Universitat de Barcelona

Master in Pure and Applied Logic

Master's thesis

Rational and Delta expansions of the Nilpotent Minimum Logic

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Abstract

The aim of this thesis is to study some expansions of the Nilpotent minimum logic (denoted by **NML**), focusing on their lattices of axiomatic and finitary extensions and, additionally, exploring the structural completeness of these logics, alongside their variants (active structural completeness, passive structural completeness, ...).

The project includes research about the rational Nilpotent minimum logic (designated by **RNML**), which is obtained by adding rational constants to the language of **NML**. Moreover, we also study the Δ-core fuzzy logic obtained by expanding the language of **NML** with the Baaz Delta connective and examine the impact of the incorporation of rational constants to the language of this logic (which is equivalent to the addition of the Baaz Delta connective to **RNML**).

The thesis culminates with the corresponding analysis of an extension of the later logic which is obtained by introducing bookkeeping axioms for the Δ operator, motivated by the aim for the algebra of constants to form a subalgebra.

In the project, through comparative analysis, the differences and similarities between the lattices of axiomatic and finitary extensions among the previously mentioned expansions are evaluated, as well as how the structural completeness results obtained may vary from one logic to another.

Resumen

El objetivo de esta tesis es estudiar algunas expansiones de la lógica del Nilpotente mínimo (denotada por **NML**), centrándonos en sus retículos de extensiones axiomáticas y finitas y, además, explorando la completitud estructural de estas lógicas, junto con sus variantes (completitud estructural activa, completitud estructural pasiva, ...).

El proyecto abarca la lógica racional del Nilpotente mínimo (designada por **RNML**), que se obtiene añadiendo constantes racionales al lenguaje de **NML**. También se estudia la lógica fuzzy Δ-core obtenida mediante la expansión del lenguaje de **NML** con el operador Delta de Baaz, y se examina el impacto de la incorporación de constantes racionales al lenguaje de esta lógica (lo que equivale a añadir el operador Delta de Baaz a **RNML**).

La tesis culmina con el correspondiente análisis de una extensión de la última lógica presentada, resultante de la introducción de bookkeeping axioms para el operador Δ, motivada por el objetivo de que el álgebra de constantes forme una subálgebra.

En el proyecto, a través de un análisis comparativo, se evalúan las diferencias y similitudes entre los retículos de extensiones axiomáticas y finitas de las distintas expansiones mencionadas anteriormente, así como la forma en que varían los resultados de completitud estructural de una lógica a otra.

Resum

L'objectiu d'aquesta tesi és estudiar algunes expansions de la lògica del Nilpotent mínim (denotada per **NML**), centrant-nos en els seus reticles d'extensions axiomàtiques i finites i, a més, explorant la completitud estructural d'aquestes lògiques, juntament amb les seves variants (completitud estructural activa, completitud estructural passiva, ...).

El projecte abasta la lògica racional del Nilpotent mínim (designada per **RNML**), que s'obté afegint constants racionals al llenguatge de **NML**. També s'estudia la lògica fuzzy Δ-core obtinguda mitjançant l'expansió del llenguatge de **NML** amb l'operador Delta de Baaz, i s'examina l'impacte de la incorporació de constants racionals al llenguatge d'aquesta lògica (el que equival a afegir l'operador Delta de Baaz a **RNML**).

La tesi culmina amb l'anàlisi corresponent d'una extensió de l'última lògica presentada, que resulta de la introducció de bookkeeping axioms per a l'operador Δ, motivada per l'objectiu que l'àlgebra de constants formi una subàlgebra.

En el projecte, mitjançant una anàlisi comparativa, s'avaluen les diferències i similituds entre els reticles d'extensions axiomàtiques i finites de les diferents expansions esmentades anteriorment, així com la manera com varien els resultats de completitud estructural d'una lògica a una altra.

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Contents

Introduction

The Nilpotent minimum logic (**NML** for short) was introduced by Esteva and Godo in [12]. This logic is obtained from the Monoidal t-norm logic, in particular, it is the axiomatic extension that consists on adding the involutive condition $\neg\neg\varphi \to \varphi$ and the nilpotent minimum condition $(\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi)$. That is, a Hilbert calculus of **NML** in the language $\{\wedge, *, \to, \bot\}$ is given by the axioms:

$$
\begin{array}{rl}
\textbf{A1)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\textbf{A2)} & (\varphi * \psi) \to \varphi \\
\textbf{A3)} & (\varphi * \psi) \to (\psi * \varphi) \\
\textbf{A4)} & (\varphi \wedge \psi) \to \varphi \\
\textbf{A5)} & (\varphi \wedge \psi) \to (\psi \wedge \varphi) \\
\textbf{A6)} & (\varphi * (\varphi \to \psi)) \to (\varphi \wedge \psi) \\
\textbf{A7a)} & (\varphi \to (\psi \to \chi)) \to ((\varphi * \psi) \to \chi) \\
\textbf{A7b)} & ((\varphi * \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \\
\textbf{A8)} & ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \\
\textbf{A9)} & \bot \to \varphi \\
\textbf{A10)} & \neg\neg\varphi \to \varphi \\
\textbf{A11)} & (\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi)
\end{array}
$$

and Modus Ponens as the only inference rule. The connectives $\neg, \vee$ that appear in the axioms and are not in the language are defined as $\neg\varphi := \varphi \to \bot$ and $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

The axiomatic extensions of **NML** are axiomatized and characterized in [15]. Moreover, the lattice of finitary extensions of the Nilpotent minimum logic is studied in [16], in addition to the structural completeness of the logic and its axiomatic extensions.
Our goal in this thesis is to delve into some expansions of the Nilpotent minimum logic and, as has been done for **NML**, investigate their lattices of axiomatic and finitary extensions alongside the structural completeness of the logics and some of their extensions.

We recall that a logic is taken to be a consequence relation among a set of formulas in a particular propositional language. The notion of a logic being structurally complete (often denoted by SC) indicates that every proper extension has to contain new logical axioms (as opposed to only containing new rules of inference). In fact, structural completeness can be interpreted as some kind of maximality condition among logics with the same theorems.

A rule $\Gamma \rhd \varphi$ is said to be derivable in a logic $\vdash$ when $\Gamma \vdash \varphi$ (that is, when it is in $\vdash$). On the other hand, a rule $\Gamma \rhd \varphi$ is admissible in a logic $\vdash$ when its addition doesn't produce new theorems. In other words, whenever for every substitution σ on the set of formulas,

$$\text{if } \varnothing \vdash \sigma(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } \varnothing \vdash \sigma(\varphi).$$

Thus, it is clear that every derivable rule in a logic will also be admissible and we can obtain an analogous definition of SC: a logic is structurally complete when all admissible rules are

also derivable. Furthermore, a logic whose extensions are all strcturally complete is called hereditarily structurally complete (HSC, for short).
Moreover, a rule $\Gamma \rhd \varphi$ is said to be passive (passive $\vdash -$ admissible) in a logic $\vdash$ if for every substitution σ the set $\sigma(\Gamma)$ is not contained in $Th(\vdash)$, this means that Γ is not unifiable. Thus, such rule is admissible but can't be applied to theorems. We say that a logic is active structurally complete (denoted ASC) when all admissible rules are either derivable or passive and, on the other hand, we say it is passive structurally complete (PSC, for short) if every passive rule is derivable in the logic.

We will study such properties for some expansions of **NML** and its extensions. In order to do that, we will take an algebraic approach, by means of the algebraization of logics explained in [5], we will use some characterizations and results about the algebraic counterpart of structural completeness and its variants. Some of these key theorems and propositions are proven in [11] and [3].
In fact, the algebraization of logics in [5] enables us to establish a dual isomorphism between the lattice of finitary extensions of a logic and the lattice of subquasivarieties of its equivalent algebraic semantics. This can be restricted to another dual isomorphism between the lattice of axiomatic extensions of a logic and the lattice of subvarieties of its equivalent semantics. Thus, concerning the lattices of axiomatic and finitary extensions of the expansions of **NML**, we will also be able to study them from an algebraic perspective.

