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COMPUTING BONUS–MALUS PREMIUMS UNDER PARTIAL PRIOR INFORMATION

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ABSTRACT

The use of classical bonus–malus systems entails very high maluses and other problems which, during recent years, have been criticised by actuaries. To avoid these problems, new bonus–malus models have been developed. For instance, it is well known that the use of an exponential loss function reduces the differences between overcharges and undercharges, solving the problem of high maluses. In order to measure the sensitivity of the exponential bonus–malus system, and according to robust Bayesian analysis, we first model the structure function by specifying a subclass of the generalised moments class. We then examine the range of relativities for each prior. Finally, we illustrate our method with a numerical example based on real data.

KEYWORDS

Bonus–Malus; Generalised Moment Class; Bayesian Robustness; Exponential Principle; Poisson-Inverse Gaussian

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1. Introduction and Motivation

The aim of the actuary is to design a tariff system that will distribute the exact weight of each risk fairly within the portfolio when policyholders provide different risks. For instance, in the automobile insurance market, the first approach to solving this problem, called tariff segmentation, consists in dividing policyholders into homogeneous classes according to some variables chosen as influencing factors (*a priori* factors): model and use of the car, age and sex of the driver, duration of driving licence, etc. Once the actuary has classified policyholders, he/she should fix the premium for each type of risk.

However, there are some factors that cannot be measured or introduced into the rates to calculate premiums according to tariff-segmentation methods. Consequently, heterogeneity still remains in every class defined with *a priori* factors. Some of these unmeasured or unknown characteristics have a significant effect on the claim amount; for instance, in automobile insurance, swiftness of reflexes, knowledge of the Highway Code or behaviour of the driver.

Given that many claims could be explained by these hidden features, it is necessary to include them in the tariff system. This is the aim of experience rating or credibility theory, the idea behind which is that past claims experience reveals information about hidden features. In automobile insurance, such methods are mostly called bonus–malus systems (BMS). The mechanism of experience rating systems (BMS) consists in rewarding discounts, called *bonuses*, to policyholders without claims, and penalising, with surcharges called *maluses*, those policyholders who have made one or more claims. This type of system is now in operation in many developed countries.

However, it is important to take into account the criticisms that these types of systems have received, and the differences in the way in which they are applied by actuaries. Although it seems correct to use as many classifying variables as possible, and then to adapt the inadequacies of the *a priori* system by using an appropriate BMS, this methodology presents some problems and errors in practice. For instance, one of these problems arises from the fact that *a priori* premiums vary from one type of risk to another, and the same bonus–malus factor is applied to all drivers. This point has previously been addressed, e.g. in Dionne & Vanasse (1992), in Gisler (1996) and in Bermúdez *et al.* (2001).

Moreover, most theoretical BMS lead to very high maluses that are not commercially understood by the insured. Consequently, the BMS used by companies are often not correct from a technical point of view, even though they are commercially accepted. Certainly, the premiums obtained with a commercial BMS are not suitable for evaluating each policyholder. In Lemaire (1979) and in Morillo & Bermúdez (2003), attempts have been made to reduce maluses with new theoretical approaches.

In recent years, as a result of this process of discussion and criticism, many new models which try to solve the problems detected have appeared in actuarial literature. However, it is difficult to prove that the changes introduced in such models do not affect their efficiency and fairness.

The aim of this paper is to illustrate notions and techniques for global, robust Bayesian analysis that enable the sensitivity of new models to be evaluated in terms of their usefulness and profitability. The most common approach to Bayesian robustness in actuarial science is through sensitivity analysis, wherein we simply interchange the prior with a class of possible prior distributions. The sensitivity is assessed by calculating the bounds of the premium when the prior runs in that class. If the range obtained is small enough, the conclusion is that the model is declared to be robust. Now, the actuary has a range of premiums, perhaps more competitive, to be charged. If we have non-robust results, further investigation will be necessary. We base our study on the global robust Bayesian analysis under the subclass of the generalised moments class (Betrò *et al.*, 1994; Moreno *et al.*, 2003).

The study is set out in the following way. First, in Section 2, we describe the exponential BMS whose robustness we have chosen to discuss. In Section 3, we propose and explain the methodology used in order to estimate and measure the sensitivity of this model. New results arise from this procedure, and in Section 4, with a numerical application, we show the results obtained. Finally, in the last section we present our conclusions.

