Letter

Universality of quantum liquids and droplets in one dimension

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We consider interacting one-dimensional bosons in the universal low-energy regime. The interactions consist of a combination of attractive and repulsive parts that can stabilize quantum gases, droplets, and liquids. In particular, we study the role of effective three-body repulsion, in systems with weak attractive pairwise interactions. Its low-energy description is often argued to be equivalent to a model including only two-body interactions with nonzero range. Here, we show that, at zero temperature, the equations of state in both theories agree quantitatively at low densities for overall repulsion, in the gas phase. However, this agreement is absent in the attractive regime, where universality only occurs in the long-distance properties of quantum droplets. We develop analytical tools to investigate the properties of the theory and obtain astounding agreement with exact numerical calculations using the density-matrix renormalization group.

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Introduction. Recent advances in the preparation, manipulation, and observation of ultracold atomic droplets and liquids [1–6] have sparked renewed interest in the physics of these systems [7–19]. This is more so due to their low densities and temperatures, allowing universal low-energy descriptions that are independent of the short-distance details of the relevant interactions [20]. The possibility to effectively confine these systems to one spatial dimension [21,22], where interaction effects are enhanced [23], makes ultracold atoms a promising platform to realize highly controllable strongly interacting droplets and liquids. With traditional quantum liquids such as ^{4}He, interatomic potentials that reproduce essentially all of their experimentally measurable properties are known accurately [24]. In deep contrast, the underlying interactions in ultracold atomic systems are highly dependent on the particular atomic species and applied external fields, such as those involved in magnetic Feshbach resonances [25] and transversal confinement [26]. Hence it is impractical, if not impossible, to attempt as accurate a description as in ^{4}He for each realization of an ultracold atomic liquid. Therefore, a universal low-energy description of these systems, within the effective field theory (EFT) paradigm [27,28], is highly desirable.

Three-body interactions are always present in a many-atom system and can be genuine or emergent from the off-shell structure of the two-body interactions [28–30]. In general, these are a combination of genuine and emergent. For repulsive one-dimensional systems, two-body effective range (on-shell) effects are typically negligible for low densities [31]. However, the effects of a nonzero physical range (off shell) are important for three-body processes, especially for large scattering lengths [29,30], which can be rigorously modeled by simple three-body forces. This can also be explained qualitatively using field redefinitions [28] that trade off-shell low-energy interactions in favor of simpler, on-shell three-particle forces. Since, in this case, the low-energy two- and three-body amplitudes are essentially identical, the S-matrix formulation of statistical mechanics [32] implies thermodynamic equivalence in the gas phase. At zero temperature this is the case for overall repulsive interactions. For attraction, where liquids may be formed, the results of Ref. [32] do not apply and, as we shall see, there is no such equivalence.

In this Letter, we investigate one-dimensional quantum gases, liquids, and droplets at zero temperature that are described by two types of low-energy theories. One includes a zero-range two-body interaction as well as an effective range. The other one features zero-range two- and three-body interactions. We demonstrate that there is no equivalence between the two theories in their liquid and droplet phases. This fact is proven by studying universal long-distance asymptotics in both models. We also develop highly nonperturbative, analytical approximations whose predictions are in excellent agreement with exact calculations using the density-matrix renormalization group (DMRG).

Low-energy theories. We start by considering two low-energy theories describing one-dimensional many-body systems of bosons in free space at zero temperature. Their respective Hamiltonians are $\hat{H}_c^{\rm 3B}$ and $\hat{H}_c^{\rm NL}$. Defining the non-interacting Hamiltonian $\hat{H}_0 = (-\hbar^2/2m)\int dx\, \hat{\phi}^\dagger \partial_x^2 \hat{\phi}$, with $\hat{\phi}$ ($\hat{\phi}^\dagger$) the bosonic annihilation (creation) field operator, these

