Meet-irreducible elements in the poset of all logics

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Resum. En aquest treball hem estudiat els elements meet-irreductibles del poset Log de totes les lògiques. A més, hem presentat les eines tècniques requerides, com la relació d'interpretabilitat entre lògiques i la construcció del producte no-indexat d'una família de lògiques, inspirades en l'anàlisi del lattice de les varietats de l'àlgebra universal. Finalment, hem treballat en criteris de meet-irreductibilitat per a alguns sub-semilattices de Log.

Paraules clau: relació d'interpretabilitat, producte no-indexat, meet-irreducibilitat, meet-primalitat, poset de totes les lògiques, jerarquia de Leibniz

Abstract. In this work we have studied the meet-irreducible elements in the poset Log of all logics. Additionally, we have presented the main technical tools required to do so, such as the interpretability relation between logics and the non-indexed product of a family of logics, inspired in the analysis of the lattice of varieties in universal algebra. Finally, we have worked out meet-irreducibility criteria for some sub-semilattices of Log.

Keywords: interpretability relation, non-indexed product, meet-irreducibility, meet-primeness, poset of all logics, Leibniz hierarchy

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1 Introduction

As it is well-known, Maltsev was the first one to have the idea of characterizing the structure of congruence lattices by means of the validity of systems of equations. More precisely, in his 1954 monograph [19], Maltsev proved that a variety V is congruence-permutable if and only if there is a term $t(x, y, z)$ such that $V \vDash t(x, x, y) \approx y$ and $V \vDash t(x, y, y) \approx x$. During the next decade, more results of this kind were proven. For example, Jónsson [17] characterized congruence-distributive varieties in a similar manner. In 1970, Grätzer gave the first definition of *Maltsev condition*, that intended to capture the general concept behind the aforementioned innovations [11]. Shortly after, Taylor provided necessary and sufficient conditions for a class of varieties to be a *Maltsev class*, i.e. a class determined by some Maltsev conditions [28] and, a few years later, Neumann [24] rephrased Taylor's theorem in terms of the *interpretability* relation between varieties [18]. This relation allowed to order the class of all varieties into a lattice, Var, studied in [9]. In this precise sense, one could then start talking about a *Maltsev hierarchy* of varieties.

On the logical side, similar phenomena took place. After the definition of the algebraizability of logics was formally settled [1], results by Blok, Pigozzi and Czelakowski, among others, showed ways of characterizing extensions or weakenings of such concept by studying the structural behaviour of the so-called *Leibniz congruence*. This proliferation of results led to the formation of the *Leibniz hierarchy* (see, e.g., [8] and [5]) and, later on, to the analogies with the Maltsev hierarchy, as motivated in [27]. The main obstacle to the desideratum of unifying, in some sense, the two hierarchies was found in the lack of a precise account of the notion of *Leibniz condition*. This difficulty has been resolved in the collection of papers [13], [15] and [14] where, following a similar approach to that of [24], an interpretability relation between logics has been introduced. Let us sketch some of the main ideas that have stemmed from these papers and that will also be an important part of the present work.

Roughly, the poset Log of all logics is obtained by ordering the proper class of all logics in the following way: we say that a logic is *interpretable into* another when there is a translation from the language of the former into the language of the latter that preserves the corresponding semantics based on the so-called Suszko models (see, e.g., [26]). Note that we do not require the logics being compared to be similar, that is, to share a fixed language. While the interpretability relation is a preorder, we will often work with the corresponding poset Log. Notably, Log has meets of families indexed by sets: these coincide with the *non-indexed products* of the elements of such families [13, Thm. 4.6] (see Theorem 3.21). This construction has its origin in the literature concerned with the study of the lattice Var that we have mentioned above; it is, in fact, the logical counterpart of the *non-indexed product* of varieties (see [28] and [9]). Intuitively, it consists in a way of syntactically combining a family of logics (each defined in a possibly different language) in order to obtain a new logic. Though *prima facie* it may seem a somewhat complicated method of mixing logics, it turns out to be an essential tool that behaves quite elegantly with the corresponding Suszko semantics [13, Prop. 4.5] (see Theorem 3.19).

As Log is a poset in which arbitrary meets exist, it is reasonable to study its meet-irreducible elements. Informally, these are the logics that cannot be represented as the non-indexed product of any other two logics and, hence, they can be viewed as 'fundamental' in a natural sense. Additionally, the problems related to these logics mirror parallel questions regarding irreducibility in the lattice Var (see, e.g., [25]). Nevertheless, and unlike what happens in Var, there is a severe lack of symmetry between the meet and the join cases, in the sense that even binary joins can fail to exist in Log [13, Thm. 5.1] (see the Remark below Theorem 3.21).

Let us briefly illustrate how some relevant ideas from [9] have been exploited in order to isolate some sufficient conditions for meet-irreducibility. The proof of our main result in this direction, Theorem 4.6, which is an improvement of a result first presented in [20], relies on the characterization given in [9] of elements of *Helly number* below k in Var (see [9, Prop. 21]). Additionally, we have studied several modifications of such theorem for the sub-semilattices of Log given by the classes of *hereditarily nontrivial* and *finitely equivalential* logics (see, e.g., [8]), respectively (see, resp., Corollary 4.13 and Theorem 4.17). Roughly, a logic is hereditarily nontrivial if its corresponding Suszko semantics satisfy some nontriviality condition (see Section 4.2). On the other hand, an alternative characterization of meet-prime elements for nested unions of varieties in Var can be found in [9, Prop. 18]. This has inspired the proof of Theorem 4.18, which provides sufficient conditions for a nested intersection of logics to be meet-irreducible when such logics are both finitely equivalential and hereditarily nontrivial. The fruitfulness of this approach is justified by the application of our main theorem to some important cases, namely, the superintuitionistic, Łukasiewicz and the modal S4 and S5 logics (for superintuitionistic as well as for the S4 and S5 logics, see [3]; for Łukasiewicz logic see [4]), in order to show that they are meet-irreducible elements of Log (see Theorems 5.2, 5.3, 5.7, resp.).

The structure of the work is the following. In Chapter 1 we introduce all the basic concepts employed in the rest of the text, as well as the crucial notion of interpretation between two logics. In Chapter 2 we present the main tools provided by [13], [15] and [14], particularly those involving the non-indexed product construction. Chapter 3 comprises the criteria for meet-irreducibility and, lastly, Chapter 4 contains the applications of such results that we have mentioned earlier.

2 Basic notions

In this chapter we will develop a notion of interpretation between logics, which generalizes a similar concept in the setting of varieties (see, e.g., [24] and [9]).

2.1 Matrices and models

Let $\mathcal{L}$ be an *algebraic* language, that is, a set of function symbols, each with a given *arity*. Then, we denote by $\mathbf{Fm}_{\mathcal{L}}(\kappa)$ the *$\mathcal{L}$-term algebra*, i.e the algebra whose universe is the set $\mathrm{Fm}_{\mathcal{L}}(\kappa)$ of formulas in such a language with variables indexed by some cardinal κ. If the context is clear, we drop the reference to $\mathcal{L}$. Moreover, a *substitution* is an endomorphism of $\mathbf{Fm}_{\mathcal{L}}(\kappa)$. From these notions we can define the concept of logic that we will use here:

Definition 2.1. A *logic* is a substitution invariant consequence relation $\vdash$ on the set of formulas $\mathrm{Fm}(\kappa)$ of some algebraic language, i.e. it is a subset of $\wp(\mathrm{Fm}(\kappa)) \times \mathrm{Fm}(\kappa)$ and verifies the following:

(i) It is *reflexive*: for every subset $\Gamma \subseteq \mathrm{Fm}(\kappa)$ and each $\varphi \in \Gamma$, it holds that $\Gamma \vdash \varphi$.

(ii) It is *transitive*: for every pair of subsets $\Gamma, \Sigma \subseteq \mathrm{Fm}(\kappa)$ and every formula $\varphi \in \mathrm{Fm}(\kappa)$, if $\Gamma \vdash \psi$, for each $\psi \in \Sigma$, and $\Sigma \vdash \varphi$, then $\Gamma \vdash \varphi$.

(iii) It is *substitution-invariant*: given $\Gamma \subseteq \mathrm{Fm}(\kappa)$ and $\varphi \in \mathrm{Fm}(\kappa)$ such that $\Gamma \vdash \varphi$, it holds that $\sigma(\Gamma) \vdash \sigma(\varphi)$, for every substitution σ.

We will denote the language associated with some logic $\vdash$ by $\mathcal{L}(\vdash)$ and the corresponding set of formulas by $\mathrm{Fm}(\vdash)$, and similarly for the term algebra. Analogously, we will indicate that the cardinal κ corresponds to $\mathrm{Fm}(\vdash)$, i.e. that the variables considered in $\mathrm{Fm}(\vdash)$ are indexed by κ, with the notation $\kappa(\vdash)$.

Now, we can define logics from a completely algebraic point of view. Recall that, given an $\mathcal{L}$-algebra $\mathcal{A}$, an *$\mathcal{A}$-valuation* is defined as a homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathcal{A}$.

Definition 2.2. Let $\mathbb{K}$ be a class of similar algebras whose language has a constant symbol, call it 1. The *assertional logic of* $\mathbb{K}$, $\vdash_{\mathbb{K}}$, is defined as follows: $\Gamma \vdash_{\mathbb{K}} \varphi$ if and only if, for every $\mathcal{A} \in \mathbb{K}$ and each $\mathcal{A}$-valuation f, it holds that $f(\Gamma) \subseteq \{1\}$ implies that $f(\varphi) = 1$.

Given a logic $\vdash$, it is natural to study sets which are, intuitively, closed under the interpretation of the rules valid in $\vdash$.

Definition 2.3. Let $\vdash$ be a logic. Let $\mathcal{A}$ be a $\mathcal{L}(\vdash)$-algebra. A set $F \subseteq A$ is said to be a *deductive filter of* $\vdash$ *on* $\mathcal{A}$ when, given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$ such that $\Gamma \vdash \varphi$, if $h(\Gamma) \subseteq F$ holds then it follows that $h(\varphi) \in F$, for every $\mathcal{A}$-valuation h.

Moreover, the set of all the deductive filters of $\vdash$ on an algebra $\mathcal{A}$ constitutes a *closure system*, i.e. a family of subsets of A closed under intersection. This allows, given a subset $S \subseteq A$, to define the deductive filter of $\vdash$ on $\mathcal{A}$ *generated by* S, which we denote by $\mathrm{Fg}^{\mathcal{A}}_{\vdash}(S)$.

Now, recall that an equivalence relation θ on the universe of a $\mathcal{L}$-algebra $\mathcal{A}$ is a *congruence* when it behaves properly with respect to the operations of $\mathcal{L}$, that is, when for each n-ary function symbol f and elements $a_1, b_1, \ldots, a_n, b_n \in A$ we have that $(f(\overline{a}), f(\overline{b})) \in \theta$ in case $(a_1, b_1), \ldots, (a_n, b_n) \in \theta$. Remember that the family of all congruences on $\mathcal{A}$ forms a lattice, that we write as $\mathrm{Con}(\mathcal{A})$ [8].

Definition 2.4. Let $\mathcal{A}$ be an algebra, $\theta \in \mathrm{Con}(\mathcal{A})$ and $F \subseteq A$. We say that θ is F*-compatible* if, for every pair of elements $a, b \in A$ it holds that:

$$\text{if } (a, b) \in \theta \text{ and } a \in F, \text{ then } b \in F.$$

From here, we can define a notorious example of congruence:

Definition 2.5. Let $\mathcal{A}$ be an algebra and $F \subseteq A$. We define the *Leibniz congruence of* F as the largest F-compatible congruence on $\mathcal{A}$. We will denote it by $\Omega^{\mathcal{A}}F$.

The following characterization of the Leibniz congruence will be useful. Recall that, given an $\mathcal{L}$-algebra $\mathcal{A}$, a map $p : A^n \to A$ is an *n-ary polynomial function* of $\mathcal{A}$ if there is an $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ and a sequence $\overline{c}$ in A such that $p(\overline{a}) = \varphi^{\mathcal{A}}(\overline{a}, \overline{c})$, for every n-tuple $\overline{a}$ in A. The next proposition can be also read as providing criteria for the existence of the Leibniz congruence:

Proposition 2.6. *Let $\mathcal{A}$ be a $\mathcal{L}$-algebra, $F \subseteq A$ and $a, b \in A$. Then, the following conditions are equivalent:*

(i) $(a, b) \in \Omega^{\mathcal{A}}F$.

(ii) For every unary polynomial function p on $\mathcal{A}$, $p(a) \in F$ if and only if $p(b) \in F$.

Proof. First, note that condition (*ii*) is equivalent to stating that, for every $\mathcal{L}$-formula $\varphi(x, \overline{z})$ and all tuples $\overline{c}$ in A, $\varphi^{\mathcal{A}}(a, \overline{c}) \in F$ if and only if $\varphi^{\mathcal{A}}(b, \overline{c}) \in F$: this condition clearly implies (*ii*) and (*ii*) implies it by structural induction on the collection of $\mathcal{L}$-formulas of the form $\varphi(x, \overline{z})$. Hence, let us see that (*i*) implies (*ii*) by checking that it implies this equivalent formulation of (*ii*). If $(a, b) \in \Omega^{\mathcal{A}}F$, again by structural induction on the construction of formulas and by the fact that $\Omega^{\mathcal{A}}F$ is a congruence, we obtain that $(\varphi^{\mathcal{A}}(a, \overline{c}), \varphi^{\mathcal{A}}(b, \overline{c})) \in \Omega^{\mathcal{A}}F$, where $\varphi(x, \overline{z})$ is an $\mathcal{L}$-formula and $\overline{c}$ is a tuple in A. Now, suppose that $\varphi^{\mathcal{A}}(a, \overline{c}) \in F$. Then, by the F-compatibility of $\Omega^{\mathcal{A}}F$, we also have that $\varphi^{\mathcal{A}}(b, \overline{c}) \in F$. The other implication is analogous, using the symmetry of $\Omega^{\mathcal{A}}F$.

Finally, let us see that (*ii*) implies (*i*). Define the relation θ on A by condition (*ii*). Then, the previous implication tells us that $\Omega^{\mathcal{A}}F \subseteq \theta$. In order to see the desired equality, it is enough to check, by definition, that θ is an F-compatible congruence on $\mathcal{A}$. That it is an equivalence relation is straightforward, so let us first see that it is a congruence: let $a_1, b_1, \ldots, a_n, b_n \in A$ such that $(a_i, b_i) \in \theta$ for each $i \leqslant n$. Consider an arbitrary n-ary operation $*$ in $\mathcal{A}$ and let p be a unary polynomial function of $\mathcal{A}$. Given an arbitrary element $c \in A$, define

$$q_1(c) := p(*^{\mathcal{A}}(c, a_2, a_3, \ldots, a_n)),$$

$$q_2(c) := p(*^{\mathcal{A}}(b_1, c, a_3, \ldots, a_n)),$$

$$\ldots$$

$$q_n(c) := p(*^{\mathcal{A}}(b_1, \ldots, b_{n-1}, c)).$$

These are unary polynomial functions of $\mathcal{A}$ and, since $(a_i, b_i) \in \theta$, for $i \leqslant n$, we have that

$$p(*^{\mathcal{A}}(a_1, \ldots, a_{n-1}, a_n)) = q_1(a_1) \in F \text{ iff } q_1(b_1) = q_2(a_2) \in F \text{ iff } q_2(b_2) = q_3(a_3) \in F \text{ iff } \ldots$$

$$\ldots \text{ iff } q_n(b_n) = p(*^{\mathcal{A}}(b_1, \ldots, b_{n-1}, b_n)) \in F,$$

and from here we obtain that $(*^{\mathcal{A}}(\overline{a}), *^{\mathcal{A}}(\overline{b})) \in \theta$ and, thus, θ is a congruence. Additionally, θ is F-compatible: applying its definition for the unary polynomial function $p(x) := x$, we have that, if $(a, b) \in \theta$ and $p(a) = a \in F$, clearly $p(b) = b \in F$.

□

The basic object that we will study in this work is the following:

Definition 2.7. A *matrix* is a pair $(\mathcal{A}, F)$ formed by an $\mathcal{L}$-algebra $\mathcal{A}$ and some subset $F \subseteq A$. It is said to be *trivial* if $\mathcal{A}$ is the trivial algebra, which we will denote by $\mathbf{1}$[1]. The *algebraic reduct* of a matrix $(\mathcal{A}, F)$ is simply the algebra $\mathcal{A}$. Given a class $\mathbb{K}$ of matrices, we denote by $\mathrm{Alg}(\mathbb{K})$ the class formed by the algebraic reducts of the elements of $\mathbb{K}$.

Now, given a matrix $(\mathcal{A}, F)$, we label the matrix $(\mathcal{A}/\Omega^{\mathcal{A}}F, F/\Omega^{\mathcal{A}}F)$ as its *reduction* and denote it by $(\mathcal{A}, F)^*$. If $\Omega^{\mathcal{A}}F$ is the identity relation, we say that $(\mathcal{A}, F)$ is *reduced*.

Note that the matrix construction allows us to consider designated subsets of a given algebra, so that, in some sense, we are working with something similar to a structure, in the model-theoretical sense (see Section 2.3). Thus, a *matrix homomorphism* $f : (\mathcal{A}, F) \to (\mathcal{B}, G)$ between two similar algebras is a *strict* homomorphism between the corresponding algebraic reducts, i.e. verifying that $a \in F$ iff $f(a) \in G$, for every $a \in A$. If there is an isomorphism between two matrices $(\mathcal{A}, F)$ and $(\mathcal{B}, G)$ (i.e. an strict isomorphism between the two algebraic reducts), we will write $(\mathcal{A}, F) \cong (\mathcal{B}, G)$.

[1] Note that we assume $\mathbf{1}$ to be an algebra in the same language as $\mathcal{A}$, namely, $\mathcal{L}$. Additionally, in virtue of the definition, we have that either $F = \{1\}$ or $F = \varnothing$.

In a similar way, given an $\mathcal{L}$-matrix $(\mathcal{A}, F)$, a *submatrix* $(\mathcal{B}, G)$ of $(\mathcal{A}, F)$ is a $\mathcal{L}$-matrix consisting of a subalgebra $\mathcal{B}$ of $\mathcal{A}$ and the set $G := F \cap B^2$. We will write, in such case, $(\mathcal{B}, G) \leqslant (\mathcal{A}, F)$[2]. On the other hand, given a family of $\mathcal{L}$-matrices $\{(\mathcal{B}_i, G_i) \,|\, i \in I\}$, we define its *direct product* as the matrix $(\prod_{i\in I} \mathcal{B}_i, \prod_{i\in I} G_i)$, where $\prod_{i\in I} \mathcal{B}_i$ is the $\mathcal{L}$-algebra of universe $\prod_{i\in I} B_i$ and where an n-ary operation $*$ is interpreted, given elements $\alpha_1, \ldots, \alpha_n \in \prod_{i\in I} B_i$, as follows:

$$*^{\prod_{i\in I} \mathcal{B}_i}(\alpha_1, \ldots, \alpha_n) := (*^{\mathcal{B}_i}(\alpha_1(i), \ldots, \alpha_n(i)))_{i\in I}.$$

The resulting matrix will be denoted by $\prod_{i\in I}(\mathcal{B}_i, G_i)$. A submatrix $(\mathcal{A}, F) \leqslant \prod_{i\in I}(\mathcal{B}_i, G_i)$ is a *subdirect product* of $\{(\mathcal{B}_i, G_i) \,|\, i \in I\}$ if it satisfies that, for each $j \in I$, the composition $p_j \circ \iota : (\mathcal{A}, F) \to (\mathcal{B}_j, G_j)$ is surjective, where $p_j : \prod_{i\in I}(\mathcal{B}_i, G_i) \to (\mathcal{B}_j, G_j)$ is the canonical projection and $\iota : (\mathcal{A}, F) \hookrightarrow \prod_{i\in I}(\mathcal{B}_i, G_i)$ is the canonical inclusion.

We will denote by $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_{\mathrm{SD}}$ the class-operators for isomorphic copies, substructures, direct products and subdirect products of matrices[3]. We will assume that with the application of the last three we obtain classes closed under $\mathbb{I}$ and that $\mathbb{P}$, as well as $\mathbb{P}_{\mathrm{SD}}$, when applied to empty sets of indexes, produce the trivial matrix $(\mathbf{1}, \{1\})$ (in the corresponding language). Additionally, given a class $\mathbb{K}$ of matrices, define the operator

$$\mathbb{R}(\mathbb{K}) := \mathbb{I}(\{(\mathcal{A}, F)^* \,|\, (\mathcal{A}, F) \in \mathbb{K}\}).$$

As in model theory, it is interesting to study structures that are models of a given theory. In our case this situation translates to the following:

Definition 2.8. Let $\vdash$ be a logic and let $\mathcal{A}$ be a $\mathcal{L}(\vdash)$-algebra. A matrix $(\mathcal{A}, F)$ is said to be a *model* of $\vdash$ if F is a deductive filter of $\vdash$ on $\mathcal{A}$. We denote by $\mathrm{Mod}(\vdash)$ the collection of all such models, and by $\mathrm{Mod}^*(\vdash)$ the collection of all the reduced ones.

In particular, we will study the following class of models:

Definition 2.9. Let $\vdash$ be a logic. By the *Suszko models of* $\vdash$ we understand the family

$$\mathrm{Mod}^{\equiv}(\vdash) := \mathbb{P}_{\mathrm{SD}}\mathbb{R}(\mathrm{Mod}(\vdash)).$$

Reciprocally, we can obtain a logic from a collection of matrices. We proceed similarly as in the case of assertional logics induced by a class of algebras, but here we already have a set of distinguished elements:

Definition 2.10. Let $\mathbb{K}$ be a class of matrices whose algebraic reducts are similar, of language $\mathcal{L}$. The *logic induced by* $\mathbb{K}$ *in* κ *variables* is the consequence relation $\vdash_{\mathbb{K}}$ on $\mathrm{Fm}_{\mathcal{L}}(\kappa)$ defined as follows: given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}(\kappa)$, the derivation $\Gamma \vdash_{\mathbb{K}} \varphi$ holds if and only if, for every $(\mathcal{A}, F) \in \mathbb{K}$ and each $\mathcal{A}$-valuation h, we have that $h(\Gamma) \subseteq F$ implies that $h(\varphi) \in F$.

Remark 2.11. The assertional logic induced by a class of algebras $\mathbb{K}$ is induced by the class of matrices $\{(\mathcal{A}, \{1\}) \,|\, \mathcal{A} \in \mathbb{K}\}$. Notably, every logic is induced by its (reduced) models [23, Thms. 1.20, 1.21].

[2]We will also follow the notation $\mathcal{B} \leqslant \mathcal{A}$ for subalgebras.

[3]Respectively, we will also follow this notation elsewhere for algebras when no confusion arises.

Let us introduce another two class operators that will come later in handy. Given a set I, a nonempty subset $H \subseteq \wp(I)$ is a *filter on I* if: (i) $I \in H$ and $\varnothing \notin H$; (ii) given $X, Y \in H$, $X \cap Y \in H$ and (iii) if $X \in H$ and $X \subseteq Y \subseteq I$ then $Y \in H$. Given a cardinal κ, a filter H on I is *κ-complete* if, whenever $\gamma < \kappa$ and $\{X_\alpha \mid \alpha < \gamma\} \subseteq H$, then $\bigcap_{\alpha<\gamma} X_\alpha \in H$. We say that a filter U on I is an *ultrafilter* if, given $X \subseteq I$, then either X or $I \setminus X$ belongs to U. A family J of subsets of a set I is said to have the *finite intersection property* (FIP) if every nonempty finite subfamily of J has nonempty intersection. It is well-known that every collection J of subsets of a set I with FIP can be extended to a filter on I and, using choice, to an ultrafilter: this is the so-called *ultrafilter lemma* (for more details see, e.g., [16]).

Consider, as before, a collection $\{(\mathcal{B}_i, G_i) \mid i \in I\}$ of similar $\mathcal{L}$-matrices and an ultrafilter U on I. Consider the relation θ_U defined on $\prod_{i\in I} B_i$ as follows:

$$(\alpha, \beta) \in \theta_U \text{ if, and only if, } \{i \in I \mid \alpha(i) = \beta(i)\} \in U.$$

It is well-known [22, Prop. 4.1] that θ_U is a congruence on $\prod_{i\in I} \mathcal{B}_i$, the quotient $\prod_{i\in I} \mathcal{B}_i/\theta_U$ is also an $\mathcal{L}$-algebra and, consequently, the following definition is coherent: an *ultraproduct* of the family $\{(\mathcal{B}_i, G_i) \mid i \in I\}$ is an $\mathcal{L}$-matrix of the form $(\prod_{i\in I} \mathcal{B}_i/\theta_U, \prod_{i\in I} G_i/\theta_U)$, where U is an ultrafilter on I[4]. Such matrix will be written as $\prod_{i\in I}(\mathcal{B}_i, G_i)/U$. The class-operator for the formation of ultraproducts will be denoted by $\mathbb{P}_{\mathrm{U}}$.