2. Exponential Bonus–Malus Model

As mentioned above, two points have traditionally been discussed regarding BMS: the maluses are very high; and they are higher still when the risks have *a priori* low frequency and the actuary uses the same bonus–malus rules for all risks. In Morillo & Bermúdez (2003) these two points were tackled using the following assumptions:

(1) First, types of risk ($\mathcal{K}_j; j = 1, \ldots, m$) according to *a priori* variables are defined, and then the premiums ($\mathcal{P}_j; j = 1, \ldots, m$) that will be charged to new policyholders classified as belonging to risk type $\mathcal{K}_j$ are calculated according to *a priori* number of claims distribution chosen.

(2) Second, a bonus–malus model is applied for each j risk type, where K_{ij} is the claim number of risk i in the period j with a risk parameter θ_i. Given θ_i, the $K_{ij}; i = 1, 2, \ldots, t$ are independent, with $K_{ij} \mid \theta_i \sim$ Poisson $(\theta_i \lambda_{ij})$.

(3) In such a bonus–malus model, the prior distribution is given by $\theta_i \sim \pi_0(\theta_i) =$ IG $(1, \beta)$, where IG denotes the inverse Gaussian distribution. The posterior distribution is therefore a generalised inverse Gaussian distribution, with updated parameters $\pi_0(\theta \mid k) = \text{GIG}\left(k - \frac{1}{2}, (1 + 2\beta\lambda_{i*})^{-\frac{1}{2}}, \beta(1 + 2\beta\lambda_{i*})\right)$, where $k = \sum_{j=1}^{t} k_{ij}$ and $\lambda_{i*} = \sum_{j=1}^{t} \lambda_{ij}$. This is the well known Poisson–Inverse Gaussian model, which has been used by Morillo & Bermúdez (2003) and Tremblay (1992), among others.

(4) However, an exponential loss function (i.e. the exponential principle) is used in the minimisation problem. Thus, the solution of the problem, under constraint $E_{\pi_0}[g_1(\theta)] = E_{\pi_0}(\theta)$, is given by:

$$\mathcal{P}_{\pi_0}(k, t) = E_{\pi_0}(g_1(\theta)) + \frac{1}{c}\left[E_j \ln E(g(\theta) \mid j) - \ln E_{\pi_0(\theta|k)}(g(\theta) \mid k)\right] \tag{1}$$

which is shown in Morillo & Bermúdez (2003) and Lemaire (1979) when $g_1(\theta) = \theta$ and $g(\theta) = e^{-c\theta}$, $c > 0$, in the bonus–malus setting.

In practical situations, and because of the segmentation of the portfolio, where $\mathcal{N}$ policyholders are divided into m classes with $\mathcal{N}_m$ policyholders in a given period t, (1) can be rewritten as:

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$$\mathcal{P}_{\pi_0}(k,t) = E_{\pi_0}(g_1(\theta)) + \frac{1}{c\mathcal{N}} \sum_{j=0}^{m} \mathcal{N}_j \ln \frac{E_{\pi_0(\theta|j)}(g(\theta) \mid j)}{E_{\pi_0(\theta|k)}(g(\theta) \mid k)} \qquad \sum_{j=0}^{m} \mathcal{N}_j = \mathcal{N}.$$

Then, it is well known that a bonus–malus premium can be obtained as:

$$\begin{aligned}
\mathcal{BMP}_{\pi_0}(k,t) &= 100 \frac{\mathcal{P}_{\pi_0}(k,t)}{\mathcal{P}_{\pi_0}(0,0)} \\
&= 100 + \frac{100}{c\mathcal{N} E_{\pi_0}(g_1(\theta))} \sum_{j=0}^{m} \mathcal{N}_j \ln \frac{E_{\pi_0(\theta|j)}(g(\theta) \mid j)}{E_{\pi_0(\theta|k)}(g(\theta) \mid k)} \\
&= 100 + \frac{100}{c\mathcal{N} E_{\pi_0}[g_1(\theta)]} \sum_{j=0}^{m} \mathcal{N}_j \ln \frac{E_{\pi_0}[g(\theta) f(j \mid \theta)] E_{\pi_0}[f(k \mid \theta)]}{E_{\pi_0}[f(j \mid \theta)] E_{\pi_0}[g(\theta) f(k \mid \theta)]}
\end{aligned}$$

$$\sum_{j=0}^{m} \mathcal{N}_j = \mathcal{N}. \tag{2}$$

Observe that this expression is a relative premium, therefore it is the premium that the policyholder has to pay if its initial premium ($t = 0$) is $\mathcal{BMP}_{\pi_0}(0,0) = 100$.

