{|z|\to 1^-} \frac{|f(z)|}{|B(z)|} = 1, \text{ and } \lim_{|z|\to 1^-} \frac{1}{|h(z)|} = \lim_{|z|\to 1^-} \frac{|B(z)|}{|f(z)|} = 1.$$

So, by the Maximum Modulus Principle A.2, both $|h(z)| \leq 1$ and $|1/h(z)| \leq 1$ for all $z \in \mathbb{D}$. This means that $|f(z)| / |B(z)| = |h(z)| = 1$, and so, f/B must be a constant function of modulus 1, i.e., $f(z)/B(z) = e^{i\theta} \in \partial\mathbb{D}$ for some $\theta \in \mathbb{R}$. In conclusion, $f(z) = e^{i\theta}B(z)$ for all $z \in \mathbb{D}$ and so f is a finite Blaschke product.

Finally, consider an arbitrary holomorphic proper map $f : \mathbb{D} \longrightarrow \mathbb{D}$ of degree k and let $(z_n)_n$ be a sequence tending to $\partial\mathbb{D}$, in other words, $|z_n| \longrightarrow 1$.

Now, if the image sequence $(f(z_n))_n \subseteq \mathbb{D}$ does not accumulate in $\partial\mathbb{D}$, then it must be contained in a compact set $K \subseteq \mathbb{D}$. In this case, since f is proper, the set $f^{-1}(K) \subseteq \mathbb{D}$ must also be compact. However, this would mean that the original sequence $(z_n)_n \subseteq f^{-1}(K)$ must have a convergent sub-sequence in $f^{-1}(K) \subseteq \mathbb{D}$, contradicting the original hypothesis. In conclusion, $(f(z_n))_n$ must accumulate in $\partial\mathbb{D}$, and so, $|f(z_n)| \longrightarrow 1$.

In other words, the limit of $|f(z)|$ as $|z| \longrightarrow 1$ is 1. Hence, by Fatou's Theorem 1.6, f must be a finite Blaschke product. □

In conclusion, we just saw that any map defined in $\mathbb{D}$ that maps sequences converging to $\partial\mathbb{D}$ to sequences converging to $\partial\mathbb{D}$ must have an analytic continuation to $\overline{\mathbb{D}}$ (and in fact, holomorphic in $\hat{\mathbb{C}}$) and has to be a finite Blaschke product.

1.3 Derivatives of finite Blaschke products in the unit circle

Since one of our goals is to study the behaviour of finite Blaschke products on the unit circle $\partial\mathbb{D}$, it is useful to study the derivative of a general FBP. For this regard, when working with large products of functions in the Riemann Sphere $f = f_1 \cdots f_k$, one easier way to compute derivatives is to look at the fraction f'/f, as we have

$$f' = \sum_{l=1}^{k} f_1 \cdots f_{l-1} f_l' f_{l+1} \cdots f_k = \sum_{l=1}^{k} f_l' \frac{f}{f_l} \Longrightarrow \frac{f'}{f} = \sum_{l=1}^{k} \frac{f_l'}{f_l}.$$

This quotient is known as the ***logarithmic derivative*** of f, and for a FBP of the form in Eq.1.2, it takes the form

$$\frac{B'(z)}{B(z)} = \sum_{l=1}^{k} \left(\frac{z - w_l}{1 - \overline{w_l} z} \right)' \frac{1 - \overline{w_l} z}{z - w_l} = \sum_{l=1}^{k} \frac{1 - |w_l|^2}{(1 - \overline{w_l} z)(z - w_l)}, \tag{1.3}$$

for all $z \in \hat{\mathbb{C}}$, where we used that the derivative of a general Möbius transformation $M(z) = \frac{az+b}{cz+d}$ is $M'(z) = \frac{ad-bc}{(cz+d)^2}$, for $ad - bc \neq 0$. This has interesting implications.

Proposition 1.7. *Let B be a finite Blaschke product. Then, for all $z \in \hat{\mathbb{C}}$*

$$B'(z) = \frac{B(z)^2}{z^2} \overline{B'(1/\overline{z})}$$

Proof. We write B in the form described in Eq. (1.2) and then evaluate Eq. (1.3) for $1/\overline{z}$. We have

$$\frac{B'(1/\overline{z})}{B(1/\overline{z})} = \sum_{l=1}^{k} \frac{1 - |w_l|^2}{(1 - \overline{w_l}/\overline{z})(1/\overline{z} - w_l)} = \overline{z}^2 \sum_{l=1}^{k} \frac{1 - |w_l|^2}{(\overline{z} - \overline{w_l})(1 - w_l \overline{z})} = \overline{z}^2 \overline{\frac{B'(z)}{B(z)}}.$$

Changing z for $1/\overline{z}$ and using the Reflection Property 1.4, we get the expression we were looking for. □

Proposition 1.8 (FBP in the unit circle)**.** *Let B be a finite Blaschke product of degree $k \geq 1$ and $\theta \in \mathbb{R}$. Then,*

(a) $B'(e^{i\theta}) \neq 0$. *Consequently, every point in $\partial\mathbb{D}$ has exactly k distinct preimages in $\partial\mathbb{D}$.*

(b) *If $B(e^{i\theta}) = e^{i\theta}$, we have $B'(e^{i\theta}) \in \mathbb{R}_{>0}$.*

(c) *If* arg *is a branch cut of the argument of the curve $B(e^{it})$ for $t \in \mathbb{R}$, we have $\frac{d}{dt} \arg B(e^{it}) = \left|B'(e^{it})\right| > 0$.*

(d) *B is a local diffeomorphism of $\partial\mathbb{D}$.*

Proof. (a) Since $B(\partial\mathbb{D}) = \partial\mathbb{D}$, we have that $\left|B(e^{i\theta})\right| = 1$ for all $\theta \in \mathbb{R}$. With this, having B as in Eq. (1.2), we can use Eq. (1.3), and thus

$$\left|B'(e^{i\theta})\right| = \left|\frac{B'(e^{i\theta})}{B(e^{i\theta})}\right| = \left|\sum_{l=1}^{k} \frac{1-|w_l|^2}{(1-\overline{w_l}e^{i\theta})(e^{i\theta}-w_l)}\right| =$$
$$= \left|\sum_{l=1}^{k} \frac{1}{e^{i\theta}} \frac{1-|w_l|^2}{(e^{-i\theta}-\overline{w_l})(e^{i\theta}-w_l)}\right| = \left|\frac{1}{e^{i\theta}} \sum_{l=1}^{k} \frac{1-|w_l|^2}{|e^{i\theta}-w_l|^2}\right| =$$
$$= \sum_{l=1}^{k} \frac{1-|w_l|^2}{|e^{i\theta}-w_l|^2} > 0.$$

since $w_l \in \mathbb{D}$ for all l.

Next, let $a \in \partial\mathbb{D}$. Due to Proposition 1.5, we know that a has k preimages in $\partial\mathbb{D}$. It is left to see that these are distinct, or, equivalently, that their multiplicity as solutions of the equation $B(z) = a$ is exactly 1. However, if one of its preimages a^* had multiplicity greater than one, we would have $B'(a^*) = 0$, which cannot happen since $a^* \in \partial\mathbb{D}$.

(b) Let $\theta \in \mathbb{R}$ such that $B(e^{i\theta}) = e^{i\theta}$. Assuming B is in the form (1.2), by Eq. (1.3), we have

$$B'(e^{i\theta}) = e^{i\theta} \sum_{l=1}^{k} \frac{1-|w_l|^2}{e^{i\theta}(e^{-i\theta}-\overline{w_l})(e^{i\theta}-w_l)} = \sum_{l=1}^{k} \frac{1-|w_l|^2}{|e^{i\theta}-w_l|^2} \in \mathbb{R}_+.$$

(c) First, note that since $B(\partial\mathbb{D}) = \partial\mathbb{D}$, the curve $B(e^{it})$ does not cross 0, and so there exists a differentiable branch cut of the argument of this curve for $t \in \mathbb{R}$. In other words, we can write $B(e^{it}) = e^{i\psi(t)}$ for some differentiable function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$. Hence, $\arg B(e^{it}) = \psi(t)$ is well-defined and differentiable.

Now, we can assume B is in the form (1.2) and use expression 1.3. We have, then,

$$\psi'(t) = e^{it}\frac{B'(e^{it})}{B(e^{it})} = e^{it} \sum_{l=1}^{k} \frac{1-|w_l|^2}{(1-\overline{w_l}e^{it})(e^{it}-w_l)} = \sum_{l=1}^{k} \frac{1-|w_l|^2}{|e^{it}-w_l|^2} = \left|B(e^{it})\right|,$$

which must be positive as we saw in (a).

(d) This result is a simple corollary of the Inverse Function Theorem, since, by (a), $B'(z) \neq 0$ for every $z \in \partial\mathbb{D}$.

□

In fact, one can see that a FBP B of degree k is what is known as a ***degree k covering*** of $\partial\mathbb{D}$, meaning that $\partial\mathbb{D}$ is divided into k arc sections, each one being sent diffeomorphically by B to the whole $\partial\mathbb{D}$. More details will be given in Section 4.2.2.

1.4 Generalisations of finite Blaschke products

Finally, we end this chapter by considering a simple generalisation of finite Blaschke products by allowing their zeros to be outside of the unit disk. In particular, we can define the following.

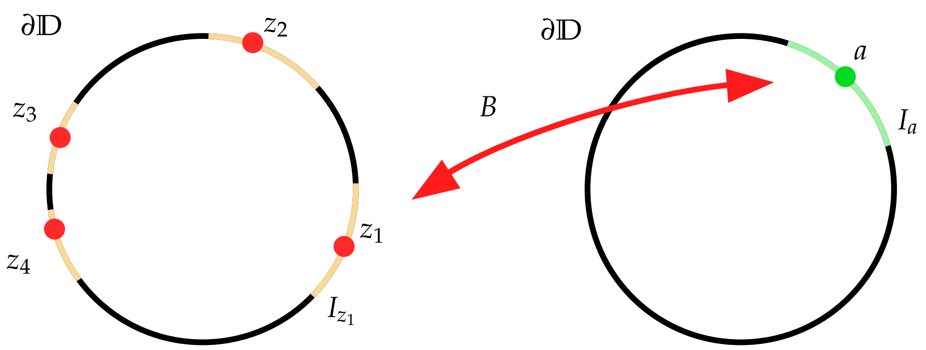


Figure 1.3: Proposition 1.8(d). Every finite Blaschke product B is diffeomorphic between neighbourhoods of any $a \in \partial\mathbb{D}$ and one of its preimages $z_j \in \partial\mathbb{D}$.

Definition 1.9 (Generalised finite Blaschke product)**.** Let $\theta \in \mathbb{R}$ and $w_1, \ldots, w_k \in \hat{\mathbb{C}}$ satisfying $w_l \overline{w_r} \neq 1$ for $l, r \in \{1, \ldots, k\}$.

We say that the ***generalised finite Blaschke product*** (GFBP) of rotation θ and zeros $w_1, \ldots, w_k$ is the function $B : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ defined as

$$B(z) = e^{i\theta} \prod_{l=1}^{k} \frac{z - w_l}{1 - \overline{w_l} z}. \tag{1.4}$$

Remark 1.10. Note that, if $w \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, we have

$$\frac{z - w}{1 - \overline{w} z} = \frac{w}{\overline{w}} \frac{(1/w)z - 1}{(1/\overline{w}) - z} = \gamma \frac{1 - \bar{a} z}{z - a}, \quad \text{for } \gamma = \frac{w}{\overline{w}} \in \partial\mathbb{D} \text{ and } a = \frac{1}{\overline{w}} \in \mathbb{D}$$

Then, if two zeros of a generalised finite Blaschke product satisfy $w_l \overline{w_r} = 1$, for $l \neq r$, we have that the factors corresponding to w_l and w_r cancel out, leaving only a constant rotation. On the other hand, if $l = r$, we have that $w_l \overline{w_l} = |w_l|^2 = 1$, and so, $w_l = e^{i\theta}$ and

$$\frac{z - e^{i\theta}}{1 - e^{-i\theta} z} = -e^{i\theta} \frac{1 - e^{-i\theta} z}{1 - e^{-i\theta} z} = -e^{i\theta} \in \partial\mathbb{D}.$$

In other words, if any $w_l \in \partial\mathbb{D}$, then its corresponding factor is a constant rotation. Then, the condition $w_l \overline{w_r} \neq 1$ guarantees that no factors cancel out and so a GFBP of the form Eq.(1.4) is a rational function of degree k.

We just saw that every factor of the product with a zero in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is the inverse of a Blaschke factor with a zero in $\mathbb{D}$. So, one can think of GFBP as the quocient of two finite Blaschke products with no common zeros.

This implies, in particular, that the Reflection Property 1.4 is still satisfied for GFBP, and they still send $\partial\mathbb{D}$ onto $\partial\mathbb{D}$. However, most other properties of finite Blaschke products are not inherited for the generalised ones, since they are a consequence of Schwarz Lemma A.1, which cannot be applied now as GFBP do not send $\mathbb{D}$ into $\mathbb{D}$. In fact, whenever we have a zero outside of $\mathbb{D}$, we have a pole in $\mathbb{D}$, and so there is a neighbourhood of this point being sent to a neighbourhood of infinity.

Nevertheless, we can still find a useful equivalence for these maps. We have the following result.

Proposition 1.11. *Let $R : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ be a rational map such that $R(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$. Then, R is a generalised Blaschke product.*

Proof. The proof of this result is analogous to the one of Fatou's Theorem 1.6, considering now a generalised FBP with the same zeros and poles inside of $\mathbb{D}$ of B. This way, the quotients R/B and B/R are holomorphic in $\overline{\mathbb{D}}$ and the same argument can be used. $\square$

Chapter 2

Complex dynamics

In broad terms, dynamical systems encompass the study of functions dependent on a ***temporal*** parameter. The most natural example, responsible for the nomenclature in this field, is the system of trajectories arising in the study of autonomous differential equations, which have been known to govern the movements of planets since the seventeenth century.

Three hundred years later, G. Julia and P. Fatou kick-started the study of complex dynamical systems. Here, the object of study shifts from differential equations to the iteration of holomorphic maps, and the continuous temporal parameter changes to a discrete index, labelling the iterates. Now, instead of stellar orbits, the robust properties of holomorphic functions create complex, intriguing structures with remarkable mathematical properties.

This chapter aims to formally describe these constructions and provide a preliminary understanding of these complex dynamical systems. The contents of this chapter can be found in [CG93, Chapters 2, 3], [Ste11, Chapters 2, 3] or [Mil90, Sections 3 to 14].

Concerning notation, given a rational map $R(z)$, we denote the n-th iterate as $R^n(z) = R(R^{n-1}(z))$, where $n \geq 1$ and we are denoting $R^0(z) = z$. This should not be confused with the n-th derivative of R, which is denoted as $R^{(n)}(z)$.

2.1 The dynamic partition of the Riemann Sphere: Fatou and Julia sets

The fundamental elements of complex dynamics are the Julia and Fatou sets, which bear the names of the early developers of this field. These divide the Riemann Sphere into two regions with different dynamical properties, and they are the natural starting point in the study of holomorphic iteration.

For our purpose, we can limit ourselves to the case of iteration of holomorphic maps in the whole Riemann Sphere, $\hat{\mathbb{C}}$, which are precisely rational maps (see Appendix A.2). All this results are discussed in [Ste11, Chapter 2].

Before starting with iteration, let us recall the concept of degree of a rational map. Given a rational map $R(z) = P(z)/Q(z)$, for P and Q relatively prime polynomials, the ***degree of*** R is the maximum between $\deg P$ and $\deg Q$. As discussed in A.5, it is straight forward to see that this is the topological degree of R, i.e., that every point in $\hat{\mathbb{C}}$ has exactly d preimages in

$\hat{\mathbb{C}}$, counted with multiplicity. In other words, R is a proper map of degree d. Notice that in the case of degree $d = 0$, R is a constant map; and in the case of degree $d = 1$, R is a Möbius transformation. Hence, we focus on rational maps of degree $d \geq 2$.

Let us start discussing rational iteration by introducing a simple but relevant example, the quadratic family, which comprises rational maps of degree $d = 2$ of the form $Q_c(z) = z^2 + c$ for some $c \in \mathbb{C}$.

Example 2.1. Let us consider the rational function $Q_0(z) = z^2$. If we study its family of iterates, we can see that $Q_0^n(z) = Q_0 \circ \overset{(n)}{\cdots} \circ Q_0(z) = z^{2^n}$. By writing z in polar coordinates $z = re^{i\theta}$, we can see that

$$Q_0^n(z) = r^{2^n} e^{i2^n\theta} \xrightarrow{n} \begin{cases} \infty, & \text{if } r > 1 \\ 0, & \text{if } r < 1, \end{cases} \tag{2.1}$$

which induces a division of the complex plane into open regions where the iterates converge to certain points, separated by a closed set.

This division in invariant open sets with the same dynamics is in fact a general trait of holomorphic maps.

Definition 2.2 (Invariant set)**.** Let R be a rational map. We say a set $U \subseteq \hat{\mathbb{C}}$ is ***forward invariant*** if $R(U) \subseteq U$, ***backward invariant*** if $R^{-1}(U) \subseteq U$ and ***completely invariant*** if U is forward and backward invariant.

Definition 2.3 (Normality)**.** Given a rational map R of degree $d \geq 2$ and let $U \subseteq \hat{\mathbb{C}}$ be an open, forward invariant set. We say that the family of iterates of R is ***normal*** in U if any partial sequence of $\{R^n\}_n$ has a sub-sequence which converges uniformly to a holomorphic map in compact subsets of U. Given $z \in \hat{\mathbb{C}}$, we say the family of iterates of R is ***normal in*** z if it is normal in some neighbourhood of z.

Remark 2.4. Note that since we are considering rational maps, $U \subseteq \hat{\mathbb{C}}$ and the iterates can converge to ∞.

One result characterising normal families with historical relevance is Montel's theorem, which served as the cornerstone for the development of the theory of rational iteration.

Theorem 2.5 (Montel's Theorem)**.** *Let R be a rational map. Let $U \subseteq \mathbb{R}$ be an open set. If $\hat{\mathbb{C}} \setminus \bigcup_{n\geq 0} R^n(U)$ contains at least 3 points, the family of iterates of R is normal in U.*

Definition 2.6 (Fatou and Julia sets)**.** Let R be a rational map of degree $d \geq 2$. We define the ***Fatou set of R*** as

$$\mathcal{F}(R) = \left\{z \in \hat{\mathbb{C}} : \text{ the iterates of } R \text{ are normal at } z\right\},$$

and the ***Julia set of R*** as its complement:

$$\mathcal{J}(R) = \left\{z \in \hat{\mathbb{C}} : \text{ the iterates of } R \text{ are not normal at } z\right\}.$$

Note that in general, we omit the dependence on R and denote $\mathcal{J} = \mathcal{J}(R), \mathcal{F} = \mathcal{F}(R)$.

Example 2.7. In our previous example, we saw that $\{Q_0^n\}_n$ converges to the constant (holomorphic) maps 0 in $\mathbb{D}$ and to ∞ in $\partial\mathbb{D}$. Hence, the iterates of Q_0 are normal in $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

However, iterates are not normal in any open set U containing points of $\partial\mathbb{D}$, since this set would contain points from $\mathbb{D}$, with iterates converging to 0, and from $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, with iterates converging to ∞. Hence, not any sub-sequence of the iterates of Q_0 can converge to a holomorphic map in U.

In conclusion, $\mathcal{J}(Q_0) = \partial\mathbb{D}$ and $\mathcal{F}(Q_0) = \hat{\mathbb{C}} \setminus \partial\mathbb{D}$.

Hence, the map Q_0 exemplifies this partition of $\hat{\mathbb{C}}$. Now, we shall introduce some basic properties of these sets.

Proposition 2.8 (Properties of Julia and Fatou sets)**.** *Let R be a rational map of degree $d \geq 2$. Let $\mathcal{J}$ and $\mathcal{F}$ be its Julia and Fatou sets, respectively. Then,*

(1) *$\mathcal{F}$ and $\mathcal{J}$ are completely invariant.*

(2) *$\mathcal{F}$ is open and $\mathcal{J}$ is closed.*

(3) *For all $k \geq 1$, $\mathcal{F}(R^k) = \mathcal{F}(R)$ and $\mathcal{J}(R^k) = \mathcal{J}(R)$.*

(4) *$\mathcal{J}$ is not empty.*

Proof. (1) As $\mathcal{F}$ and $\mathcal{J}$ form a partition of the Riemann Sphere, it is enough to see that $R^{-1}(\mathcal{F}) \subseteq \mathcal{F}$ and that $R^{-1}(\mathcal{J}) \subseteq \mathcal{J}$.

