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Facing a Risk: To Insure or Not to Insure—An Analysis with the Constant Relative Risk Aversion Utility Function

M. Mercè Claramunt †, Maite Mármol *,† and Xavier Varea †

Department of Mathematics for Economics, Finance and Actuarial Science, Faculty of Economics and Business, University of Barcelona, Avinguda Diagonal 690, 08034 Barcelona, Spain

* Correspondence: mmarmol@ub.edu; Tel.: +34-934024486

† These authors contributed equally to this work.

Abstract: The decision to transfer or share an insurable risk is critical for the decision maker's economy. This paper deals with this decision, starting with the definition of a function that represents the difference between the expected utility of insuring, with or without deductibles, and the expected utility of not insuring. Considering a constant relative risk aversion (CRRA) utility function, we provide a decision pattern for the potential policyholders as a function of their wealth level. The obtained rule applies to any premium principle, any per-claim deductible and any risk distribution. Furthermore, numerical results are presented based on the mean principle, a per-claim absolute deductible and a Poisson-exponential model, and a sensitivity analysis regarding the deductible parameter and the insurer security loading was performed. One of the main conclusions of the paper is that the initial level of wealth is the main variable that determines the decision to insure or not to insure; thus, for high levels of wealth, the decision is always not to insure regardless of the risk aversion of the decision maker. Moreover, the parameters defining the deductible and the premium only have an influence at low levels of wealth.

Keywords: decision analysis; risk analysis; CRRA utility function; deductible

MSC: 91G05; 91B05; 91B16

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Citation: Claramunt, M.M.; Marmol, M.; Varea, X. Facing a Risk: To Insure or Not to Insure—An Analysis with the Constant Relative Risk Aversion Utility Function. *Mathematics* **2023**, *11*, 1070. https://doi.org/10.3390/math11051070

Academic Editors: Eric Ulm and Budhi Surya

Received: 24 January 2023
Revised: 16 February 2023
Accepted: 17 February 2023
Published: 21 February 2023

1. Introduction

We consider risk-averse decision makers who face a random insurable loss and must decide whether to purchase insurance to cover themselves against that risk. In economics, utility functions that capture the decision maker's preferences are often used to model the decision maker's behaviour. This study is based on the analysis of the difference in the expected utility of insuring and not insuring using a one-period collective risk model that includes the possibility of per-claim deductibles.

We denote the aggregate claim amount random variable (r.v.) of a given portfolio of risks over a year as S. Using the collective risk model, S is defined as a random sum [1],

$$S = \sum_{i=1}^{N} X_i, \tag{1}$$

where $X_i, i \in \mathbb{N}$ is a non-negative r.v. that represents the cost of the i-th claim, and N is a positive counting r.v. that represents the number of claims. $X_i, i \in \mathbb{N}$ are assumed to be independent and identically distributed (i.i.d.) and also independent of N (see [2] or [3]). The r.v. $X_i, i \in \mathbb{N}$ is distributed as a strictly positive r.v. (X). Knowing the moments of X and N, the expected value and the variance of the aggregate claim amount are $E(S) = E(N)E(X)$ and $V(S) = E(N)V(X) + E(X)^2V(N)$, respectively.

Most insurance policies include a deductible so that a part of the claim is paid by the insured. Without the deductible, the aggregate claim amount coincides with the total cost covered by the insurer, and the premium paid is denoted as Π. If a deductible is applied, $A(X_i), i \in \mathbb{N}$ is the part of the cost paid by the insured, $C(X_i), i \in \mathbb{N}$ is the part of the claim paid by the insurer, the total cost for the insurer is $S_C = \sum_{i=1}^{N} C(X_i)$ and the cost assumed by the insured is $S_A = \sum_{i=1}^{N} A(X_i)$. Then, $S = S_A + S_C$. The premium paid for the deductible insurance, Π^D, considers only S_C. In the actuarial literature, one of the main topics related to deductibles is the analysis of optimal coverage through expected utility [4,5] and stochastic dominance [6]. For the insurer, a correct calculation of the premium is essential to comply with the principles of fairness, solidarity and solvency [7]. Insurance premiums can be calculated by applying different premium principles, and therefore fulfil different properties (see, for instance, [8–11]). One of these properties is that the premium includes a positive safety loading, i.e., the premium is greater than the expected cost. In this paper, we consider that the premium principle always fulfils this property. The most common is the mean principle in which a positive safety loading, symbolised by δ and proportional to the expected cost, is used, i.e., $\Pi^D = E(S_C)(1+\delta)$. For more information about premium principles and their properties, see [12,13]. The literature discussing various problems from the point of view of the insurer/insured is abundant, and includes discussions on the application of fuzzy theory to the multi-layered insurance portfolio [14], modal interval probabilities of bonus–malus systems [15], simulations related to calculating the VaR from the compound distribution of claims [16] and the impact of the Markovian bonus–malus system on insurer ruin probability [17].