On the other hand, consider once again the collection $\{(\mathcal{B}_i, G_i) \mid i \in I\}$ and let H be a proper κ-complete filter on I, where κ is a regular cardinal. Then, the relation θ_H defined as above is also a congruence and we can replicate the ultraproduct construction: a *κ-reduced product* of the family $\{(\mathcal{B}_i, G_i) \mid i \in I\}$ is a matrix of the form $\prod_{i\in I}(\mathcal{B}_i/\theta_H, G_i/\theta_H)$, that we will write as $\prod_{i\in I}(\mathcal{B}_i, G_i)/H$. We shall refer to the corresponding class-operator as $\mathbb{P}_{R,\kappa}$. The following result [13, Thm. 2.4] will be of use later:

Fact 2.12. *Given a class of matrices $\mathbb{K}$ and a regular cardinal κ, let $\vdash$ be the logic induced by $\mathbb{K}$ on* $\mathrm{Fm}(\kappa)$ *and $|\mathrm{Fm}(\kappa)| \leqslant \kappa$. Then, the following holds:*

$$\mathbb{R}(\mathrm{Mod}(\vdash)) = \mathbb{RSP}_{R,\kappa^+}(\mathbb{K}).$$

2.2 Protoalgebraic logics

In this section we wish to present some important examples of logics that we will study with detail in the future. Let us start with a very natural requirement for a propositional logic. A logic $\vdash$ *has theorems* if it admits derivations of the form $\varnothing \vdash \varphi$, for some $\varphi \in \mathrm{Fm}(\vdash)$.

Proposition 2.13. *Let $\vdash$ be a logic and $\mathcal{A}$ a $\mathcal{L}(\vdash)$-algebra. Then,*

(i) If $(\mathcal{A}, \varnothing) \in \mathbb{R}(\mathrm{Mod}(\vdash))$, then $\mathcal{A} = \mathbf{1}$.

(ii) The logic $\vdash$ has theorems if and only if $(\mathbf{1}, \varnothing) \notin \mathrm{Mod}^{\equiv}(\vdash)$ or, equivalently, if $(\mathbf{1}, \varnothing) \notin \mathbb{R}(\mathrm{Mod}(\vdash))$.

[4]In case of the members of the family $\{(\mathcal{B}_i, G_i) \mid i \in I\}$ being the same, we will talk of an *ultrapower* of the only element in such family.

Proof. For part (i) we observe that, given $a, b \in A$ and a unary polynomial function $p(x)$, it holds that $p(a) \notin \varnothing$ and $p(b) \notin \varnothing$ so, by Proposition 2.6, $(a, b) \in \Omega^{\mathcal{A}}\varnothing = id_A$ and hence $a = b$.
For part (ii) we start by noting that, by the definition of deductive filter, $(\mathbf{1}, \varnothing) \in \mathrm{Mod}(\vdash)$ if and only if $\vdash$ lacks theorems. Furthermore, since $\mathbf{1}$ has only one congruence, namely, the identity relation, it follows that $\Omega^{\mathbf{1}}\varnothing$ must be the identity, so $(\mathbf{1}, \varnothing)$ is reduced.

□

Notably, there are classes of logics that lack theorems in the sense described above. One example are the logics of the following kind:

Definition 2.14. A logic $\vdash$ is *almost inconsistent* if, for every set $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\vdash)$, $\Gamma \vdash \varphi$ holds if and only if $\Gamma \neq \varnothing$.

Another requirement that seems intuitive for a logic is that it allows us to restrict ourselves to finite derivations:

Definition 2.15. A logic $\vdash$ is *finitary* if, given $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\vdash)$ such that $\Gamma \vdash \varphi$, there is a finite subset $\Sigma \subseteq \Gamma$ such that $\Sigma \vdash \varphi$.

Given an arbitrary logic $\vdash$, there is a canonical way of obtaining a finitary logic, namely, define

$$\Gamma \vdash^* \varphi \text{ if and only if there is a finite } \Sigma \subseteq \Gamma \text{ such that } \Sigma \vdash \varphi.$$

The resulting logic $\vdash^*$ is called the *finitary companion of* $\vdash$.

Proposition 2.16. *Let $\vdash$ be the logic induced by some class $\mathbb{K}$ of similar matrices. Then, $\vdash^*$ is induced by $\mathbb{P}_{\mathrm{U}}(\mathbb{K})$.*

Proof. Assume that the derivation $\Gamma \vdash^* \varphi$ fails. This means that, for each finite subset $\Sigma \subseteq \Gamma$, $\Sigma \nvdash \varphi$ and this implies that there is some matrix $(\mathcal{A}_\Sigma, F_\Sigma) \in \mathbb{K}$ and some $\mathcal{A}_\Sigma$-valuation h_Σ such that $h_\Sigma(\Sigma) \subseteq F_\Sigma$ and $h_\Sigma(\varphi) \notin F_\Sigma$. Now, given one of these finite subsets Σ, define Σ^* as the set $\{\Sigma' \subseteq_{<\omega} \Gamma \mid \Sigma \subseteq \Sigma'\}$, where $\Sigma' \subseteq_{<\omega} \Gamma$ means that Σ' is a finite subset of Γ. The family of all such sets Σ^* has the FIP, so we know that there is an ultrafilter U extending it. Now, define the matrix

$$(\mathcal{A}, F) := \prod_{\Sigma \subseteq_{<\omega} \Gamma} (\mathcal{A}_\Sigma, F_\Sigma)/U,$$

and the $\mathcal{A}$-valuation given by $h(\psi) := (h_\Sigma(\psi))_{\Sigma \subseteq_{<\omega} \Gamma}/\theta_U$, for each $\psi \in \mathrm{Fm}(\vdash)$. Note that, by construction, given $\psi \in \Gamma$, we have that

$$h(\psi) = (h_\Sigma(\psi))_{\Sigma \subseteq_{<\omega} \Gamma}/\theta_U \in \prod_{\Sigma \subseteq_{<\omega} \Gamma} F_\Sigma/\theta_U = F,$$

so that $h(\Gamma) \subseteq F$, but

$$h(\varphi) = (h_\Sigma(\varphi))_{\Sigma \subseteq_{<\omega} \Gamma}/\theta_U \notin \prod_{\Sigma \subseteq_{<\omega} \Gamma} F_\Sigma/\theta_U = F.$$

Thus, we have seen that the derivation $\Gamma \vdash_{\mathbb{P}_{\mathrm{U}}(\mathbb{K})} \varphi$ does not hold, as we wanted.

□

Every finitary logic can be regarded as satisfying a particular case of a more general property:

Definition 2.17. Fix a regular cardinal κ. A logic $\vdash$ is *κ-compact* in case that, if $\Gamma \vdash \varphi$ holds for a set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\vdash)$, then there is a subset $\Sigma \subseteq \Gamma$ such that $|\Sigma| < \kappa$ and $\Sigma \vdash \varphi$.

In particular, the concept of finitary logic coincides with that of ω-compact logic. We will make use of the following property regarding this notion [7, Thm. 6]:

Fact 2.18. *If a logic $\vdash$ is κ-compact, for some regular cardinal κ, the class* $\mathrm{Mod}(\vdash)$ *is $\mathbb{P}_{R,\kappa}$-closed.*

The most general kind of logic that we will study in this work is the next one:

Definition 2.19. A logic $\vdash$ is *protoalgebraic* if there is a nonempty set of formulas $\Delta(x, y) \subseteq \mathrm{Fm}(\vdash)$ such that

$$\varnothing \vdash \Delta(x, x) \quad \text{and} \quad x, \Delta(x, y) \vdash y.$$

The following well-known property can be found in [23, Thm. 2.18]:

Fact 2.20. *A logic $\vdash$ is protoalgebraic if and only if* $\mathrm{Mod}^*(\vdash)$ *is closed under* $\mathbb{P}_{\mathrm{SD}}$.

Remark 2.21. As a consequence, if $\vdash$ is a protoalgebraic logic then $\mathrm{Mod}^*(\vdash) = \mathrm{Mod}^{\equiv}(\vdash)$.

An important class of protoalgebraic logics is the following:

Definition 2.22. A logic $\vdash$ is *equivalential* if there is a nonempty[5] set of formulas $\Delta(x, y) \subseteq \mathrm{Fm}(\vdash)$ such that

$$\varnothing \vdash \Delta(x, x); \qquad x, \Delta(x, y) \vdash y;$$

$$\bigcup_{i \leqslant n} \Delta(x_i, y_i) \vdash \Delta(*(\overline{x}), *(\overline{y})),$$

for every n-ary symbol $*$ of $\mathcal{L}(\vdash)$. The elements of $\Delta(x, y)$ are often called *equivalence formulas*. Equivalently [23, Thm. 3.2], a logic $\vdash$ is equivalential if there is a nonempty set $\Delta(x, y)$ of equivalence formulas such that, for every $(\mathcal{A}, F) \in \mathrm{Mod}(\vdash)$ and $a, b \in A$,

$$(a, b) \in \Omega^{\mathcal{A}} F \text{ if and only if } \Delta(a, b)^{\mathcal{A}} \subseteq F.$$

Definition 2.23. We say that a logic $\vdash$ is *finitely equivalential* when it is finitary, equivalential and its set of equivalence formulas is finite.

The next two facts are [8, Thms. 6.79, 6.81]:

Facts 2.24.

[5]The requirement that we impose to $\Delta(x, y)$ for protoalgebraic and equivalential logics, namely, that it is nonempty, is not usually required in the literature (see, e.g. [8]). Nevertheless, we will follow this usage from [13] since, in a fixed language, the unique equivalential logic with an empty $\Delta(x, y)$ is the almost inconsistent logic [8, Prop. 6.11.5].

(i) A logic $\vdash$ is finitely equivalential if and only if $\mathrm{Mod}^*(\vdash)$ *is closed under* $\mathbb{S}, \mathbb{P}$ *and* $\mathbb{P}_{\mathrm{U}}$.

(ii) Let $\vdash$ be a logic induced by some class of reduced matrices $\mathbb{K}$. Moreover, assume that $\vdash$ is finitely equivalential. Then, $\mathrm{Mod}^{\equiv}(\vdash) = \mathbb{ISPP}_{\mathrm{U}}(\mathbb{K})$.

Finally, let us present a very special class of logics. Intuitively, it contains all the logics that possess some kind of completeness theorem with respect to a class of algebras. The syntactic component is the following. Recall that an $\mathcal{L}$-*equation* is simply a pair of terms $(\epsilon, \delta) \in \mathrm{Fm}^2_{\mathcal{L}}$ that will be denoted as $\epsilon \approx \delta$. An $\mathcal{L}$-algebra $\mathcal{A}$ is said to *satisfy* an $\mathcal{L}$-equation $\epsilon \approx \delta$ if, given an arbitrary $\mathcal{A}$-valuation h, $h(\epsilon) = h(\delta)$. A class $\mathbb{K}$ of $\mathcal{L}$-algebras satisfies $\epsilon \approx \delta$ if every member of $\mathbb{K}$ does so. Moreover, given a set of $\mathcal{L}$-equations $\Theta \cup \{\epsilon \approx \delta\}$ and a class $\mathbb{K}$ of similar $\mathcal{L}$-algebras, the *equational consequence relative to* $\mathbb{K}$ is defined as follows: $\Theta \vDash_{\mathbb{K}} \epsilon \approx \delta$ holds if and only if, for every algebra $\mathcal{A} \in \mathbb{K}$ and every $\mathcal{A}$-valuation h, if $\mathcal{A}$ satisfies each element of Θ through h then $\mathcal{A}$ satisfies $\epsilon \approx \delta$ through h. Hence, in this terms, the class $\mathbb{K}$ satisfies the equation $\epsilon \approx \delta$ if $\vDash_{\mathbb{K}} \epsilon \approx \delta$ holds. The same definitions can be replicated for *quasi-equations*, i.e. formulas of the form

$$(\epsilon_1 \approx \delta_1) \& \dots \& (\epsilon_n \approx \delta_n) \Rightarrow \epsilon \approx \delta,$$

where an algebra satisfies such formula in case that, when it satisfies the equations of the premise through some valuation, it follows that it satisfies the conclusion through that same valuation (for more details see, e.g., [2]).

Additionally, recall that a class $\mathbb{K}$ of similar algebras is a *variety* if it is closed under $\mathbb{S}$, $\mathbb{P}$ and under the formation of homomorphic images. The classic result of Birkhoff (see, e.g., [2, Thm. 11.9]) states that varieties coincide with *equational classes*, i.e. classes of similar algebras axiomatized by equations. Similarly, a *quasivariety* is a class of similar algebras closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$. An analogous result (see, e.g., [2, Thm. 2.25]) tells us that quasivarieties coincide with *quasi-equational classes*, that is, classes of similar algebras that are axiomatized by quasi-equations. Given a class $\mathbb{K}$ of similar algebras, we denote by $\mathbf{V}(\mathbb{K})$ (resp., $\mathbf{Q}(\mathbb{K})$) the variety (resp., quasivariety) *generated by* $\mathbb{K}$, i.e. the smallest variety (resp., quasivariety) that contains it.

Also, remember that a *Heyting algebra* is an algebra $(A, \wedge, \vee, \to, 0, 1)$ that comprises a bounded lattice $(A, \wedge, \vee, 0, 1)$ and a binary operation $\to$ satisfying the residuation law, i.e. for every $a, b, c \in A$, $a \wedge b \leqslant c$ holds iff $a \leqslant b \to c$. A *Boolean algebra* is a Heyting algebra that verifies the excluded middle law, namely, $x \vee \neg x \approx 1$, where $\neg x := x \to 0$. Finally, a *modal algebra* is an algebra $(A, \wedge, \vee, \to, \Box, 0, 1)$ consisting in a Boolean algebra $(A, \wedge, \vee, \to, 0, 1)$ and a unary operation $\Box$ such that $\Box 1 = 1$ and, given $a, b \in A$, $\Box(a \wedge b) = \Box a \wedge \Box b$. These will be the candidates for the semantic counterparts that we mentioned before.

Definition 2.25. A logic $\vdash$ is said to be *algebraizable* if there are a quasivariety $\mathbb{K}$, a set $\tau(x) \subseteq \mathrm{Fm}(\vdash)^2$ of equations and a set $\Delta(x, y) \subseteq \mathrm{Fm}(x)$ of formulas such that, for every set of formulas $\Gamma \cup \{\varphi\}$ and every set of equations $\Psi \cup \{\epsilon \approx \delta\}$:

(i) $\Gamma \vdash \varphi$ if and only if $\tau(\Gamma) \vDash_{\mathbb{K}} \tau(\varphi)$,

(ii) $\Psi \vDash_{\mathbb{K}} \epsilon \approx \delta$ if and only if $\Delta(\Psi) \vdash \Delta(\epsilon, \delta)$,

(iii) $x \vdash \Delta(\tau(x))$ and $\Delta(\tau(x)) \vdash x$ and

(iv) $x \approx y \vDash_{\mathbb{K}} \tau(\Delta(x, y))$ and $\tau(\Delta(x, y)) \vDash_{\mathbb{K}} x \approx y$,

where $\Delta(\Psi) := \bigcup\{\Delta(\varphi, \psi) \mid \varphi \approx \psi \in \Psi\}$ and $\tau(\Gamma) := \bigcup\{\tau(\psi) \mid \psi \in \Gamma\}$. In this case, $\mathbb{K}$ is said to be an *equivalent algebraic semantics* for $\vdash$. In addition, we say that τ, Δ and $\mathbb{K}$ *witness* the algebraization of the logic $\vdash$.

Example 2.26. Let $\vdash_{\text{CPC}}$ denote the classical propositional calculus. Then, the equivalent algebraic semantics for $\vdash_{\text{CPC}}$ is the class **BA** of Boolean algebras. Additionally, the algebraization of $\vdash_{\text{CPC}}$ is witnessed by $\{x \approx 1\}, \{x \to y, y \to x\}$ and **BA**.

Example 2.27. Denote by $\vdash_{\text{IPC}}$ the intuitionistic propositional calculus. Similarly as in the previous example, the equivalent algebraic semantics for $\vdash_{\text{IPC}}$ is the class **HA** of Heyting algebras, and the algebraization of $\vdash_{\text{IPC}}$ is witnessed by $\{x \approx 1\}, \{x \to y, y \to x\}$ and **HA**.

In the case of algebraizable logics, we can obtain a simple characterization of their reduced models [8, Prop. 4.57]:

Definition 2.28. Given a set of equations $\tau(x)$ in the language of an algebra $\mathcal{A}$, the *set of solutions of* $\tau(x)$ in $\mathcal{A}$ is defined as $\tau(\mathcal{A}) := \{a \in A \mid \mathcal{A} \vDash \tau(a)\}$.

Fact 2.29. *Let* $\vdash$ *be an algebraizable logic as witnessed by* τ, Δ *and* $\mathbb{K}$. *Then,*

$$\text{Mod}^*(\vdash) = \{(\mathcal{A}, \tau(\mathcal{A})) \mid \mathcal{A} \in \mathbb{K}\}.$$

As a consequence, $\text{Mod}^{\equiv}(\vdash) = \text{Mod}^*(\vdash)$.

Hence, in particular, the reduced models of $\vdash_{\text{CPC}}$ are of the form $(\mathcal{A}, \{1\})$, where $\mathcal{A} \in \mathbf{BA}$, and, similarly, the ones of $\vdash_{\text{IPC}}$ are of the same form but with $\mathcal{A} \in \mathbf{HA}$.

2.3 Relational quasivarieties

Later on, we will make use of some essential observations regarding classes of matrices and, in particular, of Suszko models. In order to clearly motivate these remarks, we need to introduce the notion of relational quasivariety.

First, let us consider a *structure* $\mathfrak{A} := (A, \mathcal{F}, \mathcal{R})$ in the model-theoretic sense, that is, consisting in a universe A and two sets $\mathcal{F}$ and $\mathcal{R}$ of function and relation symbols, respectively. Note that an algebra is simply a structure in which $\mathcal{R} = \varnothing$. We can extend the notion of satisfiability as follows: given a set Σ of first-order formulas in some language $(\mathcal{F}, \mathcal{R})$, we write $\mathbb{K} \vDash \Sigma$ when the universal formula $\forall \overline{x}\, \Phi$ holds true in every structure from $\mathbb{K}$, for each $\Phi \in \Sigma$. On the other hand, recall that an *atomic formula* in the language $(\mathcal{F}, \mathcal{R})$ is either an equation between $\mathcal{F}$-terms or an expression $R(t_1, \dots, t_n)$, where $R \in \mathcal{R}$ is an n-ary relation symbol and $t_1, \dots, t_n$ are $\mathcal{F}$-terms.

Similarly as in the case of algebras and matrices, we can define the class-operators $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathrm{U}}$ for the closure under formation of isomorphic structures, substructures, direct products and ultraproducts of structures, respectively. A key difference here is that we do *not* impose the condition of strictness for homomorphisms. In other words, for a map $f : \mathfrak{A} \to \mathfrak{B}$ to be a homomorphism in this sense we only require that it *preserves* the function and relation symbols, while the strictness condition would also involve *reflecting* the relation symbols, i.e. given an n-ary relation symbol R from the signature of $\mathfrak{A}$ and terms $t_1, \ldots, t_n$ in such language, we only require that the weaker condition

$$R^{\mathfrak{A}}(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}}) \text{ implies } R^{\mathfrak{B}}(f(t_1^{\mathfrak{A}}), \ldots, f(t_n^{\mathfrak{A}}))$$

holds. Another relevant difference is that, here, we say that a structure $\mathfrak{A} := (A, \mathcal{F}, \mathcal{R})$ is *trivial* if $|A| = 1$ and $\mathcal{R}$ consists of non-empty relations (see, e.g., [21]).

Now, let us generalize the previously defined notions of quasi-equation and quasi-variety. A *basic Horn formula* in the language $(\mathcal{F}, \mathcal{R})$ is one having the following form:

$$(\Phi_1 \& \ldots \& \Phi_n) \Rightarrow \Phi,$$

where $\Phi_1, \ldots, \Phi_n, \Phi$ are atomic formulas from the corresponding language. A *relational quasivariety* is the model class of a set of basic Horn formulas. As before, given a class of structures $\mathbb{K}$, we denote by $\mathbf{Q}(\mathbb{K})$ the smallest relational quasivariety containing $\mathbb{K}$. In fact, it holds that $\mathbf{Q}(\mathbb{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathbb{K})$ (see, e.g., [10, Ch. 2]).

The key point here is that, given a class $\mathbb{K}$ of $\mathcal{L}$-matrices, we can see $\mathbb{K}$ as a class of structures. Indeed, consider the relational language $\mathcal{L}_R := (\mathcal{L}, \{R\})$, where R is a unary relation symbol. Then, every $\mathcal{L}$-matrix $(\mathcal{A}, F) \in \mathbb{K}$ can be regarded as a $\mathcal{L}_R$-structure by interpreting R as F. Note that the nontriviality requirement imposed on a matrix $(\mathcal{A}, F)$ seen as a relational structure only implies that it is different from $(\mathbf{1}, \varnothing)$. On the other hand, for our purposes, we require that it is also different from $(\mathbf{1}, \{1\})$, as we have explained before.

Now we are in conditions of defining the two main notions from this section:

Definition 2.30. A class $\mathbb{K}$ of $\mathcal{L}$-matrices is said to have the *joint embedding property* (JEP) if, for every set X of nontrivial elements of $\mathbb{K}$, there exists some $(\mathcal{A}, F) \in \mathbb{K}$ in which every member of X embeds.

Definition 2.31. We say that a class $\mathbb{K}$ of $\mathcal{L}$-matrices is *passively structurally complete* (PSC) if, for any pair of nontrivial elements of $\mathbb{K}$, each can be mapped homomorphically into an ultrapower of the other.

The following result due to Maltsev provides a necessary and sufficient condition for a relational quasivariety to have the JEP (for a proof, see [10, Prop. 2.1.19]):

Fact 2.32. *A class $\mathbb{K}$ of $\mathcal{L}$-matrices that is a relational $\mathcal{L}_R$-quasivariety has the JEP if and only if there is an element $(\mathcal{A}, F) \in \mathbb{K}$ such that $\mathbb{K} = \mathbf{Q}((\mathcal{A}, F))$.*

Now note that, in the case of a finitely equivalential logic $\vdash$, Fact 2.24(i) tells us that $\mathrm{Mod}^{\equiv}(\vdash)$ is indeed a relational quasivariety. Moreover:

Corollary 2.33. *If $\vdash$ is a finitely equivalential logic and $\mathrm{Mod}^{\equiv}(\vdash)$ has the JEP then there is a reduced matrix $(\mathcal{A}, F)$ such that $\vdash$ is the finitary companion of the logic induced by $(\mathcal{A}, F)$.*

Proof. By Maltsev's theorem above, we already know that $\mathrm{Mod}^{\equiv}(\vdash) = \mathbf{Q}((\mathcal{A}, F))$, for some $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$. But this means that $\mathrm{Mod}^{\equiv}(\vdash) = \mathbb{ISPP}_{\mathrm{U}}((\mathcal{A}, F))$. Let $\vdash_{\{(\mathcal{A},F)\}}$ be the logic induced by $(\mathcal{A}, F)$. Then, assume that $\Gamma \vdash^{*}_{\{(\mathcal{A},F)\}} \varphi$, for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$. Note that we may assume that Γ is finite because the finitary companion $\vdash^{*}_{\{(\mathcal{A},F)\}}$ is finitary by definition. By Proposition 2.16, and since this derivation is preserved under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$, we have that $\Gamma \vdash_{\mathbb{ISPP}_{\mathrm{U}}(\mathcal{A},F)} \varphi$, that is, $\Gamma \vdash_{\mathrm{Mod}^{\equiv}(\vdash)} \varphi$, i.e. $\Gamma \vdash \varphi$, as desired. Reciprocally, if $\Gamma \vdash_{\mathrm{Mod}^{\equiv}(\vdash)} \varphi$, since $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$, we have that $\Gamma \vdash_{\{(\mathcal{A},F)\}} \varphi$ and, since $\mathbb{P}_{\mathrm{U}}((\mathcal{A}, F)) \subseteq \mathrm{Mod}^{\equiv}(\vdash)$, we obtain that $\Gamma \vdash_{\mathbb{P}_{\mathrm{U}}(\mathcal{A},F)}$ and, by Proposition 2.16, that $\Gamma \vdash^{*}_{\{(\mathcal{A},F)\}} \varphi$.

Finally, we only have to check that $(\mathcal{A}, F)$ is reduced. But, by Remark 2.21, $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$ and this class coincides with $\mathrm{Mod}^{*}(\vdash)$, so $(\mathcal{A}, F) \in \mathrm{Mod}^{*}(\vdash)$, as desired.