For the first inclusion, let $p \in R^{-1}(\mathcal{F})$, i.e., $R(p) \in \mathcal{F}$. Then, there exists an open neighbourhood of $R(p)$, U, such that the family of iterates of R is normal in U. Now, since R is continuous, $R^{-1}(U)$ is an open neighbourhood of p. Furthermore, the family of iterates of R in $R^{-1}(U)$ coincides exactly with the family of iterates of R in U. Hence, the iterates are normal in $R^{-1}(U)$. In other words, $R^{-1}(\mathcal{F}) \subseteq \mathcal{F}$.

Second, let $p \in R^{-1}(\mathcal{J})$. Then, for every open neighbourhood U of p, $R(U)$ is an open neighbourhood of $R(p) \in \mathcal{J}$, and so, the family of iterates of R is not normal in $R(U)$ and thus cannot be normal in U. This means $R^{-1}(\mathcal{J}) \subseteq \mathcal{J}$.

(2) Let $z \in \mathcal{F}$. By definition, it exists an open neighbourhood of z, U, where the family $\{R^n\}_n$ is normal. Note that as U is open, U is a neighbourhood of all its points, and so, all their points are normal, i.e. $U \subseteq \mathcal{F}$, and $\mathcal{F}$ is open.

As $\mathcal{J} = \hat{\mathbb{C}} \setminus \mathcal{F}$, we have that $\mathcal{J}$ is closed.

(3) We shall study the normality of the families of iterates of R, $\{R^n\}_n$, and of R^k, $\left\{R^{nk}\right\}_n$. First, note that $\left\{R^{nk}\right\}_n \subseteq \{R^n\}_n$. Hence, if $\{R^n\}_n$ is normal in a domain U, $\left\{R^{nk}\right\}_n$ must be so. For the converse, note that for all $k \geq 1$, we can divide the family of iterates in the following way:

$$\{R^n\}_{n\geq 0} = \left\{R^{nk}\right\}_{n\geq 0} \cup \left\{R^1 \circ R^{nk}\right\}_{n\geq 0} \cup \cdots \cup \left\{R^{k-1} \circ R^{nk}\right\}_{n\geq 0}.$$

This way, if we assume the family of iterates of R^k, $\left\{R^{nk}\right\}_n$, is normal in a domain U and consider any sub-sequence $\left\{R^{n_l k}\right\}_l$ converging uniformly to a holomorphic map f in U, then, the sequence $\left\{R^j \circ R^{n_l k}\right\}_l$ must converge uniformly to $R^j \circ f$, since R^j is uniformly continuous for all $j \in \{0, \ldots, k-1\}$ (R^j is a continuous map defined in a compact set, the Riemann sphere). Hence, $\{R^n\}_n$ is normal in U if and only if $\left\{R^{nk}\right\}$ is normal in U. This means that the Fatou sets of R and R^n are equal, and thus, so are the Julia sets.

(4) Assume $\mathcal{J}$ is empty. Then, R is normal in $\hat{\mathbb{C}}$, and so, a sub-sequence of the iterate family, $\{R^{n_k}\}_k$, must converge uniformly to a holomorphic map $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$. Hence, f must a rational map of degree $d_0 \geq 0$.

First, assume $d_0 = 0$ and $f(z) = c$ is a constant map, for some $c \in \hat{\mathbb{C}}$. Then, since $\{R^{n_k}\}_k$ converges uniformly to c, there exists $k_0 \geq 0$ such that the image of R^{n_k} is contained in a neighbourhood of c for all $k \geq k_0$. However, this cannot happen since R^n is a rational map, and so it is onto $\hat{\mathbb{C}}$ for all $n \geq 0$.

Second, assume $d_0 \geq 1$. Then, f has d_0 zeros or d_0 poles. Consider now any differentiable curve $\gamma : (0,1) \longrightarrow \hat{\mathbb{C}}$ avoiding a neighbourhood of the zeros and poles of f. Then, there exists $\varepsilon = \inf_t f(\gamma(t)) > 0$.

Now, as $\{R^{n_k}\}_k$ tends to f uniformly, there must exist $k_0 \geq 0$ such that $|R^{n_k}(z) - f(z)| < \varepsilon \leq |f(z)|$ for all $k \geq k_0$. Then, by Rouché's Theorem A.5, the index of R^{n_k} must be equal to the index of f for $k \geq k_0$. Since γ is arbitrary, by the Argument Principle A.6, R^{n_k} must have the same number of zeros and poles as f, and so, the same degree. However, the degree of R^{n_k} is equal to $n_k d$ for each k, contradicting the hypotheses that f has a finite degree.

In conclusion, $\mathcal{J}$ cannot be empty.

□

With these general properties, we just saw that the Julia and Fatou sets indeed induce a dynamical partition of the Riemann Sphere into two completely invariant sets. Now, we give more properties further characterising this sets. Since their proofs follow a similar structure as those of Proposition 2.8 but involve wider context, we do not include them. However, they can be found in [Ste11, Theorem 2.4.3], [Ste11, Theorem 2.4.4] and [Ste11, Theorem 2.5.2].

Proposition 2.9 (Properties of Julia and Fatou sets, II)**.** *Let R be a rational map of degree $d \geq 2$ and $\mathcal{J}$ its Julia set. Then:*

(5) *For every point $a \in \mathcal{J}$, $\mathcal{J} = \overline{\bigcup_{n \geq 0} R^{-n}(a)}$*

(6) *$\mathcal{J}$ is perfect, i.e., it has no isolated points*

(7) *$\mathcal{J}$ has infinitely many elements*

(8) *(Blow-up property) For every domain D intersecting the Julia set, there exists a $n \geq 0$ such that $R^n(D \cap \mathcal{J}) = \mathcal{J}$*

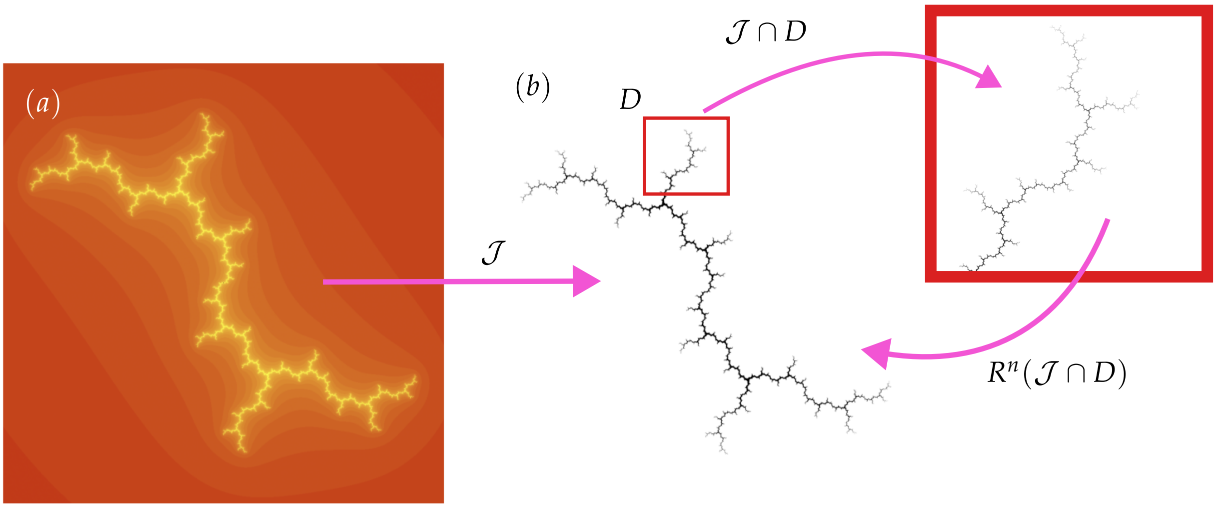


Figure 2.1: Dynamical plane of the map $Q_i(z) = z^2 + i$. The red region indicates convergence to ∞ (a). The Julia set is isolated in panel (b), and an arbitrary domain intersecting the Julia set is shown in a red square. For a certain $n \geq 0$, the image of the Julia set inside this domain generates again the whole Julia set.

2.2 Local theory

The Julia and Fatou sets we just introduced will be the basic framework for studying the dynamics of rational maps. To fully understand their characteristics, it is necessary to study the behaviour of the iterates in some sets. Our main tool are conjugacies.

Definition 2.10 (Conjugacies)**.** Let R, R' be rational maps. Let $U, V \subseteq \hat{\mathbb{C}}$ be domains. We say that R and R' are ***(conformally) conjugate*** in U or V if there exists a biholomorphic map $h : U \longrightarrow V$ such that $R' = h \circ R \circ h^{-1}$ in V. We say h is a ***(conformal) conjugacy***.

If h is not biholomorphic but it is a homeomorphism, we say R and R' are ***topologically conjugate*** and that h is a ***topological conjugacy***.

Our aim is to conjugate the dynamics of rational maps to simpler functions on a neighbourhood of certain points. We follow the classification of [CG93, Chapter II].

Definition 2.11 (Fixed point)**.** Let R be a rational map. We say $z_0 \in \hat{\mathbb{C}}$ is a ***fixed point*** of R if $R(z_0) = z_0$. In this case, we define the ***multiplier*** of z_0 as $\lambda = R'(z_0)$.

Definition 2.12 (Periodic point)**.** Let R be a rational map. We say $z_0 \in \hat{\mathbb{C}}$ is a ***periodic point of period*** p of R if z_0 is a fixed point of R^p for minimal $p \in \mathbb{Z}_{\geq 0}$. In this case, we say that $\alpha = \{z_0, R(z_0), \ldots, R^{p-1}(z_0)\}$ is a ***periodic cycle of period*** p, and its ***multiplier*** is $\lambda = (R^p)'(z_j)$ for any $j \in \{0, \ldots, p-1\}$.

Note that every element in any periodic cycle is in fact a periodic point of the same period. Furthermore, the multiplier is well-defined since, by the Chain Rule,

$$\lambda = (R^p)'(z_0) = \prod_{j=0}^{p-1} R'(R^j(z_0)) = \prod_{z \in \alpha} R'(z).$$

Multipliers determine the local behaviour of fixed points and periodic cycles, and allow us to establish a classification amongst them.

Definition 2.13 (Classification of cycles)**.** Let R be a rational map and α a periodic cycle or fixed point of R with multiplier λ. We say α is

(i) ***super-attracting*** if $\lambda = 0$,

(ii) ***attracting*** if $0 < |\lambda| < 1$,

(iii) ***indifferent*** if $|\lambda| = 1$ (if $\lambda = e^{2\pi i r}$ with $r \in \mathbb{Q}$, we say α is ***parabolic***),

(iv) ***repelling*** if $|\lambda| > 1$.

Notice that, due to Property (3) in Proposition 2.8, $\mathcal{J}(R^p) = \mathcal{J}(R)$ and $\mathcal{F}(R^p) = \mathcal{F}(R)$ for all $p \geq 1$. Since periodic points of period p of R are fixed points of R^p, we can reduce to the study of the dynamics of rational maps around fixed points.

2.2.1 Attracting and repelling case

We shall treat the attracting and repelling case together, as they are two faces of the same coin. In fact, let R be a rational map and $z_0 \in \hat{\mathbb{C}}$ a repelling fixed point with multiplier $R'(z_0) = \lambda$ with $|\lambda| > 1$. Since $\lambda \neq 0$, there exists a neighbourhood of z_0 where R has holomorphic rational inverse R^{-1}. Hence, z_0 is now an attracting fixed point of R^{-1} with multiplier $1/\lambda$.

The result characterising these fixed points is the Koenigs-Schröder Theorem.

Theorem 2.14 (Koenigs-Schröder)**.** *Let R be a rational map of degree $d \geq 2$ and $z_0 \in \hat{\mathbb{C}}$ an attracting or repelling fixed point of R. Then, R is locally conjugate to λz, where $\lambda \in \hat{\mathbb{C}}$.*

In other words, there exist U neighbourhood of z_0, V neighbourhood of 0, and a biholomorphism $\phi : U \longrightarrow V$ such that

$$\phi \circ R(z) = \lambda\phi(z), \quad \forall z \in U.$$

Moreover, ϕ is unique up to multiplicative constant.

Proof. This proof follows the outline of [Ste11, Theorem 3.4.1]. First, we consider the attracting case. Let us start by proving uniqueness. Assume $\phi : U \longrightarrow V$ is a diffeomorphism satisfying $\phi \circ R = \lambda\phi$, defined in some appropriate neighbourhoods U,V of z_0 and 0, respectively. If we evaluate the conjugacy expression on z_0, we get $\phi(z_0) = \lambda\phi(z_0)$, and so $\phi(z_0) = 0$. Furthermore, since ϕ is a diffeomorphism, we have $\phi'(z_0) \neq 0$. Hence, we can divide ϕ by the constant $\phi'(z_0)$ and get an equivalent diffeomorphism ψ with $\psi'(z_0) = 1$.

Assume then, without loss of generality, $\phi'(z_0) = 1$ and $\phi(z_0) = 0$. We have

$$\phi \circ R^n(z) = \lambda^n\phi(z) \longrightarrow 0 \text{ for } z \in U, \text{ as } |\lambda| < 1.$$

Then, as ϕ is a diffeomorphism, $R^n(z) \longrightarrow \phi^{-1}(0) = z_0$ for $z \in U$. This way, we can write

$$\phi'(0) = \lim_{z \to z_0} \frac{\phi(z) - \phi(z_0)}{z - z_0} = \lim_{n \to \infty} \frac{\phi(R^n(z)) - 0}{R^n(z) - z_0} = 1, \text{ by hypothesis.}$$

And so, by isolating $\phi(z)$ from the conjugacy equation,

$$\phi(z) = \frac{\phi(R^n(z))}{\lambda^n} \quad \forall n \Rightarrow \phi(z) = \lim_{n\to\infty} \frac{\phi(R^n(z))}{\lambda^n} = \lim_{n\to\infty} \frac{R^n(z)}{\lambda^n}.$$

Which means that ϕ is uniquely determined. This proof also gives us intuition on how to prove the existence of this conjugacy. In fact, we will prove that this limit exists and satisfies our hypotheses.

Obs. 1. ***There exists a forward invariant neighbourhood of*** z_0, U.

This can be seen by considering the definition of derivative. Indeed, since $R'(z_0) = \lambda$, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \left| \frac{R(z) - R(z_0)}{z - z_0} \right| - |\lambda| \right| < \varepsilon, \quad \text{for } z \in D(z_0, \delta).$$

Now, if we choose ρ such that $|\lambda| < \rho < 1$, we can set $\varepsilon = \rho - |\lambda| > 0$, and so, for every $z \in D(z_0, \delta)$

$$\left| \frac{R(z) - R(z_0)}{z - z_0} \right| \leq \left| \left| \frac{R(z) - R(z_0)}{z - z_0} \right| - |\lambda| \right| + |\lambda| < \rho - |\lambda| + |\lambda| = \rho.$$

Hence, for $z \in D(z_0, \delta)$, $|R(z) - R(z_0)| < \rho |z - z_0| < |z - z_0| < \delta$, i.e., $R(z) \in D(R(z_0), \delta) = D(z_0, \delta)$, since z_0 is a fixed point of R. Then, for any ρ in $(|\lambda|, 1)$, there exists a δ such that $U = D(z_0, \delta)$ is forward invariant, as we wanted to see.

Now, assume a fixed ρ and its corresponding neighbourhood of z_0, $U = D(z_0, \delta)$ for some $\delta > 0$. In this set, we can define

$$\phi_n(z) := \frac{R^n(z) - z_0}{\lambda^n}, \text{ for } z \in U.$$

Obviously, these maps are holomorphic and send U to a neighbourhood of 0, $V_n = \phi_n(U)$. Our aim is to show that these maps comprise a Cauchy sequence. For this purpose, consider

$$|\phi_{n+1}(z) - \phi_n(z)| = \left| \frac{R(R^n(z)) - z_0}{\lambda^{n+1}} - \frac{R^n(z) - z_0}{\lambda^n} \right| = \frac{|R(w) - z_0 - \lambda(w - z_0)|}{|\lambda^{n+1}|}.$$

for $w = R^n(z)$.

Obs. 2. ***There exists*** $C < 0$ ***such that*** $|R(z) - z_0 - \lambda(z - z_0)| < C |z - z_0|^2$.

This is direct if we consider the Taylor series of $R(z)$,

$$R(z) = z_0 + \lambda(z - z_0) + o(z^2) \Rightarrow |R(z) - z_0 - \lambda(z - z_0)| < C |z - z_0|^2.$$

for some $C > 0$.

Obs. 3. ***For each*** $n \geq 1$, $|R^n(z) - z_0| < (|\lambda| + C\delta)^n |z - z_0|$.

This is a direct consequence of Obs. 2. We have

$$|R(z) - z_0| \leq |R(z) - z_0 - \lambda(z - z_0)| + |\lambda(z - z_0)| < (C\delta + |\lambda|) |z - z_0|.$$

Now, if it is true for $n \geq 1$, then,

$$\left| R^{n+1}(z) - z_0 \right| = |R(R^n(z)) - z_0| < (|\lambda| + C\delta) |R^n(z) - z_0| < (|\lambda| + C\delta)^{n+1} |z - z_0|.$$

With these two observations, we get

$$|\phi_{n+1}(z) - \phi_n(z)| < \frac{C}{|\lambda|^n} |R^n(z) - z_0|^2 < \frac{C}{|\lambda|} \left(\frac{(|\lambda| + C\delta)^2}{|\lambda|} \right)^n |z - z_0|^2 .$$

Next, as we can choose δ as small as we want, and $|\lambda|^2 < |\lambda|$, we can choose $\delta < 0$ such that

$$\xi := \frac{(|\lambda| + C\delta)^2}{|\lambda|} < 1.$$

Hence, we have that $|\phi_{n+1}(z) - \phi_n(z)| < (C/|\lambda|)\xi^n |z - z_0|^2$. Finally, this means that for $m \geq n \geq 0$,

$$|\phi_m(z) - \phi_n(z)| < \frac{C}{|\lambda|}(\xi^n + \xi^{n+1} + \cdots + \xi^{m-1}) |z - z_0|^2 .$$

which must tend to 0 uniformly for all $z \in U$ since the series $\sum_{n\geq 1} \xi^n$ is convergent. Then, the sequence of maps ϕ_n must converge uniformly to a limit $\phi : U \longrightarrow \phi(U)$, which must be holomorphic since all ϕ_n are. Furthermore, since for all $n \geq 1$, $\phi'(z_0) = (R'(z_0))^n/\lambda^n = 1$, we have that ϕ is a biholomorphism in a neighbourhood of z_0 inside U.

Lastly, if z_0 is repelling, then R^{-1} is defined in some neighbourhood of z_0 and has z_0 as an attracting fixed point. So, the previous arguments can be applied. □

Therefore, around each attracting point there exists a neighbourhood in which points converge to it under iteration. In particular, attracting points are in the Fatou set, and we can define the following.

Definition 2.15 (Attracting basin)**.** Let R be a rational map of degree $d \geq 2$ and let $z_0 \in \hat{\mathbb{C}}$ be an attracting fixed point of R. We define the ***basin of attraction of*** z_0, $\mathcal{A}(z_0)$ as the set of points whose orbit converges to z_0, i.e.

$$\mathcal{A}(z_0) = \left\{ z \in \hat{\mathbb{C}} : R^n(z) \longrightarrow z_0 \right\} .$$

2.2.2 Super-attracting case

The case of super-attracting fixed points is similar to the attracting case.

Theorem 2.16. *Let R be a rational map of degree $d \geq 2$ and $z_0 \in \hat{\mathbb{C}}$ a super-attracting fixed point of R. Let $k \geq 0$ be the minimal such that $R^{(k)}(z_0) \neq 0$. Then, R is locally conjugate to az^k, where $a = R^{(k)}(z_0) \neq 0$.*

In other words, there exist $U, V \subseteq \hat{\mathbb{C}}$ neighbourhoods of z_0 and 0, respectively, and a biholomorphism $\phi : U \longrightarrow V$ such that

$$\phi \circ R(z) = a\phi(z)^k, \quad \forall z \in U.$$

Moreover, ϕ is unique up to multiplicative constant.