Faced with a risk that could cause economic damage, the decision maker considers ways of hedging against this risk. The simplest option is self-hedging, i.e., covering losses when they occur with their own resources. Insurance provides cover if the risk has a number of properties that make it insurable [1,18]. This cover may be partial if the insurance includes a deductible. The decision maker can then take out insurance that will cover all (or part) of the losses if claims occur in return for the payment of a premium calculated by the insurer. Thus, to face insurable risks, the decision maker has two options: to purchase an insurance with or without deductible, or not to insure. In this paper, we compare these two situations in order to obtain rules that help in the decision process considering a set of variables. In Situation 1, the decision maker purchases an insurance. Then, the premium Π^D is paid, and part of each claim $A(X_i)$ must be paid if a deductible is included in the insurance contract. In Situation 2, the decision maker does not insure the risk and pays each claim in full. To analyse the decision maker's choice, the literature applies utility functions related to their risk aversion. The standard model underlying such decision making under risk is the expected utility theory (see the seminal work of [19] and others such as [20,21]). Our approach to this problem was carried out using a static model (see, for instance, [22]), which analyses the situation in one period (other authors, such as [23], use a dynamic model that adapts to the decision maker's choice over time).

Considering that the initial wealth of the decision maker is $W > 0$, the wealth at the end of the period in Situation 1 is

$$W - \Pi^D - \sum_{i=1}^{N} A(X_i) = W - \Pi^D - S_A \tag{2}$$

(without deductible, $\Pi^D = \Pi$ and $S_A = 0$), whereas, in Situation 2, it is $W - S$. We define $DU(W)$ as the difference between the expected utilities in these two situations:

$$DU(W) = E\left[U\left(W - \Pi^D - S_A\right)\right] - E[U(W - S)], \tag{3}$$

where $U(x), x > 0$ is the utility function of the decision maker.

In [24], the increase in welfare generated by insurance is measured by comparing the wealth if an insurance contract is purchased or if the decision maker assumes the total cost of the claim. Specifically, the paper measures the so-called "added value of insurance", defined as the difference between sets of wealth. First, the absolute deductible is calculated such that the expected utility obtained when purchasing the insurance is maximised. Then, the first wealth is that which would provide us with the same utility, whereas the second wealth is that which would coincide with the expected utility of not insuring.

In this paper, we followed a different approach. In fact, we are interested in the sign of $DU(W)$ depending on the initial wealth of the decision maker, W; whenever $DU(W)$ is positive, the decision maker chooses to insure, whereas, if $DU(W)$ is negative, the decision maker prefers to assume the total cost of the claims. Mathematically, we use the sign function, $sgn : \mathbb{R} \to \{-1, 0, 1\}$, defined as a piece-wise function (see [25]).

$$sgn(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases} \quad (4)$$

The objective of this paper is to provide a decision pattern for the individual who is considering how to hedge against an insurable risk according to their level of wealth. The methodology used is the comparison of the expected utility obtained by the decision maker in the two previously defined situations. Other studies also focus on the decision of how to hedge against a risk, but with different methodologies [24,26]. The analysis is performed using the constant relative risk aversion (CRRA) utility function, widely used in the economic and actuarial literature to model preferences [23,24,27–29].

From the theoretical analysis in Section 2, for integer values of the relative risk aversion coefficient, we obtain the expression of $DU(W)$ as a function of the moments of S and S_A. Using the sign function, the main result of the paper is the rule that determines the decision for different wealth intervals. It should be noted that these results are generic and apply to any premium principle and any risk distribution.

After this introduction, the rest of the paper is structured as follows. In Section 2, general expressions for $DU(W)$ are obtained, and the sign of $DU(W)$ is analysed depending on the initial wealth of the insured, W. In Section 3, we present some numerical results considering an absolute deductible and the mean premium principle. A sensitivity analysis regarding the deductible's parameter and the security loading was also performed. The paper ends with some concluding remarks.

2. Theoretical Analysis of *sgn(DU(W))*

In this section, we assume the constant relative risk aversion (CRRA) utility function, sometimes called isoelastic, with the expression $U(x) = \frac{x^{1-p}-1}{1-p}, p > 0, p \neq 1$. For $p = 1$, the utility function takes the form $U(x) = ln(x)$. The value p is the coefficient of relative risk aversion, where $R(x) = \frac{-u''(x)x}{u'(x)} = p$ is constant, hence the name of this utility function. A larger value of p implies that the decision maker is more risk-averse and therefore more prudent.

Theorems 1 and 2 present the expressions of $DU(W)$ considering that $p \in \mathbb{N}^+$.

Theorem 1. *For the CRRA utility function with* $p = 1$*,*

$$DU(W) = E\left[ln\left(W - \Pi^D - S_A\right)\right] - E[ln(W - S)], W > max(S), \quad (5)$$

which can be estimated by

$$DU(W) \simeq ln\left[W - \Pi^D - E(S_A)\right] - ln[W - E(S)] + \frac{1}{2}\left[\frac{V(S)}{[W - E(S)]^2} - \frac{V(S_A)}{[W - \Pi^D - E(S_A)]^2}\right], \quad (6)$$

with $W \neq E(S)$ *and* $W \neq \Pi^D + E(S_A)$*.*

Proof of Theorem 1. Expression (5) is derived directly from the definition of $DU(W)$ using the utility function $U(x) = ln(x)$, where the $DU(W)$ domain is restricted to a wealth of greater than $max(S)$. This restriction arises from the limitation of the logarithm function to positive arguments. Considering that the no-rip-off property of the applied premium principle is fulfilled [13], $E(S_C) < \Pi_\delta < max(S_C)$, $max(S) > \Pi^D + max(S_A)$. Therefore, the function $DU(W)$ is only defined for $W > Max(S)$. Applying the second-order Taylor approximation, (6) is obtained. □

In the particular case where there is no deductible, Theorem 1 is still valid by considering that $\Pi^D = \Pi$ and $S_A = 0$ (this is also valid for the rest of the section). Specifically, (5) and (6) become

$$DU(W) = E[ln(W - \Pi)] - E[ln(W - S)], W > max(S), \tag{7}$$

and

$$DU(W) \simeq ln[W - \Pi] - ln[W - E(S)] + \frac{1}{2}\left[\frac{V(S)}{[W - E(S)]^2}\right], \tag{8}$$

respectively.

