Algebras and Representation Theory (2023) 26:1497–1517
https://doi.org/10.1007/s10468-022-10146-1

Klyachko Diagrams of Monomial Ideals

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Received: 29 October 2021 / Accepted: 2 June 2022 / Published online: 21 June 2022

Abstract
In this paper, we introduce the notion of a *Klyachko diagram* for a monomial ideal I in a certain multi-graded polynomial ring, namely the Cox ring R of a smooth complete toric variety, with irrelevant maximal ideal B. We present procedures to compute the Klyachko diagram of I from its monomial generators, and to retrieve the $B-$saturation I^{sat} of I from its Klyachko diagram. We use this description to compute the first local cohomology module $H^1_B(I)$. As an application, we find a formula for the Hilbert function of I^{sat}, and a characterization of monomial ideals with constant Hilbert polynomial, in terms of their Klyachko diagram.

Keywords Monomial ideals · Cox ring · Klyachko filtrations · Local cohomology · Hilbert function · Hilbert polynomial

1 Introduction

Lying in the crossroads of commutative algebra and combinatorics, monomial ideals play a prominent role in the study of ideals in a polynomial ring R. Indeed, many properties of arbitrary ideals $I \subset R$ are reduced to the monomial case, which can often be tackled using combinatorial tools. For instance, it is a classical result due to Macaulay in [16], that the Hilbert function of an ideal $I \subset R$ coincides with the Hilbert function of its *initial ideal* $\text{in}_>(I)$, which is itself a monomial ideal (see for instance [7, Theorem 15.3]). Since the advent of combinatorial commutative algebra, the theory of monomial ideals has been linked with various topics in discrete mathematics, such as enumerative combinatorics, graph theory, simplicial geometry or lattice polytopes (see [1, 6, 9–12, 20]).

Presented by: Michel Brion

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The aim of this paper is to introduce the *Klyachko diagram* of a monomial ideal, which can be seen as a generalization of the classical *staircase diagram*, suited to study monomial ideals inside non-standard graded polynomial rings. More precisely, we focus on the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_r]$ graded by the class group $\mathrm{Cl}(X) \cong \mathbb{Z}^l$ of a smooth complete toric variety X. That is, $r = |\Sigma(1)|$ is the number of rays of the fan Σ of X, and the degree of a variable x_i is the class in $\mathrm{Cl}(X)$ of the torus-invariant Weil divisor D_{ρ_i} corresponding to the ray ρ_i. The graded ring R can be considered as the Cox ring of the toric variety X, and it appears in the construction of a toric variety by a GIT quotient [3, Chapter 5]. For instance, if we consider $X = \mathbb{P}^{r-1}$, we recover the polynomial ring with its classical $\mathbb{Z}$-grading.

Apart from the Cox ring being a generalization of the classical $\mathbb{Z}-$graded polynomial ring, $\mathrm{Cl}(X)-$graded $R-$modules correspond to quasi-coherent sheaves on X. In particular, a $\mathrm{Cl}(X)-$graded ideal $I \subset R$ corresponds to an ideal sheaf $\mathcal{I}$ on X such that

$$H^0_*(X, \mathcal{I}) = \bigoplus_{\alpha \in \mathrm{Cl}(X)} H^0(X, \mathcal{I}(\alpha)) \cong (I : B^\infty) = I^{\mathrm{sat}},$$

where B is the *irrelevant ideal* of X. It is a monomial maximal ideal determined combinatorially by the fan Σ of the toric variety X. A $\mathrm{Cl}(X)-$graded $R-$module E gives rise to an equivariant sheaf if and only if E is $\mathbb{Z}^r-$graded (also called *fine-graded*). In particular, equivariant ideal sheaves on X are in correspondence to monomial ideals in R.

In [14] and [15], Klyachko classified equivariant torsion-free sheaves on X in terms of filtered collections of vector spaces. These filtered collections, parameterized by the cones of the fan Σ, are often referred to in the literature as *Klyachko filtrations* (see Proposition 2.6). In [19], this device was formalized by Perling, who introduced the notion of a $\Sigma-$family, obtaining a general classification of equivariant quasi-coherent sheaves. From a geometrical point of view, these methods have been used in the last two decades to study equivariant vector bundles on toric varieties (see [4, 5, 13, 18]). On the other hand, in [17], the present authors used the theory of $\Sigma-$families to study reflexive $\mathrm{Cl}(X)-$graded $R-$modules from a commutative algebra perspective.

In this note, we use this construction to introduce the Klyachko diagram of a monomial ideal $I \subset R$: a family of staircase-like diagrams parametrized by the cones of Σ encoding algebraic properties of I (see for instance Example 3.5 and Fig. 2). In particular, the Klyachko diagram is uniquely determined by the ideal I, up to $B-$saturation. We give procedures to compute the Klyachko diagram using the monomial generators of I as initial data and conversely, to determine the generators of a $B-$saturated ideal I^{sat} from a given Klyachko diagram $\{(C_I^\sigma, \Delta_I^\sigma)\}_{\sigma \in \Sigma}$. We also provide a method to compute the first local cohomology module $H^1_B(I)$ with respect to B from the diagram $\{(C_I^\sigma, \Delta_I^\sigma)\}_{\sigma \in \Sigma}$, which measures the saturatedness of I. We then use the Klyachko diagram to give a formula for the $\mathrm{Cl}(X)-$graded Hilbert function of I^{sat} in terms of lattice polytopes. Finally, we characterize monomial ideals I with constant Hilbert polynomial in terms of their Klyachko diagram.

Next we explain how this paper is organized. Section 2 contains all the preliminary results and definitions needed for the rest of this work, and it is divided in two parts. In Section 2.1, we recall the notation and basic results concerning toric varieties. In Section 2.2, we recall the theory of Klyachko filtrations.

The remaining two sections are the main body of the paper. In Section 3, we define the Klyachko diagram of a monomial ideal, and we establish its main properties. In Section 3.1, we present a procedure to obtain the Klyachko diagram from the generators of a given monomial ideal I, and we prove that it describes the collection of Klyachko filtrations of I (Proposition 3.4). As a corollary, we show how the Klyachko diagram of the sum of two monomial ideals can be computed. Conversely, in Section 3.2, we give a method to

obtain a minimal set of generators of a $B-$saturated monomial ideal corresponding to a given Klyachko diagram. Finally in Section 3.3, we use our previous results to compute the first local cohomology module $H^1_B(I)$ (Proposition 3.15) which measures how different I and I^{sat} are. In the last part of this note, we give a formula for the Hilbert function of a $B-$saturated monomial ideal in terms of its Klyachko diagram (Proposition 4.1), and we finish characterizing the Klyachko diagram of monomial ideals with a constant Hilbert polynomial (Corollary 4.4). In particular, we characterize all one dimensional monomial ideals $I \subset R$ in terms of the Klyachko diagram.

2 Preliminaries

In this section, we gather the basic notations, definitions and results about toric varieties needed in the sequel. We recall the notion of a $\Sigma-$family of an equivariant torsion-free sheaf, as introduced in [19], and we end specializing it to the setting of equivariant ideal sheaves.

2.1 Toric Varieties

Let X be an $n-$dimensional smooth complete toric variety with torus $\mathbb{T}_N \cong (\mathbb{C}^*)^n$, associated to a fan $\Sigma \subset N \otimes \mathbb{R} \cong \mathbb{R}^n$, where $N \cong \mathbb{Z}^n$ is the cocharacter lattice of $\mathbb{T}_N$. We denote by $\Sigma(k)$ (respectively $\sigma(k)$) the set of $k-$dimensional cones in Σ (respectively in σ). We refer to the cones $\rho \in \Sigma(1)$ as *rays* and we set $n(\rho) \in N$ to be the first non-zero lattice point along ρ. We denote by $M = \text{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ its *character lattice* and for $m \in M$, we set $\chi^m : \mathbb{T}_N \to \mathbb{C}^*$ the corresponding algebraic group homomorphism. For any cone $\sigma \in \Sigma$, let $\sigma^\vee$ be its *dual cone*, let $S_\sigma := \sigma^\vee \cap M$ be the associated semigroup of characters and $\mathbb{C}[S_\sigma]$ the corresponding $\mathbb{C}-$algebra. Then $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) \subset X$ is a $\mathbb{T}_N-$invariant affine subvariety of X. For any two cones $\tau \prec \sigma \in \Sigma$, there is a character $m \in M$ such that $S_\tau = S_\sigma + \mathbb{Z}\langle m \rangle$ and we have an inclusion $U_\tau \hookrightarrow U_\sigma$ given by the natural morphism of $\mathbb{C}-$algebras $\mathbb{C}[S_\sigma] \hookrightarrow \mathbb{C}[S_\sigma]_{\chi^m} = \mathbb{C}[S_\tau]$.

There is a bijection between rays $\rho \in \Sigma(1)$ and $\mathbb{T}_N-$invariant Weil divisors D_ρ. Furthermore, the $\mathbb{T}_N-$invariant Weil divisors generate the class group $\text{Cl}(X)$ of X. Indeed, we have the exact sequence

$$0 \to M \xrightarrow{\phi} \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \xrightarrow{\pi} \text{Cl}(X) \to 0, \tag{1}$$

where $\phi(m) = \text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, n(\rho) \rangle D_\rho$, for any character $m \in M$; and $\pi(D) = [D] \in \text{Cl}(X)$ the class of an invariant Weil divisor D. Hence, $\text{Cl}(X)$ is a finitely generated abelian group. (See [3, Theorem 4.1.3]).

Let $R = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ be a polynomial ring in $|\Sigma(1)|$ variables. The *Cox ring* of X is the $\mathbb{C}-$algebra R endowed with a grading, not necessarily standard, given by the class group $\text{Cl}(X)$ of X. We set $\deg(x_\rho) := [D_\rho] \in \text{Cl}(X)$, for each ray $\rho \in \Sigma(1)$. We write $R = \mathbb{C}[x_1, \ldots, x_r]$ whenever $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ is the (ordered) set of rays of Σ. For a cone σ, we set

$$x^{\hat{\sigma}} := \prod_{\rho_i \in \Sigma(1) \setminus \sigma(1)} x_i, \quad \text{and} \quad B := \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle.$$

B is called the *irrelevant ideal*. In fact, one has $B = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma_{\max} \rangle$.

Remark 2.1 In general, the Cox ring can be defined for any variety X as the ring

$$\mathcal{R}(X) = \bigoplus_{[D]\in \mathrm{Pic}(X)} H^0(X, \mathcal{O}(D)).$$

In the special case when X is a smooth toric variety it coincides with the polynomial ring we defined above.

Example 2.2 (i) $\mathbb{P}^n$ is a toric variety of dimension n. Let $\{e_1, \ldots, e_n\}$ be a basis of $N = \mathbb{Z}^n$. The fan Σ associated to $\mathbb{P}^n$ has $n+1$ rays: $\rho_0 = \mathrm{cone}(-e_1 - \cdots - e_n)$ and $\rho_i = \mathrm{cone}(e_i)$ for $1 \leq i \leq n$; and $n+1$ maximal cones $\sigma_0 := \mathrm{cone}(e_1, \ldots, e_n)$ and $\sigma(i) := \mathrm{cone}(\{e_j \mid j \neq i\} \cup \{-e_1 - \cdots - e_n\})$ for $1 \leq i \leq n$. Its associated Cox ring is $\mathbb{C}[x_0, \ldots, x_n]$ with $\deg(x_i) = 1$ for $0 \leq i \leq n$, and its irrelevant ideal is $B = \langle x_0, \ldots, x_n\rangle$.