The proof of this result can be found in [Ste11, Theorem 3.3.1]. It is similar in spirit to the Koenigs-Schröder Theorem 2.14 and we do not include it here.

Similarly to the attracting case, we have that super-attracting points have a neighbourhood where the iterates are normal.

Definition 2.17 (Super-attracting basin)**.** Let R be a rational map of degree $d \geq 2$ and let $z_0 \in \hat{\mathbb{C}}$ be a super-attracting fixed point of R. We define the ***basin of attraction of*** z_0, $\mathcal{A}(z_0)$ as the set of points iterating to z_0, i.e.

$$\mathcal{A}(z_0) = \left\{ z \in \hat{\mathbb{C}} : R^n(z) \longrightarrow z_0 \right\}.$$

Hence, super-attracting points also belong to the Fatou set.

2.2.3 Parabolic case

Indifferent fixed points have more subtleties to take into account, discussed widely in [Bea91, Section 6.5] and [Ste11, Section 3.5]. In fact, here we only consider parabolic points for which the multiplier of the fixed point is exactly 1. In other words, the rational map R has expansion near z_0 as

$$R(z) = z_0 + (z - z_0) + o(|z - z_0|^2) = z + o(|z - z_0|^2).$$

Before characterising the dynamic in a neighbourhood of z_0, we need to define some additional concepts.

Definition 2.18 (Multiplicity)**.** Let R be rational map of degree $d \geq 2$ and let $z_0 \in \hat{\mathbb{C}}$ be a parabolic fixed point. We say the ***multiplicity*** of z_0 is the minimal $n \geq 2$ such that $R^{(n)}(z_0) \neq 0$, where $R^{(n)}$ is the n-th derivative of R.

Remark 2.19. Normally, we denote the multiplicity of a fixed point as $n + 1$, to further indicate that $R^{(k)}(z_0) = 0$ for all k with $2 \leq k \leq n$.

Definition 2.20 (attracting/repelling vectors)**.** Let R be a rational map of degree $d \geq 2$ and let $z_0 \in \hat{\mathbb{C}}$ be a parabolic fixed point of multiplicity $n + 1$, with $a = R^{(n+1)}(z_0) \neq 0$.

We define the ***attracting vectors of*** R as the n different n-th roots of $-1/(an)$, and the ***repelling vectors of*** R as the n-th roots of $1/(an)$.

Note that as $R'(z_0) = 1 \neq 0$, there is a neighbourhood of z_0 where R^{-1} is well-defined and holomorphic. In fact, in this region, R^{-1} can be written as

$$R^{-1}(z) = z - a(z - z_0)^{n+1} + o(|z - z_0|^{n+2}).$$

Hence, we have that z_0 is still a parabolic fixed point of R^{-1} and, in particular, that attracting vectors of R^{-1} are repelling vectors of R, and vice-versa. Fixed an attracting vector v_0, all attracting and repelling vectors can be expressed as $v_k = e^{\pi i k/n}$, for $k \in \{0, \ldots, 2n - 1\}$, where v_k is an attracting vector for even k and a repelling vector for odd k.

Definition 2.21 (Directional convergence and parabolic basin)**.** Let R be a rational map of degree $d \geq 2$ and let $z_0 \in \hat{\mathbb{C}}$ be a parabolic fixed point of multiplicity $n + 1$. For a point $z \in \hat{\mathbb{C}}$, we say that ***the orbit of*** z ***converges to*** z_0 ***along the direction*** v if $R^m(z)$ converges to z_0 and

$$\lim_{m \to \infty} \sqrt[k]{m} R^m(z) = v.$$

Hence, we define the ***parabolic basin of attraction of*** z_0 ***at direction*** v as the set of points

$$\mathcal{A}(z_0, v) = \left\{ z \in \hat{\mathbb{C}} : z \text{ converges to } z_0 \text{ along the direction } v \right\}.$$

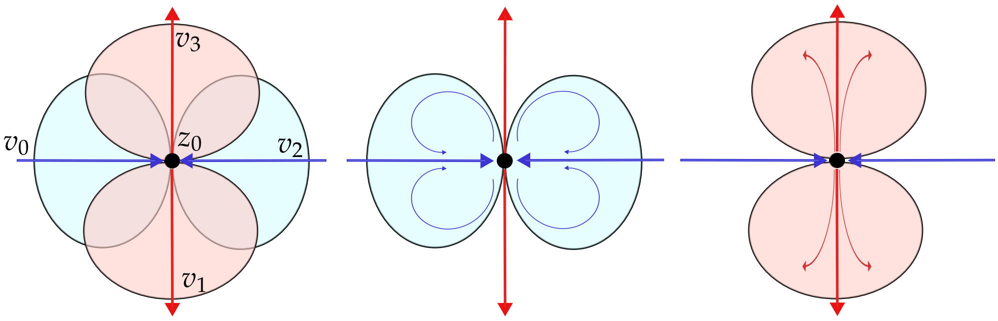


Figure 2.2: Attracting (blue) and repelling (red) petals of a parabolic fixed point z_0 of multiplicity 3. The attracting vectors are v_0, v_2 and the repelling ones v_1, v_3.

Definition 2.22 (Attracting and repelling petals)**.** Let R be a rational map of degree $d \geq 2$ and let $z_0 \in \hat{\mathbb{C}}$ be a parabolic fixed point of multiplicity $n+1$. Let v be an attracting vector of R. We say a domain P is an ***attracting petal of*** R ***for*** z_0 ***along*** v if it is forward invariant and, for every point $z \in \hat{\mathbb{C}}$, the orbit of z enters P if and only if it converges to z_0 along v.

Similarly, if R has holomorphic inverse in a domain N, and w is a repelling vector of R, we say a domain $P' \subseteq N$ is a ***repelling petal of*** R ***along*** w if it is an attracting petal for R^{-1} along w.

This is an essential difference with the attracting and super-attracting case. Instead of having one basin of attraction, we defined one for each attracting vector. This idea of convergence and repelling along a direction gets formalised by the essential result of this subsection, the Leau-Fatou Flower Theorem.

Theorem 2.23 (Leau-Fatou Flower Theorem)**.** *Let R be a rational map of degree $d \geq 2$ and $z_0 \in \hat{\mathbb{C}}$ a parabolic fixed point of R with multiplicity $n+1$. Let $v_0, \dots, v_{2n-1}$ be the attracting and repelling vectors of R.*

Then, for each attracting (repelling) vector v_k, there exists a simply connected attracting (repelling) petal P_k of R along v_k, defined in the domain between the two consecutive repelling (attracting) vectors, i.e.

$$P_k \subseteq \left\{ z \in \mathbb{C} : z = z_0 + re^{i\theta} v_k, r > 0, |\theta| < \frac{\pi}{n} \right\}.$$

Moreover, the union of these P_k is a neighbourhood of z_0. In addition, if $n \geq 2$, then each petal P_k only intersects its two direct neighbours $P_{k+1 \mod 2n}$ and $P_{k-1 \mod 2n}$ in a simply connected domain, and for $n = 1$, the corresponding repelling and attracting petals intersect each other in two simply connected domains.

This result characterises the dynamics around parabolic fixed points. A comprehensive proof is given in [Bea91, Theorem 6.5.4]. In this scenario, in contrast with attracting and super-attracting fixed points, where there is only one basin of attraction, the dynamics have a directional dependence, and we have different basins of attraction for different directions. Furthermore, now iterates in each neighbourhood of the parabolic fixed point have different dynamical behaviours, and so parabolic fixed points are in the Julia set.

2.3 Global theory: Structure of the Fatou set

We just saw that near attracting, super-attracting and parabolic fixed points we can define certain regions, the basins of attraction, where the iterates of the rational map converge to a constant function, the fixed point. From the definition, it is clear that these sets are open, disjoint from other basins of attraction, and are contained in the Fatou set with boundaries in the Julia set.

This means that the connected components of basins of attraction are in fact components of the Fatou set. So, we will now take a step back and study these in more generality. We follow the outline of [Ste11, Chapter 3].

Definition 2.24 (Fatou component)**.** Let R be a rational map of degree $d \geq 2$ and $\mathcal{F}$ its Fatou set. We say a connected component of $\mathcal{F}$ is called a ***Fatou component***.

Proposition 2.25. *Let R be a rational map of degree $d \geq 2$. Let U be a Fatou component of R. Then, $R(U)$ is a Fatou component.*

Proof. This result is direct taking into account the properties of proper maps (see Appendix A.5). If U is a Fatou component of R, since R is continuous and the Fatou set of R is completely invariant, we must have $R(U) \subseteq V$ for a Fatou component V. Furthermore, its boundary ∂U must be contained in the Julia set, hence $R(\partial U) \subseteq \partial V$, since V is contained in the Fatou set. In conclusion, U is mapped properly onto V and so $R(U) = V$. $\square$

This result has great implications regarding fixed points.

Definition 2.26 (Immediate basin of attraction, Bötcher, Schröder and Leau domains)**.** Let R be a rational map of degree $d \geq 2$ and z_0 a super-attracting or attracting fixed point of R. The ***immediate basin of attraction of*** z_0, denoted by $\mathcal{A}^*(z_0)$, is the Fatou component containing z_0. They are also called ***Bötcher domains*** if z_0 is super-attracting or ***Schröder domains*** if z_0 is attracting.

If z_0 is parabolic and v is an attracting vector, the ***immediate basin of attraction of*** z_0 ***along*** v, denoted by $\mathcal{A}^*(z_0, v)$, is the Fatou domain containing attracting petals of z_0 along v. They are also called ***Leau domains***.

Note that immediate basins of attraction must be forward invariant, since fixed points and attracting petals are so. Furthermore, they are either simply connected or infinitely connected (see [Mil90, Lemma 3.14]).

In the case of periodic points, if α is a super-attracting, attracting or parabolic periodic cycle of period p of R, we can consider the immediate basin of attraction of the cycle as the immediate basin of attraction of each point of the cycle for R^p, since $\mathcal{F}(R) = \mathcal{F}(R^p)$.

Hence, we just saw that each super-attracting, attracting or parabolic fixed or periodic point distinguishes part of the Fatou set.

In view of Proposition 2.25, we can give a first classification of general components of the Fatou set.

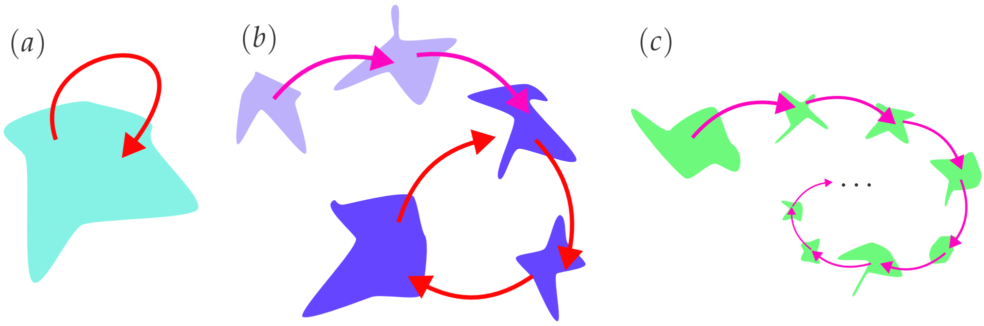


Figure 2.3: Classification of invariant Fatou components. Panel (a) sketches an invariant Fatou component, panel (b) sketches pre-periodic (light purple) and periodic domains of period 3 (blue), and panel (c) sketches (the impossible case of) wandering domains.

Definition 2.27 (Classification of Fatou components)**.** Let R be a rational map of degree $d \geq 2$, and let U be a Fatou component of R. We say U is

(a) ***periodic*** if $R^p(U) = U$ for some $p \geq 1$. If $p = 1$, we say U is an ***invariant Fatou component***;

(b) ***pre-periodic*** if $R^m(U)$ is a periodic domain, for minimal $m \geq 1$, but U is not periodic;

(c) ***wandering*** if $R^n(U) \neq R^m(U)$ for all $n \neq m$.

In other words, we classify the Fatou set components according to their orbit. If it is eventually cyclic, we have a (pre)periodic domain, or, in a more complex scenario, if the sequence is not cyclic we have a wandering domain. Fortunately, the following celebrated result ensures us that this complicated case cannot happen for rational maps.

Theorem 2.28 (Sullivan's Theorem)**.** *Let R be a rational map of degree $d \geq 2$ and let U be a Fatou component. Then, U is periodic or pre-periodic.*

The proof of this theorem is nuanced and shall be read carefully. It can be found with great detail in [Ste11, Section 3.1] and [Bea91, Chapter 8].

In view of this theorem, we know that the Fatou set of a rational map consists of different pre-periodic domains iterating eventually into periodic domains. Furthermore, note that a periodic domain of period p of R is an invariant Fatou component of R^p. Hence, it is enough to give a classification of these fixdomains.

We already studied the case where the iterate family tend to a constant point, but, we must also take into account the other possibility:

Definition 2.29 (Rotation domains)**.** Let R be a rational map of degree $d \geq 2$ and let U be an invariant Fatou component of R. We say U is a ***rotation domain*** if none of the limit functions of the iterate sequence is constant in U. Moreover, we say U is a

1. ***Siegel disc*** if U is simply connected and contains an indifferent fixed point,

2. ***Herman ring*** if U is doubly connected.

Once established these five different types of fixdomains, it is time to introduce one of the most powerful results of complex dynamics, the Classification Theorem, proved with contributions of Fatou from 1919-20 [Fat20] and Cremer (1932).

Theorem 2.30 (Classification Theorem). *Let R be a rational map of degree $d \geq 2$ and U an invariant Fatou component of R. Then, U is either a Siegel disc, a Herman Ring, a Bötcher domain, a Schröder domain or a Leau domain.*

A modern proof can be found in [Ste11, Section 3.2]. As a consequence of this result, we know that the Fatou set must be composed of different pre-periodic domains, iterating eventually into either an immediate basin of attraction (of a super-attracting, attracting or parabolic fixed point) or a rotation domain. Moreover, all but Herman rings are completely determined by fixed points.

2.4 Iteration in the unit disk and the unit circle

Having studied general properties of rational maps, we shall now consider a relevant case for our project, iteration in the unit disk $\mathbb{D}$. This will be of special interest in Chapter 3. The key result in this subject is the Wolff-Denjoy Theorem.

Theorem 2.31 (Wolff-Denjoy). *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic self-map of $\mathbb{D}$. Then, either*

(a) *f is an automorphism of $\mathbb{D}$, or*

(b) *there exists $z_0 \in \overline{\mathbb{D}}$ such that the iterate sequence $(f^n)_n$ converges to z_0 uniformly in compact subsets of $\mathbb{D}$. In this case, we say that z_0 is the* **Wolff-Denjoy point of** *f.*

For the proof, we refer to [CG93, Chapter IV.3], [Ste11, Chapter 2.7] and [Aba23, Theorem 3.2.1]. Hence, any self-map of the unit disk $f : \mathbb{D} \longrightarrow \mathbb{D}$ which can be extended to $\overline{\mathbb{D}}$ has relatively simple dynamics, since z_0 is a fixed point of f, even if $z_0 \in \partial\mathbb{D}$. A more rich scenario occurs when f cannot be extended continuously to $\partial\mathbb{D}$, but this case is out of the scope of this work.

The restriction of Blaschke products to the boundary of $\mathbb{D}$ defines an analytic map of the circle. To study these objects it is useful to consider their lifts and their rotation number. We define these concepts below, since they will be useful in Chapters 3 and 5. For the general thery of analytic maps see, for example, [dMvS12, Chapter 1].

Definition 2.32 (Lift). Let $f : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ be a self-map of the unit circle $\partial\mathbb{D}$. We say $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a ***lift*** of f if

$$e^{2\pi i F(x)} = f(e^{2\pi i x}), \quad \text{for all } x \in \mathbb{R}.$$

It is directly deduced from properties of branch cuts of the argument that for differentiable f, the lift F is differentiable. Moreover, notice that there are infinitely many lifts for every map of the circle. In particular, F_1, F_2 are lifts of a differentiable map f if and only if $F_1(x) = F_2(x) + m$ for some $m \in \mathbb{Z}$.

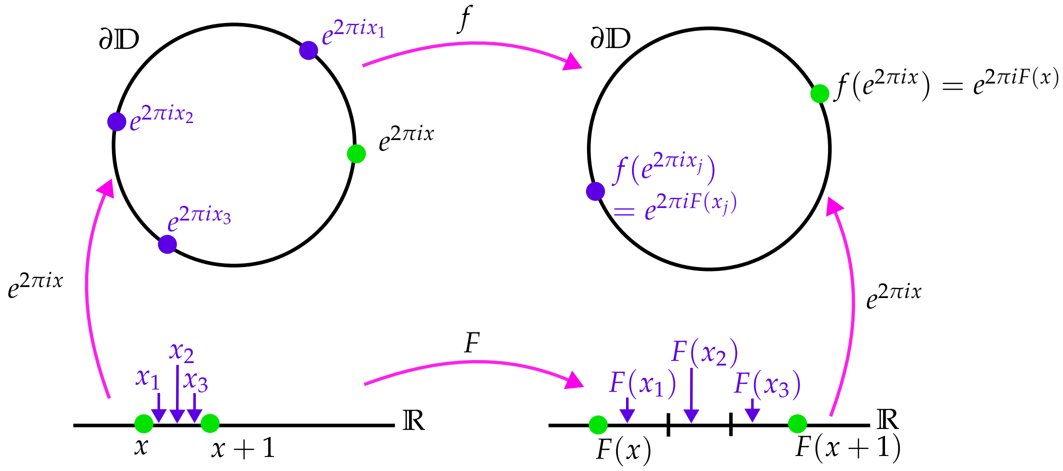


Figure 2.4: Sketch of a lift of a self-map of $\partial\mathbb{D}$ of degree 3. Note that f covers $\partial\mathbb{D}$ 3 times, and hence, $F(x+1) = F(x) + 3$.

Lifts are also useful regarding proper maps. In fact, if f is a proper map of $\partial\mathbb{D}$, then f has degree k if and only if $F(x+1) = F(x) + k$, for every lift F of f. An illustration of this property is given in Fig. 2.4. This allows us to define the rotation number.

Definition 2.33 (Rotation number)**.** Let f be a continuous self-map of the unit circle. Let F be a lift of f and let $x \in \mathbb{R}$. We say the ***rotation number of*** f is the limit

$$\rho(f) := \lim_{n\to\infty} \frac{F^n(x) - x}{n}.$$

It is not hard to see that this definition does not depend on the choice of lift F or initial point x. For a formal proof, see [dMvS12, Section 1.1].

Chapter 3

Dynamics of finite Blaschke products

In Chapter 2, we introduced a general background in complex dynamics. In this chapter, our focus shifts to the analysis of finite Blaschke products, the core subject of this project. Recall from Chapter 1 that a finite Blaschke product is a map of the form (1.2):

$$B(z) = e^{i\theta} \prod_{l=1}^{k} \frac{z - w_l}{1 - \overline{w_l} z}, \quad \text{for } \theta \in \mathbb{R}, w_l \in \mathbb{D}$$

Generally, the first step in characterising the dynamics of any rational map is determining their Julia and Fatou sets. As we saw in Chapter 1, a finite Blaschke product B can be studied both as a rational map $B : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ and as a self-map of the unit disk $B : \mathbb{D} \longrightarrow \mathbb{D}$. As a consequence, their dynamics are highly influenced by both the Wolff-Denjoy Theorem 2.31 and the Classification Theorem 2.30. The results presented in this section are a reflection of this duality, and comprise original proofs and examples from [CG93, Section III.1] and [Fle14, Section 1.4]. Let us start with some general remarks about the dynamics of FBP.

Proposition 3.1. *Let B be a finite Blaschke product of degree $d \geq 2$ and let z_0 be the Wolff-Denjoy point of B.*

(a) *The Fatou set of B contains $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Equivalently, the Julia set of B is contained in $\partial\mathbb{D}$.*

(b) *If $z_0 \in \mathbb{D}$, iterates of B tend to z_0 in $\mathbb{D}$ and to $1/\overline{z_0} \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. If $z_0 \in \partial\mathbb{D}$, then iterates of B tend to z_0 in $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.*

In particular, both $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are contained in invariant Fatou components.

