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Treball final de grau

Single-object auction analysis: theory and strategic bidding

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Abstract

The main goal of this work is to discuss the most relevant aspects regarding single-object auction theory. In particular, principal types of single-object auctions, equilibrium strategies for each type, and variations on the proposed model are disclosed. In order to accomplish these aims, a brief summary of basic terms on game theory is provided in first place so as to set the foundations of auction theory. This preliminary summary includes a series of key results on game theory, allowing to use them as a tool to apply into auction theory and thus understand the second part of the document.

Finally, a real example of an auction type is provided: the U.S Treasury Bill auction. Definitions, mechanisms and bidding behaviour are provided, along with a brief introduction to multi-object auctions.

Resum

L'objectiu principal d'aquest treball és presentar els temes més rellevants sobre la teoria de subhastes d'objecte únic. En particular, s'exposen els tipus de subhastes, estratègies d'equilibri per a cada tipus i variacions sobre el model proposat. Per a poder arribar als resultats sobre subhastes, en primer lloc s'exposa un breu resum dels principals conceptes de la teoria de jocs. Aquest resum preliminar culmina amb un conjunt de resultats importants per a utilitzar com a eina durant el document en l'anàlisi de la teoria de subhastes.

En segon lloc, s'exposa un exemple real d'un tipus de subhasta: la subhasta de deute públic a curt termini a càrrec de la Tresoreria dels Estats Units (T-bill auction). S'expliquen definicions, mecanisme de la subhasta i comportament dels jugadors, per a finalment introduir de manera breu el concepte de subhastes d'objecte múltiple.

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Contents

Chapter 1

Preliminary

This introductory chapter eases into the main concepts regarding game theory, explaining its historic origins, main definitions and strong results with the objective to further comprehend the framework used to study auction theory. Notions like dominant strategies, equilibria and game types are defined.

1.1 Introduction to Game theory

Game theory is a branch of applied mathematics that attempts to model interactions between competing agents (players) who have the power to choose actions (strategies) that influence each other, all in a battle to maximize their own rewards (payoff). The main objective of this theoretical framework is to produce optimal decision-making for every agent participating in the conflict (game), and ultimately extract these results to real-world scenarios to maximize efficiency of transactions and operations.

Game theory reduces complex situations into a simple concept defined as a **Game**. Two or more players have strategic actions in a given situation containing a set of rules and payouts for each player. The key is that one player's payout is influenced by other player's decisions. Game theory identifies all of each player's possible decisions, how they affect each other's payout and studies these interactions in an attempt to dictate how rational players should behave to maximize their payout.

The two main pioneers of game theory were John von-Neumann (1930s) and John Nash (1950s). They successfully started a field of economic behaviour study, always with a mathematical background. Initially, John von-Neumann published the paper *On the theory of games and strategy* in 1928. His best work was shown in his book *Theory of Games and Economic Behaviour*, describing a solid method to find solutions to two player zero-sum games, which are a basic type of games defined in the upcoming sections. Later on, in the 1950s, John Nash developed a tool to determine mutual consistency of strategies, called after himself Nash Equilibrium. John Nash won the Economics Nobel Prize in 1994 for his contributions to game theory.

Regarding practical matters, game theory can be applied to many fields, from economics and business to computer science or politics. It is indeed a breakthrough in the study of negotiation and bargaining situations, which is still evolving rapidly nowadays.

A well known class of games are **auctions**, which have special theoretical interest because of the vast real-life applications of the results that game theory dictates. Areas like politics bargaining, public goods allocation or even bonds and finance are subject to auction theory.

1.2 Preliminary definitions

This section provides the basic concepts and results needed to understand the upcoming chapter on auction theory.

Definition 1.1. *Player: A* ***player (Pl)*** *is an agent that participates in the game.*

Every player has some decisions to make **(strategies)** and some benefit or loss from those decisions **(payoff)**. Normally, games involve more than one player. For example, auctions can have any amount of players (minimum of 2), often referred to as bidders. We will assume a game has N players, $N \in \mathbb{N}$.

Notation: A generic player is referred to as player i, with $i = 1, ..., N$

Definition 1.2. *Strategy: A player's* ***strategy (β)*** *is any of the options which is chosen in a setting where the optimal outcome depends not only on their own actions but on the actions of others.*

Every player must submit a strategy β_i for $i = 1, ..., N$, which is their way of participating in the game. The set of all possible strategies for a player is called the **strategy set (S_i)**, which can be finite or not.

Notation: The set of every strategy set $i = 1, ..., N$ is defined as the **strategy profile** $S = S_1 \times ... \times S_N$, so a profile of strategies is $(\beta_1, ..., \beta_N) \in S$ with $\beta_i \in S_i$

for all $i = 1, ..., N$.

In an auction, the strategies for every player are called bids, which are a numerical value $b_i \in [0, \infty)$ so for $i = 1, ..., N$, thus player i's **bid (b_i)** is defined. Bidding will be discussed further on.

Depending if the strategy is played according to a probability distribution or not, a strategy can be pure or mixed:

Definition 1.3. *Pure strategy: A strategy β is said to be **pure** when it is played with full probability.*

A pure strategy is a complete and detailed decision-making rule that dictates a particular action for every possible situation. The strategy set for player i (S_i) could have also been defined as the set of all possible pure strategies.

Definition 1.4. *Mixed strategy: A strategy β_i, is said to be **mixed** if the strategy is $\beta_i = (\eta^i_1, ..., \eta^i_{m_i})$, with a probability of playing η^i_j is $P(\eta^i_j) = p_j$, for some $p_j \in \mathbb{R}$ such that $p_j \geq 0$ and $\sum_{j=1}^{m_i} p_j = 1$ (m_i denotes the cardinality of S_i).*

A pure strategy can be regarded as a mixed strategy with a certain $p_j = 1$ and all other $p_k = 0$ for $k \neq j$. Normally, mixed strategies are based on probability distributions. This document focuses solely on pure strategies.

Definition 1.5. *Payoff: A player's **payoff** is a function $\pi_i : S \longrightarrow \mathbb{R}$ which assigns a numerical value to each player for every strategy combination chosen by the players.*

The payoff quantifies the benefit (or loss) that the player gets from having played the game, once everybody has chosen their strategy. Notice that the payoff not only depends on the player's strategy set S_i (his decisions), but on all of the player's strategies (strategy portfolio S). For every different combination of strategies of the players $(\beta_1, ..., \beta_N) \in S$, there is a payoff defined for each player $\pi_i(\beta_1, ..., \beta_N)$, $i = 1, ..., N$.

These three basic notions are the main agents of what is called a **game**, the main starting point for analyzing and modelling conflict situations.

Definition 1.6. *Game: A **game** is a triplet $G = (I, (S_i)_{i \in I}, (\pi_i)_{i \in I})$ where:*

1. *$I = \{1, ..., N\}$ is the number of players.*
2. *For every $i \in I$, S_i is player i's strategy set.*
3. *For every $i \in I$, $\pi_i : S \longrightarrow \mathbb{R}$ is the payoff of player i, where $S = S_1 \times ... \times S_N$ is the strategy profile*

1.3 Types of games

There are several classifications of games, this section puts focus in the most relevant classifications for the purposes of this document.

- Symmetric or assymetric games

A game is called **symmetric** if every player has the same payoff function (transposed). Every player has the same strategies available and each player earns the same payoff when playing the same move. If not, it is called **asymmetric**. Classic examples of symmetric games include 1v1 "Matching Pennies" or "Rock, Paper Scissors". Both players have the same strategy set and payoff functions regarding each of them.

Table 1.1: **1v1 Rock, Paper, Scissors**

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
Scissors	(-1,1)	(1,-1)	(0,0)

Example 1.7. This table is called **payoff matrix**, and it is how a game with two players and finite pure strategies is normally pictured. Each row is a pure strategy for player 1, and each column is a pure strategy for player 2. The payoffs for player 1 are the first numbers on each cell, and the second numbers are player 2's payoffs. For example, if player 1 plays Paper and player 2 plays Rock, the payoff is (1,-1), so 1 for player 1 (win) and -1 for player 2 (loss). A payoff of (0,0) means a tie. Because the payoffs are the same but switched between the two players (when we change the order of the payoff numbers in each cell, we get the transposed payoff matrix), this game is symmetric.

It will be assumed that auctions are a symmetric game when modelling, as it is the most basic and relevant case, where all the players have the same information (their valuation) and can play the same moves (bid anything they want).

- Perfect or imperfect information

  A game has **perfect information** if every player knows, before playing the game, all the information that would be available at the end of the game. Some examples include Chess or Go. Games without that type of information are called **imperfect information** games. A typical example is Poker or most card games.

  Private value auctions are imperfect information games, as only each of the players knows their own valuation of the item they are bidding for.

- Zero-sum games

  A game is called **zero-sum game** if the net change in players' wealth is 0. This means that if one player loses, another player (or a set of players sharing the win) win the same amount that the first player loses, making all the payoffs sum to 0. Otherwise, the game is a **nonzero-sum game**. Auctions are normally a nonzero-sum game, meaning the wealth net change of the players is not zero.

- Simultaneous or sequential

  A game is **simultaneous** when all players execute their plays at the same time. If players take turns when playing, the game is **sequential**. This last type makes a player aware of other players' decisions before playing their move. For example, "Rock, Paper, Scissors" is simultaneous, while Chess is sequential. Depending on their type, auctions can be simultaneous or sequential.