In particular, throughout this thesis, we will focus on some specific expansions. The paper is structured as follows:
In Section 1, we present some basic notions and results about the concept of logic, its algebraization and structural completeness. Moreover, we introduce the logics **MTL** and **NML**. We exhibit some results concerning filters on MTL-algebras, the characterization of axiomatic and finitary extensions of **NML**, and its structural completeness. Additionally, we discuss the introduction of constants and connectives into the language: rational constants and the Delta Baaz connective. In the latter case, we define (Δ)-core fuzzy logics and provide some results regarding them.

In Section 2, as has been done in [18] with Łukasiewicz logic, product logic, and Gödel-Dummett logic, we will analyze the expansion of the Nilpotent minimum logic with rational constants. When constants with suitable axioms are added to a logic, we enhance its expressiveness: given a formula φ and a constant $\mathbf{c}$, assignments sending $\mathbf{c} \to \varphi$ to the top element will be the ones that evaluate φ in the upset of the value of $\mathbf{c}$. This logic will be referred to as rational Nilpotent minimum logic (**RNML**, for short) and will be algebraized by the variety of rational NM-algebras.
Just like in [18, Section 7], we begin studying the lattice of extensions of **RNML**: Theorem 2.4 specifies the structure of the lattice of subvarieties of RNM (the class of rational NM-algebras), thereby providing a total description of the lattice of axiomatic extensions of **RNML**, presented in Corollary 2.6. This previous lattice will be an uncountable chain, while the lattice of finitary extensions will contain both uncountable chains and antichains.

To finalize the section, we study structural completeness in **RNNML**, similarly to how it has been done in [18, Section 8] for the case of the rational Gödel logic: in Theorem 2.7 a characterization of passive structural completeness is presented for extensions of **RNML**, while the other variants of structural completeness (SC, ASC, HSC) turn out to be equivalent, as stated in Theorem 2.9. Finally, Theorem 2.10 provides bases for the admissible rules on all the axiomatic extensions of **RNML**.

Section 3 is dedicated to the study of the Δ-core fuzzy logic $\mathbf{NML}_\Delta$ corresponding to **NML**, and to observe how the delta connective influences the lattices of extensions and the structural completeness results.
We begin by analyzing the lattice of axiomatic extensions and observe that it is analogous to the one obtained for **NML** (as seen in Proposition 3.14 and Theorem 3.15) because NM_Δ-chains have similar properties to NM-chains. Later in the project, we move on to inspecting the lattice of finitary extensions. To study this, we use results about discriminator varieties (Theorem 3.5) and present critical algebras along with some of their properties. A characterization of these algebras is provided in Corollary 3.24, and in Corollary 3.28 we present an axiomatization of the quasivarieties generated by a single critical algebra. Since NM_Δ is a locally finite variety, every quasivariety will be generated by its critical algebras (as stated in Theorem 3.21) thus, from Lemma 3.29 and the previous results, we are able to provide more information about the lattice of finitary extensions of $\mathbf{NML}_\Delta$.

Lastly, we move on to structural completeness. In Proposition 3.35, we identify the least elements of the lattice of subquasivarieties of NM_Δ (the class of NM_Δ-algebras) and, based on this, we prove that $\mathbf{NML}_\Delta$ is not structurally complete (Theorem 3.36). Moreover, there is no nontrivial axiomatic extension that is SC.
To conclude the section, in Theorem 3.39, we demonstrate that $\mathbf{NML}_\Delta$ is hereditarily almost structurally complete and present, for any axiomatic extension, an axiomatization of all (passive) admissible quasiequations in Corollary 3.40.

In Section 4 we focus on the expansion of $\mathbf{NML}_\Delta$ with rational constants, or equivalently, the Δ-core fuzzy logic corresponding to **RNML** (denoted by $\mathbf{RNML}_\Delta$). As we did in Section 2, we study the lattice of axiomatic extensions of the logic: from Theorem 4.8 and Corollary 4.7, we obtain information about the lattice of subvarieties of RNM_Δ (the class of rational NM_Δ-algebras), which is then translated into information about the lattice of axiomatic extensions in Corollary 4.12.
Then, we proceed to study structural completeness in the logic: in Theorem 4.14 we prove structural completeness holds for some particular axiomatic extensions.

Lastly, motivated by the aim for the algebra of constants to form a subalgebra, we delve into the logic obtained from $\mathbf{RNML}_\Delta$ by adding axioms that ensure the rational constants interact appropriately with the Δ connective.
It is seen in Corollary 4.19 that this logic has no proper consistent axiomatic extensions and, as stated in Theorem 4.20, it is also structurally complete.

Finally, Section 5 is devoted to give a conclusion about the project, summarizing the main similarities and differences observed among the lattices of axiomatic and finitary extensions, as well as the structural completeness results proven for the different expansions.

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1 Preliminaries

1.1 Logic, varieties and quasivarieties

In order to be able to introduce some notions about structural completeness, its variants and the admissibility of rules we first recall some concepts from propositional logic. We will also talk about varieties, quasivarieties and algebraic counterparts.

Definition 1.1. *Given an algebraic language $\mathcal{L}$ (i.e. a set of logical connectives with ascribed arities), we denote by $Fm_{\mathcal{L}}$ the set of formulas in $\mathcal{L}$ over a denumerable set of variables. When $\mathcal{L}$ is clear from the context we can just write Fm instead of $Fm_{\mathcal{L}}$.*

Then, a logic $\vdash$ is a consequence relation on the set of formulas Fm of a given language $\mathcal{L}$ which, moreover, is substitution-invariant: for every substitution σ on Fm and every $\Gamma \cup \{\varphi\} \subseteq Fm$

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Remark 1.2. It will always be assumed that the logics $\vdash$ considered are finitary, that is, for every $\Gamma \cup \{\varphi\} \subseteq Fm$

$$\text{if } \Gamma \vdash \varphi, \text{ then } \Delta \vdash \varphi \text{ for some finite } \Delta \subseteq \Gamma.$$

Definition 1.3. *Given two logics $\vdash$ and $\vdash'$ such that the language of $\vdash'$ extends the one of $\vdash$, we say that $\vdash'$ is an expansion of $\vdash$ if, for every set of formulas $\Gamma \cup \{\varphi\}$ in the language of $\vdash$,*

$$\Gamma \vdash \varphi \text{ implies } \Gamma \vdash' \varphi.$$

Furthermore, an expansion is said to be conservative when $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash' \varphi$.

Definition 1.4. *Given two logics $\vdash$ and $\vdash'$ in the same language we say that $\vdash'$ is an extension of $\vdash$ when $\Gamma \vdash' \varphi$ for every $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash \varphi$.*
In particular, an extension can be seen as an expansion in the same language.

Moreover, an extension $\vdash'$ of $\vdash$ is said to be axiomatic when there is a set $\Sigma \subseteq Fm$ closed under substitutions such that, for all $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash' \varphi \text{ if and only if } \Gamma \cup \Sigma \vdash \varphi.$$

Definition 1.5. *A formula φ is said to be a theorem of $\vdash$ if $\varnothing \vdash \varphi$. Then, the set of theorems of some logic $\vdash$ is denoted by $Th(\vdash) = \{\varphi \in Fm : \varnothing \vdash \varphi\}$.*

Definition 1.6. *A rule is an expression of the form $\Gamma \rhd \varphi$ for some finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$.*
When $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ we sometimes write $\gamma_1, \ldots, \gamma_n \rhd \varphi$ instead of $\Gamma \rhd \varphi$.

Definition 1.7. *A basis or an axiomatization of a logic $\vdash$ is a pair (A,R) where $A \subseteq Th(\vdash)$ and $R \subseteq \vdash$ are such that $\vdash$ is the smallest consequence relation containing $R \cup \{\langle \varnothing, \alpha \rangle : \alpha \in A\}$.*
This means that $\Gamma \vdash \varphi$ iff there is a proof (also called derivation) from $A \cup \Gamma$ of φ by means of the rules from R.

