Epimorphism Surjectivity in Logic and Algebra

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CHAPTER 1

Introduction

An algebra consists of a set of elements together with certain operations. If two algebras share the same operations, we say they are *similar*. A map $f\colon \boldsymbol{A} \to \boldsymbol{B}$ between similar algebras is called a *homomorphism* if it preserves the operations of $\boldsymbol{A}$ and $\boldsymbol{B}$ (see, e.g., [8]). Notably, every surjective homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is *right cancellable* in the sense that

$$f \circ h = g \circ h \text{ implies } f = g$$

for every pair of a homomorphisms f, g with domain B. This idea is generalized by the notion of *epimorphism*. More precisely, a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ between members of a class of similar algebras K is called a K-*epimorphism* when it is right cancellable for every pair of homomorphisms $f, g\colon \boldsymbol{B} \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathsf{K}$. Similarly, h is said to be a K-*monomorphism* when it is left cancellable in K. As such, monomorphisms generalize the concept of injective maps. For more information about epimorphisms and monomorphisms, we refer the reader to [28, Sec. I.5], [1, Sec. II], and [26].

Given a class of similar algebras K, every surjective homomorphism between members of K is a K-epimorphism (see, e.g., [6]). Likewise, every injective homomorphism between members of K is a K-monomorphism. While the converses need not be true in general, to obtain that every K-monomorphism is injective, it suffices to require that K contains free objects (see, e.g., [1, Cor. 7.38]). This includes all classes of algebras closed under subalgebras and direct products, such as groups, rings, and Boolean algebras (see, e.g., [4]). In the case of epimorphisms, the situation is more complicated. There are a lot of prominent examples of classes of algebras that have non-surjective epimorphisms, including rings, as witnessed by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ (see, e.g., [23]), and distributive lattices (see, e.g., [6, Ex. 3.11]). Thus, the requirement that K-epimorphisms be surjective is a non-trivial property, called the *epimorphism surjectivity property*, for short ES property (see, e.g., [6]). As such, it has provoked the interest of the research community and has been subject to extensive study during the last decades (see, e.g., [26] and the references therein).

Besides being an interesting algebraic property in its own right, the motivation for the ES property also derives from other areas of mathematics, such as logic and model theory. Comparing the class of Boolean algebras, which has the ES property, with the class of distributive lattices, which lacks it, one might notice that the decisive difference lies in the fact that in a Boolean algebra we have a term that denotes the unique complement of a given element, whereas the language of distributive lattices lacks such a term. Translating this observation into logic, we naturally end up with the *Beth definability property* [6], which expresses the requirement that every implicit definition can be made explicit. In other words, whenever a sequence of elements satisfies a property that makes it distinguishable from any other sequence, each of its elements must be representable by a term. This correspondence between the Beth definability property in logic and the ES property in algebra was made precise by Blok and Hoogland in [6] (here Theorem 2.5.12) and implies that all advances in the study of the ES property directly translate to a deeper understanding of definability in logic. From a model-theoretic standpoint, a failure of the ES property in a class K that is closed under ultraproducts amounts to the fact that there exists a partial function in K defined by an existentially quantified finite conjunction of equations that cannot be represented by a term of the language (see [9], here Lemma 5.3).

When studying the ES property, we will restrict our attention to classes of algebras that are closed under isomorphisms, subalgebras, products, and ultraproducts, known as *quasivarieties* (see, e.g., [8]). These include most classes that are commonly studied in algebra, like groups, rings, lattices, and Boolean algebras. In addition, quasivarieties are the classes that algebraize propositional logics in the sense of [7]. In order to verify the ES property for a quasivariety K, it suffices to check that K lacks *proper epic subalgebras*, i.e., that there is no proper subalgebra $\boldsymbol{A}$ of a member $\boldsymbol{B}$ of K such that the inclusion $i\colon \boldsymbol{A} \hookrightarrow \boldsymbol{B}$ is an epimorphism (see, e.g., [6], here Lemma 2.4.5). Nevertheless, there is no easy way to determine whether a quasivariety K has the ES property. Thus, facilitating this task proves very useful. This is exactly what is achieved by the two results of Champercholi [9, Thms. 18 and 22], which form the core of this thesis (see Theorems 6.3 and 6.4). The idea is to prove that if a quasivariety K lacks the ES property, under reasonable assumptions, a failure can always be found in a well-behaved subclass of K.

In order to clarify this idea, we recall that the class K_{RFSI} of *finitely subdirectly irreducible members relative to* K consists of those $\boldsymbol{A} \in \mathsf{K}$ with the property that for every family $\{\theta_i : i \leqslant n\}$ of congruence of $\boldsymbol{A}$ such that $\boldsymbol{A}/\theta_i \in \mathsf{K}$ for every $i \leqslant n$,

$$\text{if } \mathrm{id}_{\boldsymbol{A}} = \theta_1 \cap \dots \cap \theta_n, \text{ then } \mathrm{id}_{\boldsymbol{A}} = \theta_i \text{ for some } i \leqslant n \text{ (see, e.g., [4]).}$$

Moreover, we will work in the setting of quasivarieties with a *near-unanimity term*, where a term μ of arity $n \geqslant 3$ is called a near-unanimity term for a class K when

$$\mathsf{K} \vDash \mu(y, x, \dots, x) \approx \mu(x, y, x, \dots, x) \approx \dots \approx \mu(x, \dots, x, y) \approx x.$$

For instance,

$$\mu(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

is a 3-ary near unanimity term for every class K with a lattice reduct (see, e.g., [4]). The first of Campercholi's theorems [9, Thm. 18], which we will present in Chapter 6 (see Theorem 6.3), states that a quasivariety with an $(m+1)$-ary near-unanimity term has the ES property iff every subalgebra $\boldsymbol{A} \leqslant \boldsymbol{A}_1 \times \dots \times \boldsymbol{A}_m$, where each $\boldsymbol{A}_i$ is an ultraproduct of members of K_{RFSI}, lacks subalgebras that are proper and epic in K. As a consequence, the above theorem applied to the quasivariety D of distributive lattices, where the only non-trivial RFSI member is the two-element chain $\mathbf{D}_2$ (see, e.g., [15]), yields that the failure of the ES property for D occurs in a subalgebra $\boldsymbol{A} \leqslant \mathbf{D}_2 \times \mathbf{D}_2$.

The second theorem of Campercholi ([9, Thm. 22], here Theorem 6.4) deals with *varieties*, i.e., classes of similar algebras closed under subalgebras, homomorphic images, and direct products. We recall that a variety is *arithmetical* if there exists a term $\varphi(x, y, z)$ such that

$$\mathsf{K} \vDash \varphi(x, y, x) \approx \varphi(x, y, y) \approx \varphi(y, y, x) \approx x$$

(see, e.g., [8]). For instance the varieties of Boolean algebras, Heyting algebras, and modal algebras are arithmetical (see, e.g., [8]). We will present a proof of the fact that an arithmetical variety K with the property that $\mathsf{K}_{\mathrm{RFSI}}$ is closed under isomorphic copies, subalgebras, and ultraproducts has the ES property iff the members of $\mathsf{K}_{\mathrm{RFSI}}$ lack subalgebras that are proper and epic in K ([9, Thm. 22], here Theorem 6.4). Given that the only non-trivial finitely subdirectly irreducible Boolean algebra is the two element chain, which clearly lacks proper epic subalgebras, the ES property for Boolean algebras can be derived as a straightforward consequence.

The fact that these results are highly non-trivial is reflected in the complexity of their proofs, which require extensive background theory from different mathematical fields, including universal algebra, topology and model theory. Because of this, the chapter with the main theorems (Chapter 6) is preceded by a series of preparatory chapters, where the necessary theory is developed.

In the preliminaries (Chapter 2), we start by building a foundation of basic results in general first-order logic (Section 2.1), model theory (Section 2.2), and universal algebra (Section 2.3). The last two sections of the preliminaries are devoted to the ES property (Section 2.4) and its correspondence to the Beth definability property in logic (Section 2.5). We then continue to focus on some results that play a crucial role in the proofs of Theorems 6.3 and 6.4. First, we deal with the so called *Infinitary Baker-Pixley Theorem* (see [31], here Theorem 3.11), which will be applied in the proof of Theorem 6.3. It is a generalization of the classical Baker-Pixley Theorem (see [3]) to infinite algebras and as such associates the representability of a partial function by a term with a closure property under product functions. In our reconstruction (Theorem 3.11) of the proof given in [11, Thm. 2.1], we spelled out some intermediate results in more detail and isolated a major step as a separate lemma (Lemma 3.9) with the aim to clarify the structure of the proof.

Next, we build the theory of global subdirect products (see, e.g., [27], here Chapter 4), as a topologized version of subdirect products. Global subdirect representations were introduced as a purely algebraic equivalent to the theory of sheaf representations (see [27]). Our presentation is largely based on [17]. Nonetheless, we decided to focus on proving a representation theorem for arithmetical varieties only (Theorem 4.7). As a consequence the proofs become more compact, easier to follow and do require less background theory, auxiliary results, and notation. An important feature of global subdirect products, which plays a crucial role in the proof of [9, Thm. 22] (here Theorem 6.4), is that they preserve the validity of certain formulas from their factors (see, e.g, [31, Lem. 3.1] , here Lemma 4.14).

The chapter about definability conditions (Chapter 5) is a collection of model-theoretic preservation results, asserting that the formulas preserved under certain operations can be assumed to have a particular shape (see, e.g., [10, Thm. 3.3], here Lemma 5.5). One of the main tools needed in this part are diagrams, introduced in Subsection 2.2.2. Throughout the thesis, we provided more detailed explanations compared to the original proofs, adapted some results to our specific purposes, and restructured or reformulated some arguments to help the reader follow the train of thought.

Once this preparatory work is finished, we are ready to prove the main Theorems 6.3 and 6.4. To this end, we isolated the similar strategy used in both of the proofs and the key point where they diverge. This required restructuring the two proofs and extracting a crucial observation used in both of them as a separate result (Lemma 6.1 and Corollary 6.2). In turn, the auxiliary lemma [9, Lem. 21] preceding [9, Thm. 22] in the original paper became completely incorporated into the proof of Theorem 6.4. Also, the final conclusion of the two proofs, which originally relied on some formula manipulation, is replaced by a purely algebraic argument using quotient algebras.

To conclude the thesis, we present our own result ([12, Thm. 4.3], here Theorem 7.7), which improves [9, Thm. 18] in a slightly less general setting. Instead of the ES property, in Chapter 7, we consider the *weak ES property*, which only requires epimorphisms between finitely generated algebras to be surjective (see, e.g., [20]). Studying this weaker version still yields very fruitful results. Clearly, if we can find a counterexample to the weak ES property, this also implies a failure of the ES property. Hence, the model-theoretic motivation applies to the weak ES property as well. Furthermore, there is also a weaker version of the Beth definability property, the *finite Beth definability property*, which deals with the definability of finite tuples instead of arbitrary sequences and corresponds to the weak ES property. As such, it provides an interest in studying the weak ES property from a logical perspective (see, e.g., [6]). Finally, although in general the ES property is not equivalent to its weaker version (see, e.g., [5]), there are many examples of classes of algebras that lack the ES property where we can already find a failure of the weak ES property. For example, in the variety of rings, a failure of the weak ES property is witnessed by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[1/p]$ for some prime number p, since both $\mathbb{Z}$ and $\mathbb{Z}[1/p]$ are finitely generated.

Interestingly, in establishing this improved version of [9, Thm. 18] for the weak ES property ([12, Thm. 4.3], here Theorem 7.7), we completely abandoned Campercholi's approach, following a purely algebraic proof strategy instead. As a consequence, we do not rely on the Infinitary Baker-Pixley Theorem or on any definability conditions. Besides using basic universal algebraic tools, the key lies in the introduction of *full subalgebras* and the observation that when a quasivariety lacks the weak ES property, we can always find a counterexample in the form of a full and epic subalgebra [12, Cor. 3.8] (here Corollary 7.6).

Our result [12, Thm. 4.3] (here Theorem 7.7) states that a quasivariety K with an $(m+1)$-ary near-unanimity term has the weak ES property iff every finitely generated subdirect product $\boldsymbol{A} \leqslant \boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$, with $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathsf{K}_{\text{RFSI}}$, lacks subalgebras that are full and epic in K. So, the improvement compared to Campercholi's original result is twofold: On the one hand, if a failure of the weak ES property occurs, the theorem tells us that we can find it in a subdirect product $\boldsymbol{A} \leqslant \boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$, that is, in a subalgebra with the additional property that all the projection maps $p_i \colon \boldsymbol{A} \to \boldsymbol{A}_i$, sending a tuple to its i^{th} entry, are surjective. On the other hand, instead of considering ultraproducts of members of K_{RFSI}, we can actually assume that $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$ are themselves in K_{RFSI}.

Although not part of the thesis, our submitted manuscript [12] also contains an improved version of [9, Thm. 22] in the setting of the weak ES property. Like Campercholi's result, our theorem yields that under certain assumptions the weak ES property for a variety K can be checked by considering the subclass K_{FSI} only. Again, our version is obtained by means of purely algebraic methods without relying on the theory of global subdirect representations or any definability conditions (see [12, Thm. 5.3]). To be more precise, we need to recall that a variety K is *congruence permutable* if, for every member $\boldsymbol{A}$ of K and every pair of congruences θ_1, θ_2 of $\boldsymbol{A}$, the smallest congruence of $\boldsymbol{A}$ containing both of them is given by the set

$$\{\langle a, b\rangle : \text{ there exists } c \in A \text{ such that } \langle a, c\rangle \in \theta_1 \text{ and } \langle c, b\rangle \in \theta_2\}.$$

In particular, every arithmetical variety is congruence permutable. The converse, however, is not true in general (see, e.g., [8]). Our achievement (see [12, Thm. 5.3]) was to prove that a congruence permutable variety K has the weak ES property iff every finitely generated member of K_{FSI} lacks subalgebras that are full and epic in K. So, in contrast to [9, Thm. 22], our version for the weak ES property is not limited to arithmetical varieties with some additional requirement on K_{FSI} but is applicable to every congruence permutable variety without further restrictions. This includes all varieties with a group reduct (see, e.g., [8]), which comprise a large part of the classes of algebras commonly studied. Notice that varieties of groups, for example, are in general not arithmetical and are thus not covered by [9, Thm. 22].

Further work along these lines is in progress. For example, we are exploring how our theorems can be applied to prominent quasivarieties of Heyting algebras or modal algebras.

We hope that this thesis serves as motivation to continue the path of studying the (weak) ES property and its consequences and that it will lead to interesting discoveries that provide fruitful results, fueling advances in algebra, logic, and model theory.

CHAPTER 2

Preliminaries

As the theorems we aim to prove require a variety of tools and background theory, we will start by recalling some basics of first-order logic (Section 2.1), model theory (Section 2.2), and universal algebra (Section 2.3). We will then continue presenting the protagonist of this thesis, the epimorphism surjectivity property, and making some easy but useful observations concerning this property (Section 2.4). Finally, we will establish a correspondence between the (weak) ES property in algebra and the (finite) Beth definability property in logic, providing motivation for the study of the ES property from a logical standpoint (Section 2.5).

2.1 Basics of first-order logic

In this first preliminary section, we will define the basic concepts of first-order logic and present some classical results, which will be used in the course of the following chapters. For more details, we refer the reader to [8, Section V.1], [22, Chapter 1], and [13, Chapter 1].

2.1.1 Languages, terms, and formulas

Definition 2.1.1. A *first-order language* is a quadruple $(\mathsf{Var}, \mathcal{F}, \mathcal{R}, \tau)$, where

- Var is a set of *variables*;
- $\mathcal{F}$ is a set of *function symbols*;
- $\mathcal{R}$ is a set of *relation symbols*;
- $\tau\colon \mathcal{F} \cup \mathcal{R} \to \mathbb{N}$ is a function, called the *arity function*.

When $f \in \mathcal{F}$ and $\tau(f) = n$, we say that f is an *n-ary function symbol*. Similarly, we call $R \in \mathcal{R}$ with $\tau(R) = n$ an *n-ary relation symbol*. The 0-ary function symbols are commonly referred to as the *constants* of the language $\mathcal{L}$.

Definition 2.1.2. The set $T_{\mathcal{L}}$ of *terms* of a first-order language $\mathcal{L}$ is defined recursively to be the least set such that:

- $\mathsf{Var} \subseteq T_{\mathcal{L}}$;
- $f(t_1, \dots, t_n) \in T_{\mathcal{L}}$ for every n-ary function symbol f and $t_1, \dots, t_n \in T_{\mathcal{L}}$.

Given a term $t \in T_{\mathcal{L}}$, we write $\mathsf{Var}(t)$ to refer to the set of all variables occurring in t. When $\mathsf{Var}(t) \subseteq \{x_i : i \leqslant n\}$, we will often use the notation $t(x_1, \dots, x_n)$ or $t(\vec{x})$, where $\vec{x} = \langle x_1, \dots, x_n \rangle$. Furthermore, $T_{\mathcal{L}}(x_1, \dots, x_n)$ or $T_{\mathcal{L}}(\vec{x})$ denote the set of all terms $t \in T_{\mathcal{L}}$ with variables in $\{x_i : i \leqslant n\}$. More generally, the notation $T_{\mathcal{L}}(X)$ stands for the set of all terms with variables in a possibly infinite set X. We call t a *closed term* when t does not contain any variables.

Using the *equality symbol* $\approx$ and the *logical connectives* $\neg, \wedge, \vee, \rightarrow$, we can compose $\mathcal{L}$-terms to obtain $\mathcal{L}$-formulas.

Definition 2.1.3. The set $\mathsf{Fm}_{\mathcal{L}}$ of *$\mathcal{L}$-formulas* is defined recursively to be the least set such that:

- $t_1 \approx t_2 \in \mathsf{Fm}_{\mathcal{L}}$ for every $t_1, t_2 \in T_{\mathcal{L}}$;
- $R(t_1, \dots, t_n) \in \mathsf{Fm}_{\mathcal{L}}$ for every n-ary relation symbol R and $t_1, \dots, t_n \in T_{\mathcal{L}}$;
- If $\varphi_1, \varphi_2 \in \mathsf{Fm}_{\mathcal{L}}$, then the following are $\mathcal{L}$-formulas:
  - $\neg\varphi_1$;
  - $\varphi_1 \wedge \varphi_2$;
  - $\varphi_1 \vee \varphi_2$;
  - $\varphi_1 \rightarrow \varphi_2$;
  - $\forall x \varphi_1$ for every $x \in \mathsf{Var}$;
  - $\exists x \varphi_1$ for every $x \in \mathsf{Var}$.

Given two $\mathcal{L}$-formulas φ and ψ, we abbreviate the formula $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ as $\varphi \leftrightarrow \psi$. For a set $\Sigma \subseteq \mathsf{Fm}_{\mathcal{L}}$ we will use the shorthand $\neg\Sigma$ to denote the set $\{\neg\sigma : \sigma \in \Sigma\}$.

When the language $\mathcal{L}$ is irrelevant or clear from the context, we will simply write T and Fm for the $\mathcal{L}$-terms and $\mathcal{L}$-formulas.

Definition 2.1.4. Next, we define the set of *free variables* $\mathsf{Free}(\varphi)$ of an $\mathcal{L}$-formula φ. This is also done recursively:

- If $\varphi = t_1 \approx t_2$ for some $t_1, t_2 \in T_{\mathcal{L}}$, then $\mathsf{Free}(\varphi) = \mathsf{Var}(t_1) \cup \mathsf{Var}(t_2)$;
- If $\varphi = R(t_1, \dots, t_n)$ for some $t_1, \dots, t_n \in T_{\mathcal{L}}$ and some n-ary relation symbol R, then $\mathsf{Free}(\varphi) = \bigcup_{i \leqslant n} \mathsf{Var}(t_i)$;
- If there are $\varphi_1, \varphi_2 \in T_{\mathcal{L}}$ such that
  - $\varphi = \neg\varphi_1$, then $\mathsf{Free}(\varphi) = \mathsf{Free}(\varphi_1)$;
  - $\varphi = \varphi_1 * \varphi_2$, where $* \in \{\wedge, \vee, \rightarrow\}$, then $\mathsf{Free}(\varphi) = \mathsf{Free}(\varphi_1) \cup \mathsf{Free}(\varphi_1)$;
  - $\varphi = Qx\varphi_1$, where $x \in \mathsf{Var}$ and $Q \in \{\forall, \exists\}$, then $\mathsf{Free}(\varphi) = \mathsf{Free}(\varphi_1) \smallsetminus \{x\}$.

For a tuple of variables $\vec{x} = \langle x_i : i \in I \rangle$, we write $\varphi(\vec{x})$ when we want to emphasize that $\mathsf{Free}(\varphi) \subseteq \{x_i : i \in I\}$. Notice that $\mathsf{Free}(\varphi)$ is always finite because formulas are finite strings of symbols.

Of particular interest will be the $\mathcal{L}$-formulas without free variables, which are called *$\mathcal{L}$-sentences*.

So far, this formalism consists only of strings of symbols. The next step is to fill these symbols with meaning. This idea is formalized in the following definition.

Definition 2.1.5. An *$\mathcal{L}$-structure* $\boldsymbol{A} = (A, i^{\boldsymbol{A}})$ is a non-empty set A, called the *universe of $\boldsymbol{A}$*, together with an *interpretation map $i^{\boldsymbol{A}}$* that takes

- each function symbol $F \in \mathcal{F}$ to a $\tau(F)$-ary function $f^{\boldsymbol{A}} \colon A^{\tau(F)} \to A$;
- each relation symbol R to a $\tau(R)$-ary relation $R^{\boldsymbol{A}} \subseteq A^{\tau(R)}$.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\mathcal{L}$-structures, we say they are *similar*. Likewise, we talk about a class K of *similar structures* when all of its members are structures of a common language $\mathcal{L}$.

We can evaluate an $\mathcal{L}$-formula in an $\mathcal{L}$-structure $\boldsymbol{A}$ by applying it to a tuple $\vec{a}$ of elements of A.

Definition 2.1.6. Let $\boldsymbol{A}$ be an $\mathcal{L}$-structure, $\varphi(\vec{x}) \in \mathsf{Fm}_{\mathcal{L}}$ and $\vec{a}$ a tuple of elements of A of the same length as $\vec{x}$. We say that $\boldsymbol{A}$ validates $\varphi(\vec{a})$ and write $\boldsymbol{A} \vDash \varphi(\vec{a})$ if one of the following holds:

- $\varphi(\vec{x}) = t_1(\vec{x}) \approx t_2(\vec{x})$ and $t_1^{\boldsymbol{A}}(\vec{a}) = t_2^{\boldsymbol{A}}(\vec{a})$;
- $\varphi(\vec{x}) = R(t_1(\vec{x}), \ldots, t_n(\vec{x}))$ and $\langle t_1^{\boldsymbol{A}}(\vec{a}), \ldots, t_n^{\boldsymbol{A}}(\vec{a}) \rangle \in R^{\boldsymbol{A}}$;
- $\varphi(\vec{x}) = \neg\psi(\vec{x})$ and it is not the case that $\boldsymbol{A} \vDash \psi(\vec{a})$;
- $\varphi(\vec{x}) = \psi_1(\vec{x}) \wedge \psi_2(\vec{x})$ and $\boldsymbol{A} \vDash \psi_1(\vec{a})$ and $\boldsymbol{A} \vDash \psi_2(\vec{a})$;
- $\varphi(\vec{x}) = \psi_1(\vec{x}) \vee \psi_2(\vec{x})$ and $\boldsymbol{A} \vDash \psi_1(\vec{a})$ or $\boldsymbol{A} \vDash \psi_2(\vec{a})$;
- $\varphi(\vec{x}) = \psi_1(\vec{x}) \to \psi_2(\vec{x})$ and $\boldsymbol{A} \vDash \psi_2(\vec{a})$ or it is not the case that $\boldsymbol{A} \vDash \psi_1(\vec{a})$;
- $\varphi(\vec{x}) = \forall y \psi(\vec{x}, y)$ and for every $b \in A$ we have $\boldsymbol{A} \vDash \psi(\vec{a}, b)$;
- $\varphi(\vec{x}) = \exists y \psi(\vec{x}, y)$ and there exists $b \in A$ such that $\boldsymbol{A} \vDash \psi(\vec{a}, b)$.

We will extend this interpretation in the canonical way to infinite conjunctions and disjunctions. Also, we will allow quantification over infinite sets of variables, using expressions like $\boldsymbol{A} \vDash \forall \vec{x} \bigwedge_{i \in I} \varphi_i(\vec{x})$ or $\boldsymbol{A} \vDash \bigvee_{i \in I} \varphi_i(\vec{a})$, when $\{\varphi_i : i \in I\}$ is an infinite set of formulas and $\vec{x}$ and $\vec{a}$ are infinite tuples. Notice, however, that these examples are not elements of Fm.

Definition 2.1.7. We say that two $\mathcal{L}$-formulas $\varphi(\vec{x})$ and $\psi(\vec{x})$ are *equivalent*, when for every $\mathcal{L}$-structure $\boldsymbol{A}$ and every tuple $\vec{a}$ of elements of A we have

$$\boldsymbol{A} \vDash \varphi(\vec{a}) \text{ iff } \boldsymbol{A} \vDash \psi(\vec{a}).$$

Furthermore, we will use the following notational conventions:

- Given an $\mathcal{L}$-formula $\varphi(\vec{x})$, the notation $\boldsymbol{A} \vDash \varphi(\vec{x})$ or $\boldsymbol{A} \vDash \varphi$ is to be interpreted as $\boldsymbol{A} \vDash \forall\vec{x}\varphi(\vec{x})$;
- For a set Σ of $\mathcal{L}$-sentences, we will use the shorthand $\boldsymbol{A} \vDash \Sigma$ to express that $\boldsymbol{A} \vDash \sigma$ for every $\sigma \in \Sigma$;
- The notation $\Sigma \vDash \Gamma$, where Σ and Γ are sets of $\mathcal{L}$-sentences, means that for every $\mathcal{L}$-structure $\boldsymbol{A}$:
$$\boldsymbol{A} \vDash \Sigma \text{ implies } \boldsymbol{A} \vDash \Gamma;$$
- We will write $\mathsf{K} \vDash \varphi$ for a class of $\mathcal{L}$-algebras K and an $\mathcal{L}$-formula φ, when $\boldsymbol{A} \vDash \varphi$ for every $\boldsymbol{A} \in \mathsf{K}$.

We distinguish some special types of formulas.

Definition 2.1.8. An $\mathcal{L}$-formula is called

- an *equation* if it is of the form $t_1 \approx t_2$ for some terms $t_1, t_2 \in T_{\mathcal{L}}$;
- *atomic* if it is an equation or of the form $R(t_1, \ldots, t_n)$ for some n-ary relation symbol R and some terms $t_1, \ldots, t_n \in T_{\mathcal{L}}$. The set of atomic formulas of $\mathcal{L}$ will be denoted by $\mathsf{At}(\mathcal{L})$ or simply At;
- *quantifier-free* if it does not contain any of the quantifiers $\exists$ and $\forall$;
- *positive quantifier-free* if it is built from atomic formulas using only the connectives $\wedge$ and $\vee$;
- *positive existential* if it is a positive quantifier-free formula prenexed by a sequence of existential quantifiers;
- *primitive positive* (p.p., for short) if it is a finite conjunction of atomic formulas prenexed by a sequence of existential quantifiers.

Quantifier-free formulas are particularly nice to work with since they can be assumed to be in a canonical normal form.

Definition 2.1.9. A quantifier-free formula φ is in *disjunctive normal form* when it is of the shape $\bigvee_{i \leqslant n} \bigwedge_{j \leqslant m} \alpha_{i,j}$ for some $n, m \in \mathbb{N}$, where
$$\{\alpha_{i,j} : i \leqslant n, j \leqslant m\} \subseteq \mathsf{At} \cup \neg\mathsf{At}.$$

Lemma 2.1.10. *(see, e.g., [8, Thm. V.1.20]) Every quantifier-free formula is equivalent to one in disjunctive normal form.*

Observe that every set Σ of $\mathcal{L}$-sentences defines a set of $\mathcal{L}$-structures, namely those that validate every sentence in Σ.

Definition 2.1.11. Given a set Σ of $\mathcal{L}$-sentences, we say that an $\mathcal{L}$-structure $\boldsymbol{A}$ is a *model of* Σ when $\boldsymbol{A} \vDash \Sigma$. We write $\mathsf{Mod}(\Sigma)$ for the class of models of Σ, and we call Σ *satisfiable* when $\mathsf{Mod}(\Sigma) \neq \emptyset$.

Conversely, every class K of $\mathcal{L}$-structures defines a set of $\mathcal{L}$-sentences, given by all the sentences that are valid in every member of K.

Definition 2.1.12. For a class K of $\mathcal{L}$-structures the *theory of* K is the set of all the $\mathcal{L}$-sentences φ such that $\boldsymbol{A} \vDash \varphi$ for every $\boldsymbol{A} \in \mathsf{K}$. We denote the theory of K by $\mathsf{Th}(\mathsf{K})$.

Notice that for every set of sentences Σ and every class of similar structures K we have that

$$\mathsf{K} \subseteq \mathsf{Mod}(\mathsf{Th}(\mathsf{K}))$$

and

$$\Sigma \subseteq \mathsf{Th}(\mathsf{Mod}(\Sigma)).$$

The converse inclusions, however, do not hold in general. Nevertheless, there are classes for which we do have equality. This property is captured by the following definition.

Definition 2.1.13. A class K of $\mathcal{L}$-structures is called *elementary* when

$$\mathsf{K} = \mathsf{Mod}(\mathsf{Th}(\mathsf{K})).$$

Definition 2.1.14. Two similar structures $\boldsymbol{A}$ and $\boldsymbol{B}$ are called *elementarily equivalent* when they satisfy the same sentences, that is, $\mathsf{Th}(\boldsymbol{A}) = \mathsf{Th}(\boldsymbol{B})$.

2.1.2 Class operators

We will now continue by considering relations between similar structures. This will allow us to define class operators that generate new classes of structures from given ones (see, e.g., [8, Sec. II and IV] and [4, Sec. 1]).

Definition 2.1.15. Given an $\mathcal{L}$-structure $\boldsymbol{B}$, we call a subset $A \subseteq B$ a *subuniverse* of $\boldsymbol{B}$ when $f^{\boldsymbol{B}}(a_1, \dots, a_n) \in A$ for every n-ary function symbol f and $a_1, \dots, a_n \in A$.

If, furthermore, $A \neq \emptyset$, then A is the universe of the structure $\boldsymbol{A}$, with $f^{\boldsymbol{A}} = f^{\boldsymbol{B}}{\restriction}_A$ and $R^{\boldsymbol{A}} = R^{\boldsymbol{B}}{\restriction}_A$, where $f^{\boldsymbol{B}}{\restriction}_A$ and $R^{\boldsymbol{B}}{\restriction}_A$ denote the restrictions of $f^{\boldsymbol{B}}$ and $R^{\boldsymbol{B}}$ to A^n for every n-ary function symbol f and relation symbol R. In this case $\boldsymbol{A}$ is a *substructure* of $\boldsymbol{B}$ and we write $\boldsymbol{A} \leqslant \boldsymbol{B}$. We also say that $\boldsymbol{B}$ is an *extension* of $\boldsymbol{A}$. A subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B}$ is *proper*, when $A \neq B$. Given a class of similar algebras K, the class of substructures of its members is denoted by $\mathbb{S}(\mathsf{K})$.

The smallest subuniverse of $\boldsymbol{B}$ containing s subset $A \subseteq B$ is called the *subuniverse generated by* A and is denoted by $\mathsf{Sg}^{\boldsymbol{B}}(A)$. For a finite set $A = \{a_1, \dots, a_n\}$ we will simply write $\mathsf{Sg}^{\boldsymbol{B}}(a_1, \dots, a_n)$. In this case we say that the algebra with universe $\mathsf{Sg}^{\boldsymbol{B}}(A)$ is *finitely generated with set of generators* $\{a_1, \dots, a_n\}$.

The following observation (see, e.g., [22, Lem. 1.2.2.]) is often useful when working with generated subuniverses.

Lemma 2.1.16. *Let* $\boldsymbol{B}$ *be an* $\mathcal{L}$*-structure and* $A \subseteq B$*. Then,*

$$\mathsf{Sg}^{\boldsymbol{B}}(A) = \{t^{\boldsymbol{B}}(a_1, \dots, a_n) : t \in T_{\mathcal{L}} \text{ is of arity } n \text{ for some } n \in \mathbb{N} \text{ and } a_1, \dots, a_n \in A\}.$$

Definition 2.1.17. A *homomorphisms* $h \colon \boldsymbol{A} \to \boldsymbol{B}$ between two similar structures $\boldsymbol{A}$ and $\boldsymbol{B}$ is a map $h \colon A \to B$ between their universes such that for every n-ary function symbol f, every n-ary relation symbol R, and $a_1, \dots, a_n \in A$ we have:

$$h(f^{\boldsymbol{A}}(a_1, \dots, a_n)) = f^{\boldsymbol{B}}(h(a_1), \dots, h(a_n))$$

and

$$\langle a_1, \dots, a_n \rangle \in R^{\boldsymbol{A}} \text{ implies } \langle h(a_1), \dots, h(a_n) \rangle \in R^{\boldsymbol{B}}.$$

In case that

$$\langle a_1, \dots, a_n \rangle \in R^{\boldsymbol{A}} \text{ iff } \langle h(a_1), \dots, h(a_n) \rangle \in R^{\boldsymbol{B}},$$

for every n-ary relation symbol R and every $a_1, \dots, a_n \in A$, we call h a *strong homomorphism*.

We also say that h is an $\mathcal{L}$-homomorphism when we want to emphasize the language whose function and relation symbols are preserved by h.

An injective strong homomorphism is called an *embedding* and a bijective strong homomorphism is an *isomorphism*. If $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an isomorphism, we write $\boldsymbol{A} \cong \boldsymbol{B}$ and say that $\boldsymbol{A}$ is *isomorphic* to $\boldsymbol{B}$.

Definition 2.1.18. The *kernel* of a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is the set

$$\mathsf{ker}(h) := \{\langle a_1, a_2 \rangle \in A \times A : h(a_1) = h(a_2)\}.$$

Remark 2.1.19. Observe that given a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ the image $h[A]$ is a subuniverse of B, and thus $h[\boldsymbol{A}] \leqslant \boldsymbol{B}$ is a substructure, called the *homomorphic image* of $\boldsymbol{A}$ under h. Given a class of similar structures K, the class of homomorphic images of its members is denoted by $\mathbb{H}(\mathsf{K})$. For the class of images of members of K under isomorphisms, we use the notation $\mathbb{I}(\mathsf{K})$.

Next, we will introduce some constructions that allow us to build new structures from a set of given structures. For a tuple $\vec{a}$, we use the notation $\vec{a}(i)$ to refer to the i^{th} entry of $\vec{a}$.