In other terms, the aim of $\mathcal{BMP}_{\pi_0}(k,t)$ is to adjust the amount of *a priori* premium ($\mathcal{P}_j$) according to past claim experience, in order to reduce the residual heterogeneity that remains in the different risk classes of the portfolio. Therefore, the *a posteriori* premium for the policyholder is given by:

$$P_{t+1} = \mathcal{P}_j \cdot \mathcal{BMP}_{\pi_0}(k,t) / 100.$$

The aim of the present paper is the robustness study of this model where we move the prior distribution (π_0) into a plausible class of distributions.

3. Bayesian Robustness Framework

Robustness of the prior distribution is considered in this section. Variations of the bonus–malus premium given in (2) are studied when the prior distribution belongs to a certain class; this topic is known as global robustness. Bayesian robustness consists in replacing a given prior distribution by a class Γ of prior distributions, and in analysing oscillations of a given posterior magnitude. In our case, this posterior magnitude is the bonus–malus premium given in (2). Then we focus on computing the extreme values $\inf \mathcal{BMP}_{\pi}(k,t)$ and $\sup \mathcal{BMP}_{\pi}(k,t)$ when π moves into a class of distributions.

The problem of robustness or sensitivity has always been an important element of a Bayesian framework. However, only in recent decades have attempts been introduced into actuarial settings. Eichenauer *et al.* (1988), Makov (1995), Young (1998), Ríos *et al.* (1999) and Gómez *et al.* (2001, 2002) are some examples.

A simple way of incorporating this Bayesian robustness study in a parametric model for K, $\{f(k \mid \theta), \theta \in \Theta\}$, is through a prior distribution $\pi(\theta)$ for the parameter θ, such that $\pi(\theta) \in \Gamma$, $\Gamma = \{\pi : \pi = \pi_0 + u\}$. Here π_0 is the base specified prior, and u is a signed measure, with $u(\Theta) = 0$. Then we consider u of the form $u = \varepsilon(q - \pi_0)$, where $q \in \mathcal{Q}$ and $\varepsilon \in [0, 1]$ reflects the amount of error in π_0, that is the uncertainty in the base elicited prior. This is the well known class of ε–contamination (Sivaganesan & Berger (1989) and Gómez *et al.* (2002) among others). Obviously, $\mathcal{Q}$ must be compatible with the base elicited prior π_0.

If, in expression (2), we replace π_0 by $\pi = \pi_0 + \varepsilon(q - \pi_0)$, then moment conditions in the form:

$$E_q[g(\theta)f(j \mid \theta)] \quad E_q[f(j \mid \theta)] \quad E_q[g_1(\theta)] \tag{3}$$

appear. Then the class $\mathcal{Q}$ of generalised moment conditions (Betrò *et al.*, 1994 and Moreno *et al.*, 2003), which is given by:

$$\mathcal{Q}_M = \left\{ q : \int_\Theta \mathcal{H}_i(\theta) q(\theta) d\theta = \alpha_i, \quad i = 1, 2, \ldots, n \right\} \tag{4}$$

will be used.

Now, suppose that we are interested in calculating the range of variation of the following posterior ratio quantity:

$$\rho^\pi = \frac{\rho_1^\pi}{\rho_2^\pi} = \frac{E_{\pi(\theta|\tilde{k})}[g_1(\theta)]}{E_{\pi(\theta|\tilde{k})}[g_2(\theta)]}$$

when $\pi \in \Gamma$ and $q \in \mathcal{Q}_M$, $\pi(\theta \mid \tilde{k})$ being the posterior distribution of π after observing the data $\tilde{k}$, and $g_i(\theta)$, $i = 1, 2$, functions such that expectations under π exist.

From the moment theory (Winkler, 2000), the extrema of ρ^π are attained for a discrete density that concentrates its mass in $n + 1$ points. Then we have the following result.