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are given by

$$\hat{H}_c^{3B} = \hat{H}_0 + \int dx \left[\frac{g_0}{2!} \hat{\phi}^\dagger(x)^2 \hat{\phi}(x)^2 + \frac{g_3}{3!} \hat{\phi}^\dagger(x)^3 \hat{\phi}(x)^3 \right], \quad (1)$$

$$\hat{H}_c^{NL} = \hat{H}_0 + \int dx \left[\frac{g_0'}{2!} \hat{\phi}^\dagger(x)^2 \hat{\phi}(x)^2 - g_2 \hat{\rho}(x) \partial_x^2 \hat{\rho}(x) \right]. \quad (2)$$

Above, $\hat{\rho}(x) = \hat{\phi}^\dagger(x)\hat{\phi}(x)$ is the density operator. The coupling constants of the zero-range two-body interaction are chosen to be different in the two theories ($g_0 \neq g_0'$). This freedom will allow us to ensure that the on-shell amplitudes in the two-body sector coincide, to a very good approximation, in both theories.

The three-body theory, Eq. (1), upon renormalization, requires the introduction of a three-body momentum scale Q_* [29], beyond which the three-body theory breaks down. For repulsive three-body interactions there are unphysical bound states [29,33,34] for any particle number $N > 2$. This prevents the physical ground state from being explored using ground state methods. Below, we solve this issue by discretizing the problem on a lattice near, but not in the continuum limit.

A direct discretization of the low-energy theories (1) and (2) results in the following generalized Bose-Hubbard Hamiltonians:

$$\hat{H}_c^{3B} = -J \sum_j (\hat{b}_{j+1}^\dagger \hat{b}_j + \text{H.c.}) + \frac{U_2}{2} \sum_j \hat{n}_j(\hat{n}_j - 1) + \frac{W}{6} \sum_j \hat{n}_j(\hat{n}_j - 1)(\hat{n}_j - 2) + 2J \sum_j \hat{n}_j, \quad (3)$$

$$\hat{H}_c^{NL} = -J \sum_j (\hat{b}_{j+1}^\dagger \hat{b}_j + \text{H.c.}) + \frac{U}{2} \sum_j \hat{n}_j(\hat{n}_j - 1) + V \sum_j \hat{n}_j \hat{n}_{j+1} + 2J \sum_j \hat{n}_j. \quad (4)$$

Above, $J = \hbar^2/(2md^2)$ is the hopping strength, with d the lattice spacing; $U_2 = g_0/d$ and $U = g_0'/d + 2g_2/d^3$ are the on-site two-body interaction strengths. W (>0) is a three-body coupling constant to be determined and $V = -g_2/d^3$ sets the off-shell part of the two-body interaction in the discretized model. b_j ($b_j^\dagger$) annihilates (creates) a boson at site j ($\in \mathbb{Z}$), and $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$ is the local number operator.

The discretized low-energy theories (3) and (4) allow us to numerically obtain the ground state of the respective Hamiltonians by performing DMRG simulations. To match the low-energy N-body amplitudes of both theories we consider the finite-size ground state energy of their respective Hamiltonians. We adjust the parameters (U_2, W) and (U, V) to find the best match between the corresponding energies of both theories for $2 \leqslant N \leqslant 6$ (see [35]), a well-established method in lattice and finite-size simulations [29,36].

Universality of quantum gases. We begin by demonstrating that the discretized low-energy theories (3) and (4) reproduce continuum physics and showing that universality is present in the gas phase. To do the latter, we numerically study the ground state quantum gas with pure three-body interactions, i.e., $U_2 = 0$ (scattering length $a \to \infty$) and $W > 0$. For this special case we can establish a direct connection between the three-body lattice coupling constant W and the continuum three-body momentum scale Q_* [29] by solving the three-body problem with $U_2 = 0$ at vanishing total quasimomentum. Matching the continuum and lattice amplitudes at low energies, we obtain $W/J = (\beta + \ln|Q_* d|/2\sqrt{3})^{-1}$, where $\beta = -0.1956\ldots$ is a numerical constant of no physical relevance, i.e., it is regularization dependent. Therefore, the discretized version of the three-body low-energy theory can be used to simulate a continuum repulsive three-body force—thereby avoiding unphysical bound states—at low densities with a finite lattice spacing, provided that $W/J > 0$, i.e., for $|Q_* d| > \exp(2\sqrt{3}|\beta|) \approx 8.4$. These values correspond to the weak to moderate coupling regime for the three-body interaction [29,30,33,34,37–48].