Given a basis (A,R) of some logic $\vdash$, it is interesting to know whether a formula φ is in $Th(\vdash)$, in other words, whether there is a proof of φ from A by applying rules of R. Moreover, it can be analyzed the size of such proofs. Proofs of theorems can be shortened by adding new rules and such extension of R can be done in two ways:

1. by adding derivable rules;

2. by adding admissible but non-derivable rules;

where:

Definition 1.8. *A rule $\Gamma \rhd \varphi$ is derivable in a logic $\vdash$ when $\Gamma \vdash \varphi$ (that is, when it is in $\vdash$).*

Definition 1.9. *A rule $\Gamma \rhd \varphi$ is admissible in a logic $\vdash$ when its addition doesn't produce new theorems. In other words, whenever for every substitution σ on Fm,*

$$\text{if } \varnothing \vdash \sigma(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } \varnothing \vdash \sigma(\varphi).$$

Remark 1.10. Every rule that is derivable in $\vdash$ is also admissible in $\vdash$.

Definition 1.11. *A logic $\vdash$ is said to be structurally complete (SC for short) if all admissible rules are also derivable. That is, if every proper extension has to contain some new logical axiom in its basis instead of just new rules of inference.*
Moreover, logics whose extensions are all structurally complete are called hereditarily structurally complete (also denoted by HSC).

Furthermore, every logic admits a structurally complete extension:

Proposition 1.12. [3, Proposition 1.4] *Every logic $\vdash$ has a unique structurally complete extension $\vdash^+$ with the same theorems. In fact, a rule is derivable in $\vdash^+$ precisely when it is admissible in $\vdash$.*

Hence, structurally complete logics are the maximal elements of the classes of logics having the same theorems.

Definition 1.13. *The logic $\vdash^+$ presented in the previous proposition is called structural completion of $\vdash$.*

Definition 1.14. *Moreover, since the derivable rules of $\vdash^+$ coincide with the admissible ones in $\vdash$, a set R of rules is called a base for the admissible rules on $\vdash$ if, by adding R to $\vdash$, we obtain an axiomatization of $\vdash^+$.*

In fact, structural completeness can be split in two halves:

Definition 1.15. *A rule $\Gamma \rhd \varphi$ is passive if for every substitution σ the set $\sigma(\Gamma)$ is not contained in $Th(\vdash)$, this means that Γ is not unifiable. Thus, such rule is admissible but can't be used in the proof of any theorem of the logic.*

Definition 1.16. *On the other hand, a rule $\Gamma \rhd \varphi$ is active if there exists a substitution σ such that $\varnothing \vdash \sigma(\gamma)$ for all $\gamma \in \Gamma$.*

Definition 1.17. *A logic $\vdash$ is said to be active structurally complete (ASC, for short) if every active rule that is admissible in $\vdash$ is also derivable in the logic.*
This type of logics are also called almost structurally complete and satisfy that the only admissible non-derivable rules are passive, thus, theorems can't be shortened by using method 2. presented before.

Definition 1.18. *Accordingly, a logic $\vdash$ is said to be passive structurally complete (PSC, for short) if every passive rule in the logic is also derivable in $\vdash$.*

Actually, structural completeness can also be studied from an algebraic point of view. In order to present some results on the matter we will first delve into the algebraization of logics (see [5]):

Definition 1.19. *Given $\mathbb{K}$ a class of algebras of the same type:*

$$\mathbb{I}(\mathbb{K}) = \{\boldsymbol{A} : \boldsymbol{A} \cong \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathbb{K}\}$$
$$\mathbb{S}(\mathbb{K}) = \{\boldsymbol{A} : \boldsymbol{A} \subseteq \boldsymbol{B} \text{ for some } \boldsymbol{B} \in \mathbb{K}\}$$
$$\mathbb{H}(\mathbb{K}) = \{\boldsymbol{A} : \boldsymbol{B} \twoheadrightarrow \boldsymbol{A} \text{ for some } \boldsymbol{B} \in \mathbb{K}\}$$
$$\mathbb{P}(\mathbb{K}) = \{\boldsymbol{A} : \boldsymbol{A} = \textstyle\prod_{i\in I} \boldsymbol{A}_i \text{ for some } \varnothing \neq \{\boldsymbol{A}_i : i \in I\} \subseteq \mathbb{K}\}$$
$$\mathbb{P}_U(\mathbb{K}) = \{\boldsymbol{A} : \boldsymbol{A} = \textstyle\prod_F^I \boldsymbol{A}_i \text{ for some } \varnothing \neq \{\boldsymbol{A}_i : i \in I\} \subseteq \mathbb{K} \text{ and some ultrafilter } F \text{ on } I\}$$

Throughout this project, the relations between the previous class operators are assumed to be known by the reader.

Definition 1.20. *A class of algebras of the same type $\mathbb{K}$ is a variety whenever $\mathbb{H}(\mathbb{K}) \subseteq \mathbb{K}$, $\mathbb{S}(\mathbb{K}) \subseteq \mathbb{K}$ and $\mathbb{P}(\mathbb{K}) \subseteq \mathbb{K}$.*

Moreover, since the arbitrary intersection of varieties is a variety, we can define a new operator $\mathbb{V}$ (named variety generated by) such that, for every class of algebras $\mathbb{K}$ of type τ,

$$\mathbb{V}(\mathbb{K}) = \bigcap\{\mathcal{V} : \mathcal{V} \text{ is a variety of type } \tau \text{ and } \mathbb{K} \subseteq \mathcal{V}\}.$$

Thus, $\mathbb{V}(\mathbb{K})$ is the least variety containing $\mathbb{K}$.

Theorem 1.21. [7, Theorem 11.9] *A class $\mathbb{K}$ is a variety iff it is an equational class.*

Definition 1.22. *A class of algebras of the same type $\mathbb{K}$ is a quasivariety whenever it contains the trivial algebra, $\mathbb{I}(\mathbb{K}) \subseteq \mathbb{K}$, $\mathbb{S}(\mathbb{K}) \subseteq \mathbb{K}$, $\mathbb{P}(\mathbb{K}) \subseteq \mathbb{K}$ and $\mathbb{P}_U(\mathbb{K}) \subseteq \mathbb{K}$.*

Analogously to the case of varieties, the arbitrary intersection of quasivarieties is a quasivariety. Thus, we can define a new operator $\mathbb{Q}$ (named quasivariety generated by) such that, for every class of algebras $\mathbb{K}$ of type τ,

$$\mathbb{Q}(\mathbb{K}) = \bigcap\{\mathcal{Q} : \mathcal{Q} \text{ is a quasivariety of type } \tau \text{ and } \mathbb{K} \subseteq \mathcal{Q}\}.$$

Therefore, $\mathbb{Q}(\mathbb{K})$ is the least quasivariety containing $\mathbb{K}$.

Theorem 1.23. [7, Theorem 2.25] *A class $\mathbb{K}$ is a quasivariety iff it is a quasiequational class.*

Definition 1.24. *A logic* $\vdash$ *is algebraizable by a quasivariety* $\mathbb{K}$ *if there are a finite set of equations* $\tau(x)$ *and a finite set of formulas* $\Delta(x,y)$ *in the same language such that, for every set of formulas* $\Gamma \cup \{\varphi\}$*:*

$$\begin{aligned} \Gamma \vdash \varphi \quad &\textit{if and only if} \quad \bigcup\{\tau(\gamma) : \gamma \in \Gamma\} \vDash_{\mathbb{K}} \tau(\varphi), \\ & x \approx y \Dashv\vDash_{\mathbb{K}} \bigcup\{\tau(\delta) : \delta \in \Delta(x,y)\}, \end{aligned}$$

where $\vDash_{\mathbb{K}}$ *is the equational consequence relative to* $\mathbb{K}$*. That is, for any set of tuples of formulas* $\{\langle \varphi_i, \psi_i \rangle : i \in I\} \cup \{\langle \varphi, \psi \rangle\}$ *in the language of* $\mathbb{K}$ *we have* $\{\varphi_i \approx \psi_i\}_{i \in I} \vDash_{\mathbb{K}} \varphi \approx \psi$ *if and only if, for each* $\boldsymbol{A} \in \mathbb{K}$ *and every assignment* $v_{\boldsymbol{A}}$ *in* $\boldsymbol{A}$*, it holds* $\boldsymbol{A} \vDash (\varphi \approx \psi)[v_{\boldsymbol{A}}]$ *whenever* $\boldsymbol{A} \vDash (\varphi_i \approx \psi_i)[v_{\boldsymbol{A}}]$ *is satisfied for every* $i \in I$*.*

In this case, $\mathbb{K}$ *is uniquely determined* (*see* [5, Theorem 2.15]) *and it is called the equivalent algebraic semantics of* $\vdash$*.*

Given a logic algebraized by a quasivariety $\mathbb{K}$ by means of finite sets of equations and formulas (τ and Δ, respectively) the lattice of extensions of $\vdash$ is dually isomorphic to $L_{\mathcal{Q}}(\mathbb{K})$, that is, the lattice of subquasivarieties of $\mathbb{K}$ (see [14, Corollary 3.40]).