Definition 2.1.20. Given a set of similar structures $\{\boldsymbol{A}_i : i \in I\}$, we define their *(direct) product* to be the unique structure $\boldsymbol{A} := \prod_{i \in I} \boldsymbol{A}_i$ with universe $\prod_{i \in I} A_i$, where for every n-ary function symbol f, every n-ary relation symbol R, and $\vec{a_1}, \dots, \vec{a_n} \in A$ we define:

$$f^{\boldsymbol{A}}(\vec{a_1}, \dots, \vec{a_n}) := \langle f^{\boldsymbol{A}_i}(\vec{a_1}(i), \dots, \vec{a_n}(i)) : i \in I \rangle$$

and

$$R^{\boldsymbol{A}} := \{\langle \vec{a_1}, \dots, \vec{a_n} \rangle : \langle \vec{a_1}(i), \dots, \vec{a_n}(i) \rangle \in R^{\boldsymbol{A}_i} \text{ for every } i \in I\}.$$

The notation $\mathbb{P}(\mathsf{K})$ refers to the class of all products of members of K. Furthermore, for every $m \in \mathbb{Z}^+$, we use $\mathbb{P}_m(\mathsf{K})$ to denote the class of all products of m members of K, where repetitions are allowed.

From a product, in turn, we can get a new structure by quotienting under a suitable equivalence relation. In order to carry out this construction, we need the following definition.

Definition 2.1.21. An *ultrafilter* on a set I is a proper subset $U \subset \mathcal{P}(I)$ such that:

- If $X \in U$ and $X \subseteq Y$, then $Y \in U$;
- for every $X, Y \in U$, we have $X \cap Y \in U$;
- for every $X \in \mathcal{P}(I)$, either $X \in U$ or $I \smallsetminus X \in U$.

Using ultrafilters, we are now ready to define ultraproducts.

Definition 2.1.22. Given a set of similar structures $\{\boldsymbol{A}_i : i \in I\}$ and an ultrafilter U on I, we define a relation $\sim_U$ on $\prod_{i\in I} A_i$ via

$$\vec{a} \sim_U \vec{b} :\Longleftrightarrow \{i \in I : \vec{a}(i) = \vec{b}(i)\} \in U.$$

It is easy to verify that $\sim_U$ is an equivalence relation. We will denote the equivalence class of an element $\vec{a} \in \prod_{i\in I} A_i$ by $\vec{a}/U$. Then, we define the *ultraproduct*

$$\boldsymbol{A} := \prod_{i\in I} \boldsymbol{A}_i/U$$

of the family $\{\boldsymbol{A}_i : i \in I\}$ to be the unique structure with universe $\{\vec{a}/U : \vec{a} \in \prod_{i\in I} A_i\}$, where for every n-ary function symbol f, for every n-ary relation symbol R, and $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$ we set:

$$f^{\boldsymbol{A}}(\vec{a}_1/U, \dots, \vec{a}_n/U) = \langle f^{\boldsymbol{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i)) : i \in I\rangle/U$$

and

$$R^{\boldsymbol{A}} := \{\langle \vec{a}_1/U, \dots, \vec{a}_n/U\rangle : \{i \in I : \langle \vec{a}_1(i), \dots, \vec{a}_n(i)\rangle \in R^{\boldsymbol{A}_i}\} \in U\}.$$

Using the definition of $\sim_U$ and the properties of ultrafilters, it is straightforward to see that these operations are well defined. When $\boldsymbol{A}_i = \boldsymbol{B}$ for every $i \in I$, we call $\boldsymbol{A} := \prod_{i\in I} \boldsymbol{B}/U$ an *ultrapower* of $\boldsymbol{B}$. The shorthand $\mathbb{P}_u(\mathsf{K})$ stands for the class of all ultraproducts of families of members of K.

Definition 2.1.23. We say that a class K of similar structures is *closed under* a class operator $\mathbb{O} \in \{\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_u\}$, when $\mathbb{O}(\mathsf{K}) \subseteq \mathsf{K}$.

The next lemma (see, e.g., [8, Lems. II.9.2 and V.2.22] or [18, Sec. 3.23]) provides some well-known facts about how some of these class operators interact with each other:

Lemma 2.1.24. *For a class K of similar structures, the following hold:*

1. $\mathbb{O}\mathbb{O}(\mathsf{K}) \subseteq \mathbb{I}\mathbb{O}(\mathsf{K})$ *for every* $\mathbb{O} \in \{\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_u\}$;
2. $\mathbb{O}\mathbb{I}(\mathsf{K}) \subseteq \mathbb{I}\mathbb{O}(\mathsf{K})$ *for every* $\mathbb{O} \in \{\mathbb{I}, \mathbb{H}, \mathbb{S}, \mathbb{P}, \mathbb{P}_u\} \cup \{\mathbb{P}_m : m \in \mathbb{Z}^+\}$;
3. $\mathbb{P}\mathbb{S}(\mathsf{K}) \subseteq \mathbb{S}\mathbb{P}(\mathsf{K})$;
4. $\mathbb{S}\mathbb{H}(\mathsf{K}) \subseteq \mathbb{H}\mathbb{S}(\mathsf{K})$;
5. $\mathbb{P}\mathbb{H}(\mathsf{K}) \subseteq \mathbb{H}\mathbb{P}(\mathsf{K})$.

Moreover, we have the following result for the class operators $\mathbb{P}_u$ and $\mathbb{P}_m$ (see, e.g., [19]):

Lemma 2.1.25. *Let K be a class of similar structures. Then, $\mathbb{P}_u\mathbb{P}_m(\mathsf{K}) \subseteq \mathbb{I}\mathbb{P}_m\mathbb{P}_u(\mathsf{K})$ for every $m \in \mathbb{Z}^+$.*

Proof. Let $\boldsymbol{A} \in \mathbb{P}_u\mathbb{P}_m(\mathsf{K})$. Then, $\boldsymbol{A}$ is of the form $\prod_{j\in J}(\boldsymbol{A}_{1,j} \times \cdots \times \boldsymbol{A}_{m,j})/U$ for some $\boldsymbol{A}_{1,j}, \dots, \boldsymbol{A}_{m,j} \in \mathsf{K}$ and some ultrafilter U on J. To prove the claimed inclusion, we will show that the map

$$h : \prod_{j\in J}(\boldsymbol{A}_{1,j} \times \cdots \times \boldsymbol{A}_{m,j})/U \to \prod_{j\in J} A_{1,j}/U \times \cdots \times \prod_{j\in J} A_{m,j}/U$$

defined via the assignment

$$\langle\langle a_{1,j},\ldots,a_{m,j}\rangle : j\in J\rangle/U \mapsto \langle\langle a_{1,j} : j\in J\rangle/U,\ldots,\langle a_{m,j} : j\in J\rangle/U\rangle$$

is an isomorphism. That h is surjective follows directly from the definition. To verify well-definedness and injectivity, observe the following:

$$\begin{aligned}
&\langle\langle a_{1,j},\ldots,a_{m,j}\rangle : j\in J\rangle/U = \langle\langle b_{1,j},\ldots,b_{m,j}\rangle : j\in J\rangle/U\\
\iff&\{j\in J : \langle a_{1,j},\ldots,a_{m,j}\rangle = \langle b_{1,j},...,b_{m,j}\rangle\}\in U\\
\iff&\{j\in J : a_{i,j}=b_{i,j}\}\in U \text{ for every } i\leqslant m\\
\iff&\langle a_{i,j} : j\in J\rangle/U = \langle b_{i,j} : j\in J\rangle/U \text{ for every } i\leqslant m\\
\iff&\langle\langle a_{1,j} : j\in J\rangle/U,\ldots,\langle a_{m,j} : j\in J\rangle/U\rangle = \langle\langle b_{1,j} : j\in J\rangle/U,\ldots,\langle b_{m,j} : j\in J\rangle/U\rangle.
\end{aligned}$$

We will only justify the second equivalence since the other ones are immediate consequences of the definition of ultraproducts. Observe that

$$\{j\in J : \langle a_{1,j},\ldots,a_{m,j}\rangle = \langle b_{1,j},...,b_{m,j}\rangle\} = \bigcap_{i\leqslant m}\{j\in J : a_{i,j}=b_{i,j}\}\subseteq\{j\in J : a_{i,j}=b_{i,j}\}$$

for every $i\leqslant m$. As ultrafilters are upsets and closed under finite intersections by definition, we obtain the desired equivalence

$$\{j\in J : \langle a_{1,j},\ldots,a_{m,j}\rangle = \langle b_{1,j},...,b_{m,j}\rangle\}\in U \text{ iff } \{j\in J : a_{i,j}=b_{i,j}\}\in U \text{ for every } i\leqslant m.$$

Finally, it is a straightforward consequence of the definition of ultraproducts that h is a homomorphism. Therefore, h is an isomorphisms, which proves the claimed inclusion $\mathbb{P}_u\mathbb{P}_m(\mathsf{K})\subseteq\mathbb{I}\mathbb{P}_m\mathbb{P}_u(\mathsf{K})$. ⊠

It is easy to see that the validity of any formula is preserved under isomorphisms.

Remark 2.1.26.

$$\mathsf{K}\vDash\varphi \text{ iff } \mathbb{I}(\mathsf{K})\vDash\varphi$$

for every formula φ and every class of algebras K.

Other class operators only preserve certain types of formulas, as specified in the following lemmas. Although they can be stated in a more general form, we will focus on the specific cases we will need in the course of the thesis. First, we will consider the preservation of positive existential formulas under homomorphisms (see, e.g., [22, Thm. 2.4.3.(a)]).

Lemma 2.1.27. *Let $\varphi(\vec{x})$ be a positive existential formula, $\boldsymbol{A}$ an $\mathcal{L}$-structure, and $\vec{a}$ a tuple of elements of A. Then,*

$$\boldsymbol{A}\vDash\varphi(\vec{a}) \textit{ implies } \boldsymbol{B}\vDash\varphi(h(\vec{a})),$$

for every $\mathcal{L}$-structure $\boldsymbol{B}$ and every homomorphism $h\colon \boldsymbol{A}\to\boldsymbol{B}$.

It follows as a straightforward consequence that positive existential formulas are preserved under extensions.

Corollary 2.1.28. *Let $\varphi(\vec{x})$ be a positive existential formula, $\boldsymbol{A}$ an $\mathcal{L}$-structure, and $\vec{a}$ a tuple of elements of A. Then,*

$$\boldsymbol{A}\vDash\varphi(\vec{a}) \textit{ implies } \boldsymbol{B}\vDash\varphi(\vec{a}),$$

for every $\mathcal{L}$-structure $\boldsymbol{B}$ such that $\boldsymbol{A}\leqslant\boldsymbol{B}$.

Proof. Apply Lemma 2.1.27 to the inclusion map $i\colon \boldsymbol{A} \hookrightarrow \boldsymbol{B}$. ☒

Notice that p.p. formulas and positive quantifier-free formulas are in particular positive existential formulas, which makes the above result applicable to these types of formulas as well. For p.p. formulas, we have in addition preservation under products (see, e.g., [21, Lem. 9.1.4.]).

Lemma 2.1.29. *Let $\varphi(x_1, \dots, x_n)$ be a p.p. formula, $\{\boldsymbol{A}_i : i \in I\}$ a set of $\mathcal{L}$-structures, and $a_{1,i}, \dots, a_{n,i} \in A_i$ such that $\boldsymbol{A}_i \models \varphi(a_{1,i}, \dots, a_{n,i})$ for every $i \in I$. Then,*

$$\prod_{i \in I} \boldsymbol{A}_i \models \varphi(\langle a_{1,i} : i \in I \rangle, \dots, \langle a_{n,i} : i \in I \rangle).$$

Finally, positive quantifier-free formulas are preserved under subalgebras (see, e.g., [22, Cor. 2.4.2.(a)]).

Lemma 2.1.30. *Let $\varphi(\vec{x})$ be a positive quantifier-free formula, $\boldsymbol{A}$ an $\mathcal{L}$-structure, and $\vec{a}$ a tuple of elements of A. Then,*

$$\boldsymbol{A} \models \varphi(\vec{a}) \textit{ implies } \boldsymbol{B} \models \varphi(\vec{a}),$$

for every subalgebra $\boldsymbol{B} \leqslant \boldsymbol{A}$ that contains the tuple $\vec{a}$.

2.2 Model Theory

In this section, we will build some model-theoretic background, which is needed in the following chapters. For more information in this area, see, e.g., [13] and [22].

Given a class K of $\mathcal{L}$-structures, we sometimes want to work in an expanded language $\mathcal{L}' \supseteq \mathcal{L}$. The members of K can then be turned into $\mathcal{L}'$-structures by giving interpretations for the additional function and relation symbols in the expanded language $\mathcal{L}'$.

Conversely, we can also restrict the members of K to a smaller language $\mathcal{L}' \subseteq \mathcal{L}$ by just dropping the interpretations of the function and relation symbols that do not appear in $\mathcal{L}'$. For every $\boldsymbol{A} \in \mathsf{K}$, we write $\boldsymbol{A}{\restriction}_{\mathcal{L}'}$ to denote the unique $\mathcal{L}'$-structure that coincides with $\boldsymbol{A}$ on the interpretations of all function and relation symbols in $\mathcal{L}'$. Also, for a class K, we define $\mathsf{K}{\restriction}_{\mathcal{L}'} := \{\boldsymbol{A}{\restriction}_{\mathcal{L}'} : \boldsymbol{A} \in \mathsf{K}\}$. Given an $\mathcal{L}$-homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$, we will write $h{\restriction}_{\mathcal{L}'}$ if we want to consider h as the homomorphism restricted to the $\mathcal{L}'$-structures $\boldsymbol{A}{\restriction}_{\mathcal{L}'}$ and $\boldsymbol{B}{\restriction}_{\mathcal{L}'}$.

We will often work with expansions of a language $\mathcal{L}$ by a tuple of new constants $\vec{c}$, the so called *constant expansions*. The resulting language will then be denoted by $\mathcal{L}_{\vec{c}}$. For an $\mathcal{L}$-structure $\boldsymbol{B}$, we use the notation $\boldsymbol{B}_{\vec{b}}$ to denote the unique $\mathcal{L}_{\vec{c}}$- structure with $\boldsymbol{B}_{\vec{b}}{\restriction}_{\mathcal{L}} = \boldsymbol{B}$, where the tuple of constant $\vec{c}$ is interpreted by $\vec{b}$.

When $\vec{c} = \langle c_a : a \in A \rangle$ for some $\mathcal{L}$-structure $\boldsymbol{A}$, we also write $\mathcal{L}_A$ for the enriched language $\mathcal{L} \cup \{c_a : a \in A\}$. To simplify notation, we often write a instead of c_a. If $\boldsymbol{B} \in \mathsf{K}$ contains $\boldsymbol{A}$ as a substructure, $\boldsymbol{B}$ can be canonically extended to an $\mathcal{L}_A$-structure, which we refer to as $\boldsymbol{B}_A$, by interpreting each constant as itself, i.e., $a^{\boldsymbol{B}_A} = a$.

The next lemma unravels how classes of algebras in a constant expansion relate to their counterparts in the original language.

Lemma 2.2.1. *Let K be a class of $\mathcal{L}$-structures and Σ a set of $\mathcal{L}$-formulas with free variables in $\vec{x} = \langle x_i : i \in I \rangle$. Consider the class*

$$\mathsf{K}^* := \{\boldsymbol{A} : \boldsymbol{A} \textit{ is an } \mathcal{L}_{\vec{c}} \textit{ -structure and } \boldsymbol{A}{\restriction}_{\mathcal{L}} \in \mathsf{K}\},$$

where $\vec{c} = \langle c_i : i \in I \rangle$ is a tuple of new constants. Then,

$$\mathsf{K} \vDash \Sigma(\vec{x}) \text{ iff } \mathsf{K}^* \vDash \Sigma(\vec{c}).$$

Proof. Assume first $\mathsf{K} \vDash \Sigma(\vec{x})$ and consider an $\mathcal{L}_{\vec{c}}$-structure $\boldsymbol{A}^* \in \mathsf{K}^*$. We have to show that $\boldsymbol{A}^* \vDash \Sigma(\vec{c})$. Let $\vec{a} = \langle a_i : i \in I \rangle$ be the tuple of elements of A^* that interprets the tuple of new constants $\vec{c}$ in $\boldsymbol{A}^*$, i.e., $c_i^{\boldsymbol{A}^*} = a_i$ for every $i \in I$. As $\boldsymbol{A}^*{\restriction}_{\mathcal{L}} \in \mathsf{K}$ and $\mathsf{K} \vDash \Sigma(\vec{x})$ by assumption, we conclude that $\boldsymbol{A}^*{\restriction}_{\mathcal{L}} \vDash \Sigma(\langle a_i : i \in I \rangle)$, and thus, $\boldsymbol{A}^* \vDash \Sigma(\vec{c})$.

Conversely, suppose $\mathsf{K}^* \vDash \Sigma(\vec{c})$ and consider $\boldsymbol{A} \in \mathsf{K}$ and $\vec{a} = \langle a_i : i \in I \rangle$ a tuple of elements of A. We have to verify that $\boldsymbol{A} \vDash \Sigma(\vec{a})$. Define $\boldsymbol{A}^*$ to be the unique $\mathcal{L}_{\vec{c}}$ -structure with $\boldsymbol{A}^*{\restriction}_{\mathcal{L}} = \boldsymbol{A}$ and $c_i^{\boldsymbol{A}^*} = a_i$ for every $i \in I$. Then, as $\boldsymbol{A}^* \in \mathsf{K}^*$ and $\mathsf{K}^* \vDash \Sigma(\vec{c})$ by assumption, we conclude that $\boldsymbol{A} \vDash \Sigma(\vec{a})$. ⊠

Corollary 2.2.2. *Let Σ and Γ be sets of formulas of a language $\mathcal{L}$ that contains a set of constants C. Also, let $\vec{x} = \langle c_i : i \in I \rangle$ be a tuple of variables and $\vec{c} = \langle c_i : i \in I \rangle$ a tuple of constants in C that does not appear in Σ. Then,*

$$\Sigma \vDash \Gamma(\vec{c}) \text{ iff } \Sigma \vDash \Gamma(\vec{x}).$$

Proof. Let $\mathcal{L}'$ be the language $\mathcal{L}$ without the constants $\{c_i : i \in I\}$. Assume first $\Sigma \vDash \Gamma(\vec{c})$ and let $\boldsymbol{A} \vDash \Sigma$. By assumption, it follows that $\boldsymbol{A} \vDash \Gamma(\vec{c})$. The previous Lemma 2.2.1 then implies that $\boldsymbol{A}{\restriction}_{\mathcal{L}'} \vDash \Gamma(\vec{x})$, and thus also $\boldsymbol{A} \vDash \Gamma(\vec{x})$.

Conversely, suppose $\Sigma \vDash \Gamma(\vec{x})$. As Σ is a set of $\mathcal{L}'$-sentences, $\boldsymbol{A} \vDash \Sigma$ is equivalent to $\boldsymbol{A}{\restriction}_{\mathcal{L}'} \vDash \Sigma$. By assumption, it follows that $\boldsymbol{A}{\restriction}_{\mathcal{L}'} \vDash \Gamma(\vec{x})$. Then, applying the previous Lemma 2.2.1, we conclude that $\boldsymbol{A} \vDash \Gamma(\vec{c})$, as claimed. ⊠

Our main focus will now be to establish useful properties of ultraproducts, present some compactness results, and introduce the concept of diagrams.

2.2.1 Ultraproducts and compactness

The class operator $\mathbb{P}_u$ is particularly well behaved, since the validity of a formula in an ultraproduct can be determined by looking at its factors. This is the content of a famous theorem by Łoś (see, e.g., [8, Thm. V.2.9]).

Łoś's Theorem 2.2.3. *Let $\{\boldsymbol{A}_i : i \in I\}$ be a set of $\mathcal{L}$-structures. Then, for every ultrafilter U on I and every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, we have,*

$$\prod_{i \in I} \boldsymbol{A}_i / U \vDash \varphi(\vec{a}_1/U, \ldots, \vec{a}_n/U) \text{ iff } \{i \in I : \boldsymbol{A}_i \vDash \varphi(\vec{a}_1(i), \ldots, \vec{a}_n(i))\} \in U,$$

for every $\vec{a}_1, \ldots, \vec{a}_n \in \prod_{i \in I} A_i$.

To express this tight connection between an ultraproduct and its factors, we introduce the following notions.

Definition 2.2.4. A substructure $\boldsymbol{A} \leqslant \boldsymbol{B}$ is called *elementary* if every $\mathcal{L}_A$-sentence valid in $\boldsymbol{A}_A$ is also valid in $\boldsymbol{B}_A$. In this case we also say that $\boldsymbol{B}$ is an *elementary extension* of $\boldsymbol{A}$. An embedding $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is *elementary* when $h[\boldsymbol{A}]$ is an elementary substructure of $\boldsymbol{B}$.

Remark 2.2.5. With every ultrapower $\boldsymbol{A}^I/U$ of some structure $\boldsymbol{A}$, we associate a map $f \colon \boldsymbol{A} \to \boldsymbol{A}^I/U$ defined by the rule $a \mapsto \langle a_i : i \in I \rangle / U$, where $a_i = a$ for every $i \in I$. Using Łoś's Theorem 2.2.3, we obtain that f is always an elementary embedding (see, e.g., [8, Lem. V.2.10 and Thm. V.2.11]).

An important tool in model theory is the observation, that in order to prove the validity or satisfiability of infinite sets of formulas, oftentimes it suffices to consider finite subsets. This is the content of the following compactness results. First, we will see the classical Compactness Theorem of first-order logic (see, e.g., [8, Thm. V.2.12]).

Compactness Theorem 2.2.6. *Let Σ be a set of sentences. Then Σ is satisfiable iff every finite subset $\Sigma_0 \subseteq_\omega \Sigma$ is satisfiable.*

This result can easily be transformed into a semantical statement about elementary classes.

Corollary 2.2.7. *Let K be an elementary class such that $\mathsf{K} \vDash \bigvee \Sigma$. Then there exists a finite subset $\Sigma_0 \subseteq_\omega \Sigma$ such that $\mathsf{K} \vDash \bigvee \Sigma_0$.*

Proof. Let $\Delta := \mathsf{Th}(\mathsf{K})$ and assume with a view to contradiction that $\mathsf{K} \nvDash \bigvee \Sigma_0$ for every $\Sigma_0 \subseteq_\omega \Sigma$. This implies that every finite subset of $\Delta \cup \neg\Sigma$ is satisfiable, and thus, by the Compactness Theorem 2.2.6, so is $\Delta \cup \neg\Sigma$. But this is a contradiction with the assumptions $\mathsf{K} \vDash \bigvee \Sigma$ and $\Delta = \mathsf{Th}(\mathsf{K})$. ⊠

However, one can get a more general kind of compactness theorem for a class of similar structures K under the sole assumption that K is closed under ultraproducts. Let us first recall the following property.

Definition 2.2.8. Consider a set I and family J of subsets of I. Then J is said to have the *finite intersection property*, or FIP for short, if for every finite subset $J_0 \subseteq_\omega J$, we have $\bigcap J_0 \neq \emptyset$.

The interest in this property arises from the following theorem, which tells us when a family of subsets of a given set I can be extended to an ultrafilter on I (see, e.g.,[22, Lem. 8.5.5]).

Theorem 2.2.9. *Let I be a set and assume $J \subseteq \mathcal{P}(I)$ has the FIP. Then, there exists an ultrafilter U on I with $J \subseteq U$.*

We are now ready to state the announced compactness result for classes closed under ultraproducts (see, e.g., [8, Thm. V.2.12]).

Lemma 2.2.10. *Let K be a class of $\mathcal{L}$-structures closed under ultraproducts, and let Σ be a set of $\mathcal{L}$-sentences such that $\mathsf{K} \vDash \bigvee \Sigma$. Then, there exists a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\mathsf{K} \vDash \bigvee \Sigma_0$.*

Proof. Assume, with a view to contradiction, that this was not the case. Let $\mathcal{P}_\omega(\Sigma)$ be the set of all finite subsets of Σ. For every $F \in \mathcal{P}_\omega(\Sigma)$ we define

$$J_F := \{G \in \mathcal{P}_\omega(\Sigma) : F \subseteq G\} \text{ and } J := \{J_F : F \in \mathcal{P}_\omega(\Sigma)\}.$$

Notice that $\bigcup_{i \leqslant n} F_i \in \bigcap_{i \leqslant n} J_{F_i}$ for every finite family $\{F_i : i \leqslant n\} \subseteq \mathcal{P}_\omega(\Sigma)$. Therefore, J has the FIP, which, by Theorem 2.2.9, implies that there exists an ultrafilter U on $\mathcal{P}_\omega(\Sigma)$ with $J \subseteq U$. For every $F \in \mathcal{P}_\omega(\Sigma)$, choose $\boldsymbol{A}_F \in \mathsf{K}$ such that $\boldsymbol{A}_F \vDash \neg \bigvee F$. Observe that the existence of $\boldsymbol{A}_F$ is guaranteed by the assumption that $\mathsf{K} \nvDash \bigvee F$ for every $F \in \mathcal{P}_\omega(\Sigma)$. Now, define $\boldsymbol{B} := \prod_{F \in \mathcal{P}_\omega(\Sigma)} \boldsymbol{A}_F / U$. Since K is assumed to be closed under ultraproducts, we obtain that $\boldsymbol{B} \in \mathsf{K}$. Hence, the assumption that $\mathsf{K} \vDash \bigvee \Sigma$ implies that there exists $\varphi \in \Sigma$ such that $\boldsymbol{B} \vDash \varphi$. By Łoś's Theorem 2.2.3, it follows that $\{F \in \mathcal{P}_\omega : \boldsymbol{A}_F \vDash \varphi\} \in U$. Furthermore, $J_{\{\varphi\}} \in J \subseteq U$ by construction. As ultrafilters are closed under finite intersections, we conclude that there exists $G \in \{F \in \mathcal{P}_\omega : \boldsymbol{A}_F \vDash \varphi\} \cap J_{\{\varphi\}}$. But then $\boldsymbol{A}_G \vDash \varphi \wedge \neg\varphi$, a contradiction. ⊠

Corollary 2.2.11. *Let* K *be a class of* $\mathcal{L}$*-structures closed under ultraproducts, and let* $\Sigma(\vec{x})$ *be a set of* $\mathcal{L}$*-formulas such that* $\mathsf{K} \vDash \bigvee \Sigma(\vec{x})$. *Then there exists a finite subset* $\Sigma_0 \subseteq \Sigma$ *such that* $\mathsf{K} \vDash \bigvee \Sigma_0(\vec{x})$.

Proof. Define $\mathsf{K}^* := \{\boldsymbol{A} : \boldsymbol{A} \text{ is an } \mathcal{L}_{\vec{c}}\text{-algebra and } \boldsymbol{A}\restriction_{\mathcal{L}} \in \mathsf{K}\}$, where $\vec{c}$ is a tuple of new constants . By Lemma 2.2.1 it follows that $\mathsf{K} \vDash \bigvee \Sigma(\vec{x})$ is equivalent to $\mathsf{K}^* \vDash \bigvee \Sigma(\vec{c})$. Now, we can apply Lemma 2.2.10 to the set of $\mathcal{L}_{\vec{c}}$-sentences $\Sigma(\vec{c})$ to conclude that there exists a finite subset $\Sigma_0 \subseteq_\omega \Sigma$ such that $\mathsf{K}^* \vDash \bigvee \Sigma_0(\vec{c})$, which again by Lemma 2.2.1 is equivalent to $\mathsf{K} \vDash \bigvee \Sigma_0(\vec{x})$. ⊠

Corollary 2.2.12. *Let* K *be a class of* $\mathcal{L}$*-structures closed under ultraproducts, and* Σ *and* Γ *sets of* $\mathcal{L}$*-formulas such that* $\mathsf{K} \vDash \bigwedge \Sigma \to \bigvee \Gamma$. *Then, there exist finite subsets* $\Sigma_0 \subseteq \Sigma$ *and* $\Gamma_0 \subseteq \Gamma$ *such that* $\mathsf{K} \vDash \bigwedge \Sigma_0 \to \bigvee \Gamma_0$.

Proof. Notice that $\bigwedge \Sigma \to \bigvee \Gamma$ is equivalent to $\bigvee \neg\Sigma \vee \bigvee \Gamma$. Thus, applying Corollary 2.2.11 to the set $\neg\Sigma \cup \Gamma$ yields the desired result. ⊠

Next, we will give a short introduction to diagrams. They are an important model-theoretic tool that will be applied in various proofs throughout the thesis.

2.2.2 Diagrams

Definition 2.2.13. The *diagram of* $\boldsymbol{A}$ *generated by* $\vec{a}$ is the set of all atomic and negated atomic $\mathcal{L}_{\vec{a}}$-sentences that are valid in $\boldsymbol{A}_{\vec{a}}$. It is denoted by $\mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$.

The shorthand $\mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$ refers to the set of all atomic $\mathcal{L}_{\vec{a}}$-sentences valid in $\boldsymbol{A}_{\vec{a}}$, called the *positive diagram of* $\boldsymbol{A}$ *generated by* $\vec{a}$.

If $\vec{a} = \langle a : a \in A\rangle$, we will just write $\mathsf{Diag}(\boldsymbol{A})$ or $\mathsf{Diag}^+(\boldsymbol{A})$, respectively.

The (positive) diagram of $\boldsymbol{A}$ generated by $\vec{a} = \langle a_i : i \in I\rangle$ is *satisfiable in* $\boldsymbol{B}$ when there exists a tuple $\vec{b} = \langle b_i : i \in I\rangle$ of elements of B such that defining $\vec{a}(i)^{\boldsymbol{B}} := \vec{b}(i)$ for every $i \in I$ yields that $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$ (respectively $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$). However, we will often drop the subscript $\vec{b}$. Also, on some occasions, it will be convenient to use the notation $\boldsymbol{B} \vDash \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})(\vec{b})$ (respectively $\boldsymbol{B} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})(\vec{b})$) instead. Writing $\boldsymbol{B} \vDash \neg\mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$ (respectively $\boldsymbol{B} \vDash \neg\mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$), we mean that there exists no tuple $\vec{b}$ of elements of B such that $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$ (respectively $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$).

The importance of diagrams derives from their relation to the existence of certain homomorphisms, which will be established in the next lemmas (see, e.g., [21, Lem. 1.4.2]). On the one hand, the image under a homomorphism witnesses the satisfiability of the positive diagram generated by its preimage.

Lemma 2.2.14. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be similar structures and* $\vec{a} = \langle a_i : i \in I\rangle$ *and* $\vec{b} = \langle b_i : i \in I\rangle$ *non-empty sequences of elements of* A *and* B*, respectively. If the map* $h\colon \mathsf{Sg}^{\boldsymbol{A}}(\vec{a}) \to \mathsf{Sg}^{\boldsymbol{B}}(\vec{b})$ *defined via* $t^{\boldsymbol{A}}(\vec{a}) \mapsto t^{\boldsymbol{B}}(\vec{b})$ *for every term* $t \in T$ *is a homomorphism between the corresponding subalgebras, then* $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$. *Moreover, if* h *is an embedding, then* $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$.

Proof. Consider an atomic $\mathcal{L}_{\vec{a}}$-sentence $\alpha(\vec{a}) \in \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$. As atomic formulas are preserved under homomorphisms by Lemma 2.1.27, we conclude that $\boldsymbol{B} \vDash \alpha(h(\vec{a}))$. Then, by the definition of h, it follows that $\boldsymbol{B} \vDash \alpha(\vec{b})$. This verifies that $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$.

Now, suppose that h is an embedding, and consider $\beta(\vec{a}) \in \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$. When $\beta(\vec{a})$ is an atomic $\mathcal{L}_{\vec{a}}$-sentences we have already seen that $\boldsymbol{B} \vDash \beta(\vec{b})$. Furthermore, if $\beta(\vec{a}) = \neg(t_1(\vec{a}) \approx t_2(\vec{a}))$, then $\boldsymbol{B} \vDash \beta(\vec{b})$ follows as a consequence of the injectivity of h. Finally, the fact that h is a strong homomorphism guarantees that $\boldsymbol{B} \vDash \beta(\vec{b})$ in the case that $\beta(\vec{a}) = \neg R(\vec{a})$ for some relation symbol R. ⊠

On the other hand, diagrams can be used to construct homomorphisms between two similar structures.

Lemma 2.2.15. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar structures and $\vec{a} = \langle a_i : i \in I\rangle$ a non-empty sequence of elements of A. If $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$ for some sequence $\vec{b} = \langle b_i : i \in I\rangle$ of elements of B, then the map $h \colon \mathsf{Sg}^{\boldsymbol{A}}(\vec{a}) \to \mathsf{Sg}^{\boldsymbol{B}}(\vec{b})$ defined via $t^{\boldsymbol{A}}(\vec{a}) \mapsto t^{\boldsymbol{B}}(\vec{b})$ for every term $t \in T$, is a homomorphism. Moreover, if $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$, then h is an embedding.*

Proof. First, let us see that h is well defined. Notice that, by Lemma 2.1.16, every element of $\mathsf{Sg}^{\boldsymbol{A}}(\vec{a})$ is of the form $t^{\boldsymbol{A}}(\vec{a})$ for some $t \in T$. Now, consider $t_1, t_2 \in T$ such that $t_1^{\boldsymbol{A}}(\vec{a}) = t_2^{\boldsymbol{A}}(\vec{a})$. Then, $t_1(\vec{a}) \approx t_2(\vec{a}) \in \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$, and hence the assumption that $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$ implies that $t_1^{\boldsymbol{B}}(\vec{b}) = t_2^{\boldsymbol{B}}(\vec{b})$. To verify that h is a homomorphism, consider an n-ary function symbol f and $t_1^{\boldsymbol{A}}(\vec{a}), \dots, t_n^{\boldsymbol{A}}(\vec{a}) \in \mathsf{Sg}^{\boldsymbol{A}}(\vec{a})$. We will show that

$$h(f^{\boldsymbol{A}}(t_1^{\boldsymbol{A}}(\vec{a}), \dots, t_n^{\boldsymbol{A}}(\vec{a}))) = f^{\boldsymbol{B}}(t_1^{\boldsymbol{B}}(\vec{b}), \dots, t_n^{\boldsymbol{B}}(\vec{b})).$$

Let $f^{\boldsymbol{A}}(t_1^{\boldsymbol{A}}(\vec{a}), \dots, t_n^{\boldsymbol{A}}(\vec{a})) = t_{n+1}^{\boldsymbol{A}}(\vec{a})$ for some term $t_{n+1} \in T$. Then, as $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$ and $f(t_1(\vec{a}), \dots, t_n(\vec{a})) \approx t_{n+1}(\vec{a})$ is an atomic $\mathcal{L}_{\vec{a}}$-sentence that is valid in $\boldsymbol{A}_{\vec{a}}$, we obtain the desired equality

$$f^{\boldsymbol{B}}(t_1^{\boldsymbol{B}}(\vec{b}), \dots, t_n^{\boldsymbol{B}}(\vec{b})) = t_{n+1}^{\boldsymbol{B}}(\vec{b}) = h(f^{\boldsymbol{A}}(t_1^{\boldsymbol{A}}(\vec{a}), \dots, t_n^{\boldsymbol{A}}(\vec{a}))).$$

Similarly, assume that $\langle t_1^{\boldsymbol{A}}(\vec{a}), \dots, t_n^{\boldsymbol{A}}(\vec{a})\rangle \in R^{\boldsymbol{A}}$ for some n-ary relation symbol R and $t_1^{\boldsymbol{A}}(\vec{a}), \dots, t_n^{\boldsymbol{A}}(\vec{a}) \in \mathsf{Sg}^{\boldsymbol{A}}(\vec{a})$. Then, $R(t_1(\vec{a}), \dots, t_n(\vec{a})) \in \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$. Therefore, the fact that $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}^+_{\vec{a}}(\boldsymbol{A})$ implies that $\boldsymbol{B} \vDash R(t_1(\vec{b}), \dots, t_n(\vec{b}))$. This proves that h is a homomorphism.