(c) *If $z_0 \in \mathbb{D}$, z_0 and $1/\overline{z_0}$ are fixed points of the same type.*

(d) *All fixed points of B different from z_0 and $1/\overline{z_0}$ must be in $\partial\mathbb{D}$ and must be repelling.*

Proof. (a) As both $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are forward invariant, iterates on these domains leave out more than three points of $\hat{\mathbb{C}}$. Then, by Montel's Theorem 2.5, they are contained in $\mathcal{F}$.

(b) By the Wolff-Denjoy Theorem 2.31, all iterates of B tend to z_0 in $\mathbb{D}$. Let $z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and let us consider the reflection point $z_R = 1/\overline{z} \in \mathbb{D}$. Because of the Wolff-Denjoy Theorem, the sequence $(B^n(z_R))_n$ converges to z_0, and so, $(1/\overline{B^n(z_R)})_n$ converges to $1/\overline{z_0}$. Now, thanks to the reflection property of FBP, we have that $B^n(z) = 1/\overline{B^n(1/\overline{z})} = 1/\overline{B^n(z_R)}$ for all $n \geq 0$, and so the sequence $(B^n(z))_n$ tends to $1/\overline{z_0}$ for $z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Next, as both $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are connected, they must be contained in a Fatou component. Furthermore, as they are forward invariant, by Proposition 2.25 these Fatou components must be sent onto themselves, i.e., they are invariant Fatou components. By Classification Theorem 2.30, these must be immediate basins of attraction.

(c) This is a direct consequence of Proposition 1.7.

(d) Since iterates of all points in $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ must tend to z_0 and $1/\overline{z_0}$, respectively, there cannot be any other fixed points in these domains. Hence, they all belong to $\partial\mathbb{D}$.

Moreover, if they were super-attracting, attracting or parabolic, they would have a basin of attraction intersecting $\mathbb{D}$ or $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, which cannot happen since points there would not tend to the Wolff-Denjoy point or its reflection. Since they cannot be irrationally indifferent, since its derivative is real due to 1.8(b), they must be repelling.

□

Note that z_0 must be either super-attracting, attracting, or parabolic. In fact, if $z_0 \in \mathbb{D}$, it must be in the Fatou set of B, and so, it must be either super-attracting or attracting. If $z_0 \in \partial\mathbb{D}$, due to Lemma 1.8(a) and Proposition 1.8(b), it must be either attracting or parabolic.

All these results indicate that a great part of the dynamics of finite Blaschke products is determined by the Wolff-Denjoy point. Then, it is natural to establish the following classification:

Definition 3.2 (Classification of finite Blaschke products)**.** Let B be a finite Blaschke product of degree $d \geq 2$, and let $z_0 \in \overline{\mathbb{D}}$ be its Wolff-Denjoy point. We say B is

(a) ***elliptic*** if $z_0 \in \mathbb{D}$, and so, $|B'(z_0)| < 1$,

(b) ***hyperbolic*** if $z_0 \in \partial\mathbb{D}$ and $0 < B'(z_0) < 1$,

(c) ***parabolic*** if $z_0 \in \partial\mathbb{D}$ and $B'(z_0) = 1$.

Furthermore, the parabolic case can be divided in two.

Proposition 3.3. *(Classification of parabolic FBP) Let B be a finite Blaschke product of degree $d \geq 2$, and let $z_0 \in \overline{\mathbb{D}}$ be its Wolff-Denjoy point. If z_0 is parabolic, then the Fatou set of B is composed of either:*

(a) *one Leau domain intersecting $\partial\mathbb{D}$, and z_0 has multiplicity* 2.

(b) *two Leau domains with common border $\partial\mathbb{D}$, and z_0 has multiplicity* 3.

We say that in the first case B is **simply parabolic** *and that in the second, B is* **doubly parabolic***.*

Proof. First, note that since $z_0 \in \partial\mathbb{D}$, we have $z_0 = 1/\overline{z_0}$. Hence, as z_0 is parabolic, by Proposition 3.1(b), both $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are contained in Leau domains.

In the first case, both $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are contained in the same Leau domain. Hence, as Leau domains are connected, it must have some points in $\partial\mathbb{D}$. Since there is only one Leau domain, by the Leau-Fatou Flower Theorem 2.23, z_0 must have multiplicity 2.

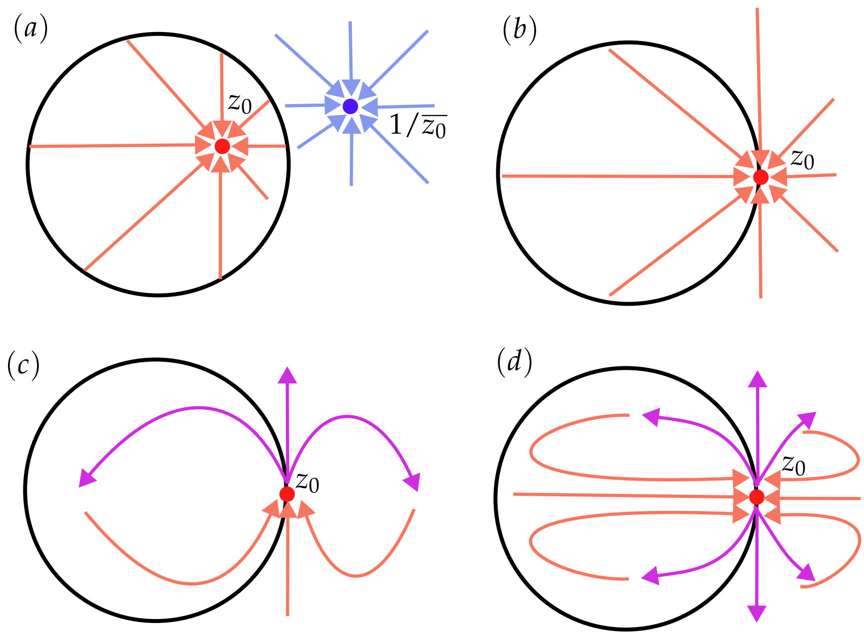


Figure 3.1: Classification of finite Blaschke products as a function of the Wolff-Denjoy point z_0: (a) is elliptic, (b) hyperbolic, and (c) and (d) are simply and doubly parabolic, respectively.

Alternatively, if $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are contained in different Leau domains, then they must be disjoint, and so, they cannot intersect $\partial\mathbb{D}$. Hence, the Fatou set consists of only two Leau domains, which are exactly $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. By the Leau-Fatou Flower Theorem 2.23, z_0 must have multiplicity 3. □

Note that this last result implies that parabolic fixed points of finite Blaschke products cannot have multiplicity greater than 3. Hence, this classification reduces all finite Blaschke products to only four cases, and we will be able to determine the dynamics in each one of them.

Lemma 3.4. *Let B be a finite Blaschke product of degree $d \geq 2$ and $\mathcal{J}$ its Julia set. Then, $\mathcal{J}$ is either $\partial\mathbb{D}$ or a Cantor subset of $\partial\mathbb{D}$, i.e., a closed, perfect and totally disconnected subset of $\partial\mathbb{D}$.*

Proof. By Proposition 3.1(a), we know that $\mathcal{J} \subseteq \partial\mathbb{D}$. Assume now that $\mathcal{J}$ is not the whole $\partial\mathbb{D}$. As the Fatou set $\mathcal{F}$ is open (Property (2) of Proposition 2.8), we can find an arc in $\partial\mathbb{D} \cap \mathcal{F}$ with endpoints in the Julia set (note that $\mathcal{J}$ is not empty, thanks to Property (4) of Proposition 2.8). Then, we can find a sequence $(a_n)_n \subseteq \mathcal{F} \cap \partial\mathbb{D}$ such that a_n converges to some $z \in \mathcal{J}$.

Next, by Proposition 1.8(d), for all $w \in B^{-1}(z)$, there exist open arcs[1] I_z and I_w with $z \in I_z$ and $w \in I_w$ such that $B : I_w \longrightarrow I_z$ is a diffeomorphism. Since the sequence $(a_n)_n$ is eventually contained in I_z, the sequence $(B|_{I_z}^{-1}(a_n))_n$ converges to w.

Since $\mathcal{F}$ is completely invariant, this new sequence must also be contained in the Fatou set. Hence, points in $B^{-1}(w)$ are also limit points of $\mathcal{F} \cap \partial\mathbb{D}$. By simple recursion, we have that for all $n \geq 0$, all points in $B^{-n}(z)$ are limit points of $\mathcal{F} \cap \partial\mathbb{D}$.

[1] An open arc is a set of $\partial\mathbb{D}$ whose lift is an open interval of $\mathbb{R}$. More details will be given in Section 4.2.2.

Then, if we denote $(\mathcal{F} \cap \partial\mathbb{D})'$ as the set of limit points of $\mathcal{F} \cap \partial\mathbb{D}$, we have that:

$$\bigcup_{n\geq 0} B^{-n}(z) \subseteq (\mathcal{F} \cap \partial\mathbb{D})' \Longrightarrow \overline{\bigcup_{n\geq 0} B^{-n}(z)} \subseteq \overline{(\mathcal{F} \cap \partial\mathbb{D})'} = (\mathcal{F} \cap \partial\mathbb{D})',$$

since $\mathcal{F} \cap \partial\mathbb{D}$ contains no isolated points. Now, by Property (5) in Proposition 2.9,

$$\mathcal{J} = \overline{\bigcup_{n\geq 0} B^{-n}(z)} \subseteq (\mathcal{F} \cap \partial\mathbb{D})'.$$

In other words, all points in the Julia set are limit points of $\mathcal{F} \cap \partial\mathbb{D}$. Hence, there can be no arcs in $\mathcal{J}$, i.e. it is totally disconnected. By Properties (2) and (6) (in Propositions 2.8 and 2.9, respectively), it is also closed and perfect, and so, $\mathcal{J}$ is a Cantor set of $\partial\mathbb{D}$. □

Therefore, we reduced the dynamics of finite Blaschke products to only two cases, according to their Fatou and Julia sets. The following theorem characterise these two cases.

Theorem 3.5 (Characterisation of the dynamics of FBP)**.** *Let B be a finite Blaschke product of degree $d \geq 2$. Then, the following holds.*

(a) If B is elliptic, then the Julia set of B is $\partial\mathbb{D}$ and the Fatou set of B consists of either two Bötcher domains or two Schröder domains.

(b) If B is hyperbolic, then the Julia set of B is a Cantor subset of $\partial\mathbb{D}$ and the Fatou set of B consists of one Schröder domain.

(c) If B is simply parabolic, the Julia set of B is a Cantor subset of $\partial\mathbb{D}$ and the Fatou set of B consists of one Leau domain.

(d) If B is doubly parabolic, the Julia set of B is $\partial\mathbb{D}$ and the Fatou set of B consists of two Leau domains.

Proof. Let $z_0 \in \overline{\mathbb{D}}$ be the Wolff-Denjoy point of B. We shall work through all cases separately.

- If B is elliptic, then iterates for all points in $\mathbb{D}$ tend to $z_0 \in \mathbb{D}$. Then, if any point in $\partial\mathbb{D}$ was in the Fatou set, their iterates would also need to converge to z_0. However, $\partial\mathbb{D}$ is forward invariant, so its iterates cannot escape from $\partial\mathbb{D}$. In other words, we must have $\mathcal{J} = \partial\mathbb{D}$.

- If B is hyperbolic, then $z_0 \in \partial\mathbb{D}$ is an attracting fixed point, which means that it must be contained in the Fatou set. Then, by Lemma 3.4, $\mathcal{J}$ is a Cantor set of $\partial\mathbb{D}$.

- The simply and doubly parabolic cases are a direct result of Proposition 3.3 and Lemma 3.4.

□

Finally, in order to dive into the local behaviour of finite Blaschke products, we can look into their fixed points.

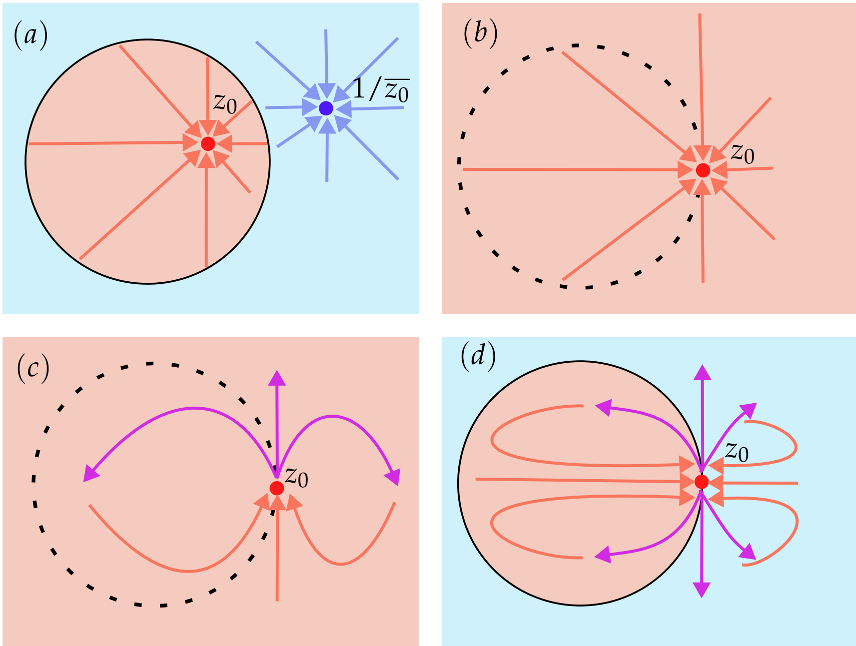


Figure 3.2: Characterisation of the dynamics of FBP. (a) shows the elliptic case, where the Fatou set has two connected components, $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. (b) and (c) show the hyperbolic and simply parabolic cases, respectively, where the Fatou set has only one connected component and the Julia set is a Cantor set of $\partial\mathbb{D}$. Finally, (d) shows the doubly parabolic case, where the Fatou set has again two connected components, $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Theorem 3.6 (Characterisation of fixed points of FBP). *Let B be a finite Blaschke product of degree $d \geq 2$. Then,*

(a) If B is elliptic, it has exactly $d-1$ different fixed points in $\partial\mathbb{D}$, one in $\mathbb{D}$ and another one in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

(b) If B is hyperbolic, it has exactly $d+1$ different fixed points, all of which are in $\partial\mathbb{D}$.

(c) If B is simply parabolic, it has exactly d different fixed points, all of which are in $\partial\mathbb{D}$.

(d) If B is doubly parabolic, it has exactly $d-1$ different fixed points, all of which are in $\partial\mathbb{D}$.

Proof. Let us recall that fixed points are the solutions of the equation $B(z) = z$, which, since the numerator and denominator of B are degree d polynomials, must have $d+1$ solutions, counted with multiplicity. Note that a fixed point z^* has multiplicity 1 if $B'(z^*) \neq 1$ and multiplicity $n \geq 2$ if $B'(z^*) = 1$ and n is minimal such that $B^{(n)}(z^*) \neq 0$.

Moreover, by Proposition 3.1, the number of solutions of $B(z) = z$ depends entirely on the Wolff-Denjoy point z_0 and its reflection $1/\overline{z_0}$. In fact, for $z_0 \in \mathbb{D}$, we have $z_0 \neq 1/\overline{z_0}$, and both are attracting or super-attracting. So, they have multiplicity 1 and there must be other $d-1$ different fixed points in $\partial\mathbb{D}$ with multiplicity 1, for a total of $d+1$ fixed points.

On the other hand, if $z_0 \in \partial\mathbb{D}$, now $z_0 = 1/\overline{z_0}$ and everything depends on the multiplicity of z_0. If z_0 has multiplicity m, then there are other $d+1-m$ different fixed points of multiplicity 1 in $\partial\mathbb{D}$. The total of fixed points in this case is $d-m+2$ in $\partial\mathbb{D}$.

If we look at each case separately:

(a) If B is elliptic, $z_0 \in \mathbb{D}$ and hence we have 1 fixed point in $\mathbb{D}$, one in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $d-1$ in $\partial\mathbb{D}$.

(b) If B is hyperbolic, $B'(z_0) < 1$, and so, it has multiplicity 1. Then, there are exactly $d+1$ fixed points in $\partial\mathbb{D}$.

(c) For simply parabolic B, by Proposition 3.3, z_0 has multiplicity 2 and so there are d different fixed points in $\partial\mathbb{D}$.

(d) Finally, if B is doubly parabolic, by Proposition 3.3, z_0 has multiplicity 3 and so there are $d-1$ different fixed points in $\partial\mathbb{D}$.

□

Chapter 4

Dynamics of finite Blaschke products in the unit circle

In the previous chapter, we just saw that due to the Wolff-Denjoy Theorem, the dynamics of finite Blaschke products in both $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is simple and well-understood. For this reason, the focus of this section is studying the behaviour of FBP in $\partial\mathbb{D}$.

To do so, we first prove the existence of a conjugacy in $\partial\mathbb{D}$ between any elliptic or doubly parabolic FBP and the map z^d, which has some remarkable dynamical properties. Afterwards, we find this conjugacy explicitly.

The results of this section include original proofs and background from [dMvS12, Section II.2] and [Dev21, Chapteres 1.6 to 1.8].

4.1 Abstract conjugacy in the unit circle

We shall start by stating a well-known result in this topic, ***Shub's Theorem***. This result allows us to partially characterise the behaviour of FBP in $\partial\mathbb{D}$, by finding a semi-conjugacy with a simple map, which can be taken to be z^d, where d is the degree of the FBP. Before starting and proving Shub's Theorem, we need to introduce some general concepts about circle maps, discussed in [dMvS12, Section II.2].

Definition 4.1 (Covering)**.** Let $f : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ be a surjective local homeomorphism of the unit circle. We say f is a ***covering of degree*** d if every point has exactly $|d|$ pre-images, where $d > 0$ if f is orientation-preserving and $d < 0$ otherwise.

Due to Propositions 1.8(a) and 1.8(d), all finite Blaschke products of degree d are degree d coverings of $\partial\mathbb{D}$, with $d > 0$.

Definition 4.2 (Expanding map)**.** Let $f : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ be a $\mathcal{C}^1$ map of $\partial\mathbb{D}$. We say f is expanding if there exist constants $C > 0$, $\lambda \geq 1$ such that:

$$\left|(f^n)'(x)\right| > C\lambda^n,$$

for all $n \geq 1$ and $x \in \mathbb{D}$.

Notice that every expanding map of $\partial\mathbb{D}$ is a degree d covering for some $|d| \geq 2$. We are now ready to state Shub's Theorem.

Theorem 4.3 (Shub's Theorem)**.** *Let $f : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ be a degree d expanding map and $g : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ a degree d covering map. Then, there exists a unique semi-conjugacy between f and g in $\partial\mathbb{D}$, i.e., a monotone, surjective and continuous map $h : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ such that*

$$h \circ g = f \circ h$$

The idea of the proof of this theorem, which can be found in [dMvS12, Theorem II.2.1] is to build a function space and a suitable operator in order to apply Banach Fixed Point Theorem A.11.

Proof of Shub's Theorem. Let us consider $F, G : \mathbb{R} \longrightarrow \mathbb{R}$ lifts of f and g, respectively. Note that as f and g are $\mathcal{C}^1$, we can choose F and G as diffeomorphisms.

Furthermore, since F^n is a lift of f^n, we have

$$e^{2\pi i F^n(x)} = f^n(e^{2\pi i x}) \Longrightarrow (F^n)'(x) = (f^n)'(e^{2\pi i x})e^{2\pi i(x - F^n(x))}.$$

And, since f is expanding, there exist $C > 0, \lambda > 1$ such that

$$\left|(F^n)'(x)\right| = \left|(f^n)'(e^{2\pi i x})\right| > C\lambda^n, \quad \forall x \in \mathbb{R}.$$

Using this and the Inverse Function Theorem,

$$\left|(F^{-n})'(x)\right| = \frac{1}{|(F^n)'(F^{-n}(x))|} < K\rho^n, \quad \forall x \in \mathbb{R}. \tag{4.1}$$

with $K = 1/C < \infty, \rho = 1/\lambda < 1$.