The dual isomorphism is given by the map that sends an extension $\vdash'$ of the logic to the quasivariety axiomatized by the quasiequations

$$\bigwedge \tau(\gamma_1) \wedge \cdots \wedge \bigwedge \tau(\gamma_n) \Rightarrow \varepsilon \approx \delta,$$

where $\gamma_1, \ldots, \gamma_n \vdash' \varphi$ and $\varepsilon \approx \delta \in \tau(\varphi)$.

On the other hand, the inverse of the dual isomorphism sends a quasivariety $\mathbb{S} \in L_{\mathcal{Q}}(\mathbb{K})$ to the logic axiomatized by the rules

$$\Delta(\varphi_1, \psi_1) \cup \cdots \cup \Delta(\varphi_n, \psi_n) \rhd \delta,$$

where $\mathbb{S} \vDash (\varphi_1 \approx \psi_1 \wedge \cdots \wedge \varphi_n \approx \psi_n) \Rightarrow \varphi \approx \psi$ and $\delta \in \Delta(\varphi, \psi)$.

In fact, this isomorphism can be restricted to a dual isomorphism between the lattice of axiomatic extensions and $L_{\mathcal{V}}(\mathbb{K})$, that is, the lattice of subvarieties of $\mathbb{K}$.
Thus, the lattice of (axiomatic) extensions of a logic can be studied in terms of $L_{\mathcal{Q}}(\mathbb{K})$ (respectively $L_{\mathcal{V}}(\mathbb{K})$).

Therefore, we obtain corresponding notions of structural completeness for classes of algebraic structures:

Definition 1.25. *Let* $\mathbb{K}$ *be a quasivariety:*

- *It holds that* $\mathbb{K}$ *is structurally complete if, for every quasivariety* $\mathbb{K}'$*,*

$$\mathbb{K}' \subset \mathbb{K} \Rightarrow \mathbb{V}(\mathbb{K}') \subset \mathbb{V}(\mathbb{K})$$

- *We obtain that* $\mathbb{K}$ *is hereditarily structurally complete if, for every quasivarieties* $\mathbb{K}_1, \mathbb{K}_2$*,*

$$\mathbb{K}_1 \subset \mathbb{K}_2 \subseteq \mathbb{K} \Rightarrow \mathbb{V}(\mathbb{K}_1) \subset \mathbb{V}(\mathbb{K}_2)$$

And we obtain the analogous result to Proposition 1.12 for quasivarieties, in which $\mathbf{F}_{\mathbb{K}}(\omega)$ denotes the denumerably free algebra of $\mathbb{K}$:

Proposition 1.26. [3, Theorem 2.3] *Let $\mathbb{K}$ be a quasivariety. There is a unique subquasivariety $\mathbb{S}$ of $\mathbb{K}$ such that $\mathbb{S}$ is structurally complete and $\mathbb{V}(\mathbb{S}) = \mathbb{V}(\mathbb{K})$. In fact, $\mathbb{S} = \mathbb{Q}(\boldsymbol{F}_{\mathbb{K}}(\omega))$.*

Definition 1.27. *The quasivariety $\mathbb{Q}(\boldsymbol{F}_{\mathbb{K}}(\omega))$ presented in the previous proposition is called the structural completion of $\mathbb{K}$.*

Thus, just as structural completeness is a maximal condition in logic, when applied to quasivarieties, it implies a kind of minimality.

Before introducing the next result we recall that a quasivariety is said to be primitive when all its subquasivarieties are relative subvarieties:

Definition 1.28. *Given a quasivariety $\mathbb{K}$, a class $\mathbb{V} \subseteq \mathbb{K}$ is said to be a relative subvariety of $\mathbb{K}$ if it is axiomatized by equations relative to $\mathbb{K}$.*
The lattice of relative subvarieties of $\mathbb{K}$ is usually denoted by $L_{\mathcal{V}}(\mathbb{K})$. Notice that, when $\mathbb{K}$ is a variety, $L_{\mathcal{V}}(\mathbb{K})$ is the lattice of subvarieties of $\mathbb{K}$.

Now, we present the following theorem that gives a purely algebraic characterization of structural completeness and its variants and is based in Proposition 1.26, [3, Proposition 2.4] and [11, Theorem 3.1 and Corollary 3.2]:

Theorem 1.29. [18, Theorem 3.2] *If a logic $\vdash$ is algebraized by a quasivariety $\mathbb{K}$, then:*

1. *$\vdash$ is SC if and only if $\mathbb{K}$ is generated as a quasivariety by $\boldsymbol{F}_{\mathbb{K}}(\omega)$;*
2. *$\vdash$ is HSC if and only if $\mathbb{K}$ is primitive;*
3. *$\vdash$ is PSC if and only if every positive existential sentence is either true in all nontrivial members of $\mathbb{K}$ or false in all of them;*
4. *$\vdash$ is ASC if and only if $\boldsymbol{A}\times\boldsymbol{F}_{\mathbb{K}}(\omega) \in \mathbb{Q}(\boldsymbol{F}_{\mathbb{K}}(\omega))$ for every relatively subdirectly irreducible algebra $\boldsymbol{A} \in \mathbb{K}$; if there is a constant symbol in the language, then we can replace "$\boldsymbol{A} \times \boldsymbol{F}_{\mathbb{K}}(\omega) \in \mathbb{Q}(\boldsymbol{F}_{\mathbb{K}}(\omega))$" by "$\boldsymbol{A} \times \boldsymbol{F}_{\mathbb{K}}(0) \in \mathbb{Q}(\boldsymbol{F}_{\mathbb{K}}(\omega))$" in this statement.*

Where, given a quasivariety $\mathbb{K}$, an algebra $\mathbf{A}$ is said to be relatively subdirectly irreducible in $\mathbb{K}$ if the congruence $Id_{\mathbf{A}}$ is completely meet-irreducible in $Con_{\mathbb{K}}\mathbf{A}$ (for $Con_{\mathbb{K}}\mathbf{A}$ the set of $\mathbb{K}$-congruences of $\mathbf{A}$, that is, the congruences θ of $\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathbb{K}$).

Finally, we use the isomorphism between the lattice of finitary extensions of a logic and $L_{\mathcal{Q}}(\mathbb{K})$ (being $\mathbb{K}$ the equivalent algebraic semantics) in the following result. The theorem presents the effects that passive structural completeness has on the lattice of extensions of an algebraizable logic:

Theorem 1.30. [25, Theorem 4.3 and Remark 5.13] *Let $\vdash$ be a logic algebraized by a quasivariety $\mathbb{K}$. If $\vdash$ is PSC, then every member of $L_{\mathcal{Q}}(\mathbb{K})$ has the JEP. Moreover, for every extension $\vdash'$ of $\vdash$ there exists an algebra $\boldsymbol{A}$ such that, for every $\Gamma \cup \{\varphi\} \subseteq Fm$,*

$$\Gamma \vdash' \varphi \quad \textit{if and only if} \quad \tau[\Gamma] \vDash_{\boldsymbol{A}} \tau(\varphi).$$

Where τ *is the set of equations witnessing the algebraization of* $\vdash$*.*

Recall that a quasivariety $\mathbb{K}$ has the joint embedding property (JEP) when every two nontrivial members of $\mathbb{K}$ can be embedded into a common element of $\mathbb{K}$. This happens iff $\mathbb{K}$ is generated as a quasivariety by a single algebra (see [19, Proposition 2.1.19]).