Theorem 1

If the system of equations:

$$\begin{bmatrix} \frac{\alpha_i - \mathcal{H}_i(\theta_1)}{f(\tilde{k}|\theta_1)} & \ldots & \frac{\alpha_i - \mathcal{H}_i(\theta_{n+1})}{f(\tilde{k}|\theta_{n+1})} \\ \mathcal{H}_i(\theta_1) & \ldots & \mathcal{H}_i(\theta_{n+1}) \\ 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_i \\ 1 \end{bmatrix} \qquad i = 1, \ldots, n \tag{5}$$

has a solution, where $\theta_j = \theta^*$ for some j, $j = 1, \ldots, n + 1$, then it is the value

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when the suprema of the function $\Phi(\theta) = \mathcal{R}_1(\theta)/\mathcal{R}_2(\theta)$, $\mathcal{R}_j(\theta) = (1-\varepsilon)\times p(\tilde{k} \mid \pi_0)\rho_j^{\pi_0}\sum_{i=1}^{n}\mathcal{H}_i(\theta) + \varepsilon f(\tilde{k} \mid \theta)g_j(\theta)\sum_{i=1}^{n}\alpha_i,\ j = 1, 2$, is attained, then:

$$\sup_{q \in \mathcal{Q}_M} \rho^{\pi} = \sup_{\theta} \Phi(\theta) = \Phi(\theta^*).$$

The same result is obtained when sup is replaced by inf.

Proof

Taking into account that $\int_\Theta \mathcal{H}_i(\theta)q(\theta)d\theta = \alpha_i,\ i = 1, \ldots, n$, then:

$$p(\tilde{k} \mid q)\sum_{i=1}^{n}\int_\Theta \frac{\mathcal{H}_i(\theta)}{f(\tilde{k} \mid \theta)}q(\theta \mid \tilde{k})d\theta$$

where $p(\tilde{k} \mid q) = \int_\Theta f(\tilde{k} \mid \theta)q(\theta)d\theta$ is the marginal distribution. Now, it is simple to show that ρ^π has the expression:

$$\rho^\pi = \frac{\int_\Theta \mathcal{R}_1(\theta)q(\theta \mid \tilde{k})d\theta}{\int_\Theta \mathcal{R}_2(\theta)q(\theta \mid \tilde{k})d\theta}$$

and, applying the well known Lemma A.1 in Sivaganesan & Berger (1989), the suprema is obtained by maximising $\Phi(\theta)$.

The first equation in system (5) ensures that $q(\theta \mid \tilde{k})$ is a probability measure, the second is the general moment conditions, and therefore the theorem is proved.

Now, the robustness of the bonus–malus premium in (2) can be studied using Theorem 1. For this purpose, we consider the following class for s fixed, with $s \in \{0, 1, \ldots, m\}$:

$$\begin{aligned}
\mathcal{Q}_M^* = \Big\{ q : \int_\Theta g(\theta)f(s \mid \theta)q(\theta)d\theta &= \int_\Theta g(\theta)f(s \mid \theta)\pi_0(\theta)d\theta = \gamma_s \\
\int_\Theta f(s \mid \theta)q(\theta)d\theta &= p(s \mid q) = p(s \mid \pi_0) = \beta_s \\
\int_\Theta g_1(\theta)q(\theta)d\theta &= \int_\Theta g_1(\theta)\pi_0(\theta)d\theta = \delta_0 \\
\int_\Theta g(\theta)f(l \mid \theta)q(\theta)d\theta &= \int_\Theta g(\theta)f(l \mid \theta)\pi_0(\theta)d\theta = \gamma_l \\
\int_\Theta f(l \mid \theta)q(\theta)d\theta &= \int_\Theta f(l \mid \theta)\pi_0(\theta)d\theta = \beta_l \\
l \in \{0, 1, \ldots, m\} \quad l &\neq s, l \neq r \Big\}
\end{aligned}$$

for some $r \in \{0, 1, \ldots, m\}$.

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This class $\mathcal{Q}^*_{\mathcal{M}}$ is a particular case of the moment generalised conditions class $\mathcal{Q}_M$, and reflects the particular moment conditions in expression (2) which appear in (3).

Now, the following proposition presents the upper and lower bound for $\mathcal{BMP}_\pi(s,t)$.