Theorem 2. *For the CRRA utility function with $p \in \mathbb{N}^+$ and $p > 1$,*

$$DU(W) = \frac{1}{1-p}\left(\frac{\sum_{s=0}^{p-1}(b_s E(S^s) - a_s E(S_A^s))}{\sum_{s=0}^{p-1}\sum_{j=0}^{p-1} a_s b_j E(S_A^s) E(S^j)}\right) \tag{9}$$

with $\sum_{s=0}^{p-1}\sum_{j=0}^{p-1} a_s b_j E(S_A^s) E(S^j) \neq 0$, being

$$a_s = (-1)^s \binom{p-1}{s} (W - \Pi^D)^{p-1-s}, \tag{10}$$

$$b_j = (-1)^s \binom{p-1}{s} W^{p-1-s}. \tag{11}$$

Proof of Theorem 2. Substituting the CRRA utility function, $U(x) = \frac{x^{1-p}-1}{1-p}$, in (3), we obtain

$$DU(W) = \frac{1}{1-p}\left[E\left[(W - \Pi^D - S_A)^{1-p}\right] - E[(W - S)^{1-p}]\right]. \tag{12}$$

We apply the binomial theorem to (12), $(a-b)^n = \sum_{s=0}^{n}(-1)^s \binom{n}{s} a^{n-s} b^s$, and we define $a_s = (-1)^s \binom{p-1}{s}(W - \Pi^D)^{p-1-s}$ and $b_j = (-1)^s \binom{p-1}{s} W^{p-1-s}$. Then, (9)–(11) are obtained. □

From (9), if $p = 2$, the difference is

$$DU(W) = \frac{E(S) - E(S_A) - \Pi^D}{[W - \Pi^D - E(S_A)][W - E(S)]}, \tag{13}$$

and, if $p = 3$, the difference is

$$DU(W) = \frac{-1}{2}\frac{[\Pi^D - E(S_C)]2W - [(\Pi^D)^2 - E(S^2) + E(S_A^2) + 2E(S_A)\Pi^D]}{[(W - \Pi^D)^2 - 2(W - \Pi^D)E(S_A) + E(S_A^2)][W^2 - 2E(S)W + E(S^2)]}. \tag{14}$$

If there is no deductible, (13) and (14) become

$$DU(W) = \frac{E(S) - \Pi}{[W - \Pi][W - E(S)]} \tag{15}$$

and

$$DU(W) = \frac{-1}{2} \cdot \frac{[\Pi - E(S)]2W - \left[\Pi^2 - E(S^2)\right]}{[(W-\Pi)^2][W^2 - 2E(S)W + E(S^2)]}. \quad (16)$$

We restrict ourselves to $p \leq 3$. This restriction is standard in macroeconomics and finance (see [23,30] and the references therein).

The decision to insure or not to insure is made on the basis of the sign of $DU(W)$. For $p = 1$, no analytical results can be obtained to determine the sign of $DU(W)$. For $p = 2$ and $p = 3$, Propositions 1 and 2 show that the sign of $DU(W)$ depends on the level of initial wealth, and the wealth intervals are defined, for which, $DU(W)$ is positive.

Proposition 1. *For $p = 2$, $DU(W)$ is not defined when $W = \Pi^D + E(S_A)$ or $W = E(S)$, and*

$$sgn(DU(W)) := \begin{cases} -1, & W < E(S) \text{ or } W > \Pi^D + E(S_A), \\ 1, & E(S) < W < \Pi^D + E(S_A). \end{cases} \quad (17)$$

Proof of Proposition 1. The numerator of (13) is always a negative value (assuming a positive safety loading), and the denominator can be rewritten as a function of W, which is a concave quadratic functions with roots $r_1 = E(S)$ and $r_2 = \Pi^D + E(S_A)$. Then, the proposition is proved. □

If the decision maker has a CRRA utility function with $p = 2$, for all kind of deductibles and different premium principles, when their initial wealth is included in the interval $E(S) < W < \Pi^D + E(S_A)$, the decision maker will prefer Situation 1, i.e., to insure the risk. For other values of the initial wealth that are greater than $E(S)$ or less than $\Pi^D + E(S_A)$, the decision maker will not insure. In fact, as $\Pi^D + E(S_A) = E(S) + L$, where $L = \Pi^D - E(S_C)$, i.e., the safety surcharge included in the premium, the decision ultimately depends on the parameters that define $E(S)$, L (if the mean principle is applied, $L = \delta E(S_C)$, where $\delta > 0$ is the security loading) and the deductible's parameters (the examples in Section 3 will be carried out for an absolute deductible of parameter a). Among all of these parameters, the decision maker only controls those of the deductible, whereas L is controlled by the insurer. The wealth interval in which $DU(W)$ is positive is widened by the security surcharge and narrowed if the insured increases their deductible.