1.2 The logic MTL and some ways to expand its language

In this subsection we will present the Monoidal t-norm logic (denoted by **MTL**) and we will study different ways to expand its language.

This logic was introduced by Esteva and Godo in [12] and constitutes a strengthening of the Monoidal logic (introduced by Höhle in [21]) and also a weakening of Basic logic (which was presented firstly by Hájek in [20]) since it is obtained by replacing the divisibility axiom by the weaker axiom $(\varphi * (\varphi \to \psi)) \to (\varphi \wedge \psi)$. That is, a Hilbert calculus of **MTL** in the language $\{\wedge, *, \to, \bot\}$ is given by the axioms:

A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
A2) $(\varphi * \psi) \to \varphi$
A3) $(\varphi * \psi) \to (\psi * \varphi)$
A4) $(\varphi \wedge \psi) \to \varphi$
A5) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
A6) $(\varphi * (\varphi \to \psi)) \to (\varphi \wedge \psi)$
A7a) $(\varphi \to (\psi \to \chi)) \to ((\varphi * \psi) \to \chi)$
A7b) $((\varphi * \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
A8) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
A9) $\bot \to \varphi$

and Modus Ponens as the only inference rule.

The Monoidal t-norm logic satisfies the following Local Deduction Theorem and substitution rule:

Proposition 1.31. [9, Proposition 2.1] *For each set of formulas* $\Gamma \cup \{\varphi, \psi, \chi\}$*, it holds:*

$$(LDT) \quad \Gamma, \varphi \vdash_{\boldsymbol{MTL}} \psi \text{ iff there is } n \in \mathbb{N} \text{ such that } \Gamma \vdash_{\boldsymbol{MTL}} \varphi^n \to \psi$$
$$(Subst.) \quad \varphi \leftrightarrow \psi \vdash_{\boldsymbol{MTL}} \chi(\varphi) \leftrightarrow \chi(\psi)$$

Where φ^n *is used as a shorthand for* $\varphi * \overset{n}{\ldots} * \varphi$*.*

And, from this, we can introduce a specific class of logics:

Definition 1.32. *We say that a finitary logic* ***L*** *in a countable language is a core fuzzy logic if*

- ***L*** *expands* ***MTL****,*

- ***L*** *satisfies (Subst.),*
- ***L*** *satisfies (LDT).*

Moreover, the logic **MTL** is algebraizable, in fact, so will be every finitary extension of the logic. The class of all MTL-algebras will be its equivalent algebraic semantics, which is called MTL and constitutes a variety (see [12, Proposition 2]):

Definition 1.33. *An MTL-algebra is a structure* $\langle S, *, \rightarrow_*, \wedge, \vee, 0, 1\rangle$ *such that the following conditions hold:*

- $\langle S, *, 1\rangle$ *is a commutative monoid.*
- $\langle S, \wedge, \vee, 0, 1\rangle$ *is a lattice with 0 and 1 as its smallest and greatest elements, respectively.*
- *For all* $a, b, c \in S$*,*
$$a \leq b \rightarrow_* c \text{ iff } a * b \leq c,$$
where $\leq$ *is the order given by the lattice structure.*
- *S satisfies the pre-linearity equation* $(x \rightarrow y) \vee (y \rightarrow x) \approx 1$*.*

The order relation we consider in this definition is clearly compatible with $*$, this follows from the first two points. Moreover, the infimum and supremum of any two elements of S always exists.

Definition 1.34. *An MTL-algebra whose order relation is total is called a linearly ordered MTL-algebra or, equivalently, an MTL-chain.*

Proposition 1.35. [12, Proposition 3] *Every MTL-algebra is a subdirect product of linearly ordered MTL-algebras.*

In fact, the logic **MTL** has strong completeness with respect to the class of MTL-chains ([12, Theorem 1]).

Moreover, we can introduce a class of linearly ordered MTL-algebras constituted by what we call standard MTL-algebras. In order to do that, we recall the definition of t-norm and left-continuous t-norm.

Definition 1.36. *A t-norm is a binary operation* $\hat{\circ}$ *on the real interval* $[0, 1]$ *which is commutative, associative, has 1 as a neutral element and is weakly decreasing (for any* $a, b, c \in [0, 1]$*, if* $a \leq b$ *then* $a \,\hat{\circ}\, c \leq b \,\hat{\circ}\, c$*).*
A t-norm is left-continuous if whenever we consider two increasing sequences $\langle a_n : n \in \mathbb{N}\rangle$*,* $\langle b_n : n \in \mathbb{N}\rangle$ *of reals in the interval* $[0, 1]$ *such that* $sup\{a_n : n \in \mathbb{N}\} = a$ *and* $sup\{b_n : n \in \mathbb{N}\} = b$*, then*
$$sup\{a_n \,\hat{\circ}\, b_n : n \in \mathbb{N}\} = a \,\hat{\circ}\, b.$$

We know, as it has been stated in [12, page 272], that the following property holds:

Proposition 1.37. *A t-norm is left-continuous if and only if it admits an associated R-implication.*

And, in this case, the R-implication is defined, for any $a, b \in [0,1]$, as follows:

$$a \Rightarrow b := sup\{c \in [0,1] : a \,\hat{\circ}\, c \leq b\}.$$

Therefore, we can use this result to study the relation that exists between left-continuous t-norms and MTL-algebras. On one hand, every left-continuous t-norm defines an MTL-chain:

Definition 1.38. *Given a left-continuous t-norm $\hat{\circ}$, if we consider the algebra $\boldsymbol{[0,1]}_{\hat{\circ}} = \langle [0,1], \hat{\circ}, \Rightarrow, \leq, 0, 1\rangle$ (where the order is the usual order on the reals) then, it is clear that we have a linearly ordered MTL-algebra. MTL-algebras of this form are called standard.*

On the other hand, every countable MTL-chain is embeddable into a standard MTL-algebra (see [22, Theorem 3.2]). Thus, the logic **MTL** is complete with respect to evaluations into standard MTL-algebras (as it is proved in [22, Theorem 3.3]).

We can now move on to the study of filters and prime filters of MTL-algebras and its consequences:

Definition 1.39. *Let $\boldsymbol{A}$ be an MTL-algebra, a non-empty set $F \subseteq A$ is said to be a filter of $\boldsymbol{A}$ if:*

- *It is upward closed: for any $a \in F$, if $a \leq b$, then $b \in F$.*
- *It is closed under multiplication: for any $a, b \in F$, $a * b \in F$.*

Moreover, a filter F of $\boldsymbol{A}$ is said to be prime when, for all $a, b \in A$, if $a \vee b \in F$ holds then, either $a \in F$ or $b \in F$.

When the set of filters of $\mathbf{A}$, $Fi(\mathbf{A})$, is ordered under the inclusion relation, it becomes a lattice. Furthermore, the following result states that the lattices $Fi(\mathbf{A})$ and $Con(\mathbf{A})$ are isomorphic:

Theorem 1.40. [20, Lemma 2.3.14] *Let $\boldsymbol{A}$ be an MTL-algebra. Then, the map $\theta(\_) : Fi(\boldsymbol{A}) \to Con(\boldsymbol{A})$, defined by the rule*

$$\theta(\_)(F) = \theta_F := \{\langle a, b\rangle \in A \times A : a \to b, b \to a \in F\},$$

is a lattice isomorphism. Furthermore, the following conditions are equivalent for a filter F of $\boldsymbol{A}$:

1. *F is prime;*
2. *$\boldsymbol{A}/\theta_F$ is a chain;*
3. *$\boldsymbol{A}/\theta_F$ is finitely subdirectly irreducible.*

Henceforth, we will write $\boldsymbol{A}/F$ as a shorthand for $\boldsymbol{A}/\theta_F$.