Having stated the types of games, it is useful to remember that auctions in this work will be supposed as symmetric and non-zero sum game, unless another type of auction is specifically stated.

1.4 Domination and equilibrium

Domination is a concept which must be introduced in order to classify strategies in a way that we can distinguish if one strategy is better than others. Like this, a strategy can be classified as "good" or in some cases optimal.

Definition 1.8. *Dominated strategy: A strategy β_i^A is said to be **strictly dominated** by another strategy β_i^B (for player i, $i \in \{1, ...N\}$) if the payoff of playing β_i^B is bigger than the payoff of playing β_i^A regardless of what other players do.*

Strict domination means, with the last notation,

$$\pi_i((\beta_1, ..., \beta_i^B, ..., \beta_N)) > \pi_i((\beta_1, ..., \beta_i^A, ..., \beta_N)) \quad \forall \beta_j \in S_j \quad j \neq i$$

Remark 1.9. The strategy β_i^A is said to be **weakly dominated** by β_i^B if we allow

$$\pi_i((\beta_1, ..., \beta_i^B, ..., \beta_N)) \geq \pi_i((\beta_1, ..., \beta_i^A, ..., \beta_N)) \quad \forall \beta_j \in S_j \quad j \neq i$$

A strictly dominated strategy β_i^A should never be played, as there is always a better strategy which will yield more payoff in every possible case than playing β_i^A. A weakly dominated strategy is "advised" not to play, as the payoff will be lower or equal than the dominant strategy.

Notation: A strategy for player i (β_i) is **strictly dominant** when it strictly dominates all other strategies. Analogously, a strategy is **weakly dominant** when it weakly dominates all other strategies

Remark 1.10. There is a simple process to discard some dominated strategies for a player, it is called **Iterated Elimination of Strictly Dominated Strategies (IESDS)**.

This algorithm consists of eliminating the strictly dominated strategies for a player, thus restricting the game to a "new" game with fewer possible strategies for every player, called a **subgame**. Iterating this process for all the players until no more strategies are eliminated, the final subgame might end up being a single strategy for every player. In that case we say the game is **solvable** by IESDS.

The order of elimination of strategies does not affect the outcome of IESDS. When we allow weakly IEDS (IEWDS), outcomes may differ. Let's illustrate how domination works.

Example 1.11. A simple yet surprising example of IESDS is **Guess the 2/3 of the average game**. This example illustrates how domination works and, although it is a simple concept, outcomes might not be trivial.

In this game, N players must choose any number $b_i \in [0, 100]$, the one that gets closest to 2/3 of the average of the b_i wins the game.

$$avg = \frac{2}{3} \frac{\sum_{i=1}^{N} b_i}{N} .$$

The possible strategies for each of the N players are $S_i = [0, 100]$. By being a symmetric game, all dominated strategies apply for all players. Because *avg* is the average of the numbers chosen times 2/3, $avg \in [0, 100\frac{2}{3}]$. By knowing this, no player would play $b_i \in (100\frac{2}{3}, 100)$ because all these strategies are strictly dominated by playing $b_i = 100\frac{2}{3}$. The new subgame restricted to this domination has $S_i \in [0, 100\frac{2}{3}]$. Applying IESDS again, no player is going to play $b_i \in (100\frac{4}{9}, 100)$, leading to a new subgame with $S_i \in [0, 100\frac{4}{9}]$. Applying IESDS infinitely, we get to the symmetric equilibrium $b_i = 0 \ \forall i = 1, ..., N$. This game is then solvable by IESDS because the only strategy surviving after IESDS is $S_i = 0$ for all $i = 1, ..., N$.

Once domination is established, one can think of defining an equilibrium strategy between all the players in a game.

Definition 1.12. *Best response: Given N-1 strategies β_j with $j \neq i$, the* ***best response*** *of player i is the strategy β_i^* that maximizes his payoff π_i when the other N-1 players choose β_j.*

$$\pi_i(\beta_{j1}, ..., \beta_i^*, ..., \beta_{jN}) \geq \pi_i(\beta_{j1}, ..., \beta_i, ..., \beta_{jN}) \ \ \forall \beta_i \in S_i$$

Remark 1.13. A best response is not necessarily unique. For example, a best response strategy that is mixed has infinite possibilities of maximizing payoff, thus allowing infinite best responses. On the other hand, if the best response is a pure strategy, it might be unique.

Applying this definition for every possible strategy of every player, it is now possible to define a notion of equilibrium.

Definition 1.14. *Nash equilibrium: We say that a set of N strategies $(\beta_1^*, ..., \beta_N^*) \in S$ is a* ***Nash Equilibrium*** *when every β_i is a best response for all β_j with $j \neq i$, for all $i = 1, ..., N$.*

In a Nash Equilibrium, the N strategies of each player are mutual best responses between them. No player wants to deviate from their chosen strategy β_i^*, as it maximizes their payoff with respect to all the other $N - 1$ strategies. Changing a player's strategy will result in the same or less payoff. We will refer to a Nash Equilibrium simply as an equilibrium to simplify notation.

1.5 Main results

This section provides some general game theory results to use as a tool to understand and prove more easily some further results on auction theory.

Definition 1.15. *Principle of rationality: This principle assumes that players' decisions are the result of maximising their own selfish payoff functions conditional on their beliefs about the other players' optimal behaviour.*

By applying the principle of rationality, every player wants to maximise their own payoff, regardless of all other factors. If two strategies yield the same payoff, the player is indifferent on which one to choose, having no interest in other players' payoff functions.

Theorem 1.16. ***Nash's Theorem*** *(1950): Any game with a finite number of players and actions has a mixed-strategy Nash Equilibrium.*

Proof. The proof of this theorem relies on a vast amount of lemmas and previous definitions that are not relevant for this work. The complete proof is quoted in [1]. The procedure consists of finding a fixed point of a continuous function $f : S \longrightarrow S$ defined in the strategy portfolio by using contraction theorems. That fixed point is the equilibrium seeked. □

This strong theorem defines an equilibrium (mixed) for almost any standard game, including auctions. This equilibrium might not be pure in the sense that all the players strategies might not be pure, but the result assures the existence of at least one equilibrium.

Theorem 1.17. *If one player of a game employs a fixed strategy, then the opponent has an optimal counterstrategy that is pure.*

Proof. Proof and in-depth explanation are quoted in [2]. □

This result helps understand the idea of best response to a set of strategies. The result is phrased for $N = 2$ players, but can be extended to N players by supposing $N - 1$ players employ a fixed strategy.

Theorem 1.18. *In a game, if one player employs a fixed strategy, then any mixture of the opponent's pure optimal counterstrategies is itself a mixed optimal counterstrategy.*

Proof. Proof and in-depth explanation are quoted in [2]. □

[1]: *A tutorial on the proof of the existence of Nash Equilibria*, University of British Columbia

This result can also be extended to N players by reasoning the same way as in the previous theorem. It illustrates the possibility of infinite mixed-strategy best responses when the player has more than one pure best-response strategy.

For example, if β_i^A and β_i^B are optimal pure counterstrategies (best responses) for player i to a set of $N-1$ strategies β_j with $j \neq i$, then the mixed strategy $\beta_i = (\beta_i^A, \beta_i^B)$ with probability of playing β_i^A, $p(\beta_i^A) = p$ and $p(\beta_i^B) = 1-p$ is an optimal mixed counterstrategy for all $p \in [0,1]$. In other words, any combination of pure-strategy best responses is a mixed-strategy best response.

Proposition 1.19. *If a game is solvable by IESDS, the only remaining strategy $(\beta_1^*, ..., \beta_N^*)$ forms an equilibrium. [3]*

Proof. The argument is quite simple, as on each iteration we are discarding dominated strategies for a player, thus always maximizing his profit with respect to all other existing strategies. Applying this for every player, in the event of one only strategy remaining, this one maximizes every player's payoff with respect to the other $N-1$ payoffs and possible stratgies. □

Corollary 1.20. *If a game is solvable by IESDS, the solution is unique and does not depend on the order of elimination of strategies. [3]*

[2]: *A gentle introduction to Game Theory*, Saul Stahl, (1999)
[3]: *Iterated Elimination and Nash Equilibria*, University of Illinois, Chicago

Chapter 2

Auction theory

This chapter addresses the main matter of the document, explaining in full context single-object auctions. For the majority of this section, the main source of information consulted is the book *Auction Theory, Second Edition* by Vijay Krishna (2010) [4].

2.1 History

Auctions have been used for a long period of time for the sale of a variety of objects, being the most early reports in Babylon around 500 B.C, where women were auctioned for marriage. Later on, during the Roman Empire, spears from the victorious warriors were auctioned, along with slaves from the defeated side. Also, romans auctioned assets from people whose property had been confiscated. In other parts of the world auctions mechanisms were also implemented. For example, in China, personal belongings of deceased monks were auctioned. Nowadays, auctions are widely spread throughout the planet with many variations. From fine art pieces and collectables, auctions evolved and today are used to sell a wide range of commodities, like fish, tobacco, real estate or livestock. Even short and long-term securities are sold in weekly auctions in most countries. One of the latest and most influential uses of auctions is to transfer assets from public to private hands, for example sale of enterprises to the public. Also, thanks to the Internet, there are auction websites where people can auction any item online.