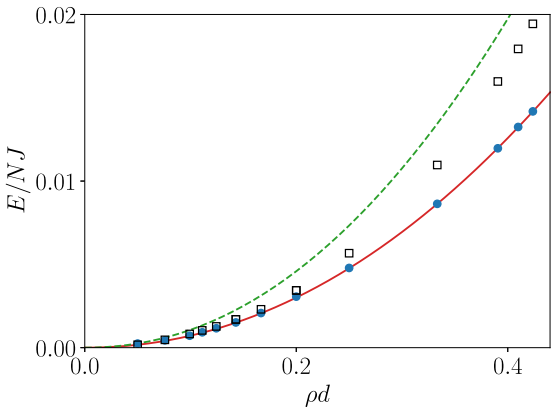

FIG. 1. Zero-temperature equation of state for Hamiltonian (3) with $U_2 = 0$ and $W/J = 1.1$ (filled blue dots) and Hamiltonian (4) at its two-body resonance (open black squares) with $(U/J, V/J) = (16, -8/5)$. Weak-coupling EOS [35], at scale $\mu = \exp(\gamma) Q_*^2/\sqrt{8}\rho$, corresponding to Pastukhov's scale [37] (green dashed line), and at the renormalization scale $\mu = 188.464/\rho d^2$ (red solid line).

For the other discretized low-energy theory, with Hamiltonian (4), imposing a two-body resonance ($a \to \infty$) implies a relation between U and V given by $U/J = -4(V/J)/(2 + V/J)$ [16,49]. To show the low-density equivalence of these two quantum gases we obtain the equation of state (EOS) for both theories using DMRG under the set of conditions just mentioned. In Fig. 1, we clearly observe that for low densities the two theories give the same EOS showing the universality of the quantum gas. This result was expected since for repulsive gases it is known that two models will have identical thermodynamic properties if their (N-body) S matrices are identical [32]. This is the case at low densities where N-body scattering processes with $N > 3$ are suppressed. Thus fixing the two- and three-body low-energy amplitudes yields similar EOS for the two models studied.

Furthermore, to test the ability of the discretized low-energy theories to recover the continuum ones we compare our DMRG results with the weak-coupling expansion of the EOS of Hamiltonian (1) with $g_0 = 0$, due to Pastukhov [35,37]. The renormalization scale $\mu = \mu(\rho) = \xi/\rho$, with ξ being a numerical constant, in the dimensionless coupling constant $g(\mu)$ [35,37] has the same ambiguity of choice, in perturbation theory, as that in the 2D Bose gas [50,51]. To obtain predictive power, we match (i.e., renormalize) Pastukhov's weak-coupling EOS [35] at only one value of the

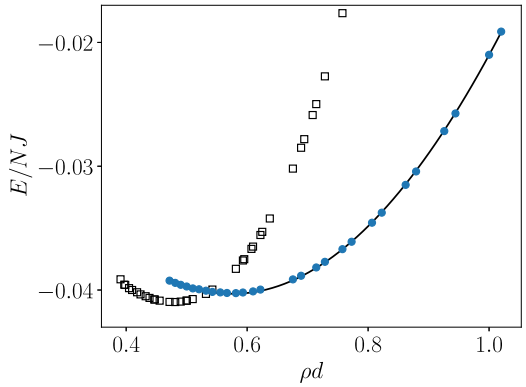

FIG. 2. Exact zero-temperature equation of state for Hamiltonian (3) with $U_2/J = -0.33$ and $W/J = 1.39$ (blue dots), and for Hamiltonian (4) with $U/J = 10$ and $V/J = -8/5$ (open black squares). Improved mean-field approximation, Eq. (7) (solid black line) with $\rho_{\rm eq} d = 0.576$, $m g_3^{\rm eq}/\hbar^2 = 0.9899$, and $\lambda = 0.2645$ (see text).

density ρ with the EOS at that density obtained with DMRG. The results are shown in Fig. 1, where astounding agreement is observed over all ranges of density.