□

We finish this section with the following observation that connects the JEP and the PSC property [21, Thm. 4.3]:

Theorem 2.34. *If a relational quasivariety $\mathbb{K}$ is PSC then it has the JEP and so do all of its subquasivarieties.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two nontrivial elements of $\mathbb{K}$. Then, there are two homomorphisms $f : \mathfrak{A} \to \mathfrak{B}_U$ and $g : \mathfrak{B} \to \mathfrak{A}_U$, where $\mathfrak{B}_U$ and $\mathfrak{A}_U$ are certain ultrapowers of $\mathfrak{B}$ and $\mathfrak{A}$, respectively. Now, we have canonical embeddings $\iota_A : \mathfrak{A} \hookrightarrow \mathfrak{A}_U$ and $\iota_B : \mathfrak{B} \hookrightarrow \mathfrak{B}_U$, and so are the maps

$$\langle \iota_A, f\rangle : \mathfrak{A} \to \mathfrak{A}_U \times \mathfrak{B}_U : a \mapsto (\iota_A(a), f(a)),$$

$$\langle g, \iota_B\rangle : \mathfrak{B} \to \mathfrak{A}_U \times \mathfrak{B}_U : b \mapsto (g(b), \iota_B(b)).$$

Thus, $\mathfrak{A}, \mathfrak{B} \in \mathbb{IS}(\mathfrak{A}_U \times \mathfrak{B}_U)$ and, since

$$\mathbb{IS}(\mathfrak{A}_U \times \mathfrak{B}_U) \subseteq \mathbf{Q}(\{\mathfrak{A}, \mathfrak{B}\}) \subseteq \mathbb{K},$$

we have that $\mathbb{K}$ has the JEP. The same argument applies for the case of its subvarieties.

□

2.4 Translations and interpretations

Translations have been quite relevant in the context of propositional and modal logics. Essentially, a translation between two languages is a map that sends each basic operation to some term. The additional condition that we impose is natural, namely, that this map preserves the arity of the basic operations.

Definition 2.35. Consider two languages $\mathcal{L}$ and $\mathcal{L}'$. A *translation of* $\mathcal{L}$ *into* $\mathcal{L}'$ is a map τ that sends each n-ary symbol $*$ in $\mathcal{L}$ to an n-ary formula $\tau(*)$ in $\mathcal{L}'$ and in variables $x_1, \dots, x_n$.

Additionally, following [13]:

Definition 2.36. In the same conditions as before, given an $\mathcal{L}'$-algebra $\mathcal{A}$, we define the *translation of* $\mathcal{A}$ as the $\mathcal{L}$-algebra $\mathcal{A}^\tau$ of universe A and with n-ary operations $*$ interpreted as:

$$*^{\mathcal{A}^\tau}(\overline{a}) := \tau(*)^{\mathcal{A}}(\overline{a}),$$

where $\overline{a}$ is a n-ary tuple of elements in A. Note that, by induction on the construction of formulas, we can extend τ to a map defined for $\mathcal{L}$-terms in general as follows. Given two infinite cardinals κ and λ such that $\kappa \leqslant \lambda$ and a formula $\varphi \in \mathrm{Fm}_{\mathcal{L}}(\kappa)$, we define the formula $\tau(\varphi) \in \mathrm{Fm}_{\mathcal{L}'}(\lambda)$ recursively: if $\varphi = x_\alpha$, for some $\alpha < \kappa$, we set $\tau(\varphi) := x_\alpha$ and, if $\varphi = *(\psi_1, \dots, \psi_n)$, for an n-ary basic operation $*$ from $\mathcal{L}$ and $\psi_1, \dots, \psi_n \in \mathrm{Fm}_{\mathcal{L}}(\kappa)$, we let $\tau(\varphi) := \tau(*)(\tau(\psi_1), \dots, \tau(\psi_n))$. Then, for every $\varphi(\overline{x}) \in \mathrm{Fm}_{\mathcal{L}}(\kappa)$ and every $\overline{a} \in A$, it makes sense to define:

$$\varphi^{\mathcal{A}^\tau}(\overline{a}) := \tau(\varphi)^{\mathcal{A}}(\overline{a}).$$

In general, given a class $\mathbb{K}$ of matrices, we denote by $\mathbb{K}^\tau$ the class formed by the translations by τ of the elements of $\mathbb{K}$, in the sense defined above. The following properties will be useful in the future:

Proposition 2.37. *Let $\tau : \mathcal{L} \to \mathcal{L}'$ be a translation.*

(i) If $\mathcal{A}$ and $\mathcal{B}$ are two $\mathcal{L}'$-algebras such that $\mathcal{A}$ is embeddable into $\mathcal{B}$ then $\mathcal{A}^\tau$ is embeddable into $\mathcal{B}^\tau$. In general, $\mathbb{S}(\mathbb{K})^\tau \subseteq \mathbb{S}(\mathbb{K}^\tau)$ for a class $\mathbb{K}$ of $\mathcal{L}'$-matrices.

(ii) For a family $\{\mathcal{A}_i \mid i \in I\}$ of $\mathcal{L}'$-algebras it holds that $(\prod_{i\in I} \mathcal{A}_i)^\tau = \prod_{i \in I} \mathcal{A}_i^\tau$. In general, $\mathbb{P}(\mathbb{K})^\tau \subseteq \mathbb{P}(\mathbb{K}^\tau)$, for a class $\mathbb{K}$ of $\mathcal{L}'$-matrices.

(iii) Let $\{\mathcal{A}_i \mid i \in I\}$ be a family of $\mathcal{L}'$-algebras, U an ultrafilter on I and denote by θ_U the corresponding congruence. Then, it holds that $(\prod_{i\in I} \mathcal{A}_i/\theta_U)^\tau = \prod_{i\in I} \mathcal{A}_i^\tau/\theta_U$. In general, $\mathbb{P}_{\mathrm{U}}(\mathbb{K})^\tau \subseteq \mathbb{P}_{\mathrm{U}}(\mathbb{K}^\tau)$, for a class $\mathbb{K}$ of $\mathcal{L}'$-matrices.

Proof.
(i) Let $f : \mathcal{A} \hookrightarrow \mathcal{B}$ be the embedding witnessing our assumption. We claim that f also witnesses the embeddability of $\mathcal{A}^\tau$ into $\mathcal{B}^\tau$. Now, note that the injectivity remains and, moreover, given a n-ary $\mathcal{L}$-term φ and arbitrary elements $a_1, \dots, a_n \in A$, we have that

$$f(\varphi^{\mathcal{A}^\tau}(a_1, \dots, a_n)) = f(\tau(\varphi)^{\mathcal{A}}(a_1, \dots a_n)) = \tau(\varphi)^{\mathcal{B}}(f(a_1), \dots, f(a_n))) = \varphi^{\mathcal{B}^\tau}(f(a_1), \dots, f(a_n)).$$

(ii) It is enough to check that terms are interpreted in the same way in both algebras (note that they share universes and their corresponding distinguished subsets). Let φ be an n-ary $\mathcal{L}'$-term and $\alpha_1, \dots, \alpha_n \in \prod_{i\in I} A_i$. Then,

$$\varphi^{(\prod_{i\in I} \mathcal{A}_i)^\tau}(\alpha_1, \dots, \alpha_n) := \tau(\varphi)^{\prod_{i\in I} \mathcal{A}_i}(\alpha_1, \dots, \alpha_n) = (\tau(\varphi)^{\mathcal{A}_i}(\alpha_1(i), \dots, \alpha_n(i)))_{i\in I},$$

and this is equal to $(\varphi^{\mathcal{A}_i^\tau}(\alpha_1(i), \dots, \alpha_n(i)))_{i\in I}$, which is in fact, equal to $\varphi^{\prod_{i\in I}\mathcal{A}_i^\tau}(\alpha_1, \dots, \alpha_n)$, as desired.

(*iii*) Similarly as before, let φ be an n-ary $\mathcal{L}'$-term and $\alpha_1/\theta_U, \dots, \alpha_n/\theta_U \in \prod_{i\in I} A_i/\theta_U$. Then,

$$\varphi^{(\prod_{i\in I}\mathcal{A}_i/\theta_U)^\tau}(\alpha_1/\theta_U, \dots, \alpha_n/\theta_U) := \tau(\varphi)^{\prod_{i\in I}\mathcal{A}_i/\theta_U}(\alpha_1/\theta_U, \dots, \alpha_n/\theta_U) = (\tau(\varphi)^{\mathcal{A}_i}(\alpha_1(i), \dots, \alpha_n(i)))_{i\in I}/\theta_U,$$

and this is equal to $(\varphi^{\mathcal{A}_i^\tau}(\alpha_1(i), \dots, \alpha_n(i)))_{i\in I}/\theta_U$. From here we can proceed similarly as before in order to see that this is actually $\varphi^{\prod_{i\in I}\mathcal{A}_i^\tau/\theta_U}(\alpha_1/\theta_U, \dots, \alpha_n/\theta_U)$.

□

Another desirable property for a translation is that it preserves the relevant semantic information, i.e. the Suszko models. This motivates the notion of interpretation [13]:

Definition 2.38. Let $\vdash, \vdash'$ be two logics. An *interpretation from* $\vdash$ *into* $\vdash'$ is a translation τ from $\mathcal{L}(\vdash)$ to $\mathcal{L}(\vdash')$ that preserves the Suszko models, that is, such that

$$\mathrm{Mod}^{\equiv}(\vdash')^{\tau} \subseteq \mathrm{Mod}^{\equiv}(\vdash).$$

If there is such an interpretation from $\vdash$ to $\vdash'$, we say that $\vdash$ is *interpretable in* $\vdash'$ and we write $\vdash \leqslant \vdash'$. We say that two logics $\vdash, \vdash'$ are *equi-interpretable* when $\vdash \leqslant \vdash' \leqslant \vdash$. We denote the equivalence class of all the logics equi-interpretable with $\vdash$ by $[\![\vdash]\!]$.

Note that this preservation property does not hold for translations in general. On the other hand, a paradigmatic example of interpretation is the following:

Example 2.39. Recall that a logic $\vdash$ is an *extension* of $\vdash'$ if they are both defined in the same language and $\vdash'$ is contained in $\vdash$ as a set. Then, given an arbitrary logic, the identity map is an interpretation of it into any of its extensions. For instance, $\vdash_{\mathrm{IPC}}$ is interpretable into $\vdash_{\mathrm{CPC}}$.

Additionally, we note that $\leqslant$ induces a preorder in the proper class of all logics and a partial order in the collection of the classes $[\![\vdash]\!]$. This allows to define the following *poset*[6]:

Definition 2.40. We define the *poset* Log *of all logics* as the class of all the equivalence classes $[\![\vdash]\!]$, for every logic $\vdash$, ordered by $\leqslant$, where $[\![\vdash]\!] \leqslant [\![\vdash']\!]$ just in case $\vdash \leqslant \vdash'$ holds.

The next proposition gives a necessary condition for a translation to be an interpretation [13, Prop. 3.3]:

Proposition 2.41. *If* τ *is an interpretation from* $\vdash$ *into* $\vdash'$ *then, for an arbitrary* $(\mathcal{A}, F) \in \mathrm{Mod}(\vdash')$ *it holds that* $(\mathcal{A}^\tau, F) \in \mathrm{Mod}(\vdash)$.

We wish to characterize the relation of interpretability in terms of two new notions. First, remember that an algebra is *term-equivalent* to another one when, roughly, each of its basic operations can be written as a term-function of the other and vice-versa. We can extend this idea to the general situation for logics:

Definition 2.42. Let $\vdash, \vdash'$ be two logics. We say that $\vdash$ and $\vdash'$ are *term-equivalent* if there are two interpretations τ and σ witnessing, respectively, $\vdash \leqslant \vdash'$ and $\vdash' \leqslant \vdash$ such that:

[6] Strictly speaking, such class is not a poset since its universe is not a set.

(i) For every $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash')$, $(\mathcal{A}, F) = (\mathcal{A}^{\tau\sigma}, F)$.

(ii) For every $(\mathcal{B}, G) \in \mathrm{Mod}^{\equiv}(\vdash)$, $(\mathcal{B}, G) = (\mathcal{B}^{\sigma\tau}, G)$.

Definition 2.43. Given two logics $\vdash$ and $\vdash'$, we say that $\vdash'$ is a *compatible expansion of* $\vdash$ if $\mathcal{L}(\vdash) \subseteq \mathcal{L}(\vdash')$ and the $\mathcal{L}(\vdash)$-reducts of the Suszko models of $\vdash'$ are Suszko models of $\vdash$.

The next result tells us that, in some sense, *interpretations come in halves* [13, Prop. 3.8]:

Fact 2.44. *Consider two logics* $\vdash, \vdash'$. *Then,* $\vdash \leqslant \vdash'$ *if and only if* $\vdash'$ *is term-equivalent to a compatible expansion of* $\vdash$.

3 Non-indexed product of logics

Non-indexed products of algebras play an important role in the study of interpretations between varieties (see, e.g., [28], [9], [18]). In this chapter we will review the basic theory of non-indexed products of logics and their relevance in the analysis of the poset of all logics [13], [15], [14]. One crucial concept for the aforementioned study is that of *non-indexed* product of varieties. In a nutshell, the non-indexed product of two varieties is a syntactical universal construction, i.e. a general way of merging both of them, even when their similarity types are different.

Let us be more precise. Given an indexed family $(\mathcal{A}_i \mid i \in I)$ of algebras (not necessarily similar), its non-indexed product is defined in [28] to be the algebra $\langle \mathcal{A}_i \mid i \in I \rangle$ with universe $\prod_{i \in I} \mathcal{A}_i$ that has an n-ary operation p corresponding to each indexed family $(p_i \mid i \in I)$ of n-ary polynomials p_i in the language of $\mathcal{A}_i$ defined as follows:

$$p(\overline{a}_1, \dots \overline{a}_n) := (p_i(\overline{a}_1(i), \dots, \overline{a}_n(i)))_{i \in I},$$

where $\overline{a}_1, \dots, \overline{a}_n \in \prod_{i \in I} \mathcal{A}_i$. From here, given an indexed family $(V_i \mid i \in I)$ of varieties (again, they are not required to share a common language), its non-indexed product is defined as the variety

$$\mathbf{V}(\{\langle \mathcal{A}_i \mid i \in I \rangle \mid \mathcal{A}_i \in V_i\}).$$

Note that all such non-indexed products have the same language by definition.

In this chapter we wish, following [13], to define the notion of non-indexed product for the case of logics. Additionally, we will present a characterization of the Suszko models of this construction first given in this same paper.

3.1 Definition and basic properties

We begin with the following construction for (algebraic) languages:

Definition 3.1. Given a family of languages $\{\mathcal{L}_i \mid i \in I\}$, we will denote by $\bigotimes_{i \in I} \mathcal{L}_i$ the language whose n-ary symbols $*$ are defined as follows:

$$* := (\varphi_i(x_1, \dots, x_n))_{i \in I},$$

where $\varphi_i(\overline{x}) \in \mathrm{Fm}_{\mathcal{L}_i}(\omega)$, for each $i \in I$.

The main concept of this chapter is the following (see [28] and [18]):

Definition 3.2. Consider a family $\{\mathcal{A}_i \mid i \in I\}$, where each $\mathcal{A}_i$ is an $\mathcal{L}_i$-algebra, for $i \in I$. The *non-indexed product of* such a family is the $\bigotimes_{i\in I} \mathcal{L}_i$-algebra $\bigotimes_{i\in I} \mathcal{A}_i$ defined as follows:

(i) The universe of $\bigotimes_{i\in I} \mathcal{A}_i$ is $\prod_{i\in I} A_i$.

(ii) Each n-ary symbol $(\varphi_i(x_1, \dots, x_n))_{i\in I}$ of $\bigotimes_{i\in I} \mathcal{L}_i$ is interpreted in $\bigotimes_{i\in I} \mathcal{A}_i$ as follows:

$$(\varphi_i(x_1,\dots,x_n))_{i\in I}^{\bigotimes_{i\in I}\mathcal{A}_i}(\overline{a}_1,\dots,\overline{a}_n) := (\varphi_i^{\mathcal{A}_i}(\overline{a}_1(i),\dots,\overline{a}_n(i)))_{i\in I}.$$

This notion can be extended naturally to matrices:

Definition 3.3. Given a family of matrices $\{(\mathcal{A}_i, F_i) \mid i \in I\}$, where each $\mathcal{A}_i$ is an $\mathcal{L}_i$-algebra, its *non-indexed product* is the matrix $(\bigotimes_{i\in I} \mathcal{A}_i, \prod_{i\in I} F_i)$. More generally, if $\mathbb{K}_i$ is a class of $\mathcal{L}_i$-matrices, for each $i \in I$, we define

$$\bigotimes_{i\in I}\mathbb{K}_i := \mathbb{I}\left\{\bigotimes_{i\in I}(\mathcal{A}_i,F_i) \mid (\mathcal{A}_i,F_i)\in\mathbb{K}_i, i\in I\right\}.$$

The next results [28, Lemmas 1.10 and 1.9, resp.] will describe some essential properties of the non-indexed product of matrices:

Lemma 3.4. *Suppose that $(\mathcal{B}, G) \leqslant (\mathcal{A}, F) := (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n, F_1 \times \cdots \times F_n)$. Then, there are some submatrices $(\mathcal{B}_i, G_i) \leqslant (\mathcal{A}_i, F_i)$, for $i = 1, \dots, n$, such that $(\mathcal{B}, G) = (\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n, G_1 \times \cdots \times G_n)$. Furthermore, every matrix of this form is a submatrix of $(\mathcal{A}, F)$.*

Proof. For $i = 1, \dots, n$, let $\mathcal{B}_i := f_i(\mathcal{B})$, where $f_i : \mathcal{B} \hookrightarrow \mathcal{A} \to \mathcal{A}_i$ and where the second arrow is the i-th projection map. Then, $\mathcal{B} \subseteq \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$ by how we have constructed these $\mathcal{B}_i$. On the other hand, let $\overline{b} := (b_1, \dots, b_n) \in B_1 \times \cdots \times B_n$. Note that, for each $i = 1, \dots, n$, there is some $\overline{a}_i \in B$ such that $\overline{a}_i(i) = b_i$. Now, consider the n-ary term $p_i(\overline{x})$ of the language of $\mathcal{A}_i$ given by $p_i(\overline{x}) := x_i$. Then, we can define the n-ary term of the language of $\mathcal{A}$ given by $(p_1, \dots, p_n)$. Now,

$$p^{\mathcal{A}}(\overline{a}_1,\dots,\overline{a}_n) = (p_i^{\mathcal{A}_i}(\overline{a}_1(i),\dots,\overline{a}_n(i)))_{i\leqslant n} = (\overline{a}_i(i))_{i\leqslant n} = (b_i)_{i\leqslant n} = \overline{b}.$$

But then, $\overline{b} \in B$, since $\mathcal{B} \leqslant \mathcal{A}$ and $\overline{a}_1, \dots, \overline{a}_n \in B$. Hence, we have seen that $\mathcal{B} = \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n$. It only remains to see the form of G, namely, $G = F \cap B = (F_1 \times \cdots \times F_n) \cap (B_1 \times \cdots \times B_n) = (F_1 \cap B_1) \times \cdots \times (F_n \cap B_n)$, so we can define $G_i := F_i \cap B_i$ in order to obtain the desired result.
The second part is proven as follows. Take a submatrix $(\mathcal{B}_i, G_i) \leqslant (\mathcal{A}_i, F_i)$, for $i = 1, \dots, n$. Now, consider a basic m-ary operation $*$ in $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ and n-tuples $\overline{b}_1, \dots, \overline{b}_m \in B_1 \times \cdots \times B_n$. By definition, we know that $*$ is in fact an n-tuple of m-ary terms, each in the corresponding language of $\mathcal{B}_i$, that is, we can write $* = (\varphi_1, \dots, \varphi_n)$. Hence,

$$*^{\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_n}(\overline{b}_1,\dots,\overline{b}_m) = (\varphi_1,\dots,\varphi_n)^{\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_n}(\overline{b}_1,\dots,\overline{b}_m) = (\varphi^{\mathcal{A}_i}(\overline{b}_1(i),\dots,\overline{b}_m(i)))_{i\leqslant n}.$$

We only need to note that $\overline{b}_1(i), \ldots, \overline{b}_m(i) \in B_i$ and remember that each $\mathcal{B}_i$ is a subalgebra of $\mathcal{A}_i$ in order to see that $\varphi^{\mathcal{A}_i}(\overline{b}_1(i), \ldots, \overline{b}_m(i)) \in B_i$. Then, $(\varphi^{\mathcal{A}_i}(\overline{b}_1(i), \ldots, \overline{b}_m(i)))_{i \leqslant n} \in B_1 \times \cdots \times B_n$, as we wanted. On the other hand, since our initial definition implies that $G_i = F_i \cap B_i$, for each $i = 1, \ldots, n$, we also obtain that $G_1 \times \cdots \times G_n = (F_1 \times \cdots \times F_n) \cap (B_1 \times \cdots \times B_n)$. Therefore, we have seen that $(\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n, G_1, \ldots, G_n) \leqslant (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n, F_1 \times \cdots \times F_n)$.

□

Lemma 3.5. *Binary direct products commute with binary non-indexed products, i.e. there is an isomorphism* $\prod_{j \in J} \bigotimes_{i \in I} (\mathcal{A}_i^j, F_i^j) \cong \bigotimes_{i \in I} \prod_{j \in J} (\mathcal{A}_i^j, F_i^j)$.

Proof. Consider the map $f : \prod_{j \in J} \bigotimes_{i \in I} (\mathcal{A}_i^j, F_i^j) \to \bigotimes_{i \in I} \prod_{j \in J} (\mathcal{A}_i^j, F_i^j)$ given by $f(\overline{a})(i)(j) := \overline{a}(j)(i)$. Let us see that it is, in fact, an isomorphism; we have to check that $\prod_{j \in J} \bigotimes_{i \in I} \mathcal{A}_i^j \to \bigotimes_{i \in I} \prod_{j \in J} \mathcal{A}_i^j$ given by the same rule as before is a strict isomorphism. First, note that it is a well-defined map. Additionally, it is injective: given $\overline{a}, \overline{b} \in \prod_{j \in J} \bigotimes_{i \in I} \mathcal{A}_i^j$ such that $f(\overline{a}) = f(\overline{b})$, we have that $\overline{a}(j)(i) = \overline{b}(j)(i)$, for each $i \in I, j \in J$. Hence, $\overline{a}(j) = \overline{b}(j)$ for each $j \in J$ and therefore $\overline{a} = \overline{b}$. On the other hand, given a tuple $\overline{c} \in \bigotimes_{i \in I} \prod_{j \in J} \mathcal{A}_i^j$, we can write $\overline{c} = ((c_i^j)_{j \in J})_{i \in I}$ and from here it is clear that we can select $((c_i^j)_{i \in I})_{j \in J} \in \prod_{j \in J} \bigotimes_{i \in I} \mathcal{A}_i^j$ to be sent to $\overline{c}$, so f is a bijection.
Now, that f preserves the operations can be seen by comparing how terms are interpreted in the two algebras. First, let $\overline{a}_1, \ldots, \overline{a}_n \in \prod_{j \in J} \bigotimes_{i \in I} \mathcal{A}_i^j$ and consider an n-ary term $*$ in this algebra. It is clear that $* = (\varphi_i)_{i \in I}$, where φ_i is an n-ary term in $\mathcal{A}_i$. Then,

$$(\varphi_i)_{i \in I}^{\prod_{j \in J} \bigotimes_{i \in I} \mathcal{A}_i^j}(\overline{a}_1, \ldots, \overline{a}_n) = ((\varphi_i)_{i \in I}^{\bigotimes_{i \in I} \mathcal{A}_i^j}(\overline{a}_1(j), \ldots, \overline{a}_n(j)))_{j \in J} = ((\varphi^{\mathcal{A}_i^j}(\overline{a}_1(j)(i), \ldots, \overline{a}_n(j)(i)))_{i \in I})_{j \in J}.$$

Now let $\overline{a}_1, \ldots, \overline{a}_n \in \bigotimes_{i \in I} \prod_{j \in J} \mathcal{A}_i^j$. An n-ary term in this algebra will consist in a tuple $(\varphi_i)_{i \in I}$ if n-ary terms in $\prod_{j \in J} \mathcal{A}_i^j$. Then,

$$(\varphi)_{i \in I}^{\bigotimes_{i \in I} \prod_{j \in J} \mathcal{A}_i^j}(\overline{a}_1, \ldots, \overline{a}_n) = (\varphi_i^{\prod_{j \in J} \mathcal{A}_i^j}(\overline{a}_1(i), \ldots, \overline{a}_n(i)))_{i \in I} = ((\varphi_i^{\mathcal{A}_i^j}(\overline{a}_1(i)(j), \ldots, \overline{a}_n(i)(j)))_{j \in J})_{i \in I}.$$

Finally, that f is strict is equivalent to stating that $\overline{a} \in \prod_{j \in J} \prod_{i \in I} F_i^j$ if and only if $f(\overline{a}) \in \prod_{i \in I} \prod_{j \in J} F_i^j$, and this is clear from the definition of f.