Now, assume that $\boldsymbol{B}_{\vec{b}} \vDash \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$ and let $t_1^{\boldsymbol{A}}(\vec{a}) \neq t_2^{\boldsymbol{A}}(\vec{a})$ be elements of $\mathsf{Sg}^{\boldsymbol{A}}(\vec{a})$. Then, $\neg(t_1(\vec{a}) \approx t_2(\vec{a})) \in \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$. Therefore, $\boldsymbol{B} \vDash \neg(t_1(\vec{b}) \approx t_2(\vec{b}))$, and thus $h(t_1^{\boldsymbol{A}}(\vec{a})) = t_1^{\boldsymbol{B}}(\vec{b}) \neq t_2^{\boldsymbol{B}}(\vec{b}) = h(t_2^{\boldsymbol{A}}(\vec{a}))$. This proves that h is injective. Furthermore, assume with a view to contradiction there were terms $t_1, \dots, t_n$ and an n-ary relation symbol R such that

$$\boldsymbol{B} \vDash R(t_1(\vec{b}), \dots, t_n(\vec{b})) \text{ and } \boldsymbol{A} \vDash \neg R(t_1(\vec{a}), \dots, t_n(\vec{a})).$$

From the latter it follows that $\neg R(t_1(\vec{a}), \dots, t_n(\vec{a})) \in \mathsf{Diag}_{\vec{a}}(\boldsymbol{A})$. But this implies that $\boldsymbol{B} \vDash \neg R(t_1(\vec{b}), \dots, t_n(\vec{b}))$, leading to a contradiction. Hence, we conclude that h is an embedding. ⊠

2.3 Universal Algebra

Definition 2.3.1. A first-order language $\mathcal{L}$ without relation symbols is called an *algebraic first-order language*. In this case an $\mathcal{L}$-structure is also called an $\mathcal{L}$-*algebra*.

From now on, if not stated otherwise, $\mathcal{L}$ will always denote an algebraic first-order language and K a class of $\mathcal{L}$-algebras. We will subsequently introduce some basic concepts and results of universal algebra. For further details see, e.g., [8], [4], and [16]. Working in this area, we are mostly interested in classes of algebras that satisfy certain closure properties.

Definition 2.3.2. A class K of similar algebras is called

- *universal*, if it is closed under $\mathbb{I}, \mathbb{S}$, and $\mathbb{P}_u$;
- a *quasivariety*, if it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$, and $\mathbb{P}_u$;
- a *variety*, if it is closed under $\mathbb{H}, \mathbb{S}$, and $\mathbb{P}$.

For a class of similar algebras K, we denote the smallest variety, quasivariety, and universal class containing K by $\mathbb{V}(\mathsf{K})$, $\mathbb{Q}\mathsf{K})$, and $\mathbb{U}(\mathsf{K})$, respectively.

Theorem 2.3.3. *For a class of similar algebras* K *we have,*

- $\mathbb{V}(\mathsf{K}) = \mathbb{HSP}_u(\mathsf{K})$ *(see, e.g., [8, Thm. II.9.5]);*
- $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_u(\mathsf{K})$ *(see, e.g., [8, Thm. V.2.25]);*
- $\mathbb{U}(\mathsf{K}) = \mathbb{ISP}_u(\mathsf{K})$ *(see, e.g., [8, Thm. V.2.20]).*

Notice that every variety is a quasivariety, and every quasivariety is a universal class. The converse inclusions do not hold in general. Furthermore, these three types of classes have the convenient property of being elementary.

Theorem 2.3.4. *The following classes are elementary:*

1. *every universal class (see, e.g., [8, Thm. V.2.20]);*
2. *every quasivariety (see, e.g., [8, Thm. V.2.25]);*
3. *every variety (see, e.g., [8, Thm. II.11.9]).*

2.3.1 Lattices

An important class of algebras is the class of *lattices* (see, e.g., [8, Chap. I]).

Definition 2.3.5. A *lattice* $\boldsymbol{A}$ is a structure in the language $\mathcal{L}$ consisting of the two binary function symbols $\wedge$ and $\vee$ such that for every $a, b, c \in A$ the following hold:

1. $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;
2. $a \wedge a = a$ and $a \vee a = a$;
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;
4. $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

Observe that on every lattice $\boldsymbol{A}$, the rule

$$a \leqslant b :\iff a \wedge b = a$$

defines a partial order on A.

Definition 2.3.6. A lattice $\boldsymbol{A}$ is called

- *distributive* if
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
or equivalently
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$$
- *bounded* if there exist elements $b, c \in A$ such that
$$b \geqslant a \text{ and } c \leqslant a \text{ for every } a \in A.$$
In this case, we will often consider $\boldsymbol{A}$ in the language expanded by the constants 0 and 1 and define $1^{\boldsymbol{A}} := b$ and $0^{\boldsymbol{A}} := c$;
- a *complete lattice* if $\bigvee_{i \in I} a_i$ and $\bigwedge_{i \in I} a_i$ exist as elements in A for every family $\{a_i : i \in I\} \subseteq A$. Observe that every complete lattice $\boldsymbol{A}$ is bounded with $1^{\boldsymbol{A}} = \bigvee_{a \in A} a$ and $0^{\boldsymbol{A}} = \bigwedge_{a \in A} a$.

Example 2.3.7. An example of a complete lattice that is distributive is the powerset $\mathcal{P}(X)$ of a given set X, where for every $A, B, C \in \mathcal{P}(X)$ we define

$$\begin{aligned} A \wedge^{\mathcal{P}(X)} B &= A \cap B; \\ A \vee^{\mathcal{P}(X)} B &= A \cup B; \\ 1^{\mathcal{P}(X)} &= X; \\ 0^{\mathcal{P}(X)} &= \emptyset. \end{aligned}$$

It even satisfies the *infinite distributive law*

$$\bigcup_{\substack{i \in I, \\ j \in J}} (A_i \cap A_j) = \bigcup_{i \in I} A_i \cap \bigcup_{j \in J} A_j$$

or equivalently

$$\bigcap_{\substack{i \in I, \\ j \in J}} (A_i \cup A_j) = \bigcap_{i \in I} A_i \cup \bigcap_{j \in J} A_j,$$

for every $\{A_i : i \in I\} \cup \{A_j : j \in J\} \subseteq \mathcal{P}(X)$.

The canonical order on $\mathcal{P}(X)$ is given by the inclusion relation $\subseteq$. To express that A is a finite subset of B, we use the notation $A \subseteq_\omega B$.

Definition 2.3.8. Let $\boldsymbol{A}$ be a lattice. An element $a \in A$ is called *meet-irreducible* when, for every finite subset $\{a_i : i \in I\} \subseteq_\omega A$, it holds that:

$$a = \bigwedge_{i \in I} a_i \text{ implies } a = a_i \text{ for some } i \in I. \tag{2.1}$$

When $\boldsymbol{A}$ is a complete lattice, we can also consider the above condition (2.1) for infinite sets of elements and call an element $1^{\boldsymbol{A}} \neq a \in A$ that satisfies it *completely meet-irreducible*.

Next, we will introduce congruences, which play an important role in universal algebra (see, e.g., [8, Sec. II.5] and [4, Sec. 1.5]).

2.3.2 Congruences

Definition 2.3.9. A *congruence* θ on an algebra $\boldsymbol{A}$ is an equivalence relation on A with the property that for every n-ary function symbol f and $a_1, \dots, a_n, b_1, \dots, b_n \in A$ such that $\langle a_i, b_i \rangle \in \theta$ for every $i \leqslant n$, we have $\langle f^{\boldsymbol{A}}(a_1, \dots, a_n), f^{\boldsymbol{A}}(b_1, \dots, b_n) \rangle \in \theta$. The set of all congruences of $\boldsymbol{A}$ is denoted by $\mathsf{Con}(\boldsymbol{A})$. Observe, that every intersection of congruences of $\boldsymbol{A}$ is again a congruence of $\boldsymbol{A}$. Thus, for a subset $X \subseteq A \times A$ we can define the *congruence generated by* X as

$$\mathsf{Cg}^{\boldsymbol{A}}(X) := \bigcap \{\theta \in \mathsf{Con}(\boldsymbol{A}) : X \subseteq \theta\}.$$

Notice that $\mathsf{Cg}(X)$ is the smallest congruence of $\boldsymbol{A}$ containing X.

Remark 2.3.10. For every algebra $\boldsymbol{A}$, the sets $A \times A$ and $\mathsf{id}_A := \{\langle a, a \rangle : a \in A\}$ are congruences of $\boldsymbol{A}$. Observe that $A \times A$ is the maximum and id_A the minimum of $\mathsf{Con}(\boldsymbol{A})$.

Remark 2.3.11. The set $\mathsf{Con}(\boldsymbol{A})$ forms a complete lattice, where for every $\theta, \phi \in \mathsf{Con}(\boldsymbol{A})$ we define

$$\begin{aligned}
\theta \wedge^{\mathsf{Con}(\boldsymbol{A})} \phi &= \theta \cap \phi; \\
\theta \vee^{\mathsf{Con}(\boldsymbol{A})} \phi &= \mathsf{Cg}^{\boldsymbol{A}}(\theta \cup \phi); \\
1^{\mathsf{Con}(\boldsymbol{A})} &= A \times A; \\
0^{\mathsf{Con}(\boldsymbol{A})} &= \mathsf{id}_A.
\end{aligned}$$

Congruences are tightly connected to kernels of homomorphisms, as we proceed to explain.

Lemma 2.3.12. *(see, e.g., [8, Thm. II.6.8]) For every homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ between similar algebras, the kernel $\mathsf{ker}(h)$ is a congruence of $\boldsymbol{A}$.*

Definition 2.3.13. Given an algebra $\boldsymbol{A}$ and a congruence $\theta \in \mathsf{Con}(\boldsymbol{A})$, we can define a *quotient algebra* $\boldsymbol{A}/\theta$ with the set of equivalence classes $A/\theta = \{a/\theta : a \in A\}$ as universe. The interpretation of an n-ary function symbol f is given by

$$f^{\boldsymbol{A}/\theta}(a_1/\theta, \dots, a_n/\theta) := f^{\boldsymbol{A}}(a_1, \dots, a_n)/\theta,$$

for every $a_1/\theta, \dots, a_n/\theta \in A/\theta$. Notice that the properties of a congruence guarantee that the algebra $\boldsymbol{A}/\theta$ is well defined.

For every algebra $\boldsymbol{A}$ and every congruence $\theta \in \mathsf{Con}(\boldsymbol{A})$ the canonical projection $\pi_\theta\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$ defined via $a \mapsto a/\theta$ is a surjective homomorphism with $\mathsf{ker}(h) = \theta$. Therefore, if $\boldsymbol{A}$ is a member of a class K of similar algebras, then $\boldsymbol{A}/\theta \in \mathbb{H}(\mathsf{K})$. In particular, if K is a variety, the above implies that $\boldsymbol{A}/\theta \in \mathsf{K}$. This is a desirable property that does not hold in general, as an arbitrary class of similar algebras need not be closed under homomorphic images. To deal with this problem, we introduce the following restricted class of congruences:

Definition 2.3.14. Let K be a class of similar algebras and $\boldsymbol{A} \in \mathsf{K}$. We define the set of K-*congruences of* $\boldsymbol{A}$ as

$$\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) := \{\theta \in \mathsf{Con}(\boldsymbol{A}) : \boldsymbol{A}/\theta \in \mathsf{K}\}.$$

If K is a variety, then $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) = \mathsf{Con}(\boldsymbol{A})$.

Definition 2.3.15. Given an algebra $\boldsymbol{A}$, a family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}(\boldsymbol{A})$ is called a *chain* when $\theta_i \subseteq \theta_j$ or $\theta_j \subseteq \theta_i$ for every $i, j \in I$.

Observe that for a quasivariety K and every family $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, we always have $\bigcap_{i \in I} \theta_i \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Therefore, we can define the K-*congruence generated by* a subset $X \subseteq A \times A$, as

$$\mathsf{Cg}_{\mathsf{K}}^{\boldsymbol{A}}(X) := \bigcap\{\theta \in \mathsf{Con}_{\mathsf{K}}(A) : X \subseteq \theta\}.$$

The union $\bigcup_{i \in I} \theta_i$, on the other hand, need not be a K-congruence, unless the family $\{\theta_i : i \in I\}$ forms a chain.

Proposition 2.3.16. *For every member* $\boldsymbol{A}$ *of a quasivariety* K *and every non-empty chain* $\{\theta_i : i \in I\} \subseteq \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$, *the union* $\bigcup_{i \in I} \theta_i$ *is also a* K*-congruence of* $\boldsymbol{A}$.

Given an algebra $\boldsymbol{A}$, and a congruence $\theta \in \mathsf{Con}(\boldsymbol{A})$ the Correspondence Theorem (see, e.g., [8, Thm. II.6.20]) gives a useful characterization of the lattice of congruences $\mathsf{Con}(\boldsymbol{A}/\theta)$ on the quotient algebra $\boldsymbol{A}/\theta$. We will state a slightly more general version that describes the lattice of K-congruences $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$, where $\boldsymbol{A}$ is a member of a quasivariety K.

Correspondence Theorem 2.3.17. *Let* K *be a quasivariety,* $\boldsymbol{A} \in \mathsf{K}$, *and* $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. *The sublattice* $\uparrow\theta$ *of* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ *with universe* $\{\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) : \theta \subseteq \phi\}$ *is isomorphic to* $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$, *where the isomorphism* $h\colon \uparrow\theta \to \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}/\theta)$ *is given by the rule*

$$\phi \mapsto \{\langle a/\theta, b/\theta\rangle : \langle a, b\rangle \in \phi\}.$$

Another useful isomorphism, is provided by the Homomorphism Theorem (see, e.g., [8, Thm. II.6.12]).

Homomorphism Theorem 2.3.18. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be two similar algebras and* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *a homomorphism between them. Then,* $\boldsymbol{A}/\mathsf{ker}(h) \cong h[\boldsymbol{B}]$.

Notice that, given a class K of similar algebras, every K-congruence θ of some $\boldsymbol{A} \in \mathsf{K}$ is realized as the kernel of the canonical projection $\boldsymbol{A} \to \boldsymbol{A}/\theta$. In the case that K is closed under isomorphic copies and subalgebras, the Homomorphism Theorem 2.3.18 implies that, on the other hand, also every kernel of a homomorphism between algebras of K is a K-congruence.

For a quasivariety K and $\boldsymbol{A} \in \mathsf{K}$, we denote the set of (completely) meet-irreducible K-congruences by $\mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$ (respectively $\mathsf{Irr}_{\mathsf{K}}^{\infty}(\boldsymbol{A})$). As a consequence of [4, Cor. 2.31], we obtain that the elements of $\mathsf{Irr}_{\mathsf{K}}^{\infty}(\boldsymbol{A})$ can be considered the basic building blocks of $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ in the following sense:

Theorem 2.3.19. *For every algebra* $\boldsymbol{A}$ *and every congruence* $\theta \in \mathsf{Con}(\boldsymbol{A})$, *there exists a set* $\{\theta_i : i \in I\} \subseteq \mathsf{Irr}_{\mathsf{K}}^{\infty}(\boldsymbol{A})$ *such that* $\theta = \bigcap_{i \in I} \theta_i$. *In particular,* $\mathsf{id}_{\boldsymbol{A}} = \bigcap \mathsf{Irr}_{\mathsf{K}}^{\infty}(\boldsymbol{A})$.

Next, we introduce a class of algebras that, in a similar manner, play the role of basic building blocks in a quasivariety. To this end, the following concept is crucial.

Definition 2.3.20. [8, Def. 8.1] We say that $\boldsymbol{A} \in \mathsf{K}$ is a *subdirect product* of a family $\{\boldsymbol{A}_i : i \in I\}$ if $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{A}_i$ and for every $i \in I$ the canonical projection $p_i \colon \boldsymbol{A} \to \boldsymbol{A}_i$ defined via $\vec{a} \mapsto \vec{a}(i)$ is surjective. An embedding $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$ is called *subdirect* if $h[\boldsymbol{A}] \leqslant \prod_{i \in I} \boldsymbol{A}_i$ is a subdirect product.

The next proposition provides a convenient method to construct subdirect embeddings (see, e.g., [8, Lem. II.8.2]).

Proposition 2.3.21. *For every algebra $\boldsymbol{A}$ and $X \subseteq \mathsf{Con}(\boldsymbol{A})$, the map*

$$h \colon \boldsymbol{A}/\bigcap X \to \prod_{\theta \in X} \boldsymbol{A}/\theta$$

given by $a/\bigcap X \mapsto \langle a/\theta : \theta \in X \rangle$ defines a subdirect embedding.

The announced building blocks of a quasivariety K are exactly the members of K, which cannot be represented as a subdirect product in a non-trivial way.

Definition 2.3.22. [8, Def. 8.3] An algebra $\boldsymbol{A}$ is called *finitely subdirectly irreducible* (FSI, for short) if, for every finite family $\{\boldsymbol{A}_i : i \leqslant n\}$ such that $\boldsymbol{A} \leqslant \prod_{i \leqslant n} \boldsymbol{A}_i$ is a subdirect product, there exists some $i \leqslant n$ such that $p_i : \boldsymbol{A} \to \boldsymbol{A}_i$ is an isomorphism.

When this is true for every family $\{\boldsymbol{A}_i : i \in I\}$ over an arbitrary index set I, we say that $\boldsymbol{A}$ is *subdirectly irreducible* (SI, for short).

Given a class of algebras K, we call $\boldsymbol{A} \in \mathsf{K}$ *(finitely) subdirectly irreducible relative to* K when the above holds for every (finite) family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}$. We refer to the subclass of (finitely) subdirectly irreducible members of K by $\mathsf{K}_{\mathrm{RSI}}$ and $\mathsf{K}_{\mathrm{RFSI}}$, respectively. Notice, that if K is a variety, the condition that the factors $\{\boldsymbol{A}_i : i \in I\}$ of the subdirect product be elements of K is always satisfied since $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathbb{H}(\boldsymbol{A}) \subseteq \mathsf{K}$. In this case, we will just write K_{SI} and $\mathsf{K}_{\mathrm{FSI}}$.

The Subdirect Decomposition Theorem (see, e.g., [8, Thm. II.8.6]) justifies the special role played by the subdirectly irreducible algebras.

Subdirect Decomposition Theorem 2.3.23. *For every algebra $\boldsymbol{A}$, there exists a family $\{\boldsymbol{A}_i : i \in I\}$ of subdirectly irreducible algebras such that $\boldsymbol{A}$ is isomorphic to a subdirect product $\boldsymbol{B} \leqslant \prod_{i \in I} \boldsymbol{A}_i$. When $\boldsymbol{A}$ is an element of a quasivariety K, we can find such a family $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{K}_{\mathrm{RSI}}$.*

As stated in the following theorem (see, e.g., [4, Thm. 3.23]), there is a tight connection between subdirectly irreducible algebras and completely meet-irreducible congruences.

Theorem 2.3.24. *Let K be a quasivariety. An algebra $\boldsymbol{A} \in \mathsf{K}$ is subdirectly irreducible relative to K iff $\mathsf{id}_{\boldsymbol{A}}$ is completely meet-irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Similarly, $\boldsymbol{A}$ is finitely subdirectly irreducible relative to K iff $\mathsf{id}_{\boldsymbol{A}}$ is meet-irreducible in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$.*

Remark 2.3.25. Using Theorems 2.3.23 and 2.3.24, it is easy to see that for a quasivariety K and every $\boldsymbol{A} \in \mathsf{K}$, we obtain a subdirect embedding

$$h \colon \boldsymbol{A} \to \prod_{\theta \in \mathsf{Irr}^{\infty}_{\mathsf{K}}(\boldsymbol{A})} \boldsymbol{A}/\theta \text{ via } a \mapsto \langle a/\theta : \theta \in \mathsf{Irr}^{\infty}_{\mathsf{K}}(\boldsymbol{A}) \rangle,$$

where the factors $\{\boldsymbol{A}/\theta : \theta \in \mathsf{Irr}^{\infty}_{\mathsf{K}}\}$ are elements of $\mathsf{K}_{\mathrm{RSI}}$.

Another special class of algebras we want to introduce are free algebras (see, e.g., [8, Sec. II.10] and [4, Sec. 4.3.]).

2.3.3 Free algebras

Definition 2.3.26. An algebra $\boldsymbol{A}$ is called *free for* K with *free generators* $\{a_i : i \in I\} \subseteq A$ when $A = \mathsf{Sg}^{\boldsymbol{A}}(\{a_i : i \in I\})$ and every map

$$f\colon \{a_i : i \in I\} \to B$$

with $\boldsymbol{B} \in \mathsf{K}$ can be extended to a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$.

We will now present the canonical example of free algebras.

Definition 2.3.27. Given a language $\mathcal{L}$ and a set of variables $X \subseteq \mathsf{Var}$, the set of terms $T(X)$ in variables X can be turned into an algebra $\mathbf{T}(X)$, called the *term algebra in variables* X, via the following interpretations: For every n-ary function symbol f and every $t_1, \ldots, t_n \in T(X)$, we define

$$f^{\mathbf{T}(X)}(t_1, \ldots, t_n) := f(t_1, \ldots, t_n).$$

Remark 2.3.28. Notice that for every $\mathcal{L}$-algebra $\boldsymbol{A}$ and every map $f\colon X \to \boldsymbol{A}$, we can canonically extend f to a homomorphism $h\colon \mathbf{T}(X) \to \boldsymbol{A}$ defining

$$h(t(x_1, \ldots, x_n)) := t^{\boldsymbol{A}}(f(x_1), \ldots, f(x_n)).$$

Therefore, $\mathbf{T}(X)$ is free for any class K of $\mathcal{L}$-algebras.

Moreover, for every algebra $\boldsymbol{A}$ with set of generators $\{a_i : i \in I\}$, where $|I| \leqslant |\mathsf{Var}|$, we can find a term algebra $\mathbf{T}(X)$ such that $\boldsymbol{A} \in \mathbb{H}(\mathbf{T}(X))$.

However, when we consider a class of algebras K, we often want to have a free algebra for K in K, which is usually not the case for the term algebras. The good news is that if K is a quasivariety, we can always find a free algebra for K in K.

Theorem 2.3.29. *(see, e.g., [16, Prop. 2.1.10]) Let* K *be a quasivariety in a language* $\mathcal{L}$ *with set of variables* Var*. Then, for every cardinal* $\lambda \leqslant |\mathsf{Var}|$ *there exists a free algebra* $\mathbf{F}_{\mathsf{K}}(X) \in \mathsf{K}$ *with set of free generators* X *of cardinality* λ*.*

We will continue by presenting some desirable properties of (quasi)varieties that are induced by properties of the lattices of congruences of their members.

2.3.4 Properties of (quasi)varieties

Definition 2.3.30. We say that a quasivariety K is *congruence distributive* if $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ is a distributive lattice for every $\boldsymbol{A} \in \mathsf{K}$.

Definition 2.3.31. If R_1, R_2 are binary relations on a set A, the *relational product* of R_1 and R_2 is the binary relation defined as:

$$R_1 \circ R_2 := \{\langle a, b\rangle \in A \times A : \text{ there exists } c \in A \text{ such that } \langle a, c\rangle \in R_1 \text{ and } \langle c, b\rangle \in R_2\}.$$

Definition 2.3.32. A variety K is called *congruence permutable* if for every $\boldsymbol{A} \in \mathsf{K}$ and $\theta_1, \theta_2 \in \mathsf{Con}(\boldsymbol{A})$ it holds that

$$\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1.$$

When working with congruence permutable varieties, it will often be convenient to use the following equivalent characterization (see, e.g., [8, Thm. II.5.9]):

Proposition 2.3.33. *A variety* K *is congruence permutable iff*

$$\theta_1 \vee^{\boldsymbol{A}} \theta_2 = \theta_1 \circ \theta_2$$

for every $\boldsymbol{A} \in \mathsf{K}$ *and* $\theta_1, \theta_2 \in \mathsf{Con}(\boldsymbol{A})$*.*

Definition 2.3.34. A variety K is called *arithmetical* if it is both congruence distributive and congruence permutable.

We will now see two equivalent conditions for a variety to be arithmetical that will be used later.

Definition 2.3.35. A term $\varphi(x, y, z)$ is called a *Pixley term* for a class of similar algebra K if

$$\mathsf{K} \vDash \varphi(x, y, x) \approx \varphi(x, y, y) \approx \varphi(y, y, x) \approx x.$$

Theorem 2.3.36. *(see, e.g., [8, Thm. II.12.5]) A variety* K *is arithmetical iff it has a Pixley term.*

Definition 2.3.37. Given an algebra $\boldsymbol{A}$, we say that a lattice $\mathbf{L} := \langle L, \wedge^{\mathbf{L}}, \vee^{\mathbf{L}} \rangle$ with $L \subseteq \mathsf{Con}(A)$ *satisfies the Chinese Remainder Theorem* when for every $\theta_1, \ldots, \theta_n \in L$ and every $a_1, \ldots, a_n \in A$ such that

$$\langle a_i, a_j \rangle \in \theta_i \vee^{\mathbf{L}} \theta_j \text{ for every } i, j \leqslant n$$

there exists $a \in A$ such that

$$\langle a_i, a \rangle \in \theta_i \text{ for every } i \leqslant n.$$

Theorem 2.3.38. *(see, e.g., [24, Thm. 2.2.1]) A variety* K *is arithmetical iff for every* $\boldsymbol{A} \in \mathsf{K}$ *the congruence lattice* $\mathsf{Con}(\boldsymbol{A})$ *satisfies the Chinese Remainder Theorem.*

To conclude this section, we will recall Jónsson's Lemma (see, e.g., [4, Lem. 5.9]), which helps to locate the FSI members inside a congruence distributive variety.

Jónsson's Lemma 2.3.39. *Let* K *be a class of similar algebras such that* $\mathbb{V}(\mathsf{K})$ *is congruence distributive. Then,*

$$\mathbb{V}(\mathsf{K})_{\text{FSI}} \subseteq \mathbb{HSP}_u(\mathsf{K}).$$

2.4 The Epimorphism Surjectivity Property

Finally, we will introduce the protagonist of this thesis: the epimomorphism surjectivity property. First, we need to clarify what an epimorphism is (see, e.g., [6, Def. 3.10]).

Definition 2.4.1. Given a class of algebras K and $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, we say that a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an *epimorphism in* K when for every $\boldsymbol{C} \in \mathsf{K}$ and every pair of homomorphisms $f, g\colon \boldsymbol{B} \to \boldsymbol{C}$ we have,

$$f \circ h = g \circ h \text{ implies } f = g.$$

A subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B}$ is *epic in* K when the inclusion map $i \colon \boldsymbol{A} \hookrightarrow \boldsymbol{B}$ is an epimorphism in K. In other words, when

$$f{\restriction}_A = g{\restriction}_A \text{ implies } f = g$$

for every $\boldsymbol{C} \in \mathsf{K}$ and every pair of homomorphisms $f, g \colon \boldsymbol{B} \to \boldsymbol{C}$. When it is clear from the context, we will just talk about epimorphisms and epic subalgebras without mentioning the class K.

In essence, a subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in some class K when every homomorphism $h \colon \boldsymbol{B} \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathsf{K}$ is completely determined by its values on $\boldsymbol{A}$. From this perspective it is not surprising that every surjective homomorphism is an epimorphism.

Lemma 2.4.2. *Let* K *be a class of similar algebras and* $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$. *Every surjective homomorphism* $h \colon \boldsymbol{A} \to \boldsymbol{B}$ *is an epimorphism in* K.

Proof. Consider a surjective homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ and an algebra $\boldsymbol{C} \in \mathsf{K}$ together with a pair of homomorphisms $f, g \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $f \circ h = g \circ h$. Now, let b be an element of B. We need to show that $f(b) = g(b)$. Since h is surjective by assumption, there exists $a \in A$ such that $h(a) = b$. Then, we obtain:

$$f(b) = f \circ h(a) = g \circ h(a) = g(b),$$

where the middle equation follows from the assumption that $f \circ h = g \circ h$. ⊠

However, not every epimorphism is surjective. A prominent example is the inclusion $i \colon \mathbb{Z} \hookrightarrow \mathbb{Q}$ (see, e.g., [23]), which clearly is a non-surjective homomorphism in the language of rings. But still, it is an epimorphism in the variety of rings (see, e.g., [23]). To see this, consider a ring $\mathbf{R}$ and two homomorphisms $f, g \colon \mathbb{Q} \to \mathbf{R}$ such that $f{\restriction}_{\mathbb{Z}} = g{\restriction}_{\mathbb{Z}}$. Now, let $q = \frac{a}{b} \in \mathbb{Q}$ for some $a, b \in \mathbb{Z}$. We have to show that $f(q) = g(q)$. This follows as

$$f(q) = f(a) \cdot f(b)^{-1} = g(a) \cdot g(b)^{-1} = g(q),$$

where the first and last equality hold because f and g are ring homomorphisms, and the middle equality is a consequence of the assumption that f and g coincide on $\mathbb{Z}$ and a and b are elements of $\mathbb{Z}$.

Nevertheless, there are a lot of examples of classes of algebras where the notions of epimorphisms and surjective homomorphisms do coincide, including sets, groups, lattices, and Boolean algebras (see, e.g., [6]). This quality is precisely what is captured by the epimorphism surjectivity property.

Definition 2.4.3. A class K of similar algebras is said to have the *epimorphism surjectivity property* when every epimorphism in K is surjective. We will refer to it using the shorthand *ES property*.

In some cases, it is of interest to consider a slightly weaker version of the ES property.

Definition 2.4.4. A class K of similar algebras is said to have the *weak epimorphism surjectivity property*, or *weak ES property* for short, when every epimorphism between finitely generated members of K is surjective.

We will now proceed to establish some easy observations about the ES property.

First, it will be useful to notice that, in the context of quasivarieties, the problem of determining whether a homomorphism is an epimorphism can always be reduced to a statement about epic subalgebras. This is the content of the following well-known lemma, which is briefly explained in [6, Def. 3.10].

Lemma 2.4.5. *Let* K *be a quasivariety. The following are equivalent:*

1. K *has the ES property;*
2. *For all* $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ *such that* $\boldsymbol{A} \leqslant \boldsymbol{B}$ *is epic in* K*, we have that* $A = B$.

Proof. First, assume that K has the ES property. Now, consider $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K. By definition this implies that the inclusion map $i \colon \boldsymbol{A} \hookrightarrow \boldsymbol{B}$ is an epimorphism in K. So, from the assumption that K has the ES property it follows that i is surjective, and thus $A = B$, as claimed.

Conversely, assume that for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K, we have that $A = B$. Consider an epimorphism $f \colon \boldsymbol{C} \to \boldsymbol{D}$ in K. Since quasivarieties are closed under subalgebras, we conclude that $f[\boldsymbol{C}] \leqslant \boldsymbol{D}$ is epic in K. Hence, the assumption implies $f[C] = D$. Therefore, f is surjective. This verifies the ES property for K. ⊠

Next, we observe that the property of being an epic subalgebra is preserved under homomorphisms.

Lemma 2.4.6. *Let* K *be a class of similar algebras and* $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ *such that* $\boldsymbol{A} \leqslant \boldsymbol{B}$ *is epic in* K*. Then,* $h[\boldsymbol{A}] \leqslant h[\boldsymbol{B}]$ *is epic in* K *for every algebra* $\boldsymbol{C}$ *and homomorphism* $h \colon \boldsymbol{B} \to \boldsymbol{C}$.

Proof. Consider $\boldsymbol{D} \in \mathsf{K}$ and a pair of homomorphisms $f, g \colon h[\boldsymbol{B}] \to \boldsymbol{D}$ such that $f{\restriction}_{h[A]} = g{\restriction}_{h[A]}$. Then, $(f \circ h){\restriction}_A = (g \circ h){\restriction}_A$. Thus, the assumption that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K implies that $f \circ h = g \circ h$. Since $h \colon \boldsymbol{B} \to h[\boldsymbol{B}]$ is surjective, and hence an epimorphism by Lemma 2.4.2, we conclude that $f = g$, as desired. ⊠

To conclude this section, we will prove the following lemma:

Lemma 2.4.7. *Let* K *be a class of similar algebras and* $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IS}(\mathsf{K})$*. Then* $\boldsymbol{A} \leqslant \boldsymbol{B}$ *is epic in* K *iff it is epic in* $\mathbb{IS}(\mathsf{K})$.

Proof. Clearly, if $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in $\mathbb{IS}(\mathsf{K})$, then so it is in $\mathsf{K} \subseteq \mathbb{IS}(\mathsf{K})$. For the converse, let $\boldsymbol{A} \leqslant \boldsymbol{B}$ be epic in K. Consider $\boldsymbol{C} \in \mathbb{IS}(\mathsf{K})$ and $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $g{\restriction}_A = h{\restriction}_A$. Assume, with a view to contradiction, that there exists $b \in \boldsymbol{B}$ such that $g(b) \neq h(b)$. Let $\boldsymbol{D} \in \mathsf{K}$ be such that $\boldsymbol{C}$ is isomorphic to a subalgebra $\boldsymbol{C}' \leqslant \boldsymbol{D}$ via an isomorphism $f \colon \boldsymbol{C} \to \boldsymbol{C}'$, and let $i \colon \boldsymbol{C}' \hookrightarrow \boldsymbol{D}$ be the inclusion map. Then,

$$(i \circ f \circ g){\restriction}_A = (i \circ f \circ h){\restriction}_A \text{ and } (i \circ f \circ g)(b) \neq (i \circ f \circ h)(b),$$

in contradiction with the assumption that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K. ⊠

2.5 ES property and Beth definability

Although the (weak) ES property is interesting in its own right, this section will provide some further motivation to study the (weak) ES property, which arises from its connection to definability in logic (see, e.g., [6]). To this end, we first need to introduce the notion of logic and establish a translation between logics and classes of algebras (see, e.g., [7]). Then, we will present the bridge theorem relating the (weak) ES property to its logical counterpart, the (finite) Beth definability property (see [6]). To avoid a lengthy construction of a more general concept of logic, we will follow [29] in using a slightly less general version of the Beth definability property to obtain the announced correspondence. As a consequence, in this section, we will restrict our attention to countable algebraic first-order languages. In particular, we assume $\mathsf{Var} = \{x_n : n \in \mathbb{N}\}$.

Definition 2.5.1. Recall that $\mathbf{T}(\mathsf{Var})$ stands for the term algebra in a given language $\mathcal{L}$ with variables Var. In the following, it will be abbreviated with $\mathbf{T}$. Notice that the elements of its universe T are the terms of the language $\mathcal{L}$, as introduced in Subsection 2.1.1. However, since they are commonly referred to as *formulas*, we will follow this convention, keeping in mind that we are not talking about first-order formulas in the sense of Definition 2.1.3.