(ii) For $a \geq 0$, the Hirzebruch surface $\mathcal{H}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ is a toric surface. Let $N = \mathbb{Z}^2$ be a lattice with $\{e, f\}$ its standard basis, and set $u_0 := -e + af$, $u_1 := e$, $v_0 := -f$ and $v_1 := f$. The fan Σ associated to $\mathcal{H}_a$ has four rays $\rho_0 = \mathrm{cone}(u_0)$, $\rho_1 = \mathrm{cone}(u_1)$, $\eta_0 = \mathrm{cone}(v_0)$ and $\eta_1 = \mathrm{cone}(v_1)$; and four maximal cones $\sigma_{00} = \mathrm{cone}(u_1, v_1)$, $\sigma_{01} = \mathrm{cone}(u_1, v_0)$, $\sigma_{10} = \mathrm{cone}(u_0, v_1)$ and $\sigma_{11} = \mathrm{cone}(u_0, v_0)$. Its Cox ring is $\mathbb{C}[x_0, x_1, y_0, y_1]$ with $\deg(x_0) = \deg(x_1) = (1, 0)$, $\deg(y_0) = (0, 1)$ and $\deg(y_1) = (-a, 1)$; and its irrelevant ideal is $B(\Sigma) = \langle x_1y_1, x_1y_0, x_0y_1, x_0y_0\rangle$.

For any cone $\sigma \in \Sigma$, the localization of R at $x^{\hat{\sigma}}$ is a $\mathrm{Cl}(X)-$graded algebra $R_{x^{\hat{\sigma}}}$. For any Weil divisor $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$, there is an isomorphism between $\mathbb{C}[S_\sigma]$ and the homogeneous $[D]-$graded piece $(R_{x^{\hat{\sigma}}})_{[D]}$ sending $\chi^m \in \mathbb{C}[S_\sigma]$ to the monomial $\underline{x}^{m+D} := \prod_{\rho\in\Sigma(1)} x_\rho^{\langle m,\rho\rangle + a_\rho} \in (R_{x^{\hat{\sigma}}})_{[D]}$. We have the following:

Proposition 2.3 (i) *For any $\alpha \in \mathrm{Cl}(X)$, there is a natural isomorphism $R_\alpha \cong \Gamma(X, \mathcal{O}_X(D))$ for any Weil divisor $D = \sum_\rho a_\rho D_\rho$ such that $\alpha = [D]$.*

(ii) *If E is a $\mathrm{Cl}(X)-$graded R-module, there is a quasi-coherent sheaf $\widetilde{E}$ on X such that $\Gamma(U_\sigma, \widetilde{E}) = (E_{x^{\hat{\sigma}}})_0$, for any $\sigma \in \Sigma$.*

(iii) *If $\mathcal{E}$ is a quasicoherent sheaf on X, there is a $\mathrm{Cl}(X)-$graded $R-$module such that $\widetilde{E} = \mathcal{E}$. $\widetilde{E}$ is coherent if and only if E is finitely generated.*

(iv) *$\widetilde{E} = 0$ if and only if $B^l E = 0$ for all $l \gg 0$.*

(v) *There is an exact sequence of $\mathrm{Cl}(X)-$graded modules*

$$0 \to H^0_B(E) \to E \to H^0_*(X, \widetilde{E}) \to H^1_B(E) \to 0.$$

Proof (i)–(iv) follow from [3, Proposition 5.3.3, Proposition 5.3.6, Proposition 5.3.7 and Proposition 5.3.10]. (v) follows from [8, Proposition 2.3]. □

The module $\Gamma E = H^0_*(X, \widetilde{E})$ is called the $B-$*saturation* of E. We say that E is $B-$*saturated* if $E \cong \Gamma E$, or equivalently if $H^0_B(E) = H^1_B(E) = 0$. If $H^0_B(E) = (0 :_E B^\infty) = 0$, we say that E is $B-$torsion free.

2.2 Equivariant Sheaves and Klyachko Filtrations

Let X be a smooth complete toric variety with fan Σ and $R = \mathbb{C}[x_1, \ldots, x_r]$ its associated $\mathrm{Cl}(X)-$graded Cox ring. In this subsection, we introduce the notion of a $\Sigma-$family to describe equivariant sheaves on X. We refer the reader to [19] and [14] for further details.

Definition 2.4 For any $t \in \mathbb{T}_N$, let $\mu_t : X \to X$ be the morphism given by the action of $\mathbb{T}_N$ on X. A quasi-coherent sheaf $\mathcal{E}$ on X is *equivariant* if there is a family of isomorphisms $\{\phi_t : \mu_t^* \mathcal{E} \cong \mathcal{E}\}_{t \in \mathbb{T}_N}$ such that $\phi_{t_1 \cdot t_2} = \phi_{t_2} \circ \mu_{t_2}^* \phi_{t_1}$ for any $t_1, t_2 \in \mathbb{T}_N$.

Notice that any $\mathbb{Z}^r-$graded $R-$module is also $\mathrm{Cl}(X)-$graded. In [2], Batyrev and Cox proved the following result:

Proposition 2.5 *Let E be a $\mathrm{Cl}(X)-$graded $R-$module. The quasi-coherent sheaf $\widetilde{E}$ is equivariant if and only if E is also $\mathbb{Z}^r-$graded.*

Proof See [2, Proposition 4.17]. □

In [14] and [15], Klyachko observed that to any equivariant torsion-free sheaf we can associate a family of filtered vector spaces, the so-called Klyachko filtration. In what follows we recall how this family can be constructed. Let $\mathcal{E}$ be an equivariant sheaf on X corresponding to a $\mathbb{Z}^r-$graded module E. For any degree $\alpha \in \mathrm{Cl}(X)$, the exact sequence (1) endows the homogeneous degree$-\alpha$ piece of E with an $M-$grading:

$$E_\alpha = \bigoplus_{m \in M} E_{z+\phi(m)}, \quad \text{for any} \quad z \in \pi^{-1}(\alpha).$$

Now, for any $\sigma \in \Sigma$ we consider the monomial $x^{\hat{\sigma}}$, and the localized $R_{x^\sigma}-$module E_{x^σ} remains $\mathbb{Z}^r-$graded. As before, for any $\alpha \in \mathrm{Cl}(X)$, $(E_{x^{\hat{\sigma}}})_\alpha$ is $M-$graded. In particular, taking $\alpha = 0$ we have:

$$E^\sigma := (E_{x^{\hat{\sigma}}})_0 = \bigoplus_{m \in M} (E_{x^{\hat{\sigma}}})_{\phi(m)} =: \bigoplus_{m \in M} E^\sigma_m. \tag{2}$$

Since $(E_{x^{\hat{\sigma}}})_0$ is isomorphic to the $\mathbb{C}[S_\sigma]-$module $\Gamma(U_\sigma, \mathcal{E})$, geometrically we can see (2) as the isotypical decomposition of $\Gamma(U_\sigma, \mathcal{E})$ into $\mathbb{T}_N-$eigenspaces of sections

$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m \in M} \Gamma(U_\sigma, \mathcal{E})_m.$$

Recall that the semigroup S_σ induces a preorder on the character lattice M: for any $m, m' \in M$ we say that $m \leq_\sigma m'$ if and only if $m' - m \in S_\sigma$, or equivalently if $\langle m' - m, u \rangle \geq 0$ for all $u \in \sigma$. For any two characters $m \leq_\sigma m'$, the multiplication by $\chi^{m'-m} \in \mathbb{C}[S_\sigma]$ yields the map

$$\chi^\sigma_{m,m'} : E^\sigma_m \to E^\sigma_{m'}.$$

For any $m \leq_\sigma m' \leq_\sigma m''$, we have

$$\chi^\sigma_{m,m} = 1 \quad \text{and} \quad \chi^\sigma_{m,m''} = \chi^\sigma_{m',m''} \circ \chi^\sigma_{m,m'}.$$

In particular, $\chi^\sigma_{m,m'}$ is an isomorphism if $m \leq_\sigma m'$ and $m' \leq_\sigma m$, or equivalently if $m' - m \in \sigma^\perp$. We call $\hat{E}^\sigma := \{E^\sigma_m, \chi^\sigma_{m,m'}\}$ a $\sigma-$*family* (see [19, Definition 4.2]).

On the other hand, let $\tau \prec \sigma$ be two cones in Σ and $m \in M$ the character such that $S_\tau = S_\sigma + \mathbb{Z}\langle m \rangle$. There are isomorphisms $\mathbb{C}[S_\tau] \cong \mathbb{C}[S_\sigma]_{\chi^m}$ and $E^\tau \cong E^\sigma_{\chi^m}$ given by the localization at χ^m. Thus, we have a morphism $i^{\sigma\tau} : E^\sigma \to E^\tau$, corresponding geometrically to the restriction map of section modules $\Gamma(U_\sigma, \mathcal{E}) \to \Gamma(U_\tau, \mathcal{E})$. For any character $m' \in M$, the morphism $i^{\sigma\tau}$ induces a linear map

$$i^{\sigma\tau}_{m'} : E^\sigma_{m'} \to E^\tau_{m'}.$$

We call $\{\hat{E}^\sigma\}_{\sigma\in\Sigma}$ a $\Sigma-$*family* (see [19, Definition 4.8]). In [19, Theorem 4.9] it is proved that $\Sigma-$families characterize equivariant sheaves on X or equivalently, $B-$saturated $R-$modules. When $\mathcal{E}$ is torsion-free, we have the following result.

Proposition 2.6 *Let $\mathcal{E}$ be an equivariant torsion-free sheaf of rank s and $\{\hat{E}^\sigma\}$ its associated $\Sigma-$family. The following holds:*

(i) *For any $m' \leq_\sigma m$, the linear map $\chi^\sigma_{m',m} : E^\sigma_{m'} \to E^\sigma_m$ is injective.*
(ii) *For any character $m \in M$, and any cones $\tau \prec \sigma$ in Σ, the linear map $i^{\sigma\tau}_m : E^\sigma_m \to E^\tau_m$ is injective.*
(iii) *There is a vector space $\mathbf{E} \cong \mathbb{C}^s$ such that $E^{\{0\}}_m \cong \mathbf{E}$ for any $m \in M$.*

We have the following commutative diagram:

$$\begin{array}{ccccc} & & E^{\{0\}}_{m'} & \longleftarrow & E^\sigma_{m'} \\ \mathbf{E} & \overset{\varphi_{m'}}{\swarrow}\ \cong\ \overset{\varphi_m}{\searrow} & \downarrow{\scriptstyle \chi^{\{0\}}_{m',m}} & & \downarrow{\scriptstyle \chi^\sigma_{m',m}} \\ & & E^{\{0\}}_m & \longleftarrow & E^\sigma_m. \end{array}$$

Moreover, for any character $m \in M$, we have

$$H^0(X,\mathcal{E})_m = \bigcap_{\sigma\in\Sigma_{\max}} E^\sigma_m.$$

Proof See [19, Section 4.4] and [15, Section 1.2 and 1.3]. □

Remark 2.7 (i) By Proposition 2.6, the $\Sigma-$family $\{\hat{E}^\sigma\}_{\sigma\in\Sigma}$ of a torsion-free sheaf $\mathcal{E}$ of rank s can be seen as a filtered collection of linear subspaces of a fixed ambient vector space $\mathbf{E}$. Geometrically, the vector space $\mathbf{E}$ can be identified with the $s-$dimensional vector space $\Gamma(\mathbb{T}_N,\mathcal{E})_m$ for any character $m \in M$.