Universality of quantum liquids. We now turn our attention to the ground state liquid phase of the low-energy theories. First, we compare their respective EOS for a particular set of values of the interaction strengths that ensure that the binding energies for all N of both theories are matched within 2% (see [35]). The results are shown in Fig. 2. We observe that the equilibrium energy is very similar for both theories but the equilibrium densities are in clear disagreement, making the two EOS very different. Therefore, we can claim that the equilibrium density of quantum liquids is a highly nonuniversal property and, therefore, so is the EOS.

The nonuniversality of the equilibrium density can be traced back to the different normalization coefficients present in the respective many-body wave functions in vacuum, as we show in the following. For an attractive system, which features N-body bound states for all N, it is possible to fix the locations of the poles of the S matrix (bound state energies) rather accurately in both models, but not the residues at the poles without further parametrization. These are related to the so-called asymptotic normalization coefficient (ANC) γ_N for a bound state [52–54]. For two particles in one dimension, defining the normalized relative bound state $\psi_2(x)$, γ_2 is defined as $|\gamma_2| = \lim_{x_{12}\to\infty}[|\psi_2(x_{12})| \exp(\sqrt{m E_2^{(B)}/\hbar^2}|x_{12}|)]$, where $E_2^{(B)}$ (>0) is the binding energy. For large and positive (attractive) scattering length a in comparison with the effective range r_e $(a/|r_e| \gg 1)$, the two-body binding energies obtained with and without including the effective range agree well with each other. The ANCs are also in rather good agreement. For example, for the two models considered here, we have $|\gamma_2^{\rm NL}/\gamma_2^{\rm 3B}|^2 \approx 1 + \{1 - 1/[1 + U/(4J)]^2\}|U_2|/(4J)$, where $\gamma_2^{\rm NL}$ and $\gamma_2^{\rm 3B}$ are, respectively, the two-body ANC of Hamiltonians (4) and (3) with identical scattering lengths. For the case studied in Fig. 2, we have $|\gamma_2^{\rm NL}/\gamma_2^{\rm 3B}|^2 \approx 1.07$, which shows a small yet non-negligible deviation from unity.

For large particle numbers, we can show that the ratios $|\gamma_N^{\rm NL}/\gamma_N^{\rm 3B}|$ become exponential in N. The density profile $\rho_N(x)$ can be obtained by computing the density with respect to a fixed center-of-mass coordinate X (set to zero for simplicity) as in Ref. [55] (see also [56]). Using the universal asymptotics for bound state wave functions in one dimension [54], we obtain [35] for N bosons, as $|x| \to \infty$,

$$\rho_N(x) \to \frac{\pi}{2}|\gamma_N|^2 A_{N-1}\frac{N}{N-1}e^{-\frac{N}{N-1}2\kappa_{1,N}|x|}. \tag{5}$$

Above, γ_N is the ANC in the $N \leftrightarrow (N-1)+1$ breakup channel, $\hbar^2\kappa_{1,N}^2/2m = |E_N - E_{N-1}|(N-1)/N$ represents the binding energy with respect to the ground state with one particle less, and A_N is a model independent normalization factor. Using Eq. (5) for the two models considered here, we obtain, as $|x| \to \infty$, $\rho_N^{\rm NL}(x)/\rho_N^{\rm 3B}(x) \to |\gamma_N^{\rm NL}/\gamma_N^{\rm 3B}|^2$. This relation has important consequences. We assume that the asymptotic normalization coefficients for the two models are different for all N, since they already are for two particles. If an N-body bound state is a quantum droplet, then most of the particle content lies within the bulk of the droplet, with a density we may consider constant. The radius R of the droplet is then $R \approx N/2\rho_N(0)$. One may approximate the exponential tails for large N as $\rho_N(x) \sim \rho_N(0)\exp(-2\kappa_{1,N}|x-R|)$ as $|x| \to \infty$, obtaining