A *substitution* is a homomorphism $\rho\colon \mathbf{T} \to \mathbf{T}$. We say that a substitution ρ *pointwise fixes* a set of variables $X \subseteq \mathsf{Var}$, if $\rho(x) = x$ for every $x \in X$. Given a set of formulas $\Delta(x_1, \dots, x_n) \subseteq \mathbf{T}$ and $\varphi_1, \dots, \varphi_n \in \mathbf{T}$, we write $\Delta(\varphi_1, \dots, \varphi_n)$ to denote the set of formulas

$$\{\rho(\delta(x_1, \dots, x_n)) : \delta \in \Delta\} \subseteq \mathbf{T},$$

where ρ is a substitution such that $\rho(x_i) = \varphi_i$ for every $i \leqslant n$.

We will define a logic as a particular relation $\vdash \subseteq \mathcal{P}(T) \times T$. To simplify notation, for a pair of formulas $\varphi, \psi \in T$ we will write $\varphi \vdash \psi$ instead of $\{\varphi\} \vdash \psi$. Also, given two sets of formulas $\Gamma, \Delta \subseteq T$, we use the shorthand $\Gamma \vdash \Delta$ to express that $\Gamma \vdash \delta$ for every $\delta \in \Delta$.

Definition 2.5.2. A binary relation $\vdash \subseteq \mathcal{P}(T) \times T$ is called a *logic*, if for every $\Gamma, \Delta \subseteq T$ and $\varphi \in T$, the following conditions hold:

1. $\Gamma \vdash \varphi$ for every $\varphi \in \Gamma$ (**Reflexivity**) ;
2. $\Delta \vdash \Gamma$ and $\Gamma \vdash \varphi$ implies $\Delta \vdash \varphi$ (**Transitivity**);
3. $\Gamma \vdash \varphi$ implies $\rho[\Gamma] \vdash \rho(\varphi)$ for every substitution ρ (**Substitution invariance**);
4. If $\Gamma \vdash \varphi$, then there exists a finite subset $\Gamma_0 \subseteq_\omega \Gamma$ such that $\Gamma_0 \vdash \varphi$ (**Finitariness**).

As a straightforward consequence of the definition, every logic $\vdash$ is *monotone* in the following sense:

Lemma 2.5.3. *Let $\vdash$ be a logic and $\Gamma, \Delta \subseteq T$ such that $\Gamma \subseteq \Delta$. Then, $\Gamma \vdash \varphi$ implies $\Delta \vdash \varphi$ for every $\varphi \in T$.*

Proof. Let $\Gamma, \Delta \subseteq T$ such that $\Gamma \subseteq \Delta$. By reflexivity it follows that $\Delta \vdash \Gamma$. Then, as $\Gamma \vdash \varphi$, the transitivity of $\vdash$ implies $\Delta \vdash \varphi$. ⊠

The connection between logics and classes of algebras will consist in a translation between a logic $\vdash$ and a suitable semantical relation $\vDash_{\mathsf{K}}$ within a quasivariety K, which we proceed to define.

Definition 2.5.4. An *equation* is a pair $\langle \varepsilon, \delta \rangle \in T \times T$, which we write as $\varepsilon \approx \delta$. The set of equations is denoted by Eq. Given a quasivariety K, the *equational consequence* on K, denoted by $\vDash_{\mathsf{K}}$, is a binary relation $\vDash_{\mathsf{K}} \subseteq \mathcal{P}(\mathsf{Eq}) \times \mathsf{Eq}$, where $\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta$ iff for every $\boldsymbol{A} \in \mathsf{K}$ and every tuple $\vec{a}$ of elements of A we have,

$$\varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}) \text{ for every } \varphi \approx \psi \in \Theta \text{ implies } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

As in the case of $\vdash$, we will write $\epsilon \approx \delta \vDash_{\mathsf{K}} \varphi \approx \psi$ instead of $\{\epsilon \approx \delta\} \vDash_{\mathsf{K}} \varphi \approx \psi$, and $\Theta \vDash_{\mathsf{K}} \Phi$, when $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ for every $\varphi \approx \psi \in \Phi$.

Remark 2.5.5. Notice that $\vDash_{\mathsf{K}}$ is transitive in the sense that

$$\Theta \vDash_{\mathsf{K}} \Phi \text{ and } \Phi \vDash_{\mathsf{K}} \varphi \approx \psi \text{ implies } \Theta \vDash_{\mathsf{K}} \varphi \approx \psi$$

for every $\Theta \cup \Phi \cup \{\varphi \approx \psi\} \subseteq \mathsf{Eq}$.

For a set of formulas $\Delta(x_1, \ldots, x_n) \subseteq T$ and a set of n-tuples $\tau \subseteq T^n$, we will refer to the set of formulas

$$\{\delta(\varphi_1, \ldots, \varphi_n) : \delta \in \Delta \text{ and } \langle \varphi_1, \ldots, \varphi_n \rangle \in \tau\} \subseteq T$$

using the shorthand $\Delta[\tau]$. Conversely, given a set of equations $\tau(x) \subseteq \mathsf{Eq}$ and a set of formulas $\Delta \subseteq T$, the notation $\tau[\Delta]$ stands for the set of equations

$$\{\langle \epsilon(\varphi), \delta(\varphi) \rangle : \epsilon(x) \approx \delta(x) \in \tau \text{ and } \varphi \in \Delta\} \subseteq \mathsf{Eq}.$$

Now, we are ready to establish the announced connection that allows us to cross the mirror between logic and algebra (see, e.g., [7]) .

Definition 2.5.6. A logic $\vdash$ is *algebraized by a quasivariety* K if there exist a finite set of equations $\tau(x) \subseteq \mathsf{Eq}$ and a finite set of formulas $\Delta(x, y) \subseteq T$ such that:

1. $\Gamma \vdash \varphi \iff \tau[\Gamma] \vDash_{\mathsf{K}} \tau(\varphi)$ (**Alg1**);
2. $\Theta \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff \Delta[\Theta] \vdash \Delta(\varepsilon, \delta)$ (**Alg2**);
3. $\varphi \vdash \Delta[\tau(\varphi)]$ and $\Delta[\tau(\varphi)] \vdash \varphi$ (**Alg3**);
4. $\varepsilon \approx \delta \vDash_{\mathsf{K}} \tau[\Delta(\varepsilon, \delta)]$ and $\tau[\Delta(\varepsilon, \delta)] \vDash_{\mathsf{K}} \varepsilon \approx \delta$ (**Alg4**);

for every $\Gamma \cup \{\varphi\} \subseteq T$ and every $\Theta \cup \{\langle \varepsilon, \delta \rangle\} \subseteq \mathsf{Eq}$. In this case, Δ is called a set of *equivalence formulas* for $\vdash$.

The next lemmas state two useful properties for a set of equivalence formulas Δ (see, e.g., [7, Lem. 2.13]). First, we will prove that Δ satisfies the following commutativity condition.

Lemma 2.5.7. *Let $\vdash$ be a logic algebraized by a quasivariety K, as witnessed by a set of equivalence formulas $\Delta \subseteq T$ and a set of equations $\tau \subseteq \mathsf{Eq}$. Then,*

$$\Gamma \vdash \Delta(\varphi_1, \varphi_2) \textit{ iff } \Gamma \vdash \Delta(\varphi_2, \varphi_1)$$

for every $\Gamma \subseteq T$ and $\varphi_1, \varphi_2 \in T$.

Proof. As the situation is symmetric, it suffices to show one implication. Assume that $\Gamma \vdash \Delta(\varphi_1, \varphi_2)$. By (Alg1) this amounts to

$$\tau[\Gamma] \vDash_{\mathsf{K}} \tau[\Delta(\varphi_1, \varphi_2)]. \tag{2.2}$$

Moreover, by (Alg4), we have $\tau[\Delta(\varphi_1, \varphi_2)] \vDash_{\mathsf{K}} \varphi_1 \approx \varphi_2$, which is equivalent to

$$\tau[\Delta(\varphi_1, \varphi_2)] \vDash_{\mathsf{K}} \varphi_2 \approx \varphi_1. \tag{2.3}$$

From conditions (2.2) and (2.3) and the transitivity of $\vDash_{\mathsf{K}}$ (see Remark 2.5.5) it follows that $\tau[\Gamma] \vDash_{\mathsf{K}} \varphi_2 \approx \varphi_1$. Then, applying (Alg2) we obtain $\Delta[\tau[\Gamma]] \vdash \Delta(\varphi_2, \varphi_1)$. Furthermore, we have $\Gamma \vdash \Delta[\tau[\Gamma]]$ by (Alg3). Thus, using the transitivity of $\vdash$, we arrive at the desired result $\Gamma \vdash \Delta(\varphi_2, \varphi_1)$. ⊠

We will now proceed by establishing a transitivity condition for sets of equivalence formulas.

Lemma 2.5.8. *Let $\vdash$ be a logic algebraized by a quasivariety K, as witnessed by a set of equivalence formulas $\Delta \subseteq T$ and a set of equations $\tau \subseteq \mathsf{Eq}$.*

$$\textit{If } \Gamma \vdash \Delta(\varphi_1, \varphi_2) \textit{ and } \Gamma \vdash \Delta(\varphi_2, \varphi_3), \textit{ then } \Gamma \vdash \Delta(\varphi_1, \varphi_3)$$

for every $\Gamma \subseteq T$ and $\varphi_1, \varphi_2, \varphi_3 \in T$.

Proof. Let $\Gamma \subseteq T$ and $\varphi_1, \varphi_2, \varphi_3 \in T$ such that $\Gamma \vdash \Delta(\varphi_1, \varphi_2)$ and $\Gamma \vdash \Delta(\varphi_2, \varphi_3)$. Using (Alg1), we obtain that $\tau[\Gamma] \vDash_{\mathsf{K}} \tau[\Delta(\varphi_1, \varphi_2)]$ and $\tau[\Gamma] \vDash_{\mathsf{K}} \tau[\Delta(\varphi_2, \varphi_3)]$. Then, by (Alg4) it follows that $\tau[\Gamma] \vDash_{\mathsf{K}} \varphi_1 \approx \varphi_2$ and $\tau[\Gamma] \vDash_{\mathsf{K}} \varphi_2 \approx \varphi_3$. By the definition of $\vDash_{\mathsf{K}}$, this amounts to $\tau[\Gamma] \vDash_{\mathsf{K}} \varphi_1 \approx \varphi_3$. Now, (Alg2) implies $\Delta[\tau[\Gamma]] \vdash \Delta(\varphi_1, \varphi_3)$. Since, by (Alg3), we also know that $\Gamma \vdash \Delta[\tau[\Gamma]]$, we can use the transitivity of $\vdash$ to conclude that $\Gamma \vdash \Delta(\varphi_1, \varphi_3)$, as desired. ⊠

Next, we will define the *(finite) Beth definability property*, which will become the logical counterpart of the (weak) ES property. Intuitively speaking, it says that every implicit definition can be made explicit (see, e.g., [6]). These notions will be made precise in the next definitions, following the slightly less general version of the Beth definability property as introduced in [29, Def. 7.3].

Definition 2.5.9. Let $\vdash$ a logic algebraized by a quasivariety K with set of equivalence formulas Δ. Given two disjoint subsets $X, Y \subseteq \mathsf{Var}$ with the property that $T(X) \neq \emptyset$ and satisfying that $|\mathsf{Var} \smallsetminus (X \cup Y)|$ is infinite and greater that $|Y|$, and a set of formulas $\Gamma \subseteq T(X \cup Y)$, we say that Y is *implicitly definable relative to Γ in terms of X* when

$$\Gamma \cup \rho[\Gamma] \vdash \Delta(y, \rho(y))$$

for every $y \in Y$ and every substitution ρ that pointwise fixes X. We call Y *explicitly definable relative to Γ in terms of X* when for every $y \in Y$, there exists $\varphi_y \in T(X)$ such that

$$\Gamma \vdash \Delta(y, \varphi_y).$$

As we will show in the next lemma, explicit definability always implies implicit definability.

Lemma 2.5.10. *Let $\vdash$ a logic algebraized by a quasivariety K with set of equivalence formulas Δ. Consider two subsets $X, Y \subseteq \mathsf{Var}$ with the property that $T(X) \neq \emptyset$ and satisfying that $|\mathsf{Var} \smallsetminus (X \cup Y)|$ is infinite and greater that $|Y|$, and a set of formulas $\Gamma \subseteq T(X \cup Y)$ such that Y is explicitly definable relative to Γ in terms of X. Then Y is implicitly definable relative to Γ in terms of X.*

Proof. Let ρ a substitution that pointwise fixes X. We have to show that, given an element $y \in Y$, we have, $\Gamma \cup \rho[\Gamma] \vdash \Delta(y, \rho(y))$. By assumption, there exists a term $\varphi_y \in T(X)$ such that

$$\Gamma \vdash \Delta(y, \varphi_y), \tag{2.4}$$

which by monotonicity of $\vdash$ (see Lemma 2.5.3) implies

$$\Gamma \cup \rho[\Gamma] \vdash \Delta(y, \varphi_y). \tag{2.5}$$

Moreover, as the logic $\vdash$ is substitution invariant, from (2.4) we conclude that

$$\rho[\Gamma] \vdash \rho[\Delta(y, \varphi_y)].$$

Since $\varphi_y \in T(X)$ and ρ pointwise fixes X, the above display amounts to

$$\rho[\Gamma] \vdash \Delta(\rho(y), \varphi_y).$$

Using Lemma 2.5.7 and monotonicity of $\vdash$ (see Lemma 2.5.3) it follows that

$$\Gamma \cup \rho[\Gamma] \vdash \Delta(\varphi_y, \rho(y)). \tag{2.6}$$

Finally, applying Lemma 2.5.8 to conditions (2.5) and (2.6) we arrive at the desired conclusion

$$\Gamma \cup \rho[\Gamma] \vdash \Delta(y, \rho(y)).$$

⊠

However, not every implicit definition can be made explicit. Therefore, it makes sense to consider this a special property of a logic. We call it the *Beth definability property* (see [29, Def. 7.3]).

Definition 2.5.11. A logic $\vdash$ algebraized by a quasivariety K with set of equivalence formulas Δ has the *Beth definability* property if

Y is implicitly definable relative to Γ in terms of X

iff

Y is explicitly definable relative to Γ in terms of X

for every pair of subsets $X, Y \subseteq \mathsf{Var}$ satisfying that $T(X) \neq \emptyset$ and $|\mathsf{Var} \smallsetminus (X \cup Y)|$ is infinite and greater that $|Y|$, and every set of formulas $\Gamma \subseteq T(X \cup Y)$.

The logic $\vdash$ is said to have the *finite Beth definability property* if the above is true whenever Y is a finite set of variables.

Finally, we are ready to state the correspondence between the (weak) ES property and the (finite) Beth definability property, originally due to Blok and Hoogland [6]. The restricted version we use can be found in [29, Cor. 7.8 and Thm. 7.9] (see also [2]).

Theorem 2.5.12. *Let $\vdash$ be a logic algebraized by a quasivariety K with set of equivalence formulas Δ. Then, the logic $\vdash$ has the (finite) Beth definability property iff K has the (weak) ES property.*

From now on, we will work on the algebraic side and focus on the (weak) ES property. However, as the above bridge theorem shows, the main theorems we aim to prove are equally interesting from a logical point of view.

CHAPTER 3

The Infinitary Baker-Pixley Theorem

The Baker-Pixley Theorem [3, Cor. 5.1] is a classical result that provides conditions under which an n-ary partial function f on some algebra $\boldsymbol{A}$ is a term function on $\boldsymbol{A}$ (see, e.g., [8, Def. 10.2]). In order to understand its statement, we need to introduce near-unanimity terms and the concept of product functions and explain what it means for a subset to be closed under a partial function.

Definition 3.1. An n-ary term $\mu(x_1, \dots, x_n)$, with $n \geqslant 3$, is called an *n-ary near-unanimity term* for a class of similar algebra K when

$$\mathsf{K} \vDash \mu(y, x, x, \dots, x) \approx \mu(x, y, x, \dots, x) \approx \cdots \approx \mu(x, \dots, x, y, x) \approx \mu(x, \dots, x, x, y) \approx x.$$

The following lemma is an easy but useful observation about near-unanimity terms.

Lemma 3.2. *If a class K has an n-ary near-unanimity term $\mu(x_1, \dots, x_n)$, then it also has a k-ary near-unanimity term for every $k \geqslant n$.*

Proof. Let $\mu_n(x_1, \dots, x_n)$ be an n-ary near-unanimity term for K and consider some $k \geqslant n$. We define

$$\mu_k(x_1, \dots, x_k) := \mu_n(x_1, \dots, x_n)$$

and claim that μ_k is a k-ary near-unanimity term for K. Let $\boldsymbol{A} \in \mathsf{K}$ and $a_1, \dots, a_k \in \boldsymbol{A}$. Assume that $a_i = a$ for every $j \neq i \leqslant k$. We distinguish two cases. If $j \leqslant n$, then

$$\mu_k(a_1, \dots, a_k) = \mu_n(a, \dots, a_j, \dots, a) = a.$$

If $n < j \leqslant k$, then

$$\mu_k(a_1, \dots, a_k) = \mu_n(a, \dots, a) = a.$$

So, in both cases, the fact that μ_n is a near-unanimity term for K implies that so is μ_k. ⊠

Definition 3.3. Let $\{\boldsymbol{A}_i : i \leqslant m\}$ be a family of similar algebras. Given a family $\{f_i \colon A_i^n \to A_i : i \leqslant m\}$ of n-ary partial functions on them, we define the *product* *function* $f_1 \times \cdots \times f_m \colon (A_1 \times \cdots \times A_m)^n \to A_1 \times \cdots \times A_m$ to be the unique n-ary partial function on $A_1 \times \cdots \times A_m$ with domain

$$\{\langle \vec{a}_1, \ldots, \vec{a}_n \rangle \in (A_1 \times \cdots \times A_m)^n : \langle \vec{a}_1(i), \ldots, \vec{a}_n(i) \rangle \in \mathsf{dom}(f_i) \text{ for every } i \leqslant m\}$$

such that for every $\langle \vec{a}_1, \ldots, \vec{a}_n \rangle \in \mathsf{dom}(f_1 \times \cdots \times f_m)$ we have,

$$(f_1 \times \cdots \times f_m)(\vec{a}_1, \ldots, \vec{a}_n) = \langle f_1(\vec{a}_1(1), \ldots, \vec{a}_n(1)), \ldots, f_m(\vec{a}_1(m), \ldots, \vec{a}_n(m)) \rangle.$$

Definition 3.4. Given an n-ary partial function f on a set A and a subset $B \subseteq A$, we say that B *is closed under* f when $f(b_1, \ldots, b_n) \in B$ for every $\langle b_1, \ldots, b_n \rangle \in B \cap \mathsf{dom}(f)$.

The classical Baker-Pixley Theorem [3, Cor. 5.1] states the following:

Baker-Pixley Theorem 3.5. *Let* $\boldsymbol{A}$ *be a finite algebra with an* $(m+1)$*-ary near-unanimity term and* f *an* n*-ary partial function on* A *such that every subuniverse of* $\boldsymbol{A}^m$ *is closed under the product function* $f \times \cdots \times f$*. Then, then there is a term* $t(x_1, \ldots, x_n) \in T$ *such that* $f(a_1, \ldots, a_n) = t^{\boldsymbol{A}}(a_1, \ldots, a_n)$ *for every* $\langle a_1, \ldots, a_n \rangle \in \mathsf{dom}(f)$*.*

The objective of this chapter is to prove the *Infinitary Baker-Pixley Theorem* due to Vaggione [31, Thm. 4.2] , which generalizes Theorem 3.5 to the setting of infinite algebras. We will follow the short proof given by Campercholi and Vaggione in [11, Thm. 2.1].

First, we need to explain what it means for a formula to define a (partial) function and introduce the notion of interpolability. The definitions we give are mainly based on [11].

Definition 3.6. Let K be a class of $\mathcal{L}$-algebras and $\varphi(x_1, \ldots, x_n, y)$ an $\mathcal{L}$-formula such that

$$\mathsf{K} \vDash \varphi(\vec{x}, y) \wedge \varphi(\vec{x}, y') \to y \approx y'$$

and

$$\mathsf{K} \vDash \exists y \varphi(\vec{x}, y).$$

Then, for every $\boldsymbol{A} \in \mathsf{K}$ we can define a function $[\varphi]^{\boldsymbol{A}}$ on A that maps every tuple $\vec{a} = \langle a_1, \ldots, a_n \rangle \in A^n$ to the unique element $b \in A$ such that $\boldsymbol{A} \vDash \varphi(\vec{a}, b)$. We say $\varphi(\vec{x}, y)$ *defines an* n*-ary function in* K.

If we drop the second condition $\mathsf{K} \vDash \exists y \varphi(\vec{x}, y)$, then we need to restrict the function $[\varphi]^{\boldsymbol{A}}$ to the subset

$$\mathsf{dom}([\varphi]^{\boldsymbol{A}}) := \{\vec{a} \in A^n : \boldsymbol{A} \vDash \exists y \varphi(\vec{a}, y)\} \subseteq A^n,$$

where $[\varphi]^{\boldsymbol{A}}$ is defined. In this case, $\varphi(\vec{x}, y)$ *defines an* n*-ary partial function in* K.

Definition 3.7. Let K be a class of $\mathcal{L}$-algebras and $\emptyset \neq T'$ a subset of $T(\vec{x})$. Consider an $\mathcal{L}$-formula $\varphi(\vec{x}, y)$ that defines a partial function in K. For any $m \in \mathbb{Z}^+$, we say that φ is m*-interpolable in* K by terms of T' when for every $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathsf{K}$ and tuples $\vec{a}_1 \in \mathsf{dom}([\varphi]^{\boldsymbol{A}_1}), \ldots, \vec{a}_m \in \mathsf{dom}([\varphi]^{\boldsymbol{A}_m})$ there exists a term $t(\vec{x}) \in T'$ such that

$$[\varphi]^{\boldsymbol{A}_i}(\vec{a}_i) = t^{\boldsymbol{A}_i}(\vec{a}_i)$$

for every $i \leqslant m$.

When T' is finite, we also say that φ is *finitely* m*-interpolable in* K and when $T' = T(\vec{x})$, we just say that φ is m*-interpolable in* K.

The next lemma provides a sufficient and necessary condition for interpolability.

Lemma 3.8. *Let* K *be a class of* $\mathcal{L}$*-algebras. For every* $\mathcal{L}$*-formula* $\varphi(x_1, \ldots, x_n, y)$ *that defines a partial function in* K*, the following are equivalent:*

1. φ *is m-interpolable in* K*;*
2. *Every subuniverse of* $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$*, with* $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathsf{K}$*, is closed under the partial function* $[\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m}$*.*

Proof. We will first show that (1) implies (2). Consider $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathsf{K}$ and a subuniverse S of $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$. Now let $\langle \vec{a}_1, \ldots, \vec{a}_n \rangle \in S \cap \mathsf{dom}([\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m})$. We need to show that $([\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m})(\vec{a}_1, \ldots, \vec{a}_n) \in S$.

By assumption we know that there exists a term $t(x_1, \ldots, x_n) \in T(x_1, \ldots, x_n)$ such that $[\varphi]^{\boldsymbol{A}_i}(\vec{a}_1(i), \ldots, \vec{a}_n(i)) = t^{\boldsymbol{A}_i}(\vec{a}_1(i), \ldots, \vec{a}_n(i))$ for every $i \leqslant m$. It follows that

$$\begin{aligned}
&([\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m})(\vec{a}_1, \ldots, \vec{a}_n) \\
&= \langle [\varphi]^{\boldsymbol{A}_1}(\vec{a}_1(1), \ldots, \vec{a}_n(1)), \ldots, [\varphi]^{\boldsymbol{A}_m}(\vec{a}_1(m), \ldots, \vec{a}_n(m)) \rangle \\
&= \langle t^{\boldsymbol{A}_1}(\vec{a}_1(1), \ldots, \vec{a}_n(1)), \ldots, t^{\boldsymbol{A}_m}(\vec{a}_1(m), \ldots, \vec{a}_n(m)) \rangle \\
&= t^{\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m}(\vec{a}_1, \ldots, \vec{a}_n) \in S,
\end{aligned}$$

as claimed.

Conversely, to prove the implication from (2) to (1), consider $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathsf{K}$ and tuples $\vec{a}_i \in (\mathsf{dom}[\varphi]^{\boldsymbol{A}_i})$ for every $i \leqslant m$. Then, the definition of the product function $[\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m}$ (see Definition 3.3) guarantees that

$$\langle \langle \vec{a}_1(1), \ldots, \vec{a}_m(1) \rangle, \ldots, \langle \vec{a}_1(n), \ldots, \vec{a}_m(n) \rangle \rangle \in \mathsf{dom}([\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m}).$$

Now, let $S := \mathsf{Sg}^{\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m}(\langle \vec{a}_1(1), \ldots, \vec{a}_m(1) \rangle, \ldots, \langle \vec{a}_1(n), \ldots, \vec{a}_m(n) \rangle)$. From the assumption that every subuniverse of $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$ is closed under the partial function $[\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m}$, we obtain

$$([\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m})(\langle \vec{a}_1(1), \ldots, \vec{a}_m(1) \rangle, \ldots, \langle \vec{a}_1(n), \ldots, \vec{a}_m(n) \rangle) \in S.$$

Hence, by Lemma 2.1.16 there must exist a term $t(x_1, \ldots, x_n) \in T(x_1, \ldots, x_n)$ such that

$$\begin{aligned}
&\langle [\varphi]^{\boldsymbol{A}_1}(\vec{a}_1(1), \ldots, \vec{a}_1(n)), \ldots, [\varphi]^{\boldsymbol{A}_m}(\vec{a}_m(1), \ldots, \vec{a}_m(n)) \rangle \\
&= ([\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m})(\langle \vec{a}_1(1), \ldots, \vec{a}_m(1) \rangle, \ldots, \langle \vec{a}_1(n), \ldots, \vec{a}_m(n) \rangle) \\
&= t^{\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m}(\langle \vec{a}_1(1), \ldots, \vec{a}_m(1) \rangle, \ldots, \langle \vec{a}_1(n), \ldots, \vec{a}_m(n) \rangle) \\
&= \langle t^{\boldsymbol{A}_1}(\vec{a}_1(1), \ldots, \vec{a}_1(n)), \ldots, t^{\boldsymbol{A}_m}(\vec{a}_m(1), \ldots, \vec{a}_m(n)) \rangle.
\end{aligned}$$

So, we conclude that $[\varphi]^{\boldsymbol{A}_i}(\vec{a}_i(1), \ldots, \vec{a}_i(n)) = t^{\boldsymbol{A}_i}(\vec{a}_i(1), \ldots, \vec{a}_i(n))$ for every $i \leqslant m$. Thus, $t(x_1, \ldots, x_n)$ is the term we were looking for. ⊠

As a second step towards the Infinitary Baker-Pixley Theorem, we will show that for classes of similar algebras that are closed under ultraproducts, interpolability implies finite interpolability.

Lemma 3.9. *Let* K *be a class of* $\mathcal{L}$*-algebras closed under ultraproducts and* $\varphi(x_1, \ldots, x_n, y)$ *an* $\mathcal{L}$*-formula, that defines a partial function in* K*. Then,* φ *is m-interpolable in* K *iff* φ *is finitely m-interpolable in* K*.*

Proof. If φ is finitely m-interpolable in K, then in particular it is m-interpolable in K. It remains to show the converse implication.

To this end, we introduce a new $(n+1)$-ary relation symbol R, which shall model φ, and define K^* to be the class of $\mathcal{L} \cup \{R\}$-algebras $\boldsymbol{A}$ with the properties that $\boldsymbol{A}{\upharpoonright}_{\mathcal{L}} \in \mathsf{K}$ and $R^{\boldsymbol{A}} = \{\langle a_1, \dots, a_n, b\rangle \in A^{n+1} : \boldsymbol{A} \vDash \varphi(\vec{a}, b)\}$. Notice that there is a one-to-one correspondence between K and K^*. On the one hand, every algebra $\boldsymbol{A} \in \mathsf{K}$ can be extended to an $\mathcal{L} \cup \{R\}$-algebra $\boldsymbol{A}^* \in \mathsf{K}^*$, by defining

$$R^{\boldsymbol{A}^*} = \{\langle a_1, \dots, a_n, b\rangle \in A^{n+1} : \boldsymbol{A} \vDash \varphi(\vec{a}, b)\}.$$

On the other hand, every element of K^* is of the form $\boldsymbol{A}^*$ for a unique $\boldsymbol{A} \in \mathsf{K}$, namely $\boldsymbol{A}^*{\upharpoonright}_{\mathcal{L}}$. As an immediate consequence of the definition of A^* we obtain that

$$\boldsymbol{A} \vDash \varphi(\vec{a}, b) \text{ iff } \boldsymbol{A}^* \vDash R(\vec{a}, b) \tag{3.1}$$

for every $\vec{a} \in A^n$ and $b \in A$ with $\boldsymbol{A} \in \mathsf{K}$. Also, it follows that for every tuple $\langle \vec{a}_1, \dots, \vec{a}_n\rangle \in (A_1 \times \dots \times A_m)^n$ and $\vec{b} \in A_1 \times \dots \times A_m$ with $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m \in \mathsf{K}$, we have

$$\boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^* \vDash R(\vec{a}_1, \dots, \vec{a}_n, \vec{b}) \text{ iff } \boldsymbol{A}_i^* \vDash R(\vec{a}_1(i), \dots, \vec{a}_n(i), \vec{b}(i)) \text{ for every } i \leqslant m. \tag{3.2}$$

First, we will show that (finite) m-interpolability in K can be phrased in terms of R.

Claim 3.10. *The following are equivalent for every non-empty subset T' of $T(\vec{x})$:*

1. *φ is m-interpolable in K by terms of T';*
2. $\mathbb{P}_m(\mathsf{K}^*) \vDash \bigvee_{t \in T'}(R(\vec{x}, y) \to R(\vec{x}, t(\vec{x})))$.

Proof. To prove the implication from (1) to (2), assume φ is m-interpolable in K by terms of T'. Recall that a generic element in $\mathbb{P}_m(\mathsf{K}^*)$ is of the form $\boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^*$, where $\boldsymbol{A}_i \in \mathsf{K}$, for every $i \leqslant m$. So, consider $\boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^* \in \mathbb{P}_m(\mathsf{K}^*)$ and let $\langle \vec{a}_1, \dots, \vec{a}_n\rangle \in (\boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^*)^n$ and $\vec{b} \in \boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^*$ such that

$$\boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^* \vDash R(\vec{a}_1, \dots, \vec{a}_n, \vec{b}).$$

We need to show that there exists a term $t \in T'$ such that

$$\boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^* \vDash R(\vec{a}_1, \dots, \vec{a}_n, t(\vec{a}_1, \dots, \vec{a}_n)).$$

By condition (3.2), we have that

$$\boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^* \vDash R(\vec{a}_1, \dots, \vec{a}_n, \vec{b}) \text{ implies } \boldsymbol{A}_i^* \vDash R(\vec{a}_1(i), \dots, \vec{a}_n(i), \vec{b}(i)) \text{ for every } i \leqslant m.$$

Using condition (3.1), the latter amounts to $\boldsymbol{A}_i \vDash \varphi(\vec{a}_1(i), \dots, \vec{a}_n(i), \vec{b}(i))$ for every $i \leqslant m$. In particular, $\langle \vec{a}_1(i), \dots, \vec{a}_n(i)\rangle \in \mathsf{dom}([\varphi]^{\boldsymbol{A}_i})$ for every $i \leqslant m$. By the assumption that φ is m-interpolable in K by terms of T', we obtain $t \in T'$ such that $\boldsymbol{A}_i \vDash \varphi(\vec{a}_1(i), \dots, \vec{a}_n(i), t(\vec{a}_1(i), \dots, \vec{a}_n(i)))$ for every $i \leqslant m$. Hence, applying condition (3.1) again, we get $\boldsymbol{A}_i^* \vDash R(\vec{a}_1(i), \dots, \vec{a}_n(i), t(\vec{a}_1(i), \dots, \vec{a}_n(i)))$ for every $i \leqslant m$. Finally, condition (3.2) yields that

$$\boldsymbol{A}_1^* \times \dots \times \boldsymbol{A}_m^* \vDash R(\vec{a}_1, \dots, \vec{a}_n, t(\vec{a}_1, \dots, \vec{a}_n)),$$

as desired.

Now, let us show that (2) implies (1). Suppose that

$$\mathbb{P}_m(\mathsf{K}^*) \vDash \bigvee_{t \in T'} (R(\vec{x}, y) \to R(\vec{x}, t(\vec{x}))).$$

Let $\boldsymbol{A}_1, \dots, \boldsymbol{A}_m \in \mathsf{K}$ and $\vec{a}_1 \in \mathsf{dom}([\varphi]^{\boldsymbol{A}_1}), \dots, \vec{a}_m \in \mathsf{dom}([\varphi]^{\boldsymbol{A}_m})$. We have to prove the existence of a term $t \in T'$ such that

$$\boldsymbol{A}_i \vDash \varphi(\vec{a}_i, t(\vec{a}_i)) \text{ for every } i \leqslant m.$$

Since $\vec{a}_i \in \mathsf{dom}([\varphi]^{\boldsymbol{A}_i})$, for every $i \leqslant m$, there exists $b_i \in A_i$ such that $\boldsymbol{A}_i \vDash \varphi(\vec{a}_i, b_i)$. By condition (3.1), it follows that $\boldsymbol{A}_i^* \vDash R(\vec{a}_i, b_i)$ for every $i \leqslant m$. Now let

$$\vec{a} = \langle \langle \vec{a}_1(1), \dots, \vec{a}_m(1) \rangle, \dots, \langle \vec{a}_1(n), \dots, \vec{a}_m(n) \rangle \rangle \text{ and } \vec{b} = \langle b_1, \dots, b_m \rangle.$$

Then, by condition (3.2), we obtain $\boldsymbol{A}_1^* \times \cdots \times \boldsymbol{A}_m^* \vDash R(\vec{a}, \vec{b})$. Thus, the assumption

$$\mathbb{P}_m(\mathsf{K}^*) \vDash \bigvee_{t \in T'} (R(\vec{x}, y) \to R(\vec{x}, t(\vec{x})))$$

yields a term $t \in T'$ such that $\boldsymbol{A}_1^* \times \cdots \times \boldsymbol{A}_m^* \vDash R(\vec{a}, t(\vec{a}))$. From condition (3.2), we conclude that $\boldsymbol{A}_i^* \vDash R(\vec{a}_i, t(\vec{a}_i))$ for every $i \leqslant m$. Finally, using condition (3.1), we arrive at the desired conclusion $\boldsymbol{A}_i \vDash \varphi(\vec{a}_i, t(\vec{a}_i))$ for every $i \leqslant m$. ⊠

So, to verify that φ is finitely m-interpolable in K, by the above Claim 3.10 and Remark 2.1.26, it suffices to find a finite subset $T_0 \subseteq_\omega T(\vec{x})$ such that

$$\mathbb{IP}_m(\mathsf{K}^*) \vDash \bigvee_{t \in T_0} (R(\vec{x}, y) \to R(\vec{x}, t(\vec{x}))).$$

To do so, we want to apply Corollary 2.2.11. Hence, we need to show that $\mathbb{IP}_m(\mathsf{K}^*)$ is closed under ultraproducts. Using Lemma 2.1.24 and Lemma 2.1.25, we get that $\mathbb{P}_u\mathbb{IP}_m(\mathsf{K}^*) \subseteq \mathbb{IP}_u\mathbb{P}_m(\mathsf{K}^*) \subseteq \mathbb{IIP}_m\mathbb{P}_u(\mathsf{K}^*) \subseteq \mathbb{IP}_m\mathbb{P}_u(\mathsf{K}^*)$. Therefore, it suffices to verify that $\mathbb{P}_u(\mathsf{K}^*) \subseteq \mathsf{K}^*$. Consider $\boldsymbol{A}^* := \prod_{i \in I} \boldsymbol{A}_i^*/U \in \mathbb{P}_u(\mathsf{K}^*)$. As K is closed under ultraproducts by assumption, it only remains to show that

$$R^{\boldsymbol{A}^*} = \{\langle \vec{a}, b \rangle \in A^{n+1} : \boldsymbol{A}^* \vDash \varphi(\vec{a}, b)\}.$$

Notice that by the definition of K^*, we have, $\boldsymbol{A}_i^* \vDash R(\vec{x}, y) \leftrightarrow \varphi(\vec{x}, y)$ for every $i \in I$. By Łoś's Theorem 2.2.3, this implies that $\boldsymbol{A}^* \vDash R(\vec{x}, y) \leftrightarrow \varphi(\vec{x}, y)$. Thus, we conclude that $\mathbb{P}_u(\mathsf{K}^*) \subseteq \mathsf{K}^*$. Hence, K^* is closed under ultraproducts and Corollary 2.2.11 yields the desired finite subset $T_0 \subseteq_\omega T$. This completes the proof. ⊠

We are now ready to state and prove the Infinitary Baker-Pixley Theorem according to [11, Thm. 2.1].