Proposition 2. *For $p = 3$, $DU(W)$ is always defined. Let $c = E\left[(S_A + \Pi^D)^2\right] - E(S^2)$. Therefore, if $c \leq 0$, $sgn(DU(W)) = -1$, and if $c > 0$,*

$$sgn(DU(W)) := \begin{cases} -1, & W > \frac{E\left[(S_A+\Pi^D)^2\right]-E(S^2)}{2\left[\Pi^D - E(S_C)\right]}, \\ 0, & W = \frac{E\left[(S_A+\Pi^D)^2\right]-E(S^2)}{2\left[\Pi^D - E(S_C)\right]}, \\ 1, & W < \frac{E\left[(S_A+\Pi^D)^2\right]-E(S^2)}{2\left[\Pi^D - E(S_C)\right]}. \end{cases} \quad (18)$$

Proof of Proposition 2. The denominator of the second factor of (14) is the product of two positive quadratic functions with respect to W without roots, and it is always a positive function. The numerator of the second factor is a linear function with a positive slope with respect to W being the root $r = \frac{c}{2\left[\Pi^D - E(S_C)\right]}$. As $\Pi^D > E(S_C)$, $sgn(r) = sgn(c)$, the sign of $DU(W)$ depends on the sign of c. If $c \leq 0$, then $r \leq 0$; therefore, the numerator of the second factor reaches positive values for any $W > 0$, and, as demonstrated in (14), DU is negative. If $c > 0$, then $r > 0$, and the sign of the numerator of the second factor depends of the value of W: if $W > r$, it is positive and $DU(W)$ is negative. If $W < r$, it is negative and $DU(W)$ is positive. □

For $p = 3$, if $c \leq 0$ or if $c > 0$ and initial wealth, W, is greater than $\frac{E\left[(S_A+\Pi^D)^2\right]-E(S^2)}{2\left[\Pi^D-E(S_C)\right]}$, the decision maker obtains a greater utility choosing Situation 2 (not insuring). If $c > 0$ and $W = r$, the two situations are indifferent, since they provide the same expected utility. Otherwise, the decision maker should choose Situation 1. As for $p = 2$, the interval of wealth values for which the sign of $DU(W)$ is positive widens with the security surcharge, but, contrary to what happens for $p = 2$, this interval also widens if the deductible is widened. If the mean premium principle is used, $\Pi^D = E(S_C)(1+\delta)$. Let us rewrite c as follows: $c = (1+\delta)^2 E(S_C)^2 + 2(1+\delta)E(S_A)E(S_C) - E(S_C^2) - 2E(S_A S_C)$. As $V(S_C) > 0$ and $Cov(S_A, S_C) > 0$ [31–33], the marginal behaviour with regard to δ is easily determined: c increases with δ, $\lim_{\delta\to\infty} c = \infty$ and $c < 0$ if $\delta = 0$.

3. Numerical Application

In this section, we obtain numerical results for $DU(W)$ and comment on the decisions that the decision maker would take. We consider that the number of claims follows a Poisson distribution $N \sim Po(\lambda)$. From now on, for reasons of simplicity, the part of the cost paid by the insured in an insurance with a deductible, $A(X)$, and the part of the claim paid by the insurer, $C(X)$, are denoted as A and C, respectively. Consequently, $E(S) = \lambda E(X)$ and $V(S) = \lambda(E(X)^2 + V(X))$. We obtain the corresponding moments of S_A and S_C by substituting the moments of X with the A and C moments.

We consider that the deductible applied is absolute. In an absolute deductible, the insured pays the first a monetary units of each claim X, and the insurer pays the excess over a, $X - a$, [34,35]. Then, if an absolute deductible with parameter $a \geq 0$ is applied, A and C are defined in Table 1.

Table 1. Absolute deductible with parameter $a \geq 0$.

X	A	C
$X \leq a$	X	0
$X > a$	a	$X - a$

The mean principle is used to calculate the premium of the deductible insurance, hereby $\Pi^D = \lambda E(C)(1+\delta)$, $\delta > 0$.

In Example 1, the CRRA utility function with $p = 1$ is used and the behaviour of $DU(W)$ is analysed depending on the values of the initial wealth (W) and the parameter of the absolute deductible (a). The same analysis is performed in Example 2 and Example 3 using the CRRA utility function when $p = 2$ and $p = 3$, respectively. If $p = 2$ or $p = 3$, the individual claim amount X is exponentially distributed, $X \sim \exp(b), b > 0$, with a probability density function of $f_X(x) = be^{-bx}$, $E(X) = 1/b$, $E(A) = \frac{1}{b}(1 - e^{-ba})$ and $E(C) = \frac{1}{b}e^{-ba}$. According to (5), the CRRA utility function refers to when $p = 1$ can only be applied to total cost distributions that have a finite maximum value. This implies, in our case, that the distribution of the individual claim amount is a mixed continuous-discrete probability distribution, symbolising its maximum value as x_{max}. In order to be able to compare the results according to the risk aversion of the decision maker, the parameter of the mixed exponential, symbolised by b_t, will be such that the expectation of the mixed exponential coincides with that of the non-mixed ($1/b$).