Proposition 1

The upper and lower bound of $\mathcal{BMP}_\pi(s,t)$ is given by:

$$\sup_{\pi\in\mathcal{Q}^*_M} \mathcal{BMP}_\pi(s,t) = 100 + \frac{100}{c\mathcal{N}\delta_0}\left\{\ln\left[\left(\frac{\beta_s}{\gamma_s}\right)^{\mathcal{N}-\mathcal{N}_r}\prod_{\substack{i=0\\ i\neq r}}^{m}\left(\frac{\gamma_i}{\beta_i}\right)^{\mathcal{N}_i}\right]\right.$$
$$\left. + \mathcal{N}_r \ln\sup_\theta \frac{(1-\varepsilon)\gamma_r\psi_r(\theta)+\varepsilon\psi_r g(\theta)}{(1-\varepsilon)\beta_r\psi_r(\theta)+\varepsilon\psi_r}\right\}$$

where:

$$\psi_r = \delta_0 + \sum_{\substack{i=0\\ i\neq r}}^{m}(\gamma_i+\beta_i)$$

$$\psi_r(\theta) = \frac{1}{f(r\mid\theta)}\left[(g(\theta)+1)\sum_{\substack{i=0\\ i\neq r}}^{m} f(i\mid\theta) + g_1(\theta)\right]$$

and $r\neq s$, $r\neq l$.

Proof

Using Theorem 1, it is easy to find that:

$$p(r\mid q) = (\gamma_s+\beta_s+\gamma_l+\beta_l+\delta_0)\left\{\int_\Theta \frac{1}{f(r\mid\theta)}[(g(\theta)+1)(f(s\mid\theta)\right.$$
$$\left. + f(l\mid\theta)) + g_1(\theta)]q(\theta\mid r)d\theta\right\}^{-1}.$$

Now, taking into account that:

$$\frac{\int_\Theta g(\theta)f(r\mid\theta)\pi(\theta)d\theta}{\int_\Theta f(r\mid\theta)\pi(\theta)d\theta} = \frac{(1-\varepsilon)\gamma_r+\varepsilon p(r\mid q)\int_\Theta g(\theta)q(\theta\mid r)d\theta}{(1-\varepsilon)\beta_r+\varepsilon p(r\mid q)}$$

and simple calculations, these provide the desired result.

The infimum is obtained when sup is replaced by inf.

Interchanging the conditions given in $\mathcal{Q}_M^*$ and using Proposition 1, we can obtain a chain of infimums and supremums in the form:

$$\left\{ \inf_{\pi \in \mathcal{Q}_M^*} {}^r\mathcal{BMP}_\pi(s,t), \quad r = 0, 1, \ldots, m \right\}$$

$$\left\{ \sup_{\pi \in \mathcal{Q}_M^*} {}^r\mathcal{BMP}_\pi(s,t), \quad r = 0, 1, \ldots, m \right\}$$

respectively.

Now, the values of infimums and supremums can be chosen by considering the maximum change of inference of the bonus–malus premium as:

$$\min_r \left\{ \inf_{\pi \in \mathcal{Q}_M^*} {}^r\mathcal{BMP}_\pi(s,t), \quad r = 0, 1, \ldots, m \right\} = \mathcal{BMP}_\pi^1$$

$$\max_r \left\{ \sup_{\pi \in \mathcal{Q}_M^*} {}^r\mathcal{BMP}_\pi(s,t), \quad r = 0, 1, \ldots, m \right\} = \mathcal{BMP}_\pi^2$$

respectively.

4. Numerical Example

In order to show the robustness of the Poisson–Inverse Gaussian model described in Section 2, we have chosen some real car insurance data from a Spanish company. The data concern the number of claims made in one year.

In order to apply a tariff segmentation *a priori*, as seen in Table 1, the observations have been categorised according to the age of the driver and the power of the car. The first risk factor (q) is divided into three categories: drivers younger than 35 years ($q = 1$); drivers between 35 and 49 ($q = 2$); and drivers older than 50 years ($q = 3$). The second risk factor (p) is divided into four categories: the power of car less than 53 cv ($p = 1$); between 54 and 75 cv ($p = 2$); between 76 and 118 cv ($p = 3$); and more than 119 cv ($p = 4$).