(ii) The description of equivariant torsion-free sheaves given above is based on [19, Section 4]. We note that our order of filtrations is reverse of that of Klyachko [14, 15]. In these references, the filtration is taken as a collection of linear subspaces of $\mathcal{E}(x_0)$, the fiber of $\mathcal{E}$ at a point in the open orbit $U_{\{0\}} = \mathbb{T}_N \subset X$ (see [19, Remark 4.25]).

Definition 2.8 Let $\mathcal{E}$ be an equivariant torsion-free sheaf, the filtered collection of vector spaces $\{E^\sigma_m \mid m \in M\}_{\sigma\in\Sigma}$ given by its $\Sigma-$family is called the collection of *Klyachko filtrations* of $\mathcal{E}$.

In this note we focus on monomial ideals I in the $\mathrm{Cl}(X)-$graded Cox ring R. Since monomial ideals are naturally $\mathbb{Z}^r-$graded they correspond to torsion-free equivariant sheaves of rank 1. Therefore, Proposition 2.6 shows that the $\Sigma-$family of a monomial ideal I is structured as a system of vector space filtrations of a $1-$dimensional vector space $\mathbf{I} \cong \mathbb{C}$, which can be identified with $I^{\{0\}}_m$ for any character $m \in M$.

Remark 2.9 Let $\{I^\sigma_m \mid m \in M\}_{\sigma\in\Sigma}$ be the collection of Klyachko filtrations of a monomial ideal I. Let $m_0 \in M$ be a character and identify $\mathbf{I}$ with $I^{\{0\}}_{m_0}$. For each $\sigma \in \Sigma$ and $m \in M$, the linear subspace $I^\sigma_m \subset \mathbf{I}$ can be either $I^\sigma_m \cong \mathbf{I}$ or $I^\sigma_m = 0$. Therefore, the collection of

Klyachko filtrations of a monomial ideal is characterized by attaching to each cone $\sigma \in \Sigma$, the set of characters $\{m \in M \mid I_m^\sigma \neq 0\}$.

We finish this preliminary section with an example which illustrates Proposition 2.6, and shows how to compute the collection of Klyachko filtrations of a monomial ideal.

Example 2.10 Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$ with fan Σ as in Example 2.2(i). Consider the monomial ideal $I = (x_2^2, x_0x_2, x_0x_1)$, we will compute the $\Sigma-$family associated to I. We present I as follows:

$$R(0,0,-2) \oplus R(-1,0,-1) \oplus R(-1,-1,0) \xrightarrow{(x_2^2\ x_0x_2\ x_0x_1)} I \to 0. \tag{3}$$

Next, we localize at $x^{\widehat{\{0\}}} = x_0x_1x_2$ and we set $R^{\{0\}} := R_{x^{\widehat{\{0\}}}}$ the localized ring. For any multidegree $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3$, $R^{\{0\}}_{(\alpha_0,\alpha_1,\alpha_2)} = \mathbb{C}\langle x_0^{\alpha_0}x_1^{\alpha_1}x_2^{\alpha_2}\rangle$, the vector space spanned by the monomial $x_0^{\alpha_0}x_1^{\alpha_1}x_2^{\alpha_2}$. On the other hand, any character $m = (d_1, d_2)$ is embedded as $m = (-d_1-d_2, d_1, d_2)$ in $\mathbb{Z}^3$ via the exact sequence (1). To compute $I_m^{\{0\}}$ we take the degree m component of (3). This yields the following exact sequence of vector spaces

$$R^{\{0\}}(0,0,-2)_m \oplus R^{\{0\}}(-1,0,-1)_m \oplus R^{\{0\}}(-1,-1,0)_m \xrightarrow{(x_2^2\ x_0x_2\ x_0x_1)} I_m^{\{0\}} \to 0.$$

Thus, $I_m^{\{0\}} = \mathbb{C}\langle x_0^{-d_1-d_2}x_1^{d_1}x_2^{d_2}\rangle$ and there are isomorphisms $\phi_m^{\{0\}} : I_m^{\{0\}} \cong \mathbf{I}$. Let us now fix the ray $\rho_0 \in \Sigma(1)$ and compute $I_m^{\rho_0}$ for any character $m = (d_1, d_2) \in \mathbb{Z}^2$. As before, we set $R^{\rho_0} := R_{x^{\widehat{\rho_0}}}$ the localization at $x^{\widehat{\rho_0}} = x_1x_2$. Now, for any multidegree $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3$,

$$R^{\rho_0}_{(\alpha_0,\alpha_1,\alpha_2)} = \begin{cases} \mathbb{C}\langle x_0^{\alpha_0}x_1^{\alpha_1}x_2^{\alpha_2}\rangle, & \text{if} \quad \alpha_0 \geq 0 \\ 0, & \text{if} \quad \alpha_0 \leq -1 \end{cases}$$

and restricting the exact sequence (3) to degree $m = (d_1, d_2)$, we have

$$I_m^{\rho_0} = \begin{cases} \mathbb{C}\langle x_0^{-d_1-d_2}x_1^{d_1}x_2^{d_2}\rangle \cong \mathbf{I}, & \text{if} \quad -d_1-d_2 \geq 0 \\ 0, & \text{if} \quad -d_1-d_2 \leq -1. \end{cases}$$

Similarly, we obtain

$$I_m^{\rho_1} \cong \begin{cases} \mathbf{I}, & \text{if} \quad d_1 \geq 0 \\ 0, & \text{if} \quad d_1 \leq -1 \end{cases} \qquad I_m^{\rho_2} \cong \begin{cases} \mathbf{I}, & \text{if} \quad d_2 \geq 0 \\ 0, & \text{if} \quad d_2 \leq -1. \end{cases}$$

It only remains to compute the components in the $\Sigma-$family associated to the two dimensional cones in Σ. Let us consider $\sigma_0 \in \Sigma(2)$ with rays $\sigma_0(1) = \{\rho_1, \rho_2\}$. We set $R^{\sigma_0} := R_{x^{\widehat{\sigma_0}}}$ the localization at $x^{\widehat{\sigma_0}} = x_0$ and for any multidegree $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}^3$,

$$R^{\sigma_0}_{(\alpha_0,\alpha_1,\alpha_2)} = \begin{cases} \mathbb{C}\langle x_0^{\alpha_0}x_1^{\alpha_1}x_2^{\alpha_2}\rangle, & \text{if} \quad \alpha_1 \geq 0,\ \alpha_2 \geq 0 \\ 0, & \text{if} \quad \alpha_1 \leq -1 \text{ or } \alpha_2 \leq -1. \end{cases}$$

As before, taking the component of degree $m = (d_1, d_2)$ of (3) we obtain

$$I_m^{\sigma_0} \cong \begin{cases} \mathbf{I}, & \text{if } d_1 = 0 \text{ and } d_2 \geq 1, \text{ or} \\ & d_1 \geq 1 \text{ and } d_2 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we obtain the remaining components of the $\Sigma-$family:

$$I_m^{\sigma_1} \cong \begin{cases} \mathbf{I}, & \text{if } -d_1 - d_2 = 0 \text{ and } d_2 \geq 2, \text{ or} \\ & -d_1 - d_2 \geq 1 \text{ and } d_2 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$I_m^{\sigma_2} \cong \begin{cases} \mathbf{I}, & \text{if } -d_1 - d_2 \geq 0 \text{ and } d_1 \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

3 Klyachko Diagrams of Monomial Ideals

In this section, we focus our attention on monomial ideals I in the Cox ring R of a smooth complete toric variety X. Using the theory of Klyachko filtrations, we define the *Klyachko diagram* of I, and we show how it is determined combinatorially by the monomials generating I. Conversely, we give a method to compute a minimal set of generators of a $B-$saturated monomial ideal I from its Klyachko diagram. Finally, we compute the first local cohomology module $H^1_B(I)$ for any monomial ideal I using its Klyachko diagram.

From now on, we fix a smooth complete toric variety X with fan Σ. We set $r = |\Sigma(1)|$, we denote by $R = \mathbb{C}[x_1, \ldots, x_r]$ its associated $\mathrm{Cl}(X)-$graded Cox ring and by B its irrelevant ideal.

3.1 From a Monomial Ideal to a Klyachko Diagram

Let $I = (m_1, \ldots, m_t)$ be a monomial ideal. We write the monomials

$$m_i = x_1^{k_1^i} \cdots x_r^{k_r^i} =: \underline{x}^{\underline{k}^i}, \quad \text{for} \quad \underline{k}^i := (k_1^i, \ldots, k_r^i) \in \mathbb{Z}^r_{\geq 0}, \quad \text{and} \quad 1 \leq i \leq t.$$

and we present I as the image of a $\mathbb{Z}^r-$graded map as follows:

$$\bigoplus_{i=1}^{t} R(-\underline{k}^i) \xrightarrow{(m_1, \ldots, m_t)} I \longrightarrow 0. \tag{4}$$

Thus, for any character $m = (d_1, \ldots, d_n)$, $I_m^{\{0\}} \cong \mathbb{C}\langle x_1^{\langle m, n(\rho_1)\rangle} \cdots x_r^{\langle m, n(\rho_r)\rangle}\rangle$ and there are isomorphisms $\phi_m^{\{0\}} : I_m^{\{0\}} \cong \mathbf{I}$. Our first objective is to describe the subspaces $I_m^\sigma \subset \mathbf{I}$ for any character $m = (d_1, \ldots, d_n)$ and any cone $\sigma \in \Sigma$. As observed in Remark 2.9, we want to characterize the sets of characters $\{m \in M \mid I_m^\sigma \neq 0\}$ for each cone $\sigma \in \Sigma$. Each of this sets can be seen as the staricase diagram for the inclusion $I^\sigma \subset R^\sigma$ as $\mathbb{C}[S_\sigma]-$modules.

Lemma 3.1 *Let $I = (\underline{x}^{\underline{k}}) \subset R$, with $\underline{k} \in \mathbb{Z}^r_{\geq 0}$, be an ideal generated by a single monomial. Then, for any cone $\sigma = \mathrm{cone}(\rho_{i_1}, \ldots, \rho_{i_c}) \in \Sigma$,*

$$I_m^\sigma \cong \begin{cases} \mathbf{I}, & \textit{if } m \in \mathcal{C}_{\underline{k}}^\sigma \\ 0, & \textit{otherwise}, \end{cases}$$

where $\mathcal{C}_{\underline{k}}^\sigma := \{m \in M \mid \langle m, \rho_{i_j}\rangle \geq k_{i_j}, \textit{ for } 1 \leq j \leq c\}$.