$$\left|\frac{\gamma_N^{\rm NL}}{\gamma_N^{\rm 3B}}\right|^2 \sim \frac{\rho_N^{\rm NL}(0)}{\rho_N^{\rm 3B}(0)}\exp\left[\kappa_{1,N}N\left(\frac{1}{\rho_N^{\rm NL}(0)} - \frac{1}{\rho_N^{\rm 3B}(0)}\right)\right]. \tag{6}$$

The above relation indicates that we require algebraically small $[O(N^{-1/2-|\epsilon|})]$ differences between the ANCs in order to achieve algebraically $[O(N^{-1-|\delta|})]$ small differences in the equilibrium densities of quantum droplets and liquids, and vice versa. A finite difference in the large-N densities of the two models immediately implies an exponential disagreement between the ANCs and therefore the residues at the poles of the S matrix. Equation (6) shows our claim that, in order for two different models of a quantum liquid to share thermodynamical properties, not only the bound state energies but also the residues at these energies must be matched and that the matching must have algebraic precision. To gain some analytical insight into the EOS of the system with two- and three-body interactions, we present an improved version of the mean-field theory (IMF), by allowing the three-body coupling constant g_3 to depend logarithmically on the density $g_3(\rho) = g_3^{\rm eq}/[\lambda \ln|\rho/\rho_{\rm eq}| + 1]$, so that it can account for the anomaly nonperturbatively. The value $g_3^{\rm eq}$ is given by the coupling constant at equilibrium and is fixed by $e_{\rm eq}$. λ is a dimensionless parameter that is fixed by the equilibrium density $\rho_{\rm eq}$, given $e_{\rm eq}$. To lowest order, integrating the chemical potential gives for the energy per particle $e_{\rm IMF}$ in this approximation

$$e_{\rm IMF}(\rho) \approx \tfrac{1}{2}g_0\rho + \tfrac{1}{6}g_3(\rho)\rho^2. \tag{7}$$

Given the freedom of scale in logarithmic running of the coupling constant, the above approximation is sufficient. This IMF theory is able to recover the EOS of the three-body interacting system for all densities; see Fig. 2. Finally, we demonstrate the inequivalence between the EOS of the low-energy theories at any value of the coupling strengths. In Fig. 3 we present the ratio of the equilibrium energy and equilibrium density for different values of the coupling constants. We observe that this ratio does not coincide in both theories at any

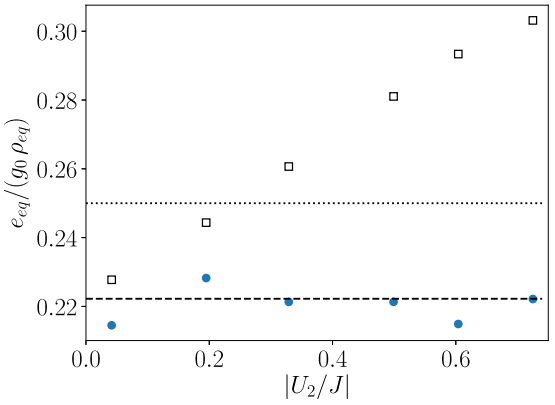

FIG. 3. Ratio $e_0/g_0\rho_{\rm eq}$ calculated numerically for Hamiltonian (3) as function of two-body interaction strength (blue dots). The three-body interaction strengths for points labeled with $U_2/J = -0.195, -0.329, -0.499, -0.604$, and -0.725 are given by $W/J = 1.28$, 1.4, 1.48625, 1.506, and 1.508, respectively. Open black squares correspond to the same ratio but for Hamiltonian (4), with $V/J = -8/5$. From left to right, $U/J = 12$, 10, 8, 7, and 6. Dashed line ($2/9 = 0.222\ldots$) is the prediction from Eq. (8) and dotted line ($1/4$) is the standard mean field prediction; see text.