The word auction comes from the Latin *augere*, which means "to increase", mirroring the rise in the price of the desired object caused by the inherent competition of the players participating in the auction.

2.2 Types of auctions

Auctions can be single-object (bidders compete for one object), which is the topic of the document, or multiple-object (more than one object in a single auction).

Regarding single-object auctions, the classic types of auctions are the following:

- English Auction

  The English Auction (often called open ascending price auction) is the oldest and most common type of auction. The auctioneer (conductor of the sale) calls out a low price. Then, players start bidding in small increments, until there is only one interested player left. In that moment the auction ends and the object goes to the only interested bidder. In other words, the highest bidder wins the auctioned object.

  When modelling an English auction, the object's price starts from 0 and rises continuously, and each player indicates an interest to purchase at the current price. Once the price is high enough to only get one player interested, the auction ends. Last bidder wins the object. This type of auction is the most theatrical and common one in the movies.

- Dutch Auction

  The Dutch Auction (also called open descending price auction) is the counterpart to the English Auction. The auction begins with the object being called at a high price so that no player is interested in buying the object, then the price gets gradually lowered until the first player bids. Object is sold to the first bidder at the according price.

  This auction is generally used when fisherman sell the fish they have caught every day in what is called the Fish Market. Although it might not be so dramatic as the English, it has some theoretical interest.

- Sealed-bid first-price auction

  Players enter their bids simultaneously in sealed envelopes, the highest bid wins the object and pays the price written in their bid. This is the other most common type of auction.

- Sealed-bid second-price auction

  Players bid simultaneously in sealed envelopes, the highest bid wins the object but, instead of paying their own bid, the winner pays the second highest bid.

- Other

  Many selling formats fall under the scope of being defined as an "auction". For example, there are mixed Dutch-English formats, where price gets lowered until somebody bids, then with the possibility of bidding upwards. The Internet's common type of auction is called "deadline", where a bidder wins the object when he has the highest bid when a fixed time period ends. There are "candle" auctions, which are random "deadline" auctions. The candle auction was introduced in England during the 17th century, and it ended whenever the flame of the candle expired. When more objects are included, many more possibilities open up as a multi-object auction. All of these are less common and do not have much theoretical interest.

Remark 2.1. One can think of the occurrence of a tie, where 2 or more players bid the same. If the auction format allows ties, then the auctioned object goes to one of the winning bids with probability p, where $p = 1/m$ with m being the number of winning bids.

Four main different types of auctions have been pointed out. The interest is in studying what is the optimal behaviour of the bidders for each type of auction. This job cam be cut in half, as there is an equivalence between the Dutch and first-price auctions, and another equivalence between the English and the second-price auctions, despite being different formats at a first glance.

Proposition 2.2. *The Ducth open descending price auction is strategically equivalent to the first-price sealed-bid auction.*

Proof. The only information available to bidders in the Dutch auction is that one player has agreed to buy the object at the current price, which does not come in handy as the auction has already ended when the information is revealed. In the first-price sealed-bid auction, the player reflects their private information into a bid. Placing a bid in the first-price sealed-bid auction is equivalent to offering to buy at that bid in the Dutch, if the object has not been already bought. □

This proposition tells us that strategic behaviour and optimal strategies are the same in both types of auctions. For every strategy in a first-price auction there is an equivalent strategy in the Dutch auction and vice versa.

Proposition 2.3. *The English open ascending price auction is equivalent to the second-price sealed-bid auction with privative values.*

Proof. In an English auction, a player gets information when other players "drop out", meaning they are not willing to bid higher than the current price. But with privative values, this does not help. We can argue it by reasoning that, in an English auction, it is not optimal to stay in the auction if the price rises further than our valuation, only causing a potential loss. In the second-price auction, it is best to bid one's valuation (demonstrated later). $\square$

It is concluded that, with privative values, the optimal strategy for a player is bidding up to one's valuation. Privative values are defined in the next section, they refer to privacy among each player's monetary valuation of the object.

Remark 2.4. This equivalence is "weaker" than the first one because it is not a strategic equivalence and the hypothesis of privative values is required, while the Dutch - First-price equivalence does not rely on privative values.

Thanks to these equivalences, our study will be mainly focused on the first-price (I) and second-price (II) auctions from now on.

2.3 Formal definitions

Definition 2.5. ***Auction***: *An auction is a game $A = (I, (S_i)_{i \in I}, (\pi_i)_{i \in I})$ with:*

1. *$I = [1, ..., N]$ being the number of players, referred to as bidders.*
2. *$S_i = [0, \infty)$ where $b_i \in S_i$ for $i = 1, ..., N$ are called bids, which is a monetary amount agreed to pay for the object by each player.*
3. *$\pi_i : S \longrightarrow \mathbb{R}$, $i = 1, ..., N$, payoff which reflects the monetary gain or loss of each player resulting of buying (or not) the object.*

Auctions are used because the seller is unsure about the valuations that each player attaches to the object being sold. This uncertainty, from both the buyer's and seller's point of view is an inherent characteristic of auctions.

Definition 2.6. ***Valuation***: *Maximum price each player is willing to pay for the auctioned object (v_i).*

After having introduced valuations, it is fairly logical to conclude that no player would ever want to bid above their own valuation. In first-price auctions, if the bid is higher than the player's valuation, in the event of winning the auction, the player pays more than their valuation, resulting in a monetary loss (negative payoff). On the other hand, if losing, the payoff is 0. However, by bidding their own valuation, the payoff is always 0 no matter if the player wins or loses. We have just demonstrated that bidding one's valuation weakly dominates bidding above the valuation (in first-price auctions), but should nobody bid above their valuation in all auction formats?

This is not entirely true, as for example, suppose we have a second-price sealed bid auction. Suppose player i bids $b_i > v_i$ and suppose all other bids b_j with $j \neq i$ are such that $b_j < v_i < b_i$. This causes that the player to win the object with a positive payoff $v_i - b_k$, where $b_k = \max_{j \neq i} b_j$. If the player had bid $b_i \in (b_k, \infty)$ he would have had the same positive payoff $v_i - b_k$. So the strategy of bidding $b_i > v_i$ is not "entirely wrong", despite the fact that it is weakly dominated by other strategies.

Definition 2.7. ***Privative values***: *Situation where every player knows their own valuation v_i but does not know with certainty other players valuations v_j with $j \neq i$, and knowledge of other bidders' valuations will not affect their own valuation.*

Remark 2.8. Privative values will be assumed for the majority of the cases, as it is the most common situation. This leads to assuming that the valuations of the other players are identically and independently distributed (iid) in an interval $[0, W]$, according to a continuous distribution function F with density $f = F'$. It is supposed that players know the distribution function F of the valuations v_i with $i = 1, ...N$.

Sometimes, even the does not know their own valuation v_i, so the player must estimate it and may be affected by other players' valuations and information.

Definition 2.9. ***Interdependent valuations***: *Situation where a player's valuation is estimated based on other players' valuations.*

Interdependent valuations are often correlated with the true value of the object because other players have more information available, causing a big influence on players decisions.

A special case is when every player has the same valuation, called common valuation. This situation does not have much theoretical interest.

2.4 Equilibrium bids

This section presents and studies the optimal behaviours for the two main types of auctions, the First-price Auction (I) and the Second-price Auction (II). Because of the equivalence with the Dutch and the English auctions, it is sufficient to study (I) and (II). In order to model the uncertainty regarding valuations and other aspects that could affect the outcome, the following symmetrical model of an auction is defined.

Definition 2.10. ***Symmetrical model of an auction*** *A:*

- *Single-object auction (A) with N players.*
- *Players know their own valuations of the object v_i, realizations of V_i with $i = 1, ..., N$ are iid (independent and identically distributed) on some interval $[0, W]$ according to a distribution function F, which admits a continuous density $f = F'$.*
- *It is required that $E[V_i] < \infty$ for $i = 1, ..., N$.*
- *Privative values are supposed.*

Remark 2.11. Some clarifications for the symmetrical model. Each bidder knows their own v_i and that others v_j with $j \neq i$ are iid according to F. Bidders seek to maximize expected profit (risk-neutral). It is also assumed that each bidder does not have a budget constraint (they can pay up any amount), and all bidders want is to maximize their own selfish profit (principle of rationality).

Definition 2.12. *A strategy for a player (bidder) in an auction is a function*

$$\beta_i : [0, W] \longrightarrow \mathbb{R}^+$$

that determines a bid b_i for any given valuation v_i.

Notation: $\beta_i(v_i) = b_i$ notes the bid of player i when his valuation is v_i.

An equilibrium between all the players is searched, a set of bids that no person would want to move from their decision (mutual best response). Because of the symmetric model, the equilibrium will be a symmetric equilibrium strategy for each player. To study equilibriums, it is first required to model the two types of auctions: first-price (I) and second-price (II).