In fact, by adapting the result presented in [24, Lemma 2.3], we obtain a lemma that states the existence of prime filters on MTL-algebras:

Lemma 1.41. *Let $\boldsymbol{A}$ be an MTL-algebra, and let $I \subseteq A \setminus \{1\}$ be such that $a \vee c \in I$, whenever $a, c \in I$. Then, there is a prime filter F of $\boldsymbol{A}$ disjoint from I.*

Now, we move on to studying how we can expand the language of **MTL** with the Baaz Delta connective (which will be denoted by Δ). We will also introduce the class of core fuzzy logics, its Δ-expansions and some properties of these types of logics.

Definition 1.42. *Given a core fuzzy logic $\boldsymbol{L}$, the logic $\boldsymbol{L}_\Delta$ is the expansion of $\boldsymbol{L}$ obtained by enriching the language with the unary connective Δ and adding to its axiomatization the following rules and axioms:*

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$

(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$

(Δ3) $\Delta\varphi \to \varphi$

(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$

(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$ *(Necessitation rule)* $\dfrac{\varphi}{\Delta\varphi}$

Definition 1.43. *Given a core fuzzy logic, an L_Δ-algebra is a structure* **A** *satisfying that it is an L-algebra expanded by the operation Δ. Moreover, $\mathbf{A} \vDash \alpha \approx \overline{1}$ for each $\alpha \in \{\Delta1, \dots, \Delta5\}$, and $\mathbf{A} \vDash \Delta(\overline{1}) \approx \overline{1}$.*

We have already seen that, given any core fuzzy logic **L**, the Local Deduction Theorem is satisfied. However, the logic $\mathbf{L}_\Delta$ will enjoy a different form of Deduction Theorem:

Theorem 1.44. (Δ-Deduction Theorem) [10, Theorem 2.2.1] *Let $\boldsymbol{L}$ be a core fuzzy logic. Then, for any set of formulas $\Gamma \cup \{\varphi, \psi\}$ of $\boldsymbol{L}_\Delta$, the following equivalence holds:*

$$(DT_\Delta) \qquad \Gamma, \varphi \vdash_{\boldsymbol{L}_\Delta} \psi \textit{ iff } \Gamma \vdash_{\boldsymbol{L}_\Delta} \Delta\varphi \to \psi.$$

Now, we can introduce a particular class of logics:

Definition 1.45. *We say that a finitary logic $\boldsymbol{L}$ in a countable language is a Δ-core fuzzy logic if*

- *$\boldsymbol{L}$ expands $\boldsymbol{MTL}_\Delta$,*
- *$\boldsymbol{L}$ satisfies (Subst.),*
- *$\boldsymbol{L}$ satisfies (DT_Δ).*

The following result, in particular, lets us know which expansions of $\mathbf{MTL}_\Delta$ are Δ-core fuzzy logics:

Proposition 1.46. [9, Proposition 2.11] *Let $\boldsymbol{L}$ be an expansion of $\boldsymbol{MTL}$ (respectively of $\boldsymbol{MTL}_\Delta$) satisfying (Subst.). Then, $\boldsymbol{L}$ is a (Δ-)core fuzzy logic if and only if it is an axiomatic expansion of $\boldsymbol{MTL}$ ($\boldsymbol{MTL}_\Delta$).*

Moreover, from the axioms of Δ, we easily obtain:

Proposition 1.47. *Let $\boldsymbol{L}_\Delta$ be a Δ-core fuzzy logic and* $\mathbf{B}$ *an $\boldsymbol{L}_\Delta$-chain. Then, $\Delta^{\mathbf{B}}x = 1^{\mathbf{B}}$ if $x = 1^{\mathbf{B}}$ and $\Delta^{\mathbf{B}}x = 0^{\mathbf{B}}$ otherwise.*

Remark 1.48. In the previous case, the operator Δ will give us a notion of full truth and, furthermore, the ordering of truth values will be internalized in $\mathbf{L}_\Delta$ since $\Delta(x \to y) = \overline{1}$ iff $x \leq y$.

Given a Δ-core fuzzy logic $\mathbf{L}_\Delta$, the standard L_Δ-algebras will be the standard L-algebras expanded by the operation Δ defined as

$$\Delta(x) = \begin{cases} 1 & \text{if } x = 1; \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

due to Proposition 1.47, since they are linearly ordered.

Finally, we present some propositions collecting basic properties of (Δ-)core fuzzy logics:

Proposition 1.49. [9, Proposition 2.14] *Let $\boldsymbol{L}$ be a ($\Delta-$)core fuzzy logic.*

- *$\boldsymbol{L}$ is algebraizable with the same translations as* ***MTL****.*
- *The class of L-algebras is the equivalent algebraic semantics of $\boldsymbol{L}$ and it is a variety.*
- *Every L-algebra is representable as a subdirect product of L-chains.*
- *Subdirectly irreducible L-algebras are L-chains.*
- *For every set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\boldsymbol{L}} \varphi$ if and only if $\Gamma \vDash_{\{L\text{-}chains\}} \varphi$.*

Proposition 1.50. [10, page 37] *Given a core fuzzy logic $\boldsymbol{L}$, the logic $\boldsymbol{L}_\Delta$ is strong (respectively, finite strong) standard complete if and only if $\boldsymbol{L}$ is strong (respectively, finite strong) standard complete.*

Proposition 1.51. [9, Proposition 2.15] *For every core fuzzy logic $\boldsymbol{L}$, $\boldsymbol{L}_\Delta$ is a conservative expansion of $\boldsymbol{L}$. That is, for any set of formulas $\Gamma \cup \{\varphi\}$ in the language of $\boldsymbol{L}$, if $\Gamma \vdash_{\boldsymbol{L}_\Delta} \varphi$ then $\Gamma \vdash_{\boldsymbol{L}} \varphi$.*

Where, for each core fuzzy logic $\mathbf{L}$, the logic $\mathbf{L}_\Delta$ is its corresponding Δ-core fuzzy logic.

Finally, let $\mathbf{L}$ be an arbitrary extension of **MTL** such that, for some t-norm $*$, $\mathbb{V}([\mathbf{0,1}]_*)$ algebraizes $\mathbf{L}$ and $[0,1] \cap \mathbb{Q}$ is the universe of a subalgebra of $[\mathbf{0,1}]_*$ (which we will denote by $[\mathbf{0,1}]_* \cap \mathbf{Q}$). We can consider the expansion of $\mathbf{L}$ obtained by adding to the language a set of rational constants:

$$\mathscr{C} = \{\mathbf{c}_q : q \in [0,1] \cap \mathbb{Q}\},$$

where $\mathbb{Q}$ denotes the set of rational numbers.
During the project we will focus on the case where $\mathbf{L}$ is a particular logic, hence, $*$ is a specific t-norm (under this circumstances we will omit the subscript of the standard algebra).

Definition 1.52. *We call bookkeeping axioms of $\boldsymbol{[0,1]}_*$ the axioms in the language of $\boldsymbol{[0,1]}_*$ expanded with the constants in $\mathscr{C}$ that are of the form:*

$$\boldsymbol{c}_p * \boldsymbol{c}_q \leftrightarrow \boldsymbol{c}_{p *^{[0,1]} q}, \qquad \boldsymbol{c}_p \rightarrow \boldsymbol{c}_q \leftrightarrow \boldsymbol{c}_{p \rightarrow^{[0,1]} q}, \qquad \boldsymbol{c}_0 \leftrightarrow 0, \qquad \boldsymbol{c}_1 \leftrightarrow 1,$$

for all $p, q \in [0,1] \cap \mathbb{Q}$.

We can also express them as equations of the form: $\boldsymbol{c}_p * \boldsymbol{c}_q \approx \boldsymbol{c}_{p *^{[0,1]} q}$, $\boldsymbol{c}_p \rightarrow \boldsymbol{c}_q \approx \boldsymbol{c}_{p \rightarrow^{[0,1]} q}$, $\boldsymbol{c}_0 \approx 0$, $\boldsymbol{c}_1 \approx 1$, where will denote by $\mathscr{B}(\boldsymbol{[0,1]}_*)$ the set constituted by these equations, for all $p, q \in [0,1] \cap \mathbb{Q}$.

We remark that we do not include bookkeeping axioms for $\neg$, $\wedge$ and $\vee$ since $\neg$ can be expressed in terms of $\rightarrow$ and 0 and $\wedge, \vee$ correspond to the minimum and the maximum respectively.