Infinitary Baker-Pixley Theorem 3.11. *(Vaggione) Let K be a class of $\mathcal{L}$-algebras closed under ultraproducts with an $(m+1)$-ary near-unanimity term. Then, for every formula $\varphi(x_1, \dots, x_n, y)$ that defines a partial function in K, the following are equivalent:*

1. *There exists a term $t(x_1, \dots, x_n) \in T$ such that for every $\boldsymbol{A} \in \mathsf{K}$ and every tuple $\vec{a} \in \mathsf{dom}([\varphi]^{\boldsymbol{A}})$ we have $[\varphi]^{\boldsymbol{A}}(\vec{a}) = t^{\boldsymbol{A}}(\vec{a})$;*

2. *Every subuniverse of* $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$*, with* $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathsf{K}$*, is closed under the partial function* $[\varphi]^{\boldsymbol{A}_1} \times \cdots \times [\varphi]^{\boldsymbol{A}_m}$*.*

Proof. Recall from Lemma 3.8 that condition (2) is equivalent to the statement that φ is m-interpolable in K. Clearly, this holds under the assumption of (1). Therefore, it only remains to prove the converse implication.

Suppose that φ is m-interpolable in K. We need to verify that there exists a term $t \in T$ such that for every $\boldsymbol{A} \in \mathsf{K}$ and every $\vec{a} \in A^n$ with $\boldsymbol{A} \vDash \exists y \varphi(\vec{a}, y)$, we have $\boldsymbol{A} \vDash \varphi(\vec{a}, t(\vec{a}))$. Equivalently, we will show the existence of a term $t \in T$ such that $\mathsf{K} \vDash \varphi(\vec{x}, y) \to \varphi(\vec{x}, t(\vec{x}))$.

Notice that by Lemma 3.9 there exists a finite subset $T_0 \subseteq_\omega T$ such that φ is m-interpolable in K by terms of T_0. Then, for each subset $S \subseteq T_0$ we define the formula

$$\psi_S(\vec{x}) := \bigwedge_{s \in S} \varphi(\vec{x}, s(\vec{x})) \wedge \bigwedge_{s \in T_0 \smallsetminus S} \neg\varphi(\vec{x}, s(\vec{x}))$$

and the set

$$X := \{S \subseteq T_0 : S \neq \emptyset \text{ and there is } \boldsymbol{A} \in \mathsf{K} \text{ such that } \boldsymbol{A} \vDash \exists \vec{x} \psi_S(\vec{x})\}.$$

The proof now proceeds through a series of claims. First, observe the following:

Claim 3.12. $\mathsf{K} \vDash \varphi(\vec{x}, y) \to \bigvee_{S \in X} \psi_S(\vec{x})$.

Proof. Consider $\boldsymbol{A} \in \mathsf{K}$, $\vec{a} \in A^n$ and $b \in A$ such that $\boldsymbol{A} \vDash \varphi(\vec{a}, b)$. We have to show that there exists $S \in X$ such that $\boldsymbol{A} \vDash \psi_S(\vec{a})$. Define $S := \{s \in T_0 : \boldsymbol{A} \vDash \varphi(\vec{a}, s(\vec{a}))\}$. Then clearly $\boldsymbol{A} \vDash \psi_S(\vec{a})$, and in particular also $\boldsymbol{A} \vDash \exists \vec{x} \psi_S(\vec{x})$. It remains to show that $S \neq \emptyset$. Since by assumption φ is m-interpolable in K by terms of T_0, there exists a term $t(\vec{x}) \in T_0$ such that $\boldsymbol{A} \vDash \varphi(\vec{a}, t(\vec{a}))$. As φ defines a partial function in K, from $\boldsymbol{A} \vDash \varphi(\vec{a}, b)$ and $\boldsymbol{A} \vDash \varphi(\vec{a}, t(\vec{a}))$ it follows that $b = t(\vec{a})$. So, $b = t(\vec{a})$ witnesses that $S \neq \emptyset$, which concludes the proof of the claim. ⊠

Claim 3.13. *For each subset* $Y \subseteq X$*, there exists a term* $t_Y(\vec{x}) \in T$ *such that*

$$\mathsf{K} \vDash \bigwedge_{S \in Y} (\psi_S(\vec{x}) \to \varphi(\vec{x}, t_Y(\vec{x}))).$$

Proof. Notice that $X \subseteq \mathcal{P}(T_0)$ is finite, since T_0 is finite. Then, so is $Y \subseteq X$, which allows us to proceed by induction on $|Y|$. For the base case, assume $|Y| \leqslant m$ and let $\boldsymbol{A} \in \mathsf{K}$. Observe that for every $S \in Y$, we have $\boldsymbol{A} \vDash \psi_S(\vec{x}) \to \bigwedge_{s \in S} \varphi(\vec{x}, s(\vec{x}))$, by the definition of $\psi_S(\vec{x})$. So, if there exists $t(\vec{x}) \in \bigcap Y$, we can take $t_Y = t$. Thus, it suffices to show that $\bigcap Y \neq \emptyset$. Since $|Y| \leqslant m$, we can assume that $Y = \{S_1, \ldots, S_m\}$, possibly with repetitions. As $Y \subseteq X$, we know that $S_i \neq \emptyset$ for every $i \leqslant m$. Furthermore, for every $i \leqslant m$, there exist $\boldsymbol{A}_i \in \mathsf{K}$ and $\vec{a}_i \in A_i^n$ such that $\boldsymbol{A}_i \vDash \psi_{S_i}(\vec{a}_i)$, and hence, in particular, $\vec{a}_i \in \mathsf{dom}([\varphi]^{\boldsymbol{A}_i})$. Now, the assumption that φ is m-interpolable in K by terms of T_0 implies that there exists $t \in T_0$ such that $[\varphi]^{\boldsymbol{A}_i}(\vec{a}_i) = t^{\boldsymbol{A}_i}(\vec{a}_i)$, and thus $\boldsymbol{A}_i \vDash \varphi(\vec{a}_i, t(\vec{a}_i))$ for every $i \leqslant m$. Finally, from the fact that $\boldsymbol{A}_i \vDash \psi_{S_i}(\vec{a}_i)$, it follows that $\boldsymbol{A}_i \vDash \bigwedge_{s \in T_0 \smallsetminus \{S_i\}} \neg\varphi(\vec{a}_i, s(\vec{a}_i))$ for each $i \leqslant m$. Therefore, we conclude that $t \in \bigcap Y$, which establishes the base case.

For the inductive step, let $k \geqslant m$ and assume the claim holds for every $Y \subseteq X$ with $|Y| = k$. Now, consider the case $Y = \{S_1, \dots, S_{k+1}\}$. By the inductive hypothesis, we know that for each $i \leqslant k+1$, there exists a term t_i such that

$$\mathsf{K} \vDash \bigwedge_{S \in Y \smallsetminus \{S_i\}} (\psi_S(\vec{x}) \to \varphi(\vec{x}, t_i(\vec{x}))). \tag{3.3}$$

Recall that, by assumption, K has an $(m+1)$-ary near-unanimity term. Given that $k+1 \geqslant m+1$, Lemma 3.2 guarantees that the class K also has a $(k+1)$-ary near-unanimity term $\mu(x_1, \dots, x_{k+1})$. Now, define $t_Y(\vec{x}) := \mu(t_1(\vec{x}), \dots, t_{k+1}(\vec{x}))$. We claim that

$$\mathsf{K} \vDash \bigwedge_{S \in Y} (\psi_S(\vec{x}) \to \varphi(\vec{x}, t_Y(\vec{x}))).$$

To see this, fix some $i \leqslant k+1$ and consider $\boldsymbol{A} \in \mathsf{K}$ and $\vec{a} \in A^n$ such that $\boldsymbol{A} \vDash \psi_{S_i}(\vec{a})$. Then, for each $j \leqslant k+1$ with $j \neq i$, by condition (3.3), we know that $[\varphi]^{\boldsymbol{A}}(\vec{a}) = t_j^{\boldsymbol{A}}(\vec{a})$. This yields

$$t_Y(\vec{a}) = \mu^{\boldsymbol{A}}(t_1^{\boldsymbol{A}}(\vec{a}), \dots, t_i^{\boldsymbol{A}}(\vec{a}), \dots, t_{k+1}^{\boldsymbol{A}}(\vec{a})) = \mu^{\boldsymbol{A}}([\varphi]^{\boldsymbol{A}}(\vec{a}), \dots, t_i^{\boldsymbol{A}}(\vec{a}), \dots, [\varphi]^{\boldsymbol{A}}(\vec{a})).$$

Moreover, as $\mu(x_1, \dots, x_{k+1})$ is a near-unanimity term, we have

$$\mu^{\boldsymbol{A}}([\varphi]^{\boldsymbol{A}}(\vec{a}), \dots, t_i^{\boldsymbol{A}}(\vec{a}), \dots, [\varphi]^{\boldsymbol{A}}(\vec{a})) = [\varphi]^{\boldsymbol{A}}(\vec{a}).$$

So, we obtain that $t_Y(\vec{a}) = [\varphi]^{\boldsymbol{A}}(\vec{a})$, and thus $\boldsymbol{A} \vDash \varphi(\vec{a}, t_Y(\vec{a}))$. This concludes the inductive argument. ⊠

In particular, the above Claim 3.13 also holds for X itself, meaning there exists a term $t \in T$ such that

$$\mathsf{K} \vDash \bigwedge_{S \in X} (\psi_S(\vec{x}) \to \varphi(\vec{x}, t(\vec{x}))). \tag{3.4}$$

Finally, to show that $\mathsf{K} \vDash \varphi(\vec{x}, y) \to \varphi(\vec{x}, t(\vec{x}))$, consider $\boldsymbol{A} \in \mathsf{K}$, $\vec{a} \in A^n$ and $b \in A$ such that $\boldsymbol{A} \vDash \varphi(\vec{a}, b)$. Recall that $\mathsf{K} \vDash \varphi(\vec{x}, y) \to \bigvee_{S \in X} \psi_S(\vec{x})$, by Claim 3.12. Thus, there exists some $S \in X$ such that $\boldsymbol{A} \vDash \psi_S(\vec{a})$. So, from condition (3.4), it follows that $\boldsymbol{A} \vDash \varphi(\vec{a}, t(\vec{a}))$.

⊠

CHAPTER 4

Global subdirect products

This chapter is dedicated to the concept of a global subdirect product as a refinement of the classical subdirect product construction (see Definition 2.3.20), which inherits the validity of a larger class of formulas from its factors. A first approach to achieving a tighter connection between the factors of a subdirect product and the subdirect product itself was the construction of the so-called sheaf representations (see, e.g., [32]). The concept of global subdirect products is the result of rephrasing the rather complex notion of a sheaf representation in the language of universal algebra, thus obtaining a simpler formulation, as carried out by Krauss and Clark in [27]. Starting from a classical subdirect product, a global subdirect product is obtained by endowing the index set with a suitable topology satisfying appropriate gluing requirements. Remarkably, this concept of global subdirect products also comes with a generalized version of Birkhoff's subdirect decomposition theorem, established in [17, Thm. 2.1], and indeed it has strong preservation properties, as witnessed, for example, by Lemma 4.14.

We will start this chapter by introducing some concepts that are vital for the construction of global subdirect products and verifying some auxiliary lemmas before presenting the announced global subdirect representation theorem and the preservation lemma.

Further details on the topological notions we will proceed to define can be found, e.g., in [25].

Definition 4.1. A *topology* on a set X is a subset $\tau \subseteq \mathcal{P}(X)$ such that:

- $X, \emptyset \in \tau$;
- $O \cap U \in \tau$ for every $O, U \in \tau$;
- $\bigcup Y \in \tau$ for every $Y \subseteq \tau$.

In this case, we say that (X, τ) is a *topological space*. The elements of τ are called *open sets*.

A family $\{O_i : i \in I\} \subseteq \tau$ is called an *open cover* for X when $X = \bigcup_{i \in I} O_i$. Given an open cover $\{O_i : i \in I\}$, a *subcover* of $\{O_i : i \in I\}$ is a family $\{O_j : j \in J\}$ with $J \subseteq I$ such that $X = \bigcup_{j \in J} O_j$, and it is a *finite subcover* when $J \subseteq_\omega I$ is a finite subset.

If a topological space (X, τ) has the property that every open cover has a finite subcover, it is called *compact*.

Definition 4.2. Let (X, τ) be a topological space. We call $\mathcal{B} \subseteq \tau$ a *basis* for τ when every element of τ is a union of elements of $\mathcal{B}$. A subset $\mathcal{S} \subseteq \tau$ is called a *subbasis* for τ when the set of all finite intersections of elements of $\mathcal{S}$ is a basis for τ.

Given a family of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$, the topology *generated by* $\mathcal{S}$ is the smallest topology τ on X containing $\mathcal{S}$. Notice that in this case, $\mathcal{S}$ forms a subbasis for τ.

Definition 4.3. Let $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{A}_i$ be a subdirect product and $a, b \in A$. We call the set

$$E(a,b) := \{i \in I : p_i(a) = p_i(b)\} = \{i \in I : \langle a, b\rangle \in \mathsf{ker}(p_i)\}$$

the *equalizer* of a and b, where $p_i \colon \boldsymbol{A} \to \boldsymbol{A}_i$ denotes the canonical projection on the i^{th} coordinate. More generally, for $\vec{a}, \vec{b} \in A^n$ the notation $E(\vec{a}, \vec{b})$ is to be understood componentwise, meaning that $E(\vec{a}, \vec{b}) = \bigcap_{k=1}^{n} E(\vec{a}(k), \vec{b}(k))$.

The topology τ^{eq} on I generated by $\{E(a,b) : a, b \in A\}$ is called the *equalizer topology* on I. When $\boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$ is a subdirect embedding, we will also use the term *equalizer topology* to refer to the topology on I generated by $\{E(a,b) : a, b \in h[A]\}$.

Definition 4.4. Let $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{A}_i$ be a subdirect product and $\mathcal{S}$ a family of subsets of I. We say that $\boldsymbol{A}$ *patches* over $\mathcal{S}$ if, for every $S_1, \ldots, S_n \in \mathcal{S}$ and $a_1, \ldots, a_n \in A$ such that

- $S_1 \cup \cdots \cup S_n = I$;
- $S_j \cap S_k \subseteq E(a_j, a_k)$ for every $j, k \leqslant n$,

there exists $a \in A$ such that $S_j \subseteq E(a, a_j)$ for every $j \leqslant n$. Recalling the definition of the equalizers, this means that a is an element of A that coincides with a_j on all indices in S_j.

If $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$ is a subdirect embedding, we also say that $\boldsymbol{A}$ patches over $\mathcal{S}$, instead of $h[\boldsymbol{A}]$ patches over $\mathcal{S}$.

For our purposes it is convenient to work with the following equivalent version of the patching property.

Lemma 4.5. *Let $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{A}_i$ be a subdirect product and $\mathcal{S}$ a family of subsets of I. Then, $\boldsymbol{A}$ patches over $\mathcal{S}$ if, for every $S_1, \ldots, S_n \in \mathcal{S}$ and tuples $\vec{a}_1, \ldots, \vec{a}_n \in A^m$ for some $m \in \mathbb{Z}^+$ such that*

- *$S_1 \cup \cdots \cup S_n = I$;*
- *$S_j \cap S_k \subseteq E(\vec{a}_j, \vec{a}_k)$ for every $j, k \leqslant n$,*

there exists a tuple $\vec{a} \in A^m$ such that $S_j \subseteq E(\vec{a}, \vec{a}_j)$ for every $j \leqslant n$.

Proof. Clearly, the original Definition 4.4 of patching is obtained from the above version by considering tuples of length 1. For the converse assume that $\boldsymbol{A}$ patches over $\mathcal{S}$ according to Definition 4.4 and consider $S_1, \ldots, S_n \in \mathcal{S}$ and $\vec{a}_1, \ldots, \vec{a}_n \in A^m$ such that $S_1 \cup \cdots \cup S_n = I$ and $S_j \cap S_k \subseteq E(\vec{a}_j, \vec{a}_k)$ for every $j, k \leqslant n$. Observe that the latter can be rephrased as $S_j \cap S_k \subseteq E(\vec{a}_j(l), \vec{a}_k(l))$ for every $l \leqslant m$. As, by assumption, $\boldsymbol{A}$ patches over $\mathcal{S}$, for every $l \leqslant m$ we obtain an element $a_l \in A$ such that $S_j \subseteq E(a_l, \vec{a}_j(l))$ for every $j \leqslant n$. Then, $\vec{a} := \langle a_1, \ldots, a_m \rangle$ has the desired property that $S_j \subseteq E(\vec{a}, \vec{a}_j)$ for every $j \leqslant n$.

⊠

Now, we are ready to introduce the notion of a global subdirect product.

Definition 4.6. We say that a subdirect product $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{A}_i$ is a *global subdirect product* when

- (I, τ^{eq}) is a compact topological space;
- $\boldsymbol{A}$ patches over τ^{eq}.

As in the case of subdirect products, we call an embedding $h \colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$ a *global subdirect embedding* when $h[\boldsymbol{A}] \leqslant \prod_{i \in I} \boldsymbol{A}_i$ is a global subdirect product.

For a class of algebras K, we will use the shorthand $\mathbb{P}_G(\mathsf{K})$ to denote the class of all global subdirect products with factors in K.

The main result of this section is the following generalized version of the Subdirect Decomposition Theorem 2.3.23.

Global Subdirect Representation Theorem 4.7. *[17, Thm. 2.1] Let K be an arithmetical variety and $\mathsf{M} \subseteq \mathsf{K}$ a universal class containing all the SI members of K. Then, $\mathsf{K} \subseteq \mathbb{IP}_G(\mathsf{M})$.*

To prove this theorem, given some $\boldsymbol{A} \in \mathsf{K}$, we will construct a subdirect embedding $h \colon A \to \prod_{\theta \in \Sigma} A/\theta$, defined by $a \mapsto \langle a/\theta : \theta \in \Sigma \rangle$ for every $a \in A$, where the index set $\Sigma := \{\theta \in \mathsf{Con}(\boldsymbol{A}) : \boldsymbol{A}/\theta \in \mathsf{M}\}$ is equipped with the equalizer topology τ^{eq}. The main work will consist in verifying compactness of (Σ, τ^{eq}) and the patching property. So, we will first consider these two properties on their own and then join the obtained results to obtain a proof of the above Theorem 4.7. We start with the following theorem that establishes a sufficient condition for the compactness of (Σ, τ^{eq}).

Theorem 4.8. *Let K be a variety, $\boldsymbol{A} \in \mathsf{K}$, and $\mathsf{M} \subseteq \mathsf{K}$ a universal class. Define*

$$\Sigma := \{\theta \in \mathsf{Con}(\boldsymbol{A}) : \boldsymbol{A}/\theta \in \mathsf{M}\}$$

and let $h \colon \boldsymbol{A} \to \prod_{\theta \in \Sigma} \boldsymbol{A}/\theta$ be a subdirect embedding, where $h(a) = \langle a/\theta : \theta \in \Sigma \rangle$. Then, (Σ, τ^{eq}) is compact.

To prove this result, we will make use of the Alexander Subbasis Theorem (see, e.g., [25, Thm. 5.6]), which simplifies the verification of compactness.

Alexander Subbasis Theorem 4.9. *Let $\mathsf{X} = (X, \tau)$ be a topological space with a subbasis $\mathcal{B}$. Then, X is compact iff every open cover $\mathcal{C}$ of X with $\mathcal{C} \subseteq \mathcal{B}$ has a finite subcover.*

Proof of Theorem 4.8. To verify the compactness of (Σ, τ^{eq}), we need to show that every open cover of Σ has a finite subcover. By the Alexander Subbasis Theorem 4.9, we can assume that the open cover consists of equalizers, as they form a subbasis for τ^{eq}. So, let $\Sigma = \bigcup_{j \in J} E(a_j, b_j)$ and assume, with a view to contradiction, that $\Sigma \neq \bigcup_{j \in F} E(a_j, b_j)$ for every finite subset $F \subseteq J$.

Now, consider the set of $\mathcal{L}_A$-formulas

$$\Delta = \mathsf{Th}(\mathsf{M}) \cup Diag^+(\boldsymbol{A}) \cup \{\neg(a_j \approx b_j) : j \in J\}.$$

We claim that Δ is satisfiable. By the Compactness Theorem 2.2.6 it suffices to show that $\Delta_F = \mathsf{Th}(\mathsf{M}) \cup Diag^+(\boldsymbol{A}) \cup \{\neg(a_j \approx b_j) : j \in F\}$ is satisfiable for every finite subset $F \subseteq_\omega J$.

So, fix some $F \subseteq_\omega J$. The assumption that $\Sigma \neq \bigcup_{j \in F} E(a_j, b_j)$ implies that there exists $\theta_F \in \Sigma$ such that $\langle a_j, b_j \rangle \notin \theta_F$ for every $j \in F$. We claim that $\boldsymbol{A}/\theta_F \vDash \Delta_F$, where $\boldsymbol{A}/\theta_F$ is considered an $\mathcal{L}_A$-algebra, via the canonical interpretation $a^{\boldsymbol{A}/\theta_F} = a/\theta_F$ for every $a \in A$.

First, observe that $\theta_F \in \Sigma$ implies $A/\theta_F \in \mathsf{M}$, by the definition of Σ. Thus, we conclude that $\boldsymbol{A}/\theta_F \vDash \mathsf{Th}(\mathsf{M})$. Furthermore, by Lemma 2.2.14, the canonical projection $\boldsymbol{A} \to \boldsymbol{A}/\theta_F$ witnesses that $A/\theta_F \vDash Diag^+(\boldsymbol{A})$. Finally, the assumption that $\langle a_j, b_j \rangle \notin \theta_F$ for every $j \in F$ guarantees that $A/\theta_F \vDash \{\neg(a_j \approx b_j) : j \in J\}$. So, we conclude that there exists a model $\boldsymbol{B}$ of Δ. As M is universal, and thus, by Theorem 2.3.4, an elementary class, it follows that $\mathsf{Mod}(\mathsf{Th}(\mathsf{M})) = \mathsf{M}$. Hence, from $\boldsymbol{B} \vDash \mathsf{Th}(\mathsf{M})$, we conclude that $\boldsymbol{B}{\restriction}_{\mathcal{L}} \in \mathsf{M}$. Furthermore, by Lemma 2.2.15, $\boldsymbol{B} \vDash Diag^+(\boldsymbol{A})$ implies that there exists a homomorphism $h \colon \boldsymbol{A}_A \to \boldsymbol{B}$. Thus, $\boldsymbol{A}/\mathsf{ker}(h) \in \mathbb{IS}(\mathsf{M})$. The assumption that M is a universal class then yields $\boldsymbol{A}/\mathsf{ker}(h) \in \mathsf{M}$, and thus $\mathsf{ker}(h) \in \Sigma$ by the definition of Σ.

Now, recall that, by assumption, $\Sigma = \bigcup_{j \in J} E(a_j, b_j)$. Hence, there exists $k \in J$ such that $\langle a_k, b_k \rangle \in \mathsf{ker}(h)$, which implies that $a_k^{\boldsymbol{B}} = h(a_k) = h(b_k) = b_k^{\boldsymbol{B}}$, and thus $\boldsymbol{B} \vDash a_k \approx b_k$. But this is a contradiction because $\boldsymbol{B} \vDash \{\neg(a_j \approx b_j) : j \in J\}$. This completes the proof that the open cover $\{E(a_j, b_j) : j \in J\}$ must have a finite subcover. Therefore, (Σ, τ^{eq}) is compact.

⊠

It turns out that the patching property is tightly connected to the Chinese Remainder Theorem (see Theorem 4.11). To establish this connection, we first need to do some preliminary work.

Definition 4.10. Let $\boldsymbol{A}$ be an algebra and $\Sigma \subseteq \mathsf{Con}(\boldsymbol{A})$. We define

$$\Sigma_\cap := \{\bigcap \Gamma : \Gamma \subseteq \Sigma\},$$

which is the closure of Σ under arbitrary intersections. For $\Delta \subseteq A \times A$, the smallest congruence in $\Sigma_\cap$ containing Δ is

$$\mathsf{Cg}^{\Sigma_\cap}(\Delta) := \bigcap\{\theta \in \Sigma : \Delta \subseteq \theta\}.$$

Notice that $(\Sigma_\cap, \wedge^{\Sigma_\cap}, \vee^{\Sigma_\cap})$ is a complete lattice, where

$$\theta_1 \wedge^{\Sigma_\cap} \theta_2 := \theta_1 \cap \theta_2,$$

which is well defined since $\Sigma_\cap$ is closed under intersections, and

$$\theta_1 \vee^{\Sigma_\cap} \theta_2 := \mathsf{Cg}^{\Sigma_\cap}(\theta_1 \cup \theta_2).$$

The announced connection between the patching property and the Chinese Remainder Theorem is made precise in the following theorem. A more general version of this result, using slightly different terminology, can be found, e.g., in [27, Lem. 4.35].

Theorem 4.11. *Let $\Sigma \subseteq \mathsf{Con}(\boldsymbol{A})$ and $h \colon \boldsymbol{A} \to \prod_{\theta \in \Sigma} \boldsymbol{A}/\theta$ a subdirect embedding such that (Σ, τ^{eq}) is a compact topological space and $h(a) = \langle a/\theta : \theta \in \Sigma \rangle$ for every $a \in A$. If $\Sigma_\cap$ satisfies the Chinese Remainder Theorem, then $\boldsymbol{A}$ patches over τ^{eq}, and thus h is a global subdirect embedding.*

In order to prove Theorem 4.11, we will make use of the following auxiliary lemma, which establishes a translation between intersections of sets of equalizers and congruences. It is a rephrasing of the statement of [30, Lem. 1.1].

Lemma 4.12. *Let $h\colon \boldsymbol{A} \to \prod_{\theta\in\Sigma} \boldsymbol{A}/\theta$ be a subdirect embedding, where $\Sigma \subseteq \mathsf{Con}(\boldsymbol{A})$ and $h(a) = \langle a/\theta : \theta \in \Sigma\rangle$ for every $a \in A$. Consider $\Delta := \{\langle c_i, d_i\rangle : i \leqslant n\} \subseteq A \times A$. Then, for every $a, b \in A$ we have,*

$$\langle a, b\rangle \in \mathsf{Cg}^{\Sigma_\cap}(\Delta) \quad \textit{iff} \quad \bigcap_{i\leqslant n} E(c_i, d_i) \subseteq E(a, b).$$

Proof. Observe that for every $\theta \in \Sigma$ we have, $\theta \in E(c, d)$ iff $\langle c, d\rangle \in \theta$. Keeping this in mind, we will first establish the following:

Claim 4.13.

$$\bigcap_{i\leqslant n} E(c_i, d_i) = \{\theta \in \Sigma : \Delta \subseteq \theta\}.$$

Proof. To verify the inclusion $\bigcap_{i\leqslant n} E(c_i, d_i) \subseteq \{\theta \in \Sigma : \Delta \subseteq \theta\}$, notice that for every $\phi \in \bigcap_{i\leqslant n} E(c_i, d_i)$, we have $\Delta = \{\langle c_i, d_i\rangle : i \leqslant n\} \subseteq \phi$, and thus $\phi \in \{\theta \in \Sigma : \Delta \subseteq \theta\}$. Conversely, consider $\phi \in \Sigma$ such that $\Delta \subseteq \phi$. Then, from the definition of Δ, it follows that $\langle c_i, d_i\rangle \in \phi$ for every $i \leqslant n$. So, we conclude that $\phi \in \bigcap_{i\leqslant n} E(c_i, d_i)$, as desired. ⊠

Now, from the the definition of $\mathsf{Cg}^{\Sigma_\cap}$, it follows that

$$\langle a, b\rangle \in \mathsf{Cg}^{\Sigma_\cap}(\Delta) \text{ iff } \langle a, b\rangle \in \theta \text{ for every } \theta \in \Sigma \text{ such that } \Delta \subseteq \theta.$$

By Claim 4.13 this is equivalent to $\langle a, b\rangle \in \theta$ for every $\theta \in \bigcap_{i\leqslant n} E(c_i, d_i)$. This last condition, in turn, is equivalent to $\bigcap_{i\leqslant n} E(c_i, d_i) \subseteq E(a, b)$. ⊠

We are now ready to proceed with the proof of Theorem 4.11.

Proof of Theorem 4.11. Let $S_1, \dots, S_n \in \tau^{eq}$ and $a_1, \dots, a_n \in A$ such that $\bigcup_{i\leqslant n} S_i = \Sigma$ and $S_i \cap S_j \subseteq E(a_i, a_j)$ for every $i, j \leqslant n$. We have to show that there exists $a \in A$ such that $S_i \subseteq E(a, a_i)$ for every $i \leqslant n$.

Consider the set of finite intersections of equalizers

$$\mathcal{B} := \{\bigcap_{k\leqslant m} E(b_k, c_k) : m \in \mathbb{N} \text{ and } b_k, c_k \in A \text{ for } k \leqslant m\}.$$

First, we will deal with the case that $S_i \in \mathcal{B}$ for every $i \leqslant n$. Then, there exist $b^i_1, \dots, b^i_{m_i}, c^i_1, \dots, c^i_{m_i} \in A$ such that $S_i = \bigcap_{k\leqslant m_i} E(b^i_k, c^i_k)$ for all $i \leqslant n$.

Now, for every $i \leqslant n$ define $\delta_i := \mathsf{Cg}^{\Sigma_\cap}(\{\langle b^i_k, c^i_k\rangle : k \leqslant m_i\})$. Observe that

$$\delta_i \vee^{\Sigma_\cap} \delta_j = \mathsf{Cg}^{\Sigma_\cap}(\{\langle b^i_k, c^i_k\rangle : k \leqslant m_i\} \cup \{\langle b^j_k, c^j_k\rangle : k \leqslant m_j\})$$

for every $i, j \leqslant n$. By Lemma 4.12 the assumption that

$$\bigcap_{k\leqslant m_i} E(b^i_k, c^i_k) \cap \bigcap_{k\leqslant m_j} E(b^j_k, c^j_k) = S_i \cap S_j \subseteq E(a_i, a_j)$$

implies that $\langle a_i, a_j\rangle \in \delta_i \vee^{\Sigma_\cap} \delta_j$.

Thus, since $\Sigma_\cap$ is assumed to satisfy the Chinese Remainder Theorem, there exists $a \in A$ such that $\langle a, a_i \rangle \in \delta_i = \mathsf{Cg}^{\Sigma_\cap}(\{\langle b^i_k, c^i_k \rangle : k \leqslant m_i\})$ for every $i \leqslant n$. Lemma 4.12 then entails that $S_i = \bigcap_{k \leqslant m_i} E(b^i_k, c^i_k) \subseteq E(a, a_i)$ for every $i \leqslant n$. This shows that $\boldsymbol{A}$ patches over τ^{eq} when $S_i \in \mathcal{B}$ for every $i \leqslant n$.

Let us now consider the generic case. Since the equalizers form a subbasis of τ^{eq}, we obtain that $\mathcal{B}$ is a basis of τ^{eq}. Hence, for every $i \leqslant n$ we have, $S_i = \bigcup_{m \in M_i} B_m$ for some index set M_i and $B_m \in \mathcal{B}$ for all $m \in M_i$. Notice that, by the infinite distributive law, the assumption that $S_i \cap S_j \subseteq E(a_i, a_j)$ for every $i, j \leqslant n$ can be equivalently stated as

$$\bigcup_{\substack{m \in M_i, \\ k \in M_j}} (B_m \cap B_k) \subseteq E(a_i, a_j). \tag{4.1}$$

In other words, $B_m \cap B_k \subseteq E(a_i, a_j)$ for every $i, j \leqslant n$ and $\langle m, k \rangle \in M_i \times M_j$. Recall that

$$\Sigma = S_1 \cup \cdots \cup S_n = \bigcup_{m \in M_1} B_m \cup \cdots \cup \bigcup_{m \in M_n} B_m.$$

Hence, by the assumed compactness of (Σ, τ^{eq}), it follows that for every $i \leqslant n$, there exists a finite subset $M'_i \subseteq M_i$, such that $\Sigma = \bigcup_{m \in M'_1} B_m \cup \cdots \cup \bigcup_{m \in M'_n} B_m$. Now, consider the finite family $\mathcal{S} := \{B_m : m \in M'_i \text{ and } i \leqslant n\} \subseteq \mathcal{B}$, and recall that by condition (4.1) for every $i, j \leqslant n$ and $\langle m, k \rangle \in M_i \times M_j$ we have, $B_m \cap B_k \subseteq E(a_i, a_j)$. So, the family $\mathcal{S}$ satisfies the requirements of the patching property. Since, moreover, $\mathcal{S} \subseteq \mathcal{B}$ we can use the previous observation to obtain $a \in A$ such that

$$B_m \subseteq E(a, a_i) \text{ for every } i \leqslant n \text{ and every } m \in M'_i. \tag{4.2}$$

To conclude the proof, it remains to show that $S_i = \bigcup_{m \in M_i} B_m \subseteq E(a, a_i)$ for every $i \leqslant n$. Consider $\theta \in S_i$. Then, there exists $m \in M_i$ such that $\theta \in B_m$. On the other hand, the fact that $\Sigma = \bigcup_{m \in M'_1} B_m \cup \cdots \cup \bigcup_{m \in M'_n} B_m$ implies that there exists $j \leqslant n$ and $k \in M'_j$ such that $\theta \in B_k$. By condition (4.1) we know that

$$B_m \cap B_k \subseteq E(a_i, a_j) \tag{4.3}$$

Moreover, as $k \in M'_j$, condition (4.2) yields that

$$B_k \subseteq E(a, a_j). \tag{4.4}$$

Since $B_m \cap B_k \subseteq B_k$, from conditions (4.3) and (4.4), it follows that

$$\theta \in B_m \cap B_k \subseteq E(a_i, a_j) \cap E(a, a_j) \subseteq E(a, a_i),$$

as desired. ⊠

Putting together the previous results from Theorems 4.8 and 4.11, we now set out to prove the Global Subdirect Representation Theorem 4.7.