- **Second-price auction equilibrium**

Each player submits a bid b_i with $i = 1, ..., N$ according to a given valuation v_i, and their payoffs are:

$$\pi_i(b_i) = \begin{cases} v_i - max_{b_j \neq b_i} b_j & if \quad b_i > max_{b_j \neq b_i} b_j \\ 0 & if \quad b_i < max_{b_j \neq b_i} b_j \end{cases} . \tag{2.1}$$

If $b_i = max_{b_j \neq b_i} b_j$, there is a tie and the object goes to each winning bid with equal probability. If the winning bid is made by m players, $m = \#\{j | b_j = b_i\}$, then the expected payoff of a winning bidder is:

$$\frac{(v_i - max_{b_j \neq b_i} b_j)}{m} .$$

Proposition 2.13. *In a second-price sealed-bid auction, bidding one's own valuation*

$$\beta_i^{II}(v_i) = v_i$$

for $i = 1, ..., N$, is a weakly dominant strategy for every player.

Proof. Consider player 1 with a valuation v_1, and suppose $m_1 = max_{j \neq 1} b_j$ is the highest of the other players bids. By bidding b_1, player 1 wins if $b_1 > m_1$ with payoff $v_1 - m_1$ and loses the auction if $b_1 < m_1$ with payoff 0, (if $b_1 = m_1$, the player does not care if he wins or he loses, as payoff is 0 in both cases).

Suppose now player 1 bids $b_1 < v_1$. If $v_1 > b_1 \geq m_1$, player 1 wins the auction and the payoff is still $v_1 - m_1$. If $m_1 > v_1 > b_1$, player still loses. But if $v_1 > m_1 > b_1$, then player loses, whereas if he had bid v_1, he would have won and made a positive payoff $v_1 - m_1$, thus bidding $b_1 = v_1$ weakly dominates bidding $b_1 < v_1$.

Suppose now player 1 bids $b_1 > v_1$. If $b_1 > v_1 \geq m_1$, the player wins the auction, with payoff $v_1 - m_1$. If $m_1 > b_1 > v_1$, player loses the auction with payoff 0. But if $b_1 > m_1 > v_1$, the player wins the auction with payoff $v_1 - m_1 < 0$, so the player is better off by bidding v_1 and making 0 payoff, thus bidding $b_1 = v_1$ weakly dominates bidding $b_1 > v_1$.

Following this argument for every player, a symmetric equilibrium of the Second-price auction is for every player to bid their own valuation $\beta_i^{II}(v_i) = v_i$. $\square$

- **First-price auction equilibrium**

Each player submits a bid b_i and their payoffs are:

$$\pi_i(b_i) = \begin{cases} v_i - b_i & if \quad b_i > max_{j\neq i}b_j \\ 0 & if \quad b_i < max_{j\neq i}b_j \end{cases} . \tag{2.2}$$

If $b_i = max_{j\neq i}b_j$, there is a tie and object goes to each winning bid with equal probability. If the winning bid is made by m players, $m = \#\{j|b_j = b_i\}$, then the expected payoff of a winning bidder is:

$$\frac{(v_i - b_i)}{m} .$$

Clearly, no bidder would bid an amount equal to v_i, as their payoff would be guaranteed to be 0. Also, no player would bid higher than v_i as it would guarantee negative or zero payoff. We can conclude that bidding $b_i < v_i$ weakly dominates bidding $b_i \geq v_i$, as it yields 0 payoff or a positive payoff if a win is achieved.

In this type of auction, bidders face a trade-off between increasing the probability of winning the auction while reducing their potential gain.

We now know that the distribution function satisfies $b_i \leq \beta(V)$ and $\beta(0) = 0$, meaning that a player with valuation 0 will bid 0. Player 1 wins if

$$max_{i\neq 1}\beta(v_i) < b_1$$

Because β is an increasing function,

$$max_{i\neq 1}\beta(v_i) = \beta(max_{i\neq 1}v_i) = \beta(Y_1)$$

being Y_1 the highest of $N-1$ values iid in $[0, W]$. G will note the distribution function of Y_1. Player 1 wins if $\beta(Y_1) < b_1$, same as $Y_1 < \beta^{-1}(b_1)$. The expected payoff for player 1 is the probability of winning times the payoff if winning (if he loses, the payoff is 0 so we do not need to add it):

$$G(\beta^{-1}(b_1)) \times (v_1 - b_1)$$

The player wants to maximize this payoff, so we differentiate the last expression with respect to his bid b_1, resulting in:

$$\frac{g(\beta^{-1}(b_1))}{\beta'(\beta^{-1}(b_1))}(v_1 - b_1) - G(\beta^{-1}(b_1)) = 0$$

where $g = G'$ is the density of Y_1, which exist because it is the maximum of N-1 values of a continuous distribution function with density.

Supposing a symmetric equilibrium, $b_i = \beta(v_i)$, so $\beta^{-1}(b_i) = v_i$. To simplify notation, we will refer to a valuation as v and to a bid as b:

$$\begin{gathered}\frac{g(\beta^{-1}(b_1))}{\beta'(\beta^{-1}(b_1))}(v_1 - b_1) - G(\beta^{-1}(b_1)) = 0 \\ \Leftrightarrow \frac{g(v)}{\beta'(v)}(v - b) = G(v) \\ \Leftrightarrow g(v)v = G(v)\beta'(v) + g(v)\beta(v)\ .\end{gathered}$$

This results in the following differential equation:

$$\frac{d}{dv}G(v)\beta(v) = vg(v)\ .$$

and by applying $\beta(0) = 0$ we can specifically solve the differential equation:

$$\beta(v) = \frac{1}{G(v)}\int_0^v yg(y)\,dy = E[Y_1 | Y_1 < v]\ .$$

Notation: For the rest of the chapter, F indicates the distribution function of the valuations, iid in an interval $[0, W]$. G is the distribution function of Y_1, which is the highest of N-1 iid valuations in $[0, W]$. In particular, $G = F^{N-1}$. Both F and G accept densities $f = F'$ and $g = G'$.

This clarification of the strategy will help us prove the following result.

Proposition 2.14. *Symmetric equilibrium strategies in a first-price auction are given by*

$$\beta_i^I(v_i) = E[Y_1|Y_1 < v_i]$$

for $i = 1, ..., N$, where Y_1 is the highest of N-1 iid values in $[0, W]$.

Proof. We are going to show that following the strategy in the proposition is optimal.

Because β is increasing and continuous, in equilibrium the player with the highest valuation will bid the highest and win the auction. Bidding $b > \beta(W)$ is not optimal as shown earlier in the chapter.

The expected payoff of a player when bidding $b \leq \beta(W)$ is the following. Naming $z = \beta^{-1}(b)$ the value for which b is an equilibrium bid, bidding $\beta(z)$ will yield an expected payoff (Notation: $\Pi(b, v)$ is the expected payoff of a bidder with valuation v that bids b):

$$\begin{aligned}
\Pi(b, v) &= G(z)[v - \beta(z)] \\
&= G(z)v - G(z)E[Y_1|Y_1 < z] \\
&= G(z)v - \int_0^z yg(y)\,dy \\
&= G(z)v - G(z)z + \int_0^z G(y)\,dy \\
&= G(z)(v - z) + \int_0^z G(y)\,dy \; .
\end{aligned}$$

We have obtained that

$$\Pi(\beta(v), v) - \Pi(\beta(z), v) = G(z)(z - v) - \int_v^z G(y)\,dy \geq 0 \; .$$

for all $z \geq v$ and all $z \leq v$.

This shows that if all bidders follow the strategy β, a bidder with valuation v_i cannot benefit from bidding anything different than $\beta(v_i) = b_i$, implying a symmetric equilibrium. □

Corollary 2.15. *The optimal bid $\beta_i^I(v_i)$ can be expressed as:*

$$\beta_i^I(v_i) = v_i - \int_0^{v_i} \frac{G(y)}{G(v_i)}\,dy$$

Proof. The conditional expectation $E[Y_1|Y_1 < v]$ is:

$$\beta^I(v) = E[Y_1|Y_1 < v] = \frac{1}{G(v)} \int_0^v y g(y)\, dy$$

$$\Leftrightarrow G(v)E[Y_1|Y_1 < v] = \int_0^v y g(y)\, dy$$

$$\Leftrightarrow G(v)E[Y_1|Y_1 < v] = vG(v) - \int_0^v G(y)\, dy$$

$$\Leftrightarrow E[Y_1|Y_1 < v] = v - \int_0^v \frac{G(y)}{G(v)}\, dy\ .$$

□

This remark is really useful to easily calculate equilibrium strategies.

This section has shown symmetric equilibrium strategies for both types of auctions. Because of the equivalence of the Dutch auction to the first-price, and the English auction to the second-price, these equilibria are the same for the strategically equivalent formats (with the assumptions made: symmetrical model, no budget constraint, attempt to maximize profit, valuations are IID).

Example 2.16. A classic example of studying a first-price auction is the uniform distribution $v_i \sim U(0, W)$ with IID valuations and $N \geq 2$ players.

The distribution function of the valuations is:

$$F(v) = \frac{v}{W}$$

$$G(v) = (\frac{v}{W})^{N-1}$$

is then the distribution function of the maximum of N-1 IID values in [0,W]. This models the probability of all the other N-1 bids are lower than the bid b_i

The optimal bid is, applying the last remark:

$$\beta_i^I(v) = v - \int_0^v \frac{G(y)}{G(v)}\, dy$$

$$= v - \int_0^v (\frac{y}{v})^{N-1}\, dy$$

$$= v - \frac{v}{N}$$

$$= \frac{N-1}{N} v$$

resulting in the equilibrium strategy

$$\beta^I(v) = \frac{N-1}{N}v$$

Makes sense that every player wants to bid just a little lower than their valuation, to try to win the object keeping a competitive bid, while attempting to make a strictly positive profit by not bidding v_i or more. Notice that this bid does not depend on W, so this result is valid for any uniform distribution of the valuations in any interval $[0, W]$. Also notice that when the number of players is increased, the bids b_i tend to v_i, minimizing every player's payoff in case of winning.