STEP 1: Definition of the function space. Consider the space of continuous bounded maps of $\mathbb{R}$, denoted by $\mathcal{C}_B(\mathbb{R})$, with the supremum distance

$$|\phi_1 - \phi_2| := \sup_{x \in \mathbb{R}} |\phi_1(x) - \phi_2(x)| < \infty, \quad \forall \phi_1, \phi_2 \in \mathcal{C}_B(\mathbb{R}).$$

Endowed with this distance, $\mathcal{C}_B(\mathbb{R})$ is a complete metric space. Now, let us define

$$\mathcal{E} := \{\phi : \mathbb{R} \longrightarrow \mathbb{R} : \phi(x+1) = \phi(x) + 1, \phi \text{ continuous}\}.$$

Note that $\mathcal{E} \subseteq \mathcal{C}_B(\mathbb{R})$, since for all $\phi \in \mathcal{E}$, $\sup_{x \in \mathbb{R}} |\phi(x)| = \sup_{x \in [0,1]} |\phi(x)|$ and $[0,1]$ is compact.

Furthermore, $\mathcal{E}$ is a closed subset of $\mathcal{C}_B(\mathbb{R})$. Indeed, if we have $(\phi_n)_n \subseteq \mathcal{E}$ converging to a map $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ with the supremum distance, then the convergence as maps of $\mathbb{R}$ is uniform. In particular, ϕ must be continuous and satisfy $\phi(x+1) = \phi(x) + 1$, i.e. $\phi \in \mathcal{E}$.

Since $\mathcal{E}$ is a closed subset of a complete metric space, it is also a complete metric space. This means that we can apply Banach Fixed Point Theorem in $\mathcal{E}$, as long as we find a suitable contractible operator.

STEP 2: Define a contractible operator. Consider the operator $T : \mathcal{E} \longrightarrow \mathcal{E}$, defined as

$$T(\phi) = T\phi := F^{-1} \circ \phi \circ G, \quad \forall \phi \in \mathcal{E}.$$

Observe that $T\phi$ is continuous, and

$$T\phi(x+1) = F^{-1}(\phi(G(x+1))) = F^{-1}(\phi(G(x)+d)) = \\ = F^{-1}(\phi \circ G(x) + d) = F^{-1} \circ \phi \circ G(x) + 1 = T\phi(x) + 1, \quad \forall x \in \mathbb{R}.$$

since F and G are lifts of degree d covering maps. Thus, the operator $T : \mathcal{E} \longrightarrow \mathcal{E}$ is well-defined.

Next, we see that, for n large enough, T^n is contractible. Indeed, let $n \geq 1$ and $\phi_1, \phi_2 \in \mathcal{E}$. Then, for all $x \in \mathbb{R}$,

$$\begin{aligned} |T^n\phi_1(x) - T^n\phi_2(x)| &= \left|F^{-n}(\phi_1 \circ G(x)) - F^{-n}(\phi_2 \circ G(x))\right| = \\ &= \left|(F^{-n})'(\xi)\right| |\phi_1 \circ G(x) - \phi_2 \circ G(x)| \leq \\ &\leq K\rho^n \sup_{x \in \mathbb{R}} |\phi_1(x) - \phi_2(x)| = K\rho^n |\phi_1 - \phi_2|, \end{aligned}$$

where we applied Lagrange Mean Value Theorem for some $\xi \in \mathbb{R}$ and Eq. (4.1). Hence, since $\rho < 1$, there exists n large enough so that $\rho^n < K$, and

$$|T^n\phi_1 - T^n\phi_2| < |\phi_1 - \phi_2|.$$

In other words, T^n is contractible for $n \geq 1$ large enough. Hence, by Banach Fixed Point Theorem A.11, there exists $H \in \mathcal{E}$ such that $TH = H$ and, furthermore, $(T^n\phi)_n$ converges to H, for all $\phi \in \mathcal{E}$. Since $H \in \mathcal{E}$, H is continuous and satisfies $H(x+1) = H(x) + 1$. Thus, H is the lift of a continuous circle map $h : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ of degree 1, i.e. $h(e^{ix}) = e^{iH(x)}$. We shall see that this map h satisfies the requirements of the theorem.

First, note that since H is a fixed point of T, we have $TH = F^{-1} \circ H \circ G = H$. Thus, $H \circ G = F \circ H$. Applying the exponential function in both sides, we get $h \circ g = f \circ h$, as we wanted to see.

Now, it is left to see that h is in fact a semi-conjugacy. We already have that it is continuous and surjective. We claim it is monotonous. Indeed, using that H is the limit of $T^n\phi$ as n tends to ∞ for every map $\phi \in \mathcal{E}$, we have that it is, in particular, the limit of $T^n\mathrm{id} = F^{-n} \circ \mathrm{id} \circ G^n = F^{-n} \circ G^n$. As F and G are diffeomorphisms, we have that $F^{-n} \circ G^n = T^n\mathrm{id}$ must be so for every $n \geq 1$. Thus, H is the uniform limit of a sequence of strictly monotonous maps. Hence, it is monotonous. As a straight-forward consequence, h is monotonous as well.

In conclusion, the map $h : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ is continuous, monotonous, surjective, and satisfies $h \circ g(z) = f \circ h(z)$ for all $z \in \partial\mathbb{D}$, as we wanted to see.

Finally, the uniqueness of h comes directly from the uniqueness of H. □

Now, back to our matter of study, for any finite Blaschke product B of degree $d \geq 2$ we can consider the semi-conjugacy to the expanding map $f(z) = z^d$ (note $f^n(z) = z^{d^n}$, so $(f^n)'(z) = d^n$). The relevant question now is whether this semi-conjugacy is a topological conjugacy or not. We have the following result, illustrated in Fig. 4.1.

Theorem 4.4 (Shub's conjugacy)**.** *Let B be a finite Blaschke product of degree $d \geq 2$ and $h : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ its semi-conjugacy to $f(z) = z^d$ given by Shub's Theorem 4.3. Then, h is a conjugacy if and only if B is elliptic or doubly parabolic.*

Proof. Let B be elliptic or doubly parabolic. First, let us assume h is not a conjugacy. By Shub's Theorem 4.3, the only possibility is that h is not strictly monotonous but just monotonous. In other words, there exist $z_1, z_2 \in \partial\mathbb{D}$ such that $z_1 \neq z_2$ and $h(z_1) = h(z_2)$. Since h is monotonous, it must also be constant in the whole arc I between z_1 and z_2. In other words, $h(I) = p$ for some $p \in \partial\mathbb{D}$.

Second, consider any domain D such that $D \cap \partial\mathbb{D} = I$. By Theorem 3.5, since B is elliptic or doubly parabolic, the Julia set of B, $\mathcal{J}$, is equal to $\partial\mathbb{D}$. Hence, by the blow-up property (8) from 2.9, there exists $n \geq 0$ such that $B^n(D \cap \mathcal{J}) = B^n(D \cap \partial\mathbb{D}) = B^n(I) = \mathcal{J} = \partial\mathbb{D}$. Then, we have

$$\begin{aligned} &h \circ g^n(I) = h(\partial\mathbb{D}) = \partial\mathbb{D}, \quad \text{since } h \text{ is onto } \partial\mathbb{D} \\ &h \circ g^n(I) = (h \circ g) \circ g^{n-1} = (f \circ h) \circ g^{n-1}(I) = \cdots = f^n \circ h(I) = f^n(p) \end{aligned}$$

which is impossible, since $f^n(p)$ is a single point and cannot be the whole $\partial\mathbb{D}$. In conclusion,

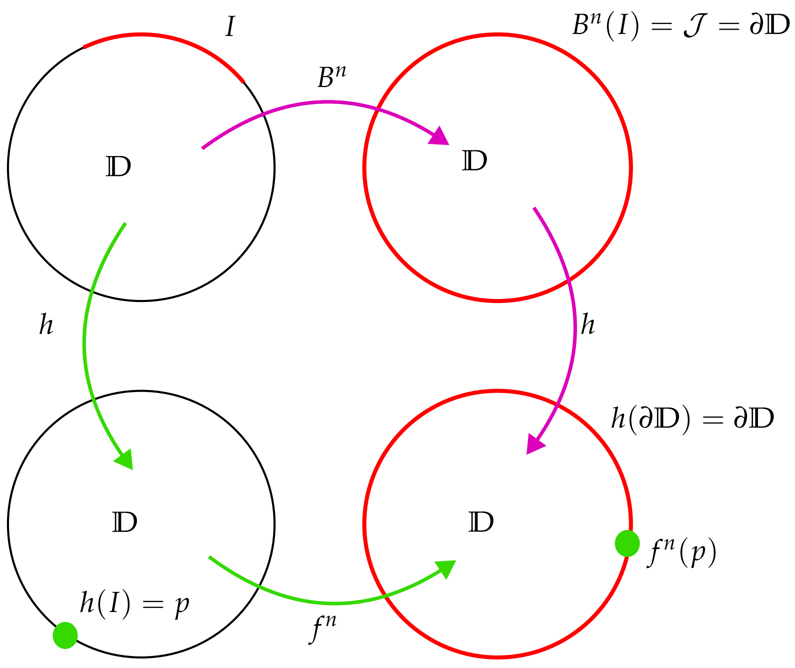


Figure 4.1: Contradiction of Theorem 4.4 for an elliptic or doubly parabolic finite Blaschke product.

h must be strictly monotonous. Now, since h is bijective and continuous, and $\partial\mathbb{D}$ is compact and Hausdorff, it must be a homeomorphism.[1] In other words, h is a topological conjugacy.

Finally, if B is hyperbolic or simply parabolic, by Theorem 3.6 it must have $d+1$ or d distinct fixed points in $\partial\mathbb{D}$, respectively. Since z^d has exactly $d-1$ fixed points in $\partial\mathbb{D}$, there cannot be any topological conjugacy between B and z^d. □

[1] This is a well-known result from topology, detailed in every basic course of metric spaces. For instance, see [Sut75, Proposition 13.26].

4.2 Study of Shub's conjugacy

We just saw that the dynamics of elliptic and doubly parabolic finite Blaschke products is relatively simple. Outside of the unit circle, all points tend to the Wolff-Denjoy point (or its reflected point, in the elliptic case), and in the unit circle, the function is topologically conjugate to z^d where $d \geq 2$ is the degree of the Blaschke product. The only loose end left to tie is the study of this conjugacy. In this section, we find it explicitly by taking an intermediate step: symbolic dynamics.

4.2.1 Symbolic dynamics in the unit circle

In order to characterise the topological conjugacy given in Shub's Conjugacy Theorem 4.4 between an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$, and the map z^d, we shall introduce the framework of symbolic dynamics. More background can be found in [KH95, Section 1.9], [Dev21, Chapters 1.6, 1.7].

Definition 4.5 (Symbolic metric space)**.** For an integer $d \geq 2$, let us define the symbolic metric space (of d symbols) as $(\Sigma_d, \mathcal{D})$, where Σ_d is the set of sequences of d symbols

$$\Sigma_d := \{\underline{s} = s_0 s_1 \ldots | s_n \in \{1, \ldots, d\}\, \forall n \geq 0\},$$

and $\mathcal{D}$ is defined as

$$\mathcal{D}(\underline{s}, \underline{t}) := \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{d^n}, \quad \forall \underline{s}, \underline{t} \in \Sigma_d, \underline{s} = s_0 s_1 \ldots, \underline{t} = t_0 t_1 \ldots$$

We note that $\mathcal{D}$ is well-defined and it is indeed a distance in Σ_d. Indeed, since for each $n \geq 0$, $s_n, t_n \in \{1, \ldots, d\}$, their difference is smaller than $d-1$. Thus, the series in the definition of $\mathcal{D}$ can be bounded by $(d-1)\sum_n (1/d^n) = d$. Taking into account the properties of the absolute value, it is straight-forward that $\mathcal{D}$ is a distance in Σ_d.

Remark 4.6. We denote the elements of Σ_d as $\underline{s}$, without indicating every time that $\underline{s} = s_0 s_1 \ldots$. Moreover, if any sequence is eventually constant, i.e. $\underline{s}$ satisfies $s_n = c$ for all $n \geq n_0 + 1$, we denote $\underline{s} = s_0 \ldots s_{n_0} \overline{c}$.

Proposition 4.7. *Let $(\Sigma_d, \mathcal{D})$ be the symbolic metric space for $d \geq 2$, and let $k \geq 0$. For $\underline{s}, \underline{t} \in \Sigma_d$, the distance $\mathcal{D}$ satisfies the following.*

- *If $s_n = t_n$ for $n \in \{0, \ldots, k\}$, we have $\mathcal{D}(\underline{s}, \underline{t}) \leq \frac{1}{d^k}$.*
- *If $\mathcal{D}(\underline{s}, \underline{t}) < \frac{1}{d^k}$, then $s_n = t_n$ for $n \in \{0, \ldots, k\}$.*

Proof. First, assume $s_n = t_n$ for $n \in \{0, \ldots, k\}$. Then,

$$\mathcal{D}(\underline{s}, \underline{t}) = \sum_{n=k+1}^{\infty} \frac{|s_n - t_n|}{d^n} \leq \sum_{n=k+1}^{\infty} \frac{(d-1)}{d^n} = (d-1) \sum_{n=k+1}^{\infty} \frac{1}{d^n} = (d-1) \frac{(1/d)^{k+1}}{1-(1/d)} = \frac{1}{d^k}.$$

Second, for the other statement, assume that there exists $n_0 \in \{0, \ldots, k\}$, such that $s_{n_0} \neq t_{n_0}$. We have

$$\mathcal{D}(\underline{s}, \underline{t}) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{d^n} \geq \frac{|s_{n_0} - t_{n_0}|}{d^{n_0}} \geq \frac{1}{d^{n_0}} \geq \frac{1}{d^k}.$$

Hence, conversely, if we have $k \geq 0$ such that $\mathcal{D}(\underline{s}, \underline{t}) < 1/d^k$, then $s_n = t_n$ for $n \in \{0, \ldots, k\}$, as we wanted to see. □

Definition 4.8 (Shift map)**.** Let $(\Sigma_d, \mathcal{D})$ be the symbolic metric space for $d \geq 2$. The ***shift map*** of Σ_d is defined by

$$\sigma(s_0 s_1 s_2 \ldots) = s_1 s_2 \ldots, \quad \forall \underline{s} = s_0 s_1 s_2 \cdots \in \Sigma_d.$$

Proposition 4.9. *Let $(\Sigma_d, \mathcal{D})$ be the symbolic metric space for $d \geq 2$ and let σ be the shift map of Σ_d. Then, σ is uniformly continuous.*

Proof. Let $\varepsilon > 0$. Choose $k \geq 1$ such that $(1/d^{k-1}) < \varepsilon$. Then, for $\delta \leq (1/d^{k+1})$, and $\underline{s}, \underline{t} \in \Sigma_d$, we have that if $\mathcal{D}(\underline{s}, \underline{t}) < \delta \leq (1/d^k)$, then $s_n = t_n$ for $n \in \{0, \ldots, k\}$, by Proposition 4.7.

In other words, the first n symbols of the sequences $\sigma(\underline{s})$ and $\sigma(\underline{t})$ coincide. Hence, by Proposition 4.7, we have $\mathcal{D}(\underline{s}, \underline{t}) \leq \frac{1}{d^{k-1}} < \varepsilon$, so $\mathcal{D}$ is uniformly continuous. □

4.2.2 Finding the conjugacy explicitly

Finally, we shall find Shub's conjugacy between any elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$ and z^d explicitly. To do so, we will take an intermediate step and find a relationship between the finite Blaschke product and the shift map.

To achieve this, we divide the unit circle into d numbered regions and study which of these are visited by any point in $\partial\mathbb{D}$ under iteration. Keeping track on this list of visited regions, we are able to assign a sequence in Σ_d to each point of the unit circle. While it is well-known, as mentioned in [IU23, Section 8.1], that finite Blaschke products in the unit circle admit this construction, it has not yet been formally written.

First of all, we briefly discuss the concept of arcs in $\partial\mathbb{D}$ with some formality.

Definition 4.10 (Ordering in $\partial\mathbb{D}$)**.** Let $p \in \partial\mathbb{D}$ and let φ_p denote a branch cut of the argument in $\partial\mathbb{D} \setminus \{p\}$. We define the ***ordering of*** $\partial\mathbb{D}$ ***with origin*** p as the order relation $\leq_p$ defined by φ_p. That is,

- $z \leq_p w$ if and only if $\varphi_p(z) \leq \varphi_p(w)$, for all $z, w \in \partial\mathbb{D} \setminus \{p\}$,
- $p \leq_p z$ for all $z \in \partial\mathbb{D}$.

.

Note that the definition does not depend in the branch cut of the argument, since two different branch cuts are equal up to additive constant.

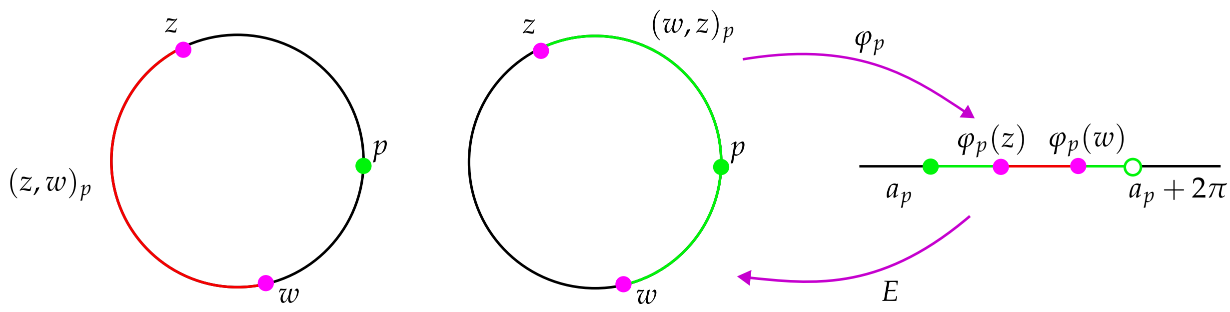


Figure 4.2: Sketch of the p-ordering of $\partial\mathbb{D}$, where $p \leq_p z \leq_p w$, and of the arcs $(z,w)_p$, in red, and $(w,z)_p$, in green.

Definition 4.11. Let $p, z, w \in \partial\mathbb{D}$ and consider the ordering of $\partial\mathbb{D}$ of origin p, $\leq_p$. Let $\varphi_p : \partial\mathbb{D} \longrightarrow [a_p, a_p + 2\pi)$ be any branch cut of the argument with $\varphi_p(p) = a_p$, for some $a_p \in \mathbb{R}$. Let $E(t) = e^{it}$. We define the ***(open) arc of endpoints*** z, w ***in the ordering of*** p as the arc

$$(z,w)_p := \begin{cases} E((z,w)), & \text{if } z \leq_p w \\ E((z, a_p + 2\pi]) \cup E([a_p, w)), & \text{if } z >_p w. \end{cases}$$

We can also define the ***closed arc*** $[z,w]_p := (z,w)_p \cup \{z,w\}$ and the ***semi-open arcs*** $[z,w)_p := [z,w]_p \setminus \{w\}$ and $(z,w]_p := [z,w]_p \setminus \{z\}$.

Observation 4.12. It is straight-forward to see that the definition of these arcs does not depend in the branch cut chosen.

Back to finite Blaschke products, we shall start discussing their behaviour through the itineraries of points. If we are working in degree $d \geq 2$, it follows from Proposition 3.6 that elliptic and doubly parabolic FBP have $d - 1 \geq 1$ fixed points in $\partial\mathbb{D}$, and, by Proposition 1.8(a), each fixed point has exactly d different preimages. This is our starting point.

Definition 4.13. *(Induced dynamical partition)* Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$. Let $p \in \partial\mathbb{D}$ be a fixed point of B. Let $B^{-1}(p) = \{z_1 = p, z_2, \dots z_d\}$ be the set of preimages of p, with $z_1 = p <_p z_2 <_p z_3 <_p \cdots <_p z_d$.

Let us consider the set of disjoint semi-open arcs $I_k = [z_k, z_{k+1})$ for $k \in \{1, \dots, d-1\}$, and $I_d = [z_d, p)$. We say $I_1, \dots, I_d$ is the ***dynamical partition of*** $\partial\mathbb{D}$ ***induced by*** B ***with origin*** p.