Proof of Theorem 4.7. Consider $\boldsymbol{A} \in \mathsf{K}$ and let $\Sigma := \{\theta \in \mathsf{Con}(\boldsymbol{A}) : \boldsymbol{A}/\theta \in \mathsf{M}\}$. We will show that the map $h\colon \boldsymbol{A} \to \prod_{\theta \in \Sigma} \boldsymbol{A}/\theta$, where $h(a) = \langle a/\theta : \theta \in \Sigma \rangle$ for every $a \in A$, is a global subdirect embedding.

First, notice that, as $\mathsf{K}_{\mathrm{SI}} \subseteq \mathsf{M}$, the Subdirect Decomposition Theorem 2.3.23 yields that $h\colon \boldsymbol{A} \to \prod_{\theta \in \Sigma} \boldsymbol{A}/\theta$ is a subdirect embedding. So, it only remains to check compactness and the patching property.

Since our assumptions include the ones of Theorem 4.8, the compactness of (Σ, τ^{eq}) is a direct consequence of that result.

Given that (Σ, τ^{eq}) is compact, to see that $\boldsymbol{A}$ patches over τ^{eq}, by Theorem 4.11, it suffices to verify that $\Sigma_\cap$ satisfies the Chinese Remainder Theorem. Consider $\theta_1, \dots, \theta_n \in \Sigma_\cap$ such that $\langle a_i, a_j \rangle \in \theta_i \vee^{\Sigma_\cap} \theta_j$ for every $i, j \leqslant n$. We need to show that there exists $a \in A$ such that $\langle a, a_i \rangle \in \theta_i$ for every $i \leqslant n$. By assumption K is an arithmetical variety, and hence, by Theorem 2.3.38, $\mathsf{Con}(\boldsymbol{A})$ satisfies the Chinese Remainder Theorem. In order to be able to apply it, we will now prove that $\langle a_i, a_j \rangle \in \theta_i \vee \theta_j$ for every $i, j \leqslant n$, where $\theta_i \vee \theta_j$ denotes the join of $\mathsf{Con}(\boldsymbol{A})$. Assume, with a view to contradiction, that there exist $i, j \leqslant n$ such that $\langle a_i, a_j \rangle \notin \theta_i \vee \theta_j$. By Theorem 2.3.19 there exists a family $(\phi_k)_{k \in I}$ of completely meet-irreducible congruences of $\boldsymbol{A}$, such that $\theta_i \vee \theta_j = \bigcap_{k \in I} \phi_k$. Then, $\theta_i \cup \theta_j \subseteq \phi_k$ for every $k \in I$. On the other hand, our assumption $\langle a_i, a_j \rangle \notin \theta_i \vee \theta_j$ implies that there exists $k \in I$ such that $\langle a_i, a_j \rangle \notin \phi_k$. As ϕ_k is completely meet-irreducible, Theorem 2.3.24 allows us to conclude that $\boldsymbol{A}/\phi_k \in \mathsf{K}_{\mathrm{SI}} \subseteq \mathsf{M}$, and thus $\phi_k \in \Sigma$. But this is impossible because $\langle a_i, a_j \rangle \in \theta_i \vee^{\Sigma_\cap} \theta_j = \bigcap\{\theta \in \Sigma : \theta_i \cup \theta_j \subseteq \theta\}$ and $\phi_k \in \{\theta \in \Sigma : \theta_i \cup \theta_j \subseteq \theta\}$. We conclude that $\langle a_i, a_j \rangle \in \theta_i \vee \theta_j$. Hence, the Chinese Remainder Theorem for $\mathsf{Con}(\boldsymbol{A})$ yields an element $a \in A$ such that $\langle a, a_i \rangle \in \theta_i$ for every $i \leqslant n$. This verifies the Chinese Remainder Theorem for $\Sigma_\cap$, and thus concludes the proof. ⊠

The following lemma will be instrumental in the proof of Theorem 6.4 and an example of the usefulness of the notion of global subdirect products. It states that the validity of certain formulas is preserved under global subdirect products [31, Lem. 3.1].

Lemma 4.14. *Let φ be an $\mathcal{L}$-formula of the form $\forall \vec{x} \exists! \vec{y} \bigwedge_{j \leqslant n} (\varepsilon_j \approx \delta_j)(\vec{x}, \vec{y})$ and M a class of $\mathcal{L}$-algebras. Then,*

$$\mathsf{M} \vDash \varphi \text{ implies } \mathbb{IP}_G \vDash \varphi.$$

Proof. Let $\boldsymbol{A} \leqslant \prod_{i \in I} \boldsymbol{A}_i$ be a global subdirect product with $\{\boldsymbol{A}_i : i \in I\} \subseteq \mathsf{M}$ such that $\mathsf{M} \vDash \forall \vec{x} \exists! \vec{y} \bigwedge_{j \leqslant n} (\varepsilon_j \approx \delta_j)(\vec{x}, \vec{y})$. Now, consider a tuple $\vec{a}$ of elements of A of the same length as $\vec{x}$. We need to show that $\boldsymbol{A} \vDash \exists! \vec{y} \bigwedge_{j \leqslant n} (\varepsilon_j \approx \delta_j)(\vec{a}, \vec{y})$. Let $p_i\colon \boldsymbol{A} \to \boldsymbol{A}_i$ be the canonical projection. Then, by assumption, for each $i \in I$ there exists a unique tuple $\vec{b}_i$ of elements of A_i such that $\boldsymbol{A}_i \vDash \bigwedge_{j \leqslant n} (\varepsilon_j \approx \delta_j)(p_i(\vec{a}), \vec{b}_i)$. Since p_i is surjective for each $i \in I$, there exists a tuple $\vec{c}_i$ of elements of A such that $p_i(\vec{c}_i) = \vec{b}_i$. Thus, for every $i \in I$, we obtain

$$\boldsymbol{A}_i \vDash \bigwedge_{j \leqslant n} (\varepsilon_j \approx \delta_j)(p_i(\vec{a}), p_i(\vec{c}_i)).$$

In other words, for every $i \in I$, we have,

$$i \in \bigcap_{j \leqslant n} E(\varepsilon_j(\vec{a}, \vec{c}_i), \delta_j(\vec{a}, \vec{c}_i)). \tag{4.5}$$

We want to use the fact that $\boldsymbol{A}$ patches over τ^{eq}. To this end, for every $i \in I$, we define

$$S_i := \bigcap_{j \leqslant n} E(\varepsilon_j(\vec{a}, \vec{c}_i), \delta_j(\vec{a}, \vec{c}_i)) \in \tau^{eq}.$$

Notice that condition (4.5) implies that $\bigcup_{i\in I} S_i = I$. Then, since (I, τ^{eq}) is compact, there exists a finite subset $F \subseteq I$ such that $I = \bigcup_{i\in F} S_i$. We claim that for all $k, l \in F$, we have $S_k \cap S_l \subseteq E(\vec{c}_k, \vec{c}_l)$. To see this, let $q \in S_k \cap S_l$. Then,

$$\boldsymbol{A}_q \vDash \bigwedge_{j\leqslant n} (\varepsilon_j \approx \delta_j)(p_q(\vec{a}), p_q(\vec{c}_k)) \wedge \bigwedge_{j\leqslant n} (\varepsilon_j \approx \delta_j)(p_q(\vec{a}), p_q(\vec{c}_l)).$$

From the assumption that $\boldsymbol{A}_q \vDash \exists! \vec{y} \bigwedge_{j\leqslant n}(\varepsilon_j \approx \delta_j)(p_q(\vec{a}), \vec{y})$, we thus conclude that $\boldsymbol{A}_q \vDash p_q(\vec{c}_k) \approx p_q(\vec{c}_l)$, which implies $q \in E(\vec{c}_k, \vec{c}_l)$. Now, the patching property for tuples (see Lemma 4.5) yields a tuple $\vec{c}$ of elements of A such that

$$S_i \subseteq E(\vec{c}, \vec{c}_i) \text{ for every } i \in I.$$

We claim that $\boldsymbol{A} \vDash \bigwedge_{j\leqslant n}(\varepsilon_j \approx \delta_j)(\vec{a}, \vec{c})$. Equivalently, we will verify that

$$\bigcap_{j\leqslant n} E(\varepsilon_j(\vec{a}, \vec{c}), \delta_j(\vec{a}, \vec{c})) = I.$$

So, consider some $q \in I$. We have to show that $q \in \bigcap_{j\leqslant n} E(\varepsilon_j(\vec{a}, \vec{c}), \delta_j(\vec{a}, \vec{c}))$. Recall that $I = \bigcup_{i\in F} S_i$. Thus, $q \in S_i = \bigcap_{j\leqslant n} E(\varepsilon_j(\vec{a}, \vec{c}_i), \delta_j(\vec{a}, \vec{c}_i))$ for some $i \in F$. On the other hand, we know that $S_i \subseteq E(\vec{c}, \vec{c}_i)$. Hence, we arrive at the desired conclusion

$$q \in \bigcap_{j\leqslant n} E(\varepsilon_j(\vec{a}, \vec{c}_i), \delta_j(\vec{a}, \vec{c}_i)) \cap E(\vec{c}, \vec{c}_i) \subseteq \bigcap_{j\leqslant n} E(\varepsilon_j(\vec{a}, \vec{c}), \delta_j(\vec{a}, \vec{c})).$$

Finally, it remains to show that the tuple $\vec{c}$ of elements of A, which we found, is unique. Assume, with a view to contradiction, there was a tuple $\vec{d}$ of elements of A such that $\vec{c} \neq \vec{d}$ and $\boldsymbol{A} \vDash \bigwedge_{j\leqslant n}(\varepsilon_j \approx \delta_j)(\vec{a}, \vec{c}) \wedge \bigwedge_{j\leqslant n}(\varepsilon_j \approx \delta_j)(\vec{a}, \vec{d})$. Since $\boldsymbol{A} \leqslant \prod_{i\in I} \boldsymbol{A}_i$ is a subdirect embedding, $\vec{c} \neq \vec{d}$ implies that there exists $i \in I$ such that $p_i(\vec{c}) \neq p_i(\vec{d})$. As equations are preserved under homomorphisms by Lemma 2.1.27, we obtain

$$\boldsymbol{A}_i \vDash \bigwedge_{j\leqslant n} (\varepsilon_j \approx \delta_j)(p_i(\vec{a}), p_i(\vec{c})) \wedge \bigwedge_{j\leqslant n} (\varepsilon_j \approx \delta_j)(p_i(\vec{a}), p_i(\vec{d})).$$

But this is a contradiction with the assumption that $\boldsymbol{A}_i \vDash \forall \vec{x} \exists! \vec{y} \bigwedge_{j\leqslant n}(\varepsilon_j \approx \delta_j)(\vec{x}, \vec{y})$. Hence, we must have $\vec{c} = \vec{d}$, which establishes the claimed uniqueness. ⊠

We will conclude this section with the following useful result, which is a consequence of the above preservation Lemma 4.14.

Theorem 4.15. *[10, Lem. 7.8] Let* K *be a variety in the language* $\mathcal{L}$ *and* $\mathsf{M} \subseteq \mathsf{K}$ *such that* $\mathsf{K} \subseteq \mathbb{IP}_G(\mathsf{M})$. *If* $\mathsf{M} \vDash \forall \vec{x} \exists! y \varphi(\vec{x}, y)$, *where* $\varphi(\vec{x}, y)$ *is a conjunction of equations, then there exists an* $\mathcal{L}$*-term* $t(\vec{x})$ *such that* $\mathsf{K} \vDash \varphi(\vec{x}, t(\vec{x}))$.

Proof. By Lemma 4.14, we obtain that $\mathsf{K} \vDash \forall \vec{x} \exists! y \varphi(\vec{x}, y)$. Now, let $\vec{x} = \langle x_1, \dots, x_n \rangle$ and consider $\mathbf{F}_{\mathsf{K}}(\vec{x})$, the free algebra in K with free generators $\{x_i : i \leqslant n\}$, which exists by Lemma 2.3.29. Then, $\mathbf{F}_{\mathsf{K}}(\vec{x}) \vDash \exists! y \varphi(\vec{x}, y)$. Hence, by Lemma 2.1.16, there exists an $\mathcal{L}$-term $t(\vec{x})$ such that $\mathbf{F}_{\mathsf{K}}(\vec{x}) \vDash \varphi(\vec{x}, t(\vec{x}))$. Now, consider $\boldsymbol{A} \in \mathsf{K}$ and $\vec{a} = \langle a_1, \dots, a_n \rangle \in A^n$. In order to complete the proof, we need to show that $\boldsymbol{A} \vDash \varphi(\vec{a}, t(\vec{a}))$. As $\mathbf{F}_{\mathsf{K}}(\vec{x})$ is free for K, the map sending $x_i \mapsto a_i$ for every $i \leqslant n$ extends to a homomorphism $h \colon \mathbf{F}_{\mathsf{K}}(\vec{x}) \to \boldsymbol{A}$. Since conjunctions of equations are preserved under homomorphisms by Lemma 2.1.27, the desired conclusion $\boldsymbol{A} \vDash \varphi(\vec{a}, t(\vec{a}))$ follows from $\mathbf{F}_{\mathsf{K}}(\vec{x}) \vDash \varphi(\vec{x}, t(\vec{x}))$. ⊠

CHAPTER 5

Definability conditions

Given a class of similar algebras K and $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, sometimes it is useful to express certain semantic relations, like $\boldsymbol{A} \leqslant \boldsymbol{B}$ being epic in K, in terms of syntactic properties, like existence or preservation of special formulas (see Theorem 5.3). Conversely, verifying semantic conditions for K might allow us to transform formulas into a particularly nice shape, which is convenient to work with, like positive quantifier-free formulas (see Theorem 5.5) or conjunctions of equations (see Corollary 5.9). To establish these connections, we will mainly rely on model-theoretic tools.

The first lemma of this chapter relates the validity of p.p. sentences to the existence of homomorphisms to ultrapowers.

Lemma 5.1. *[21, Thm. 6.5.7] The following conditions are equivalent for every pair of similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$:*

1. *Every p.p. sentence that holds in $\boldsymbol{A}$ also holds in $\boldsymbol{B}$;*
2. *There exists $\boldsymbol{C} \in \mathbb{P}_u(\boldsymbol{B})$ and a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{C}$.*

Proof. To prove the implication (1) $\Rightarrow$ (2), suppose that every p.p. sentence that holds in $\boldsymbol{A}$ also holds in $\boldsymbol{B}$, and assume with a view to contradiction that there is no homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{C}$ with $\boldsymbol{C} \in \mathbb{P}_u(\boldsymbol{B})$. Then, Lemma 2.2.15 yields that $\mathsf{Diag}^+(\boldsymbol{A})$ is not satisfiable in any $\boldsymbol{C} \in \mathbb{P}_u(\boldsymbol{B})$. Using Remark 2.1.26, it follows that

$$\mathbb{IP}_u(\boldsymbol{B}) \vDash \bigvee \neg \mathsf{Diag}^+(\boldsymbol{A}).$$

Observe that $\mathbb{IP}_u(\boldsymbol{B})$ is closed under ultraproducts by Lemma 2.1.24. So, we can apply Corollary 2.2.11 to obtain a finite subset $\Delta \subseteq \mathsf{Diag}^+(\boldsymbol{A})$ such that

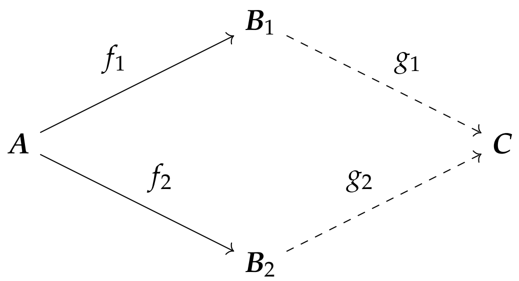
$$\mathbb{IP}_u(\boldsymbol{B}) \bigvee \neg \Delta. \tag{5.1}$$

On the other hand, $\exists \vec{x} \bigwedge \Delta(\vec{x})$ is a p.p. sentence valid in $\boldsymbol{A}$, since $\Delta \subseteq \mathsf{Diag}^+(\boldsymbol{A})$ is a finite set of atomic formulas. So, the assumption that every p.p. sentence that holds in $\boldsymbol{A}$ also holds in $\boldsymbol{B}$ implies that $\boldsymbol{B} \vDash \exists \vec{x} \bigwedge \Delta(\vec{x})$. But this is a contradiction with condition (5.1), since $\boldsymbol{B} \in \mathbb{IP}_u(\boldsymbol{B})$.

For the converse implication (2) $\Rightarrow$ (1), let $\boldsymbol{C} \in \mathbb{P}_u(\boldsymbol{B})$ and $h\colon \boldsymbol{A} \to \boldsymbol{C}$ a homomorphism. Now, consider a p.p. sentence φ valid in $\boldsymbol{A}$. We need to show that $\boldsymbol{B} \vDash \varphi$. Recall that p.p. sentences are preserved under homomorphisms by Lemma 2.1.27. Thus, $\boldsymbol{A} \vDash \varphi$ implies $\boldsymbol{C} \vDash \varphi$. Finally, as $\boldsymbol{C} \in \mathbb{P}_u(\boldsymbol{B})$, we can apply Łoś's Theorem 2.2.3 to obtain that $\boldsymbol{B} \vDash \varphi$, as claimed. ⊠

The next result will be instrumental in the proof of Theorem 5.3. It states that for every two elementary extensions of some algebra, there exists one in which both of them embed. A version of this result, similar to the one we will present, can be found in [21, Thm. 6.4.1].

Elementary Amalgamation Theorem 5.2. *Let K be a class of $\mathcal{L}$-algebras and $\boldsymbol{A}, \boldsymbol{B}_1, \boldsymbol{B}_2 \in \mathsf{K}$. Moreover, let $f_1 \colon \boldsymbol{A} \to \boldsymbol{B}_1$ and $f_2 \colon \boldsymbol{A} \to \boldsymbol{B}_2$ be elementary embeddings. Then, there exist an algebra $\boldsymbol{C} \in \mathbb{ISP}_u(\mathsf{K})$ and a pair of embeddings $g_1 \colon \boldsymbol{B}_1 \to \boldsymbol{C}$ and $g_2 \colon \boldsymbol{B}_2 \to \boldsymbol{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*



Proof. Since $f_1 \colon \boldsymbol{A} \to \boldsymbol{B}_1$ and $f_2 \colon \boldsymbol{A} \to \boldsymbol{B}_2$ are elementary embeddings, we can consider $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ as $\mathcal{L}_A$ algebras via the canonical interpretations $a^{\boldsymbol{B}_i} := f_i(a)$ for $i = 1, 2$. Now, consider the languages $\mathcal{L}_{B_i} = \mathcal{L}_A \cup \{b : b \in B_i \smallsetminus f_i[A]\}$ for $i = 1, 2$. Without loss of generality, we can assume that $B_1 \smallsetminus f_1[A]$ and $B_2 \smallsetminus f_2[A]$ are disjoint. Let

$$\Sigma := \mathsf{Th}(\mathbb{ISP}_u(\mathsf{K})) \cup \mathsf{Diag}_{\mathcal{L}_{B_1}}(\boldsymbol{B}_1) \cup \mathsf{Diag}_{\mathcal{L}_{B_2}}(\boldsymbol{B}_2).$$

We claim that Σ is satisfiable. By the Compactness Theorem 2.2.6, it suffices to show that for every finite subset $\Delta \subseteq_\omega \mathsf{Diag}_{\mathcal{L}_{B_2}}(\boldsymbol{B}_2)$, the set $\Sigma_\Delta := \mathsf{Th}(\mathbb{ISP}_u(\mathsf{K})) \cup \mathsf{Diag}_{\mathcal{L}_{B_1}}(\boldsymbol{B}_1) \cup \Delta$ is satisfiable.

Assume this was not the case. Then, there exist tuples $\vec{a}$ of constants of A and $\vec{b}$ of constants of $B_2 \smallsetminus f_2[A]$ such that

$$\boldsymbol{B}_2 \vDash \bigwedge \Delta(\vec{a}, \vec{b}) \tag{5.2}$$

and

$$\mathsf{Th}(\mathbb{ISP}_u(\mathsf{K})) \cup \mathsf{Diag}_{\mathcal{L}_{B_1}}(\boldsymbol{B}_1) \vDash \neg \bigwedge \Delta(\vec{a}, \vec{b}).$$

Since by assumption the constants $\vec{b}$ do not appear in $\mathsf{Th}(\mathbb{ISP}_u(\mathsf{K})) \cup \mathsf{Diag}_{\mathcal{L}_{B_1}}(\boldsymbol{B}_1)$, by Corollary 2.2.2, the latter is equivalent to

$$\mathsf{Th}(\mathbb{ISP}_u(\mathsf{K})) \cup \mathsf{Diag}_{\mathcal{L}_{B_1}}(\boldsymbol{B}_1) \vDash \forall \vec{y} \neg \bigwedge \Delta(\vec{a}, \vec{y}).$$

As $\boldsymbol{B}_1 \vDash \mathsf{Th}(\mathbb{ISP}_u(\mathsf{K})) \wedge \mathsf{Diag}_{\mathcal{L}_{B_1}}(\boldsymbol{B}_1)$, we conclude that $\boldsymbol{B}_1 \vDash \forall \vec{y} \neg \bigwedge \Delta(\vec{a}, \vec{y})$. Recalling that $f_1 \colon \boldsymbol{A} \to \boldsymbol{B}_1$ is an elementary embedding, this yields $\boldsymbol{A} \vDash \forall \vec{y} \neg \bigwedge \Delta(\vec{a}, \vec{y})$. Finally, since $f_2 \colon \boldsymbol{A} \to \boldsymbol{B}_1$ is also an elementary embedding, we obtain $\boldsymbol{B}_2 \vDash \forall \vec{y} \neg \bigwedge \Delta(\vec{a}, \vec{y})$. But this is a contradiction with condition (5.2).

So, there exists a model $\boldsymbol{C} \in \mathsf{Mod}(\Sigma)$. In particular, this implies

$$\boldsymbol{C} \in \mathsf{Mod}(\mathsf{Th}(\mathbb{ISP}_u(\mathsf{K}))) = \mathbb{ISP}_u(\mathsf{K}),$$

where the equality is a consequence of the fact that $\mathbb{ISP}_u(\mathsf{K})$ is a universal class by Theorem 2.3.3, and universal classes are elementary by Theorem 2.3.4(1). Furthermore, the two embeddings $g_1 \colon \boldsymbol{B}_1 \to \boldsymbol{C}$ and $g_2 \colon \boldsymbol{B}_2 \to \boldsymbol{C}$ exist by Lemma 2.2.15 because $\boldsymbol{C} \vDash \mathsf{Diag}_{\mathcal{L}_{B_1}}(\boldsymbol{B}_1)$ and $\boldsymbol{C} \vDash \mathsf{Diag}_{\mathcal{L}_{B_2}}(\boldsymbol{B}_2)$. The additional requirement that $g_1 \circ f_1 = g_2 \circ f_2$ is a straightforward consequence of the facts that f_1 and f_2 are elementary embeddings and $\boldsymbol{C} \vDash \mathsf{Diag}_{\mathcal{L}_{B_1}}(\boldsymbol{B}_1) \cup \mathsf{Diag}_{\mathcal{L}_{B_2}}(\boldsymbol{B}_2)$.

⊠

The two previous results will now be applied to prove the next theorem, which gives a characterization of epic subalgebras using p.p. formulas and is a key ingredient in the proofs of the Theorems 6.3 and 6.4 in Chapter 6.

Theorem 5.3. *[9, Thm. 3] Let K be a class of $\mathcal{L}$-algebras closed under ultraproducts. For every $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathsf{K}$, the following are equivalent:*

1. *$\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K;*
2. *For every $b \in B$, there exist a p.p. formula $\varphi(\vec{x}, y)$ and a tuple $\vec{a}$ of elements of A such that φ defines a partial function in K and $\boldsymbol{B} \vDash \varphi(\vec{a}, b)$.*

Proof. To prove that (1) implies (2), let $b \in B$ and define

$$\Sigma(y) := \{\varphi(y) \in Fm_{\mathcal{L}_A} : \varphi \text{ is a p.p. formula such that } \boldsymbol{B}_A \vDash \varphi(b)\}.$$

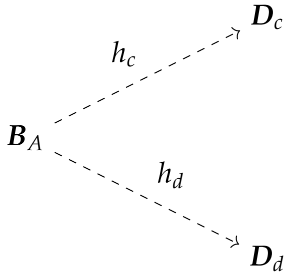
Now, we introduce two new constants c and d and consider the class

$$\mathsf{K}^* := \{\boldsymbol{C} : \boldsymbol{C} \text{ is an } \mathcal{L}_A \cup \{c, d\}\text{-algebra and } \boldsymbol{C}{\restriction}_{\mathcal{L}} \in \mathsf{K}\}.$$

Our first goal is to show the following:

Claim 5.4. $\mathsf{K}^* \vDash (\bigwedge \Sigma(c) \wedge \bigwedge \Sigma(d)) \to c \approx d$.

Proof. Consider $\boldsymbol{C} \in \mathsf{K}^*$ such that $\boldsymbol{C} \vDash \bigwedge \Sigma(c) \wedge \bigwedge \Sigma(d)$. We claim that there are $\boldsymbol{D}_c, \boldsymbol{D}_d \in \mathbb{P}_u(\boldsymbol{C})$ and a pair of homomorphisms $h_c \colon \boldsymbol{B}_A \to \boldsymbol{D}_c$ and $h_d \colon \boldsymbol{B}_A \to \boldsymbol{D}_d$ such that $h_c(b) = c^{\boldsymbol{D}_c}$ and $h_d(b) = d^{\boldsymbol{D}_d}$.



Let $\boldsymbol{B}_A(c)$ be the unique $\mathcal{L}_A \cup \{c\}$-algebra with $\boldsymbol{B}_A(c){\restriction}_{\mathcal{L}_A} = \boldsymbol{B}_A$, where the constant c is interpreted as $c^{\boldsymbol{B}_A(c)} := b$. Also, define $\boldsymbol{C}(c) := \boldsymbol{C}{\restriction}_{\mathcal{L}_A \cup \{c\}}$. Since $\boldsymbol{C} \vDash \Sigma(c)$, we conclude that all p.p. $\mathcal{L}_A \cup \{c\}$-sentences valid in $\boldsymbol{B}_A(c)$ also hold in $\boldsymbol{C}(c)$. Therefore, Lemma 5.1 yields an $\mathcal{L}_A \cup \{c\}$-homomorphism $h'_c \colon \boldsymbol{B}_A(c) \to \boldsymbol{D}'_c$, for some $\boldsymbol{D}'_c \in \mathbb{P}_u(\boldsymbol{C}(c))$, which restricts to the claimed homomorphism

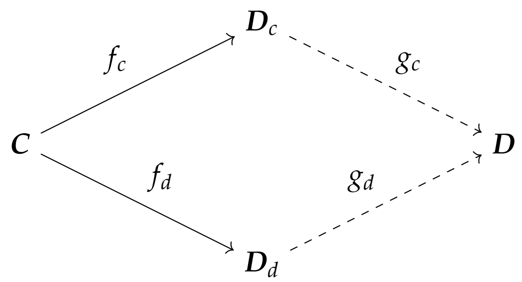
$$h_c \colon \boldsymbol{B}_A \to \boldsymbol{D}_c \text{ with } \boldsymbol{D}_c := \boldsymbol{D}'_c{\restriction}_{\mathcal{L}_A} \in \mathbb{P}_u(\boldsymbol{C}) \text{ and } h_c(b) = c^{\boldsymbol{D}_c}.$$

Repeating the same argument with d instead of c yields a homomorphism

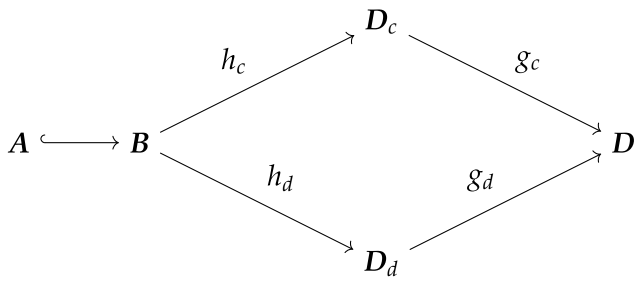
$$h_d \colon \boldsymbol{B}_A \to \boldsymbol{D}_d \text{ with } \boldsymbol{D}_d \in \mathbb{P}_u(\boldsymbol{C}) \text{ and } h_d(b) = d^{\boldsymbol{D}_d}.$$

On the other hand, we also have the canonical elementary embeddings $f_c \colon \boldsymbol{C} \to \boldsymbol{D}_c$ and $f_d \colon \boldsymbol{C} \to \boldsymbol{D}_d$ (see Remark 2.2.5). Now, notice that K^* is closed under ultraproducts, since so is K, by assumption. Hence, as $\boldsymbol{C} \in \mathsf{K}^*$, we conclude that also $\boldsymbol{D}_c, \boldsymbol{D}_d \in \mathsf{K}^*$. Thus, we can apply Lemma 5.2 to obtain $\boldsymbol{D} \in \mathbb{ISP}_u(\mathsf{K}^*)$ and a pair of embeddings $g_c \colon \boldsymbol{D}_c \to \boldsymbol{D}$ and $g_d \colon \boldsymbol{D}_d \to \boldsymbol{D}$ such that

$$g_c \circ f_c = g_d \circ f_d. \tag{5.3}$$



Recall that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K and $\mathbb{P}_u(\mathsf{K}) \subseteq \mathsf{K}$ by assumption. Therefore, using Lemma 2.4.7, it follows that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is also epic in $\mathbb{ISP}_u(\mathsf{K})$. Since the homomorphisms $g_c \circ h_c, g_d \circ h_d \colon \boldsymbol{B} \to \boldsymbol{D}{\restriction}_{\mathcal{L}}$, with $\boldsymbol{D}{\restriction}_{\mathcal{L}} \in \mathbb{ISP}_u(\mathsf{K})$, agree on A, from the fact that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in $\mathbb{ISP}_u(\mathsf{K})$ we conclude that $g_c \circ h_c = g_d \circ h_d$.



In particular,

$$g_c \circ h_c(b) = g_d \circ h_d(b). \tag{5.4}$$

So, we obtain

$$g_c \circ f_c(c^{\boldsymbol{C}}) = g_c(c^{\boldsymbol{D}_c}) = g_d(d^{\boldsymbol{D}_d}) = g_d \circ f_d(d^{\boldsymbol{C}}) = g_c \circ f_c(d^{\boldsymbol{C}}), \tag{5.5}$$

where the first and third equality follow because f_c and f_d preserve the constants c and d, respectively. The second equality is due to condition (5.4) together with the fact that $h_c(b) = c^{\boldsymbol{D}_c}$ and $h_d(b) = d^{\boldsymbol{D}_d}$, by construction. Finally, condition (5.3) yields the last equality. To conclude the proof of the claim, recall that g_c and f_c are embeddings, and thus so is their composition $g_c \circ f_c$. Hence, condition (5.5) implies $c^{\boldsymbol{C}} = d^{\boldsymbol{C}}$, as desired. ⊠

Now, as K^* is closed under ultraproducts, we can apply Corollary 2.2.12 to Claim 5.4 and obtain a finite subset $\Sigma_0 \subseteq_\omega \Sigma$ such that

$$\mathsf{K}^* \vDash (\bigwedge \Sigma_0(c) \wedge \bigwedge \Sigma_0(d)) \to c \approx d. \tag{5.6}$$

Recall that $\Sigma(y)$ is a set of p.p. $\mathcal{L}_A$-formulas. Since finite conjunctions of p.p. formulas are equivalent to p.p. formulas, there exist a p.p. $\mathcal{L}$-formula $\varphi(\vec{x}, y)$ and a tuple $\vec{a}$ of constants in A such that $\bigwedge \Sigma_0(y)$ is equivalent to $\varphi(\vec{a}, y)$. Moreover, the definition of Σ implies that $\boldsymbol{B} \vDash \varphi(\vec{a}, b)$.

Then condition (5.6) is equivalent to $\mathsf{K}^* \vDash \varphi(\vec{a}, c) \wedge \varphi(\vec{a}, d) \to c \approx d$. Restricting to the original language $\mathcal{L}$ without the constants, Lemma 2.2.1 yields that

$$\mathsf{K} \vDash \varphi(\vec{x}, y) \wedge \varphi(\vec{x}, y') \to y \approx y'.$$

So, φ defines a partial function in K. Hence, together with the previous observations, we have verified that $\varphi(\vec{x}, y)$ has all the claimed properties. We have thus shown that (1) implies (2).

Conversely, assume condition (2). We want to prove that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K. To this end, consider $\boldsymbol{C} \in \mathsf{K}$ and $g, h \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $f{\restriction}_A = g{\restriction}_A$, and let $b \in B$. We have to show that $g(b) = h(b)$. By assumption there exist a p.p. formula $\varphi(\vec{x}, y)$ and a tuple $\vec{a}$ of elements of A such that φ defines a partial function in K and $\boldsymbol{B} \vDash \varphi(\vec{a}, b)$. As p.p. formulas are preserved under homomorphisms by Lemma 2.1.27, we conclude that $\boldsymbol{C} \vDash \varphi(g(\vec{a}), g(b)) \wedge \varphi(h(\vec{a}), h(b))$. Since g and h agree on A, this amounts to $\boldsymbol{C} \vDash \varphi(g(\vec{a}), g(b)) \wedge \varphi(g(\vec{a}), h(b))$, which implies $g(b) = h(b)$ because φ defines a partial function in K. This establishes the implication (2) ⇒ (1) and thus concludes the proof.

⊠

The remaining part of the chapter focuses on semantic conditions that allow us to work with formulas of a particularly nice shape.

First, we will prove a lemma that tells us when a formula is equivalent to a quantifier-free one.

Lemma 5.5. *[10, Thm. 3.3] Let K be a class of similar algebras that is closed under ultraproducts. Then for every formula $\varphi(\vec{x})$ the following are equivalent:*

1. *There exists a positive quantifier-free formula $\psi(\vec{x})$ such that $\mathsf{K} \vDash \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$;*

2. *For every $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ with subalgebras $\boldsymbol{A}' \leqslant \boldsymbol{A}$ and $\boldsymbol{B}' \leqslant \boldsymbol{B}$ and every homomorphism $h \colon \boldsymbol{A}' \to \boldsymbol{B}'$, we have,*
$$\boldsymbol{A} \vDash \varphi(\vec{a}) \text{ implies } \boldsymbol{B} \vDash \varphi(h(\vec{a}))$$
for every tuple $\vec{a}$ of elements of A'.