Unless other values are indicated, such as $E(X) = 2$, $\lambda = 1$ and $\delta = 0.2$.

Example 1. *Case $p = 1$. As mentioned before, X follows a truncated exponential distribution:*

$$f_X(x) = \begin{cases} b_t e^{-b_t x}, x \in (0, x_{max}), \\ e^{-b_t x_{max}}, x = x_{max}, \end{cases} \tag{19}$$

$b_t > 0$. Hereby, $E(X) = \frac{1}{b_t}(1 - e^{-b_t x_{max}})$, $E(A) = \frac{1}{b_t}(1 - e^{-b_t a})$; $E(C) = \frac{1}{b_t}(e^{-b_t a} - e^{-b_t x_{max}})$, $V(X) = \frac{1}{b_t^2}\left(1 - 2b_t x_{max} e^{-b_t x_{max}} - e^{-2b_t x_{max}}\right)$; and $V(A) = -2b_t e^{-b_t a} - a^2 e^{-b_t x_{max}} + \frac{1-e^{-2b_t a}}{b_t^2}$. If $x_{xmax} = 4$ and $b_t = 0.398406$, $E(X) = 2$ as desired. The value of the deductible ´s parameter must be less than the value of the maximum of the claim amount, x_{max}. In the example, $a < 4$. In order to calculate $max(S)$, we can consider that the maximum number of claims is six (as it accumulates a probability of 99.9917%).

In Figures 1–3, $DU(W)$ is plotted for different values of $W > 24$ and $a < 4$ for several values of a and for different values of δ. We can observe that the sign of $DU(W)$ is always negative.

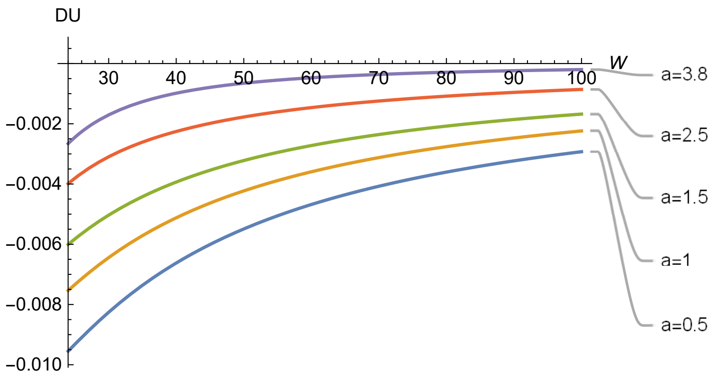

Figure 1. $DU(W)_\delta$ for $a = 0.5, 1, 1.5, 2.5, 3.8$.

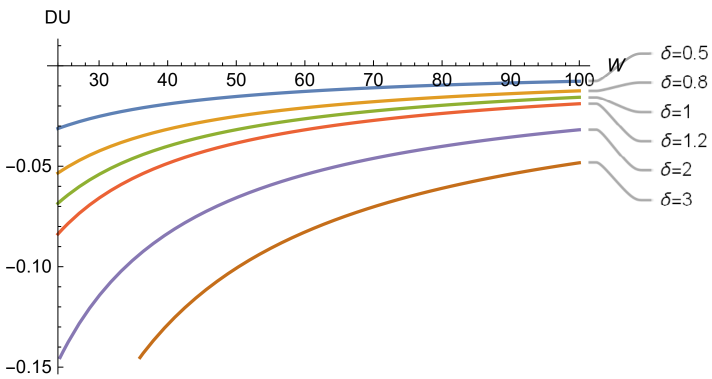

Figure 2. $DU(W)_{a=0.5}$ for $\delta = 0.5, 0.8, 1, 1.2, 2, 3$.

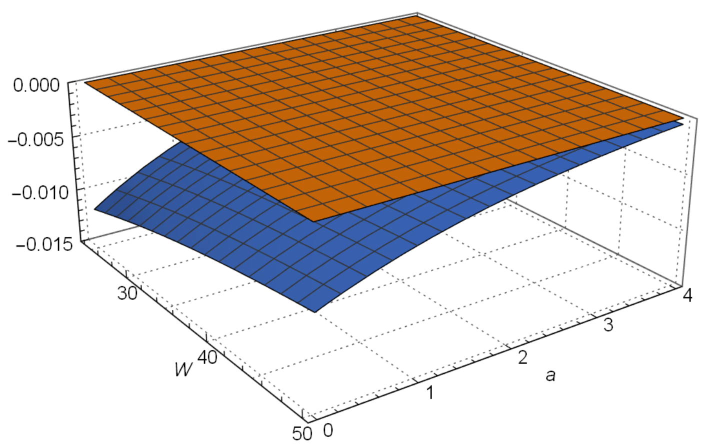

Figure 3. $DU(W, a)$.

We performed a sensitivity analysis, where changes in $E(X)$, λ, δ, x_{max} or b_t did not alter the result; $DU(W)$ is always negative.