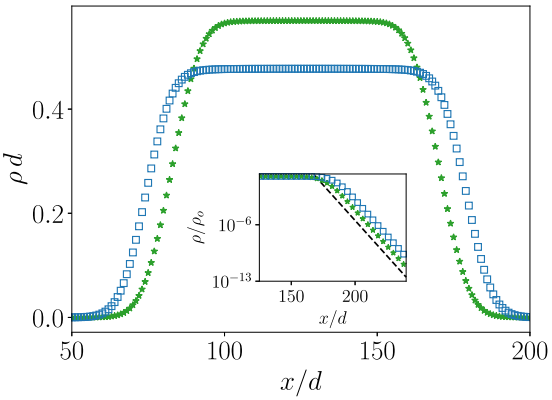

FIG. 4. Main panel: density profiles with $N = 50$ particles for Hamiltonian (3) with $U_2/J = -0.33$ and $W/J = 1.39$ (open squares) and Hamiltonian (4) with $U/J = 10$ and $V/J = -1.6$ (stars). Inset panel: density tails, rescaled by their respective saturation densities. Dashed line indicates the analytical result Eq. (5).

value of the couplings. In fact, the three-body theory (3) shows an approximately constant ratio while the theory with nonzero range (4) shows a nearly linear relation. The constant relation for the three-body theory can be explained qualitatively by performing a mean-field approximation of theory (1). By computing the chemical potential $\mu = g_0\rho + g_3\rho^2/2$ we obtain $e_{\rm eq}/|g_0|\rho_{\rm eq} = 1/4$; see Fig. 3. To improve the mean-field prediction we perform a two-body decoupling approximation. This consists of decomposing the three-body interaction into a set of two-body interactions dependent on the ground state of the system [35]. Once we reduce the three-body interaction to a state-dependent two-body one, we solve the problem self-consistently employing the exact solution of the attractive Lieb-Liniger model [57] due to MacGuire [58]. This procedure leads to the equilibrium relation

$$e_{\rm eq} = \tfrac{2}{9} g_0 \rho_{\rm eq}. \tag{8}$$

Our theory, although approximate, is highly nonperturbative. It predicts a strongly constrained, linear relation between the equilibrium energy per particle and density for fixed two-body interaction strength $g_0 < 0$. In Fig. 3 we show that the DMRG results of the discrete three-body model (3) are in excellent agreement with our prediction given by Eq. (8).

Universality of quantum droplets. The disagreement in the EOS between the two models has important consequences for the density profiles of the respective quantum droplets. Even though both models exhibit the same equilibrium energy, their equilibrium densities, and thus the droplet saturation densities, are different. This implies that droplets in the two models with equal particle number have different sizes, see Fig. 4, which is a direct consequence of the ANCs γ_N, Eq. (6). On the other hand, the decay of the density far away from the center of the droplet has to be identical in both models if the equilibrium energy is the same, as predicted by Eq. (5). In the inset panel of Fig. 4 we show that both droplets exhibit the same exponential density decay far away from the respective centers. Moreover, the decay is dictated by Eq. (5) which shows that it is directly related to the chemical potential at equilibrium for large number of particles. Therefore, the tails of quantum droplets are universal for different models with identical binding energies (in free space), as opposed to the saturation density.

Conclusions. We have investigated one-dimensional quantum liquids at zero temperature and droplets using universal low-energy theories. We have shown that, while in the repulsive case different models with (almost) identical two- and three-body scattering amplitudes have identical low-density equations of state, small deviations in densities in the few-particle sector grow exponentially in the many-body limit for droplets and liquids. This implies the lack of equivalence of the zero-temperature equations of state. We have developed theoretical techniques that yield quantitatively accurate predictions in all density regimes, when compared to exact results obtained with DMRG.

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