We have used the estimator proposed by Gisler (1996) for the structural parameter β based on available data: $\widehat{\beta} = (\widehat{\sigma}^2_{K_{ij}} - \widehat{\lambda}_j - \widehat{\sigma}^2_{\Lambda_j}) / (\widehat{\lambda}_j^2 + \widehat{\sigma}^2_{\Lambda_j})$, obtaining $\widehat{\beta} = 0.9290279582$ for these data.

Table 2 shows the distribution of the policyholders in all the subportfolios of the portfolio, where we have the subportfolios in the same position as in Table 1.

Table 1. Number of policyholders in subportfolio and claim averages

	$q = 1$	$q = 2$	$q = 3$
$p = 1$	3,945 0.1850	9,023 0.1564	11,758 0.1277
$p = 2$	11,947 0.2639	25,719 0.2252	27,287 0.1969
$p = 3$	8,447 0.2917	19,609 0.2495	18,688 0.2345
$p = 4$	1,486 0.3135	5,762 0.2769	5,812 0.2424

Using Proposition 1, Table 3 shows the range of the relative premiums for $\varepsilon = 0$ (in bold) and $\varepsilon = 0.1$ for the class $p = 4$, $q = 1$. In this table, the infimum, the base premium, the supremum and RS factor — which appears in Gómez *et al.* (2002) — are shown in this order. Relative sensitivity (RS) is a standarised factor, defined as:

$$\mathrm{RS} = 100\left[2\mathcal{BMP}_{\pi_0}(k,t)\right]^{-1}\left\{\mathcal{BMP}_{\pi}^{2} - \mathcal{BMP}_{\pi}^{1}\right\}.$$

This RS factor represents the amount of variation, in percent, in $\mathcal{BMP}_{\pi}$ as π varies over $\mathcal{Q}_M^*$. For example, in Table 3 for $(k, t) = (1, 1)$, the RS factor 12.75 means that $\left\{\mathcal{BMP}_{\pi},\ \pi \in \mathcal{Q}^*\right\}$ can vary 12.75% on either side of the centre of its range over $\mathcal{Q}^*$. If this RS factor is small enough, the conclusion is that the model is declared to be robust. If not, further elicitation, data collection or analysis is necessary. Obviously, the particular situation for $\varepsilon = 0$ corresponds to that when no errors arise in the process of elicitation, i.e. we obtain the premium given in (2).

The RS factor is particulary low for class $k = 0$, therefore we have robust results in this class. However, the RS factor takes a higher value for the rest of the classes, so we can conclude that we have no robust results for the classes $k > 0$. This particular situation is similar to that which occurs in the rest of the classes of the portfolio, as can be seen in Figure 1, which shows us the RS factor for the twelve classes of the portfolio. These appear in the same position as in Table 1. The graphics correspond to $t = 1$: – –; $t = 2$: **–**; $t = 3$: – · – and $t = 4$: –.

In all classes, the results show that the lower RS factor is always given for $k = 0$ and the higher for $k = 1$, decreasing smoothly from here. Furthermore, the RS factor decreases when t increases, except for the $k = 0$ class.

Looking at each subportfolio in Figure 1, we can now see that the RS factor decreases when the value of λ_{ij} increases. This is so, because the lower value of λ_{ij} is less compatible with the prior distribution than with the higher

Table 2. Distribution of policyholders in the subportfolios of the portfolio

	k					k					k			
t	0	1	2	$\geqslant 3$	t	0	1	2	$\geqslant 3$	t	0	1	2	$\geqslant 3$
0	3,945				0	9,023				0	11,758			
1	3,316	548	61	20	1	7,797	1,063	140	23	1	10,437	1,159	143	19
2	2,756	965	182	42	2	6,668	1,952	344	59	2	9,185	2,193	331	48
3	2,290	1,226	339	90	3	5,702	2,562	625	135	3	8,354	2,693	604	107
4	1,903	1,371	503	168	4	4,876	2,954	937	257	4	7,114	3,517	923	204