□

As a consequence, we obtain the following:

Proposition 3.6. *Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two classes of similar matrices (respectively, algebras) which are closed under subdirect products, then so is the class $\mathbb{K}_1 \otimes \mathbb{K}_2$.*

Proof. Let $(\mathcal{A}, F) \leqslant \prod_{i \in I} (\mathcal{B}_i, G_i)$ be a subdirect product, where $(\mathcal{B}_i, G_i) \in \mathbb{K}_1 \otimes \mathbb{K}_2$. Then, for each $i \in I$, $(\mathcal{B}_i, G_i) = (\mathcal{B}_i^1, G_i^1) \otimes (\mathcal{B}_i^2, G_i^2)$, where $(\mathcal{B}_i^j, G_i^j) \in \mathbb{K}_j$, for $j = 1, 2$. Then, by the previous Lemma,

$$\prod_{i \in I} (\mathcal{B}_i, G_i) = \prod_{i \in I} (\mathcal{B}_i^1, G_i^1) \otimes (\mathcal{B}_i^2, G_i^2) \cong \prod_{i \in I} (\mathcal{B}_i^1, G_i^1) \otimes \prod_{i \in I} (\mathcal{B}_i^2, G_i^2).$$

Hence, by Lemma 3.4, $(\mathcal{A}, F) = (\mathcal{A}_1, F_1) \otimes (\mathcal{A}_2, F_2)$, where $(\mathcal{A}_j, F_j) \leqslant \prod_{i \in I} (\mathcal{B}_i^j, G_i^j)$ for $j = 1, 2$. Therefore, our initial assumption can be read as follows: for every $i \in I$, the composition

$$p_i \circ \iota : (\mathcal{A}_1, F_1) \otimes (\mathcal{A}_2, F_2) \to (\mathcal{B}_i^1, G_i^1) \otimes (\mathcal{B}_i^2, G_i^2)$$

is surjective. Now, if there were some $i \in I$ such that $p_i^1 \circ \iota : (\mathcal{A}_1, F_1) \to (\mathcal{B}_i^1, G_i^1)$ failed to be surjective, so it would fail to be $p_i \circ \iota$, and the same argument applies for the case $j = 2$. Hence, each of these subalgebras is a subdirect product and thus, by the hypothesis on $\mathbb{K}_j$, we have seen that $(\mathcal{A}_j, F_j) \in \mathbb{K}_j$. Therefore, $(\mathcal{A}, F) \in \mathbb{K}_1 \otimes \mathbb{K}_2$, as desired.

□

Remark 3.7. Suppose that we have two logics $\vdash_1$ and $\vdash_2$. Then, we consider the *canonical translations*

$$\iota_1 : \mathcal{L}(\vdash_1) \to \mathcal{L}(\vdash_1 \otimes \vdash_2) : \varphi(\overline{x}) \mapsto (\varphi(\overline{x}), x_1) := \varphi_L(\overline{x}),$$

$$\iota_2 : \mathcal{L}(\vdash_2) \to \mathcal{L}(\vdash_1 \otimes \vdash_2) : \varphi(\overline{x}) \mapsto (x_1, \varphi(\overline{x})) =: \varphi_R(\overline{x}),$$

that map basic n-ary operations of $\mathcal{L}(\vdash_1)$ and $\mathcal{L}(\vdash_2)$ to n-ary terms of $\mathcal{L}(\vdash_1 \otimes \vdash_2)$, respectively. We also have the following *dot operation* in $\mathcal{L}(\vdash_1 \otimes \vdash_2)$ [14]:

$$(x_1, y_1) \cdot (x_2, y_2) := (\pi_1^2(x_1, y_1), \pi_2^2(x_2, y_2)),$$

where, given an n-tuple $\overline{x}$ of variables, we define $\pi_i^n(\overline{x}) := x_i$, for each $i \leqslant n$. Now, in each case, let φ be n-ary basic operations of $\mathcal{L}(\vdash_1)$ and $\mathcal{L}(\vdash_2)$, respectively. Then, given an $\mathcal{L}(\vdash_1)$-algebra $\mathcal{A}$, an $\mathcal{L}(\vdash_2)$-algebra $\mathcal{B}$, and elements $(a_1, b_1), \ldots, (a_{n+m}, b_{n+m}) \in A \times B$, we have that

$$\varphi_L^{\mathcal{A} \otimes \mathcal{B}}((a_1, b_1), \ldots, (a_n, b_n)) = (\varphi^{\mathcal{A}}(a_1, \ldots, a_n), b_1),$$

$$\varphi_R^{\mathcal{A} \otimes \mathcal{B}}((a_1, b_1), \ldots, (a_m, b_m)) = (a_1, \varphi^{\mathcal{B}}(b_1, \ldots, b_m)),$$

$$(a_1, b_1) \cdot^{\mathcal{A} \otimes \mathcal{B}} (a_2, b_2) = (a_1, b_2).$$

Analogous dot operations can be defined for greater arities. For example, consider:

$$(x_1, \ldots, x_n) \cdot_i (y_1, \ldots, y_n) := (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n),$$

for some $i \leqslant n$, which is, coordinate by coordinate, a projection. Hence, we can define the corresponding basic operation in, say, $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$, as follows:

$$(a_1, \ldots, a_n) \cdot_i^{\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n} (b_1, \ldots, b_n) := (b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_n).$$

With the aid of these operations, we can prove the following fact:

Lemma 3.8. *Fix some $n \in \mathbb{N}$. For each $i \leqslant n$, let $\mathcal{A}_i, \mathcal{B}_i$ be two $\mathcal{L}_i$-algebras and denote by $\mathbf{1}_i$ the trivial $\mathcal{L}_i$-algebra. If, for some $i \leqslant n$, $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ embeds into $\mathcal{B}_1 \otimes \cdots \otimes \mathbf{1}_i \otimes \cdots \otimes \mathcal{B}_n$, where $\mathbf{1}_i$ occurs in the i-th position, then $\mathcal{A}_i = \mathbf{1}_i$.*

Proof. Assume that $f : \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \hookrightarrow \mathcal{B}_1 \otimes \cdots \otimes \mathbf{1}_i \otimes \cdots \otimes \mathcal{B}_n$ and let us see that $\mathcal{A}_i$ is trivial. Suppose that we have two elements $a, b \in A_i$. It is enough to check that a and b are equal. Select the elements $a_j \in A_j$ where $j \neq i$. Now, consider the dot-operation defined above:

$$(x_1, \ldots, x_n) \cdot_i (y_1, \ldots, y_n) := (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n).$$

Then,

$$(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \cdot_i (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n).$$

Now, such equation, being $\cdot_i$ a basic operation, must be preserved by the embedding f. Denote by $(c_1, \ldots, c_{i-1}, 1, c_{i+1}, \ldots, c_n)$ and $(d_1, \ldots, d_{i-1}, 1, d_{i+1}, \ldots, d_n)$ the images by f of $(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n)$ and $(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$, respectively. Then, using the previous equality, we obtain:

$$f((a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)) = (d_1, \ldots, d_{i-1}, 1, d_{i+1}, \ldots, d_n) =$$

$$= (c_1, \ldots, c_{i-1}, 1, c_{i+1}, \ldots, c_n) \cdot_i (d_1, \ldots, d_{i-1}, 1, d_{i+1}, \ldots, d_n) =$$

$$= f((a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n)) \cdot_i f((a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)) =$$

$$= f((a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \cdot_i (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n))$$

$$= f((a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n)),$$

so we have that $f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n)$. By the injectivity of f, $(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) = (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$ and thus $a = b$.

□

3.2 Non-indexed subdirect products of matrices

Here we will describe certain properties of the non-indexed product that allow us to compare its behaviour to that of the subdirect product.

Definition 3.9. A submatrix $(\mathcal{A}, F) \leqslant \bigotimes_{i \in I} (\mathcal{A}_i, F_i)$ is said to be a *non-indexed subdirect product of the family* $\{(\mathcal{A}_i, F_i) \mid i \in I\}$ if the projection maps $\pi_j : A \to A_j$ are surjective. In this case, we write $(\mathcal{A}, F) \subseteq_{\mathrm{sd}} \bigotimes_{i \in I} (\mathcal{A}_i, F_i)$. If $(\mathcal{A}, F)$ is isomorphic to a non-indexed subdirect product of $\{(\mathcal{A}_i, F_i) \mid i \in I\}$, we write $(\mathcal{A}, F) \leqslant_{\mathrm{sd}} \bigotimes_{i \in I} (\mathcal{A}_i, F_i)$.

We start with a basic property of the notion that we have just defined:

Lemma 3.10. *If* $(\mathcal{A}, F) \subseteq_{\mathrm{sd}} \bigotimes_{i \in I} (\mathcal{A}_i, F_i)$ *and* $F \neq \varnothing$*, then for arbitrary* $a, b \in A$*,*

$$(a, b) \in \Omega^{\mathcal{A}} F \text{ if and only if } (a(i), b(i)) \in \Omega^{\mathcal{A}_i} F_i, \text{ for every } i \in I.$$

Proof. We will constantly use the characterization from Proposition 2.6. The implication from right to left is clear: given $a, b \in A$, a unary polynomial function $p(x)$ of $\mathcal{A}$ is a formula $\varphi(x, \overline{c})$, where $\overline{c} := (c_1, \ldots, c_m)$

is a sequence from A. Since $\varphi(x,\overline{c})$ is a formula in the language of $\bigotimes_{i\in I}\mathcal{A}_i$, we also know that it is of the form $(\psi_i)_{i\in I}(x,\overline{c})$, where ψ_i is a term, for each $i\in I$. Moreover, a variable x in this language is a tuple of variables $(x_i)_{i\in I}$, each corresponding to the language of each factor of the product. Hence, each ψ_i is of the form $\psi_i(x_i,c_1(i),\dots,c_m(i))$, that is, a unary polynomial function $q_i(x_i)$ in the corresponding language. Then, $p(a)\in F$ if and only if $p(a)\in A\cap\prod_{i\in I}F_i$, that is, iff $q_i(a(i))\in F_i$, and from here we can apply our hypothesis and follow the same steps in order to obtain that $p(b)\in F$.

Let us see the implication from left to right. Suppose that $(a,b)\in\Omega^{\mathcal{A}}F$. We have to see that, given an arbitrary unary polynomial function $p(x)$ of $\mathcal{A}_j$, it holds that $p_j(a(j))\in F_j$ iff $p_j(b(j))\in F_j$, for each $j\in I$. Hence, consider a formula $\varphi(x,\overline{y})$ in $\mathcal{A}_j$ and an n-tuple tuple $\overline{e}$ in A_j such that $\varphi^{\mathcal{A}_j}(\overline{a}(j),\overline{e})\in F_j$. Now, since the projection map $\pi_j:A\to A_j$ is surjective by assumption, we can find tuples $\overline{e}_1,\dots,\overline{e}_n$ in A such that their j-th components are e_j, respectively. Since $F\neq\varnothing$, it contains some element, say $\overline{e}$. Consider the basic operation of $\mathcal{A}$ given by

$$\psi(x,\overline{y},z):=(\psi_i(x,\overline{y},z)\,|\,i\in I),$$

where $\psi_i:=z$ if $i\neq j$ and $\psi_j:=\varphi$. Then, by definition, $\psi(a,\overline{e}_1,\dots,\overline{e}_n,\overline{e})(i)$ equals $\varphi^{\mathcal{A}_j}(a(j),\overline{e})$ if $i=j$ and equals $\overline{e}(i)$ otherwise. Now, since $\varphi^{\mathcal{A}_j}(a(j),\overline{e})\in F_j$ and $\overline{e}\in F$, we have that each i-th component of $\psi(a,\overline{e}_1,\dots,\overline{e}_n,\overline{e})$ belongs to F_i and, thus,

$$\psi(a,\overline{e}_1,\dots,\overline{e}_n,\overline{e})\in F.$$

But remember that $(a,b)\in\Omega^{\mathcal{A}}F$, so we have that $\psi(b,\overline{e}_1,\dots,\overline{e}_n,\overline{e})\in F$ and, moreover,

$$\varphi^{\mathcal{A}_j}(b(j),\overline{e})=\psi(b,\overline{e}_1,\dots,\overline{e}_n,\overline{e})(j)\in F_j,$$

and this implies that $(a(j),b(j))\in\Omega^{\mathcal{A}_j}F_j$, as desired.

□

Proposition 3.11. *Let $(\mathcal{A},F)\subseteq_{\mathrm{sd}}\bigotimes_{i\in I}(\mathcal{A}_i,F_i)$ and $F\neq\varnothing$. Then, the following holds:*

(i) If $(\mathcal{A}_i,F_i)$ is reduced, for each $i\in I$, then so is $(\mathcal{A},F)$.

(ii) $(\mathcal{A},F)^*\leqslant_{\mathrm{sd}}\bigotimes_{i\in I}(\mathcal{A}_i,F_i)^*$.

Proof. For (i), the assumption gives us that $\Omega^{\mathcal{A}_i}F_i=id_{A_i}$, for each $i\in I$. Hence, given $(a,b)\in\Omega^{\mathcal{A}}F$, the previous Lemma tells us that, for each $i\in I$, $(a(i),b(i))\in\Omega^{\mathcal{A}_i}F_i=id_{A_i}$ so $a(i)=b(i)$. Therefore, $a=b$ and $\Omega^{\mathcal{A}}F=id_A$, as desired.

For (ii), consider the map $f:(\mathcal{A},F)^*\to\bigotimes_{i\in I}(\mathcal{A}_i,F_i)^*$ defined by the assignation $a/\Omega^{\mathcal{A}}F\mapsto(a(i)/\Omega^{\mathcal{A}_i}F_i\,|\,i\in I)$. Note that the previous Lemma tells us that f is well-defined and an embedding. But then, since $(\mathcal{A},F)\subseteq_{\mathrm{sd}}\bigotimes_{i\in I}(\mathcal{A}_i,F_i)$, we also have that $(\mathcal{A},F)^*\leqslant_{\mathrm{sd}}\bigotimes_{i\in I}(\mathcal{A}_i,F_i)^*$.

□

Now we can prove the following important fact:

Proposition 3.12. *Let $\{\vdash_i \mid i \in I\}$ be a family of logics and $(\mathcal{A}, F)$ a matrix with $F \neq \varnothing$. Then, $(\mathcal{A}, F) \in \mathbb{R}(\mathrm{Mod}(\bigotimes_{i\in I} \vdash_i))$ if and only if we can find, for each $i \in I$, a model $(\mathcal{A}_i, F_i) \in \mathbb{R}(\mathrm{Mod}(\vdash_i))$ such that $(\mathcal{A}, F) \leqslant_{\mathrm{sd}} \bigotimes_{i\in I}(\mathcal{A}_i, F_i)$.*

Proof. Let us start with the implication from right to left. By the definition of $\bigotimes_{i\in I} \vdash_i$ it follows that $\bigotimes_{i\in I}(\mathcal{A}_i, F_i)$ is a model of $\bigotimes_{i\in I} \vdash_i$. But submatrices of models are still models: hence, $(\mathcal{A}, F) \in \mathrm{Mod}(\bigotimes_{i\in I} \vdash_i)$. Now, we only need to apply the previous Corollary to the assumption that each $(\mathcal{A}_i, F_i)$, for $i \in I$, is reduced in order to obtain that $(\mathcal{A}, F)$ is also reduced, as desired.

Now, conversely, consider $(\mathcal{A}, F) \in \mathbb{R}(\mathrm{Mod}(\bigotimes_{i\in I} \vdash_i))$, define $\kappa := \prod_{i\in I} |\mathrm{Fm}(\vdash_i)|$ and let $\mathrm{Fm}(\kappa)$ be the set of formulas of $\bigotimes_{i\in I} \vdash_i$ in κ variables. Therefore, $|\mathrm{Fm}(\kappa)| \leqslant \kappa$ by the definition of the non-indexed product of languages. Moreover, $\bigotimes_{i\in I} \vdash_i$ is, by definition, the logic induced in $\mathrm{Fm}(\kappa)$ by $\bigotimes_{i\in I} \mathrm{Mod}^{\equiv}(\vdash_i)$, so we are in conditions of Fact 2.12. Thus, $(\mathcal{A}, F) \in \mathbb{RSP}_{R,\kappa^+}(\bigotimes_{i\in I} \mathrm{Mod}^{\equiv}(\vdash_i))$. This means that there is a matrix $(\mathcal{B}, G)$, a family of matrices $\{(\mathcal{B}_i^j, G_i^j) \mid i \in I, j \in J\}$ and a κ^+-complete filter H on J such that

$$(\mathcal{B}, G)^* \cong (\mathcal{A}, F); \quad (\mathcal{B}_i^j, G_i^j) \in \mathrm{Mod}^{\equiv}(\vdash_i), \text{ for each } i \in I; \quad (\mathcal{B}, G) \leqslant \left(\prod_{j\in J}\bigotimes_{i\in I}(\mathcal{B}_i^j, G_i^j)\right)/H.$$

Now, remember from Lemma 3.5 that the map $f : \prod_{j\in J}\bigotimes_{i\in I}(\mathcal{B}_i^j, G_i^j) \to \bigotimes_{i\in I}\prod_{j\in J}(\mathcal{B}_i^j, G_i^j)$ given by $f(\overline{a})(i)(j) := \overline{a}(j)(i)$ is an isomorphism.

Claim 1. The map $g : \prod_{j\in J}(\bigotimes_{i\in I}(\mathcal{B}_i^j, G_i^j))/H \to \bigotimes_{i\in I}(\prod_{j\in J}(\mathcal{B}_i^j, G_i^j)/H)$ defined by $g(\overline{a}/H)(i) := f(\overline{a})(i)/H$ is an isomorphism.

Proof of the claim. Let us see that it is well-defined. Let $\overline{a}, \overline{b} \in \prod_{j\in J}(\bigotimes_{i\in I} \mathcal{B}_i^j)$ be such that $\overline{a}/H = \overline{b}/H$, i.e. such that $\{j \in J \mid \overline{a}(j) = \overline{b}(j)\} \in H$. Now, for each $i \in I$, $g(\overline{a}/H)(i) = g(\overline{b}/H)(i)$ holds if and only if $f(\overline{a})(i)/H = f(\overline{b})(i)/H$ does or, in other words, iff

$$\{j \in J \mid f(\overline{a})(i)(j) = f(\overline{b})(i)(j)\} = \{j \in J \mid \overline{a}(j)(i) = \overline{b}(j)(i)\} \in H.$$

But note that this last set contains $\{j \in J \mid \overline{a}(j) = \overline{b}(j)\}$, which belongs to H by hypothesis, so it also belongs to H. Therefore, $g(\overline{a}/H) = g(\overline{b}/H)$, as we wanted.

On the other hand, g is also a surjective map. Suppose that we have $\overline{a} \in \bigotimes_{i\in I}(\prod_{j\in J} \mathcal{B}_i^j/H)$. This means that $\overline{a} = (\overline{b}_i/H)_{i\in I}$. Take now the tuple $(\overline{b}_i)_{i\in I} \in \bigotimes_{i\in I}\prod_{j\in J} \mathcal{B}_i^j$. Because f is an isomorphism, there is some $\overline{c} \in \prod_{j\in J}\bigotimes_{i\in I} \mathcal{B}_i^j$ such that $f(\overline{c}) = (\overline{b}_i)_{i\in I}$. Then, for each $i \in I$,

$$g(\overline{c}/H)(i) = f(\overline{c})(i)/H = (\overline{b}_i)_{i\in I}(i)/H = \overline{b}_i/H,$$

so $g(\overline{c}/H) = \overline{a}$, as desired.

That g preserves the operations and is strict is straightforward keeping in mind that these properties hold for f.

Let us finally check the injectivity. Let $\overline{a}, \overline{b} \in \prod_{j\in J}(\bigotimes_{i\in I}(\mathcal{B}_i^j, G_i^j))$ such that $g(\overline{a}/H) = g(\overline{b}/H)$. This means that $f(\overline{a})(i)/H = f(\overline{b})(i)/H$ for every $i \in I$. Now, $|I| \leqslant \kappa$ by definition so, together with the

κ^+-completeness of H,

$$\{j \in J \mid \overline{a}(j) = \overline{b}(j)\} = \bigcap_{i \in I} \{j \in J \mid \overline{a}(j)(i) = \overline{b}(j)(i)\} = \bigcap_{i \in I} \{j \in J \mid f(\overline{a})(i)(j) = f(\overline{b})(i)(j)\} \in H.$$

But this implies that $\overline{a}/H = \overline{b}/H$, as we wanted to see. ■

This Claim, together with the fact that $(\mathcal{B}, G) \leqslant \left(\prod_{j \in J} \bigotimes_{i \in I} (\mathcal{B}_i^j, G_i^j)\right)/H$, gives us that

$$(\mathcal{B}, G) \leqslant \bigotimes_{i \in I} (\prod_{j \in J} (\mathcal{B}_i^j, G_i^j)/H).$$

Now, this implies that there are $(\mathcal{A}_i, F_i) \in \mathbb{SP}_{R,\kappa^+}(\mathrm{Mod}^{\equiv}(\vdash_i))$ such that $(\mathcal{B}, G) \leqslant_{\mathrm{sd}} \bigotimes_{i \in I} (\mathcal{A}_i, F_i)$. By part (ii) of the previous Corollary, $(\mathcal{A}, F) \leqslant_{\mathrm{sd}} \bigotimes_{i \in I} (\mathcal{A}_i, F_i)^*$, where $(\mathcal{A}_i, F_i)^* \in \mathbb{RSP}_{R,\kappa^+}(\mathrm{Mod}^{\equiv}(\vdash_i))$. It remains to check that $(\mathcal{A}_i, F_i)^* \in \mathbb{R}(\mathrm{Mod}(\vdash_i))$, for each $i \in I$. Clearly, it suffices to prove that $(\mathcal{A}_i, F_i) \in \mathrm{Mod}(\vdash_i)$, for every $i \in I$, but this holds because $\mathrm{Mod}(\vdash_i)$ is $\mathbb{S}$-closed (always) and it is also $\mathbb{P}_{R,\kappa^+}$-closed by Fact 2.18, since $\vdash_i$ is κ^+-compact: the set of formulas of $\vdash_i$ has cardinality below κ by definition and, thus, strictly below κ^+.

□

3.3 Description of the Suszko models

The notion of non-indexed product can be extended to logics as follows [13]:

Definition 3.13. Let $\{\vdash_i \mid i \in I\}$ be a family of logics, each one in language $\mathcal{L}_i$, for $i \in I$. We define the *non-indexed product of* $\{\vdash_i \mid i \in I\}$ as the logic $\bigotimes_{i \in I} \vdash_i$ in the language $\bigotimes_{i \in I} \mathcal{L}_i$ with $\prod_{i \in I} |\mathrm{Fm}(\vdash_i)|$ variables and induced by the class of matrices $\bigotimes_{i \in I} \mathrm{Mod}^{\equiv}(\vdash_i)$.

Remark 3.14. If $I = \varnothing$, we assume that $\bigotimes_{i \in I} \vdash_i$ is the logic in the empty language, with countably many variables and induced by the matrix $(\mathbf{1}, \{1\})$, where $\mathbf{1}$ is the trivial algebra.

The next observation shows that the class of logics with theorems is closed under non-indexed products:

Proposition 3.15. *Let $\{\vdash_i \mid i \in I\}$ be a family of logics. Then, the logic $\bigotimes_{i \in I} \vdash_i$ has theorems if and only if each $\vdash_i$ has theorems.*

Proof. If each $\vdash_i$ has some theorem φ_i, by substitution invariance we can assume that $\varphi_i = \varphi_i(x)$, for every $i \in I$. Then, the formula $\varphi(x) := (\varphi_i(x))_{i \in I}$ is a theorem of $\bigotimes_{i \in I} \vdash_i$. Indeed, $\bigotimes_{i \in I} \vdash_i \varphi(x)$ is equivalent, by definition, to the assertion that, for every $\bigotimes_{i \in I} (\mathcal{A}_i, F_i)$, where $(\mathcal{A}_i, F_i) \in \mathrm{Mod}^{\equiv}(\vdash_i)$, and every valuation $h : \mathrm{Fm}(\bigotimes_{i \in I} \vdash_i) \to \bigotimes_{i \in I} (\mathcal{A}_i, F_i)$, it holds that $h(\varphi) \in \prod_{i \in I} F_i$. Now, we only need to note that $h(\varphi)$ is, component by component, the valuation of a theorem of the corresponding logic, so that each of these components belongs to the corresponding deductive filter.

Now, if $\bigotimes_{i \in I} \vdash_i$ has a theorem $\varphi(\overline{x})$, we know that it is of the form $(\varphi_i(\overline{x}))_{i \in I}$, where $\varphi_i(\overline{x}) \in \mathrm{Fm}(\vdash_i)$. By definition this means, as before, that for an arbitrary $(\mathcal{A}_i, F_i) \in \mathrm{Mod}^{\equiv}(\vdash_i)$ and an arbitrary valuation

$h : \mathrm{Fm}(\bigotimes_{i\in I} \vdash_i) \to \bigotimes_{i\in I}(\mathcal{A}_i, F_i)$, we have that $h(\varphi(\overline{x})) \in \prod_{i\in I} F_i$ and this means that the corresponding valuation of each component $\varphi_i(\overline{x})$ belongs to the deductive filter F_i, so it is in fact a theorem.