Proof The lemma follows from (4), when $t = 1$ and using that

$$R_m^\sigma(-\underline{k}) = \begin{cases} \mathbb{C}\langle x_1^{\langle m, \rho_1\rangle - k_1} \cdots x_r^{\langle m, \rho_r\rangle - k_r}\rangle, \text{ if } \langle m, \rho_{i_j}\rangle - k_{i_j} \geq 0, \text{ for } 1 \leq j \leq c \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

□

We set $\mathcal{C}_0^\sigma := \mathcal{C}^\sigma_{(0,\dots,0)}$, and notice the inclusion $\mathcal{C}^\sigma_{\underline{k}} \subset \mathcal{C}_0^\sigma$ corresponding to $(\underline{x}^{\underline{k}})^\sigma \subset R^\sigma$ for any $\underline{k} \in \mathbb{Z}^r$. Applying Lemma 3.1, repeatedly, we have:

Proposition 3.2 *Let $I = (m_1, \dots, m_t) \subset R$ be a monomial ideal with $m_i = \underline{x}^{\underline{k}^i}$ for $\underline{k}^i \in \mathbb{Z}^r_{\geq 0}$ and $1 \leq i \leq t$. Then, for any cone $\sigma \in \Sigma$,*

$$I_m^\sigma \cong \begin{cases} \mathbf{I}, & \text{if } m \in \bigcup_{i=1}^t \mathcal{C}^\sigma_{\underline{k}^i} \\ 0, & \text{otherwise.} \end{cases}$$

Proof It follows from (4) that $I_m^\sigma = 0$ if and only if $R_m^\sigma(-\underline{k}^i) = 0$ for all $1 \leq i \leq t$. By Lemma 3.1 this occurs if and only if $m \in M \setminus \bigcup_{i=1}^t \mathcal{C}^\sigma_{\underline{k}}$, and the result follows. □

Notice that Proposition 3.2 already gives a description of the collection of Klyachko filtrations of a monomial ideal. However, the information on the inclusion $I^\sigma \subset R^\sigma$ is encoded in

$$\mathcal{C}^0 \setminus \bigcup_{i=1}^t \mathcal{C}^\sigma_{\underline{k}^i} = \bigcap_{i=1}^t (\mathcal{C}_0^\sigma \setminus \mathcal{C}^\sigma_{\underline{k}^i}) = \bigcap_{i=1}^t \bigcup_{j=1}^c \{m \in M \mid 0 \leq \langle m, \rho_{i_j} \rangle < k^i_{i_j}\},$$

which is the union of c^t sets. Indeed, for each $1 \leq j_1, \dots, j_t \leq c$,

$$P_{j_1,\dots,j_t} := \{m \in M \mid 0 \leq \langle m, \rho_{i_{j_1}} \rangle < k^1_{j_1}, \dots, 0 \leq \langle m, \rho_{i_{j_t}} \rangle < k^t_{j_t}\}.$$

The *Klyachko diagram* defined below is used in Proposition 3.4 to give a more compact alternative characterization of the collection of Klyachko filtrations of a monomial ideal. We attach to the monomial ideal I a collection of pairs $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma \in \Sigma}$ constructed as follows.

For any ray $\rho_j \in \Sigma(1)$, we set $s_j = s_{\rho_j} := \min\{\deg_{\rho_j}(m_1), \dots, \deg_{\rho_j}(m_t)\}$. We write $\underline{s} := (s_1, \dots, s_r)$, and for any $c-$dimensional cone $\sigma = \text{cone}(\rho_{i_1}, \dots, \rho_{i_c})$, we set

$$\mathcal{C}_I^\sigma := \mathcal{C}^\sigma_{\underline{s}} = \{m \in M \mid \langle m, \rho_{i_p} \rangle \geq s_{i_p},\ 1 \leq p \leq c\} = \bigcap_{j=1}^c \mathcal{C}_I^{\rho_{i_j}}.$$

Next, we construct Δ_I^σ. First, for any subset of monomials $\mathcal{S} = \{n_1, \dots, n_s\} \subset \{m_1, \dots, m_t\}$ with $0 \leq s \leq t$, we define $\Delta_I^\sigma(\mathcal{S}) \subset \mathcal{C}_I^\sigma$ recursively on s:

If $s = 0$, then $\mathcal{S} = \emptyset$ and we set:

$$\Delta_I^\sigma(\emptyset) := \{m \in M \mid s_{i_1} \leq \langle m, \rho_{i_1} \rangle, \dots, s_{i_c} \leq \langle m, \rho_{i_c} \rangle\}.$$

Otherwise, $s \geq 1$ and there is a permutation $\epsilon_{i_c} \in \mathfrak{S}_s$ such that

$$\deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(1)}) \leq \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(2)}) \leq \cdots \leq \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(s)}).$$

- If $c = 1$ (and thus σ is a ray), then

$$\Delta_I^\sigma(\mathcal{S}) := \{m \in M \mid s_{i_1} \leq \langle m, \rho_{i_1} \rangle < \deg_{\rho_{i_1}}(n_{\epsilon_{i_1}(1)})\}.$$

- Otherwise, $\Delta_I^\sigma(\mathcal{S}) := \bigcup_{j=0}^{s} \Delta_I^\sigma(\mathcal{S})_j$ where:

$$\Delta_I^\sigma(\mathcal{S})_0 := \{m \in M \mid s_{i_c} \leq \langle m, \rho_{i_c}\rangle < \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(1)})\} \cap \Delta_I^{\sigma'}(\emptyset),$$

$$\Delta_I^\sigma(\mathcal{S})_j := \{m \in M \mid \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(j)}) \leq \langle m, \rho_{i_c}\rangle < \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(j+1)}))\} \cap \Delta_I^{\sigma'}(\{n_{\epsilon_{i_c}(1)}, \ldots, n_{\epsilon_{i_c}(j)}\}),\ 1 \leq j \leq s-1,$$

$$\Delta_I^\sigma(\mathcal{S})_s := \{m \in M \mid \deg_{\rho_{i_c}}(n_{\epsilon_{i_c}(s)}) \leq \langle m, \rho_{i_c}\rangle\} \cap \Delta_I^{\sigma'}(\{n_{\epsilon_{i_c}(1)}, \ldots, n_{\epsilon_{i_c}(s)}\}),$$

with $\sigma' = \text{cone}(\rho_{i_1}, \ldots, \rho_{i_{c-1}})$.

Finally, we define $\Delta_I^\sigma := \Delta_I^\sigma(\{m_1, \ldots, m_t\})$.

Definition 3.3 We call the collection of pairs $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$ the *Klyachko diagram* of I.

Observe that each of the pairs $(\mathcal{C}_I^\sigma, \Delta_I^\sigma)$ depicts a staircase diagram of the inclusion $I^\sigma \subset R^\sigma$ (see also Example 3.5 below). Precisely, we have the following proposition showing that the Klyachko diagram characterizes the $\Sigma-$family of I.

Proposition 3.4 *Let $I = (m_1, \ldots, m_t) \subset R$ be a monomial ideal with $m_i = \underline{x}^{\underline{k}^i}$ for $\underline{k}^i \in \mathbb{Z}^r_{\geq 0}$ and $1 \leq i \leq t$. Let $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$ be the Klyachko diagram of I. Then, for any cone $\sigma \in \Sigma$,*

$$I_m^\sigma \cong \begin{cases} \mathbf{I}, & m \in \mathcal{C}_I^\sigma \setminus \Delta_I^\sigma \\ 0, & \textit{otherwise.} \end{cases}$$

In particular it holds

$$\bigcup_{i=1}^{t} \mathcal{C}^\sigma_{\underline{k}^i} = \mathcal{C}_I^\sigma \setminus \Delta_I^\sigma \quad \textit{and} \quad \mathcal{C}_0^\sigma \setminus \bigcup_{i=1}^{t} \mathcal{C}^\sigma_{\underline{k}^i} = \Delta_I^\sigma \cup (\mathcal{C}_0^\sigma \setminus \mathcal{C}_I^\sigma). \tag{5}$$

Before proving this result let us see an example that illustrate Proposition 3.4, and shows how to compute the Klyachko diagram of a monomial ideal.

Example 3.5 Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$ with fan Σ as in Example 2.2(i), and let $I = (x_2^2, x_0x_2, x_0x_1)$ be the monomial ideal of Example 2.10. First, we compute $s_0 = s_1 = s_2 = 0$ and we have

$$\mathcal{C}_I^{\rho_0} = \{(d_1, d_2) \mid d_1 + d_2 \leq 0\}\ \mathcal{C}_I^{\rho_1} = \{(d_1, d_2) \mid d_1 \geq 0\}$$
$$\mathcal{C}_I^{\rho_2} = \{(d_1, d_2) \mid d_2 \geq 0\}.$$

We compute $\Delta_I^{\sigma_0}$. We order the monomials with respect to ρ_2: $\deg_{\rho_2}(x_0x_1) = 0 < \deg_{\rho_2}(x_0x_2) = 1 < \deg_{\rho_2}(x_2^2) = 2$. We obtain

$$\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_0 = \emptyset$$
$$\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_1 = \{(d_1, d_2) \mid d_2 = 0\} \cap \Delta^{\rho_1}(\{x_0x_1\})\}$$
$$\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_2 = \{(d_1, d_2) \mid d_2 = 1\} \cap \Delta^{\rho_1}(\{x_0x_1, x_0x_2\})$$
$$\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_3 = \emptyset.$$

Since $\Delta_I^{\rho_1}(\{x_0x_1\}) = \{(d_1, d_2) \mid d_1 = 0\}$ and $\Delta_I^{\rho_1}(\{x_0x_1, x_0x_2\}) = \emptyset$, we get $\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_1 = \{(0, 0)\}$ and $\Delta_I^{\sigma_0}(\{x_2^2, x_0x_2, x_0x_1\})_2 = \emptyset$. Therefore, $\Delta_I^{\sigma_0} =$

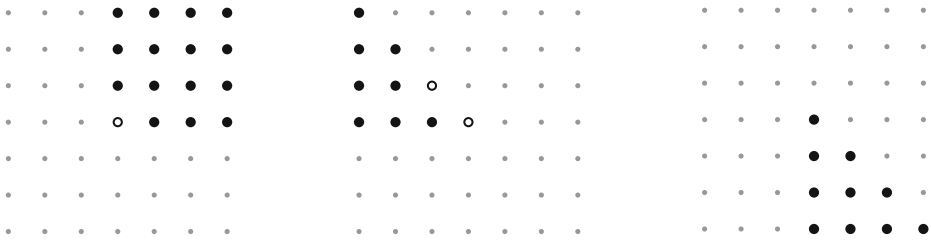

Fig. 1 ∘ stand for points of $\Delta_I^{\sigma_0}$ (respectively $\Delta_I^{\sigma_1}$ and $\Delta_I^{\sigma_2}$) inside the points • of $\mathcal{C}_I^{\sigma_0}$ (respectively $\mathcal{C}_I^{\sigma_1}$ and $\mathcal{C}_I^{\sigma_2}$)

$\{(0,0)\}$. Similarly, for the remaining cones we compute $\Delta_I^{\sigma_1} = \{(0,0),(-1,1)\}$ and $\Delta_I^{\sigma_2} = \emptyset$. (See Figs. 1 and 2).

By Proposition 3.4, we have

$$I_m^{\rho_0} \cong \begin{cases} \mathbf{I},\ m \in \{d_1 + d_2 \leq 0\} \\ 0, \text{otherwise.} \end{cases} \qquad I_m^{\rho_1} \cong \begin{cases} \mathbf{I},\ m \in \{d_1 \geq 0\} \\ 0, \text{otherwise.} \end{cases}$$

$$I_m^{\rho_2} \cong \begin{cases} \mathbf{I},\ m \in \{d_2 \geq 0\} \\ 0, \text{otherwise.} \end{cases} \qquad I_m^{\sigma_0} \cong \begin{cases} \mathbf{I},\ m \in \begin{cases} d_1 \geq 0 \\ d_2 \geq 0 \end{cases} \backslash \{(0,0)\} \\ 0, \text{otherwise.} \end{cases}$$

$$I_m^{\sigma_1} \cong \begin{cases} \mathbf{I},\ m \in \begin{cases} d_1 + d_2 \leq 0 \\ d_2 \geq 0 \end{cases} \backslash \begin{cases} (0,0) \\ (-1,1) \end{cases} \\ 0, \text{otherwise.} \end{cases} \qquad I_m^{\sigma_2} \cong \begin{cases} \mathbf{I},\ m \in \begin{cases} d_1 \geq 0 \\ d_1 + d_2 \leq 0 \end{cases} \\ 0, \text{otherwise} \end{cases}$$

which coincides with the $\Sigma-$family computed in Example 2.10.