Example 2. *Case* $p = 2$. *The sign of* $DU(W)$ *depends on* $E(S)$ *and* $\Pi^D + E(S_A)$. *The first one does not depend on the deductible parameter,* $E(S) = \lambda/b$, *whereas the second one does,* $\Pi^D + E(S_A) = \frac{\lambda+\delta\lambda e^{-ab}}{b}$. *These values for* $a = 0, 0.5, 1, 1.5, 2, 2.5$ *are shown in Table 2.*

Table 2. $E(S)$ and $\Pi^D + E(S_A)$ for different values of a.

a	$E(S)$	$\Pi^D + E(S_A)$
0	2	2.4
0.5	2	2.31152
1	2	2.24261
1.5	2	2.18894
2	2	2.14715
2.5	2	2.11460

As stated in Proposition 1, $DU(W)$ is always different from zero and is positive when the initial wealth is between $E(S)$ and $\Pi^D + E(S_A)$, with these values being the asymptotes of $DU(W)$. $DU(W)$ is represented in Figure 4 as a function of only one variable, the initial wealth, and, in Figure 5, as a function of two variables, the initial wealth and the deductible's parameter.

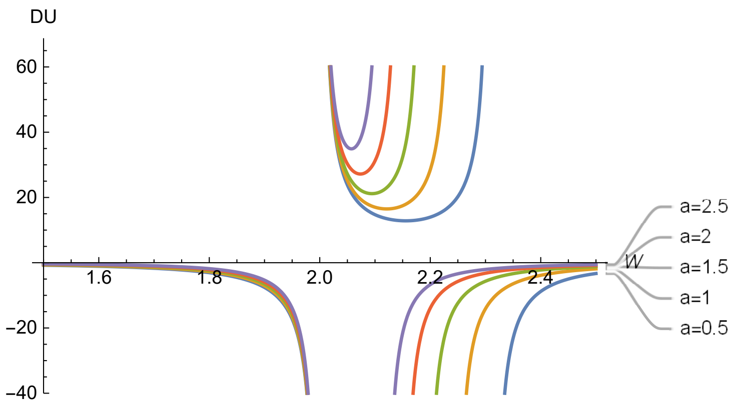

Figure 4. $DU(W)_\delta$ for $a = 0.5, 1, 1.5, 2, 2.5$.

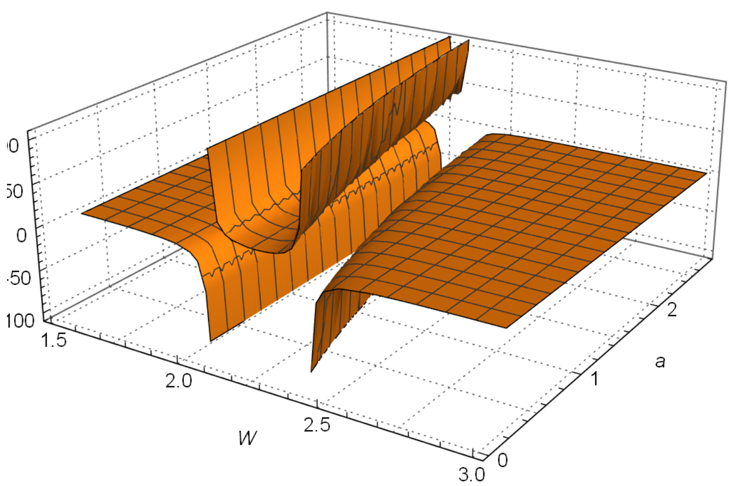

Figure 5. $DU(W, a)$.

Example 3. *Case $p = 3$. From Proposition 2, $DU(W)$ is always defined and its sign depends on the sign of c. In this case, $c(\delta, a) = e^{-2ab}\lambda(\delta^2\lambda - 2e^{ab}(1 + ab - \delta\lambda))/b^2$ and $r = (-1 - ab + \delta\lambda + 0.5\delta^2\lambda e^{-ab})/b\delta$. It is easy to see that*

$$sgn(c(\delta,a)) = \begin{cases} -1, & \delta \in (0,\delta^*) \text{ or } (\delta \in (\delta^*,\infty) \text{ and } a > a^*(\delta)), \\ 0, & (\delta = \delta^*, a = 0) \text{ or } (\delta \in (\delta^*,\infty) \text{ and } a = a^*(\delta)), \\ 1, & \delta \in (\delta^*,\infty) \text{ and } a < a^*(\delta), \end{cases} \quad (20)$$

where δ^* is such that $c(\delta^*, 0) = 0$ and $a^*(\delta)$ is such that $c(\delta, a^*(\delta)) = 0$ for $\delta > \delta^*$. In our example, $\delta^* = 0.73205$ and Table 3 includes the values of $a^*(\delta)$ for different values of $\delta > \delta^*$. In Figure 6, for $c(a)$, several values of the security loading are represented and Figure 7 shows the contour plots of $c(\delta, a)$. From both figures, the behaviour of the sign of $c(\delta, a)$ depending on $a^*(\delta)$ can be verified.

Table 3. $a^*(\delta)$ for different values of δ.