□

In this section we wish to present a characterization of the Suszko models of the non-indexed product of a family of logics. This result is crucial for the rest of the work. We will need first to present some technical ideas:

Definition 3.16. Consider a family $\{\mathcal{L}_i \,|\, i \in I\}$ of languages and let $(\mathcal{A}, F)$ be a $\mathcal{L}_j$-algebra, for some $j \in I$. The *product expansion of* $(\mathcal{A}, F)$ is the $\bigotimes_{i\in I} \mathcal{L}_i$-matrix $(\mathcal{A}, F)^\flat := \bigotimes_{i\in I}(\mathcal{A}_i^-, F_i^-)$, where

$$(\mathcal{A}_i^-, F_i^-) := \begin{cases} (\mathcal{A}, F), & \text{if } i = j, \\ (\mathbf{1}_i, \{1_i\}), & \text{the } \mathcal{L}_i\text{-trivial algebra, otherwise.} \end{cases}$$

Remark 3.17. We observe that, if $(\mathcal{A}, F)$ is reduced, then so is $(\mathcal{A}, F)^\flat$. Indeed, if $\Omega^{\mathcal{A}} F = id_A$ then, in particular, we obtain that $F \neq \varnothing$ (otherwise, given $(a, b) \in A^2$, since $p(a) \notin F$ and $p(b) \notin F$ for every polynomial function p, we would have that $\Omega^{\mathcal{A}} F = A^2$) and hence we are in the conditions of Proposition 3.11, because $\prod_{i\in I} F_i^- \neq \varnothing$. Since $(\mathcal{A}, F)^\flat \subseteq_{\mathrm{sd}} (\mathcal{A}, F)^\flat$, the aforementioned Corollary tells us that $(\mathcal{A}, F)^\flat$ is reduced when so is each $(\mathcal{A}_i^-, F_i^-)$, for $i \in I$. But it remains to check that $\Omega^{\mathbf{1}_i}\{1_i\} = id_{\mathbf{1}_i}$ holds for $i \neq j$, which is clear.

Now we can prove the auxiliary result needed for the main result of this section:

Lemma 3.18. *If* $\{\vdash_i \,|\, i \in I\}$ *is a family of logics, then*

$$\mathbb{R}(\mathrm{Mod}(\bigotimes_{i\in I} \vdash_i)) \subseteq \mathbb{P}_{\mathrm{SD}}(\bigotimes_{i\in I} \mathbb{R}(\mathrm{Mod}(\vdash_i))) \subseteq \mathbb{P}_{\mathrm{SD}}\mathbb{R}(\mathrm{Mod}(\bigotimes_{i\in I} \vdash_i)).$$

Proof. Let us start by checking the first inclusion, since the second one is very similar. Let $(\mathcal{A}, F) \in \mathbb{R}(\mathrm{Mod}(\bigotimes_{i\in I} \vdash_i))$. We can distinguish the following cases:

(a) Suppose that $F = \varnothing$. Since we assume that $(\mathcal{A}, F)$ is reduced, Proposition 2.13(i) tells us that $\mathcal{A} = \mathbf{1}$. Now, since this means that $(\mathbf{1}, \varnothing) \in \mathbb{R}(\mathrm{Mod}(\bigotimes_{i\in I} \vdash_i))$, Proposition 2.13(ii) implies, moreover, that the logic $\bigotimes_{i\in I} \vdash_i$ has no theorems. Therefore, by Proposition 3.15, there is some $j \in I$ such that $\vdash_j$ has no theorems. By Proposition 2.13(ii) we obtain that the $\mathcal{L}_j$-matrix $(\mathbf{1}, \varnothing)$ belongs to $\mathbb{R}(\mathrm{Mod}(\vdash_j))$. By Remark 3.17,

$$(\mathcal{A}, F) = (\mathbf{1}, \varnothing)^\flat \in \bigotimes_{i\in I} \mathbb{R}(\mathrm{Mod}(\vdash_i)),$$

as desired.

(b) If $F \neq \varnothing$, we are in the conditions of Proposition 3.12. Hence, there are some $(\mathcal{A}_i, F_i) \in \mathbb{R}(\mathrm{Mod}(\vdash_i))$ such that $(\mathcal{A}, F) \leqslant_{\mathrm{sd}} \bigotimes_{i\in I}(\mathcal{A}_i, F_i)$.

Claim. The map $f : \prod_{i\in I}(\mathcal{A}_i, F_i)^\flat \to \bigotimes_{i\in I}(\mathcal{A}_i, F_i)$ defined by $f(\overline{a})(i) := \overline{a}(i)(i)$, for $i \in I$, is an isomorphism.

Proof of the claim. Note that, by definition, $\prod_{i\in I}(\mathcal{A}_i, F_i)^\flat$ is precisely the product matrix

$$\prod_{j\in I}\bigotimes_{i\in I}(\mathcal{A}_i^{j-}, F_i^{j-}),$$

where $(\mathcal{A}_i^{i-}, F_i^{i-}) = (\mathcal{A}_i, F_i)$ and $(\mathcal{A}_i^{j-}, F_i^{j-}) = (\mathbf{1}_j, \{1_j\})$ otherwise (here $(\mathbf{1}_j, \{1_j\})$ is the corresponding trivial $\mathcal{L}_j$-algebra). Now, given $\overline{a}, \overline{b} \in \prod_{j\in I}\bigotimes_{i\in I} A_i^{j-}$, suppose that $f(\overline{a}) = f(\overline{b})$. Then, for each $i \in I$, $f(\overline{a})(i) = f(\overline{b})(i)$, that is, $\overline{a}(i)(i) = \overline{b}(i)(i)$. But note that $\overline{a}(i)(j) = 1_j = \overline{b}(i)(j)$ if $j \neq i$, so we have seen that $\overline{a} = \overline{b}$ and f is therefore injective. On the other hand, given $\overline{c} \in \prod_{i\in I} A_i$, consider the element $(\langle\overline{c}(i)\rangle)_{i\in I} \in \prod_{j\in I}\prod_{i\in I} A_i^{j-}$, where $\langle\overline{c}(i)\rangle$ is the tuple whose j-th components are 1_j when $j \neq i$ and $c(i)$ otherwise. Then, for $j \in I$,

$$f((\langle\overline{c}(i)\rangle)_{i\in I})(j) = (\langle\overline{c}(i)\rangle)_{i\in I}(j)(j) = \langle\overline{c}(j)\rangle(j) = \overline{c}(j),$$

so $f((\langle\overline{c}(i)\rangle)_{i\in I}) = \overline{c}$, and f is surjective. Finally, let us see that f preserves the operations. Consider the n-ary term $(\varphi_i)_{i\in I}$ in $\prod_{j\in I}\bigotimes_{i\in I}\mathcal{A}_i^{j-}$ and $\alpha_1, \dots, \alpha_n \in \prod_{j\in I}\prod_{i\in I} A_i^{j-}$. Then,

$$f((\varphi_i)_{i\in I}^{\prod_{j\in I}\bigotimes_{i\in I}\mathcal{A}_i^{j-}}(\alpha_1, \dots, \alpha_n)) = f(((\varphi_i)_{i\in I}^{\bigotimes_{i\in I}\mathcal{A}_i^{j-}}(\alpha_1(j), \dots, \alpha_n(j)))_{j\in I})$$

$$= f(((\varphi^{\mathcal{A}_i^{j-}}(\alpha_1(j)(i), \dots, \alpha_n(j)(i)))_{i\in I})_{j\in I}),$$

so, given $i \in I$, $f((\varphi_i)_{i\in I}^{\prod_{j\in I}\bigotimes_{i\in I}\mathcal{A}_i^{j-}}(\alpha_1, \dots, \alpha_n))(i) = ((\varphi^{\mathcal{A}_i^{j-}}(\alpha_1(j)(i), \dots, \alpha_n(j)(i)))_{i\in I})_{j\in I}(i)(i)$ and this is precisely $\varphi^{\mathcal{A}_i^{i-}}(\alpha_1(i)(i), \dots, \alpha_n(i)(i))$, i.e. $\varphi^{\mathcal{A}_i}(\alpha_1(i)(i), \dots, \alpha_n(i)(i))$, by definition of the product expansion. Now,

$$(\varphi_i)_{i\in I}^{\bigotimes_{i\in I}\mathcal{A}_i}(f(\alpha_1), \dots, f(\alpha_n)) = (\varphi^{\mathcal{A}_i}(f(\alpha_1)(i), \dots, f(\alpha_n(i))))_{i\in I} = (\varphi^{\mathcal{A}_i}(\alpha_1(i)(i), \dots, \alpha_n(i)(i)))_{i\in I},$$

as desired. That f is strict is straightforward: if $\alpha \in \prod_{j\in I}\prod_{i\in I} F_i^{j-}$, then, given $i \in I$, $f(\alpha)(i) = \alpha(i)(i) \in F_i^{i-} = F_i$ and $f(\alpha) \in \prod_{i\in I} F_i$, and the converse is analogous. ■

Therefore, $(\mathcal{A}, F) \leqslant \prod_{i\in I}(\mathcal{A}_i, F_i)^\flat$ is a subdirect product, so $(\mathcal{A}, F) \in \mathbb{P}_{\mathrm{SD}}(\bigotimes_{i\in I}\mathbb{R}(\mathrm{Mod}(\vdash_i)))$, as we wanted.

For the second inclusion, let $(\mathcal{A}, F) \in \mathbb{P}_{\mathrm{SD}}(\bigotimes_{i\in I}\mathbb{R}(\mathrm{Mod}(\vdash_i)))$. This means that there are matrices $(\mathcal{B}_i^j, G_i^j) \in \mathbb{R}(\mathrm{Mod}(\vdash_i))$, for each $i \in I$, such that $(\mathcal{A}, F) \leqslant \prod_{j\in J}\bigotimes_{i\in I}(\mathcal{B}_i^j, G_i^j)$ is a subdirect product. Now, since $(\mathcal{B}_i^j, G_i^j) \in \mathbb{R}(\mathrm{Mod}(\vdash_i))$, we already know that $(\mathcal{B}_i^j, G_i^j)^\flat \in \mathbb{R}(\bigotimes_{i\in I}\mathrm{Mod}(\vdash_i))$. By the previous claim,

$$(\mathcal{A}, F) \in \mathbb{IP}_{\mathrm{SD}}\left(\left\{\prod_{i\in I}(\mathcal{B}_i^j, G_i^j)^\flat \mid j \in J\right\}\right) \subseteq \mathbb{ISPP}\left(\left\{(\mathcal{B}_i^j, G_i^j)^\flat \mid i \in I, j \in J\right\}\right)$$

$$\subseteq \mathbb{ISP}\left(\left\{(\mathcal{B}_i^j, G_i^j)^\flat \mid i \in I, j \in J\right\}\right).$$

Moreover, is can be easily checked that the following composition is surjective (f denotes the isomorphism from the claim):

$$(\mathcal{A}, F) \xrightarrow{p_j \circ f^{-1} \circ \iota} \prod_{i \in I} (\mathcal{B}_i^j, G_i^j)^\flat \xrightarrow{p_i} (\mathcal{B}_i^j, G_i^j)^\flat.$$

Hence, $(\mathcal{A}, F) \in \mathbb{P}_{\mathrm{SD}}\mathbb{R}(\mathrm{Mod}(\bigotimes_{i \in I} \vdash_i))$, as desired.

□

Finally, we present the main theorem of this chapter [13, Prop. 4.5]:

Theorem 3.19. *Given a family $\{\vdash_i \mid i \in I\}$ of logics, the following holds:*

$$\mathrm{Mod}^{\equiv}\left(\bigotimes_{i \in I} \vdash_i\right) = \mathbb{P}_{\mathrm{SD}}\left(\bigotimes_{i \in I} \mathrm{Mod}^{\equiv}(\vdash_i)\right).$$

Proof. Let us begin with the first statement. First, we note that, using Lemma 3.18, together with the definition of the Suszko models,

$$\mathrm{Mod}^{\equiv}(\bigotimes_{i \in I} \vdash_i) = \mathbb{P}_{\mathrm{SD}}\mathbb{R}(\bigotimes_{i \in I} \vdash_i) \subseteq \mathbb{P}_{\mathrm{SD}}\mathbb{P}_{\mathrm{SD}}(\bigotimes_{i \in I} \mathbb{R}(\mathrm{Mod}(\vdash_i))) = \mathbb{P}_{\mathrm{SD}}(\bigotimes_{i \in I} \mathbb{R}(\mathrm{Mod}(\vdash_i))) \subseteq \mathbb{P}_{\mathrm{SD}}(\bigotimes_{i \in I} \mathrm{Mod}^{\equiv}(\vdash_i)).$$

Hence, it only remains to check the converse inclusion. Moreover, since the class of Suszko models is closed under $\mathbb{P}_{\mathrm{SD}}$, it is enough to see that

$$\bigotimes_{i \in I} \mathrm{Mod}^{\equiv}(\vdash_i) \subseteq \mathrm{Mod}^{\equiv}(\bigotimes_{i \in I} \vdash_i)$$

holds. For each $i \in I$, let $(\mathcal{A}_i, F_i) \in \mathrm{Mod}^{\equiv}(\vdash_i)$. This, by definition, implies that $(\mathcal{A}_i, F_i) \leqslant \prod_{j \in J_i} (\mathcal{A}_i^j, F_i^j)$ is a subdirect product, for a family $\{(\mathcal{A}_i^j, F_i^j) \mid j \in J_i\} \subseteq \mathbb{R}(\mathrm{Mod}(\vdash_i))$. Without loss of generality we may assume that $J_i = J = J_j$ for every $i, j \in I$ because we can add trivial matrices in each product, if necessary. But then, it follows that

$$\bigotimes_{i \in I} (\mathcal{A}_i, F_i) \leqslant \prod_{j \in J} \bigotimes_{i \in I} (\mathcal{A}_i^j, F_i^j)$$

is a subdirect product. Again by Lemma 3.18,

$$\bigotimes_{i \in I} (\mathcal{A}_i, F_i) \in \mathbb{P}_{\mathrm{SD}}(\bigotimes_{i \in I} \mathbb{R}(\mathrm{Mod}(\vdash_i))) \subseteq \mathbb{P}_{\mathrm{SD}}\mathbb{R}(\mathrm{Mod}(\bigotimes_{i \in I} \vdash_i)) = \mathrm{Mod}^{\equiv}(\bigotimes_{i \in I} \vdash_i).$$

Hence, we have seen the desired inclusion.

□

The binary (and, with some extra steps, finite) case of the previous theorem allows us to simplify the semantics of the Taylorian product even more:

Corollary 3.20. *Given two logics* $\vdash$ *and* $\vdash'$*, the following holds:*

$$\mathrm{Mod}^{\equiv}(\vdash \otimes \vdash') = \mathrm{Mod}^{\equiv}(\vdash) \otimes \mathrm{Mod}^{\equiv}(\vdash').$$

Proof. By Proposition 3.6 we know that $\mathrm{Mod}^{\equiv}(\vdash) \otimes \mathrm{Mod}^{\equiv}(\vdash')$ is closed under subdirect products. Then, by Theorem 3.19, $\mathrm{Mod}^{\equiv}(\vdash \otimes \vdash') = \mathbb{P}_{\mathrm{SD}}(\mathrm{Mod}^{\equiv}(\vdash) \otimes \mathrm{Mod}^{\equiv}(\vdash')) \subseteq \mathrm{Mod}^{\equiv}(\vdash) \otimes \mathrm{Mod}^{\equiv}(\vdash')$.

□

3.4 Infima in Log

The relevance of non-indexed products in the context of the study of the poset of all logics is due to the next observation [13, Thm. 4.6]:

Theorem 3.21. *The infimum of a subset* $\{[\![\vdash_i]\!] \mid i \in I\}$ *of* Log *is* $[\![\bigotimes_{i\in I} \vdash_i]\!]$*. Therefore,* Log *is a set-complete* $\wedge$*-semilattice, i.e., infima of subsets of* Log *exist.*

Proof. First, we have to show that there is an interpretation of $\bigotimes_{i\in I} \vdash_i$ into $\vdash_j$, for each $j \in I$. Fix an arbitrary $j \in I$. Consider the map $\tau : \mathcal{L}(\bigotimes_{i\in I} \vdash_i) \to \mathcal{L}(\vdash_j)$ that sends each n-ary basic operation to its j-th component, which is a n-ary term of $\mathcal{L}(\vdash_j)$. Let $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash_j)$.

Claim 1. $(\mathcal{A}^{\tau}, F) \cong \bigotimes_{i\in I}(\mathcal{A}_i, F_i)$, where $(\mathcal{A}_i, F_i)$ is the trivial $\mathcal{L}(\vdash_i)$-matrix if $i \neq j$ and is $(\mathcal{A}, F)$ otherwise.

Proof of the claim. Consider the map $f : \mathcal{A}^{\tau} \to \bigotimes_{i\in I} \mathcal{A}_i$ that sends each element a to the tuple $\langle a\rangle$ whose components are the only element 1_i of $\mathcal{A}_i$ if $i \neq j$, and a if $i = j$. Clearly, this well-defined map is bijective. That f is strict is also immediate. Let us check that it preserves the operations. Take an n-ary term in $\mathcal{L}(\bigotimes_{i\in I} \vdash_i)$, that is, a tuple $(\varphi_i)_{i\in I}$ of n-ary terms, each one in $\mathcal{L}_i$, and elements $a_1, \ldots, a_n \in A$. Then,

$$f((\varphi_i)_{i\in I}^{\mathcal{A}^{\tau}}(a_1, \ldots, a_n)) = f(\tau((\varphi_i)_{i\in I})^{\mathcal{A}}(a_1, \ldots, a_n)) = f(\varphi_j^{\mathcal{A}}(a_1, \ldots, a_n)) = \langle \varphi_j^{\mathcal{A}}(a_1, \ldots, a_n)\rangle,$$

and

$$(\varphi_i)_{i\in I}^{\bigotimes_{i\in I} \mathcal{A}_i}(f(a_1), \ldots, f(a_n)) = (\varphi_i^{\mathcal{A}_i}(\langle a_1\rangle(i), \ldots, \langle a_n\rangle(i)))_{i\in I} = \langle \varphi_j^{\mathcal{A}}(a_1, \ldots, a_n)\rangle,$$

as we wanted. ■

Now, by Claim 1 and Theorem 3.19,

$$(\mathcal{A}^{\tau}, F) \cong \bigotimes_{i\in I}(\mathcal{A}_i, F_i) \in \bigotimes \mathrm{Mod}^{\equiv}(\vdash_i) \subseteq \mathrm{Mod}^{\equiv}(\bigotimes_{i\in I} \vdash_i).$$

Therefore, τ is an interpretation from $\bigotimes_{i\in I} \vdash_i$ into $\vdash_j$. Thus, we have seen that $[\![\bigotimes_{i\in I} \vdash_i]\!]$ is a lower bound of $\{[\![\vdash_i]\!] \mid i \in I\}$ in Log. It remains to check that it is, in fact, the greatest lower bound of this family.

Suppose that a logic $\vdash$ verifies that $\vdash \leqslant \vdash_i$, for every $i \in I$. Let $\tau_i : \mathcal{L}(\vdash) \to \mathcal{L}(\vdash_i)$ the interpretation that witnesses this, for each $i \in I$. Now, define the map τ as follows: if $*$ is a n-ary symbol in $\mathcal{L}(\vdash)$, let

$$\tau(*) := (\tau_i(*))_{i \in I},$$

where the latter is a n-ary term of $\mathcal{L}(\bigotimes_{i\in I} \vdash_i)$. We claim that τ is an interpretation of $\vdash$ into $\bigotimes_{i\in I} \vdash_i$. Note that it is clearly a translation. On the other hand, let $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\bigotimes_{i\in I} \vdash_i)$. We want to see that $(\mathcal{A}^\tau, F) \in \mathrm{Mod}^{\equiv}(\vdash)$. By Theorem 3.19, we know that $(\mathcal{A}, F) \leqslant \prod_{j\in J}(\bigotimes_{i\in I}(\mathcal{A}_i^j, F_i^j))$ can be put as a subdirect product, with $(\mathcal{A}_i^j, F_i^j) \in \mathrm{Mod}^{\equiv}(\vdash)$.

Claim 2. $(\mathcal{A}^\tau, F) \leqslant \prod_{j\in J}\prod_{i\in I}((\mathcal{A}_i^j)^{\tau_i}, F_i^j)$ is also a subdirect product.
Proof of the claim. First, remember that if $(\mathcal{A}, F) \leqslant \prod_{j\in J}(\bigotimes_{i\in I}(\mathcal{A}_i^j, F_i^j))$ is a subdirect product then each map $p_j \circ \iota : (\mathcal{A}, F) \to \bigotimes_{i\in I}(\mathcal{A}_i^j, F_i^j)$ is surjective. Note that, by Proposition 2.37,

$$(\mathcal{A}, F)^\tau \leqslant \left(\prod_{j\in J}(\bigotimes_{i\in I}(\mathcal{A}_i^j, F_i^j))\right)^\tau.$$

Now, given an n-ary term and $\alpha_1, \ldots, \alpha_n \in \prod_{j\in J}\prod_{i\in I} A_i^j$,

$$\varphi^{(\prod_{j\in J}\bigotimes_{i\in I}\mathcal{A}_i^j))^\tau}(\alpha_1, \ldots, \alpha_n) := \tau(\varphi)^{\prod_{j\in J}\bigotimes_{i\in I}\mathcal{A}_i^j}(\alpha_1, \ldots, \alpha_n) = (\tau(\varphi)^{\bigotimes_{i\in I}\mathcal{A}_i^j}(\alpha_1(j), \ldots, \alpha_n(j)))_{j\in J} =$$

$$= (((\tau_i(\varphi))_{i\in I})^{\bigotimes_{i\in I}\mathcal{A}_i^j}(\alpha_1(j), \ldots, \alpha_n(j)))_{j\in J} = ((\tau_i(\varphi)^{\mathcal{A}_i^j}(\alpha_1(j)(i), \ldots, \alpha_n(j)(i)))_{i\in I})_{j\in J} =$$

$$= ((\varphi^{(\mathcal{A}_i^j)^{\tau_i}}(\alpha_1(j)(i), \ldots, \alpha_n(j)(i))_{i\in I})_{j\in J},$$

and this last term is actually $\varphi^{\prod_{j\in J}\bigotimes_{i\in I}(\mathcal{A}_i^j)^\tau}(\alpha_1, \ldots, \alpha_n)$, as it can be checked by going through similar steps as the ones described above. From here we can clearly see the desired isomorphism with $\prod_{j\in J}\bigotimes_{i\in I}((\mathcal{A}_i^j)^{\tau_i}, F_i^j)$. That it is strict is clear. Moreover, checking that each $p_j \circ \iota$ remains surjective is easy. ■

Now, since each τ_i is an interpretation of $\vdash$ into $\vdash_i$, we know that $((\mathcal{A}_i^j)^{\tau_i}, F_i^j) \in \mathrm{Mod}^{\equiv}(\vdash)$, for each i and each j. Then, by Claim 2,

$$(\mathcal{A}^\tau, F) \in \mathbb{P}_{\mathrm{SD}}\mathbb{P}(\mathrm{Mod}^{\equiv}(\vdash)) = \mathbb{P}_{\mathrm{SD}}(\mathrm{Mod}^{\equiv}(\vdash)),$$

and, since $\mathrm{Mod}^{\equiv}(\vdash)$ is closed under $\mathbb{P}_{\mathrm{SD}}$, we have that $(\mathcal{A}^\tau, F) \in \mathrm{Mod}^{\equiv}(\vdash)$, as desired.

□

Remark 3.22. On the other hand, the existence of suprema in Log is not guaranteed in general [13, Thm. 5.1]. Although it is known that every poset whose universe is a set and in which infima of sets exist is a complete lattice, the proof of this fact relies on the assumption that the universe of such poset is a set and, therefore, cannot be applied to Log.

4 Meet-irreducibility criteria

In this chapter we will study the meet-irreducible elements of Log. The main motivation behind this notion is that a meet-irreducible logic will be, in certain sense, fundamental. Specifically, such a logic cannot be recovered as the non-indexed product of any pair of logics, none of which is equi-interpretable to it. Hence, it seems desirable to obtain some kind of criterion for meet-irreducibility. Let us start by making this notion more precise. Recall that, in virtue of Theorem 3.21, for every pair of logics $\vdash_1$ and $\vdash_2$, the meet of $[\![\vdash_1]\!]$ and $[\![\vdash_2]\!]$ in Log is precisely $[\![\vdash_1 \otimes \vdash_2]\!]$.

Definition 4.1. We say that a logic $\vdash$ is *meet-irreducible* when $[\![\vdash]\!]$ is a meet-irreducible element of Log, i.e., for every pair of logics $\vdash_1$ and $\vdash_2$,

$$[\![\vdash_1 \otimes \vdash_2]\!] = [\![\vdash]\!] \text{ implies that either } \vdash_1 \leqslant \vdash \text{ or } \vdash_2 \leqslant \vdash .$$

In a similar way, $\vdash$ is *meet-prime* when, for every pair of logics $\vdash_1$ and $\vdash_2$,

$$[\![\vdash_1 \otimes \vdash_2]\!] \leqslant [\![\vdash]\!] \text{ implies that either } \vdash_1 \leqslant \vdash \text{ or } \vdash_2 \leqslant \vdash .$$

We can provide more general notions of irreducibility and primeness relativized to a given positive integer k that can be found in [9]:

Definition 4.2. Let $k \in \mathbb{N}$. We say that a logic $\vdash$ *has irreducibility degree* $\leqslant k$ if, given logics $\vdash_0, \dots, \vdash_k$,

$$[\![\vdash_0]\!] \wedge \cdots \wedge [\![\vdash_k]\!] = [\![\vdash]\!] \text{ implies that } [\![\vdash_0]\!] \wedge \cdots \wedge \cancel{[\![\vdash_i]\!]} \wedge \cdots \wedge [\![\vdash_k]\!] \leqslant [\![\vdash]\!], \text{ for some } i \leqslant k.$$

Respectively, a logic $\vdash$ *has Helly number* $\leqslant k$ if, given arbitrary logics $\vdash_0, \dots, \vdash_k$,

$$[\![\vdash_0]\!] \wedge \cdots \wedge [\![\vdash_k]\!] \leqslant [\![\vdash]\!] \text{ implies that } [\![\vdash_0]\!] \wedge \cdots \wedge \cancel{[\![\vdash_i]\!]} \wedge \cdots \wedge [\![\vdash_k]\!] \leqslant [\![\vdash]\!], \text{ for some } i \leqslant k.$$

The *Helly number* of $\vdash$ is the least k such that the above condition holds. Clearly, a logic $\vdash$ is meet-irreducible (resp. meet-prime) if and only if it has irreducibility number (resp. Helly number) 1.