Proof of Proposition 3.4 First, we recall that for $1 \leq j \leq t$,

$$R_m^{\rho_j} = \begin{cases} \mathbb{C}\langle x_1^{\langle m, n(\rho_1)\rangle} \cdots x_r^{\langle m, n(\rho_r)\rangle}\rangle, & \langle m, n(\rho_j)\rangle \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

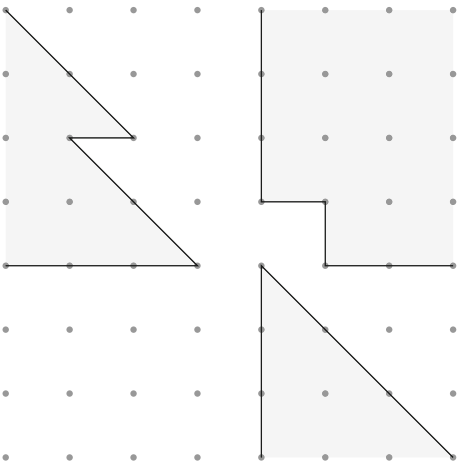

Fig. 2 The Klyachko diagram of Fig. 1 represented together in a single figure. The shadowed region corresponds to each set $\mathcal{C}_I^{\sigma_i} \setminus \Delta_I^{\sigma_i}$ for $i = 0, 1, 2$

If $\sigma = \rho_j$ is a ray, then we have

$$I_m^{\rho_j} \cong \begin{cases} \mathbf{I}, & \langle m, n(\rho_j)\rangle \geq s_j \\ 0, & \langle m, n(\rho_j)\rangle < s_j. \end{cases}$$

Otherwise, $\sigma = \operatorname{cone}(\rho_{i_1}, \ldots, \rho_{i_c})$ for some $2 \leq c \leq r$. Assume that $m \in \Delta_I^\sigma(\{m_1, \ldots, m_t\})_j$ for some $1 \leq j \leq t$ and $\epsilon_{i_c} \in \mathfrak{S}_t$ as above. Therefore,

$$\deg_{\rho_{i_c}}(m_{\epsilon_{i_c}(j)}) \leq \langle m, \rho_{i_c}\rangle < \deg_{\rho_{i_c}}(m_{\epsilon_{i_c}(j+1)})).$$

In particular, if $j < t$, we have $R_m^\sigma(-\underline{k}^{\epsilon_{i_c}(j+1)}) = \cdots = R_m^\sigma(-\underline{k}^{\epsilon_{i_c}(t)}) = 0$. It suffices to see that $R_m^\sigma(-\underline{k}^{\epsilon_{i_c}(p)}) = 0$ for $1 \leq p \leq j$. On the other hand, $m \in \Delta_I^{\sigma'}(\{m_{\epsilon_{i_c}(1)}, \ldots, m_{\epsilon_{i_c}(j)}\})$. We repeat the same argument for $\sigma' = \operatorname{cone}(\rho_{i_1}, \ldots, \rho_{i_{c-1}})$ and the set of monomials $\{m_{\epsilon_{i_c}(1)}, \ldots, m_{\epsilon_{i_c}(j)}\}$, and so on. This procedure stops either when $\dim(\sigma') = 1$ or when the set of monomials is empty. In the latter case, it follows straightforward that we arrived at $R_m^\sigma(-\underline{k}^{\epsilon_{i_1}(1)}) = \cdots = R_m^\sigma(-\underline{k}^{\epsilon_{i_1}(t)}) = 0$ and hence $I_m^\sigma = 0$. In the case $\dim(\sigma') = 1$, assume the set of monomials is $\{n_{\epsilon_{i_1}(1)}, \ldots, n_{\epsilon_{i_1}(s)}\} \subset \{m_1, \ldots, m_t\}$ with $s \leq j$. Since we have

$$s_{i_1} \leq \langle m, \rho_{i_1}\rangle < \deg_{\rho_{i_1}}(n_{\epsilon_{i_1}(1)}),$$

then $R_m^\sigma(-\underline{k}^{\epsilon_{i_1}(1)}) = \cdots = R_m^\sigma(-\underline{k}^{\epsilon_{i_1}(s)}) = 0$, and we obtain $I_m^\sigma = 0$. Analogously, if $m \in \mathcal{C}_I^\sigma \setminus \Delta_I^\sigma$ we get $R_m^\sigma(-\underline{k}^p) \neq 0$ for some $1 \leq p \leq t$, so $I_m^\sigma \cong \mathbf{I}$.

Finally, (5) follows from a comparison with the description of I_m^σ in Proposition 3.2. □

Combining Propositions 3.2 and 3.4, the next result shows how to obtain the Klyachko diagram of the sum of two monomial ideals.

Corollary 3.6 *Let $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}$ and $\{(\mathcal{C}_J^\sigma, \Delta_J^\sigma)\}$ be the Klyachko diagrams of two monomial ideals I and J, respectively. Then, the Klyachko diagram of $I + J$ is given by*

$$\begin{cases} \mathcal{C}_{I+J}^\sigma = \{m \in M \mid \langle m, \rho\rangle \geq \min\{s_\rho^I, s_\rho^J\},\ \rho \in \sigma(1)\} \\ \Delta_{I+J}^\sigma = (\Delta_I^\sigma \cap \Delta_J^\sigma) \cup (\Delta_I \cap (\mathcal{C}_{I+J}^\sigma \setminus \mathcal{C}_J^\sigma)) \cup (\Delta_J \cap (\mathcal{C}_{I+J}^\sigma \setminus \mathcal{C}_I^\sigma)) \cup \\ \qquad (\mathcal{C}_{I+J}^\sigma \setminus (\mathcal{C}_I^\sigma \cup \mathcal{C}_J^\sigma)). \end{cases}$$

Proof We write $I = (\underline{x}^{\underline{k}^1}, \ldots, \underline{x}^{\underline{k}^t})$ and $J = (\underline{x}^{\underline{l}^1}, \ldots, \underline{x}^{\underline{l}^s})$. Then, $s_j^I = \min\{k_j^1, \ldots, k_j^t\}$, $s_j^J = \min\{l_j^1, \ldots, l_j^s\}$ and $I + J = (\underline{x}^{\underline{k}^1}, \ldots, \underline{x}^{\underline{k}^t}, \underline{x}^{\underline{l}^1}, \ldots, \underline{x}^{\underline{l}^s})$. It follows that $s_j^{I+J} = \min\{s_j^I, s_j^J\}$ and then

$$\mathcal{C}_{I+J}^\sigma = \{m \in M \mid \langle m, \rho\rangle \geq \min\{s_\rho^I, s_\rho^J\},\ \rho \in \sigma(1)\}.$$

In particular, $\mathcal{C}_I^\sigma, \mathcal{C}_J^\sigma$ are contained in $\mathcal{C}_{I+J}^\sigma$. By Propositions 3.2 and 3.4 we have $\mathcal{C}_{I+J}^\sigma \setminus \Delta_{I+J}^\sigma = (\mathcal{C}_I^\sigma \setminus \Delta_I^\sigma) \cup (\mathcal{C}_J^\sigma \setminus \Delta_J^\sigma)$. Taking complementaries with respect to $\mathcal{C}_{I+J}^\sigma$ it yields

$$\begin{aligned} \Delta_{I+J}^\sigma &= \mathcal{C}_{I+J}^\sigma \setminus ((\mathcal{C}_I^\sigma \setminus \Delta_I^\sigma) \cup (\mathcal{C}_J^\sigma \setminus \Delta_J^\sigma)) \\ &= (\mathcal{C}_{I+J}^\sigma \setminus (\mathcal{C}_I^\sigma \setminus \Delta_I^\sigma)) \cap (\mathcal{C}_{I+J}^\sigma \setminus (\mathcal{C}_J^\sigma \setminus \Delta_J^\sigma)) \\ &= (\Delta_I^\sigma \cup (\mathcal{C}_{I+J}^\sigma \setminus \mathcal{C}_I^\sigma)) \cap (\Delta_J^\sigma \cup (\mathcal{C}_{I+J}^\sigma \setminus \mathcal{C}_J^\sigma)), \end{aligned}$$

and the result follows. □

The following example illustrates Corollary 3.6.

Example 3.7 Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$ with fan Σ as in Example 2.2(i), and let $I = (x_1^2x_2^4, x_1^3x_2, x_1^5)$ and $J = (x_2^4, x_1x_2^3, x_1^4x_2^2)$ be two monomial ideals. Notice

that $(s_0^I, s_1^I, s_2^I) = (0, 2, 0)$ and $(s_0^J, s_1^J, s_2^J) = (0, 0, 2)$, so $(s_0^{I+J}, s_1^{I+J}, s_2^{I+J}) = (0, 0, 0)$. Hence, $\mathcal{C}_I^{\sigma_0} = \{d_1 \geq 2, d_2 \geq 0\}$, $\mathcal{C}_J^{\sigma_0} = \{d_1 \geq 0, d_2 \geq 2\}$ and $\mathcal{C}_{I+J}^{\sigma_0} = \{d_1 \geq 0, d_2 \geq 0\}$; while $\mathcal{C}_I^{\sigma_1} = \mathcal{C}_J^{\sigma_1} = \mathcal{C}_{I+J}^{\sigma_1} = \{d_1 + d_2 \leq 0, d_2 \geq 0\}$ and $\mathcal{C}_I^{\sigma_2} = \mathcal{C}_J^{\sigma_2} = \mathcal{C}_{I+J}^{\sigma_2} = \{d_1 + d_2 \leq 0, d_1 \geq 0\}$.

Computing the remaining Klyachko diagrams of I and J we obtain:

$$\begin{array}{ll} \Delta_I^{\sigma_0} = \{(2,0), (2,1), (2,2), (2,3), (3,0), (4,0)\}, & \Delta_I^{\sigma_1} = \Delta_I^{\sigma_2} = \emptyset \\ \Delta_J^{\sigma_0} = \{(0,2), (0,3), (1,2), (2,2), (3,2)\}, & \Delta_J^{\sigma_1} = \Delta_J^{\sigma_2} = \emptyset. \end{array}$$

Thus, applying Corollary 3.6 we get $\Delta_{I+J}^{\sigma_1} = \Delta_{I+J}^{\sigma_2} = \emptyset$ and

$$\begin{aligned} \Delta_{I+J}^{\sigma_0} = \{&(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2), \\ &(3,0), (4,0)\}. \end{aligned}$$

Figure 3 illustrates this example.

We end this subsection with more examples on the compution of Klyachko diagrams.