δ	$a^*(\delta)$
0.8	0.18380
1	0.70347
1.2	1.19304
2	2.92611
3	4.81170

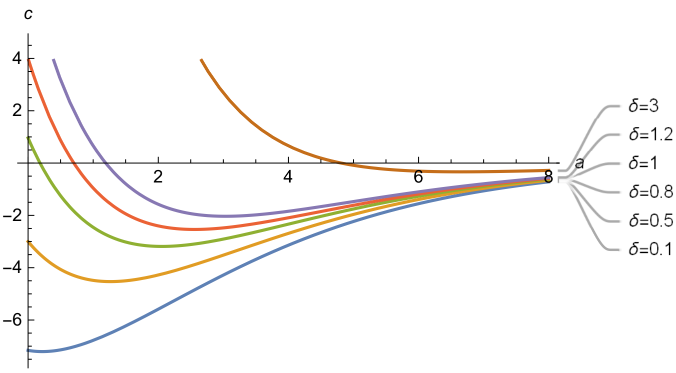

Figure 6. $c(a)$ for $\delta = 0.1, 0.5, 0.8, 1, 1.2, 3$.

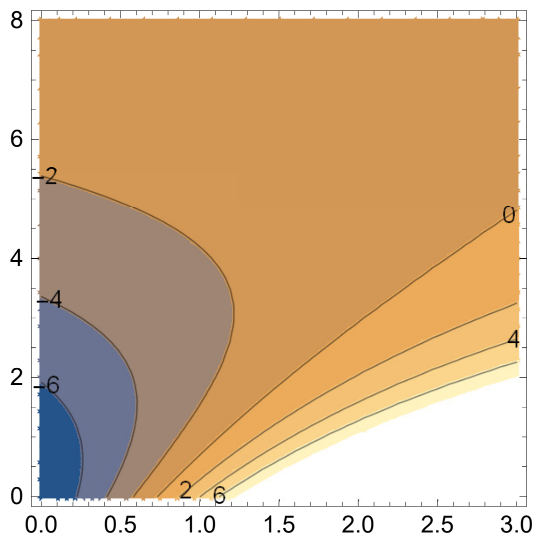

Figure 7. Contour plots of $c(\delta, a)$.

Regarding $DU(W)$, using Proposition 1, if $\delta \in (0, \delta^*)$ or $(\delta = \delta^*, a = 0)$ or $(\delta \in (\delta^*, \infty)$ and $a \geq a^*(\delta))$, $sgn(DU(W)) < 0$ for all W, and if $(\delta \in (\delta^*, \infty)$ and $a < a^*(\delta))$,

$$sgn(DU(W)) = \begin{cases} 1, & W < r, \\ 0, & W = r, \\ -1, & W > r. \end{cases} \quad (21)$$

In Figures 8–10, $DU(W)$ is plotted in the three mentioned cases and it can be realised that the sign of $DU(W)$ is almost always negative; it is positive in only a few situations.

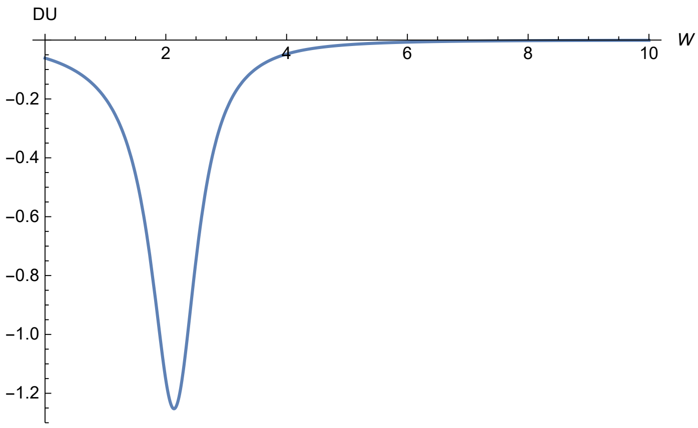

Figure 8. $DU(W)_{\delta=0.1,a=0.5}$.

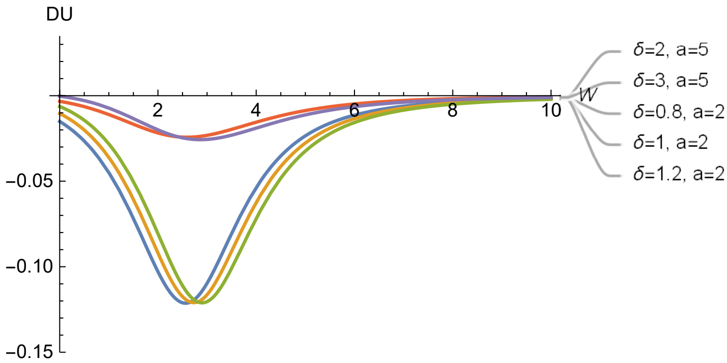

Figure 9. $DU(W)$ for several combinations of δ and a such that $\delta > 0.73205$ and $a > a^*(\delta)$.

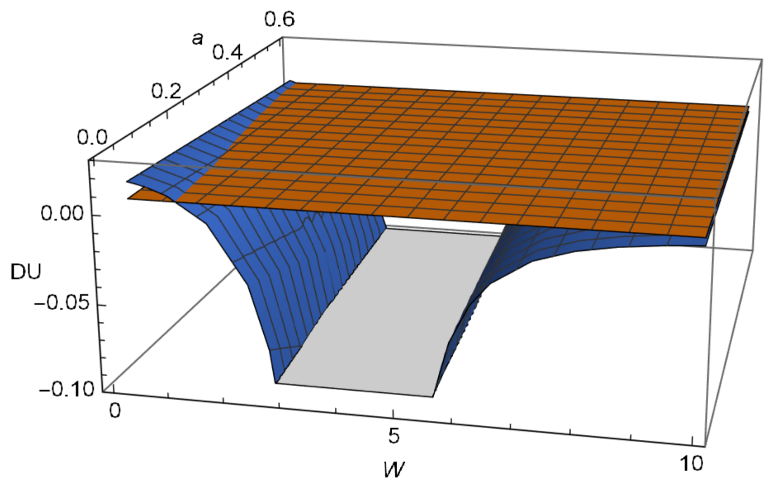

Figure 10. $DU(W, a)_{\delta=1}$ for $a \in (0, 0.70347)$.