Sometimes, we will restrict these notions to a specific sub-semilattice of Log. More precisely, let L be a sub-semilattice of Log and $\vdash$ a logic such that $[\![\vdash]\!]$ belongs to L. We say that $\vdash$ is meet-prime *in* L when the meet-primeness condition of $\vdash$ holds for arbitrary logics $\vdash_1$ and $\vdash_2$ such that $[\![\vdash_1]\!]$ and $[\![\vdash_2]\!]$ are elements of L.

4.1 Criteria for Log

In this section we will prove our main theorem regarding meet-irreducibility in Log. We will make use of some technical lemmas:

Lemma 4.3. *Suppose that a logic with theorems $\vdash$ satisfies the following conditions:*

(1) There are logics $\vdash_0, \dots, \vdash_k$ such that $[\![\vdash_0 \otimes \cdots \otimes \vdash_k]\!] = [\![\vdash]\!]$.

(2) Every nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash)$ has a substructure of cardinality $p_1 \cdots p_t$ with $t \leqslant k$ and p_i prime.

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ do not have trivial substructures.

Moreover, let τ be the interpretation witnessing that $\vdash_0 \otimes \cdots \otimes \vdash_k \leqslant \vdash$. Then, for every nontrivial matrix $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$ there is some $i \leqslant k$ such that, for every $j \leqslant k$ different from i, there is a model $(\mathcal{A}_j, F_j) \in \mathrm{Mod}^{\equiv}(\vdash_j)$ such that

$$(\mathcal{A}^{\tau}, F) \cong (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_{i-1} \otimes \mathbf{1} \otimes \mathcal{A}_{i+1} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_{i-1} \times \{1\} \times F_{i+1} \times \cdots \times F_k).$$

Proof. Let $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$ be nontrivial. By our assumption on the interpretation τ, we have that $(\mathcal{A}^{\tau}, F) \in \mathrm{Mod}^{\equiv}(\vdash_0 \otimes \cdots \otimes \vdash_k)$. Moreover, by Corollary 3.20 (applied a finite number of times) we may assume that $(\mathcal{A}^{\tau}, F) = (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_k)$, for some $(\mathcal{A}_j, F_j) \in \mathrm{Mod}^{\equiv}(\vdash_j)$, with $j = 0, \dots, k$.

By condition (2), we know that there is some $(\mathcal{B}, G) \leqslant (\mathcal{A}, F)$ of cardinality $p_1 \cdots p_t$, with each p_i prime and $t \leqslant k$. But then, by Proposition 2.37, we have that $(\mathcal{B}^{\tau}, G) \leqslant (\mathcal{A}^{\tau}, F)$ and, thus, that

$$(\mathcal{B}^{\tau}, G) \leqslant (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_k).$$

Now, by Lemma 3.4, there are some matrices $(\mathcal{B}_j, G_j)$, with $j = 0, \dots, k$ such that $(\mathcal{B}_j, G_j) \leqslant (\mathcal{A}_j, F_j)$, for $j = 0, \dots, k$, and

$$(\mathcal{B}^{\tau}, G) = (\mathcal{B}_0 \otimes \cdots \otimes \mathcal{B}_k, G_0 \times \cdots \times G_k).$$

Since condition (2) tells us that $|B| = p_1 \cdots p_t$ with $t \leqslant k$ and, on the other hand, we have seen that $|B| = |B_0| \cdots |B_k|$, we must have that some $\mathcal{B}_i$ is trivial on cardinality grounds.

Now, as $\mathcal{B}_i$ is trivial, the matrix $(\mathcal{A}_i, F_i)$ has a trivial submatrix. But note that $\vdash \leqslant \vdash_0 \otimes \cdots \otimes \vdash_k \leqslant \vdash_i$ holds by Theorem 3.21, so there is an interpretation ρ witnessing that $\vdash \leqslant \vdash_i$. Moreover, $(\mathcal{A}_i, F_i)$ is trivial since, otherwise, $(\mathcal{A}_i^{\rho}, F_i)$ would be a nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash)$ with a trivial submatrix $(\mathcal{B}_i^{\rho}, G_i)$, by Proposition 2.37, which contradicts condition (3). Hence, we obtain that

$$(\mathcal{A}^{\tau}, F) = (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_{i-1} \otimes \mathbf{1} \otimes \mathcal{A}_{i+1} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_{i-1} \times F_i \times F_{i+1} \times \cdots \times F_k).$$

On the other hand, as $\mathcal{A}_i = \mathbf{1}$, we have that either $F_i = \{1\}$ or $F_i = \varnothing$. We will show that $F_i = \{1\}$. Indeed, if $F_i = \varnothing$, we have that $(\mathcal{A}_i^\rho, \varnothing)$ belongs to $\mathrm{Mod}^{\equiv}(\vdash)$, which contradicts the assumption that $\vdash$ has theorems by Proposition 2.13. But then, $(\mathcal{A}^\tau, F) = (\mathcal{A}_0 \otimes \cdots \otimes \mathbf{1} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times \{1\} \times \cdots \times F_k)$, as we wanted to see.

□

For the next Lemma, we strengthen the conditions from the previous one:

Lemma 4.4. *Suppose that a logic with theorems $\vdash$ satisfies the following conditions:*

(1) There are logics $\vdash_0, \ldots, \vdash_k$ such that $[\![\vdash_0 \otimes \cdots \otimes \vdash_k]\!] = [\![\vdash]\!]$.

(2) Every nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash)$ has a substructure of cardinality $p_1 \cdots p_t$ with $t \leqslant k$ and p_i prime.

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ do not have trivial substructures.

(4) $\mathrm{Mod}^{\equiv}(\vdash)$ has the JEP.

As before, let τ the interpretation witnessing that $\vdash_0 \otimes \cdots \otimes \vdash_k \leqslant \vdash$. Then, there is some $i \leqslant k$ such that, for every matrix $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$ and each $j \leqslant k$ different from i, there are some models $(\mathcal{A}_j, F_j) \in \mathrm{Mod}^{\equiv}(\vdash_j)$ such that

$$(\mathcal{A}^\tau, F) \cong (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_{i-1} \otimes \mathbf{1} \otimes \mathcal{A}_{i+1} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_{i-1} \times \{1\} \times F_{i+1} \times \cdots \times F_k).$$

Proof. We reason by contradiction. Suppose the contrary. Then, there are $k+1$ matrices $(\mathcal{B}_0, G_0), \ldots,$ $(\mathcal{B}_k, G_k) \in \mathrm{Mod}^{\equiv}(\vdash)$ that falsify the stated condition, i.e. such that each $(\mathcal{B}_i^\tau, G_i)$ is isomorphic to a non-indexed product of $k+1$ matrices, $(\mathcal{A}_0^i, F_0^i), \ldots, (\mathcal{A}_k^i, F_k^i)$, such that $(\mathcal{A}_j^i, F_j^i) \in \mathrm{Mod}^{\equiv}(\vdash_j)$ for each $j \leqslant k$ and where $(\mathcal{A}_i^i, F_i^i)$ is nontrivial. In short, each $(\mathcal{B}_i^\tau, G_i)$ is isomorphic to a non-indexed product of matrices that has a nontrivial factor in position i, for each $i \leqslant k$.

Moreover, the matrices $(\mathcal{B}_0, G_0), \ldots, (\mathcal{B}_k, G_k)$ must be nontrivial. If, say, $(\mathcal{B}_i, G_i)$ were trivial, this means that $\mathcal{B}_i = \mathbf{1}$. Therefore, either $G_i = \varnothing$ or $G_i = \{1\}$ would hold. Recall that, again by Theorem 3.21 and our initial assumption, $\vdash \leqslant \vdash_0 \otimes \cdots \otimes \vdash_k \leqslant \vdash_i$, as witnessed by, say, ρ. Hence, in the first case, we would have that $(\mathcal{B}_i^\rho, \varnothing) \in \mathrm{Mod}^{\equiv}(\vdash)$, which contradicts by Proposition 2.13 that $\vdash$ has theorems. Therefore, $(\mathcal{B}_i, G_i)$ would have to be of the form $(\mathbf{1}, \{1\})$. But then, letting $(\mathcal{A}_j^i, F_j^i) := (\mathbf{1}, \{1\})$, for each $j \leqslant k$, we would have that

$$(\mathcal{B}_i^\tau, F_i) = (\mathbf{1}, \{1\}) \cong (\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \{1\} \times \cdots \times \{1\} \times \cdots \times \{1\}) = (\mathbf{1} \otimes \cdots \otimes \mathcal{A}_i^i \otimes \cdots \otimes \mathbf{1}, \{1\} \times \cdots \times F_i^i \times \cdots \times \{1\}),$$

that is, $(\mathcal{A}_i^i, F_i^i) = (\mathbf{1}, \{1\})$, which contradicts how we have originally selected $(\mathcal{B}_i, G_i)$. Hence, we conclude that each $(\mathcal{B}_i, G_i)$ is nontrivial, as we wanted.

The nontriviality of each of the $(\mathcal{B}_i, G_i)$ grants that we are in the conditions of Lemma 4.3. Thus, for each $i \leqslant k$, there are some $t_i \leqslant k$ and matrices $(\mathcal{A}^i_j, F^i_j) \in \mathrm{Mod}^{\equiv}(\vdash_i)$ for $j \neq t_i$ such that

$$(\mathcal{B}_i^\tau, G_i) \cong (\mathcal{A}_0^i \otimes \cdots \otimes \mathbf{1} \otimes \cdots \otimes \mathcal{A}_k^i, F_0^i \times \cdots \times \{1\} \times \ldots F_k^i),$$

where the trivial matrix occurs at position t_i. Note that all we know is that $t_i \neq i$, for each $i \leqslant k$, by our assumption on the nontrivial matrices $(\mathcal{B}_0, G_0), \ldots, (\mathcal{B}_k, G_k)$.

Now, by condition (4), there is some $(\mathcal{B}, G) \in \mathrm{Mod}^{\equiv}(\vdash)$ such that $(\mathcal{B}_i, G_i)$ can be embedded in $(\mathcal{B}, G)$, for $i = 0, \ldots, k$. Note that the nontriviality of each $(\mathcal{B}_i, G_i)$ implies the nontriviality of $(\mathcal{B}, G)$. Once again, by Lemma 4.3 we have that there are some $s \leqslant k$ and some matrices $(\mathcal{A}_j, F_j) \in \mathrm{Mod}^{\equiv}(\vdash_j)$, with $j \neq s$, satisfying that

$$(\mathcal{B}^\tau, G) \cong (\mathcal{A}_0 \otimes \cdots \otimes \{1\} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times \{1\} \times \cdots \times F_k),$$

the trivial algebra occurring in the s-th position. On the other hand, since $(\mathcal{B}_i, G_i) \hookrightarrow (\mathcal{B}, G)$, for every i, we can select $i = s$ so that $(\mathcal{B}_s^\tau, G_s) \hookrightarrow (\mathcal{B}^\tau, G)$ by Proposition 2.37. But then, Lemma 3.8 tells us that the s-th factor of $(\mathcal{B}_s^\tau, G_s)$ is trivial, contradicting our assumption on the $(\mathcal{B}_i, G_i)$, namely, that the i-th factor of $(\mathcal{B}_i^\tau, G_i)$ is nontrivial.

□

Lemmas 4.3 and 4.4 establish the a similar condition regarding the triviality of one factor from the non-indexed product decomposition of a certain algebra: the former provides for each algebra a particular position for the trivial algebra, the latter gives the same position for every element of $\mathrm{Mod}^{\equiv}(\vdash)$.

Proposition 4.5. *Let τ be a translation of $\vdash_0 \otimes \cdots \otimes \vdash_k$ into $\vdash$ and let $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$. Assume that there is some $i \leqslant k$ such that, for every $j \leqslant k$ different from i, there is a model $(\mathcal{A}_j, F_j) \in \mathrm{Mod}^{\equiv}(\vdash_j)$ such that*

$$(\mathcal{A}^\tau, F) \cong (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_{i-1} \otimes \mathbf{1} \otimes \mathcal{A}_{i+1} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_{i-1} \times \{1\} \times F_{i+1} \times \cdots \times F_k).$$

Moreover, define the translation $e : \mathcal{L}(\vdash_0 \otimes \cdots \otimes \not\vdash_i \otimes \cdots \otimes \vdash_k) \to \mathcal{L}(\vdash_0 \otimes \cdots \otimes \vdash_k)$ that sends each n-ary operation $(\varphi_j(\overline{x}))_{1 \leqslant j \leqslant k}$ to the n-ary term given by the tuple that is equal, component-wise, to $(\varphi_j(\overline{x}))_{1 \leqslant j \leqslant k}$ except in the i-th position, where a variable x_i appears. Additionally, consider the translation given by $\tau_e := \tau \circ e$[1]. Then, in these conditions, it holds that

$$(\mathcal{A}^{\tau_e}, F) \cong (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_{i-1} \otimes \mathcal{A}_{i+1} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_{i-1} \times F_{i+1} \times \cdots \times F_k).$$

Proof. Let us suppose, by symmetry, that $i = 0$ (this is merely done in order to make the reading easier). First, note that the isomorphism between $(\mathcal{A}^\tau, F)$ and $(\mathbf{1} \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k, \{1\} \times F_1 \times \cdots \times F_k)$ already gives us, using Proposition 2.37, that

$$(\mathcal{A}^{\tau_e}, F) = (\mathcal{A}^{\tau \circ e}, F) = ((\mathcal{A}^\tau)^e, F) \cong ((\mathbf{1} \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k)^e, \{1\} \times F_1 \times \cdots \times F_k).$$

[1]It can be readily seen that e is a translation and then observed that τ_e is a composition of translations.

Let us see that the map $f : ((\mathbf{1} \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k)^e, \{1\} \times F_1 \times \cdots \times F_k) \to (\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k, F_1 \times \cdots \times F_k)$ given by the rule $(1, a_1, \ldots, a_k) \mapsto (a_1, \ldots, a_k)$ is an isomorphism. Note that it is clearly strict, injective and surjective. Moreover, given $k+1$-tuples $\overline{a}_1, \ldots, \overline{a}_n$ in $\{1\} \times A_1 \times \cdots \times A_k$ and an n-ary term in $\mathcal{L}(\vdash_1 \otimes \cdots \otimes \vdash_k)$, i.e. a tuple $(\varphi_1(\overline{x}), \ldots, \varphi_k(\overline{x}))$ of n-ary terms from each $\mathcal{L}(\vdash_j)$, where $1 \leqslant j \leqslant k$, we have that

$$f((\varphi_1(\overline{x}), \ldots, \varphi_k(\overline{x}))^{(\mathbf{1}\otimes\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_k)^e}(\overline{a}_1, \ldots, \overline{a}_n)) = f(e(\varphi_1(\overline{x}), \ldots, \varphi_k(\overline{x}))^{\mathbf{1}\otimes\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_k}(\overline{a}_1, \ldots, \overline{a}_n)) =$$

$$f((x_0, \varphi_1(\overline{x}), \ldots, \varphi_k(\overline{x}))^{\mathbf{1}\otimes\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_k}(\overline{a}_1, \ldots, \overline{a}_n)) = f((1, \varphi_1^{\mathcal{A}_1}(\overline{a}_1(1), \ldots, \overline{a}_n(1)), \ldots, \varphi_k^{\mathcal{A}_k}(\overline{a}_1(k), \ldots, \overline{a}_n(k))) =$$

$$(\varphi_1^{\mathcal{A}_1}(\overline{a}_1(1), \ldots, \overline{a}_n(1)), \ldots, \varphi_k^{\mathcal{A}_k}(\overline{a}_1(k), \ldots, \overline{a}_n(k))),$$

and, on the other hand,

$$(\varphi_1(\overline{x}), \ldots, \varphi_k(\overline{x})^{\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_k}(f(\overline{a}_1), \ldots, f(\overline{a}_n)) =$$

$$(\varphi_1(\overline{x}), \ldots, \varphi_k(\overline{x})^{\mathcal{A}_1\otimes\cdots\otimes\mathcal{A}_k}((\overline{a}_1(1), \ldots, \overline{a}_1(k)), \ldots, (\overline{a}_n(1), \ldots, \overline{a}_n(k))) =$$

$$(\varphi_1^{\mathcal{A}_1}(\overline{a}_1(1), \ldots, \overline{a}_n(1)), \ldots, \varphi_k^{\mathcal{A}_k}(\overline{a}_1(k), \ldots, \overline{a}_n(k))),$$

so f preserves the operations. Hence, we have seen that it is an isomorphism, as desired.

□

In [9, Prop. 21], the following sufficient condition for meet-primeness in the lattice Var can be found: a variety V *spanned* by a finite algebra $\mathcal{A}$ has Helly number $\leqslant k$ if the cardinality of $\mathcal{A}$ has exactly k prime factors in its factorization (counting repeated primes)[2]. This reasoning has inspired the proof for the main theorem of this chapter. We have restricted ourselves to irreducibility degree below k since, as it can be seen by the statements of Lemmas 4.3 and 4.4, we make full use of the assumption $[\![\vdash_0 \otimes \cdots \otimes \vdash_k]\!] = [\![\vdash]\!]$ and not merely of the weaker condition $[\![\vdash_0 \otimes \cdots \otimes \vdash_k]\!] \leqslant [\![\vdash]\!]$.

Theorem 4.6. *Every logic with theorems $\vdash$ satisfying the following conditions has irreducibility degree $\leqslant k$ in* Log*:*

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ *has the JEP.*

(2) Every nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash)$ *has a substructure of cardinality $p_1 \cdots p_t$ with $t \leqslant k$ and p_i prime.*

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *do not have trivial substructures.*

Proof. Suppose that we have such a logic $\vdash$. Let $\vdash_0, \ldots, \vdash_k$ be logics verifying that $[\![\vdash_0 \otimes \cdots \otimes \vdash_k]\!] = [\![\vdash]\!]$. This means, in particular, that $\vdash_0 \otimes \cdots \otimes \vdash_k$ is interpretable into $\vdash$, so let us denote by τ the witnessing interpretation.

Now, note that we are in the conditions of Lemma 4.4, so let i be the positive integer given by such Lemma. Again, by symmetry, we may assume that $i = 0$. Then, we will see that the non-indexed

[2]The precise notion of variety spanned by an algebra can be found in [9, Ch. 2].

product of logics $\vdash_1 \otimes \cdots \otimes \vdash_k$ (i.e. deleting $\vdash_0$) is interpretable into $\vdash$. Consider the translation $e : \mathcal{L}(\vdash_1 \otimes \cdots \otimes \vdash_k) \to \mathcal{L}(\vdash_0 \otimes \cdots \otimes \vdash_k)$ determined by the rule $(\varphi_j(\overline{x}))_{1 \leqslant j \leqslant k} \mapsto (x_0, \varphi_1(\overline{x}), \ldots, \varphi_k(\overline{x}))$ and define $\tau_e := \tau \circ e$. Our claim is that the translation τ_e is the desired interpretation, that is, we need to check that τ_e preserves the Suszko models. Let $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$ and let us see that $(\mathcal{A}^{\tau_e}, F) \in \mathrm{Mod}^{\equiv}(\vdash_1 \otimes \cdots \otimes \vdash_k)$.

By our initial assumption on $i = 0$ we have that, for every such model $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$ and for each $0 < j \leqslant k$, there exists $(\mathcal{A}_j, F_j) \in \mathrm{Mod}^{\equiv}(\vdash_j)$ such that $(\mathcal{A}^{\tau}, F) \cong (\mathbf{1} \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k, \{1\} \times F_1 \times \cdots \times F_k)$. Now, by Proposition 4.5, we know that $(\mathcal{A}^{\tau_e}, F)$ is isomorphic to $(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k, F_1 \times \cdots \times F_k)$ and therefore, by Corollary 3.20, that $(\mathcal{A}^{\tau_e}, F) \in \mathrm{Mod}^{\equiv}(\vdash_1 \otimes \cdots \otimes \vdash_k)$, as desired.

□

Remark 4.7. Note that the relevance of Lemma 4.4 resides in the fact that the same component is trivial for *every* model, so we can deduce that the corresponding translation of *each* of them lies in $\mathrm{Mod}^{\equiv}(\vdash_1 \otimes \cdots \otimes \vdash_k)$. Otherwise, if we were only able to apply Lemma 4.3, the translation of each model would belong to a possibly different class in each case.

In particular, we have proven this result for meet-irreducibility, which is simply the case $k = 1$[3]. The first formulation of such consequence was announced in [20]:

Corollary 4.8. *Every logic with theorems $\vdash$ satisfying the following conditions is meet-irreducible in* Log*:*

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ *has the JEP.*

(2) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *have substructures of prime cardinality.*

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *do not have trivial substructures.*

4.2 Criteria for hereditarily nontrivial logics

In this section we wish to improve our previous results for meet-irreducibility to analogous results for meet-primeness. It turns out that, in fact, we need to restrict ourselves to certain sub-semilattices of Log.

Take a look at the statement of Theorem 4.6 and, in particular, at property (3). We say that a logic $\vdash$ is *hereditarily nontrivial* if this property (3) property holds for $\vdash$, i.e. the nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ do not have trivial substructures. Consider the subposet HNT of Log whose universe consists in all the $[\![\vdash]\!]$ where $\vdash$ is a hereditarily nontrivial logic. Let us check that HNT is, in fact, a sub-semilattice, so that it makes sense to talk about finite meets in this case:

Proposition 4.9. *If $\vdash_1$ and $\vdash_2$ are two hereditarily nontrivial logics, so is $\vdash_1 \otimes \vdash_2$. Moreover, if $\vdash$ is hereditarily nontrivial and $\vdash \leqslant \vdash'$, so is $\vdash'$. As a consequence,* HNT *is a filter of* Log*.*

[3]Lemmas 4.3 and 4.4 can also be easily translated for this case.

Proof. Let $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash_1 \otimes \vdash_2)$ be nontrivial. We already know, by Corollary 3.20, that $(\mathcal{A}, F) = (\mathcal{A}_1 \otimes \mathcal{A}_2, F_1 \times F_2)$, with $(\mathcal{A}_i, F_i) \in \mathrm{Mod}^{\equiv}(\vdash_i)$, for $i = 1, 2$. Suppose that there is a trivial submatrix $(\mathcal{B}, G) \leqslant (\mathcal{A}, F)$. We also know, by Lemma 3.4, that $(\mathcal{B}, G) = (\mathcal{B}_1 \otimes \mathcal{B}_2, G_1 \times G_2)$, where $(\mathcal{B}_i, G_i) \leqslant (\mathcal{A}_i, F_i)$ and $i = 1, 2$. Now, since $1 = |B| = |B_1| \cdot |B_2|$, we must have that $\mathcal{B}_1 = \mathcal{B}_2 = \mathbf{1}$, which contradicts the assumption on $\vdash_1$ and $\vdash_2$. On the other hand, let $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash')$ be nontrivial. We know that $(\mathcal{A}^{\tau}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$ is also nontrivial so, if there were some trivial submatrix $(\mathcal{B}, G) \leqslant (\mathcal{A}, F)$, we would have a trivial submatrix $(\mathcal{B}^{\tau}, G) \leqslant (\mathcal{A}^{\tau}, F)$, contradicting the assumption that $\vdash$ is hereditarily nontrivial.

□

We can prove a similar result for a subclass of HNT. Denote by HNT* the class of all the $[\![\vdash]\!]$ such that $\vdash$ is hereditarily nontrivial and has theorems. Then, the following proposition shows that it also makes sense to talk about finite meets in this more restricted setting:

Proposition 4.10. *If $\vdash_1$ and $\vdash_2$ are two hereditarily nontrivial logics with theorems, so is $\vdash_1 \otimes \vdash_2$. Moreover, if $\vdash$ is hereditarily nontrivial with theorems and $\vdash \leqslant \vdash'$, so is $\vdash'$. As a consequence,* HNT* *is a filter of* Log.