On the other hand, an equilibrium in the second-price auction with uniform valuations is straightforward: $\beta^{II}(v) = v$ for every player.

Corollary 2.17. *Equilibrium bids in second-price auctions are higher than in first-price auctions.*

$$\beta^{II}(v) > \beta^I(v)$$

Proof. This result is fairly simple, but crucial observation. $\beta^{II}(v) = v$ is the optimal bid for II, while $\beta^I(v) = E[Y_1|Y_1 < v] < v$ is the optimal bid for I, hence proved. □

2.5 Revenue and efficiency

There are two main different ways to measure the degree of performance of an auction. From the point of view of the seller, the **revenue**, or expected selling price, is the main tool to compare different auction formats. From the society's point of view (players), **efficiency**, meaning that the object goes to the player that values it the most, is the way to determine how good the auction format performs by what is called the expected payment. This efficiency takes an important role when in public held asset auctions. The government must choose the most "efficient" auction format so the object gets allocated in the most optimal way, although the revenue for the government might be higher in another format. In the optimal situation, the auctioned object should be won by the player who values it the most (has the biggest valuation v_i).

However, should efficiency be questioned, or should we leave the market to reallocate the object efficiently even after the auction takes place? Post-auction transactions (reselling) typically involve less agents and bargaining, thus getting far from efficiency. Also, resales might carry higher transaction costs, which leads to concluding that resale operations are not efficient.

- **Revenue and efficiency comparison between second-price and first-price auctions**

The expected payment of the first-price auction for a bidder with value v is their bid times the chance of winning:

$$m^I(v) = P(Win) \times Bid = G(v) \times E[Y_1|Y_1 < v] \; .$$

This expected payment is the same as in the second-price auction. Suppose a player bids $b_i = v_i$ in the second-price auction, then the expected payment of the player is:

$$m^{II}(v) = P(Win) \times E[2nd\ highest\ bid \mid b_i\ is\ the\ highest\ bid] \; .$$

and because of the equilibrium $\beta_i^{II} = v_i$,

$$\begin{aligned} m^{II}(v) &= P(Win) \times E[2nd\ highest\ valuation \mid v_i\ is\ the\ highest\ valuation] \\ &= G(v) \times E[Y_1|Y_1 < v_i] \end{aligned}$$

with G and Y_1 being the same as in the last chapter.

Notation: The expected payment of a player in both auctions with valuation v is then defined as:

$$m^I(v) = m^{II}(v) = m(v) \; .$$

Proposition 2.18. *The expected revenues of a first-price auction and a second-price auction are the same when valuations v_i are iid and privative $i = 1, ..., N$*

Proof. The expected revenue of the seller is the sum of the expected payments of each bidder before knowing their values (ex ante).

The ex-ante expected payment of a bidder (for both auctions because $m^I(v) = m^{II}(v) = m(v)$) is:

$$\begin{aligned} E[m(v)] &= \int_0^W m(v) f(v)\, dv = \int_0^W (\int_0^v yg(y)\, dy) f(v)\, dv \\ &= \int_0^W (\int_y^W f(v)\, dv) yg(y)\, dy = \int_0^W y(1 - F(y)) g(y)\, dy \; . \end{aligned}$$

The distribution functions F, G, and their densities $f = F'$ and $g = G'$ are the same as in the previous sections. The expected revenue of the seller $E[R_s]$ is N times (number of players) the ex-ante expected payment of a bidder:

$$E[R_s] = N \times E[m(v)] = N \times \int_0^W y(1 - F(y))g(y)\,dy\ .$$

Now, we can spot that the density of Y_2^N (second highest of N values) is

$$f_2^N = N(1 - F(y))f_1^{N-1}(y)$$

and because $f_1^{N-1}(y) = g(y)$, we get the following:

$$E[R_s] = \int_0^W y f_2^N(y)\,dy = E[Y_2^N]$$

for both types of auctions. □

This proposition concludes that, **on average**, the expected revenue of both auctions are the same, but this does not mean that the two formats are indifferent to the seller. Let's illustrate this case with a familiar example.

Example 2.19. Suppose $N = 2$ (2 bidders) and values are uniformly distributed. In the last section we demonstrated that the symmetrical equilibrium strategy for the first-price auction (for uniformly distributed values, see Example 2.11) is $\beta^I = \frac{1}{2}v$. If the realized values are such that $\frac{1}{2}v_1 > v_2$, we will have $R_s^I > R_s^{II}$. If $\frac{1}{2}v_1 < v_2$, then $R_s^I < R_s^{II}$.

We can go a bit more in depth to compare price distributions.

Remark 2.20. Revenues of the second-price auction are more variable than revenues of the first-price auction

This is quite a simple observation, but helps to understand the next results. Supposing players behave optimally, bids in the second-price auction can go from 0 to W (all valuations are in $[0, W]$, and a player behaving optimally would never bid above their valuation). On the other hand, bids in the first-price auction can go from 0 to $E[Y_1] < W$ (equilibrium bids), thus giving more variation in revenue for the seller in the second-price.

This observation is reflected in the following more general result, but first let's introduce a simple concept:

Definition 2.21. ***Mean-preserving spread:*** *Let F and G be distribution functions, we say G is a mean-preserving spread (MPS) of F, if G is formed spreading out one or more portions of F's density function and their expected value is the same.*

Proposition 2.22. *Let L^I and L^{II} be the distributions of prices (bids) in both auctions, then L^{II} is a* ***mean-preserving spread*** *of L^I.*

The last result shows us that from the point of view of the seller, the second-price is riskier than the first-price auction. If the seller is risk-averse, he is going to opt for the first-price auction in an attempt to raise an assured amount of money (assuming bidders are risk-neutral). On the other hand, if the seller is not risk-averse, he will prefer the second-price to try to gain more money (although he could also lose more. The following result applies for equilibrium prices, being L^I_* and L^{II}_* equilibrium distribution of prices:

Proposition 2.23. *Assuming iid and privative values, L^{II}_* is a mean-preserving spread of L^I_*, being the distribution of equilibrium prices of second and first-price auctions respectively.*

Proof. The revenue in a second-price auction is $R_s^{II} = Y_2^N$, while in the first-price it is $R_s^I = \beta(Y_1^N)$ with $\beta = \beta^I$. We have that

$$E[R_s^{II}|R_s^I = p] = E[Y_2^N|Y_1^N = \beta^{-1}(p)]$$

and now for all y,

$$E[Y_2^N|Y_1^N = y] = E[Y_1^{N-1}|Y_1^{N-1} < y] \ .$$

Using this equality, we have

$$E[R_s^{II}|R_s^I = p] = E[Y_1^{N-1}|Y_1^{N-1} < \beta^{-1}(p)] = \beta(\beta^{-1}(p)) = p \ .$$

Because $E[R_s^{II}|R_s^I = p] = p$, there exists a random variable Z such that the distribution of R_s^{II} is the same as the distribution of $R_s^I + Z$ and $E[Z|R_s^I = p] = 0$. Like this, we have that L^{II}_* is a mean-preserving spread of L^I_*. □

Let's now focus on explaining the equality of expected revenues of the two types of auctions.

Definition 2.24. ***Standard auction:*** *An auction is said to be standard if the person who bids the highest wins the auction (gets the auctioned object).*

Standard auctions can have many formats. For example, first-price and second-price are obviously standard. Even third-price, fourth-price... are standard. An example of a nonstandard auction is **lottery**, where the chance of winning is the ratio of the amount bid by the player between the total amount bid by all players. The person who bids the most (buys more lottery tickets) has more chance to win, but might not win the prize in the end.

Theorem 2.25. ***Revenue-equivalence principle:*** *Suppose valuations are iid and bidders are risk-neutral. Any symmetric and increasing equilibrium β^A of any standard auction A, such as $m^A(0) = 0$ (equilibrium expected payment of a bidder with value 0 is 0), yields the same expected revenue to the seller.*

Proof. Consider a standard auction A, its symmetric equilibrium β^A and $m^A(v)$ the equilibrium expected payment of a player with value v. Suppose $m^A(0) = 0$. Suppose all players but player 1 follow β^A. Let's consider player 1's expected payoff (with value v) when he bids $\beta(z)$ instead of $\beta(v)$ (equilibrium). Player 1 wins if $\beta(z) > \beta(Y_1)$, so wins if $z > Y_1$. His expected payoff is:

$$\pi(z,v) = G(z)x - m^A(z)$$

with $G(z) = F(z)^{N-1}$ distribution function of Y_1. Maximizing the payoff with respect to z, we get:

$$\frac{d}{dz}\pi(z,v) = g(z)x - \frac{d}{dz}m^A(z)\ .$$

and applying that in equilibrium $z = v$, we get that for all y

$$\frac{d}{dy}m^A(y) = g(y)y$$

so we can obtain the following:

$$m^A(v) = m^A(0) + \int_0^v yg(y)\,dy$$

$$= \int_0^v yg(y)\,dy = G(x) \times E[Y_1|Y_1 < v]\ .$$

where we have used our hypothesis of $m^A(0) = 0$. We can see that the right hand side of the equality does not depend on A (the auction form), hence proved. □

Example 2.26. Let's work on the familiar example, with values uniformly distributed on $[0, W]$, to find out the expected revenue.