Then, the rational **L** logic (also denoted by **RL**) is the one obtained by adding to the axiomatization of **L** the bookkeeping axioms of $\mathbf{[0,1]}_*$.

We proceed to define its equivalent algebraic semantics:

Definition 1.53. *An algebra $\boldsymbol{A}$ in the language of MTL-algebras expanded with the constants in $\mathscr{C}$ is said to be a rational L-algebra if the MTL-reduct of $\boldsymbol{A}$ is an L-algebra and $\boldsymbol{A}$ validates the bookkeeping axioms $\mathscr{B}(\boldsymbol{[0,1]}_*)$.*

We denote by RL the variety of rational L-algebras: If L is a variety, that is, an equational class (let's say $L = Mod(\Sigma)$ for Σ some set of equations), we are considering RL to be $Mod(\Sigma \cup \mathscr{B}(\mathbf{[0,1]}_*))$ thus, it will also be a variety.

Definition 1.54. *The canonical rational L-algebra can be obtained by expanding the standard L-algebra $\boldsymbol{[0,1]}_*$ with the natural interpretation of the constants in $\mathscr{C}$ (interpreting $\boldsymbol{c}_q$ as the rational q, for every $q \in [0,1] \cap \mathbb{Q}$). We will denote this algebra by $\boldsymbol{[0,1]}_*^{\boldsymbol{Q}}$ and its subalgebra with universe $[0,1] \cap \mathbb{Q}$ by $\boldsymbol{[0,1]}_*^{\boldsymbol{Q}} \cap \boldsymbol{Q}$.*

For readability's sake we will usually omit the superscript **Q** from here onwards.

Remark 1.55. We notice that Theorem 1.40 will also apply to rational L-algebras since **L** is an extension of **MTL** (L-algebras will be, in particular, MTL-algebras) and the addition of constants to a given algebra does not change its congruences and filters.

From the viewpoint of logic, the variety RL algebraizes the rational **L** logic, that is, for every set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathbf{RL}} \varphi \text{ if and only if } \tau[\Gamma] \vDash_{RL} \tau(\varphi)$$

where $\tau := \{x \approx 1\}$.

1.3 The Nilpotent Minimum Logic

Now that we have presented the logic **MTL** and some important results about it, we can continue by studying one of its extensions:

The Nilpotent minimum logic (denoted by **NML**), is the logic obtained from the Monoidal t-norm logic by adding the involutive condition $\neg\neg\varphi \to \varphi$ (where $\neg\varphi := \varphi \to \bot$) and the nilpotent minimum condition $(\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \varphi * \psi)$.

Since **NML** is an axiomatic extension of **MTL**, all the formulas presented in [12, Proposition 1] are also provable in **NML**. Thus:

Proposition 1.56. *For each set of formulas $\Gamma \cup \{\varphi, \psi\}$, it holds:*

$$(LDT) \quad \Gamma, \varphi \vdash_{\boldsymbol{NML}} \psi \textit{ iff there is } n \in \mathbb{N} \textit{ such that } \Gamma \vdash_{\boldsymbol{NML}} \varphi^n \to \psi.$$

Furthermore, in the Nilpotent minimum logic it is clear that $x^3 \equiv x^2$. Therefore, we can modify the previous proposition and obtain a Deduction Theorem for **NML**:

Theorem 1.57. *For each set of formulas $\Gamma \cup \{\varphi, \psi\}$, it holds:*

$$(DT) \quad \Gamma, \varphi \vdash_{\boldsymbol{NML}} \psi \textit{ iff } \Gamma \vdash_{\boldsymbol{NML}} \varphi^2 \to \psi.$$

The logic will be algebraized by the class of all NM-algebras:

Definition 1.58. *A nilpotent minimum algebra (NM-algebra) is a bounded residuated lattice $\boldsymbol{A} = \langle A; *, \to, \wedge, \vee, \neg, 0, 1\rangle$ where $\wedge$, $\vee$ are the meet and join respectively, $\langle *, \to\rangle$ is a residuated pair, $\neg$ is the negation associated to $\to$ (for any $a \in A$, $\neg a := a \to 0$) and 0,1 are the lower and upper bound respectively.*
Furthermore, nilpotent minimum algebras satisfy the following equations:

- *Pre-linearity:* $(x \to y) \vee (y \to x) \approx 1$.
- *Involutivity:* $\neg\neg x \approx x$.
- *Nilpotent minimum:* $(x * y \to 0) \vee (x \wedge y \to x * y) \approx 1$.

That is, they are MTL-algebras satisfying the last two conditions.

Definition 1.59. *We name standard NM-algebra the NM-algebra whose universe is $[0, 1]$. Esteva and Godo* ([12, Proposition A.6]) *proved that there is only one nilpotent minimum algebra defined on $[0, 1]$ up to isomorphism. We take it to be $\boldsymbol{[0,1]} = \langle [0, 1]; *, \to, \wedge, \vee, \neg, 0, 1\rangle$ where $\wedge$ and $\vee$ are the meet and join respectively with the usual order and*

$$\neg x = 1 - x,$$
$$x * y = \begin{cases} 0 & \text{if } y \le \neg x; \\ x \wedge y & \text{otherwise.} \end{cases}$$
$$x \to y = \begin{cases} 1 & \text{if } x \le y; \\ \neg x \vee y & \text{otherwise.} \end{cases}$$

Remark 1.60. In fact, given a totally ordered set A with upper bound 1 and lower bound 0 equipped with an involutive negation $\neg$ which is order preserving, $\wedge$ and $\vee$ defined as the meet and join and

$$a * b = \begin{cases} 0 & \text{if } b \leq \neg a; \\ a \wedge b & \text{otherwise.} \end{cases}$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b; \\ \neg a \vee b & \text{otherwise.} \end{cases}$$

for every $a, b \in A$, we obtain that $\mathbf{A} = \langle A; * \to, \wedge, \vee, \neg, 0, 1 \rangle$ is an NM-chain:

Clearly $\mathbf{A}$ is bounded, totally ordered, satisfies the involutive equation and it holds that $\wedge$ and $\vee$ are the meet and join. Since 0 is the lower bound, by definition of $\to$, it is clear that $\neg$ is the associated negation ($\neg x = x \to 0$) and since we have a total order (it either holds $x \leq y$ or $y \leq x$) we obtain that the pre-linearity equation holds.
It is easily checked that $\langle *, \to \rangle$ is a residuated pair and it is also clear that $\langle A, *, 1 \rangle$ is a commutative monoid where 1 is the neutral element of $*$ (because it is the upper bound of A).
Finally, by definition of $*$ we know it either holds $x * y = 0$ or $x * y = x \wedge y$ therefore, the nilpotent minimum equation is satisfied.
Thus, we have seen that $\mathbf{A}$ is a NM-chain.

Moreover, every NM-chain is of this form.

We have previously defined the concept of NM-algebra and, now, we can present some results and examples of NM-chains (totally ordered NM-algebras):

Proposition 1.61. [15, Proposition 1] *Each NM-algebra is representable as a subdirect product of NM-chains.*

Example 1.62. For every $n \in \omega$ we consider $\mathbf{A}_{2n+1} = \langle [-n, n] \cap \mathbb{Z}; *, \to, \wedge, \vee, \neg, -n, n \rangle$ where $\wedge$ and $\vee$ are the meet and join with the usual order, the negation is also defined as usual ($\neg x := -x$) and, for every $m, k \in \{-n, \dots, 0, \dots, n\}$ we have:

$$m * k = \begin{cases} -n & \text{if } k \leq -m; \\ m \wedge k \ (= min\{m, k\}) & \text{otherwise.} \end{cases}$$

$$m \to k = \begin{cases} n & \text{if } m \leq k; \\ -m \vee k \ (= max\{-m, k\}) & \text{otherwise.} \end{cases}$$

Note that $\mathbf{A}_1$ is the trivial algebra.