Proof. If $\vdash_1$ and $\vdash_2$ are hereditarily nontrivial then so is $\vdash_1 \otimes \vdash_2$, by Proposition 4.9. On the other hand, $\vdash_1 \otimes \vdash_2$ has theorems by Proposition 3.15. Now, for the second part, we already know by the preceding Proposition that, if $\vdash$ is hereditarily nontrivial and $\vdash \leqslant \vdash'$, then so is $\vdash'$. Now, suppose that $\vdash$ has theorems and that $\vdash \leqslant \vdash'$ is witnessed by τ. If $\vdash'$ lacks theorems, this means that $(\mathbf{1}, \varnothing) \in \mathrm{Mod}^{\equiv}(\vdash')$, so $(\mathbf{1}^{\tau}, \varnothing) \in \mathrm{Mod}^{\equiv}(\vdash)$, which is impossible (note that the algebra $\mathbf{1}^{\tau}$ is precisely the trivial algebra in the language of $\vdash$).

□

Now, observe that in the proof of Lemma 4.3 we only used the assumption $\vdash \leqslant \vdash_0 \otimes \cdots \otimes \vdash_k$ once, in order to obtain the triviality of a Suszko model of one of these $k+1$ logics from the triviality of one of its submatrices. Similarly, this assumption is used once in Lemma 4.4. We can replace the argument using the premise of meet-irreducibility by another only assuming that the logics $\vdash_0, \ldots, \vdash_k$ are hereditarily nontrivial. In other words, we can weaken the premise of irreducibility to that of primeness:

Lemma 4.11. *Suppose that a logic with theorems $\vdash$ satisfies the following conditions:*

(1) There are logics $\vdash_0, \ldots, \vdash_k$ in HNT *such that $[\![\vdash_0 \otimes \cdots \otimes \vdash_k]\!] \leqslant [\![\vdash]\!]$.*

(2) Every nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash)$ has a substructure of cardinality $p_1 \cdots p_t$ with $t \leqslant k$ and p_i prime.

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ do not have trivial substructures.

Moreover, let τ be the interpretation witnessing that $\vdash_0 \otimes \cdots \otimes \vdash_k \leqslant \vdash$. Then, for every nontrivial matrix $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$ there is some $i \leqslant k$ such that, for every $j \leqslant k$ different from i, there is a model $(\mathcal{A}_j, F_j) \in \mathrm{Mod}^{\equiv}(\vdash_j)$ such that

$$(\mathcal{A}^{\tau}, F) \cong (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_{i-1} \otimes \mathbf{1} \otimes \mathcal{A}_{i+1} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_{i-1} \times \{1\} \times F_{i+1} \times \cdots \times F_k).$$

Proof. The proof of Lemma 4.3 can be followed until the point in which one obtains that $\mathcal{B}_i$ is a trivial subalgebra of $\mathcal{A}_i$, so that $(\mathcal{A}_i, F_i)$ has a trivial submatrix, call it $(\mathcal{B}_i, G_i)$, where either $G_i = \{1\}$ or $G_i = \varnothing$. What we wanted to see here is that $(\mathcal{A}_i, F_i)$ must be, in fact, trivial. Note that here, in principle, we do not have an interpretation from $\vdash$ into $\vdash_0 \otimes \cdots \otimes \vdash_k$: we can only use Theorem 3.21 in order to obtain an interpretation ρ witnessing that $\vdash_0 \otimes \cdots \otimes \vdash_k \leqslant \vdash_i$. Now, if $(\mathcal{A}_i, F_i)$ is nontrivial, this means that $(\mathcal{A}_i^\rho, F_i) \in \mathrm{Mod}^\equiv(\vdash_0 \otimes \cdots \otimes \vdash_k)$ is also nontrivial. Since $\vdash_0 \otimes \cdots \otimes \vdash_k$ is an element of HNT by Proposition 4.9, this means that $(\mathcal{A}_i^\rho, F_i)$ has no trivial substructure. But, by Proposition 2.37, $(\mathcal{B}_i^\rho, G_i) \leqslant (\mathcal{A}_i^\rho, F_i)$ and $(\mathcal{B}_i^\rho, G_i)$ is trivial because so is $(\mathcal{B}_i, G_i)$. Hence, $(\mathcal{A}_i, F_i)$ must be trivial, and from here we can follow the rest of the proof as in Lemma 4.3.

□

Lemma 4.12. *Suppose that a logic with theorems $\vdash$ satisfies the following conditions:*

(1) There are logics $\vdash_0, \ldots, \vdash_k$ in HNT* *such that $[\![\vdash_0 \otimes \cdots \otimes \vdash_k]\!] \leqslant [\![\vdash]\!]$.*

(2) Every nontrivial element of $\mathrm{Mod}^\equiv(\vdash)$ *has a substructure of cardinality $p_1 \cdots p_t$ with $t \leqslant k$ and p_i prime.*

(3) Nontrivial elements of $\mathrm{Mod}^\equiv(\vdash)$ *do not have trivial substructures.*

(4) $\mathrm{Mod}^\equiv(\vdash)$ *has the JEP.*

As before, let τ be the interpretation witnessing that $\vdash_0 \otimes \cdots \otimes \vdash_k \leqslant \vdash$. Then, there is some $i \leqslant k$ such that, for every matrix $(\mathcal{A}, F) \in \mathrm{Mod}^\equiv(\vdash)$ and each $j \leqslant k$ different from i, there are some models $(\mathcal{A}_j, F_j) \in \mathrm{Mod}^\equiv(\vdash_j)$ such that

$$(\mathcal{A}^\tau, F) \cong (\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_{i-1} \otimes \mathbf{1} \otimes \mathcal{A}_{i+1} \otimes \cdots \otimes \mathcal{A}_k, F_0 \times \cdots \times F_{i-1} \times \{1\} \times F_{i+1} \times \cdots \times F_k).$$

Proof. We can follow the proof from Lemma 4.4 until the point in which we want to prove that the matrices $(\mathcal{B}_0, G_0), \ldots, (\mathcal{B}_k, G_k)$ must be nontrivial. Suppose that some $(\mathcal{B}_i, G_i)$ is trivial, so that $\mathcal{B}_i = \mathbf{1}$. Therefore, either $G_i = \varnothing$ or $G_i = \{1\}$ would hold. In this case, Theorem 3.21 only gives us that $\vdash_0 \otimes \cdots \otimes \vdash_k \leqslant \vdash_i$, as witnessed by, say, ρ. In the first case, that is, where $(\mathcal{B}_i, G_i) = (\mathbf{1}, \varnothing)$, we would have that $(\mathbf{1}^\rho, \varnothing) \in \mathrm{Mod}^\equiv(\vdash_0 \otimes \cdots \otimes \vdash_k)$, which contradicts by Proposition 2.13 that $\vdash_0 \otimes \cdots \otimes \vdash_k$ has theorems. Therefore, $(\mathcal{B}_i, G_i)$ must be of the form $(\mathbf{1}, \{1\})$, and from here one can follow the proof from Lemma 4.4.

□

From here, with the aid of Proposition 4.5, we can prove the desired variation of Theorem 4.6 for HNT*:

Corollary 4.13. *Every logic (with theorems) $\vdash$ satisfying the following conditions has Helly number $\leqslant k$ in* HNT**:*

(1) $\mathrm{Mod}^\equiv(\vdash)$ *has the JEP.*

(2) Every nontrivial element of $\mathrm{Mod}^\equiv(\vdash)$ *has a substructure of cardinality $p_1 \cdots p_t$ with $t \leqslant k$ and p_i prime.*

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *do not have trivial substructures*[4].

And, in particular:

Corollary 4.14. *Every logic (with theorems)* $\vdash$ *satisfying the following conditions is meet-prime in* HNT^**:*

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ *has the JEP.*

(2) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *have substructures of prime cardinality.*

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *do not have trivial substructures.*

4.2.1 The subposet FinEq

Let FinEq be the subposet of Log that contains the classes $[\![\vdash]\!]$ where $\vdash$ is a finitely equivalential logic. The following proposition [14, Lemma 2.3] allows us to observe that the poset FinEq has binary meets and that, therefore, it makes sense talking about meet-primeness and irreducibility:

Proposition 4.15. *If* $\vdash_1$ *and* $\vdash_2$ *are two finitely equivalential logics then* $\vdash_1 \otimes \vdash_2$ *is also finitely equivalential. Moreover, if* $\vdash$ *is finitely equivalential and* $\vdash \leqslant \vdash'$*, so is* $\vdash'$.

Proof. Let us first check that $\vdash_1 \otimes \vdash_2$ is finitary. Suppose that we have formulas $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\vdash_1 \otimes \vdash_2)$ such that $\Gamma \vdash_1 \otimes \vdash_2 \varphi$. This, by definition, means that for each $(\mathcal{A}_1 \otimes \mathcal{A}_2, F_1 \times F_2) \in \mathrm{Mod}^{\equiv}(\vdash_1) \otimes \mathrm{Mod}^{\equiv}(\vdash_2)$ and each valuation $h : \mathrm{Fm}(\vdash_1 \otimes \vdash_2) \to \mathcal{A}_1 \otimes \mathcal{A}_2$, $h(\Gamma) \subseteq F_1 \times F_2$ implies that $h(\varphi) \in F_1 \times F_2$. Taking the corresponding projection for $i = 1, 2$, we obtain two derivations $\Gamma_i \vdash_i \varphi_i$, with $\Gamma_i \cup \{\varphi_i\} \subseteq \mathcal{L}(\vdash_i)$. Hence, since the $\vdash_i$ are finitary, we may select a finite $\Sigma_i \subseteq \Gamma_i$ in each case such that $\Sigma_i \vdash_i \varphi_i$ and therefore $\Sigma_1 \times \Sigma_2 \vdash_1 \otimes \vdash_2 \varphi$.

Now, let us see that $\vdash_1 \otimes \vdash_2$ is finitely equivalential. The assumption already gives us two finite sets $\Delta_1(x, y)$ and $\Delta_2(x, y)$ of equivalence formulas for $\vdash_1$ and $\vdash_2$, respectively. Define $\Delta(x, y) := \Delta_1(x, y) \times \Delta_2(x, y)$, which is also a finite set. Moreover, $\Delta(x, y)$ verifies the properties of a set of equivalence formulas. For example, $\varnothing \vdash_1 \otimes \vdash_2 \Delta(x, y)$ holds if $\varnothing \vdash_1 \Delta_1(x, y)$ and $\varnothing \vdash_2 \Delta_2(x, y)$, by the definition of $\vdash_1 \otimes \vdash_2$, and this is one of our assumptions. The other properties are verified similarly.

On the other hand, let $\vdash$ be finitely equivalential and $\vdash'$ such that $\vdash \leqslant \vdash'$, say, by the interpretation τ. Denote by $\Delta(x, y)$ the finite set of equivalence formulas for $\vdash$. Define the (finite) set $\Delta'(x, y) := \tau(\Delta(x, y))$ of $\mathcal{L}(\vdash')$-formulas. We want to check that, in fact, it constitutes a set of equivalence formulas for $\vdash'$. Since it holds that $\varnothing \vdash \Delta(x, x)$, this means that, in particular, for every $(\mathcal{B}, G) \in \mathrm{Mod}(\vdash)$ and every $\mathcal{B}$-valuation h, $h(\Delta(x, x)) \subseteq G$. Let $(\mathcal{A}, F) \in \mathrm{Mod}(\vdash')$ be an arbitrary model and take an arbitrary $\mathcal{A}$-valuation f'. We note that the map $f' \circ \tau$ is, in fact, an $\mathcal{A}^\tau$-valuation. Since $(\mathcal{A}^\tau, F) \in \mathrm{Mod}(\vdash)$ by Proposition 2.41, our observations above tell us that $f' \circ \tau(\Delta(x, x)) = f'(\Delta'(x, x)) \subseteq F$. Therefore, by the completeness of every logic with respect to its models, we have seen that $\varnothing \vdash' \Delta'(x, x)$. The case $x, \Delta'(x, y) \vdash' y$ can be

[4] Note that this condition is satisfied by $\vdash$ in virtue of the definition of HNT^*. We write it along the others for the shake of symmetry with respect to the main theorem.

justified similarly.

It remains to see that, given an n-ary connective $*$ of $\vdash'$, $\bigcup_{i\leqslant n}\Delta'(x_i,y_i)\vdash'\Delta(*(\overline{x}),*(\overline{y}))$. Consider an arbitrary n-ary connective $*$ and a model $(\mathcal{A},F)\in\mathrm{Mod}^{\equiv}(\vdash')$. Let $a_1,b_1,\dots,a_n,b_n\in A$ be such that $\bigcup_{i\leqslant n}\Delta'(a_i,b_i)^{\mathcal{A}}\subseteq F$. Since $\Delta'(x,x)=\tau(\Delta(x,x))$, it follows that $\bigcup_{i\leqslant n}\Delta(a_i,b_i)^{\mathcal{A}^\tau}\subseteq F$. Now, $\Delta(x,y)$ is a set of equivalence formulas for $\vdash$ and $(\mathcal{A}^\tau,F)\in\mathrm{Mod}^{\equiv}(\vdash)=\mathrm{Mod}^*(\vdash)$ (being $\vdash$ finitely equivalential), so the previous condition is equivalent to $(a_i,b_i)\in\Omega^{\mathcal{A}^\tau}F=id_A$, and therefore $a_i=b_i$ for each $i\leqslant n$. Hence, $*^{\mathcal{A}}(\overline{a})=*^{\mathcal{A}}(\overline{b})$. Now, we know that $\varnothing\vdash'\Delta'(x,x)$, so $\Delta'(*(\overline{a},*(\overline{b})))^{\mathcal{A}}\subseteq F$. Hence, we have obtained the desired result.

□

In [9, Prop. 18] is proven that, if a variety V is a countable nested union of varieties V_n, that is, $V=\bigcup_{n\in\omega}V_n$ where $V_i\subseteq V_j$ if $i\leqslant j$, and each of these is spanned by a finite algebra $\mathcal{A}_n$, then V is a meet-prime element of Var. In our case, while we did not succeed in proving a variation of this claim, we were able to recover it for the restricted setting of FinEq. Note that talking about a countable nested intersection of logics is coherent, since the duality between logics and their models (which are playing the role of the varieties for us) swaps the order of inclusion: we say that a logic $\vdash$ in some language $\mathcal{L}$ is a nested intersection of a countable family of $\mathcal{L}$-logics $\{\vdash_i\ \mid i\in\omega\}$ if $\vdash$ can be written as $\bigcap_{i\in\omega}\vdash_i$, where $\vdash_i\subseteq\vdash_j$ (i.e. $\vdash_j$ is an extension of $\vdash_i$) if $j\leqslant i$.

Remark 4.16. Since, by definition, every finitely equivalential logic has theorems, we can rephrase the statement of 4.3, 4.4 and 4.6 so that they require $\vdash$ to be finitely equivalential instead.

Now, we can prove the following particular case using a different argument. The key point is that we can make additional assumptions regarding our class of Suszko models in order to recover the argument of Lemma 4.4 (for case $k=1$) in an alternative way:

Theorem 4.17. *Every finitely equivalential logic $\vdash$ satisfying the following conditions is meet-irreducible in* FinEq*:*

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ *has the JEP.*

(2) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *have substructures of prime cardinality.*

(3) Nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$ *do not have trivial substructures.*

Proof. As before, we may assume that $\vdash$ has a nonempty set of equivalence formulas and, consequently, that it has theorems. Let $\vdash_1,\vdash_2$ be two finitely equivalential logics verifying that $[\![\vdash]\!]=[\![\vdash_1\otimes\vdash_2]\!]$. Again, in particular, we have an interpretation τ from $\vdash_1\otimes\vdash_2$ into $\vdash$. Note that we are in conditions to apply Lemma 4.3.

Now, since $\vdash$ is finitely equivalential and verifies (1), by Corollary 2.33 we have that there is a reduced matrix $(\mathcal{A},F)$ such that $\vdash$ is the finitary companion of the logic induced by $(\mathcal{A},F)$. This means, in virtue of Proposition 2.16, that $\vdash$ is the logic induced by $\mathbb{P}_{\mathrm{U}}(\{(\mathcal{A},F)\})$. By Fact 2.24(i), $\mathrm{Mod}^*(\vdash)$ is $\mathbb{P}_{\mathrm{U}}$-closed and, since $(\mathcal{A}, F) \in \mathrm{Mod}^*(\vdash)$, we have that $\mathbb{P}_{\mathrm{U}}(\{(\mathcal{A}, F)\}) \subseteq \mathrm{Mod}^*(\vdash)$. Therefore, we are in conditions of Fact 2.24(ii) and, thus, we have that $\mathrm{Mod}^{\equiv}(\vdash) = \mathbb{ISPP}_{\mathrm{U}}(\{(\mathcal{A}, F)\})$.

Now, by Lemma 4.3 (for $k = 1$), without loss of generality we may assume that $(\mathcal{A}^{\tau}, F) \cong (\mathcal{A}_1 \otimes \mathbf{1}, F_1 \times \{1\})$, for some $(\mathcal{A}_1, F_1) \in \mathrm{Mod}^{\equiv}(\vdash_1)$. Therefore, by Proposition 4.5, considering the translation $\tau_e : \mathcal{L}(\vdash_1) \to \mathcal{L}(\vdash)$, we know that $(\mathcal{A}^{\tau_e}, F) \cong (\mathcal{A}_1, F_1) \in \mathrm{Mod}^{\equiv}(\vdash_1)$. Let us see that τ_e is in fact an interpretation witnessing that $\vdash_1 \leqslant \vdash$. Note that this is not obvious: it is precisely the content of Theorem 4.6, where we had the aid of Lemma 4.4. This is the part of the argument that we wish to recover.

Let $(\mathcal{B}, G) \in \mathrm{Mod}^{\equiv}(\vdash)$. This class is, by our observation above, precisely $\mathbb{ISPP}_{\mathrm{U}}(\{(\mathcal{A}, F)\})$. Hence, $(\mathcal{B}, G)$ can be embedded into a product of ultrapowers of $(\mathcal{A}, F)$ and, by Proposition 2.37, $(\mathcal{B}^{\tau_e}, G)$ can be embedded in a product of ultrapowers of $(\mathcal{A}^{\tau_e}, F)$. Therefore, $(\mathcal{B}^{\tau_e}, G) \in \mathbb{ISPP}_{\mathrm{U}}(\{(\mathcal{A}^{\tau_e}, F)\}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathrm{Mod}^{\equiv}(\vdash_1))$. But remember that $\vdash_1$ is finitely equivalential by assumption so, by Fact 2.24(i), we have that $(\mathcal{B}^{\tau_e}, G) \in \mathrm{Mod}^{\equiv}(\vdash_1)$. Therefore, we conclude that $\vdash_1 \leqslant \vdash$, as desired.

□

Now, as we announced, in this case we are able to derive a special form of Theorem 4.17:

Theorem 4.18. *Let $\{\vdash_n \mid n \in \omega\}$ be a collection of finitely equivalential logics (with theorems) such that $\vdash_n$ is an extension of $\vdash_m$ if $n < m$. Moreover, assume that each of these logics verifies the conditions (1)-(3) from Theorem 4.17. Then, if $\bigcap_{n\in\omega} \vdash_n$ is an element of* $\mathrm{FinEq} \cap \mathrm{HNT}^*$*, it is in fact meet-irreducible in* $\mathrm{FinEq} \cap \mathrm{HNT}^*$.

Proof. First, let us denote $\vdash := \bigcap_{n\in\omega} \vdash_n$. Note that all the logics $\vdash_n$ share the same language. Suppose that there are two logics $\vdash_a$ and $\vdash_b$ from $\mathrm{FinEq} \cap \mathrm{HNT}^*$ such that $\vdash_a \otimes \vdash_b$ can be interpreted into $\vdash$ through, say, τ. First, note that we have the canonical translations $\iota_a : \mathcal{L}(\vdash_a) \to \mathcal{L}(\vdash_a \otimes \vdash_b) : \varphi(\overline{x}) \mapsto (\varphi(\overline{x}), x_1)$ and $\iota_b : \mathcal{L}(\vdash_b) \to \mathcal{L}(\vdash_a \otimes \vdash_b) : \varphi(\overline{x}) \mapsto (x_1, \varphi(\overline{x}))$. Moreover, since $\vdash_n$ is an extension of $\vdash$, we have that $\mathrm{Mod}^{\equiv}(\vdash_n) \subseteq \mathrm{Mod}^{\equiv}(\vdash)$ and, hence, that the identity $id : \mathcal{L}(\vdash) \to \mathcal{L}(\vdash_n)$ is an interpretation of $\vdash$ into $\vdash_n$, for each $n \in \omega$.

Therefore, τ can be seen as an interpretation of $\vdash_a \otimes \vdash_b$ into $\vdash_n$, for each $n \in \omega$, because $\tau = id \circ \tau : \mathcal{L}(\vdash_a \otimes \vdash_b) \to \mathcal{L}(\vdash) \to \mathcal{L}(\vdash_n)$, for every such $n \in \omega$. But note that, by assumption, each of the logics $\vdash_n$ satisfies the conditions (1)-(3) and, consequently, each one of them is meet-prime in HNT^* by Corollary 4.14. Hence, since they are finitely equivalential, they are meet-prime in $\mathrm{FinEq} \cap \mathrm{HNT}^*$. That is, following the proof of Theorem 4.6, we have that either $\sigma_a := \tau \circ \iota_a$ or $\sigma_b := \tau \circ \iota_b$ witnesses that $\vdash_a \leqslant \vdash_n$ or $\vdash_b \leqslant \vdash_n$, respectively, for every $n \in \omega$[5].

Let us now justify the following: there is one $x = a, b$ such that, for every $n \in \omega$, the interpretation σ_x witnesses that $\vdash_x \leqslant \vdash_n$. Suppose otherwise. Then, we can find $n, m \in \omega$ such that σ_a does not witness that $\vdash_a \leqslant \vdash_n$ and σ_b does not witness that $\vdash_b \leqslant \vdash_m$. By our observations above, this implies that

[5]Note that ι_a and ι_b are both translations of the same form as the one from Proposition 4.5. This is why the proof of Theorem 4.6 can be mirrored here.

σ_b witnesses that $\vdash_b \leqslant \vdash_n$ and that σ_a witnesses that $\vdash_a \leqslant \vdash_m$. On the other hand, we can assume by symmetry that $n \leqslant m$. But then, this implies that $\mathrm{Mod}^{\equiv}(\vdash_n) \subseteq \mathrm{Mod}^{\equiv}(\vdash_m)$ and, taking the composition with the canonical inclusion seen as an interpretation, we would have that, in fact $\vdash_a \leqslant \vdash_n$ is witnessed but σ_a, which contradicts our initial supposition. Therefore, our claim holds. We may assume that the interpretation thus obtained is σ_a. Now, we want to see that σ_a is, in fact, an interpretation from $\vdash_a$ into $\vdash$.

First, note that that $\vdash$ is the logic induced by the class of reduced matrices $\mathbb{K} := \bigcup_{n\in\omega} \mathrm{Mod}^*(\vdash_n)$. The condition $\Gamma \vdash_{\mathbb{K}} \varphi$ means that, for every $(\mathcal{A}, F) \in \mathbb{K}$ and every $\mathcal{A}$-valuation h, it holds that $h(\Gamma) \subseteq F$ implies that $h(\varphi) \in F$. But, given $n \in \mathbb{N}$, $\Gamma \vdash_n \varphi$ is true when such condition holds for elements of $\mathrm{Mod}^*(\vdash)$, since every logic is induced by its reduced models. Hence, we have seen that $\Gamma \vdash_n \varphi$ for every $n \in \mathbb{N}$, that is, that $\Gamma \vdash \varphi$. Conversely, suppose that $\Gamma \vdash \varphi$ and that we have an arbitrary $(\mathcal{A}, F) \in \mathrm{Mod}^*(\vdash_n)$. Then, since $\Gamma \vdash_n \varphi$, by the same reason as before we have that for every $\mathcal{A}$-valuation h, $h(\Gamma) \subseteq F$ implies that $h(\varphi) \in F$. Therefore, we have seen this condition for the elements of each $\mathrm{Mod}^*(\vdash)$ and, thus, $\Gamma \vdash_{\mathbb{K}} \varphi$.

Then, again by Fact 2.24(ii), we have that $\mathrm{Mod}^{\equiv}(\vdash) = \mathbb{ISPP}_{\mathrm{U}}(\mathbb{K})$. From here, we can follow the proof of the previous theorem, using again Proposition 2.37 and that both $\vdash_a$ and $\vdash_b$ are finitely equivalential. Indeed, let $(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash)$, that is, $(\mathcal{A}, F) \in \mathbb{ISPP}_{\mathrm{U}}(\mathbb{K})$. By Proposition 2.37, $(\mathcal{A}^{\sigma_a}, F) \in \mathbb{ISPP}_{\mathrm{U}}(\mathbb{K}^{\sigma_a})$. Now, as $\mathbb{K} \subseteq \bigcup_{n\in\omega} \mathrm{Mod}^{\equiv}(\vdash_n)$ by our previous observation and σ_a witnesses that $\vdash_a \leqslant \vdash_n$ for every $n \in \omega$, we get, again using Proposition 2.37, that

$$\mathbb{K}^{\sigma_a} \subseteq \left(\bigcup_{n\in\omega} \mathrm{Mod}^{\equiv}(\vdash_n)\right)^{\sigma_a} \subseteq \mathrm{Mod}^{\equiv}(\vdash_a).$$

Now, from here we can infer that

$$(\mathcal{A}^{\sigma_a}, F) \in \mathbb{ISPP}_{\mathrm{U}}(\mathbb{K}^{\sigma_a}) \subseteq \mathbb{ISPP}_{\mathrm{U}}(\mathrm{Mod}^{\equiv}(\vdash_a)) \subseteq \mathrm{Mod}^{\equiv}(\vdash_a),$$

where we have used Fact 2.24(i). Therefore, we have seen that σ_a is an interpretation from $\vdash_a$ into $\vdash$, as desired.