Example 3.8 Let $R = \mathbb{C}[x_0, x_1, x_2, x_3]$ be the Cox ring of $\mathbb{P}^3$ with fan Σ as in Example 2.2(i), and let $I = (x_0x_1, x_1x_2x_3^2, x_2^2)$ be a monomial ideal. We have $s_0 = s_1 = s_2 = s_3 = 0$ and

$$\begin{array}{ll} \mathcal{C}_I^{\rho_0} = \{(d_1, d_2, d_3) \mid d_1 + d_2 + d_3 \leq 0\} & \mathcal{C}_I^{\rho_1} = \{(d_1, d_2, d_3) \mid d_1 \geq 0\} \\ \mathcal{C}_I^{\rho_2} = \{(d_1, d_2, d_3) \mid d_2 \geq 0\} & \mathcal{C}_I^{\rho_3} = \{(d_1, d_2, d_3) \mid d_3 \geq 0\}. \end{array}$$

We compute $\Delta_I^{\sigma_0}$. We order the monomials with respect to ρ_3: $\deg_{\rho_3}(x_0x_1) = \deg_{\rho_3}(x_2^2) = 0 < \deg_{\rho_3}(x_1x_2x_3^2) = 2$, and we obtain

$$\begin{aligned} \Delta_I^{\sigma_0}(\mathcal{G})_0 &= \emptyset \\ \Delta_I^{\sigma_0}(\mathcal{G})_1 &= \emptyset \\ \Delta_I^{\sigma_0}(\mathcal{G})_2 &= \{(d_1, d_2, d_3) \mid 0 \leq d_3 < 2\} \cap \Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\}) \\ \Delta_I^{\sigma_0}(\mathcal{G})_3 &= \{(d_1, d_2, d_3) \mid 2 \leq d_3\} \cap \Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2, x_1x_2x_3^2\}), \end{aligned}$$

where $\mathcal{G} = \{x_0x_1, x_1x_2x_3^2, x_2^2\}$ and $\sigma_0' = \text{cone}(\rho_1, \rho_2)$. We proceed computing $\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\})$, ordering the two monomials with respect to ρ_2: $\deg_{\rho_2}(x_0x_1) = 0 <$

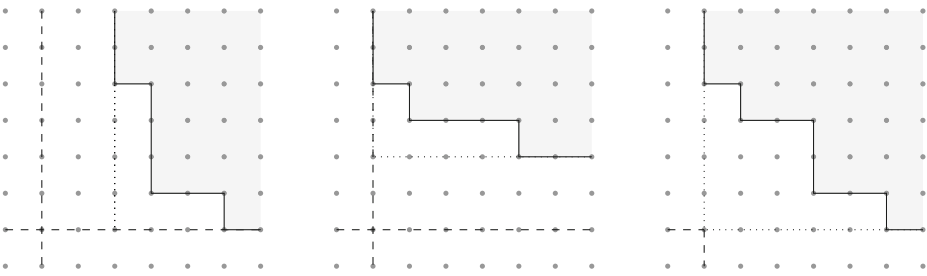

Fig. 3 The part of the Klyachko diagram associated to the cone σ_0 of I, J and $I + J$ respectively. The dotted part corresponds to $\mathcal{C}_\bullet^{\sigma_0}$ and the shadowed part to $\mathcal{C}_\bullet^{\sigma_0} \setminus \Delta_\bullet^{\sigma_0}$

$\deg_{\rho_2}(x_2^2) = 2$. We get

$$\begin{aligned}
&\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\})_0 = \emptyset\\
&\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\})_1 = \{(d_1, d_2, d_3) \mid 0 \le d_2 < 2\} \cap \Delta_I^{\rho_1}(\{x_0x_1\})\\
&\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\})_2 = \{(d_1, d_2, d_3) \mid 2 \le d_2\} \cap \Delta_I^{\rho_1}(\{x_0x_1, x_2^2\}),
\end{aligned}$$

and $\Delta_I^{\rho_1}(\{x_0x_1\}) = \{(d_1, d_2, d_3) \mid d_1 = 0\}$, while $\Delta_I^{\rho_1}(\{x_0x_1, x_2^2\}) = \emptyset$. Hence, $\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2\}) = \{(d_1, d_2, d_3) \mid 0 \le d_2 < 2,\ d_1 = 0\}$, and

$$\Delta_I^{\sigma_0}(\mathcal{G})_2 = \{(d_1, d_2, d_3) \mid 0 \le d_3 < 2,\ 0 \le d_2 < 2,\ d_1 = 0\}.$$

Similarly, we obtain $\Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2, x_1x_2x_3^2\}) = \Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2, x_1x_2x_3^2\})_1 \cup \Delta_I^{\sigma_0'}(\{x_0x_1, x_2^2, x_1x_2x_3^2\})_2 = \{(d_1, d_2, d_3) \mid 0 \le d_2 \le 1,\ d_1 = 0\}$. Hence,

$$\Delta_I^{\sigma_0}(\mathcal{G})_3 = \{(d_1, d_2, d_3) \mid 2 \le d_3,\ 0 \le d_2 \le 1,\ d_1 = 0\},\ \text{and}$$

$$\Delta_I^{\sigma_0} = \{(0, d_2, d_3) \mid 0 \le d_3 \le 1,\ 0 \le d_2 \le 1\} \cup \{(0, d_2, d_3) \mid 2 \le d_3,\ 0 \le d_2 \le 1\}.$$

Applying the same procedure for the remaining cones, we get

$$\begin{aligned}
&\Delta_I^{\sigma_1} = \{(-d_2 - d_3, d_2, d_3) \mid 0 \le d_2, d_3 \le 1\} \cup \{(-d_3, 0, d_3) \mid 2 \le d_3\}\\
&\Delta_I^{\sigma_2} = \emptyset\\
&\Delta_I^{\sigma_3} = \{(0, d_2, d_3) \mid 0 \le d_2 \le 1,\ d_3 \le -d_2\} \cup \{(d_1, 0, -d_1) \mid 0 \le d_1\}.
\end{aligned}$$

We notice that in this example $\Delta_I^{\sigma_1}$ and $\Delta_I^{\sigma_3}$ are unbounded.

Example 3.9 Let $R = \mathbb{C}[x_0, x_1, y_0, y_1]$ be the Cox ring of the Hirzebruch surface $\mathcal{H}_3$ with fan Σ as in Example 2.2(ii). R is endowed with a $\mathbb{Z}^2-$grading such that $\deg(x_0) = \deg(x_1) = (1, 0)$, $\deg(y_0) = (0, 1)$ and $\deg(y_1) = (-3, 1)$. We consider the monomial ideal $I = (x_1, x_0^3y_1)$, so $s_{\rho_0} = s_{\rho_1} = s_{\eta_0} = s_{\eta_1} = 0$. Thus,

$$\begin{aligned}
&\mathcal{C}_I^{\rho_0} = \{(d_1, d_2) \mid d_1 \le 3d_2\} \quad \mathcal{C}_I^{\rho_1} = \{(d_1, d_2) \mid d_1 \ge 0\}\\
&\mathcal{C}_I^{\eta_0} = \{(d_1, d_2) \mid d_2 \le 0\} \quad\ \ \mathcal{C}_I^{\eta_1} = \{(d_1, d_2) \mid d_2 \ge 0\}.
\end{aligned}$$

We compute $\Delta^{\sigma_{00}}$. We order the monomials with respect to η_1: $\deg_{\eta_1}(x_1) = 0 < \deg_{\eta_1}(x_0^3y_1) = 1$. We have

$$\begin{aligned}
&\Delta_I^{\sigma_{00}}(\mathcal{G})_0 = \emptyset\\
&\Delta_I^{\sigma_{00}}(\mathcal{G})_1 = \{(d_1, d_2) \mid d_2 = 0\} \cap \Delta_I^{\rho_1}(\{x_1\})\\
&\Delta_I^{\sigma_{00}}(\mathcal{G})_2 = \{(d_1, d_2) \mid d_2 = 1\} \cap \Delta_I^{\rho_1}(\{x_1, x_0^3y_1\}),
\end{aligned}$$

where $\mathcal{G} = \{x_1, x_0^3y_1\}$. Since $\Delta_I^{\rho_1}(\{x_1\}) = \{(d_1, d_2) \mid d_1 \ge 0\}$ and $\Delta_I^{\rho_1}(\{x_1, x_0^3y_1\}) = \emptyset$, we obtain that $\Delta_I^{\sigma_{00}} = \{(0, 0)\}$. Applying the same procedure, we arrive at $\Delta_I^{\sigma_{01}} = \Delta_I^{\sigma_{10}} = \Delta_I^{\sigma_{11}} = \emptyset$. (See Fig. 4).

3.2 From a Klyachko Diagram to a Monomial Ideal

Our next goal is to find the minimal set of monomials generating the saturated ideal I associated to a Klyachko diagram $\{(\mathcal{C}_I^{\sigma}, \Delta_I^{\sigma})\}_{\sigma\in\Sigma}$. We may assume that $\Delta^{\sigma} \neq \emptyset$ for some cone $\sigma \in \Sigma$. Otherwise, by Proposition 3.4 and Lemma 3.1 the $\Sigma-$family of I would be the $\Sigma-$family of a principal monomial ideal. Then, $I = (x_1^{s_1} \cdots x_r^{s_r})$, where $s_i \in \mathbb{Z}$ such that $\mathcal{C}_I^{\rho_j} = \{m \in M \mid \langle m, \rho_j\rangle \ge s_j\}$.

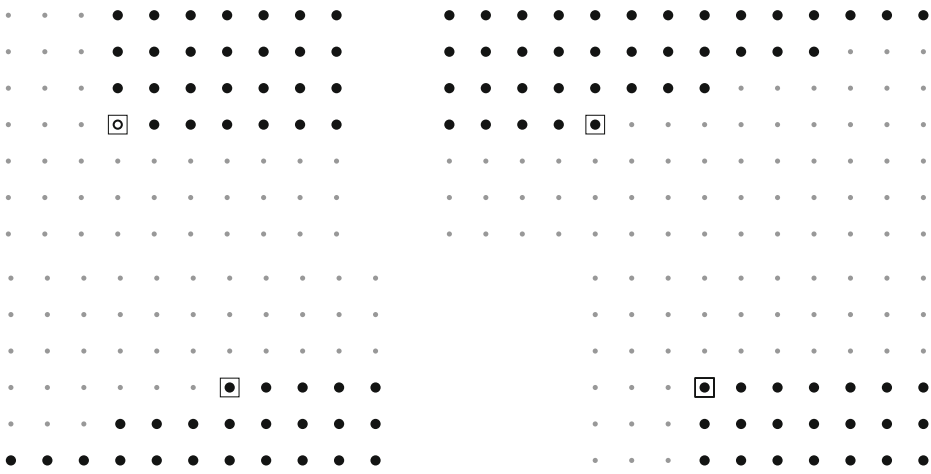

Fig. 4 Klyachko diagram of Example 3.5 (iii). It displays $\Delta_I^{\sigma_{ij}}$ ($\circ$) inside $\mathcal{C}_I^{\sigma_{ij}}$ ($\bullet$) for $(i, j) = (0, 0), (1, 0), (0, 1), (1, 1)$, clockwise. In each picture $\square$ places the origin $(0, 0) \in M \cong \mathbb{Z}^2$

Since I is $\mathrm{Cl}(X)-$graded and finitely generated, the monomials minimally generating I belong in a finite number of homogeneous pieces. For any $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$, we denote by $[D] = \sum_{\rho\in\Sigma(1)} a_\rho[D_\rho] \in \mathrm{Cl}(X)$ the class of the corresponding Weil divisor in X. We first start by providing a monomial basis of $I_{[D]} \subset R_{[D]}$.

Lemma 3.10 *Let $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$ be a Klyachko diagram of a $B-$saturated monomial ideal I, and $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$. Then,*

$$I_{[D]} = \mathbb{C}\left\langle \underline{x}^{m+D} | m \in \bigcap_{\sigma\in\Sigma_{\max}} (\mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D)) \right\rangle.$$

Proof For any cone $\sigma \in \Sigma$ we have that

$$I_{[D]}^\sigma = I_0^\sigma(D) = \bigoplus_{m\in M} I(D)_m, \text{ and}$$

$$R_{[D]}^\sigma = \mathbb{C}\langle \underline{x}^{m+D} \mid m \in \mathcal{C}_0^\sigma(D)\rangle.$$

On the other hand, for any $m \in M$ we have

$$I(D)_m \cong H^0(X, \tilde{I}(D))_m \cong \bigcap_{\sigma\in\Sigma_{\max}} I^\sigma(D)_m.$$

By Proposition 3.4, for any cone $\sigma \in \Sigma$, $I_m^\sigma(D) \neq 0$ if and only if $m \in \mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D)$. Hence,

$$I_{[D]} = \bigoplus_{m\in M} \bigcap_{\sigma\in\Sigma_{\max}} I_m^\sigma(D) = \bigoplus_{m\in \bigcap_{\sigma\in\Sigma_{\max}} (\mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D))} \mathbb{C}\langle \underline{x}^{m+D}\rangle$$

and the lemma follows. □

The following remark shows how shifting by a multidegree $D \in \mathbb{Z}^r$ affects the Klyachko diagram.