Proof. To prove the implication (1) ⇒ (2), let $\psi(\vec{x})$ be a positive and quantifier-free formula such that $\mathsf{K} \vDash \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$. Now, consider $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, two subalgebras $\boldsymbol{A}' \leqslant \boldsymbol{A}$ and $\boldsymbol{B}' \leqslant \boldsymbol{B}$, and a homomorphism $h \colon \boldsymbol{A}' \to \boldsymbol{B}'$ such that $\boldsymbol{A} \vDash \varphi(\vec{a})$ for some tuple $\vec{a}$ of elements of A'. Then the assumption $\mathsf{K} \vDash \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$ implies that $\boldsymbol{A} \vDash \psi(\vec{a})$. Given that A' contains the tuple $\vec{a}$ and since positive quantifier-free formulas are preserved under subalgebras, homomorphisms and extensions by Lemmas 2.1.30, 2.1.27, and Corollary 2.1.28, respectively, we conclude that

$$\boldsymbol{A} \vDash \psi(\vec{a}) \Rightarrow \boldsymbol{A}' \vDash \psi(\vec{a}) \Rightarrow \boldsymbol{B}' \vDash \psi(h(\vec{a})) \Rightarrow \boldsymbol{B} \vDash \psi(h(\vec{a})).$$

Finally, using the assumption $\mathsf{K} \vDash \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$ again, we obtain the claimed result $\boldsymbol{B} \vDash \varphi(h(\vec{a}))$.

For the converse, assume condition (2) holds and let $\{(\boldsymbol{A}_i, \vec{a}_i) : i \in I\}$ be the collection of all tuples $(\boldsymbol{A}, \vec{a})$ with $\boldsymbol{A} \in \mathsf{K}$ such that $\boldsymbol{A} \vDash \varphi(\vec{a})$, where $\vec{a}$ is a tuple of elements of A. First, observe that

Claim 5.6. $\mathsf{K} \vDash \varphi(\vec{x}) \to \bigvee_{i \in I} \bigwedge \mathsf{Diag}^{+}_{\vec{a}_i}(\boldsymbol{A}_i)(\vec{x})$.

Proof. Consider $\boldsymbol{A} \in \mathsf{K}$ and a tuple $\vec{a}$ of elements of A such that $\boldsymbol{A} \vDash \varphi(\vec{a})$. Then, there exists $i \in I$ such that $(\boldsymbol{A}, \vec{a}) = (\boldsymbol{A}_i, \vec{a}_i)$. Therefore, $\boldsymbol{A} \vDash \mathsf{Diag}^{+}_{\vec{a}_i}(\boldsymbol{A}_i)(\vec{a})$, as desired. ⊠

Since K is closed under ultraproducts by assumption, we can apply Corollary 2.2.12 to Claim 5.6, to obtain a finite subset $I_0 \subseteq_\omega I$ such that

$$\mathsf{K} \vDash \varphi(\vec{x}) \to \bigvee_{i \in I_0} \bigwedge \mathsf{Diag}^{+}_{\vec{a}_i}(\boldsymbol{A}_i)(\vec{x}). \tag{5.7}$$

Next, we will establish that

Claim 5.7. $\mathsf{K} \vDash \bigwedge \mathsf{Diag}^{+}_{\vec{a}_i}(\boldsymbol{A}_i)(\vec{x}) \to \varphi(\vec{x})$*, for every* $i \in I$.

Proof. Let $\boldsymbol{A} \in \mathsf{K}$ and $\vec{a}$ a tuple of elements of A such that $\boldsymbol{A} \vDash \mathsf{Diag}^{+}_{\vec{a}_i}(\boldsymbol{A}_i)(\vec{a})$. Now, consider the subalgebras $\boldsymbol{A}_i' \leqslant \boldsymbol{A}_i$ and $\boldsymbol{A}' \leqslant \boldsymbol{A}$ with universes $A_i' := \mathsf{Sg}^{\boldsymbol{A}_i}(\vec{a}_i)$ and $A' := \mathsf{Sg}^{\boldsymbol{A}}(\vec{a})$, respectively. As $\boldsymbol{A} \vDash \mathsf{Diag}^{+}_{\vec{a}_i}(\boldsymbol{A}_i)(\vec{a})$, Lemma 2.2.15 implies that there exists a homomorphism $h_i \colon \boldsymbol{A}_i' \to \boldsymbol{A}'$ such that $t(\vec{a}_i) \mapsto t(\vec{a})$, for every term t. Since $\boldsymbol{A}_i \vDash \varphi(\vec{a}_i)$, our assumption yields that $\boldsymbol{A} \vDash \varphi(h(\vec{a}_i))$, and hence $\boldsymbol{A} \vDash \varphi(\vec{a})$, by the definition of h. This verifies Claim 5.7. ⊠

Given that K is closed under ultraproducts by assumption, applying Corollary 2.2.12 to Claim 5.7, for every $i \in I$, we obtain a finite subset $\Delta_i \subseteq_\omega \mathsf{Diag}^{+}_{\vec{a}_i}(\boldsymbol{A}_i)$ such that

$$\mathsf{K} \vDash \bigwedge \Delta_i(\vec{x}) \to \varphi(\vec{x}). \tag{5.8}$$

Furthermore, as condition (5.8) holds for every $i \in I_0 \subseteq I$, we conclude that

$$\mathsf{K} \vDash \bigvee_{i \in I_0} \bigwedge \Delta_i(\vec{x}) \to \varphi(\vec{x}). \tag{5.9}$$

Observe that

$$\psi(\vec{x}) := \bigvee_{i \in I_0} \bigwedge \Delta_i(\vec{x})$$

is a positive quantifier-free formula, since $\Delta_i \subseteq \mathsf{Diag}^{+}_{\vec{a}_i}(\boldsymbol{A}_i)$ is a finite set of atomic formulas for every $i \in I_0$. We will now verify that it satisfies $\mathsf{K} \vDash \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$. By condition (5.9), we know that

$$\mathsf{K} \vDash \psi(\vec{x}) \to \varphi(\vec{x}).$$

For the converse, recall from condition (5.7) that $\mathsf{K} \vDash \varphi(\vec{x}) \to \bigvee_{i \in I_0} \bigwedge \mathsf{Diag}^+_{\vec{a}_i}(\boldsymbol{A}_i)(\vec{x})$. As $\Delta_i \subseteq \mathsf{Diag}^+_{\vec{a}_i}(\boldsymbol{A}_i)$ for every $i \in I_0$, it follows that $\mathsf{K} \vDash \varphi(\vec{x}) \to \bigvee_{i \in I_0} \bigwedge \Delta_i(\vec{x})$, and thus

$$\mathsf{K} \vDash \varphi(\vec{x}) \to \psi(\vec{x}).$$

This establishes the claimed equivalence $\mathsf{K} \vDash \psi(\vec{x}) \leftrightarrow \varphi(\vec{x})$. ⊠

We will conclude this chapter with the observation that, under reasonable assumptions, every positive quantifier-free formula is equivalent to a conjunction of equations. The key to this result lies in the next theorem. We introduce the following abbreviation to simplify the notation: For tuples of variables $\vec{x} = \langle x_1, \dots, x_n \rangle$ and $\vec{y} = \langle y_1, \dots, y_n \rangle$ we will write $\vec{x} \approx \vec{y}$ as a shorthand for $\bigwedge_{i \leqslant n} x_i \approx y_i$.

Theorem 5.8. *[14, Thm. 2.3] Let K be a congruence distributive quasivariety such that $\mathsf{K}_{\mathrm{RFSI}}$ is a universal class. Then, there exists a finite set of equations*

$$\{p_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \approx q_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) : i \in I\}$$

such that

$$\mathsf{K}_{\mathrm{RFSI}} \vDash \bigwedge_{i \in I} p_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \approx q_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \longleftrightarrow (\vec{x} \approx \vec{y}) \vee (\vec{w} \approx \vec{z}).$$

Proof. Let $\mathbf{F} := \mathbf{F}_{\mathsf{K}}(\vec{x}, \vec{y}, \vec{w}, \vec{z})$ be the free algebra in K with free generators $\vec{x}, \vec{y}, \vec{w}$ and $\vec{z}$. We define

$$\theta(\vec{x}, \vec{y}) := \mathsf{Cg}^{\mathbf{F}}_{\mathsf{K}}(\{\langle x_i, y_i \rangle : i \leqslant n\}) \text{ and } \theta(\vec{w}, \vec{z}) := \mathsf{Cg}^{\mathbf{F}}_{\mathsf{K}}(\{\langle w_i, z_i \rangle : i \leqslant n\}).$$

Now, let $\{\langle p_j, q_j \rangle : j \in J\}$ be a set of generators for $\theta(\vec{x}, \vec{y}) \cap \theta(\vec{w}, \vec{z}) \in \mathsf{Con}_{\mathsf{K}}(\mathbf{F})$. Consider $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RFSI}}$ with $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ tuples of elements of A and a homomorphism $h \colon \mathbf{F} \to \boldsymbol{A}$ such that $h(\vec{x}) = \vec{a}, h(\vec{y}) = \vec{b}, h(\vec{z}) = \vec{c}$, and $h(\vec{w}) = \vec{d}$.

We will first establish the following:

$$\mathsf{K}_{\mathrm{RFSI}} \vDash \bigwedge_{j \in J} p_j(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \approx q_j(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \to (\vec{x} \approx \vec{y}) \vee (\vec{w} \approx \vec{z}). \tag{5.10}$$

So, assume that $p_j(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = q_j(\vec{a}, \vec{b}, \vec{c}, \vec{d})$, which by the definition of h implies $h(p_j(\vec{x}, \vec{y}, \vec{w}, \vec{z})) = h(q_j(\vec{x}, \vec{y}, \vec{w}, \vec{z}))$ for all $j \in J$. We have to show that $\vec{a} = \vec{b}$ or $\vec{c} = \vec{d}$.

Notice that, for every $j \in J$, the condition $h(p_j(\vec{x}, \vec{y}, \vec{w}, \vec{z})) = h(q_j(\vec{x}, \vec{y}, \vec{w}, \vec{z}))$ is equivalent to $\langle p_j(\vec{x}, \vec{y}, \vec{w}, \vec{z}), q_j(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \rangle \in \mathsf{ker}(h)$. Since the set $\{\langle p_j, q_j \rangle : j \in J\}$ generates $\theta(\vec{x}, \vec{y}) \cap \theta(\vec{w}, \vec{z})$, this implies that $\theta(\vec{x}, \vec{y}) \cap \theta(\vec{w}, \vec{z}) \subseteq \mathsf{ker}(h)$, which is equivalent to $(\theta(\vec{x}, \vec{y}) \cap \theta(\vec{w}, \vec{z})) \vee \mathsf{ker}(h) = \mathsf{ker}(h)$. Then, using the assumed congruence distributivity, we obtain

$$(\theta(\vec{x}, \vec{y}) \vee \mathsf{ker}(h)) \cap (\theta(\vec{w}, \vec{z}) \vee \mathsf{ker}(h)) = \mathsf{ker}(h). \tag{5.11}$$

Recall that $\boldsymbol{A} \in \mathsf{K}_{\mathrm{RFSI}}$. As $\mathbf{F}/\mathsf{ker}(h) \in \mathbb{IS}(\boldsymbol{A})$ and $\mathsf{K}_{\mathrm{RFSI}}$ is a universal class by assumption, we conclude that $\mathbf{F}/\mathsf{ker}(h) \in \mathsf{K}_{\mathrm{RFSI}}$. Thus, $\mathsf{ker}(h)$ is finitely meet-irreducible in $\mathsf{Con}_{\mathsf{K}}(\mathbf{F})$ by Theorem 2.3.24. Hence, condition (5.11) implies that $\mathsf{ker}(h) = \theta(\vec{x}, \vec{y}) \vee \mathsf{ker}(h)$ or $\mathsf{ker}(h) = \theta(\vec{w}, \vec{z}) \vee \mathsf{ker}(h)$. We will assume that $\mathsf{ker}(h) = \theta(\vec{x}, \vec{y}) \vee \mathsf{ker}(h)$. The other case can be dealt with analogously. Then, we get that $\theta(\vec{x}, \vec{y}) \subseteq \mathsf{ker}(h)$, and thus

$h(\vec{x}) = h(\vec{y})$. Therefore, by the definition of h it follows that $\vec{a} = \vec{b}$. This verifies condition (5.10).

The next step is now to pass from J to a finite subset $I \subseteq_\omega J$ such that

$$\mathsf{K}_{\text{RFSI}} \vDash \bigwedge_{i \in I} p_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \approx q_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \to (\vec{x} \approx \vec{y}) \vee (\vec{w} \approx \vec{z}).$$

Since K is a quasivariety, and thus in particular closed under ultraproducts, this is an immediate consequence of Corollary 2.2.12.

Finally, it remains to show the other implication

$$\mathsf{K}_{\text{RFSI}} \vDash (\vec{x} \approx \vec{y}) \vee (\vec{w} \approx \vec{z}) \to \bigwedge_{i \in I} p_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \approx q_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}). \tag{5.12}$$

To this end, suppose that $\vec{a} = \vec{b}$ or $\vec{c} = \vec{d}$, and thus $h(\vec{x}) = h(\vec{y})$ or $h(\vec{z}) = h(\vec{w})$. We have to show that $p_i(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = q_i(\vec{a}, \vec{b}, \vec{c}, \vec{d})$, for every $i \in I$. Since

$$\{\langle p_i, q_i \rangle : i \in I\} \subseteq \theta(\vec{x}, \vec{y}) \cap \theta(\vec{w}, \vec{z}),$$

from $h(\vec{x}) = h(\vec{y})$ or $h(\vec{z}) = h(\vec{w})$ it follows that $h(p_i(\vec{x}, \vec{y}, \vec{w}, \vec{z})) = h(q_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}))$, and hence $p_i(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = q_i(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ for every $i \in I$. This establishes condition (5.12) and concludes the proof. ⊠

As a consequence, we obtain the announced result on positive quantifier-free formulas (see, e.g., [10, Prop. 5.7]).

Corollary 5.9. *If a quasivariety* K *is congruence distributive and* K_{RFSI} *forms a universal class, then for every positive quantifier-free formula* φ *there exists a conjunction of equations* α *such that* $\mathsf{K}_{\text{RFSI}} \vDash \varphi \leftrightarrow \alpha$.

Proof. Since φ is positive and quantifier-free, by Lemma 2.1.10 we may assume, without loss of generality, that φ is in disjunctive normal form. Thus, there exist some $n \in \mathbb{Z}^+$ and a set $\{\varepsilon_i : i \leqslant n\}$, where each ε_i is a finite conjunction of equations, such that $\varphi = \bigvee_{i \leqslant n} \varepsilon_i$. We will proceed by induction on n to show that there exists a conjunction of equations α such that $\mathsf{K}_{\text{RFSI}} \vDash \bigvee_{i \leqslant n} \varepsilon_i \leftrightarrow \alpha$. The base case $n = 1$ is trivial, as we can just take $\alpha := \varepsilon_1$. So, assume the claim holds for some $n \in \mathbb{N}$ and let

$$\varphi = \bigvee_{i \leqslant n+1} \varepsilon_i = \bigvee_{i \leqslant n} \varepsilon_i \vee \varepsilon_{n+1}.$$

By the inductive hypothesis, we know that $\mathsf{K}_{\text{RFSI}} \vDash \bigvee_{i \leqslant n} \varepsilon_i \leftrightarrow \beta$, where β is a conjunction of equations. Hence, we obtain

$$\mathsf{K}_{\text{FSI}} \vDash \varphi \leftrightarrow (\beta \vee \varepsilon_{n+1}). \tag{5.13}$$

Let $\beta = \vec{t} \approx \vec{s}$ and $\varepsilon_{n+1} = \vec{v} \approx \vec{u}$ for some finite tuples of $\mathcal{L}$-terms $\vec{t}, \vec{s}, \vec{v}$, and $\vec{u}$. By Theorem 5.8, there exists a finite set of equations $\{p_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \approx q_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) : i \in I\}$ such that

$$\mathsf{K}_{\text{RFSI}} \vDash (\vec{x} \approx \vec{y}) \vee (\vec{w} \approx \vec{z}) \longleftrightarrow \bigwedge_{i \in I} p_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}) \approx q_i(\vec{x}, \vec{y}, \vec{w}, \vec{z}).$$

This implies that

$$\mathsf{K}_{\text{RFSI}} \vDash (\vec{t} \approx \vec{s}) \vee (\vec{v} \approx \vec{u}) \longleftrightarrow \bigwedge_{i \in I} p_i(\vec{t}, \vec{s}, \vec{v}, \vec{u}) \approx q_i(\vec{t}, \vec{s}, \vec{v}, \vec{u}). \tag{5.14}$$

Now, let α be the finite conjunction of equations $\bigwedge_{i \in I} p_i(\vec{t}, \vec{s}, \vec{v}, \vec{u}) \approx q_i(\vec{t}, \vec{s}, \vec{v}, \vec{u})$ and recall that $\beta = \vec{t} \approx \vec{s}$ and $\varepsilon_{n+1} = \vec{v} \approx \vec{u}$. Then, the desired conclusion

$$\mathsf{K}_{\text{RFSI}} \vDash \varphi \leftrightarrow \alpha$$

follows from conditions (5.13) and (5.14).

⊠

CHAPTER 6

Two theorems on the ES property

We have now reached the core of the thesis. This chapter will put into practice the previously developed theory and apply the results we established so far, with the objective of proving two theorems by Campercholi [9, Thms. 18 and 22] that facilitate the detection of failures of the ES property in (quasi)varieties with some special properties. They state that, under some reasonable assumptions, in order to verify the ES property, it suffices to look at particular types of algebras and check whether they lack proper epic subalgebras.

Before going into the details of the proofs, we will consider the following technical lemma and a useful corollary of it, which constitute an important ingredient of the proofs for both of the main theorems.

Lemma 6.1. *Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra and consider three $\mathcal{L}_A$-algebras $\mathbf{B}, \mathbf{D}$, and $\mathbf{E}$ such that $\boldsymbol{A}_A \leqslant \boldsymbol{B}$ and $\mathbf{E} \in \mathbb{P}_u(\boldsymbol{D})$. Moreover, let $h\colon \boldsymbol{B} \to \mathbf{E}$ be a homomorphism such that $h[A] = h[B]$. Then, for each $\boldsymbol{C} \in \mathbb{S}(\mathbf{D})$, there exists a homomorphism $g_C\colon \boldsymbol{B} \to \boldsymbol{C}$ such that $g_C[A] = g_C[B]$.*

Proof. Since $\boldsymbol{A}_A \leqslant \boldsymbol{B}$ and $h[A] = h[B]$, we have that $h\colon \boldsymbol{B} \to \mathbf{E}$ restricts to a homomorphism $h\colon \boldsymbol{B} \to h[\boldsymbol{A}_A]$. Thus, by Lemma 2.2.14, there exists a sequence of constants $\vec{a}$ of elements of A such that

$$h[\boldsymbol{A}_A] \vDash \mathsf{Diag}^+(\boldsymbol{B})(h(\vec{a})). \tag{6.1}$$

As $h\colon \boldsymbol{B} \to h[\boldsymbol{A}_A]$ is an $\mathcal{L}_A$-homomorphism, it preserves the constants of A. So, the above display (6.1) amounts to the fact that $h[\boldsymbol{A}_A]$ validates the set of positive quantifier-free $\mathcal{L}_A$-sentences $\mathsf{Diag}^+(\boldsymbol{B})(\vec{a})$. Since $h[\boldsymbol{A}_A] = h[\boldsymbol{B}] \leqslant \mathbf{E}$ and positive quantifier-free sentences are preserved under extensions by Corollary 2.1.28, we conclude that also $\mathbf{E} \vDash \mathsf{Diag}^+(\boldsymbol{B})(\vec{a})$. Moreover, recall that $\mathbf{E} \in \mathbb{P}_u(\boldsymbol{D})$. Thus, Łoś's Theorem 2.2.3 implies that $\mathbf{D} \vDash \mathsf{Diag}^+(\boldsymbol{B})(\vec{a})$. Now, consider $\boldsymbol{C} \leqslant \boldsymbol{D}$. As positive quantifier-free sentences are also preserved under subalgebras by Lemma 2.1.30, it follows that $\boldsymbol{C} \vDash \mathsf{Diag}^+(\boldsymbol{B})(\vec{a})$. Using Lemma 2.2.15, we obtain a homomorphism $g_C\colon \boldsymbol{B} \to \boldsymbol{C}$ such that $g_C[B] \subseteq \{a^{\boldsymbol{C}} : a \in A\} = g_C[A]$, where the last equality is a consequence of the fact that g_C preserves the constants of A. On the other hand, since $A \subseteq B$, we also have $g_C[A] \subseteq g_C[B]$, and hence $g_C[A] = g_C[B]$. So, g_C is the desired homomorphism. ⊠

Corollary 6.2. *Let $\boldsymbol{A}$ be an $\mathcal{L}$-algebra and consider three $\mathcal{L}_A$-algebras $\mathbf{B}, \mathbf{D}$, and $\mathbf{E}$ such that $\boldsymbol{A}_A \leqslant \boldsymbol{B}$ and $\mathbf{E} \in \mathbb{P}_u(\boldsymbol{D})$. Moreover, let $h\colon \boldsymbol{B} \to \mathbf{E}$ be a homomorphism such that $h[A] = h[B]$.*

Then, for every positive existential $\mathcal{L}_A$-formula φ, the following holds: If $\boldsymbol{B} \vDash \varphi(\vec{b})$ for some tuple $\vec{b}$ of elements of B, then there exists a tuple $\vec{a}$ of constants of A such that $\mathbb{S}(\boldsymbol{D}) \vDash \varphi(\vec{a})$. In particular, for every positive existential $\mathcal{L}_A$-sentence ψ, we get that $\boldsymbol{B} \vDash \psi$ implies $\mathbb{S}(\boldsymbol{D}) \vDash \psi$.

Proof. Let φ be a positive existential formula such that $\boldsymbol{B} \vDash \varphi(\vec{b})$, and consider an algebra $\boldsymbol{C} \in \mathbb{S}(\boldsymbol{D})$. We have to show that there exists a tuple $\vec{a}$ of constants of A such that $\boldsymbol{C} \vDash \varphi(\vec{a})$. From Lemma 6.1 we obtain a homomorphism $g_C \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $g_C[A] = g_C[B]$. As positive existential formulas are preserved under homomorphisms by Lemma 2.1.27, the assumption $\boldsymbol{B} \vDash \varphi(\vec{b})$ implies that $\boldsymbol{C} \vDash \varphi(g_C(\vec{b}))$. Now, as $g_C[A] = g_C[B]$ there exists a tuple $\vec{a}$ of elements of A, such that $g_C(\vec{a}) = g_C(\vec{b})$. Finally, since g_C is an $\mathcal{L}_A$-homomorphism, we have that $g_C(\vec{a}) = \vec{a}^{\boldsymbol{C}}$, and thus $\boldsymbol{C} \vDash \varphi(\vec{a})$, as claimed. ⊠

Since both Theorems 6.3 and 6.4 are similar in what concerns their statement and proof strategy, we will consider them simultaneously up to the point where their proofs diverge. As they are mosaics of various different observations and results, it is easy to get lost in the details of the numerous claims and partial results, which constitute the actual proof of the theorems. So, this approach is also an attempt to clarify the outline and the core ideas of the proofs.

Theorem 6.3. *[9, Thm. 18] Let K be a quasivariety with an $(m+1)$-ary near-unanimity term. Then, the following are equivalent:*

1. *K has the ES property;*
2. *Every subalgebra $\boldsymbol{A} \leqslant \boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$, with $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m \in \mathbb{P}_u(\mathsf{K}_{\text{RFSI}})$, lacks subalgebras that are proper and epic in K.*

Theorem 6.4. *[9, Thm. 22] Let K be an arithmetical variety whose class of FSI members is universal. Then, the following are equivalent:*

1. *K has the ES property;*
2. *Every member of K_{FSI} lacks subalgebras that are proper and epic in K.*

Recall from Lemma 2.4.5 that K has the ES property iff its members lack proper epic subalgebras. Then, in particular, the members of every subclass of K lack proper subalgebras that are epic in K. So, in both theorems the implication $(1) \Rightarrow (2)$ is straightforward. To establish the implication $(2) \Rightarrow (1)$, in both cases we will proceed as follows:

Proof of $(2) \Rightarrow (1)$ in Theorems 6.3 and 6.4. Assume condition (2). By Lemma 2.4.5, it suffices to show that the members of K lack proper epic subalgebras. So, consider a subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B}$ that is epic in K and an element $b \in B$. We will prove that $b \in A$. To this end, we use Theorem 5.3 to obtain a p.p. formula $\varphi(\vec{x}, y)$ that defines a partial function in K and a tuple $\vec{a}$ of elements of A such that $\boldsymbol{B} \vDash \varphi(\vec{a}, b)$. Now, we define the set of p.p. $\mathcal{L}_A$-sentences

$$\Sigma := \{\psi \in Fm_{\mathcal{L}_A} : \psi \text{ is a p.p. sentence and } \boldsymbol{B}_A \vDash \psi\}$$

and the class of $\mathcal{L}_A$-algebras

$$\mathsf{K}^* := \{\boldsymbol{C} \in \mathsf{Mod}(\Sigma) : \boldsymbol{C}{\restriction}_{\mathcal{L}} \in \mathbb{IP}_u(\mathsf{K}_{\text{RFSI}})\}.$$

We will use the following observation:

Claim 6.5. *The $\mathcal{L}_A$-formula $\psi(y) := \varphi(\vec{a}, y)$ defines a 0-ary function in K^*.*

Proof. To see that it defines a partial function in K^*, consider $\boldsymbol{C} \in \mathsf{K}^*$ and $c, d \in C$ such that $\boldsymbol{C} \vDash \psi(c) \wedge \psi(d)$. Then, $\boldsymbol{C}{\upharpoonright}_{\mathcal{L}} \vDash \varphi(\vec{a}^{\boldsymbol{C}}, c) \wedge \varphi(\vec{a}^{\boldsymbol{C}}, d)$. Now, notice that $\boldsymbol{C}{\upharpoonright}_{\mathcal{L}} \in \mathsf{K}$ by the definition of K^*. Recalling that $\varphi(\vec{x}, y)$ defines a partial function in K, we conclude that $c = d$, which verifies that $\psi(y)$ defines a partial function in K^*. Moreover, observe that $\exists y \psi(y)$ is a p.p. sentence that holds in $\boldsymbol{B}$, as $\boldsymbol{B} \vDash \varphi(\vec{a}, b)$ by assumption. It follows that $\exists y \psi(y) \in \Sigma$. Since $\mathsf{K}^* \subseteq \mathsf{Mod}(\Sigma)$, this implies that $\mathsf{K}^* \vDash \exists y \psi(y)$, which concludes the proof that $\psi(y)$ defines a total function in K^*. ⊠

The next step is to establish the following observation. This will be done in two different ways: one for Theorem 6.3 and one for Theorem 6.4.

Claim 6.6. *There exists a closed $\mathcal{L}_A$-term t such that $\mathsf{K}^* \vDash \psi(t)$.*

Once we have obtained this term, we will assume, with a view to contradiction, that $b \notin A$. Observe that $a := t^{\boldsymbol{B}_A} \in A$, as t is a closed term in $\mathcal{L}_A$. So, $b \notin A$ in particular implies $b \neq a$. Then, $\langle b, a \rangle \notin \mathrm{id}_{\boldsymbol{B}} = \bigcap \mathsf{Irr}^{\infty}_{\mathsf{K}}(\boldsymbol{B})$, where the last equality is due to Theorem 2.3.19. Hence, there exists $\theta \in \mathsf{Irr}^{\infty}_{\mathsf{K}}(\boldsymbol{B})$ such that $a/\theta \neq b/\theta$. Now, notice that $\boldsymbol{B}/\theta \in \mathsf{K}_{\text{RSI}} \subseteq \mathsf{K}_{\text{RFSI}}$ by Theorem 2.3.24. Moreover, recall that p.p. formulas are preserved under homomorphisms by Lemma 2.1.27. So, on the one hand, from $\boldsymbol{B}_A \vDash \Sigma$, it follows that $\boldsymbol{B}_A/\theta \vDash \Sigma$, and thus $\boldsymbol{B}_A/\theta \in \mathsf{K}^*$. Thus, Claim 6.6 implies that $\boldsymbol{B}_A/\theta \vDash \psi(t^{\boldsymbol{B}_A/\theta})$. On the other hand, recall that $\boldsymbol{B}_A \vDash \varphi(\vec{a}, b)$, and hence $\boldsymbol{B}_A \vDash \psi(b)$, by the definition of ψ. Therefore, as $\psi(y)$ is a p.p. $\mathcal{L}_A$-formula and the canonical surjection $\pi \colon \boldsymbol{B}_A \to \boldsymbol{B}_A/\theta$ is an $\mathcal{L}_A$-homomorphism, using Lemma 2.1.27, it follows that $\boldsymbol{B}_A/\theta \vDash \psi(b/\theta)$. Since $\boldsymbol{B}_A \in \mathsf{K}^*$ and $\psi(y)$ defines a function in K^* by Claim 6.5, from $\boldsymbol{B}_A/\theta \vDash \psi(t^{\boldsymbol{B}_A/\theta})$ and $\boldsymbol{B}_A/\theta \vDash \psi(b/\theta)$ we conclude that: $t^{\boldsymbol{B}_A/\theta} = b/\theta$. As by definition $a = t^{\boldsymbol{B}_A}$, we obtain $a/\theta = t^{\boldsymbol{B}_A/\theta} = b/\theta$. But this is a contradiction with the assumption that $a/\theta \neq b/\theta$. So, we must have that $b \in A$, as claimed. ⊠

The key to completing the proofs of the two Theorems 6.3 and 6.4 now lies in verifying Claim 6.6.

This is also the point where the two proofs diverge. For Theorem 6.3, we will apply the Infinitary Baker-Pixley Theorem 3.11, whereas in order to prove Theorem 6.4, we will make use of the theory of global subdirect products developed in Chapter 4.

We will first deal with the situation of Theorem 6.3.

Proof of Claim 6.6 in the case of 6.3. Recall that

$$\mathsf{K}^* := \{\boldsymbol{C} \in \mathsf{Mod}(\Sigma) : \boldsymbol{C}{\upharpoonright}_{\mathcal{L}} \in \mathbb{IP}_u(\mathsf{K}_{\text{RFSI}})\}.$$

To prove Claim 6.6, we want to apply the Infinitary Baker-Pixley Theorem 3.11 to the class K^* and the 0-ary function defined by $\psi(y)$ (see Claim 6.5). So, we need to check that K^* satisfies the assumptions of Theorem 3.11.

Claim 6.7. *K^* is closed under ultraproducts and has an $(m+1)$-ary near-unanimity term.*

Proof. Let $\boldsymbol{C} \in \mathbb{P}_u(\mathsf{K}^*)$. Then,

$$\boldsymbol{C}{\upharpoonright}_{\mathcal{L}} \in \mathbb{P}_u(\mathsf{K}^*{\upharpoonright}_{\mathcal{L}}) \subseteq \mathbb{P}_u\mathbb{IP}_u(\mathsf{K}_{\text{RFSI}}) \subseteq \mathbb{IP}_u(\mathsf{K}_{\text{RFSI}}),$$

where the first inclusion follows from the definition of K^* and the second one is a consequence of Lemma 2.1.24. Furthermore, Łoś's Theorem 2.2.3 implies that $\boldsymbol{C} \vDash \Sigma$, and thus $\boldsymbol{C} \in \mathsf{K}^*$.

Also notice that the $(m+1)$-ary near-unanimity term of K serves as an $(m+1)$-ary near-unanimity term for K^* as well, since $\mathsf{K}^*{\restriction}_{\mathcal{L}} \subseteq \mathsf{K}$. ⊠

This puts us in a position to apply the Infinitary Baker-Pixley Theorem 3.11. To this end, consider $\boldsymbol{C}_1, \dots, \boldsymbol{C}_m \in \mathsf{K}^*$. We will first verify the following observation.

Claim 6.8. *There exists a homomorphism* $h\colon \boldsymbol{B}_A \to \boldsymbol{D}$, *with* $\boldsymbol{D} \in \mathbb{P}_u(\boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m)$, *such that* $h[A] = h[B]$.

Proof. Since $\boldsymbol{C}_i \vDash \Sigma$ for each $i \leqslant m$ and p.p. formulas are preserved under direct products by Lemma 2.1.29, we obtain $\boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m \vDash \Sigma$. Thus, Lemma 5.1 yields a homomorphism $h\colon \boldsymbol{B}_A \to \boldsymbol{D}$, with $\boldsymbol{D} \in \mathbb{P}_u(\boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m)$. It remains to show that $h[A] = h[B]$. Observe that $h[\boldsymbol{A}_A], h[\boldsymbol{B}_A] \in \mathbb{SP}_u\mathbb{P}_m(\mathsf{K}^*) \subseteq \mathbb{SIP}_m\mathbb{P}_u(\mathsf{K}^*)$, where the inclusion follows from Lemma 2.1.25. So, $h[\boldsymbol{A}], h[\boldsymbol{B}] \in \mathbb{SIP}_m\mathbb{P}_u(\mathsf{K}_{\mathrm{RFSI}}) \subseteq \mathbb{ISP}_m\mathbb{P}_u(\mathsf{K}_{\mathrm{RFSI}})$, by the definition of K^* and Lemma 2.1.24. Furthermore, since $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K, so is $h[\boldsymbol{A}] \leqslant h[\boldsymbol{B}]$ by Lemma 2.4.6. Hence, we can apply our assumption (2) that the members of $\mathbb{SP}_m\mathbb{P}_u(\mathsf{K}_{\mathrm{RFSI}})$, and thus, by Lemma 2.4.7 also the members of $\mathbb{ISP}_m\mathbb{P}_u(\mathsf{K}_{\mathrm{RFSI}})$, lack subalgebras that are proper and epic in K, to conclude that $h[A] = h[B]$. ⊠

Now, let S be a subuniverse of $\boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m$ in the language $\mathcal{L}_A$. In order to obtain the desired $\mathcal{L}_A$-term t from the Infinitary Baker-Pixley Theorem, we need to prove the following claim.

Claim 6.9. *S is closed under* $[\psi]^{\boldsymbol{C}_1} \times \cdots \times [\psi]^{\boldsymbol{C}_m}$.