We know $F(v) = \frac{v}{W}$, then $G(v) = (\frac{v}{W})^{N-1}$ (calculated earlier). Now, by applying the revenue-equivalence principle, if we have a standard auction A with $m^A(0) = 0$, then

$$m^A(v) = \frac{N-1}{N}v(\frac{v}{W})^{N-1}$$

and applying expected values we get the expected payment of each player:

$$E[m^A(v)] = \frac{N-1}{N(N+1)}W$$

Now, the expected revenue of the seller is the sum of each expected payment, but since they are the same for the N players,

$$E[R^A] = E[m^A(v)] \times N = \frac{N-1}{N+1}W$$

The revenue-equivalence principle shows the reason why the expected selling prices of symmetric first-price and second-price are the same. This powerful result can also help to point out equilibrium strategies in auctions that seem a bit strange and that, at first glance, seem difficult to decipher. Let's discuss an example of a different auction format:

Example 2.27. All-pay auction: An all-pay auction is a first-price sealed bid auction, but with an extra rule: all players pay what they bid, no matter they win or lose. It is a quite effective way of modelling lobbying activities, so it has a lot of theoretical interest. Lobbying activities consist on different parties (players) spending money (bids) to influence the government into selecting or directing some specific policies into a specific way (winning the "auction").

This type of auction is indeed standard, as the highest bidder wins the object (policy in favor of the highest bidder). Also, a bidder with value 0 is not interested in bidding, so $m(0) = 0$. Searching for a symmetric equilibrium strategy with IID and privative values, we can apply the Revenue-Equivalence principle to find a symmetric, increasing equilibrium of the All-pay auction. By observing that, in this specific type of auction, the expected payment of a bidder with value $m(v)$ is the same as his bid, the following equality stands:

$$\beta^{AllPay}(v) = m^{AllPay}(v) = \int_0^v yg(y)\,dy$$

Let's prove that this strategy is a symmetric equilibrium. Suppose all players but one employ the strategy β^{AllPay}. By bidding $\beta(z)$ and having value v, is expected payoff is:

$$G(z)v - \beta(z) = G(z)v - \int_0^v yg(y)\,dy = G(z)(v-z) + \int_0^z G(y)\,dy$$

where G and $g = G'$ are the same as in the last chapter and integration by parts is used in the second equality.

This expected payoff is the same situation as in the first-price auction equilibrium proof (stated earlier in this document), so by reasoning in the same way, the expected payoff is maximized when $z = v$, giving the **symmetric equilibrium of the all-pay auction**:

$$\beta_i^{AllPay}(v_i) = E[Y_1 | Y_1 < v_i]$$

for $i = 1, ..., N$.

Remark 2.28. Because of the revenue-equivalence principle, any single-object auction which allocates the object to the highest bidder (standard auction) and $m^A(0) = 0$, will have the same expected revenue for the seller. These include firs-price, second-price, all-pay, third-price, n-th price... but not lottery, as it is not standard.

The seller might think of ways of raising the expected revenue, this can be done by establishing entry fees or reserve prices, which are discussed in the next section. These modifications might break the revenue-equivalence principle and might not give the object to the party valuing it the most, thus breaking efficiency.

2.6 Further considerations

This section takes on some specific notions in auction theory that affect bidder behaviour and expected revenues in both first-price and second price auctions. Symmetric model assumptions are questioned and the consequence of elliminating some of them are studied.

2.6.1 Reserve prices

Definition 2.29. *A **reserve price** $r > 0$ is a fixed monetary amount set by the selling party so that the object auctioned will not be sold if the price determined after the auction is lower than r.*

Reserve prices are a useful tool for the seller when searching for a minimum gain. This concept changes the expected payment of each bidder thus the expected revenue for the seller. One can ask if symmetric equilibrium bids depend on the introduction of a reserve price.

In a second-price auction, any bidder with valuation $v_i < r$ will not be able to earn a positive payoff, as the object is sold at a minimum price r. Like this, the optimal bidding strategy for the second-price auction still holds $\beta_i^{II}(v_i, r) = v_i$. On the other hand, the expected payment of a bidder when $v_i \geq r$ is:

$$m^{II}(v_i, r) = rG(r) + \int_r^{v_i} yg(y)\, dy \; .$$

Notice that if $v_i = r$, then $m^{II}(v_i, r) = rG(r)$. Also notice that it is necessary that $v_i \geq r$ because of the impossibility of obtaining a gain when $v_i < r$.

In a first-price auction, the same reasoning about bidders with $v_i < r$ stands: they cannot make a positive payoff. Also, a bidder with $v_i = r$ will win the auction only if all other $v_j < r$, $j \neq i$. Then this bidder can win the auction bidding $\beta_i^I(r) = r$. The equilibrium strategy remains unaltered regarding other aspects, so the symmetric equilibrium strategy when $v_i \geq r$ is the following:

$$\beta_i^I(v_i, r) = E[max Y_1, r | Y_1 < v_i] = r\frac{G(r)}{G(v_i)} + \frac{1}{G(v_i)} \int_r^{v_i} yg(y)\, dy \; .$$

As a consequence, the expected payment when $v_i \geq r$ is:

$$m^I(v_i, r) = G(v_i)\beta_i^I(v_i, r) = rG(r) + \int_r^{v_i} yg(y)\, dy \; .$$

To sum up, a reserve price $r > 0$ eliminates all bidders with valuations $v_i < r$, as they are indifferent in participating in the auction or not.

Remark 2.30. The expected payments of both auctions are the same when introducing a reserve price r, just as in the previous sections when no reserve price was introduced ($m^I(v, r) = m^{II}(v, r)$).

2.6.2 Entry fees

Definition 2.31. *An **entry fee** $t > 0$ is a fixed monetary amount set by the selling party which bidders must pay if they want to submit any bid.*

One can picture an entry fee as a ticket for participating in the auction.

This entry fee also leaves out all bidders with valuation $v_i < t$, as they will not be able to make a positive payoff by any means in the event of participating in the auction.

Remark 2.32. By choosing a reserve price of r, the effect on bidder behaviour (strategy and expected payment) is the same than by setting an entry fee $t = G(r)r$. The same set of bidders will be excluded.

This remark is easily proven by substituting $t = G(r)r$ in the expected payment function stated earlier.

It is concluded that introducing an entry fee has an equivalent effect of a reserve price (although they may not be the same monetary amount).

Remark 2.33. When in absence of entry fees (or an equivalent reserve price), the object is always sold to the highest bidder, which is, in the symmetric model of both auctions, the bidder with the highest valuation. This shows that both formats (I and II) are efficient because they allocate the object to the party that values it the most. On the other hand, efficiency when introducing entry fees (or reserve prices) is often questioned. Suppose the value of the object for the seller is $v_s > 0$ and sets a reserve price of $r > v_s$ (same for entry fee). If all valuations are such that $v_i < r$ but one valuation is such that $v_k > v_s$, then the object will stay in hands of the bidder, meaning inefficiency because there was a bidder that valued it more. This observation causes a trade-off between efficiency and revenue.

2.6.3 Budget constraints

During all this document, it has been assumed that all bidders can pay up to their valuation v_i, but this may not always be the case.

Definition 2.34. *A **budget constraint** for bidder i is a monetary amount c_i which indicates the maximum quantity that the player can pay for the auctioned object.*

Supposing every bidder $i = 1, ..., N$ has a pair of valuation and budget constraint (v_i, c_i), one can ask which is the best strategy to follow on the two main types of auctions (Ic and IIc noting c for budget constraint), supposing the symmetrical model.

- Budget constraints in second-price auctions

Proposition 2.35. *Budget constraints in second-price auctions induce a (weakly) dominant strategy which is to bid:*

$$\beta_i^{IIc}(v_i, c_i) = min\{v_i, c_i\}$$

for $i = 1, ...N$

Proof. Two simple observations can discard some cases. Firstly, bidding $b_i > c_i$ is dominated by bidding $b_i = c_i$. If a player wins by bidding $b_i > c_i$ and the second highest bid is below c_i, bidding $b_i = c_i$ would have also resulted in a win and same payoff. But, if the second highest bid is above c_i, the payoff is negative thus it is better to bid $b_i = c_i$.

Also, if $v_i \leq c_i$ the budget constraint does not affect the behaviour of the bidder, so it is the same case as without constraints: the (weakly) dominant strategy is to bid one's valuation $b_i = v_i$. So the only case missing is when $v_i > c_i$, that is when the budget constraint affects a bidder's behaviour.

Notice that if $v_i > c_i$, applying that bidding $b_i = c_i$ dominates bidding $b_i > c_i$, the only possible dominating strategies are those with $b_i \leq c_i$, but applying the same argument, $b_i = c_i$ weakly dominates $b_i < c_i$, thus resulting in the strategy $b_i = c_i$. □

- Budget constraints in second-price auctions

Proposition 2.36. *Budget constraints in first-price auctions produce a symmetric equilibrium strategy of:*

$$\beta_i^{Ic}(v_i, c_i) = min\{\gamma(v_i), c_i\}$$

for some increasing function $\gamma(v_i)$ and all $i = 1, ...N$.