Let us now summarise the decision to insure or not to insure depending on the initial wealth level, W, taking into account that the analysis of the decision is divided into three intervals: the low wealth interval for $W < 8$, which corresponds to $W < 4E(S)$, the interval of intermediate wealth corresponding to values of W between $4E(S)$ and $12E(S)$, and the high wealth interval for values of $W > 12E(S)$ (see Figure 11).

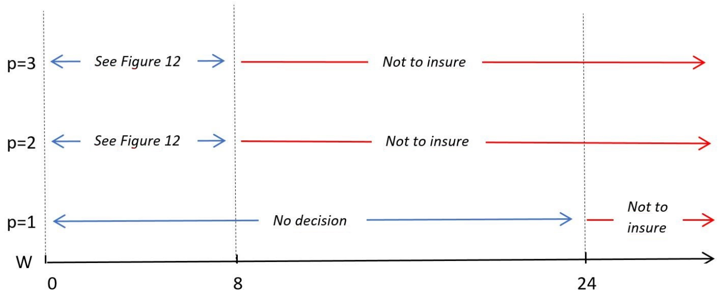

Figure 11. Sign of $DU(W)$ for W.

This decision depends on the sign of the function $DU(W)$ that is different for different risk aversions ($p = 1, 2$ or 3). If $p = 1$, we can only make decisions for $W > 24$, and the decision is always not to insure. For $p = 2$ or $p = 3$, the decision is also always not to insure for $W > 8$; in other cases, the decision depends on a, δ and W, and is represented in Figure 12 (red, not to insure; green, to insure).

Figure 12. Sign of $DU(W)$ for $W \in (0, 8)$: negative (red), positive (green). The first (second) row of each block of two rows corresponds to $p = 2$ ($p = 3$). The white column and row separate blocks for several combinations of $a = 0.1, 0.5, 1, 2, 4$ (row) and $\delta = 0.5, 0.8, 1, 1.2, 2, 3$ (column).

Analysing the low wealth interval, $0 < W < 8$, regarding the marginal behaviour with respect to δ and a, we observe that an increase in δ increases the range of wealth for which the decision is to insure (green in Figure 12). On the other hand, an increase in a (which implies that the insured assumes a larger share of each loss) decreases this range.

For values of the initial wealth close to zero, the increase in risk aversion does lead to the decision to insure. On the other hand, for values of W close to 8, the increase in risk aversion (from $p = 2$ to $p = 3$) is not sufficient to make the decision to insure.

The numerical application presented has made it possible to obtain a pattern of decision maker behaviour as a function of the different parameters that define the model. Although this pattern corresponds to a specific set of parameter values, these rules for deciding whether or not to insure are logically applicable for other combinations of parameters, and the main conclusions on the effect that W, δ, a and risk aversion p have on the decision remain valid.

4. Discussion

To analyse the decision problem of whether or not to insure an insurable risk faced by an individual, in this paper, we used the CRRA utility function to define the difference between the expected utilities of the two choices: to insure (with or without deductible) or not to insure. A theoretical analysis of the sign of this difference permits us to provide a decision pattern as a function of wealth level, when the risk aversion parameter is a natural number less than three, which is a standard restriction in macroeconomics and finance.

To determine the decision maker´s choice, we work with expected utilities (see the seminal work of [19]). Although, as discussed by [36], the dual utility characterised by [37] could also be used, the expected utility is usually applied as the preference relation for individuals [38]. Several utility functions have been used in the economic and actuarial literature, of which, the exponential is the simplest, but the CRRA is a more standard formalisation of individual preferences within the field of personal financial decision making [39]. We consider a one-period model so that the decision is made at the beginning of the period considering only the expected utility of wealth at the end of the period. We leave open for future research the analysis with a dynamic model, so that decisions made in one period affect decisions in successive periods.

The theoretical results obtained (see Propositions 1 and 2) show us, given some values of the parameters of the model, the decision to be taken by the individual according to their initial level of wealth. This decision pattern is valid for any premium calculation criterion, any type of deductible and any compound loss distribution, which makes it particularly robust. One of the main conclusions of the paper is that the initial level of wealth is the main variable that determines the decision to insure or not to insure so that, for high levels of wealth, the decision is always not to insure regardless of the risk aversion of the decision maker. Moreover, the parameters defining the deductible and the premium only have an influence at low levels of wealth.

Author Contributions: Conceptualization, M.M.C., M.M. and X.V.; Formal analysis, M.M.C., M.M. and X.V.; Writing—original draft, M.M.C., M.M. and X.V.; Writing—review & editing, M.M.C., M.M. and X.V. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: We would like to thank reviewers for taking the time and effort necessary to review the manuscript. We sincerely appreciate all valuable comments and suggestions, which helped us to improve the quality of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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