Proof. Recall that $\psi(y)$ defines a 0-ary function in K^* by Claim 6.5. So, we need to show that $\langle d_1, \dots, d_m \rangle \in S$, where by the definition of the product function (see Definition 3.3) $d_i \in C_i$ is such that $\boldsymbol{C}_i \vDash \psi(d_i)$ for each $i \leqslant m$. As p.p. formulas are preserved under direct products by Lemma 2.1.29, from $\boldsymbol{C}_i \vDash \psi(d_i)$ for each $i \leqslant m$, we get that $\boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m \vDash \psi(\vec{d})$, where $\vec{d} = \langle d_1, \dots, d_m \rangle$. On the other hand, the homomorphism $h\colon \boldsymbol{B}_A \to D$ that exists by Claim 6.8 allows us to apply Corollary 6.2 to obtain $a \in A$ such that $\boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m \vDash \psi(a)$. Now, recall that p.p. formulas are preserved under homomorphisms, in particular under the canonical projections $p_i\colon \boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m \to \boldsymbol{C}_i$ for $i \leqslant m$. This implies that $\boldsymbol{C}_i \vDash \psi(a)$ for every $i \leqslant m$. Since $\boldsymbol{C}_i \vDash \psi(d_i)$ and $\psi(y)$ defines a function in K^* by Claim 6.5, it follows that $d_i = a^{\boldsymbol{C}_i}$ for every $i \leqslant m$. But then,

$$[\psi]^{\boldsymbol{C}_1} \times \cdots \times [\psi]^{\boldsymbol{C}_m} = \langle d_1, \dots, d_m \rangle = \langle a^{\boldsymbol{C}_1}, \dots, a^{\boldsymbol{C}_m} \rangle = a^{\boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m}.$$

As S is an $\mathcal{L}_A$-subuniverse of $\boldsymbol{C}_1 \times \cdots \times \boldsymbol{C}_m$, this yields that $[\psi]^{\boldsymbol{C}_1} \times \cdots \times [\psi]^{\boldsymbol{C}_m} \in S$, which proves the claim. ⊠

So, the Infinitary Baker-Pixley Theorem 3.11 provides the desired closed $\mathcal{L}_A$-term t such that $\mathsf{K}^* \vDash \psi(t)$. This verifies Claim 6.6, and thus completes the proof of Theorem 6.3. ⊠

Next, we turn to the proof of Theorem 6.4, which is a bit more involved.

Proof of Claim 6.6 in the case of Theorem 6.4. Notice that, as K is a variety and K_{FSI} is closed under $\mathbb{I}$ and $\mathbb{P}_u$ by assumption, we get that

$$\mathsf{K}^* = \{\boldsymbol{C} \in \mathsf{Mod}(\Sigma) : \boldsymbol{C}{\restriction}_{\mathcal{L}} \in \mathsf{K}_{\text{FSI}}\}.$$

As a first step, we will establish some useful observations about K^*.

Claim 6.10. *$\mathbb{V}(\mathsf{K}^*)$ is an arithmetical variety.*

Proof. Using Theorem 2.3.36, this is an immediate consequence of the assumption that K is arithmetical, because the Pixley term of K also serves as a Pixley term for $\mathbb{V}(\mathsf{K}^*)$. ⊠

Claim 6.11. *K^* is a universal class.*

Proof. That K^* is closed under isomorphic copies is straightforward. Next, observe that K^* is closed under ultraproducts, since K_{FSI} is by assumption and the closure of $\mathsf{Mod}(\Sigma)$ under ultraproducts follows from Łoś's Theorem 2.2.3. Now, consider a subalgebra $\boldsymbol{C} \leqslant \boldsymbol{D}$ for some $\boldsymbol{D} \in \mathsf{K}^*$. Clearly, $\boldsymbol{C}{\restriction}_{\mathcal{L}} \in \mathbb{S}(\boldsymbol{D}{\restriction}_{\mathcal{L}}) \subseteq \mathsf{K}_{\text{FSI}}$, as by assumption K_{FSI} is a universal class and thus closed under subalgebras. It remains to show that $\boldsymbol{C} \vDash \Sigma$. This part follows the same strategy as the proof of Claims 6.8 and 6.9 in Theorem 6.3.

By Lemma 5.1 we obtain $\boldsymbol{E} \in \mathbb{P}_u(\boldsymbol{D})$ and a homomorphism $h\colon \boldsymbol{B}_A \to \boldsymbol{E}$. Now, $h[\boldsymbol{A}]$ and $h[\boldsymbol{B}]$ are subalgebras of $\boldsymbol{E}{\restriction}_{\mathcal{L}}$, which, in turn, is an ultrapower of $\boldsymbol{D}{\restriction}_{\mathcal{L}} \in \mathsf{K}$. Since K_{FSI} is closed under both subalgebras and ultrapowers by assumption, we conclude that $h[\boldsymbol{A}], h[\boldsymbol{B}] \in \mathsf{K}_{\text{FSI}}$. The fact that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K, by Lemma 2.4.6 implies that also $h[\boldsymbol{A}] \leqslant h[\boldsymbol{B}]$ is epic in K. So, we can apply our assumption (2) to conclude that $h[A] = h[B]$. Now, consider a p.p. $\mathcal{L}_A$-sentence $\alpha \in \Sigma$. By the definition of Σ it follows that $\boldsymbol{B}_A \vDash \alpha$. Then Corollary 6.2 yields $\boldsymbol{C} \vDash \alpha$, which concludes the proof of Claim 6.11. ⊠

Claim 6.12. $\mathbb{V}(\mathsf{K}^*)_{\text{FSI}} = \mathsf{K}^*$.

Proof. First, consider $\boldsymbol{C} \in \mathbb{V}(\mathsf{K}^*)_{\text{FSI}}$. Then, clearly, $\boldsymbol{C}{\restriction}_{\mathcal{L}} \in \mathbb{V}(\mathsf{K}^*{\restriction}_{\mathcal{L}}) \subseteq \mathsf{K}$ by the definition of K^* and the fact that K is a variety. Since the additional constants in the language $\mathcal{L}_A$ do not change the congruences of $\boldsymbol{C}$, the fact that $\boldsymbol{C}$ is an FSI member of $\mathbb{V}(\mathsf{K}^*)$ implies $\boldsymbol{C}{\restriction}_{\mathcal{L}} \in \mathsf{K}_{\text{FSI}}$. It remains to show that $\boldsymbol{C} \vDash \Sigma$. To this end, we apply Jónsson's Lemma 2.3.39 to get $\mathbb{V}(\mathsf{K}^*)_{\text{FSI}} \subseteq \mathbb{HSP}_u(\mathsf{K}^*)$. As K^* is universal by the previous Claim 6.11 and thus closed under subalgebras and ultraproducts, it suffices to show that $\mathbb{H}(\mathsf{K}^*) \vDash \Sigma$. But this is immediate since Σ consists of p.p. sentences, which are preserved under homomorphic images by Lemma 2.1.27. Therefore, $\boldsymbol{C} \vDash \Sigma$, and hence, $\boldsymbol{C} \in \mathsf{K}^*$.

For the converse inclusion, let $\boldsymbol{C} \in \mathsf{K}^*$. Clearly, $\mathsf{K}^* \subseteq \mathbb{V}(\mathsf{K}^*)$. Furthermore, we know that $\boldsymbol{C}{\restriction}_{\mathcal{L}} \in \mathsf{K}_{\text{FSI}}$ by the definition of K^*. As the additional constants in the language $\mathcal{L}_A$ do not change the congruences of $\boldsymbol{C}$, we conclude that $\boldsymbol{C} \in \mathbb{V}(\mathsf{K}^*)_{\text{FSI}}$. ⊠

So, we have verified that $\mathbb{V}(\mathsf{K}^*)$ is an arithmetical variety (Claim 6.10), K^* is a universal class (Claim 6.11), and $\mathbb{V}(\mathsf{K}^*)_{\text{SI}} \subseteq \mathbb{V}(\mathsf{K}^*)_{\text{FSI}} = \mathsf{K}^*$ (Claim 6.12). By Theorem 4.7, we thus obtain that

$$\mathbb{V}(\mathsf{K}^*) \subseteq \mathbb{IP}_G(\mathsf{K}^*). \tag{6.2}$$

We now aim to apply Theorem 4.15 to obtain the desired closed $\mathcal{L}$-term t and thus verify Claim 6.6. In order to accomplish this, we will need to transform ψ into a conjunction of equations up to equivalence in K^*. This will be done in two steps. Our first goal is to obtain a positive quantifier-free formula equivalent to $\psi(y)$.

Claim 6.13. *There exists a positive quantifier-free formula $\beta(y)$ such that*

$$\mathsf{K}^* \vDash \psi(y) \leftrightarrow \beta(y).$$

Proof. Notice that K^* is closed under ultraproducts by Claim 6.11 and $\psi(y)$ defines a function in K^* by Claim 6.5. We want to apply Lemma 5.5 to obtain the claimed quantifier-free formula. So, we need to show that for every $\boldsymbol{C}, \boldsymbol{D} \in \mathsf{K}^*$ and homomorphism $h\colon \boldsymbol{C}' \to \boldsymbol{D}'$ between subalgebras $\boldsymbol{C}' \leqslant \boldsymbol{C}$ and $\boldsymbol{D}' \leqslant \boldsymbol{D}$ we have $\boldsymbol{D} \vDash \psi(h(c))$ for each $c \in C'$ such that $\boldsymbol{C} \vDash \psi(c)$. Since K^* is universal and $\boldsymbol{C} \in \mathsf{K}^*$, we conclude that $\boldsymbol{C}' \in \mathsf{K}^*$, and thus $\boldsymbol{C}' \vDash \exists y \psi(y)$, because ψ defines a function in K^* by Claim 6.5. So, let $c' \in C'$ be such that $\boldsymbol{C}' \vDash \psi(c')$. Now, as $\psi(y)$ is a p.p. formula and thus preserved under extensions by Corollary 2.1.28, this implies that $\boldsymbol{C} \vDash \psi(c')$. On the other hand, we also have $\boldsymbol{C} \vDash \psi(c)$. From the fact that $\psi(y)$ defines a function in K^*, we conclude that $c = c'$, and thus $\boldsymbol{C}' \vDash \psi(c)$. As p.p. formulas are preserved under homomorphisms and extensions, from $\boldsymbol{C}' \vDash \psi(c)$, we obtain $\boldsymbol{D}' \vDash \psi(h(c))$, and finally $\boldsymbol{D} \vDash \psi(h(c))$, as desired. Hence, Lemma 5.5 yields a positive quantifier-free formula $\beta(y)$ such that $\mathsf{K}^* \vDash \psi(y) \leftrightarrow \beta(y)$. ⊠

As a second step, from the positive quantifier-free formula obtained in Claim 6.13, we will now pass to a conjunction of equations.

Claim 6.14. *There exists a conjunction of equations $\alpha(y)$ such that $\mathsf{K}^* \vDash \alpha(y) \leftrightarrow \beta(y)$.*

Proof. Recall from Claim 6.12 that $\mathbb{V}(\mathsf{K}^*)_{\text{FSI}} = \mathsf{K}^*$, and from Claim 6.11 that K^* is universal. Moreover, $\mathbb{V}(\mathsf{K}^*)$ is arithmetical by Claim 6.10, and thus, in particular, congruence distributive. So, Corollary 5.9 yields the desired conjunction of equations $\alpha(y)$. ⊠

Taking together Claim 6.13 and Claim 6.14, we thus obtain a conjunction of equations $\alpha(y)$ such that

$$\mathsf{K}^* \vDash \psi(y) \leftrightarrow \alpha(y). \tag{6.3}$$

Now, since $\psi(y)$ defines a function in K^* by Claim 6.5, we conclude that $\mathsf{K}^* \vDash \exists! y \psi(y)$, and thus, by condition (6.3), also $\mathsf{K}^* \vDash \exists! y \alpha(y)$. As displayed in condition (6.2), we have verified that $\mathbb{V}(\mathsf{K}^*) \subseteq \mathbb{IP}_G(\mathsf{K}^*)$. So, we can apply Theorem 4.15, which provides a closed $\mathcal{L}_A$-term t such that $\mathbb{V}(\mathsf{K}^*) \vDash \alpha(t)$. In particular, this implies that $\mathsf{K}^* \vDash \alpha(t)$, which is equivalent to $\mathsf{K}^* \vDash \psi(t)$ by condition (6.3). This proves Claim 6.6, and thus concludes the proof of Theorem 6.4.

⊠

CHAPTER 7

The weak ES property

This chapter is based on [12] and provides an improvement of Theorems 6.3 in the setting of the weak ES property. To carry out the proof, we use an alternative characterization of the weak ES property in terms of almost total subalgebras (see, e.g., [12, Prop. 3.2] and [29, Thm. 5.4]).

Definition 7.1. A subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B}$ is called *almost total* when there exists $b \in B$ such that $B = \mathsf{Sg}^{\boldsymbol{B}}(A \cup \{b\})$.

Lemma 7.2. *Let* K *be a quasivariety. Then, the following are equivalent:*

1. K *has the weak ES property;*
2. *The finitely generated members of* K *lack proper subalgebras that are epic in* K*;*
3. *The finitely generated members of* K *lack proper subalgebras that are almost total and epic in* K*.*

Proof. For the implication from (1) to (2), we will argue by contradiction. Assume that K has the weak ES property, and let $\boldsymbol{A} \leqslant \boldsymbol{B}$ be a proper subalgebra that is epic in K, where $\boldsymbol{B} \in \mathsf{K}$ is finitely generated by $\{b_i : i \leqslant n\}$. As $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K, by Theorem 5.3, for every $i \leqslant n$, there exists a p.p. formula $\varphi_i(\vec{x}, y)$ that defines a function in K and a tuple $\vec{a}_i$ of elements of A such that $\boldsymbol{B} \vDash \varphi_i(\vec{a}_i, b_i)$. Now, define $\boldsymbol{A}' := \mathsf{Sg}^{\boldsymbol{A}}(\{\vec{a}_i : i \leqslant n\})$. Notice that $\boldsymbol{A}'$ is finitely generated. We aim to show that $\boldsymbol{A}' \leqslant \boldsymbol{B}$ is epic in K. Consider $\boldsymbol{C} \in \mathsf{K}$ and a pair of homomorphisms $f, g \colon \boldsymbol{B} \to \boldsymbol{C}$ such that $f{\upharpoonright}_{A'} = g{\upharpoonright}_{A'}$. It suffices to show that $f(b_i) = g(b_i)$ for every $i \leqslant n$, because f and g are homomorphisms and $\{b_i : i \leqslant n\}$ is a set of generators for B. So, fix $i \leqslant n$. As p.p. formulas are preserved under homomorphisms by Lemma 2.1.30, from $\boldsymbol{B} \vDash \varphi(\vec{a}_i, b_i)$, it follows that $\boldsymbol{C} \vDash \varphi(f(\vec{a}_i), f(b_i)) \wedge \varphi(g(\vec{a}_i), g(b_i))$. Recall that $f{\upharpoonright}_{A'} = g{\upharpoonright}_{A'}$. Since $\vec{a}_i$ is a tuple of elements in A', this implies $f(\vec{a}_i) = g(\vec{a}_i)$. Then, as φ defines a function on K, we conclude that $f(b_i) = g(b_i)$. Thus, we have proven that $\boldsymbol{A}', \boldsymbol{B}$ are finitely generated members in K such that the inclusion $\boldsymbol{A}' \hookrightarrow \boldsymbol{B}$ is a non-surjective epimorphism, contradicting our assumption that K has the weak ES property. The implication from (2) to (3) is straightforward.

Finally, for the implication (3) $\Rightarrow$ (1), assume that the finitely generated members of K lack proper subalgebras that are almost total and epic in K, and let $f \colon \boldsymbol{A} \to \boldsymbol{B}$

be an epimorphism between finitely generated members $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$. We have to show that f is surjective. Suppose this was not the case. Then $f[\boldsymbol{A}] \leqslant \boldsymbol{B}$ is a proper epic subalgebra. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are finitely generated, there exists a finite set $X \subseteq B$ and an element $b \in B$ such that

$$B = \mathsf{Sg}^{\boldsymbol{B}}(f[A] \cup X \cup \{b\}) \text{ and } b \notin \mathsf{Sg}^{\boldsymbol{B}}(f[A] \cup X). \tag{7.1}$$

Now, let $\boldsymbol{C} \leqslant \boldsymbol{B}$ be the algebra with universe $\mathsf{Sg}^{\boldsymbol{B}}(f[A] \cup X)$. By the above display (7.1), we conclude that $\boldsymbol{C} \leqslant \boldsymbol{B}$ is proper and almost total. Finally, as $\boldsymbol{A} \leqslant \boldsymbol{C} \leqslant \boldsymbol{B}$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K by assumption, it follows that $\boldsymbol{C} \leqslant \boldsymbol{B}$ is also epic in K. This establishes the claimed implication and concludes the proof. ⊠

In contrast to Campercholi's approach in [9], the techniques used in our version of [9, Thm. 18] for the weak ES property are purely algebraic and do not rely on the Infinitary Baker-Pixley Theorem (Chapter 3) or require any model-theoretic background (Chapter 5). Besides the standard universal algebraic concepts introduced in the preliminaries (Section 2.3), we will only make use of the following new notion, which turns out to be our crucial tool.

Definition 7.3. [12, Def. 3.4] Let $\boldsymbol{B}$ be a member of a quasivariety K. A subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B}$ is called *full* in K when it is proper, almost total, and for every $\mathsf{id}_B \neq \theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ and every $b \in B$, there exists $a \in A$ such that $\langle a, b\rangle \in \theta$. When $\boldsymbol{A} \leqslant \boldsymbol{B}$ is both epic and full in K, we say that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is *fully epic* in K.

The next lemma is a useful observation that facilitates the verification of fullness.

Lemma 7.4. *Let K be a quasivariety and $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is proper and almost total with $B = \mathsf{Sg}^{\boldsymbol{B}}(A \cup \{b\})$. Then, $\boldsymbol{A} \leqslant \boldsymbol{B}$ is full in K iff for every $\mathsf{id}_B \neq \theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ there exists $a \in A$ such that $\langle a, b\rangle \in \theta$.*

Proof. Consider $\mathsf{id}_B \neq \theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$. If $\boldsymbol{A} \leqslant \boldsymbol{B}$ is full in K, then for every $c \in B$ there exists $a \in A$ such that $\langle a, c\rangle \in \theta$. So, in particular, this also holds for b.

Conversely, let $c \in B$. As $B = \mathsf{Sg}^{\boldsymbol{B}}(A \cup \{b\})$, from Lemma 2.1.16, it follows that there exist a term t and $a_1, \dots, a_n \in A$ such that $c = t^{\boldsymbol{B}}(a_1, \dots, a_n, b)$. Now, by assumption, there exists $a \in A$ such that $\langle a, b\rangle \in \theta$. Thus, we conclude that $\langle t^{\boldsymbol{B}}(a_1, \dots, a_n, b), t^{\boldsymbol{B}}(a_1, \dots, a_n, a)\rangle \in \theta$. Therefore, $a' := t^{\boldsymbol{B}}(a_1, \dots, a_n, a) \in A$ satisfies $\langle c, a'\rangle \in \theta$. ⊠

The usefulness of the notion of fullness is justified by the observation that we can find a fully epic counterexample in any quasivariety lacking the weak ES property. This will be an immediate consequence of the next proposition.

Proposition 7.5. *[12, Prop. 3.7] Let K be a quasivariety, $\boldsymbol{B} \in \mathsf{K}$, and $\boldsymbol{A} \leqslant \boldsymbol{B}$ proper and almost total. Then, there exists $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ such that $\boldsymbol{A}/\theta \leqslant \boldsymbol{B}/\theta$ is full in K.*

Proof. As $\boldsymbol{A} \leqslant \boldsymbol{B}$ is proper and almost total, there exists an element $b \in B \smallsetminus A$ such that $B = \mathsf{Sg}^{\boldsymbol{B}}(A \cup \{b\})$. Consider the poset

$$X := \{\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B}) : \text{there exists no } a \in A \text{ such that } \langle a, b\rangle \in \theta\}$$

ordered under the inclusion relation. We will apply Zorn's Lemma to obtain a maximal element of X. Clearly, X contains id_B. Furthermore, the definition of X and Proposition

2.3.16 guarantee that X is closed under unions of non-empty chains. Therefore, there exists a maximal element θ of X.

From $\theta \in X$ it follows that b/θ does not belong to A/θ. Therefore, $\boldsymbol{A}/\theta \leqslant \boldsymbol{B}/\theta$ is proper. Moreover, $B = \mathsf{Sg}^{\boldsymbol{B}}(A \cup \{b\})$ implies that $B/\theta = \mathsf{Sg}^{\boldsymbol{B}/\theta}(A/\theta \cup \{b/\theta\})$. Therefore, $\boldsymbol{A}/\theta \leqslant \boldsymbol{B}/\theta$ is almost total. It only remains to prove that $\boldsymbol{A}/\theta \leqslant \boldsymbol{B}/\theta$ is full in K. To this end, consider $\phi \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B}/\theta) \smallsetminus \{\mathsf{id}_{B/\theta}\}$. Using Lemma 7.4 it suffices to show that there exists $a/\theta \in A/\theta$ such that $\langle a/\theta, b/\theta \rangle \in \phi$. By the Correspondence Theorem 2.3.17, there exists a K-congruence $\eta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ such that

$$\theta \subsetneq \eta \text{ and } \phi = \{\langle a/\theta, c/\theta \rangle : \langle a, c \rangle \in \eta\}.$$

Since θ is a maximal element of X, from $\theta \subsetneq \eta$, it follows that $\eta \notin X$. Therefore, there exists $a \in A$ such that $\langle a, b \rangle \in \eta$. In view of the above display, this yields $\langle a/\theta, b/\theta \rangle \in \phi$, as desired. ⊠

Corollary 7.6. *[12, Cor. 3.8] A quasivariety K has the weak ES property iff its finitely generated members lack subalgebras that are fully epic in K.*

Proof. The implication from left to right follows from Lemma 7.2. To prove the other implication, suppose by contraposition that K lacks the weak ES property. By Lemma 7.2 there exist a finitely generated algebra $\boldsymbol{B} \in \mathsf{K}$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$ proper, almost total, and epic in K. Therefore, we can apply Proposition 7.5 obtaining $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ such that $\boldsymbol{A}/\theta \leqslant \boldsymbol{B}/\theta$ is full in K. Furthermore, from the assumption that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is epic in K by Lemma 2.4.6, it follows that so is $\boldsymbol{A}/\theta \leqslant \boldsymbol{B}/\theta$. ⊠

We are now ready to present a proof of the improved version of Theorem 6.3, which simplifies the verification of the weak ES property or its failure in the presence of a near-unanimity term.

Theorem 7.7. *[12, Thm. 4.3] Let K be a quasivariety with an $(m+1)$-ary near-unanimity term. Then, the following are equivalent:*

1. *K has the weak ES property;*
2. *Every finitely generated subdirect product $\boldsymbol{A} \leqslant \boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$, where the factors $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$ are elements of K_{RFSI}, lacks subalgebras that are fully epic in K.*

Notice that, in comparison to Theorem 6.3, a failure of the weak ES property does not only occur in an arbitrary subalgebra $\boldsymbol{A} \leqslant \boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_m$, but $\boldsymbol{A}$ can be assumed to be a subdirect product. Moreover, the factors $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$ are themselves RFSI members of K, whereas in Theorem 6.3 we need to consider members of the class $\mathbb{P}_u(\mathsf{K}_{\text{RFSI}})$.

The key of the proof is to verify that in the presence of an $(m+1)$-ary near-unanimity term, given a full subalgebra $\boldsymbol{A} \leqslant \boldsymbol{B}$, we can always find a subdirect embedding of $\boldsymbol{B}$ into $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_m$ for some $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_m \in \mathsf{K}_{\text{RFSI}}$. This will be the content of Proposition 7.10.

To carry out that proof, it is convenient to introduce the following concept:

Definition 7.8. [12, Def. 4.4] Let K be a quasivariety, $\boldsymbol{A} \in \mathsf{K}$, and $\theta \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. Given a positive integer m, we say that θ is *m-irreducible* in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ when for every family $\theta_1, \ldots, \theta_m \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$

$$\theta = \theta_1 \cap \cdots \cap \theta_m \text{ implies } \theta = \theta_1 \cap \cdots \cap \theta_{i-1} \cap \theta_{i+1} \cap \cdots \cap \theta_m \text{ for some } i \leqslant m.$$

When K is clear from the context, we will simply say that θ is *m-irreducible*.

Notice that the only 1-irreducible K-congruence of $\boldsymbol{A}$ is $A \times A$. Moreover, a K-congruence θ of $\boldsymbol{A}$ is 2-irreducible iff either $\theta \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$ or $\theta = A \times A$.

Establishing the following observation about m-irreducible congruences will be instrumental for the proof of Proposition 7.10.

Proposition 7.9. *[12, Prop. 4.5] Let* K *be a quasivariety,* $\boldsymbol{A} \in \mathsf{K}$, *and* θ *an* $(m+1)$*-irreducible* K*-congruence of* A. *Then there exist* $\phi_1, \dots, \phi_m \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$ *such that* $\theta = \phi_1 \cap \dots \cap \phi_m$.

Proof. Let k be the least positive integer such that θ is k-irreducible. Since we allow repetitions among the $\phi_1, \dots, \phi_m$ in the statement and $k \leqslant m+1$, it is sufficient to show that there exist $\phi_1, \dots, \phi_{k-1} \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta = \phi_1 \cap \dots \cap \phi_{k-1}$. If $k = 1$, then θ is 1-irreducible. Thus, $\theta = A \times A$, and θ can be written as the intersection of an empty family of members of $\mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$. So, we may assume that $k \geqslant 2$.

As θ is not $(k-1)$-irreducible, there exist $\theta_1, \dots, \theta_{k-1} \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that

$$\theta = \theta_1 \cap \dots \cap \theta_{k-1} \text{ and } \theta \neq \theta_1 \cap \dots \cap \theta_{i-1} \cap \theta_{i+1} \cap \dots \cap \theta_{k-1} \text{ for every } i \leqslant k-1. \tag{7.2}$$

Consider the poset

$$X := \{\langle \phi_1, \dots, \phi_{k-1} \rangle : \theta_i \subseteq \phi_i \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A}) \text{ for every } i \leqslant k-1 \text{ and } \theta = \phi_1 \cap \dots \cap \phi_{k-1}\}$$

ordered under the relation given by $\langle \phi_1, \dots, \phi_{k-1} \rangle \leqslant \langle \eta_1, \dots, \eta_{k-1} \rangle$ iff $\phi_i \subseteq \eta_i$ for every $i \leqslant k-1$. We will apply Zorn's Lemma to obtain a maximal element of X. Clearly, X contains $\langle \theta_1, \dots, \theta_{k-1} \rangle$. Consider a non-empty chain C in X. For each $i \leqslant k-1$ let C_i be the projection of C on the i^{th} coordinate. Observe that C_i is a non-empty chain in $\mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$ because C is a non-empty chain in X. We will prove that $\langle \bigcup C_1, \dots, \bigcup C_{k-1} \rangle$ is an upper bound of C in X. Proposition 2.3.16 implies that each $\bigcup C_i$ is a K-congruence of $\boldsymbol{A}$. Furthermore, as θ_i is contained in every member of C_i and C_i is non-empty, we have $\theta_i \subseteq \bigcup C_i$. Lastly, observe that

$$\left(\bigcup C_1\right) \cap \dots \cap \left(\bigcup C_{k-1}\right) = \bigcup \{\phi_1 \cap \dots \cap \phi_{k-1} : \phi_i \in C_i \text{ for } i \leqslant k-1\} = \theta, \tag{7.3}$$

where the first equality holds by the infinite distributive law and the second one can be established as follows. Let $\phi_1 \in C_1, \dots, \phi_{k-1} \in C_{k-1}$. Then, there exists a tuple $\langle \phi'_1, \dots, \phi'_{k-1} \rangle \in C$ such that $\phi_i \subseteq \phi'_i$ for every $i \leqslant k-1$ because C is a chain. As a consequence,

$$\theta_1 \cap \dots \cap \theta_{k-1} \subseteq \phi_1 \cap \dots \cap \phi_{k-1} \subseteq \phi'_1 \cap \dots \cap \phi'_{k-1} = \theta, \tag{7.4}$$

where the the first inclusion holds because $\phi_i \in C_i$ for every $i \leqslant k-1$, and the last equality holds because $\langle \phi'_1, \dots, \phi'_{k-1} \rangle \in C$. As $\theta = \theta_1 \cap \dots \cap \theta_{k-1}$, condition (7.4) implies that $\phi_1 \cap \dots \cap \phi_{k-1} = \theta$. This establishes condition (7.3) and shows that $\langle \bigcup C_1, \dots, \bigcup C_{k-1} \rangle$ is an upper bound of C in X. By Zorn's Lemma the poset X has a maximal element $\langle \phi_1, \dots, \phi_{k-1} \rangle$. In particular, $\theta = \phi_1 \cap \dots \cap \phi_{k-1}$.

It only remains to show that $\phi_1, \dots, \phi_{k-1} \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$. First, observe that for every $i \leqslant k-1$ we have

$$\theta \neq \phi_1 \cap \dots \cap \phi_{i-1} \cap \phi_{i+1} \cap \dots \cap \phi_{k-1},$$

because otherwise $\theta = \theta_1 \cap \dots \cap \theta_{i-1} \cap \theta_{i+1} \cap \dots \cap \theta_{k-1}$ as $\theta \subseteq \theta_j \subseteq \phi_j$ for every $j \leqslant k-1$, which contradicts condition (7.2). It follows that $\phi_i \neq A \times A$ for every

$i \leqslant k-1$ because $\theta = \phi_1 \cap \cdots \cap \phi_{k-1}$. Now, suppose that $\phi_i = \eta_1 \cap \eta_2$ for some $\eta_1, \eta_2 \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{A})$. We have

$$\theta = \phi_1 \cap \cdots \cap \phi_{k-1} = \phi_1 \cap \cdots \cap \phi_{i-1} \cap \eta_1 \cap \eta_2 \cap \phi_{i+1} \cap \cdots \cap \phi_{k-1}.$$

As θ is k-irreducible and $\theta \neq \phi_1 \cap \cdots \cap \phi_{i-1} \cap \phi_{i+1} \cap \cdots \cap \phi_{k-1}$ for every $i \leqslant k-1$, we obtain

$$\theta = \phi_1 \cap \cdots \cap \phi_{i-1} \cap \eta_j \cap \phi_{i+1} \cap \cdots \cap \phi_{k-1}$$

for $j = 1$ or $j = 2$. Since $\theta_i \subseteq \phi_i \subseteq \eta_1, \eta_2$, the maximality of $\langle \phi_1, \ldots, \phi_{k-1} \rangle$ in X implies that $\phi_i = \eta_1$ or $\phi_i = \eta_2$. Thus, $\phi_i \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{A})$ as desired. ⊠

Now we are ready to prove the announced proposition. The proof of Theorem 7.7 will then be a straightforward consequence.

Proposition 7.10. *[12, Prop. 4.6] Let K be a quasivariety with an $(m+1)$-ary near-unanimity term. Moreover, let $\boldsymbol{B} \in \mathsf{K}$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$ full in K. Then, there exists a subdirect embedding of $\boldsymbol{B}$ into $\boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_m$ for some $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_m \in \mathsf{K}_{\text{RFSI}}$.*

Proof. We first show that id_B is $(m+1)$-irreducible. Let $\theta_1, \ldots, \theta_{m+1} \in \mathsf{Con}_{\mathsf{K}}(\boldsymbol{B})$ be such that $\mathsf{id}_B = \theta_1 \cap \cdots \cap \theta_{m+1}$. Define $\phi_i := \theta_1 \cap \cdots \cap \theta_{i-1} \cap \theta_{i+1} \cdots \cap \theta_{m+1}$ for each $i \leqslant m+1$. We will show that $\phi_i = \mathsf{id}_B$ for some $i \leqslant m+1$. Suppose the contrary, with a view to contradiction. Now, recall that $\boldsymbol{A} \leqslant \boldsymbol{B}$ is proper and almost total. Therefore, there exists $b \in B$ such that $b \notin A$ and $B = \mathsf{Sg}^{\boldsymbol{B}}(A \cup \{b\})$. Since $\boldsymbol{A} \leqslant \boldsymbol{B}$ is full in K, there exist $a_1, \ldots, a_{m+1} \in A$ such that $\langle a_i, b \rangle \in \phi_i$ for every $i \leqslant m+1$. By assumption K has a near-unanimity term $\mu(x_1, \ldots, x_{m+1})$. We will prove that

$$\langle \mu^{\boldsymbol{B}}(a_1, \ldots, a_{m+1}), b \rangle \in \theta_j$$

for every $j \leqslant m+1$. To this end, consider $j \leqslant m+1$. As $\langle a_i, b \rangle \in \phi_i \subseteq \theta_j$ for every $i \leqslant m+1$ such that $i \neq j$, we obtain $\langle \mu^{\boldsymbol{B}}(a_1, \ldots, a_{m+1}), \mu^{\boldsymbol{B}}(b, \ldots, b, a_j, b, \ldots, b) \rangle \in \theta_j$. Furthermore, since μ is a near-unanimity term, we have $\mu^{\boldsymbol{B}}(b, \ldots, b, a_j, b, \ldots, b) = b$. Hence, $\langle \mu^{\boldsymbol{B}}(a_1, \ldots, a_{m+1}), b \rangle \in \theta_j$. This establishes the above display. Together with the assumption that $\mathsf{id}_B = \theta_1 \cap \cdots \cap \theta_{m+1}$, this implies $b = \mu^{\boldsymbol{B}}(a_1, \ldots, a_{m+1})$. As $a_1, \ldots, a_{m+1} \in A$ and $\boldsymbol{A} \leqslant \boldsymbol{B}$, we conclude that $b \in A$, which is false. Hence, id_B is $(m+1)$-irreducible, as desired.

By Proposition 7.9 there exist $\theta_1, \ldots, \theta_m \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{B})$ such that $\mathsf{id}_B = \theta_1 \cap \cdots \cap \theta_m$. Therefore, we can apply Proposition 2.3.21 obtaining a subdirect embedding

$$f \colon \boldsymbol{B} \to \boldsymbol{B}/\theta_1 \times \cdots \times \boldsymbol{B}/\theta_m.$$

Furthermore, from Theorem 2.3.24 and $\theta_1, \ldots, \theta_m \in \mathsf{Irr}_{\mathsf{K}}(\boldsymbol{B})$ it follows that each $\boldsymbol{B}/\theta_i$ belongs to K_{RFSI}. ⊠

Proof of Theorem 7.7. The implication from (1) to (2) holds by Lemma 7.2. We will prove the other implication by contraposition. Suppose that K lacks the weak ES property. By Corollary 7.6 there exist $\boldsymbol{B} \in \mathsf{K}$ finitely generated and $\boldsymbol{A} \leqslant \boldsymbol{B}$ fully epic in K. Then, in view of Proposition 7.10, we may assume that $\boldsymbol{B} \leqslant \boldsymbol{B}_1 \times \cdots \times \boldsymbol{B}_m$ is a subdirect product for some $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_m \in \mathsf{K}_{\text{RFSI}}$. ⊠

Exploiting the concept of full subalgebras, we could also achieve the following improvement of Theorem 6.4 for the weak Es property:

Theorem 7.11. *[12, Thm. 5.3] Let* K *be a congruence permutable variety. Then, the following are equivalent:*

1. K *has the weak ES property;*
2. *Every finitely generated member of* K_{FSI} *lacks subalgebras that are fully epic in* K*.*

Observe that the statement of this theorem is exactly the statement of Theorem 6.4, translated to the case of the weak ES property, but under significantly weaker assumptions. Indeed, we only require the variety to be congruence permutable instead of arithmetical, and we can completely dispense with the requirement that K_{FSI} must be a universal class.

Again the proof, which can be found in [12], only requires universal algebraic tools and does not depend on the theory of global subdirect representations (Chapter 4) or on any definability conditions (Chapter 5).

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