□

5 | Applications

In this chapter we will apply our results on meet-irreducibility to intermediate logics [3], the Łukasiewicz logic [4] and the modal logics $S4$ and $S5$ [3]. One key aspect shared by these logics is that they are algebraizable, so we can apply Fact 2.29 in order to obtain a simple description of their Suszko models, namely, that these coincide with matrices whose algebraic reducts belong to some quasivariety and whose designated subsets are the corresponding sets of solutions. As we will see, this allows us to check the conditions of Theorem 4.6 for the case $k = 1$ quite immediately.

5.1 Intermediate logics

Recall that a *superintuitionistic logic* is an axiomatic extension of the intuitionistic propositional calculus IPC. An *intermediate logic* is a nontrivial superintuitionistic logic. Denote by Ext(IPC) the lattice of axiomatic extensions of IPC and by $V(\mathbf{HA})$ the lattice of subvarieties of the variety of Heyting algebras $\mathbf{HA}$. Remember that IPC is algebraizable, as witnessed by the class $\mathbf{HA}$ and the sets $\tau(x) := \{x \approx 1\}$ and $\Delta(x, y) := \{x \to y, y \to x\}$.

Then, a well-known result [3, Thm. 7.54] tells us that there is a dual lattice isomorphism $\Lambda : \mathrm{Ext}(\mathrm{IPC}) \to V(\mathbf{HA})$ such that $\Lambda(\vdash)$ is the class of all Heyting algebras $\mathcal{A}$ verifying that $\mathcal{A} \vDash \tau(\varphi)$, where $\varnothing \vdash \varphi$. On the other hand, $\Lambda^{-1}(V)$ is precisely the assertional logic $\vdash_V$ of V. In particular, Λ sends each superintuitionistic logic to its algebraic equivalent semantics. Therefore, for an intermediate logic $\vdash$, such semantic is nontrivial. Moreover, by Fact 2.29,

$$\mathrm{Mod}^{\equiv}(\vdash) = \{(\mathcal{A}, \{1\}) \mid \mathcal{A} \in \Lambda(\vdash)\}. \tag{5.1}$$

We start by verifying that the class of the algebraic reducts of the Suszko models satisfies the JEP:

Lemma 5.1. *The class* $\mathbf{HA}$ *is PSC. Therefore, by Proposition 2.34,* $\mathbf{HA}$ *has the JEP and so do its subvarieties.*

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathbf{HA}$ be nontrivial. Let us see, without loss of generality, that there is a surjective homomorphism $\mathcal{A} \to \mathbf{B}_2$, where $\mathbf{B}_2$ is the Boolean algebra of two elements. Indeed, this algebra can be embedded into every nontrivial Heyting algebra, so our desired conclusion would follow. Take the family of proper lattice filters on $\mathcal{A}$, ordered by inclusion. Note that it is nonempty since $\{1\}$ occurs in it and that we can always generate a filter by a family of subsets. Therefore, by Zorn's lemma, there is a maximal proper filter F. Remember that F corresponds to a congruence θ_F on $\mathcal{A}$, which is divided in two blocks, namely, F and $A \setminus F$. Moreover, the map $f : \mathcal{A}/\theta_F \to \mathbf{B}_2$ that sends the former to 1 and the latter to 0 is an isomorphism. Thus, the composition $f \circ \pi_{\theta_F} : \mathcal{A} \to \mathcal{A}/\theta_F \to \mathbf{B}_2$, where π_{θ_F} is the canonical projection, provides us the homomorphism we wanted.

□

From here, we can prove the desired result:

Theorem 5.2. *Every intermediate logic is meet-irreducible in* Log.

Proof. Let $\vdash$ be an intermediate logic. Note that we are under the conditions of Theorem 4.6: notice that $\vdash$ has theorems because it is an axiomatic extension of IPC. Let us check that the conditions (1), (2) and (3) from this theorem are fulfilled by $\vdash$.

First, note that (2) is satisfied by $\vdash$ because every nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash)$ has a submatrix of prime cardinality, namely, $(\mathbf{B}_2, \{1\})$, where $\mathbf{B}_2$ is the Boolean algebra of two elements. Similarly with condition (3): every nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash)$ contains $(B_2, \{1\})$ and each one of its submatrices does too. Therefore, we are left with proving that $\vdash$ verifies condition (1), in other words, that $\mathrm{Mod}^{\equiv}(\vdash)$ has the JEP. Let $(\mathcal{A}, F), (\mathcal{B}, G)$ be nontrivial elements of $\mathrm{Mod}^{\equiv}(\vdash)$. By 5.1, we know that they are of the form $(\mathcal{A}, \{1\})$ and $(\mathcal{B}, \{1\})$, where $\mathcal{A}, \mathcal{B} \in \Lambda(\vdash)$. By the previous Lemma, $\Lambda(\vdash)$ has the JEP, so there is some $\mathcal{C} \in \Lambda(\vdash)$ such that $\mathcal{A}, \mathcal{B} \hookrightarrow \mathcal{C}$. But it is clear that these embeddings respect the designated element, i.e. they can be viewed as embeddings of matrices, so $(\mathcal{A}, \{1\}), (\mathcal{B}, \{1\}) \hookrightarrow (\mathcal{C}, \{1\})$ and $(\mathcal{C}, \{1\}) \in \mathrm{Mod}^{\equiv}(\vdash)$, as we wanted.

□

5.2 Łukasiewicz logic

Let us briefly introduce Łukasiewicz propositional logic (again, see [4] and [12] for more details). The logic Ł can be defined by the Hilbert calculus formed by the following axioms, together with the rule of *modus ponens*:

(Ł$_1$) $\varphi \to (\psi \to \varphi)$,

(Ł$_2$) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(Ł$_3$) $(\neg\varphi \to \neg\psi) \to (\varphi \to \psi)$,

(Ł$_4$) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

It is well-known that Ł is algebraized by the class **MV** of *MV-algebras*, i.e. algebras of type $(A, \oplus, \neg, 0)$ verifying the following identities:

(1) $(x \oplus y) \oplus z \approx x \oplus (y \oplus z)$,

(2) $x \oplus 0 \approx x$,

(3) $x \oplus \neg 0 \approx \neg 0$,

(4) $x \oplus y \approx y \oplus x$,

(5) $\neg\neg x \approx x$,

(6) $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$.

One relevant example of an MV-algebra is the *standard MV-algebra*, $[0,1]_{Ł}$, which has as universe the unit interval $[0,1]$ and the operations $\neg x := 1 - x$ and $x \oplus y := \min\{x + y, 1\}$. Another example is the two-element chain, $Ł_2$, which has these same operations and universe $\{0,1\}$. We remark that $Ł_2$ can be embedded into every nontrivial MV-algebra. Additionally, a theorem due to Chang [12, Lemma 3.2.11] tells us that $\mathbf{MV} = \mathbf{Q}([0,1]_{Ł})$.

On the other hand, since Ł is algebraized by the class $\mathbf{MV}$ and the sets $\tau(x) := \{x \approx 1\}$ and $\Delta(x,y) := \{x \to y, y \to x\}$, where 1 is a shorthand for $\neg 0$, we have by Fact 2.29 that

$$\mathrm{Mod}^{\equiv}(\vdash) = \{(\mathcal{A}, \{1\}) \mid \mathcal{A} \in \mathbf{MV}\}. \tag{5.2}$$

This already allows us to prove the following theorem:

Theorem 5.3. $\vdash_{Ł}$ *is meet-irreducible in* Log.

Proof. First, note that, since Ł has theorems, it is enough to check that the requirements (1), (2) and (3) from Theorem 4.6 for $k = 1$ hold in this case. Remember that the JEP, in virtue of Theorem 2.32, is equivalent to stating that there is some $(\mathcal{A}, \{1\}) \in \mathrm{Mod}^{\equiv}(\vdash_{Ł})$ such that $\mathrm{Mod}^{\equiv}(\vdash_{Ł}) = \mathbf{Q}((\mathcal{A}, \{1\}))$. Now, by Chang's theorem we know that $\mathbf{MV} = \mathbf{Q}([0,1]_{Ł}) = \mathbb{ISPP}_{\mathrm{U}}([0,1]_{Ł})$. But then, by 5.2,

$$\mathrm{Mod}^{\equiv}(\vdash_{Ł}) = \{(\mathcal{B}, \{1\}) \mid \mathcal{B} \in \mathbb{ISPP}_{\mathrm{U}}([0,1]_{Ł})\} = \mathbb{ISPP}_{\mathrm{U}}(([0,1]_{Ł}, \{1\})) = \mathbf{Q}(([0,1]_{Ł}, \{1\})).$$

On the other hand, requirements (2) and (3) are verified by noting that $(Ł_2, \{1\})$ can be embedded into every nontrivial element of $\mathrm{Mod}^{\equiv}(\vdash_{Ł})$. Indeed, every nontrivial element of such class contains a submatrix of prime cardinality, namely, the matrix $(Ł_2, \{1\})$, and every submatrix of a nontrivial element from $\mathrm{Mod}^{\equiv}(\vdash_{Ł})$ will contain $(Ł_2, \{1\})$ and, hence, be nontrivial.

□

5.3 Modal logics

Recall that the *least normal modal logic* is the set K of modal formulas, i.e. those in type $(\wedge, \vee, \to, \Box, 0, 1)$, such that: (i) contains all the tautologies from classical logic; (ii) contains the formula $\Box(x \to y) \to (\Box x \to \Box y)$; (iii) is closed under substitutions: for every substitution σ and every $\psi \in K$, we have that $\sigma(\psi) \in K$; (iv) is closed under modus ponens: if φ and $\varphi \to \psi$ belong to K then ψ belongs to K and (v) is closed under the necessitation rule: for each $\psi \in K$, it hols that $\Box\psi \in K$. This set K is not a logic in our sense. However, we can associate with it the *global consequence* $\vdash_K$ that can be axiomatized by the Hilbert calculus whose axioms are the formulas in K and whose sole rules are modus ponens (from φ and $\varphi \to \psi$

we may infer ψ) and the necessitation rule (from φ we may infer $\Box\varphi$).

More generally, a *normal modal logic* $L := K + \Gamma$ will be the smallest set of modal formulas that contains K and Γ and that is closed under modus ponens, substitutions and the necessitation rule. The global consequence associated with L will be, therefore, an axiomatic extension of $\vdash_K$ obtained by adding the new axioms from Γ to $\vdash_K$. In particular, we can consider the normal modal logics $S4 := K + T + 4$ and $S5 := K + T + 5$, where the axioms T, 4 and 5 are, respectively, $\Box\varphi \to \varphi$, $\Box\varphi \to \Box\Box\varphi$ and $\Diamond\varphi \to \Box\Diamond\varphi$. Then, the global consequence logics $\vdash_{S4}$ and $\vdash_{S5}$ are the axiomatic extensions of $\vdash_K$ with common axiom (T) $\Box\varphi \to \varphi$ and with respective axioms (4) $\Box\varphi \to \Box\Box\varphi$ and (5) $\Diamond\varphi \to \Box\Diamond\varphi$[1].

On the other hand, an *S4-algebra* is a modal algebra $(A, \wedge, \vee, \to, \Box, 0, 1)$ that verifies $\Box a \leqslant a$ and $\Box a \leqslant \Box\Box a$, for every $a \in A$. Similarly, an *S5-algebra* is a modal algebra $(A, \wedge, \vee, \to, \Box, 0, 1)$ verifying that $\Box a \leqslant a$ and $\Diamond a \leqslant \Box\Diamond a$, for every $a \in A$. Let us denote by **S4** and **S5** the classes of S4 and S5-algebras, respectively. Then, since once again these classes, together with the sets $\tau(x) := \{x \approx 1\}$ and $\Delta(x, y) := \{x \to y, y \to x\}$, witness the algebraization for $\vdash_{S4}$ and $\vdash_{S5}$, respectively, Fact 2.29 gives us that

$$\mathrm{Mod}^{\equiv}(\vdash_{S4}) = \{(\mathcal{A}, \{1\}) \,|\, \mathcal{A} \in \mathbf{S4}\} \quad \text{and} \quad \mathrm{Mod}^{\equiv}(\vdash_{S5}) = \{(\mathcal{A}, \{1\}) \,|\, \mathcal{A} \in \mathbf{S5}\}. \tag{5.3}$$

The idea of this section is, then, to replicate the previous arguments for $\vdash_{S4}$ and $\vdash_{S5}$. Nevertheless, first we need to study requirement (2) from Theorem 4.6 in the context of modal algebras. In this direction we have the next result. First, let us recall a basic observation on Boolean algebras [3, Cor. 7.34]:

Fact 5.4. *Up to isomorphism, the finite Boolean algebras are precisely the finite powerset Boolean algebras, i.e. algebras of the form* $(\wp(X), \cap, \cup, \to, \varnothing, X)$*, where*

$$U \to V := (X \setminus U) \cup V,$$

and where, additionally, X *is a finite set.*

Then, our result reads as follows:

Proposition 5.5. *Let* $V \subseteq \mathbf{MA}$ *be a variety of modal algebras. The following conditions are equivalent:*

- *(1) Every nontrivial element of* V *has a subalgebra of prime cardinal.*
- *(2) For every nontrivial algebra* $\mathcal{A} \in V$*, the subalgebra* $\mathbf{B}_2^{\mathcal{A}}$ *generated by* $\{0^{\mathcal{A}}, 1^{\mathcal{A}}\}$ *in* $\mathcal{A}$ *is exactly* $\{0^{\mathcal{A}}, 1^{\mathcal{A}}\}$*.*
- *(3) It holds that either* $V \vDash \Box 0 \approx 0$ *or* $V \vDash \Box 0 \approx 1$*.*

Proof. The direction (2) $\to$ (1) is clear: given a nontrivial algebra $\mathcal{A} \in V$ it is enough to select the subalgebra $\mathbf{B}_2^{\mathcal{A}}$. Let us see that (1) implies (2). Let $\mathcal{A} \in V$ be nontrivial and consider the subalgebra $\mathbf{B}_2^{\mathcal{A}}$ again. Then, $\mathbf{B}_2^{\mathcal{A}}$ is a nontrivial element of V, so it contains a subalgebra of prime cardinality. But note that $\mathbf{B}_2^{\mathcal{A}}$ lacks proper subalgebras for, given one, it should contain the interpretations of 0 and 1, and by

[1] Equivalently, $\vdash_{S5}$ can be seen as extending $\vdash_{S4}$ with the axiom $\varphi \to \Box\Diamond\varphi$.

definition it should contain $\mathbf{B}_2^{\mathcal{A}}$. Now, it is easy to note that $\mathbf{B}_2^{\mathcal{A}}$ is in fact a Boolean algebra and finite (by merely using the definition of modal algebras). By the previous Fact, there is a finite set X such that

$$(\mathbf{B}_2^{\mathcal{A}}, \wedge, \vee, \to, 0, 1) \cong (\wp(X), \cap, \cup, \to, \varnothing, X).$$

Hence, $|\mathbf{B}_2^{\mathcal{A}}| = 2^{|X|}$. But we have seen before that $|\mathbf{B}_2^{\mathcal{A}}|$ must be, simultaneously, prime. Therefore, we conclude that the only possible value for the cardinal of $\mathbf{B}_2^{\mathcal{A}}$ is 2. On the other hand, since $\{0^{\mathcal{A}}, 1^{\mathcal{A}}\} \subseteq \mathbf{B}_2^{\mathcal{A}}$, we have seen that $\mathbf{B}_2^{\mathcal{A}} = \{0^{\mathcal{A}}, 1^{\mathcal{A}}\}$, as desired.

Now, (2) $\to$ (3) holds: suppose that $V \nvDash \Box 0 \approx 0$ and $V \nvDash \Box 0 \approx 1$. This means that there are two algebras $\mathcal{A}, \mathcal{B} \in V$ such that $\mathcal{A} \nvDash \Box 0 \approx 0$ and $\mathcal{B} \nvDash \Box 0 \approx 1$. Consider the algebra $\mathcal{A} \times \mathcal{B} \in V$. Then is easy to see that $\mathcal{A} \times \mathcal{B} \nvDash \Box 0 \approx 0$ and $\mathcal{A} \times \mathcal{B} \nvDash \Box 0 \approx 1$. Therefore, $\{0^{\mathcal{A}\times\mathcal{B}}, 1^{\mathcal{A}\times\mathcal{B}}\}$ cannot be a subalgebra of $\mathcal{A} \times \mathcal{B}$ and, thus, differs from $\mathbf{B}_2^{\mathcal{A}\times\mathcal{B}}$, which contradicts (2).

Finally, (3) implies (2) because, given a nontrivial algebra $\mathcal{A} \in V$, $\{0^{\mathcal{A}}, 1^{\mathcal{A}}\}$ is closed under operations and verifies the properties of a modal algebra. This implies, as it can be readily seen, that either $\Box 0^{\mathcal{A}} = 0^{\mathcal{A}}$ or $\Box 0^{\mathcal{A}} = 1^{\mathcal{A}}$ for every $\mathcal{A} \in V$ and, hence, that either $V \vDash \Box 0 \approx 0$ or $V \vDash \Box 0 \approx 1$, as we wanted.

$\Box$

In the proof of the main theorem of this section we will rely on the following construction. Recall that a *Kripke frame* is a pair (X, R) formed by a set X and a binary relation R on X. Now, given a modal algebra $\mathcal{A}$, we can construct a Kripke frame $K(\mathcal{A})$ canonically associated with it as follows: we take the universe of $K(\mathcal{A})$ to be the family $U(\mathcal{A})$ of all ultrafilters of $\mathcal{A}$ and define the relation

$$(U, V) \in R_{\mathcal{A}} \text{ if and only if, for every } a \in A, \text{ it holds that } \Box a \in U \text{ implies that } a \in V.$$

Now, it is well-known that given an S_4-algebra $\mathcal{A}$, the relation $R_{\mathcal{A}}$ will turn out to be reflexive (T) and transitive (4). Analogously, if $\mathcal{A}$ is an S_5-algebra, the relation $R_{\mathcal{A}}$ will be reflexive and euclidean (5)[2]. It is well-known that these conditions are, in fact, equivalent [3, Props. 3.30, 3.31, 3.37].

Conversely, given a Kripke frame $\mathcal{X} := (X, R)$, we can associate a modal algebra $\wp(\mathcal{X}) := (\wp(X), \cap, \cup, \to, \Box, \varnothing, X)$, called the *complex algebra* of $\mathcal{X}$ (see, e.g., [3] and [22]), where $(\wp(X), \cap, \cup, \to, \varnothing, X)$ is a powerset Boolean algebra and where, given $U \subseteq X$,

$$\Box U := \{x \in X \mid \text{if } xRy \text{ then } y \in U\}.$$

Moreover, recall the following representation theorem [22, Thm. 2.42]:

Fact 5.6. *Let $\mathcal{A}$ be a modal algebra. Then, the map $\varepsilon : \mathcal{A} \to \wp(K(\mathcal{A}))$ defined by the rule*

$$\varepsilon(a) := \{U \in U(\mathcal{A}) \mid a \in U\},$$

[2] A relation S is said to be *euclidean* if, given elements x, y, z, it holds that xSy and xSz implies that ySz.

is in fact an embedding of $\mathcal{A}$ into the complex algebra $\wp(K(\mathcal{A}))$.

Now we have all the required tools for proving the main theorem of this section:

Theorem 5.7. *The logics $\vdash_{S4}$ and $\vdash_{S5}$ are meet-irreducible in* Log.

Proof. It is enough, as previously announced, to check the conditions (1), (2) and (3) of Theorem 4.6, since both $\vdash_{S4}$ and $\vdash_{S5}$ have theorems. Note that condition (3) holds directly: given a nontrivial element from $\mathrm{Mod}^{\equiv}(\vdash_{S4})$ and an arbitrary submatrix of it, since the signature of modal algebras includes the constant symbols 0 and 1, it is clear that such submatrix will contain the matrix whose universe is the subalgebra generated by the interpretations of 0 and 1, which is nontrivial. Hence, every submatrix of a nontrivial element from $\mathrm{Mod}^{\equiv}(\vdash_{S4})$ will be nontrivial. The case of $\vdash_{S5}$ is completely analogous.

Additionally, condition (2) holds in virtue of Proposition 5.5. Indeed, $\mathbf{S4} \vDash \Box 0 \approx 0$ and $\mathbf{S5} \vDash \Box 0 \approx 0$ are both true, since for every $S4$-algebra it holds that $\Box 0 \leqslant 0$ and no element is below 0 in these algebras, apart from 0 itself (and the same applies for $S5$-algebras).

Finally, let us check condition (1). We will present the proof for the case of $S4$, since the reasoning for $S5$ is completely analogous. By 5.3 we can consider two nontrivial elements of the form $(\mathcal{A}, \{1\}), (\mathcal{B}, \{1\}) \in \mathrm{Mod}^{\equiv}(\vdash_{S4})$. Consider $K(\mathcal{A}) \times K(\mathcal{B})$ as product of directed graphs, that is, the underlining relation (we shall call it R) will connect (x, y) with (z, w) if and only if $xR_{\mathcal{A}}z$ and $yR_{\mathcal{B}}w$. Then, we have the projection maps $\pi_{\mathcal{A}} : K(\mathcal{A}) \times K(\mathcal{B}) \to K(\mathcal{A})$ and $\pi_{\mathcal{B}} : K(\mathcal{A}) \times K(\mathcal{B}) \to K(\mathcal{B})$, respectively. Let us check that the map $\pi_{\mathcal{A}}^{-1} : \wp(K(\mathcal{A})) \to \wp(K(\mathcal{A}) \times K(\mathcal{B}))$ defined by the rule $S \mapsto \pi_{\mathcal{A}}^{-1}(S)$ is an embedding of modal algebras.

Clearly, it is an injective map: if $\pi_{\mathcal{A}}^{-1}(S) = \pi_{\mathcal{A}}^{-1}(T)$ and $x \in S$, this means that $\pi_{\mathcal{A}}(x, y) = x \in S$, so $(x, y) \in \pi_{\mathcal{A}}^{-1}(S) = \pi_{\mathcal{A}}^{-1}(T)$ and then $\pi_{\mathcal{A}}(x, y) = x \in T$. By symmetry, $S = T$. On the other hand, let us see that $\pi_{\mathcal{A}}^{-1}(\Box S) = \Box \pi_{\mathcal{A}}^{-1}(S)$, since the other operations are easily seen to be preserved. Suppose that $(x, y) \in \pi_{\mathcal{A}}^{-1}(\Box S)$. This means that $\pi_{\mathcal{A}}(x, y) = x \in \Box S$ so, for every z holds that, if $xR_{\mathcal{A}}z$ then $z \in S$. Now, take (a, b) such that $(x, y)R(a, b)$. Then, in particular, $xR_{\mathcal{A}}a$ and hence $a \in S$, so $\pi_{\mathcal{A}}(a, b) = a \in S$ and therefore $(a, b) \in \pi_{\mathcal{A}}^{-1}(S)$. Thus, we have seen that $(x, y) \in \Box \pi_{\mathcal{A}}^{-1}(S)$, so $\pi_{\mathcal{A}}^{-1}(\Box S) \subseteq \Box \pi_{\mathcal{A}}^{-1}(S)$. The other inclusion is very similar: let $(x, y) \in \Box \pi_{\mathcal{A}}^{-1}(S)$ and let us see that $(x, y) \in \pi_{\mathcal{A}}^{-1}(\Box S)$, that is, that $\pi_{\mathcal{A}}(x, y) = x \in \Box S$. Let a be such that $xR_{\mathcal{A}}a$. Then, $(x, y)R(a, y)$ and hence $(a, y) \in \pi_{\mathcal{A}}^{-1}(S)$, so $a \in S$, as we wanted.

Then, by Fact 5.6, there is an embedding $\varepsilon : \mathcal{A} \hookrightarrow \wp(K(\mathcal{A}))$ and, composing with $\pi_{\mathcal{A}}^{-1}$, we obtain the embedding $\pi_{\mathcal{A}}^{-1} \circ \varepsilon : \mathcal{A} \hookrightarrow \wp(K(\mathcal{A}) \times K(\mathcal{B}))$. The same reasoning applies to $\pi_{\mathcal{B}}$, so we have already seen that $\mathcal{A}$ and $\mathcal{B}$ can be embedded into $\wp(K(\mathcal{A}) \times K(\mathcal{B}))$. Finally, we have to check that, in fact, $\wp(K(\mathcal{A}) \times K(\mathcal{B}))$ is an element of $\mathbf{S4}$. It is enough to verify that R is reflexive and transitive, but this is straightforward from the fact that both $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$ are. Note that the previous composition of maps is strict so we have in fact that $(\mathcal{A}, \{1\})$ and $(\mathcal{B}, \{1\})$ can be embedded, as matrices, into $(\wp(K(\mathcal{A}) \times K(\mathcal{B})), \{1\})$, as we wanted. □

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