	k					k					k			
t	0	1	2	$\geqslant 3$	t	0	1	2	$\geqslant 3$	t	0	1	2	$\geqslant 3$
0	11,947				0	25,719				0	27,287			
1	9,470	1,916	445	116	1	21,031	3,775	720	193	1	22,788	3,766	591	142
2	7,273	3,391	984	299	2	16,789	6,796	1,680	455	2	18,714	6,778	1,457	337
3	5,585	4,079	1,638	645	3	13,402	8,444	2,903	969	3	15,369	8,594	2,591	733
4	4,289	4,265	2,234	1,159	4	10,699	9,151	4,108	1,761	4	12,622	9,543	3,763	1,360

	k					k					k			
t	0	1	2	$\geqslant 3$	t	0	1	2	$\geqslant 3$	t	0	1	2	$\geqslant 3$
0	8,447				0	19,609				0	18,688			
1	6,570	1,423	321	133	1	15,702	3,112	603	92	1	15,158	2,848	510	172
2	4,908	2,495	759	286	2	12,234	5,478	1,456	441	2	11,988	5,065	1,262	373
3	3,666	2,933	1,266	582	3	9,532	6,647	2,496	934	3	9,482	6,230	2,198	778
4	2,738	2,990	1,701	1,017	4	7,426	7,032	3,469	1,681	4	7,499	6,686	3,101	1,402

	k					k					k			
t	0	1	2	$\geqslant 3$	t	0	1	2	$\geqslant 3$	t	0	1	2	$\geqslant 3$
0	1,486				0	5,762				0	5,812			
1	1,125	274	69	18	1	4,554	902	224	82	1	4,680	900	187	45
2	822	458	154	52	2	3,452	1,640	492	178	2	3,672	1,597	426	117
3	601	523	247	115	3	2,617	1,968	817	359	3	2,881	1,952	723	256
4	439	520	322	205	4	1,984	2,041	1,109	628	4	2,261	2,080	1,005	466

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Table 3. Range of relative premiums, base premiums and RS factor for $p = 4$, $q = 1$

	k			
t	0	1	2	3
0	100			
	95.457	107.118	123.491	143.841
1	**95.716**	**108.441**	**124.814**	**144.019**
	95.935	134.780	151.154	170.430
	0.25	12.75	11.08	9.22
	92.189	102.676	117.012	133.839
2	**92.375**	**103.674**	**118.010**	**134.837**
	92.843	122.032	136.369	153.195
	0.35	9.33	8.20	7.17
	89.314	99.079	111.942	126.994
3	**89.599**	**99.845**	**112.708**	**127.760**
	90.340	112.734	125.597	140.649
	0.57	6.83	6.05	5.34
	86.942	96.148	107.872	121.554
4	**87.318**	**96.737**	**108.461**	**122.143**
	88.284	105.819	117.543	131.224
	0.76	5.00	4.45	3.95

Table 4. Lower and higher RS factor in all the subportfolios

0.12 - 14.11	0.11 - 14.19	0.09 - 14.88
0.20 - 13.34	0.16 - 13.74	0.14 - 14.02
0.13 - 13.10	0.18 - 13.41	0.12 - 13.64
0.24 - 12.75	0.20 - 13.31	0.29 - 13.54

value. Observe that the unconditional mean of the number of claims is $E[K_{ij}] = E\big[E\big[K_{ij} \mid \theta_i\big]\big] = \lambda_{ij}$, and the mean of the prior distribution is one.

Because a low value of the RS factor indicates robustness, and a high value lack of robustness, the lower and higher values of the RS factor for each subportfolio can be summarised as in Table 4. These values are always achieved in period $t = 1$ for $k = 0$ and $k = 1$, respectively.

We can conclude that the actuary may feel reassured when tariffing an insured person of any class who belongs to position $(k, t) = (0, 1)$. On the other hand, the actuary must tariff very carefully when the person insured is at position $(k, t) = (1, 1)$. The rest of positions are obviously intermediary situations.