Observation 4.14. Since, by Proposition 1.8(d), B is a local diffeomorphism in $\partial\mathbb{D}$, we have that $B : I_k \longrightarrow \partial\mathbb{D}$ is holomorphic and one-to-one, for $k \in \{1, \dots, d\}$.

This partition, dependent on B, allows us to "follow" each point through the different iterates of B.

Definition 4.15 (Itinerary of a point and sequence map). Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$, let $p \in \partial\mathbb{D}$ be a fixed point of B, and let $I_1, \dots, I_d$ be the dynamical partition of $\partial\mathbb{D}$ induced by B with origin p. Let Σ_d be the symbolic space of d symbols.

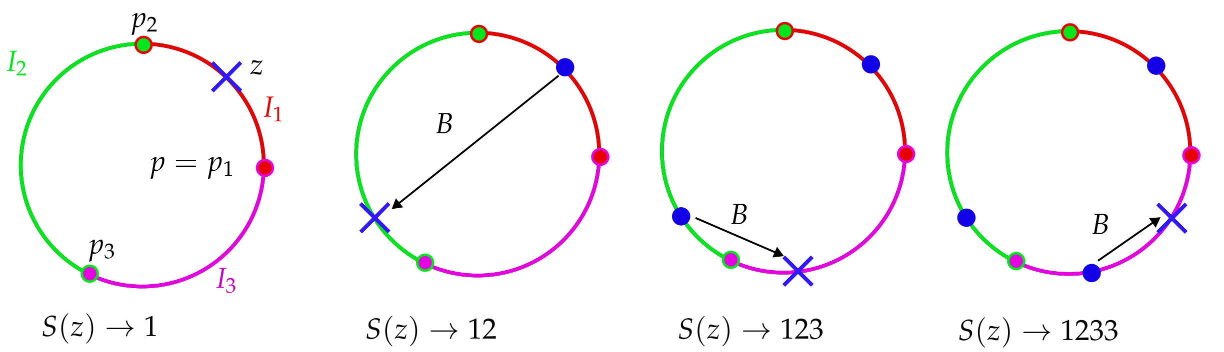


Figure 4.3: Sketch of the itinerary of a point $z \in \partial\mathbb{D}$. Green, red and magenta arcs represent different arcs of the dynamical partition with origin in p.

We say that the sequence $\underline{s} \in \Sigma_d$ is the ***itinerary*** of $z \in \partial\mathbb{D}$ if $B^n(z) \in I_{s_n}$, for each $n \geq 0$, and define the map

$$\begin{aligned} S : \partial\mathbb{D} &\longrightarrow \Sigma_d \\ z &\longmapsto S(z) := \underline{z}, \end{aligned}$$

where $\underline{z}$ is the itinerary of z. We say S is the ***sequence map of*** B (with origin p).

Example 4.16. For instance, the itinerary of the fixed point p is $\underline{p} = 111\cdots = \overline{1}$, since $B^n(p) = p \in I_1$ for all $n \geq 0$.

Therefore, we divided $\partial\mathbb{D}$ into d arcs, which allow us to assign to each point on $\partial\mathbb{D}$ a sequence in Σ_d, its itinerary. Hence, a natural question is which sequences of Σ_d can be realised as itineraries, and whether two points can share the same itinerary or not. In other words, our immediate goal is determining whether the sequence map S is one-to-one or not. For this purpose we define the following.

Definition 4.17 (Arc of a given sequence)**.** Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$, let $p \in \partial\mathbb{D}$ be a fixed point of B, and let $I_1, \ldots, I_d$ be the dynamical partition of $\partial\mathbb{D}$ induced by B with origin p. Given $s_0, \ldots s_n \in \{1, \ldots, d\}$, we define the ***arc of sequence*** $s_0, \ldots, s_n$ as

$$I_{s_0 \ldots s_n} := I_{s_0} \cap B^{-1}(I_{s_1}) \cap \cdots \cap B^{-n}(I_{s_n}).$$

With this definition, $z \in I_{s_0 \ldots s_n}$ if and only if the itinerary of z starts with $s_0 \ldots s_n$. It can be seen, similarly to Obs. 4.14, that B^{n+1} is holomorphic and one-to-one between any arc $I_{s_0 \ldots s_n}$ and $\partial\mathbb{D}$, for $n \geq 0$.

The following proposition describes precisely the shape of any such arc.

Proposition 4.18. *Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$, let $p \in \partial\mathbb{D}$ be a fixed point of B, and let $I_1, \ldots, I_d$ be the dynamical partition of $\partial\mathbb{D}$ induced by B with origin p. Then, for each finite sequence $s_0 \ldots s_n$ with $s_0, \ldots, s_n \in \{1, \ldots, d\}$, the following is satisfied.*

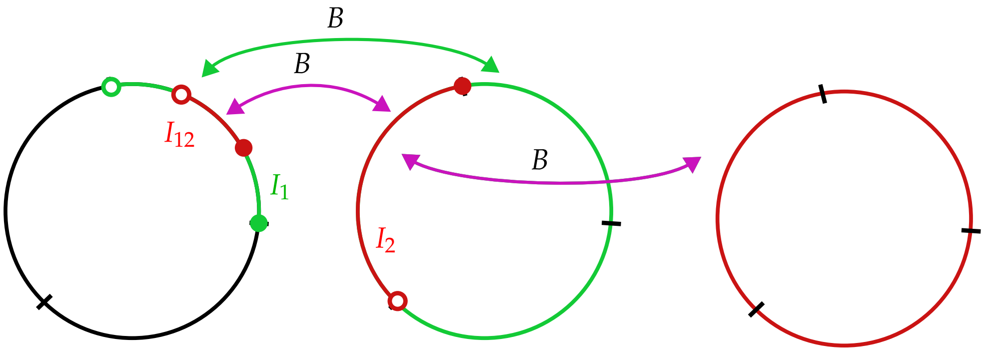


Figure 4.4: Idea behind arcs of a given sequence. B maps I_1 to $\partial\mathbb{D}$ as a diffeomorphism. Then, there is a preimage of I_2 in I_1, which is I_{12}. Hence, B maps I_{12} to I_2 diffeomorphically, and so does B^2 from I_{12} to $\partial\mathbb{D}$.

(a) *$I_{s_0\dots s_n}$ is an arc of the form $[z_n, w_n)_p$, with $z_n, w_n \in \partial\mathbb{D}$ satisfying $B^{n+1}(z_n) = B^{n+1}(w_n) = p$. Moreover, every point in the backward orbit of p is the endpoint of an arc of a certain sequence.*

(b) *$I_{s_0\dots s_n} \subseteq I_{s_0\dots s_{n-1}}$. In other words, for a given sequence, its arc is nested inside the arc of their sub-sequence.*

Proof. (a) We shall prove the statement by induction on the length of the sequences, n. For $n = 0$, we already saw that $I_k = [z, w)_p$ for $z, w \in B^{-1}(p)$.

Next, fix $n \geq 1$ and assume it is true for all $t_0, \dots, t_{n-1} \in \{1, \dots, d\}$. Let $s_0, \dots, s_n \in \{1, \dots, d\}$. We have

$$I_{s_0\dots s_n} = I_{s_0} \cap B^{-1}(I_{s_1\dots s_n})$$

Now, note that, by the induction hypothesis, $I_{s_1\dots s_n} = [z, w)_p$ for $z, w \in \partial\mathbb{D}$ with $B^n(z) = B^n(w) = p$.

In addition, as remarked in Obs. 4.14, $B : I_k \longrightarrow \partial\mathbb{D}$ is holomorphic and one-to-one for $k \in \{1, \dots, d\}$. Then, the preimage $B^{-1}(I_{s_1\dots s_n})$ comprises exactly d semi-open arcs $[z_k, w_k)_p$, each contained in a different I_k, with $B(z_k) = z$ and $B(w_k) = w$. Thus, the intersection $I_{s_0} \cap B^{-1}(I_{s_1\dots s_n})$ must be $[z_{s_0}, w_{s_0})_p$, where $B^{n+1}(z_{s_0}) = B^n(z) = p$ and $B^{n+1}(w_{s_0}) = B^n(w) = p$, as we wanted to see.

Conversely, if $z \in \partial\mathbb{D}$ satisfies $B^n(z) = p$ for some $n \geq 0$, then the itinerary of z is of the form $z_0 \dots z_{n-1}\overline{1}$ for some $z_0, \dots, z_{n-1} \in \{1, \dots, d\}$. Hence, z must be the endpoint of the arc of sequence $I_{z_0\dots z_{n-1}}$, since $B^n : I_{z_0\dots z_{n-1}} \longrightarrow \partial\mathbb{D}$ is bijective.

(b) The proof can be deduced straight from Def. 4.17. We have

$$\begin{aligned} I_{s_0\dots s_n} &= I_{s_0} \cap B^{-1}(I_{s_1}) \cap \dots \cap B^{-(n-1)}(I_{s_{n-1}}) \cap B^{-n}(I_{s_n}) = \\ &= I_{s_0\dots s_{n-1}} \cap B^{-n}(I_{s_n}) \subseteq I_{s_0\dots s_{n-1}}. \end{aligned}$$

□

Therefore, this new concept induces smaller and smaller divisions of $\partial\mathbb{D}$, each one containing points with similar itineraries. Notice that

$$z \in \partial\mathbb{D} \text{ has itinerary } \underline{s} \text{ if and only if } z \in \bigcap_{n=0}^{\infty} I_{s_0\dots s_n}.$$

In particular, if two different points share itinerary, they must be in the same infinite intersection of these kind of arcs. Hence, in order to study the possible itineraries of points of the unit disk, it is useful to study these infinite intersections. For this purpose, it is more convenient to work with closed sets, this means, the closure of the arcs of a given sequence.

Proposition 4.19. *Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$, let $p \in \partial\mathbb{D}$ be a fixed point of B, and let $I_1, \ldots, I_d$ be the dynamical partition of $\partial\mathbb{D}$ induced by B with origin p. Let Σ_d be the symbolic space of d symbols.*

Then, the **intersection map**

$$\begin{aligned} \mathcal{I} : \Sigma_d &\longrightarrow \partial\mathbb{D} \\ \underline{s} &\longmapsto \bigcap_{n=0}^{\infty} \overline{I_{s_0\ldots s_n}}, \end{aligned}$$

is well-defined, where $\overline{I_{s_0\ldots s_n}}$ denotes the closure of the arc of sequence $s_0 \ldots s_n$, for each $n \geq 0$. In particular, each infinite intersection contains exactly one point of $\partial\mathbb{D}$, and, if $z \in \partial\mathbb{D}$ has itinerary $\underline{z}$, we have $\mathcal{I}(\underline{z}) = z$.

Proof. We need to see that this infinite intersection contains exactly one element for each sequence. The basic idea is to apply the Nested Interval Theorem to the closed arcs of sequence.

So, let $\underline{s} \in \Sigma_d$. By Proposition 4.18, for each $n \geq 0$ we have $I_{s_0\ldots s_n} = [z_n, w_n)_p$ for some $z_n, w_n \in \partial\mathbb{D}$. Furthermore, if $\leq_p$ is the ordering of $\partial\mathbb{D}$ with origin p, then by Proposition 4.18, we have $z_n \leq_p z_{n+1} \leq_p w_{n+1} \leq_p w_n$ for each $n \geq 0$. Hence, there exist $z_\infty = \sup_n z_n$ and $w_\infty = \inf_n w_n$ and $z_\infty \leq_p w_\infty$. Thus, the infinite intersection of the closed arcs of sequence is exactly $[z_\infty, w_\infty]_p \neq \emptyset$.

To see that it contains exactly one point, we have to see $z_\infty = w_\infty$. Assume, on the contrary, $z_\infty <_p w_\infty$. In this case, $I = (z_\infty, w_\infty)_p$ contains infinitely many points, and we can find a domain $D \subseteq \hat{\mathbb{C}}$ such that $D \cap \partial\mathbb{D} = I$. Since B is elliptic or doubly parabolic, by Theorem 3.5, the Julia set of B, $\mathcal{J}$, is exactly $\partial\mathbb{D}$. So, by the blow-up property (8), there exists $n_0 \geq 0$ such that $B^{n_0}(D \cap \mathcal{J}) = B^{n_0}(I) = \mathcal{J} = \partial\mathbb{D}$.

Now, we consider two points on $\partial\mathbb{D}$ in different arcs of the dynamical partition: $y_1 \in I_1$ and $y_2 \in I_2$. Since $B^{n_0}(I) = \partial\mathbb{D}$, there exist $x_1, x_2 \in I$ with $B^{n_0}(x_1) = y_1 \in I_1$ and $B^{n_0}(x_2) = y_2 \in I_2$. Hence, the itineraries of x_1 and x_2 contain 1 and 2 in the n_0-th position, respectively. However, $(z_\infty, w_\infty)_p \subseteq [z_{n_0}, w_{n_0}]_p = \overline{I_{s_0\ldots s_{n_0}}}$, so all points in (z_∞, w_∞) must have the same itinerary. Hence, we have reached a contradiction, implying that $z_\infty = w_\infty$. Therefore, the map $\mathcal{I}$ is well-defined, as we wanted to see.

For the last statement, simply note that if $z \in \partial\mathbb{D}$ has itinerary $\underline{z}$, then

$$z \in \bigcap_{n=0}^{\infty} I_{z_0\ldots z_n} \subseteq \bigcap_{n=0}^{\infty} \overline{I_{z_0\ldots z_n}} = \{\mathcal{I}(\underline{z})\},$$

and thus $\mathcal{I}(\underline{z}) = z$. □

We observe next that the intersection map $\mathcal{I}$ is precisely the inverse of the sequence map S.

Proposition 4.20. *Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$, $p \in \partial\mathbb{D}$ a fixed point of B. Let S and $\mathcal{I}$ be the sequence and intersection maps of B with origin in p, respectively. Then, they satisfy*

$$\mathcal{I} \circ S = id.$$

Hence, the sequence map S is injective and the intersection map $\mathcal{I}$ is surjective.

Proof. This is a direct consequence of Proposition 4.19. In fact, let $z, w \in \partial\mathbb{D}$. Assume $S(z) = S(w)$, that is, $\underline{z} = \underline{w}$. Since the intersection map $\mathcal{I}$ is well-defined, we have $z = \mathcal{I}(\underline{z}) = \mathcal{I}(\underline{w}) = w$, as desired. □

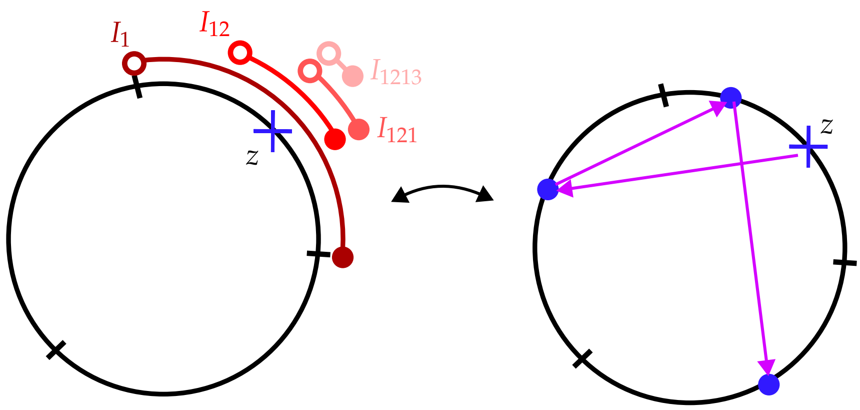


Figure 4.5: Certain arcs of finite sub-sequences of a given sequence in Σ_d end up intersecting in a single point. Then, each itinerary determines exactly one point.

Note that the previous proposition only gives us the injectivity of S and the surjectivity of I. A natural question to ask is whether these maps are indeed one-to-one. The answer turns out to be negative, unless restricted to appropriate subsets of Σ_d and $\partial\mathbb{D}$. We show this in the following proposition.

Proposition 4.21. *Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$, $p \in \partial\mathbb{D}$ a fixed point of B. Let S and $\mathcal{I}$ be the sequence and intersection maps of B with origin in p, respectively. Then, the following holds.*

(a) *$B \circ \mathcal{I}(\underline{s}) = \mathcal{I} \circ \sigma(\underline{s})$ and $\sigma(S(z)) = S(B(z))$, for all $\underline{s} \in \Sigma_d$ and $z \in \partial\mathbb{D}$.*

(b) *If $\underline{s} \in \Sigma_d$ satisfies $B^{n+1}(\mathcal{I}(\underline{s})) = p$, then either $\underline{s} = s_0 \dots s_n\overline{1}$ or $\underline{s} = s_0 \dots s_n\overline{d}$.*

(c) *$S \circ \mathcal{I}(\underline{s}) \neq \underline{s}$ if and only if $\underline{s} = s_0 \dots s_{n-1}\overline{d}$, for some $n \geq 0$.*

(d) *S is not continuous at any point in the backward orbit of p.*

Proof. (a) The proof is straight-forward, considering that, if $z \in \partial\mathbb{D}$ has itinerary $z_0 z_1 \dots$, then $B^n(z)$ has itinerary $z_n z_{n+1} \dots$, and that $\overline{B^{-n}(I_k)} = B^{-n}\left(\overline{I_k}\right)$ for each $n \geq 0$, $k \in \{1, \dots, d\}$.

(b) Simply note that fixed $n \geq 0$, the semi-open arcs $I_{s_0 \ldots s_n}$ form a partition of $\partial\mathbb{D}$, for all finite sequences with $s_0, \ldots, s_n \in \{1, \ldots, d\}$. Hence, any point can be at most adherent to two different $I_{s_0 \ldots s_n}$. For the case of p, it is clearly in $\overline{I_{1 \overset{(n)}{\ldots} 1}}$ and in $I_{d \overset{(n)}{\ldots} d}$. Hence, $\mathcal{I}(\underline{s}) = p$ if and only if $\underline{s} = \overline{1}$ or $\underline{s} = \overline{d}$. The general case is deduced directly from Propositions 4.18 and item (a).

(c) First, let $\underline{s} \in \Sigma_d$ be a sequence which is not any itinerary, i.e. $S(\mathcal{I}(\underline{s})) \neq \underline{s}$. Then, if we consider the point $z = \mathcal{I}(\underline{s})$, $\underline{s}$ cannot be the itinerary of z, which we denote by $\underline{z}$. Hence, there exists some $n \in \mathbb{Z}_{\geq 0}$ such that $s_n \neq z_n$. Consequently, $z \notin I_{s_0 \ldots s_n}$.

Now, since $z = \mathcal{I}(\underline{s})$, z must be in the closure $\overline{I_{s_0 \ldots s_n}}$. Since $\overline{I_{s_0 \ldots s_n}}$ and $I_{s_0 \ldots s_n}$ only differ in one point, the endpoint, we must have that z is the endpoint of $I_{s_0 \ldots s_n}$. By Proposition 4.18, this means that $B^{n+1}(z) = p$, and so, by item (b), we must have either $\underline{s} = s_0 \ldots s_n \overline{1}$ or $\underline{s} = s_0 \ldots s_n \overline{d}$. Since one of these two must be the itinerary of z, and itineraries of preimages of p must contain the sub-sequence $\overline{1}$, we conclude that the former sequence is the itinerary of z. Hence, by hypothesis, $\underline{s} = s_0 \ldots s_n \overline{d}$.

The converse case is derived directly from item (b).

(d) Let $z \in \partial\mathbb{D}$ be such that $B^{n+1}(z) = p$, for some $n \geq 0$. Then, $S(z) = s_0 \ldots s_n \overline{1}$ for some $s_0, \ldots, s_n \in \{1, \ldots, d\}$. By Proposition 4.18, z must be the endpoint of two different arcs of different sequences, $I_{s_0 \ldots s_n}$ and $I_{t_0 \ldots t_n}$, with $s_n \neq t_n$. Hence, any neighbourhood of z, say V, must contain points from both arcs. In other words, there exist $w_1, w_2 \in V$ such that $w_1 \in I_{s_0 \ldots s_n}$ and $w_2 \in I_{t_0 \ldots t_n}$. But, if we consider their sequences, we have that $S(w_1)$ starts with $s_0 \ldots s_n$ and that $S(w_2)$ starts with $t_0 \ldots t_n$, with $t_n \neq s_n$. Thus, by Proposition 4.7, $\mathcal{D}(S(w_1), S(w_2)) > 1/d^n$. In other words, S is not continuous at z.