It is obvious that if $c_i \geq \gamma(v_i)$, the budget constraint makes no effect, so the function $\gamma(v_i)$ is the equilibrium strategy in the standard first-price auction case. Also, it is required that $\gamma(v_i) < v_i$, because a bidder with $v_i < c_i$ would then make more profit by bidding less. The only case where the function γ is not specified is when $c_i < \gamma(v_i)$, which the most that can be said is that the equilibrium strategy exists by the assumptions of the model.

2.6.4 Uncertain number of bidders

In the majority of real-life held auctions, bidders can usually determine how many other bidders they are facing by counting people in the room or by checking invites. However, the number of bidders in online auctions or auctions with massive demand is virtually impossible to determine, making bidders uncertain on how to bid (if the equilibrium bids depend on the number of attendees). All other assumptions from the symmetrical model still hold (iid valuations, risk neutrality...)

Notation: The set $\Omega = \{1, ..., N\}$ refers to the potential bidders of the auction. The subset $\Theta \subseteq \Omega$ denotes the actual bidders of the auction.

An actual bidder $i \in \Theta$ assigns a probability of facing n bidders (p_n), which is their belief that there is $n + 1$ actual bidders in the auction. This process will be supposed to be symmetrical, so every symmetrical bidder has the same beliefs. This causes each p_n for $n = 1, ..., N - 1$ to be independent of the bidder and the valuations.

Proposition 2.37. *The revenue equivalence principle is valid with uncertain number of bidders.*

Proof. The notation used for this proof is the one defined above and in earlier chapters. Having shown earlier the principle holds when certainty on the number of participants, suppose now a standard auction A and its symmetric equilibrium β with uncertain number of actual bidders. Because of this, bidders equilibrium bids β do not depend on n.

If v is a bidder's valuation and $\beta(v)$ his equilibrium bid, suppose they bid $\beta(z)$. He will win if $Y_{1,n} < z$ with probability $G_n(z) = F(z)^n$, where $Y_{1,n}$ is the maximum of n values iid along a distribution function F. Doing this for each $n = 0, ..., N-1$, the probability of winning when bidding $\beta(z)$ is:

$$P(z) = \sum_{n=0}^{N-1} p_n \times G_n(z) \ .$$

As a consequence, the expected payoff of a bidder with value v and bid $\beta(z)$ is:

$$\pi(z, v) = P(z)v - m(z)$$

with $m(z)$ being the expected payment. This reasoning is applicable for any standard auction, and since this equality is the same as in the proof of the revenue equivalence principle, following the same steps the result stands. □

- Uncertain number of bidders in second-price auctions

Uncertainty in the number of bidders in (II) does not affect the equilibrium bids, leaving the dominant strategy unaltered $\beta_i^{II,uncertain}(v_i) = v_i$ for every actual bidder.

The expected payment of an actual bidder with value v is:

$$m^{II}(v) = \sum_{n=0}^{N-1} p_n \times G_n(v) \times E[Y_{1,n} | Y_{1,n} < v] \ .$$

- Uncertain number of bidders in first-price auctions

In a first-price auction, the expected payment of an actual bidder with value v is:

$$m^I(v) = P(v) \times \beta(v)$$

with P being defined in the last proposition.

Because of the revenue equivalence principle, $m^I(v) = m^{II}(v)$ for all v:

$$P(v) \times \beta^{I,uncertain}(v) = \sum_{n=0}^{N-1} p_n \times G_n(v) \times E[Y_{1,n}|Y_{1,n} < v]$$

$$\Leftrightarrow \beta^{I,uncertain}(v) = \sum_{n=0}^{N-1} \frac{p_n \times G_n(v)}{P(v)} \times E[Y_{1,n}|Y_{1,n} < v]$$

$$\Leftrightarrow \beta^{I,uncertain}(v) = \sum_{n=0}^{N-1} \frac{p_n \times G_n(v)}{P(v)} \times \beta_n^I(v) \ .$$

being $\beta_n^I(v) = E[Y_{1,n}|Y_{1,n} < v]$ the symmetric equilibrium strategy of a first-price auction with $n+1$ actual bidders.

This shows that the symmetric equilibrium strategy in a first-price auction with uncertain number of bidders is the weighted average of the equilibrium bids where the number of bidders are certain.

2.6.5 Asymmetric valuations

During all this document, it has been supposed that all valuations v_i are iid from the same distribution function F. Let's suppose now that bidders are asymmetric in the sense that their valuations are obtained from different distribution functions $F_1, ..., F_N$.

- Asymmetries in second-price auctions

The equilibrium bids for asymmetric bidders still hols, it is (weakly) dominant to bid one's valuation $\beta_i^{II,asymmetric}(v_i) = v_i$

- Asymmetries in first-price auctions

It can be shown that in asymmetric first-price auction an equilibrium situation exists, but a specific universal expression of the equilibrium bid cannot be found, thus making comparisons between both formats impossible in the majority of the cases. Proof of these results with further explanations can be found in the bibliography [6].

Also, II with asymmetries is still efficient, but I is not. This breaks the revenue equivalence principle in the case of asymmetries among bidders.

[6]: *Bernard Lebrun, Université Laval, Canada, First-price auctions in the asymmetric N bidder case*

Chapter 3

Bill Auctions

This chapter discusses a particular type of auction which is carried out by most public administrations on a regular basis, the bill auction. This auction format is mainly used by the governments to raise money from the public by promising some returns in a short period of time. Definitions, mechanisms and bidder behaviour will be presented. The section will be focused on the U.S Treasury, and all monetary quantities are expressed in U.S. dollars ($). Because of its complexity, some assumptions are made in order to model the auction with the tools showed in past chapters. The main sources of information consulted are [7], [8] and [9].

3.1 Introduction

Bill auctions are the main way of publicly selling securities by the U.S. Treasury (and most government treasuries in other countries). Security selling by the U.S. government started around the 1930s. Securities sold at short-term are called Treasury bills (T-bills), which are federal debt obligations with maturities ranging from one month to one year (although securities for 5, 10 or 30 years called notes/bonds may also be auctioned as mid-term or long-term securities). T-bills are sold at a discount (with face value $1000) every week in an electronic bill auction, resembling a Dutch format. T-bills are considered to be one of the safest investments because of the low maturity and the backing from the government treasury (lowest risk). Because of this, they tend to have low returns, which range around 5%.

[7]: Bill Auction basics, *https://www.investopedia.com*

[8]: U.S. Treasury Bill Auctions, *https://www.treasurydirect.gov/auctions/*

[9]: *Auction theory: a summary with applications to treasury markets*, Sanjiv Ranjan Das, Rangarajan K. Sundaram

When buying debt, the price the bidder pays is lower than the face value of the bill (below par) to gain some interest at the maturity date. For example, if a 52-week T-bill is bought at a 5% discount rate, the buyer will pay $950 (price) and then receive $1000 (face value) at maturity, which is after 52 weeks. The return for the buyer will be $\frac{100-95}{95} = 5.26\%$ approximately. The formula to easily calculate a T-bill price is:

$$P = FV * (1 - d \times \frac{r}{360})$$

Where P is the price paid, FV is the face value (normally $1000), d is the discount rate (divided by 100) and r is the maturity of the bill in days. Note that this formula is for T-bills only, which are coupon-zero (only payment the buyer receives is at the end of the maturity). For notes and bonds (>1 year), a different formula must be applied.

Bidders (players) in the bill auction are divided in two categories, and bid in increments of $1000:

- Competitive bids: They determine the discount rate at which the T-bill will be paid. These are institutional investors, limited to bid 35% each of the total amount auctioned. 24 primary dealers (including financial institutions and brokerages) are required to participate for the auction to be valid.
- Non-competitive bids: They are guaranteed to get the securities they bid, but at the discount rate decided by the competitive bids. These are normally smaller investors. Non-competitive bids cannot surpass $10 million each.

Competitive bidders are not guaranteed to receive any T-bills, as they depend on the amount of non-competitive bids and the interest rate proposed by the other competitive bids.

3.2 Mechanism of the auction

Firstly, the government releases an announcement stating that a bill auction is going to be held. This announcement includes the type of securities, the date of the auction, the issuing date, participation eligibility... but most importantly, how much debt is going to be auctioned (how many T-bills with face value $1000).

Notation: The total amount of debt auctioned is D (in $ million).

Once the auction starts, competitive bids are made, part of them by the primary dealers, who are authorized and obliged to participate. This bids contain an amount of money $b_i^C \leq 0.35D$ and a discount rate d_i with $i = 1, ..., N$, stating how much debt and at what interest yield each player would like to buy debt. For notation, (b_i^C, d_i) will be a competitive bid.

At the same time, non-competitive bids are also made, which only include an amount of debt desired to buy b_i^{NC} lower than \$10 million for $i = 1, ..., M$.

After all bids have been made, to make sure all non-competitive bids get their securities, the total amount of non-competitive bids is subtracted from the total debt offered:

$$D_1 = D - \sum_{i=1}^{M} b_j^{NC}$$

Then, all competitive bids (b_i^C, d_i) are ranked from lower discount rates (higher price) to higher discount rates (lower price). The bids with lower discount rates (thus higher price) will be accepted first because the government will prefer to pay lower yields. T-bills are allocated starting from the higher price in descending order (in ascending order of interest yield) until an amount of D_1 has been covered, where the auction finishes because there is no more debt to be auctioned. The discount rate of the last accepted bid d^* is applied to all players whose bid has been accepted. This format is called the **uniform-price bill auction**.