Remark 3.11 Since X is smooth, for any $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$ and any cone $\sigma \in \Sigma$, there is a character $\tau_\sigma \in M$ such that $\langle \tau_\sigma, \rho\rangle = a_\rho$ for all $\rho \in \sigma(1)$. Then, for any $m \in M$ and $\rho \in \sigma(1)$, $\langle m, \rho\rangle + a_\rho = \langle m + \tau_\sigma, \rho\rangle$, so $R^\sigma_m(D) = R^\sigma_{m+\tau_\sigma}$. It follows from (4) that $I^\sigma_m(D) = I^\sigma_{m+\tau_\sigma}$. Therefore, the Klyachko diagram of the shifted monomial ideal $I(D)$ is given by

$$\begin{cases} \mathcal{C}^\sigma_{I(D)} = \mathcal{C}^\sigma_I(D) := \mathcal{C}^\sigma_I + \tau_\sigma \\ \Delta^\sigma_{I(D)} = \Delta^\sigma_I(D) := \Delta^\sigma_I + \tau_\sigma, \end{cases}$$

which is obtained applying translations to the original Klyachko diagram.

By Lemma 3.10 we already know a basis of each homogeneous piece of I. Our next task is to characterize which monomials in a homogeneous piece $I_{[E]}$ are divisible by a single monomial x^{m+D}. The following Lemma answers this question.

Lemma 3.12 *Let $\{(\mathcal{C}^\sigma_I, \Delta^\sigma_I)\}_{\sigma\in\Sigma}$ be a Klyachko diagram of a $B-$saturated monomial ideal I, and $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$. Let $m \in \bigcap_{\Sigma_{\max}} (\mathcal{C}^\sigma_I(D) \setminus \Delta^\sigma_I(D))$ and let $E = (b_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$ be such that $b_\rho \geq a_\rho$ for all $\rho \in \Sigma(1)$. The set of monomials in $I_{[E]}$ which are divisible by $\underline{x}^{m+D}$ is*

$$T_E(\underline{x}^{m+D}) := \left\{ \underline{x}^{m'+E} \,\middle|\, \langle m', \rho\rangle \geq \langle m, \rho\rangle + a_\rho - b_\rho, \rho \in \Sigma(1) \right\}.$$

Proof For any $m' \in M$, the monomial $\underline{x}^{m'+E}$ is divisible by $\underline{x}^{m+D}$ if and only if $\langle m', \rho\rangle + b_\rho \geq \langle m, \rho\rangle + a_\rho$ for any $\rho \in \Sigma(1)$, and the lemma follows. □

Now, let $\mathcal{G} = \{\underline{x}^{m_1+\underline{k}^1}, \ldots, \underline{x}^{m_t+\underline{k}^t}\}$ be a finite set of monomials of possibly different degrees. For any $E = (b_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$ such that $b_\rho \geq k^i_\rho$ for $\rho \in \Sigma(1)$ and $1 \leq i \leq t$, we define

$$T_E\mathcal{G} := T_E(x^{m_1+\underline{k}^1}) \cup \cdots \cup T_E(\underline{x}^{m_t+\underline{k}^t}),$$

which describes the span of the monomials of $\mathcal{G}$ inside $I_{[E]}$.

Finally, we can describe a finite set of generators of a $B-$saturated monomial ideal I corresponding to a given Klyachko diagram. Since X is smooth we can assume that $\mathrm{Cl}(X) \cong \mathbb{Z}\langle [D_{\rho_{i_1}}], \ldots, [D_{\rho_{i_\ell}}]\rangle \cong \mathbb{Z}^\ell$. Up to permutation of variables, we may also assume that $i_1 = 1, \ldots, i_\ell = \ell$, and for any $a = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell$ we set $\overline{a} := (a_1, \ldots, a_\ell, 0, \ldots, 0) \in \mathbb{Z}^r$. For any $u, v \in \mathbb{Z}^l$ we say that $v \preceq u$, if $u_i \geq v_i$ for $1 \leq i \leq l$, which defines a partial order. We set $\mathcal{G}_0 = \emptyset$ and for any $u \in \mathbb{Z}^l$, we define

$$\mathcal{G}_u := \left\{ \underline{x}^{m+\overline{u}} \mid m \in \bigcap_{\sigma\in\Sigma_{\max}} \mathcal{C}^\sigma_I(\overline{u}) \setminus \Delta^\sigma_I(\overline{u}) \right\} \setminus \bigcup_{v\preceq u} T_{\overline{u}}\mathcal{G}_v$$

assuming we have determined $\mathcal{G}_v$ for any $v \preceq u$. Since I is finitely generated, there are only finitely many degrees $u \in \mathbb{Z}^l$ such that $\mathcal{G}_u \neq \emptyset$.

If R is the Cox ring of $\mathbb{P}^{r-1}$, and so R has the standard $\mathbb{Z}-$grading, then by construction that method gives directly a minimal set of monomial generators for I. The following example illustrates the method:

Example 3.13 (i) Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$ (See Example 2.2(i)). Consider the following Klyachko diagram $\{(\mathcal{C}_I^i, \Delta_I^i)\}_{0\leq i\leq 2}$, where $\mathcal{C}_I^i$ and Δ_I^i stand for $\mathcal{C}_I^{\sigma_i}$ and $\Delta_I^{\sigma_i}$:

$$\begin{array}{l} \mathcal{C}_I^0 = \{(d_1, d_2) \mid d_1 \geq 0,\ d_2 \geq 0\} \\ \mathcal{C}_I^1 = \{(d_1, d_2) \mid d_1 + d_2 \leq 0,\ d_2 \geq 0\} \end{array} \quad \mathcal{C}_I^2 = \{(d_1, d_2) \mid d_1 + d_2 \leq 0,\ d_1 \geq 0\},$$

$\Delta_I^0 = \{(0,0), (1,0)\}$ and $\Delta_I^1 = \Delta_I^2 = \{(0,0)\}$. Applying the above procedure, we obtain that $\mathcal{G}_0 = \mathcal{G}_1 = \emptyset$, $\mathcal{G}_2 = \{x_0x_2, x_1x_2\}$, $\mathcal{G}_3 = \{x_0x_1^2\}$ and $\mathcal{G}_j = \emptyset$ for $j \geq 4$. Hence, the saturated monomial ideal corresponding to this Klyachko diagram is $I = (x_0x_2, x_1x_2, x_0x_1^2)$.

(ii) Let $R = \mathbb{C}[x_0, x_1, x_2, x_3]$ be the Cox ring of $\mathbb{P}^3$ (See Example 2.2(i)). Consider the following Klyachko diagram $\{(\mathcal{C}_I^i, \Delta_I^i)\}_{0\leq i\leq 3}$, where $\mathcal{C}_I^i$ and Δ_I^i stand for $\mathcal{C}_I^{\sigma_i}$ and $\Delta_I^{\sigma_i}$:

$$\begin{array}{l} \mathcal{C}_I^0 = \{(d_1, d_2, d_3) \mid d_1 \geq 0,\ d_2 \geq 0, d_3 \geq 0\} \\ \mathcal{C}_I^1 = \{(d_1, d_2, d_3) \mid d_1 + d_2 + d_3 \leq 0,\ d_2 \geq 0, d_3 \geq 0\} \\ \mathcal{C}_I^2 = \{(d_1, d_2, d_3) \mid d_1 + d_2 + d_3 \leq 0,\ d_1 \geq 0, d_3 \geq 0\} \\ \mathcal{C}_I^3 = \{(d_1, d_2, d_3) \mid d_1 + d_2 + d_3 \leq 0,\ d_1 \geq 0, d_2 \geq 0\}, \end{array}$$

$\Delta_I^0 = \{(0,0,d_3) \mid d_3 \geq 0\} \cup \{(0,1,0)\}$ and $\Delta_I^1 = \Delta_I^2 = \Delta_I^3 = \emptyset$. Applying the above procedure, we obtain that $\mathcal{G}_0 = \emptyset$, $\mathcal{G}_1 = \{x_1\}$, $\mathcal{G}_2 = \{x_0^2x_1, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3\} \setminus T_{(2,0,0,0)}\{x_1\} = \{x_2^2, x_2x_3\}$ and $\mathcal{G}_j = \emptyset$ for $j \geq 3$. Hence, the saturated monomial ideal corresponding to this Klyachko diagram is $I = (x_1, x_2^2, x_2x_3)$.

In more general gradings, we cannot assure that this method gives a *minimal* set of generators of I, but a finite set of monomials generating I. However, we can extract from it a minimal set of monomials generating I by using suitable monomial divisions. The following example illustrates this situation:

Example 3.14 Let $R = \mathbb{C}[x_0, x_1, y_0, y_1]$ be the Cox ring of the Hirzebruch surface $\mathcal{H}_3$ (see Example 2.2(ii)). In particular, R is $\mathbb{Z}^2-$graded with $\deg(x_0) = \deg(x_1) = (1,0)$, $\deg(y_0) = (0,1)$ and $\deg(y_1) = (-3,1)$. Let $\{(\mathcal{C}_I^{\sigma_{ij}}, \Delta_I^{\sigma_{ij}})\}_{0\leq i,j\leq 1}$ be the Klyachko diagram of Example 3.5 (iii). Applying the above procedure we obtain that $\mathcal{G}_u = \emptyset$ for any $u \in \mathbb{Z}^2 \setminus \{(1,0), (0,1)\}$, and $\mathcal{G}_{(1,0)} = \{x_1\}$ and $\mathcal{G}_{(0,1)} = \{x_0^3y_1, x_0^2x_1y_1, x_0x_1^2y_1, x_1^3y_1\}$. Hence $\{x_1, x_0^3y_1, x_0^2x_1y_1, x_0x_1^2y_1, x_1^3y_1\}$ is a set of generators for a saturated monomial ideal I corresponding to this Klyachko diagram. However, the first monomial divides the three last monomials. Therefore, I is *minimally* generated by $\{x_1, x_0^3y_1\}$.

3.3 Non-saturated Monomial Ideals

The previous subsections have shown that the theory of Klyachko diagrams is well suited to describe saturated monomial ideals, but we cannot retrieve directly information of non-saturated monomial ideals. In this subsection, we describe the quotient $I^{\text{sat}}/I \cong H_B^1(I)$ using the Klyachko diagram $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$ and the generators of I.