□

Observe that we have just proved that, if we define the ***reduced set of*** Σ_d, as

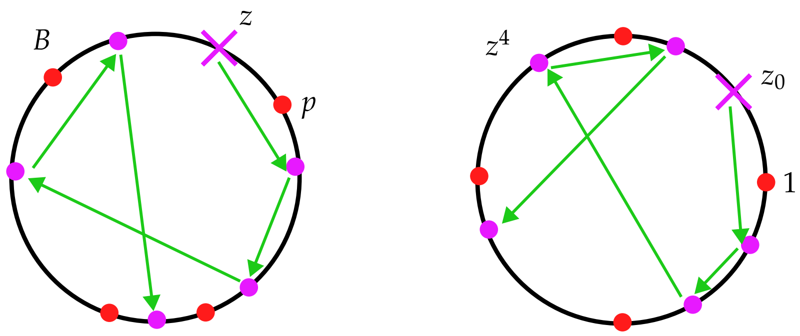
$$\widetilde{\Sigma}_d := \Sigma_d \setminus \left\{ \underline{s} \in \Sigma_d : \underline{s} = s_0 \ldots s_n \overline{d}, \text{ for some } n \geq 0 \right\}$$

then $S \circ \mathcal{I} = \text{id}$ in this set. Hence, S and $\mathcal{I}$ are one-to-one, and satisfy

$$S \circ B(z) = \sigma \circ S(z), \quad \forall z \in \partial\mathbb{D},$$
$$\mathcal{I} \circ \sigma(\underline{s}) = B \circ \mathcal{I}(\underline{s}), \quad \forall \underline{s} \in \widetilde{\Sigma}_d.$$

Note that S and $\mathcal{I}$ are not conjugacies, since they are not homeomorphisms. However, they serve as an intermediate step for a topological conjugacy between B and z^d. The key idea for building this conjugacy is realising that z^d is indeed an elliptic finite Blaschke product of degree d with fixed point 1 (and the other $(d-1)$-th roots of 1), and noting that the reduced symbolic space does not depend in B nor p.

Theorem 4.22. *Let B be an elliptic or doubly parabolic finite Blaschke product of degree $d \geq 2$ and let $f(z) = z^d$. Let $p \in \partial\mathbb{D}$ be a fixed point of B, and consider S and $\mathcal{I}$ the sequence and intersection maps of B with origin p, respectively. Similarly, consider S_0 and $\mathcal{I}_0$ the sequence and intersection maps of f with origin 1, respectively.*



$S(z) = S_0(z_0) = 144213\ldots$

Figure 4.6: Sketch of the conjugacy between an elliptic or doubly parabolic FBP B of degree 4 and z^4. Each point z is identified with the one with the same itinerary, z_0.

Then, $\mathcal{I} \circ S_0$ is a topological conjugacy between B and z^d. In other words, $\mathcal{I} \circ S_0$ is a homeomorphism with inverse $\mathcal{I}_0 \circ S$, and

$$(\mathcal{I} \circ S_0) \circ B(z) = f \circ (\mathcal{I} \circ S_0)(z), \quad \forall z \in \partial\mathbb{D}.$$

Proof. For the sake of clarity, we shall denote $\phi = \mathcal{I} \circ S_0$. First of all, notice that the map ϕ is well-defined. Indeed, the reduced space $\widetilde{\Sigma_d}$ is independent on the finite Blaschke product and the origin point. Thus, it is the same for B and for $f(z) = z^d$, so ϕ is well-defined.

Second, by Proposition 4.20, ϕ is bijective with inverse $\phi^{-1} = \mathcal{I}_0 \circ S$. Hence, it is left to study the continuity of this map and its inverse.

For this purpose, we shall lay standard notation. We denote the arc of sequence $s_0 \ldots s_n$ of B with origin in p as $I_{s_0\ldots s_n}$. When working with the map f with origin 1, we refer to the arc of such sequence as $I^0_{s_0\ldots s_n}$.

It is also important to note the following.

Obs. 1. $z \in \partial\mathbb{D}$ is in the backward orbit of 1 by f if and only if $\phi(z)$ is in the backward orbit of p by B.

This observation is immediate, as the itineraries of points in the backward orbit of these fixed points are exactly those ending in $\overline{1}$, taking into account Proposition 4.21.

At this point, we are ready to prove that ϕ is continuous in $\partial\mathbb{D}$. Thus, for $z \in \partial\mathbb{D}$, our goal is to prove that for every neighbourhood of $\phi(z)$, say V, there exists a neighbourhood of z, say U, such that $\phi(w) \in V$ if $w \in U$. For this purpose, we consider three cases, two of which are illustrated in Fig. 4.7.

Case 1. Let us assume z is not in the backward orbit of 1 by f. Hence, if $S_0(z) = z_0 z_1 \ldots$ is the itinerary of z by f, we have that z is not the endpoint of any arc $I^0_{z_0\ldots z_n}$ for $n \geq 0$, by Proposition 4.18. Consequently, the arcs $\{I^0_{z_0\ldots z_n}\}_{n\geq 0}$ are a basis of neighbourhoods of z in $\partial\mathbb{D}$.

Similarly, $\{I_{z_0\ldots z_n}\}_{n\geq 0}$ is also a basis of neighbourhoods of $\mathcal{I} \circ S_0(z)$ in $\partial\mathbb{D}$, since $\mathcal{I} \circ S_0(z)$ is not in the backward orbit of p by B, due to Obs. 1.

This way, for every neighbourhood of $\phi(z)$, say V, we can consider $n \geq 1$ large enough so that $I_{z_0\ldots z_n} \subseteq V$. This way, choosing $U = I^0_{z_0\ldots z_n}$, we have that, for $w \in U$, the itinerary of w by f, $S_0(w)$ starts with $z_0 \ldots z_n$. Hence, the point $\phi(w) = \mathcal{I} \circ S_0(w)$ belongs in the arc $I_{z_0\ldots z_n} \subseteq V$. Thus, we have proved ϕ is continuous at z.

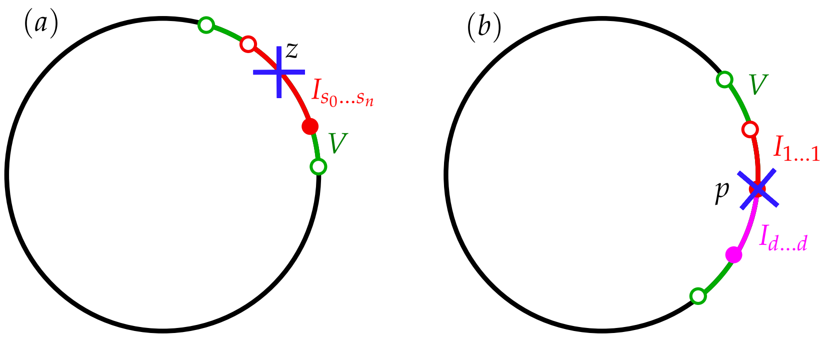


Figure 4.7: Sketch of the basis of neighbourhoods in proof of Theorem 4.22, for the first case (a), and for the second case (b).

Case 2. Let us assume now $z = 1$. In this case, 1 is exactly at one endpoint of all arcs of the form $I^0_{1\dots1}$ and $I^0_{d\dots d}$ for any number of 1s and ds. Similarly, $\phi(1) = p$ is at the endpoint of all arcs $I_{1\dots1}$ and $I_{d\dots d}$.

Thus, we can consider the union of these arcs, $\widetilde{I}^0_n := I^0_{1\overset{(n)}{\dots}1} \cup I^0_{d\overset{(n)}{\dots}d}$ and $\widetilde{I}_n := I_{1\overset{(n)}{\dots}1} \cup I_{d\overset{(n)}{\dots}d}$. These conform bases of neighbourhoods of 1 and p, respectively.

Thus, for any neighbourhood of $\phi(1) = p$, say V, we can find $n \geq 1$ large enough so that $I_n \subseteq V$. Thus, choosing $U = I^0_n$, we have that, for $w \in U$, the itinerary of w by f, $S_0(w)$, starts by either n 1s or n ds. In both cases, the point $\phi(w) = \mathcal{I} \circ S_0(w)$ belongs to $I_k \subseteq V$. Hence, ϕ is continuous at 1.

Case 3. Finally, let us assume that z is in the backward orbit of 1, i.e. there exists $n \geq 1$ such that $f^n(z) = 1$. Note that this implies that $B^n(\phi(z)) = p$.

Thus, consider any neighbourhood of $\phi(z)$, say V. Without loss of generality, we can change V if needed by a smaller neighbourhood so that B^n is diffeomorphic between V and $W = B^n(V)$, since B is a local diffeomorphism (Proposition 1.8(d)). Moreover, W is a neighbourhood of $p = \phi(1)$, as $B^n(\phi(z)) = p$, and ϕ is continuous at 1. Thus, there exists some neighbourhood of 1, $\widetilde{U}$ such that $\phi(w) \in W$ if $w \in \widetilde{U}$.

Taking into account that f^n is also a local diffeomorphism and that $f^n(z) = 1$, we have that there exists a neighbourhood of z, say U, such that $f^n : U \longrightarrow \widetilde{U}$ is a diffeomorphism, changing $\widetilde{U}$ by a smaller neighbourhood of 1 if needed.

This way, we have that for $w \in U$, $f^n(w) \in \widetilde{U}$. By continuity of ϕ in 1, we have that $\phi(f^n(w)) \in W$. Next, by Proposition 4.21(a), we have that

$$\phi(f^n(w)) = \mathcal{I} \circ S_0(f^n(w)) = \mathcal{I}(\sigma^n(S_0(w))) = B^n(\mathcal{I} \circ S_0(w)),$$

where σ is the shift map of $\widetilde{\Sigma_d}$. Hence, $B^n(\phi(w)) \in W$. As $B^n : V \longrightarrow W$ is diffeomorphic, we get that $w \in V$. In conclusion, ϕ is continuous at z, as we wanted to see.

Thus, we have that $\mathcal{I} \circ S_0 : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ is continuous and one-to-one. Since $\partial\mathbb{D}$ is a compact metric space, continuous bijective maps are homeomorphisms.[2]. Hence, $\phi = \mathcal{I} \circ S_0$ is

[2] As previously discussed, this is a well-known result from topology. See e.g. [Sut75, Theorem 13.26].

a homeomorphism. Furthermore, from Proposition 4.21,

$$S_0 \circ f(z) = \sigma \circ S_0(z), \quad \forall z \in \partial\mathbb{D},$$
$$\mathcal{I} \circ \sigma(\underline{s}) = B \circ \mathcal{I}(\underline{s}), \quad \forall \underline{s} \in \widetilde{\Sigma_d}.$$

And so, we conclude:

$$(\mathcal{I} \circ S_0)(z^d) = B \circ (\mathcal{I} \circ S_0)(z), \quad \forall z \in \partial\mathbb{D},$$

as we wanted to see. □

In conclusion, we have that any elliptic or doubly parabolic FBP is topologically conjugate to z^d. One might ask if the previous conjugacy has any higher degree of regularity. The answer to this question is negative. Specifically, we have the following result by Hamilton, from [Ham96, Theorem 1].

Theorem 4.23. *Let B be an elliptic or doubly parabolic finite Blaschke product. Then, B has no absolutely continuous conjugacy to another Blaschke product unless $\tilde{B} = M \circ B \circ M^{-1}$, with $M \in Aut_{\mathcal{H}}(\mathbb{D})$.*

4.3 Chaos in finite Blaschke products

Now that we have proved that elliptic and doubly parabolic finite Blaschke products are topologically conjugate to z^d, and having studied this conjugacy extensively, we shall take a brief look into the implications of this conjugacy.

Since this conjugacy is topological, an elliptic or doubly parabolic FBP and z^d share the same topological properties. This is, properties preserved by homeomorphisms, such as fixed points, periodic points, or, notably, chaos. In this last section we will delve into this last concept. It should be mentioned that, in the context of dynamical systems, chaos can be defined following many different approaches. In our case, we use the convention of Devaney in [Dev21, Chapter 1.8].

Definition 4.24 (Topological transitivity)**.** Let (X, d) be a metric space and $f : X \longrightarrow X$ a continuous map. We say f is ***topologically transitive*** if for any pair of open sets U, V, there exists $n \geq 0$ such that

$$f^n(U) \cap V \neq \emptyset.$$

Definition 4.25 (Sensitive dependence on initial conditions)**.** Let (X, d) be a metric space and $f : X \longrightarrow X$ a continuous map. We say f has ***sensitive dependence on initial conditions*** if there exists $\delta > 0$ such that, for all $x \in X$ and any neighbourhood of x, say U, there exist $y \in U$ and $n \geq 0$ such that

$$|f^n(x) - f^n(y)| > \delta.$$

Definition 4.26 (Chaos)**.** Let (X, d) be a metric space and $f : X \longrightarrow X$ a continuous map. We say f is ***chaotic*** in X if periodic points of f are dense in X, f is topologically transitive and f has sensitive dependence on initial conditions.

In our case, it is straight-forward to see that the map z^d is chaotic. Indeed,

- It has sensitive dependence on initial conditions, since z^d multiplies the angle between two nearby points by d.
- It is topologically transitive, since, as we have previously argued, z^d is an elliptic finite Blaschke product. Thus, its Julia set is exactly $\partial\mathbb{D}$, and the blow-up property (8) from Proposition 2.9 directly implies topological transitivity.
- Finally, periodic points of z^d are dense in $\partial\mathbb{D}$. Indeed, periodic points satisfy $z^{d^n} = z$ for some $n \geq 1$. Thus, they are exactly those of the form

$$p_{n,k} = e^{2\pi i r_{n,k}}, \text{ for } r_{n,k} = \frac{k}{d^n - 1}, \quad \text{where } k \in \mathbb{Z} \text{ and } n \geq 1.$$

  Hence, given any point $z = e^{2\pi i\alpha} \in \partial\mathbb{D}$ for $\alpha \in \mathbb{R}$ any neighbourhood of z, say U, we can consider an contained in U, spanning an angle of $\varepsilon > 0$, say A_ε. We claim that we can always find a periodic point inside this arc. Indeed, if we choose $n > (1/\varepsilon)$, then periodic points of the form $p_{n,k}$ for $k \in \{1, \ldots, d^n - 1\}$ are separated an angular distance of $2\pi n < 2\pi\varepsilon$. Thus, there must be some periodic point $p_{n,k}$ inside $A_\varepsilon \subseteq U$. In other words, periodic points of z^d are dense in $\partial\mathbb{D}$.

At this point, the only loose end left to tie is whether this definition of chaos is preserved or not through topological conjugacies. In the case of compact metric spaces, such as our case, $\partial\mathbb{D}$, it is straight-forward to see that chaos is in fact a topological property. The general case is also true, even though it is more subtle. A wider discussion can be found in [BBC$^+$92]. Thus, we have the following.

Theorem 4.27. *Let B be an elliptic or doubly parabolic finite Blaschke product. Then, B is chaotic in $\partial\mathbb{D}$.*

Chapter 5

Dynamics of Generalised Blaschke products: an example

Having studied the dynamics of finite Blaschke products in Chapter 3, we have seen that the structure of the Fatou and Julia sets is relatively simple in those cases. The Julia set restricts to either $\partial\mathbb{D}$ or a Cantor subset of $\partial\mathbb{D}$, and the Fatou set is either composed of only one or two Fatou domains, exhibiting Bötcher, Schröder or Leau domains in different cases.

Taking this into account, in this chapter we shall finish our work by showing the high phenomenology of generalised finite Blaschke products. By only allowing the zeros of Blaschke products to be outside the unit disk, the simple dynamical properties studied in Chapter 3 vanish and more complex structures such as rotation domains appear. Of course, this change in behaviour can be ultimately understood with the existence of poles in $\mathbb{D}$: the unit disk is no longer forward invariant and our two main tools, Schwarz Lemma A.1 and the Wolff-Denjoy Theorem 2.31 are no longer valid.

More background in the results of this section can be found in [dMvS12, Chapter 4].

5.1 Existence of rotation domains

Our goal in this section is to introduce some results characterising the existence of rotation domains. Recall from section 2.3 that, given a rational map, a rotation domain is an invariant Fatou component where the iterates do not converge to a constant map. Fatou and Julia showed that in this case, the component must be either a topological disk (Siegel disk) or a topological annulus (Herman ring), and in both cases, the rational map is conjugate to an irrational rigid rotation, hence their name.

Historically, rotation domains, and in particular Herman rings, were first proved to exist in [Her79] with a careful study of the analytic properties of diffeomorphisms of $\partial\mathbb{D}$. Years later, Shishikura introduced a new and complex method, quasi-conformal surgery, in [Shi87].

In our case, we shall follow Herman's approach: First, we restrict our GFBP to the unit circle and show that its iterates in this set are conjugated to an irrational rotation. Afterwards, we extend this conjugacy to a neighbourhood of the circle and conclude it must be contained in a rotation domain.

5.1.1 Conjugating to an irrational rotation

Our first step is to introduce the Herman-Yoccoz Theorem, which establishes a conjugacy between a diffeomorphism of the unit circle and an irrational rotation. Previously, we shall define Diophantine numbers.

Definition 5.1 (Diophantine number)**.** We say an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is ***Diophantine*** if there exist $K > 0, \mu < \infty$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{K}{q^{2+\mu}}, \qquad \text{for all } p \in \mathbb{Z}, q \in \mathbb{Z}_{>0}.$$

Theorem 5.2 (Herman-Yoccoz)**.** *Let f be a diffeomorphism of the unit circle $\partial\mathbb{D}$. Let $\alpha = \rho(f)$ be the rotation number of f. If α is Diophantine, then f is conjugate to a rotation of angle α, $\rho_\alpha(z) = e^{i\alpha}z$.*

The proof of this result is well-established but involves deep mathematics. Since the purpose of this chapter is just to give a short insight in the existence of rotation domains, we do not include its proof. However, it can be found in [dMvS12, Theorem I.3.2] and [PM92, Section 3.2.b].

5.1.2 Finding a Diophantine rotation number

Herman and Yoccoz Theorem 5.2 guarantees the existence of a conjugacy from a diffeomorphism to an irrational rotation. However, it is necessary to find a diffeomorphism with a Diophantine rotation number. Hence, we now give a short insight in irrational number theory and rotation numbers.

First, in order to find a Diophantine rational number, we present one of the most important results in number theory, Liouville's Theorem.

Theorem 5.3 (Liouville)**.** *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an algebraic irrational number of degree. In other words, there exists an irreducible polynomial $f \in \mathbb{Z}[X]$ of degree $d \geq 2$ and integer coefficients such that $f(\alpha) = 0$.*

Then, there exists $K > 0$ such that for every $p, q \in \mathbb{Z}$ with $q > 1$,

$$\left|\alpha - \frac{p}{q}\right| > \frac{K}{q^d}.$$

The proof of this result as well as a deeper background can be found in [Sim07, Theorem B.18.1]. In particular, it states that every algebraic irrational number is Diophantine.

Hence, we have that for every diffeomorphism of the unit circle with algebraic irrational rotation number, we can find a conjugacy to an irrational rotation in the unit circle.

The only thing left to see is whether we can find an appropriate diffeomorphism with this rotation number. This will be given by the following result.

Theorem 5.4. *Let $f : \partial\mathbb{D} \longrightarrow \partial\mathbb{D}$ be a homeomorphism of the unit circle. Let $\rho_\alpha(z) = e^{i\alpha}z$ be the rotation of angle α. Then, the rotation function*

$$\begin{aligned} \rho : \mathbb{R} &\longrightarrow [0,1) \\ \alpha &\longmapsto \rho(\rho_\alpha \circ f), \end{aligned}$$

where $\rho(f)$ stands for the rotation number of f, assumes every irrational value in $[0,1)$ exactly once.

A detailed proof of this result can be found in [dMvS12, Chapter 4].

In conclusion, due to these results, we have that, given a diffeomorphism of the unit circle f, we can find appropriate α such that $\rho_\alpha \circ f$ has Diophantine rotation number, and hence, is conjugate to an irrational rotation in $\partial\mathbb{D}$.

5.1.3 Extending the conjugacy

Now that we are able to conjugate an appropriate diffeomorphism to an irrational rotation in the unit circle, the last ingredient needed to prove the existence of rotation domains is seeing that this conjugacy can be extended to a neighbourhood of $\partial\mathbb{D}$.