A second aspect should also be taken into account by the actuary. As mentioned above, the subportfolios with *a priori* low values of λ_{ij} present a higher RS factor. Furthermore, as is well known, for *a priori* low frequency

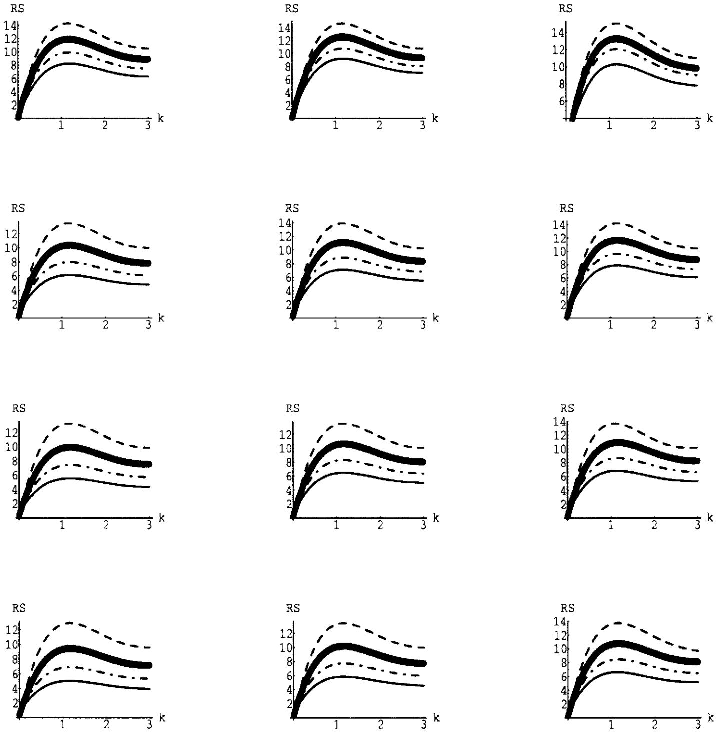

Figure 1. RS in all the subportfolios of the portfolio

risks the maluses are even higher than for those risks with a high frequency, which is not understood by the customer. Although theoretically correct, such a point must still be treated carefully by the actuary because of the chosen prior distribution.

Observe that this situation occurs when using, as we have done, a very reduced class, which incorporates many conditions which must be elicited by the practitioner. As Moreno *et al.* (2003) point out: "the range of the posterior inference on a given quantity of interest as the prior varies over the generalised moments class is generally too large."

Nevertheless, if the actuary has enough information to incorporate into class $\mathcal{Q}_M$, he can achieve robustness. In any case, the model described in Section 2, which has an inverse–Gaussian as a prior distribution and exponential premium principle, is much more robust than that applied in Gómez *et al.* (2002), which has a gamma as a prior distribution and net premium principle.

5. Conclusions and Extensions

This paper deals with the advantages of combining the most commonly used methods of robust Bayesian methodology, and presents a practical situation in a BMS using the Poisson–inverse Gaussian model extended to expression (2). We study how the choice of the prior can critically affect the relative premiums.

Obviously, the approach used here can be applied to other distributions; for example, Lemaire (1979), among others, uses the Poisson–Gamma model and Meng & Whitmore (1999) use the Negative Binomial–Pareto model.

The specification of a concrete class of prior distributions seeks to be a model about the uncertainty in which the actuary has a base prior distribution. If RS is small, the actuary can be satisfied with his conclusions about uncertainty on the prior distribution. If RS is large, the actuary must be careful about the ratemaking procedure. Otherwise, he/she should follow some other strategy, such as obtaining a more robust model by considering other premium principles, other distributions, and the consideration of properties which allow him/her to use a more reduced class, including expert opinion and historical data.

A matter which deserves treatment is the incorporation of the unimodality condition to the class $\mathcal{Q}_M$. This is a characteristic which can easily be elicited by an actuary, but, for the moment, it is intractable in this class.

Of course, an extension of the result presented here can be obtained for those premium principles which are expressed as a ratio of later expectations, such as the Esscher and variance principles (see Gómez *et al.*, 2001). Although the common approach to assessing sensitivity or robustness analysis is to measure the size of the class of later expectations (referred to as a global sensitivity analysis), local sensitivity analysis is also possible.

The idea of local sensitivity is to examine the rate at which the posterior changes, relative to the prior (Gustafson, 1996a). This technique is appropriate for our problem, and it again has connections with the ε–contamination class (Gustafson, 1996a). In this regard, Gustafson (1996b) states: "When studying sensitivity to prior distribution, there is always a trade-off: classes of priors that are too large have poor behaviour, and classes of priors that are too small may understate sensitivity. This is especially so for local diagnostics."

Although local analysis involves a complicated mathematical treatment in the field of functional analysis, we nevertheless believe that its use in our model is appropriate and should be used here, leaving further consideration of this point for a forthcoming paper.

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