Proposition 3.15 *Let $I = (\underline{x}^{m_1+\underline{k}^t}, \ldots, \underline{x}^{m_t+\underline{k}^t})$ be a monomial ideal with Klyachko diagram $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$, such that for $1 \leq i \leq t$, $m_i \in M$ and $\underline{k}^i = (k_\rho^i)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$ satisfy $\langle m_i, \rho\rangle + k_\rho^t \geq 0$ for all $\rho \in \Sigma(1)$. Then, for any $D = (a_\rho)_{\rho\in\Sigma(1)} \in \mathbb{Z}^r$,*

$$H_B^1(I)_{[D]} \cong \mathbb{C}\left\langle \left\{ \underline{x}^{m+D} \,\middle|\, m \in \bigcap_{\sigma\in\Sigma_{\max}} (\mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D)) \right\} \setminus \bigcup_{i=1}^t T_D(\underline{x}^{m_i+\underline{k}^i}) \right\rangle.$$

Proof From Lemma 3.12, $I_{[D]} = \bigcup_{i=1}^t T_D(\underline{x}^{m_i+\underline{k}^i}) \subset I_{[D]}^{sat}$. By Proposition 3.4, the Klyachko diagram characterizes the saturation of I, and by Lemma 3.10 we have $I_{[D]}^{sat} = \mathbb{C}\langle \underline{x}^{m+D} | m \in \bigcap_{\sigma\in\Sigma_{\max}} (\mathcal{C}_I^\sigma(D) \setminus \Delta_I^\sigma(D))\rangle$, and the result follows. □

Example 3.16 Let $R = \mathbb{C}[x_0, x_1, x_2]$ be the Cox ring of $\mathbb{P}^2$, $B = (x_0, x_1, x_2)$ and let $I = (x_0^3x_1, x_0x_1x_2^2, x_2^3, x_1^3)$ be a monomial ideal. Computing its Klyachko diagram we obtain $s_0 = s_1 = s_2 = 0$, $\Delta^0 = \{(0,0), (0,1), (0,2)\}$ and $\Delta^1 = \Delta^2 = \emptyset$. From Proposition 3.15 we obtain:

$$\begin{array}{ll} H_B^1(I)_0 = 0 & H_B^1(I)_3 = \mathbb{C}\langle x_0^2x_1, x_0x_1^2, x_0x_1x_2, x_1^2x_2, x_1x_2^2\rangle \\ H_B^1(I)_1 = \mathbb{C}\langle x_1\rangle & H_B^1(I)_4 = \mathbb{C}\langle x_0^2x_1^2, x_0^2x_1x_2, x_0x_1^2x_2, x_1^2x_2^2\rangle \\ H_B^1(I)_2 = \mathbb{C}\langle x_0x_1, x_1^2, x_1x_2\rangle & H_B^1(I)_5 = \mathbb{C}\langle x_0^2x_1^2x_2\rangle \\ & H_B^1(I)_j = 0 \quad \text{for} \quad j \geq 6. \end{array}$$

4 Application: Hilbert Function of Monomial Ideals

In this section, we show how to compute the Hilbert function and Hilbert polynomial of a $B-$saturated monomial ideal from its Klyachko diagram. As a consequence we develop a formula for the Hilbert polynomial in terms of the Klyachko diagram.

For any ray $\rho \in \Sigma(1)$, recall that $\mathcal{C}_0^\rho := \{m \in M | \langle m, \rho\rangle \geq 0\}$, $\mathcal{C}_0^\sigma = \bigcap_{\rho\in\sigma(1)} \mathcal{C}^\rho$ for $\sigma \in \Sigma$. Recall by Lemma 3.1 that $\{\mathcal{C}_0^\sigma, \emptyset)\}_{\sigma\in\Sigma}$ is the Klyachko diagram of the monomial ideal (1), or equivalently of R. For any multidegree $D \in \mathbb{Z}^r$, we define $\mathcal{C}_0(D) := \bigcap_{\sigma\in\Sigma_{\max}} \mathcal{C}_0^\sigma(D)$, such that $\mathcal{C}_0(D)$ gives a monomial basis of $R_{[D]}$ (see Proposition 3.10). The following result tells us how to compute the value of the Hilbert function of I from this description.

Proposition 4.1 *Let I be a $B-$saturated monomial ideal with Klyachko diagram $\{(\mathcal{C}_I^\sigma, \Delta_I^\sigma)\}_{\sigma\in\Sigma}$. Then, the Hilbert function of I is given by*

$$h_{R/I}(\alpha) = \left| \bigcup_{\sigma\in\Sigma_{\max}} \left((\Delta_I^\sigma(\overline{\alpha}) \cap \mathcal{C}_0(\overline{\alpha})) \cup (\mathcal{C}_0(\overline{\alpha}) \setminus \mathcal{C}_I^\sigma(\overline{\alpha})) \right) \right|$$

for any $\alpha \in \mathrm{Cl}(X)$.

Proof By Lemma 3.10, there is a bijection between a monomial basis of I_α (respectively of R_α) and $\bigcap_{\sigma\in\Sigma_{\max}}(\mathcal{C}_I^\sigma(\overline{\alpha})\setminus\Delta_I^\sigma(\overline{\alpha}))$ (respectively $\mathcal{C}_0(\overline{\alpha})$). Thus,

$$\begin{aligned} h_{R/I}(\alpha) &= \left|\mathcal{C}_0(\overline{\alpha})\setminus\left(\bigcap_{\sigma\in\Sigma_{\max}}(\mathcal{C}_I^\sigma(\overline{\alpha})\setminus\Delta_I^\sigma(\overline{\alpha}))\right)\right| \\ &= \left|\bigcup_{\sigma\in\Sigma_{\max}}\mathcal{C}_0(\overline{\alpha})\setminus(\mathcal{C}_I^\sigma(\overline{\alpha})\setminus\Delta_I^\sigma(\overline{\alpha}))\right| \\ &= \left|\bigcup_{\sigma\in\Sigma_{\max}}\left((\Delta_I^\sigma(\overline{\alpha})\cap\mathcal{C}_0(\overline{\alpha}))\cup(\mathcal{C}_0(\overline{\alpha})\setminus\mathcal{C}_I^\sigma(\overline{\alpha}))\right)\right|. \end{aligned}$$

□

In the following we remark how the above formula simplifies when I is an ideal generated by monomials with no common factor.

Remark 4.2 Let $I\subset R$ be a monomial ideal and $x^{\underline{k}}\in R$ a monomial. We recall that the Hilbert function of $J=x^{\underline{k}}I$ is

$$h_{R/J}(\alpha)=h_R(\alpha)-h_R(\alpha-[\underline{k}])+h_{R/I}(\alpha-[\underline{k}]). \tag{6}$$

Thus, we can assume that the Klyachko diagram of I has $s_\rho=0$ for any $\rho\in\Sigma(1)$ and, its Hilbert function is

$$h_{R/I}(\alpha)=\left|\bigcup_{\sigma\in\Sigma_{\max}}\left(\Delta^\sigma(\overline{\alpha})\cap\mathcal{C}_0(\overline{\alpha})\right)\right|. \tag{7}$$

Otherwise, $I=\left(\prod_{\rho\in\Sigma(1)}x_\rho^{s_\rho}\right)I_0$ where I_0 is a monomial ideal with $s_\rho=0$ for any $\rho\in\Sigma(1)$, and we can compute the Hilbert function of I using (6).

Example 4.3 Let $R=\mathbb{C}[x_0,x_1,x_2]$ be the Cox ring of $\mathbb{P}^2$ and $I=(x_2^2,\,x_0x_2,x_0x_1)$ as in Example 3.5 (i). For any $a\in\mathbb{Z}$, we set $\overline{a}=(a,0,0)$ and

$$\Delta_I^{\sigma_0}(\overline{a})=\{(0,0)\}\quad\Delta_I^{\sigma_1}(\overline{a})=\{(a,0),(a-1,1)\}\quad\Delta_I^{\sigma_2}(\overline{a})=\emptyset.$$

Since $s_0=s_1=s_2=0$, by (7) we have the following Hilbert function:

$$h_{R/I}(t)=\begin{cases}0, & t\le -1\\ 1, & t=0\\ 3, & t\ge 1.\end{cases}$$

In particular, the Hilbert polynomial of R/I is $P_{R/I}\equiv 3$ constant.

In the following result we characterize the Klyachko diagram of a monomial ideal I with constant Hilbert polynomial. In particular, notice that I is necessarily generated by monomials without common factors.

Corollary 4.4 *Let I be a monomial ideal with Klyachko diagram $\{(\mathcal{C}_I^\sigma,\Delta_I^\sigma)\}_{\sigma\in\Sigma}$. Then, the Hilbert polynomial $P_{R/I}$ of I is constant if and only if $s_\rho=0$ for any $\rho\in\Sigma(1)$ and Δ_I^σ is finite for any $\sigma\in\Sigma_{\max}$. Moreover,*

$$P_{R/I}(\alpha)=\sum_{\sigma\in\Sigma_{\max}}|\Delta_I^\sigma|.$$

Proof The left implication follows directly from Proposition 4.1 and Remark 4.2. Conversely, if $s_\rho > 0$ for some $\rho \in \Sigma(1)$, then there is $\sigma \in \Sigma_{\max}$ such that $\rho \in \sigma(1)$ and $\mathcal{C}_0(\overline{\alpha}) \setminus \mathcal{C}_I^\sigma(\overline{\alpha})$ increases with α, and $P_{R/I}$ would not be constant. Now, assume that there is some $\sigma \in \Sigma_{\max}$ such that Δ_I^σ is not finite. By construction, Δ_I^σ contains a set Δ' of the form

$$\{m \in M \mid \langle m, \rho_{i_1}\rangle = k_1, \ldots, \langle m, \rho_{i_l}\rangle = k_l, \langle m, \rho_{i_{l+1}}\rangle \geq k_{l+1}, \ldots, \langle m, \rho_{i_c}\rangle \geq k_c\}.$$

Since $\Delta'(\overline{\alpha})$ is not bounded in $\mathcal{C}_0^\sigma(\overline{\alpha})$, the number of points in $\Delta'(\overline{\alpha}) \cap \mathcal{C}_0(\overline{\alpha})$ increases with α for $\alpha \gg 0$. Therefore, it follows from (7) that the Hilbert function $h_{R/I}(\alpha)$ of I increases with α for $\alpha \gg 0$. □

Remark 4.5 Notice that by Corollary 4.4 we have characterized all monomial ideals $I \subset R$ with $\dim R/I = 1$, in terms of the Klyachko diagram.

We finish by illustrating Corollary 4.4 with the following example.

Example 4.6 Let $R = \mathbb{C}[x_0, x_1, x_2, x_3]$ be the Cox ring of $\mathbb{P}^3$ and $I = (x_0x_1, x_2^2, x_1x_2x_3^2)$ as in Example 3.5 (ii). For any $a \in \mathbb{Z}$ we set $\overline{a}$ and we have,

$$\begin{aligned}
\Delta_I^{\sigma_0}(\overline{a}) &= \{(0, d_2, d_3) \mid 0 \leq d_2, d_3 \leq 1\} \cup \{(0, d_2, d_3) \mid d_3 \geq 2, 0 \leq d_2 \leq 1\}\\
\Delta_I^{\sigma_1}(\overline{a}) &= \{(a - d_2 - d_3, d_2, d_3) \mid 0 \leq d_2, d_3 \leq 1\} \cup \{(a - d_3, 0, d_3) \mid d_3 \geq 2\}\\
\Delta_I^{\sigma_2}(\overline{a}) &= \emptyset\\
\Delta_I^{\sigma_3}(\overline{a}) &= \{(0, d_2, d_3) \mid 0 \leq d_2 \leq 1, d_3 \leq a - d_2\} \cup \{(d_1, 0, a - d_1) \mid d_1 \geq 1\}
\end{aligned}$$

Counting the number of different points in $\bigcup_{i=0}^3 \Delta_I^{\sigma_i}(\overline{a})$, we obtain that $h_{R/I}(0) = 1$, $h_{R/I}(1) = 4$, $h_{R/I}(2) = 8$, and for $a \geq 3$, $h_{R/I}(a) = 3(a+1)$, for $a \geq 3$. Thus, the Hilbert polynomial of I is $P_{R/I}(a) = 3(a+1)$.

Acknowledgements The first author was partially supported by PID2019-104844GB-I00. The second author is partially supported by MDM-2014-0445-18-2.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of Interests The authors have no competing interests to declare that are relevant to the content of this article.

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