Competitive parties offering a higher discount yield $d_i > d^*$ will not get any debt, whereas all non-competitive bids and competitive bids with $d_i < d^*$ will get allocated the amount requested at the discount rate d^*. In an event of having more amount of non-competitive money bid than the amount offered ($D_1 \leq 0$), then a sufficient amount of competitive bids are accepted to determine the discount rate, but this is an unusual situation.

Example 3.1. Let's illustrate how a bill auction works with a simple example. Suppose the government wants to raise \$10 million in one-year T-bills at 5% discount rate. They decide to conduct a bill auction.

The total amount of non-competitive bids adds up to \$2 million.

Competitive bids (ranked by ascending rates) are the following:

Bid 1: (\$2 million, 4.79%)
Bid 2: (\$1.5 million, 4.87%)

Bid 3: (\$1 million, 4.96%)
Bid 4: (\$0.5 million, 5.02%)
Bid 5: (\$1 million, 5.06%)
Bid 6: (\$2.5 million, 5.16%)
Bid 7: (\$2 million, 5.20%)
Bid 8: (\$2 million, 5.45%)

Note that maybe more than one bidder wants the same discount rate, so they are jointly counted as one.

The allocation of the $D = 10$ million in debt is \$2 million for the non-competitive bids, so $D_1 = 8$ million for the competitive bids. Bid 1 gets \$2 million, bid 2 gets \$1.5 million... until bid 6 gets \$2 million out of \$2.5 million requested, because $D_1 = 8$ million cannot be surpassed. The best discount rate is then $d^* = 5.16\%$ which is the discount rate requested by bid 6.

Bid 7 and bid 8 will not get any money because they bid a price that was too low (a discount rate too high). Meanwhile, all the competitive bids (\$2 million) will get the discount rate $d^* = 5.16\%$. The government will gain $10 * (1 - \frac{5.16}{100}) = \$9.484$ million, and in 52 weeks will have to pay back $D = \$10$ million, thus the interest being $D \times d^* = 10 \times 5.16\% = 516,000$ for the buyers. The government had initially planned for the bills to be at a discount rate of 5%, thus ideally getting \$9.5 million, but they ended up raising a bit less than expected because $d^* > 5\%$.

It has been shown that in the T-bill auction, competitive bidders face a trade-off between biding a high discount yield (paying less for the same return after 52 weeks, thus more gain), and the chance of getting their bid accepted. If a player bids a discount rate that is too high, it is less probable for him to get the debt allocated. On the other hand, players do not want to bid too low because of lower gains mixed with cost of capital, inflation...

From now on, the section will be centered in competitive bidders, particularly in behavior regarding discount rate bidding, as the non-competitive bidders can be left out by simply subtracting their bid, playing more of a passive role in the T-bill auction.

3.3 Competitive bidding

This section discusses how competitive bidders (institutional parties) should behave in terms of what discount rate should they bid.

T-bill auctions are common value auctions, where bidders can be assumed to be symmetric and risk-neutral, with no budget constraint and seeking to maximize expected payoff.

Suppose, after non-competitive players have been allocated their requested money, the remaining amount of debt to allocate is $D_1 > 0$. N competitive players participate in the auction, so their bids will be $B_i = (b_i, d_i)$ for $i = 1, ..., N$. The payoff of player i (at the maturity of the bill) will be:

$$\pi_i(b_i, d_i) = \begin{cases} b_i \times d^* \times \frac{r}{360} & if \quad d_i < d^* \\ 0 & if \quad d_i > d^* \end{cases} .$$

Recall that $d*$ is the highest discount rate of a winning competitive bidder.

This payoff illustrates the difference between the payment P at the moment of the auction and the return of the face value at the maturity date. If $d_i = d^*$, payoff will be $c * b_i \times d^* \times \frac{r}{360}$, with $c \in [0, 1]$ depending on how much debt is left to allocate when arriving to d^* by the mechanism shown in the last section.

It will be supposed that all competitive bidders bid the same amount $b_i = b$ and $\frac{D_1}{b} = M \in \mathbb{N}$ (m is the number of players that get allocated debt), so the study is centered in the discount rate d_i.

Two cases are derived from this assumption:

1. If $b \times N \leq D_1$, the number of winners $M = \frac{D_1}{b} \geq N$, then all competitive players (and the non-competitive) will be awarded to buy debt b at the highest discount rate proposed by a player (d^*). This would never happen in practice as players would tend to bid more quantity or more players would join the auction.

2. If $b \times N > D_1$, the number of winners $M = \frac{D_1}{b} < N$, then some competitive players do not get allocated debt. This case is the most common situation and the one with theoretical interest, as there is more demand than supply, causing prices to raise thus the discount rates bid to be lower. However, institutional parties that invest in T-bills do not want to bid a really low discount rate because of inflation, cost of capital...

Supposing $M < N$ (more debt demanded than offered), players must bid the discount rate d_i competitively if they want to maximize their payoff. Using the equation from the last section:

$$P_i = FV * (1 - d_i \times \frac{r}{360}) \Leftrightarrow d_i = (1 - \frac{P_i}{FV}) \times \frac{360}{r}.$$

Because the face value (\$1000) and the maturity of the T-bill (r days) are already determined in the auction rules, they are fixed and common for all N players, causing the discount rate to depend solely on the valuation of the price of the bill for each player at the moment of the auction.

A good estimation of the bill price is known publicly as a common value (noted as P^v or d^v if focusing on the equivalent discount rate) taking as a guide past auctions and economical situation, but analysts from institutional parties may try to forecast its fluctuation, giving valuations which will be assumed to be distributed in an interval $[P^v - k, P^v + k]$ according to a distribution function F. Like this, valuations of the price are distributed in an interval $v_i^P \in [P^v - k, P^v + k]$ so equivalently, valuations of the discount rate $v_i^d \in [d^v - l, d^v + l]$. As a conclusion, bidding a discount rate d_i is equivalent to bidding a price of the T-bill P_i.

Applying auction theory, this type of bill auction is in fact a uniform-price multi-object auction with M winners and $N - M$ losers (regarding competitive bidders). In other words, the T-bill auction has M competitive players that get allocated debt at the same price as the highest winning competitive bidder $(d*)$, and $N - M$ competitive players that do not get allocated debt, so their payoff will be 0.

When $M = 1$, there is only one winner and $N - 1$ losers, which is indeed a first-price auction. In this case, the optimal bid for a competitive player is:

$$\beta_i(v_i^P) = E[Y_1 | Y_1 < v_i^P]$$

where Y_1 is the maximum of $N - 1$ iid values in $[P^v - k, P^v + k]$.

For $M \geq 2$, the auction has more than one winner, so it is not possible with the tools shown on single-object auction theory to derive the optimal behaviour. T-bill auctions with $M \geq$ fall under the scope of what are called multiunit auctions. If there was only one winner, the uniform-price auction would be an "n-th price auction"

Definition 3.2. *A **multiunit auction** (or multi-object) is an auction game where two or more homogeneous objects are auctioned simultaneously.*

In multiunit auctions, bidders bid two quantities: The amount of homogeneous goods they want and at what price. This is exactly how players bid in the T-bill auction, (b_i, d_i). However, there is another format of bill auctions carried out by other governments, which was more popular in the last century.

3.4 Uniform-price and Discriminatory-price auctions

This section briefly explains the two most important types of multiunit auctions: the Uniform-price and the Discriminatory-price auctions.

In the **Uniform-price** auction, multiple identical goods are sold at the same price $(p*)$. This is the case of T-bill auctions in the U.S., where it has been shown that all debt allocated is "sold" at a discount rate $d*$, thus at the same price $p*$.

On the other hand, the **Discriminatory-price** auction sells multiple identical goods at different prices to each winning bid. In the T-bill auction case, the payoff of a winning competitive bid would be:

$$\pi_i(b_i, d_i) = \begin{cases} b_i \times d_i \times \frac{r}{360} & if \quad d_i < d^* \\ 0 & if \quad d_i > d^* \end{cases} .$$

Notice that the payoff still depends on having $d_i < d*$ as before, but in the event of winning, the discount rate proposed by the own player applies to their payoff function, instead of every winner having the same discount $d*$. If $d_i = d*$, it works the same way as in the Uniform-price auction. A real example of a Discriminatory-price auction is the Dutch Flower Auction, the largest flower auction worldwide.

These two auction formats are more complex to model and study their equilibrium strategies, but it is still possible with multiunit auction theory. Thanks to the works of Sushil Bikhchandani and Chi-fu Huang ([10], [11]), a result shows that Uniform-price auction is superior (in terms of efficiency) to the Discriminatiry-price auction, even with forward/resale markets (secondary pre or post-auction markets). Nevertheless, this result is broken when the goods auctioned are divisible, meaning they can be partitioned in smaller amounts. Securities auctions allow this division of goods, so the result does not stand.

[10]: *The Economics of Treasury Securities Markets*, Sushil Bikhchandani and Chi-fu Huang, UCLA, MIT

[11]: *Auctions with Resale Markets: An Exploratory Model of Treasury Bill Markets*, Sushil Bikhchandani and Chi-fu Huang, UCLA, MIT

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