Theorem 5.5. *Let R be a rational map of degree $d \geq 2$ such that $R|_{\partial\mathbb{D}}$ is a diffeomorphism of the unit circle with Diophantine rotation number.*

Then, $\partial\mathbb{D}$ is contained in a rotation domain of R.

Proof. Let $\alpha = \rho(R|_{\partial\mathbb{D}})$ be the rotation number of R in $\partial\mathbb{D}$. By the Herman-Yoccoz Theorem 5.2, there exists a diffeomorphism φ such that $\varphi \circ R(z) = \rho_\alpha \circ \varphi(z)$ for $z \in \partial\mathbb{D}$, where $\rho_\alpha(z) = e^{i\alpha}z$ is the rotation of angle α.

Since φ is holomorphic, it is also analytic. Hence, it can be extended to a neighbourhood of $\partial\mathbb{D}$, say U. Similarly, its holomorphic inverse φ^{-1} can be extended to a neighbourhood of $\partial\mathbb{D}$, V. Let $\Phi : U \longrightarrow U$ and $\Phi^{-1} : V \longrightarrow V$ be these extensions.

First, note that $\Phi^{-1} \circ \Phi(z) = \varphi^{-1} \circ \varphi(z) = z$ for all $z \in \partial\mathbb{D}$. Then, by the Identity Theorem A.3, Φ^{-1} is the inverse of Φ in $U \cap V$, so they are biholomorphisms in $U \cap V$.

Second, we also have that $\Phi \circ B(z) = \varphi \circ B(z) = \rho_\alpha \circ \varphi(z) = \rho_\alpha \circ \Phi(z)$ for all $z \in \partial\mathbb{D}$. Hence, by the Identity Theorem $\Phi \circ B = \rho_\alpha \circ \Phi$ in $U \supseteq U \cap V$.

This means that the conjugacy from R to ρ_α extends to $U \cap V$, an open neighbourhood of $\partial\mathbb{D}$. In this neighbourhood we can write $B(z) = \Phi(e^{i\alpha}\Phi(z))$. In particular, we have that this neighbourhood is forward invariant by B, and hence, by Montel's Theorem 2.5, it must be contained in the Fatou set. Since it is a domain, it must be an invariant Fatou component. Hence, by Classification Theorem 2.30, it must either be an immediate basin of attraction or a rotation domain. However, since B is conjugate to ρ_α in this domain, its iterates cannot converge to a constant function. Hence, this neighbourhood of $\partial\mathbb{D}$ must be contained in a rotation domain. $\square$

5.2 Herman rings in the family of degree 3 GFBP

In this section, we apply the results from Section 5.1 to deduce the existence of rotation domains in the family of generalised finite Blaschke products of degree 3 defined as

$$B_{\theta,a}(z) = e^{i\theta} z^2 \frac{z-a}{1-az}, \quad \text{for } \theta, a \in \mathbb{R}, a > 1.$$

This family comprises one of the first studied examples of Herman rings, see for instance, [Shi87, Section 6]. Recall that we are limiting ourselves to the case $a > 1$, since for $0 < a < 1$ is exactly the one studied in Chapter 3, and the case $a = 1$ restricts to $-e^{i\theta}z^2$.

Our goal is to find appropriate θ and a such that $B_{\theta,a}$ is a diffeomorphism of Diophantine rotation number in $\partial\mathbb{D}$. We start by studying the critical points of $B_{\theta,a}$, in other words, the points where the derivative vanishes. Evaluating the derivative of the case $\theta = 0$, we get

$$B'_{0,a}(z) = z\frac{-2az^2 + (3+a^2)z - 2a}{(1-az)^2}.$$

And hence, the critical points in $\mathbb{C}^*$ are

$$c_{\pm} = \frac{1}{4a}\left(3 + a^2 \pm \sqrt{(a^2-1)(a^2-9)}\right).$$

So, for $1 < a < 3$, we have that $c_{\pm}$ belong in $\partial\mathbb{D}$, and for $a = 3$, $c_+ = c_- = 1$ a critical point of multiplicity 2 . The case of interest is $a > 3$, where these two critical points are outside the unit circle, and hence, $B'_{0,a}(z) \neq 0$ for $z \in \partial\mathbb{D}$.

Moreover, for $\theta \in \mathbb{R}$, we have that $B'_{\theta,a}(z) = B'_{0,a}(z) \neq 0$ for $z \in \partial\mathbb{D}$, when $a > 3$. Hence, we have the following.

Proposition 5.6. *Let $\theta \in \mathbb{R}$ and $a < 3$. Then, $B_{\theta,a}$ is a diffeomorphism of $\partial\mathbb{D}$.*

Proof. Recall that $B_{\theta,a}$ is a proper map of degree 3. In other words, every point has exactly 3 preimages, counting with multiplicity. In particular, the unit circle $\partial\mathbb{D}$ must have 3 preimages, which must also be connected compact sets.

It is direct to see that $B(z) \in \partial\mathbb{D}$ for $z \in \partial\mathbb{D}$. Hence, we have that at least one of the three preimages of the unit disk is the unit disk itself.

Furthermore, notice that the unit disk $\mathbb{D}$ contains both a zero, $z = 0$, and a pole of $B_{\theta,a}$, $z = 1/a < 1/3$. Hence, the segment joining 0 and $1/a$ must contain a preimage of a point of $\partial\mathbb{D}$. Since preimages of $\partial\mathbb{D}$ must be connected, there must be a preimage of $\partial\mathbb{D}$ containing this point. Note that if this new preimage intersected $\partial\mathbb{D}$, the crossing points would have to be critical points, since they would be preimages of points with multiplicity greater than 1. In other words, there is a preimage of $\partial\mathbb{D}$ contained inside $\mathbb{D}$. By the reflection property, there must exist another preimage of $\partial\mathbb{D}$ contained in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

In other words, the three preimages of $\partial\mathbb{D}$ are disjoint and contained in $\mathbb{D}$, $\partial\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Hence, $B_{\theta,a}$ must cover $\partial\mathbb{D}$ exactly once. Since $B'(z) \neq 0$ for all $z \in \partial\mathbb{D}$, by the Inverse Function Theorem, we have that B is a biholomorphism of the unit disk. $\square$

Next, we shall find appropriate $\theta \in \mathbb{R}$ such that $B_{\theta,a}$ has Diophantine rotation number in $\partial\mathbb{D}$ for a certain $a \in (0, \infty)$. The common way to do this is numerically, computing the rotation number for different values of θ with a certain tolerance. In fact, we can evaluate the rotation number for different values of the two parameters, $a \in (3, \infty)$ and $\theta \in (0, 2\pi)$, and choose a pair that gives the rotation number of interest. The result is shown in Fig. 5.1, where ***rational Arnold tongues*** $\mathcal{T}_r$ are shown. These are the set of parameters that give a rational rotation number r. Two ***irrational tongues*** $\mathcal{T}_\alpha$, the set of parameters giving an irrational rotation number α, are also shown. We did evaluate these calculations, but due to the high computational power needed to create high resolution pictures, we took Fig. 5.1 from [BFGH05]. In the same article, a general discussion of the same family of generalised finite Blaschke products can be found.

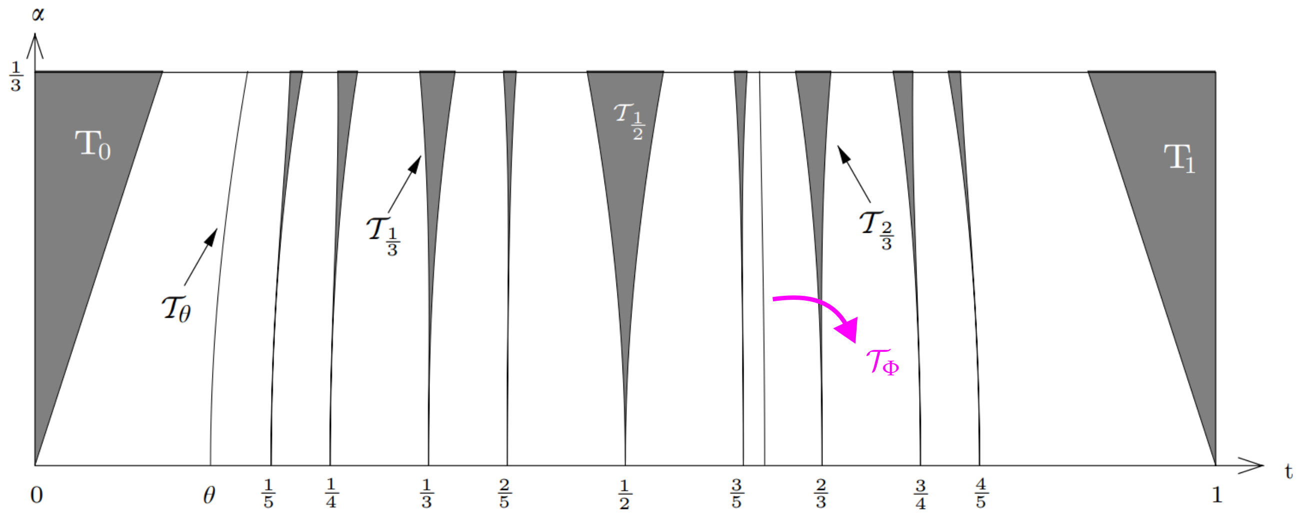


Figure 5.1: Tongues for the family of GFBP $B_{\theta,a}$, for the parameters $\alpha = 1/a$ and $t = \theta/2\pi$. In grey, rational Arnold tongues for some rational numbers, alongside with the irrational tongue for $\theta = \sqrt[5]{2} - 1$ and for $\Phi = (\sqrt(5) - 1)/2$, remarked in magenta. Figure taken from [BFGH05, Figure 1].

For historical reasons, we will use rotation number $\Phi = (1 + \sqrt{5})/2 \mod 1 = (\sqrt(5) - 1)/2$, the golden ratio. We found that this rotation number occurs, for instance, with parameters $a = 4$ and $\theta_0 = 0.615$. For these parameters, we have that $B_{\theta_0,4}$ is a diffeomorphism of the unit disk with Diophantine rotation number. Hence, by Theorem 5.5, $\partial\mathbb{D}$ must be contained in a rotation domain, H. Moreover, we can consider the following fact.

Proposition 5.7. *For every $\theta \in \mathbb{R}$ and $a \in \mathbb{R}$, 0 is a super-attracting point of $B_{\theta,a}$.*

Proof. This is direct, taking into account that z^2 divides $B_{\theta,a}$. □

Thus, 0 must have a Bötcher domain around it. This means that the rotation domain H cannot contain the whole unit disk $\mathbb{D}$. Hence, it cannot be simply connected, and so, it must be a Herman ring. See Fig. 5.2.

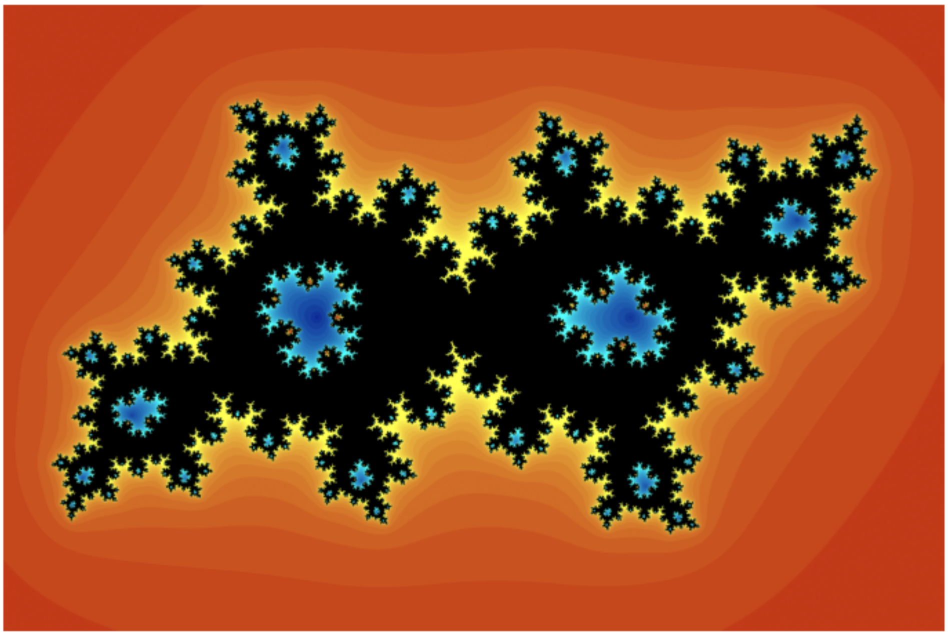


Figure 5.2: Dynamical plane of $B_{\theta_0,a}$, for $\theta_0 = 2\pi \cdot 0.615, a = 4$. Blue and red regions indicate convergence to the super-attracting fixed points 0 and ∞, respectively. Darker regions indicate faster convergence. Black regions indicate the domains where the iterates do not converge to any point. In particular, the bigger annulus on the left is a Herman ring, and other annuli correspond to its successive preimages.

Appendix A

Appendix: Tools from complex analysis

In this appendix, we shall recall some common-knowledge concepts and results from complex analysis and topology that have been used throughout our work. Most of these results can be found in any standard book of complex analysis. For instance, we refer to [Ahl66].

A.1 Basic concepts from complex analysis

Given the matter of study of this work are maps of the unit disk, the first result we shall recall is Schwarz Lemma, detailed in [Ahl66, Theorem 4.13].

Lemma A.1 (Schwarz Lemma). *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic self-map of $\mathbb{D}$ such that $f(0) = 0$. Then, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.*

Moreover, if either $|f(z^*)| = |z^*|$ for some $z^* \in \mathbb{D} \setminus \{0\}$ or $|f'(0)| = 1$, then $f(z) = e^{i\theta}z$ for all $z \in \mathbb{D}$, for some $\theta \in \mathbb{R}$.

Next, we shall recall two basic theorems capturing the essential characteristic of holomorphic functions, the Maximum Modulus Principle and the Identity Theorem, which can be found in [Ahl66, Theorem 4.12] and [Ahl66, Section 4.3.2].

Theorem A.2 (Maximum modulus Principle). *Let $f : \Omega \longrightarrow \mathbb{C}$ be a non-constant holomorphic map defined in some domain $\Omega \subseteq \mathbb{C}$. Then, the function $|f|$ has no local maxima in Ω.*

Theorem A.3 (Identity Theorem). *Let $f, g : \Omega \longrightarrow \mathbb{C}$ be holomorphic functions in some domain $\Omega \subseteq \hat{\mathbb{C}}$. Let $S \subseteq \Omega$ be the set*

$$S := \{z \in \Omega : f(z) = g(z)\}.$$

If S has an accumulation point in Ω, then $f(z) = g(z)$ for all $z \in \Omega$.

As a consequence of the Identity Theorem, we shall also review the following, discussed extensively in [Ahl66, Theorem 4.24].

Theorem A.4 (Schwarz Reflection Principle). *Let $f : \mathbb{H}_+ \longrightarrow \mathbb{C}$ be a holomorphic function defined in the upper half plane, $\mathbb{H}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \subseteq \hat{\mathbb{C}}$. If f can be extended continuously to the real line $\mathbb{R}$, then f has a unique holomorphic extension to $\hat{\mathbb{C}}$, satisfying:*

$$f(z) = \overline{f(\overline{z})}, \quad \text{for } z \in \mathbb{C} \setminus \mathbb{H}_+.$$

Next, we shall discuss two fundamental results related to meromorphic functions and the concept of index, detailed in [Ahl66, Theorem 4.18].

Theorem A.5 (Rouché's Theorem). *Let $\gamma_1 : \mathbb{R} \longrightarrow \Omega, \gamma_2 : \mathbb{R} \longrightarrow \Omega$ be two continuous curves defined in some domain $\Omega \subseteq \mathbb{C}$. Let $z_0 \in \Omega$. If f and g satisfy*

$$|f(z) - g(z)| < |f(z) - z_0|,$$

then $Ind(f, z_0) = Ind(g, z_0)$.

Theorem A.6 (Argument Principle). *Let $f : \Omega \longrightarrow \mathbb{C}$ be a meromorphic map defined in some domain $\Omega \subseteq \mathbb{C}$. Let γ be any differentiable simple closed curve in Ω with interior domain D which does not intersect any zero or pole of f. Let Z be the number of zeros of f in D counted with multiplicity and E be the number of poles of f in D, counted taking into account its order. Then,*

$$Ind(f \circ \gamma, z) = Z - E, \quad \text{for all } z \in D.$$

A.2 The Riemann sphere

Now, it is convenient to review the natural extension of the complex plane, the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Formally, the Riemann sphere is a complex Riemann manifold of dimension 1. In particular, it has a holomorphic atlas composed of two local charts, the stereographic projections centred in opposite poles of the sphere. Each one of these charts covers the whole Riemann sphere except one point, and the metric in the sphere is the one induced by the usual metric in $\mathbb{C}$ with these charts.

Thus, in our case, whenever we are working in the Riemann sphere, we can work as we normally would in the complex plane, taking into account that rational functions can be extended to ∞ holomorphically.

Working in the Riemann sphere, we shall also recall the concept of Möbius transformations, detailed in [Ahl66, Section 3.3].

Definition A.7. *We say a Möbius transformation is a map $M : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ of the form*

$$M(z) = \frac{az + b}{cz + d}, \quad \text{for all } z \in \hat{\mathbb{C}},$$

for some $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc \neq 0$.

Proposition A.8. *Möbius transformations are the only biholomorphisms of the Riemann sphere, $\hat{\mathbb{C}}$.*

Proposition A.9. *Möbius transformations send circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$, taking into account that straight lines in $\mathbb{C}$ are circles in $\hat{\mathbb{C}}$ crossing the infinity point.*

Finally, let us recall the Riemann mapping theorem, a key result in iteration in the unit disk.

Theorem A.10 (Riemann mapping Theorem). *Let $U \subseteq \mathbb{C}$ be a simply connected, non-emtpy, open set of $\hat{\mathbb{C}}$, such that the boundary of U contains at least 2 points.*
Then, it exists a **biholomorphic map** *between U and the unit disk $\mathbb{D}$.*

A.3 Operator theory

We shall also review one key result in operator theory.

Theorem A.11 (Banach Fixed Point). *Let $(\mathcal{X}, d)$ be a complete metric space. Let $T : \mathcal{X} \longrightarrow \mathcal{X}$ be eventually contractible, i.e., for some $k \geq 1$:*

$$d(T^k x, T^k y) < d(x, y), \quad \forall x, y \in \mathcal{X}$$

Then, there exists a unique $x_0 \in \mathcal{X}$ such that $T(x_0) = x_0$. Moreover, for any $x \in \mathcal{X}$, the sequence $(T^n(x))_n$ converges to x_0.

More background on this well-known theorem can be found in [AJS18, Theorem 1.1].

A.4 Uniformization theorem

We shall also mention the fundamental result classifying simply connected Riemann surfaces.

Theorem A.12 (Uniformization theorem). *Let S be a simply connected Riemann surface. Then, S is conformally equivalent to either the unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$, or the Riemann sphere $\hat{\mathbb{C}}$.*

The relevance of this theorem in complex dynamics is notable, since every non-simply connected Riemann surface can be covered by a simply connected one.

A.5 Proper maps

Finally, we end this appendix by reviewing a purely topological concept which can be studied in the framework of complex analysis, proper maps, introduced formally in [Ste11, Section 1.2].

Definition A.13 (Proper map)**.** Let us consider domains $U, V \subseteq \hat{\mathbb{C}}$. We say that a function $f : U \longrightarrow V$ is proper if, for every compact set $K \subseteq V$, the preimage $f^{-1}(K) \subseteq U$ is compact.

Proposition A.14. *Let us consider domains $U, V \subseteq \hat{\mathbb{C}}$ and an analytic map $f : U \longrightarrow V$. Then, the following are equivalent:*

(i) f is proper.

(ii) f has finite topological degree k, i.e., every point $y \in V$ has exactly k preimages in U, counted with multiplicity.

(iii) $f(z)$ tends to ∂V as z tends to ∂U, in the following sense: if the sequence $(z_n)_n \subseteq U$ is not contained in any compact set in U, then the sequence $(f(z_n))_n$ is not contained in any compact set in V.

For instance, rational maps of degree k are proper maps of the same degree.

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