Definition 1.63. *Given an NM-algebra* $\boldsymbol{A}$ *and some element* $a \in A$*:*

- *If* $a > \neg a$ *then we say that* a *is a positive element of* $\boldsymbol{A}$*.*
- *If* $a < \neg a$ *then we say that* a *is a negative element of* $\boldsymbol{A}$*.*

- *We call a a negation fixpoint iff $a = \neg a$.*

In [21] Höhle proves that, given any NM-algebra, the negation fixpoint, if it exists, is unique.

Proposition 1.64. *Given an NM-chain $\boldsymbol{C}$ with negation point $c \in C$, $C \setminus \{c\}$ is the universe of a subalgebra of $\boldsymbol{C}$ which we denote by $\boldsymbol{C}^{-}$.*

Proof. It is enough to see that $C \setminus \{c\}$ is closed under $\to, *, \neg, 0, 1$ (it is trivially closed under $\wedge$ and $\vee$ since they are min and max respectively):

- Closed under 0: The fixpoint c satisfies $\neg c = c \to 0 = c$. By Remark 1.60 we know how $\to$ and $*$ are defined, thus, we know $0 \to 0 = 1$ which means that 0 can't be the fixpoint and $0 \in C \setminus \{c\}$.

- Closed under 1: We assume $1 = c \to 0$ is the fixpoint c and we will arrive to a contradiction. If this holds then $1 \leq c \to 0$ and, by the residuation law, $c = 1 * c \leq 0$ which can't be the case because 0 is the lower bound and we have seen that $c \neq 0$. Therefore, $1 \in C \setminus \{c\}$.

- Closed under $\neg$: If for some $a \in C \setminus \{c\}$ we have $\neg a = c$, then $a = \neg\neg a = \neg c = c$ holds. We were assuming $a \neq c$, therefore we have showed that we have closure under $\neg$.

- Closed under $*$: For any $a, b \in C \setminus \{c\}$, by Remark 1.60, we know either $a * b = 0$ or $a * b = a \wedge b$ and, in both cases, $a * b \neq c$ since $\wedge$ is defined as the minimum and we have seen $0 \neq c$. Thus, $a * b \in C \setminus \{c\}$.

- Closed under $\to$: It is seen analogously to the previous case.
  For any $a, b \in C \setminus \{c\}$, by Remark 1.60, we know either $a \to b = 1$ or $a \to b = \neg a \vee b$ and, in both cases, $a \to b \neq c$ since $\vee$ is defined as the maximum, we know we have closure under $\neg$ and we have seen $1 \neq c$.

Therefore, it is clear what we wanted to prove. □

Thanks to this proposition, we can give more examples of NM-chains:

Example 1.65. For every natural number $n > 0$ we consider $\mathbf{A}_{2n} = \langle [-n, n] \cap (\mathbb{Z} \setminus \{0\}); *, \to, \wedge, \vee, \neg, -n, n \rangle$, where $\mathbf{A}_{2n}$ is the subalgebra of $\mathbf{A}_{2n+1}$ obtained by removing the negation fixpoint 0 from its universe ($\mathbf{A}_{2n} := \mathbf{A}^{-}_{2n+1}$).

Thus, by the definitions we have given, the following relations between NM-chains hold:

Proposition 1.66. [15, Proposition 2]

- *$\boldsymbol{A}_{2m+1}$ is a subalgebra of $\boldsymbol{A}_{2n+1}$ for every $m \leq n$,*

- *$\boldsymbol{A}_{2m}$ is a subalgebra of $\boldsymbol{A}_{2n+1}$ for every $0 < m \leq n$,*

- *$\boldsymbol{A}_{2m}$ is a subalgebra of $\boldsymbol{A}_{2n}$ for every $0 < m \leq n$,*

- *$\boldsymbol{A}_m$ is embeddable into **[0,1]** for every $m > 1$,*

- $\boldsymbol{A}_{2m}$ *is embeddable into* $[\mathbf{0,1}]^-$ *for every* $m > 0$.

Remark 1.67. It is easy to see that $\mathbf{A}_{2n}$ and $\mathbf{A}_{2n+1}$ are, up to isomorphism, the only nilpotent minimum chains with exactly 2n and 2n+1 elements, respectively. Moreover, any finitely generated subalgebra of a nontrivial NM-chain is finite and, therefore, isomorphic to $\mathbf{A}_{2n}$ or $\mathbf{A}_{2n+1}$ for some $n > 0$.

Esteva and Godo proved that **NML** is complete with respect to the class of NM-algebras and, moreover, standard complete, that is, complete with respect to the standard NM-algebra (see [12, Theorem 4]). In fact, $NM = \mathbb{V}([\mathbf{0,1}]) = \mathbb{Q}([\mathbf{0,1}])$, since every countable NM-chain is embeddable into $[\mathbf{0,1}]$ (where we denote by NM the class of all NM-algebras). Furthermore:

Proposition 1.68. [15, Corollary 2] *Let* $\boldsymbol{A}$ *be an infinite NM-chain containing the negation fixpoint. Then* $\mathbb{V}(\boldsymbol{A}) = \mathbb{Q}(\boldsymbol{A}) = NM$.

Finally, Theorem 1.40 and Lemma 1.41 will also be valid for NM-algebras (since they are, in particular, MTL-algebras) and they will be useful to prove some important results in the next sections.

In order to be able to study, later on, how some results change when expanding the language of **NML**; first, we will speak about the lattice of axiomatic extensions of the Nilpotent minimum logic and we will present some statements about it:

Definition 1.69. *For each* $n > 0$ *we consider the following formulas:*

$$\boldsymbol{A12_n} : S_n(x_0, \dots, x_n) := \bigwedge_{i<n}((x_i \to x_{i+1}) \to x_{i+1}) \to \bigvee_{i<n+1} x_i$$

$$\boldsymbol{A13} : BP(x) := \neg(\neg x^2)^2 \leftrightarrow (\neg(\neg x)^2)^2$$

Where it can be proved (see [15, Theorem 2]*) that an NM-chain satisfies* $S_n(x_0, \dots, x_n) \approx \overline{1}$ *if and only if it has less than* $2n+2$ *elements and a nontrivial NM-chain satisfies* $BP(x) \approx \overline{1}$ *if and only if it does not contain the negation fixpoint.*

Theorem 1.70. [15, Theorem 3] *Every proper nontrivial subvariety of NM is of one of the following types:*

1. $NM^- := \mathbb{V}([\mathbf{0,1}]^-) = \mathbb{V}(\{\boldsymbol{A}_{2k} : k \in \omega\})$
2. $\mathbb{V}(\boldsymbol{A}_{2m+1})$ *for some* $m > 0$
3. $\mathbb{V}(\boldsymbol{A}_{2n})$ *for some* $n > 0$
4. $\mathbb{V}([\mathbf{0,1}]^-, \boldsymbol{A}_{2m+1})$ *for some* $m > 0$
5. $\mathbb{V}(\boldsymbol{A}_{2n}, \boldsymbol{A}_{2m+1})$ *for some* $m, n \in \omega$ *such that* $0 < m < n$

Moreover, if Σ *is any set of equations axiomatizing NM, then*

1. $\mathbb{V}([\mathbf{0,1}]^-)$ *is axiomatized by* Σ *plus the equation* $BP(x) \approx \overline{1}$,
2. $\mathbb{V}(\boldsymbol{A}_{2m+1})$ *is axiomatized by* Σ *plus the equation* $S_m(x_0, \dots, x_m) \approx \overline{1}$,

3. $\mathbb{V}(\boldsymbol{A}_{2n})$ *is axiomatized by* Σ *plus the equations* $S_n(x_0, \ldots, x_n) \approx \overline{1}$ *and* $BP(x) \approx \overline{1}$,
4. $\mathbb{V}(\boldsymbol{[0,1]}^-, \boldsymbol{A}_{2m+1})$ *is axiomatized by* Σ *plus the equation* $BP(x) \vee S_m(x_0, \ldots, x_m) \approx \overline{1}$,
5. $\mathbb{V}(\boldsymbol{A}_{2n}, \boldsymbol{A}_{2m+1})$ *with* $m < n$ *is axiomatized by* Σ *plus* $(BP(x) \wedge S_n(x_0, \ldots, x_n)) \vee S_m(x_0, \ldots, x_m) \approx \overline{1}$.

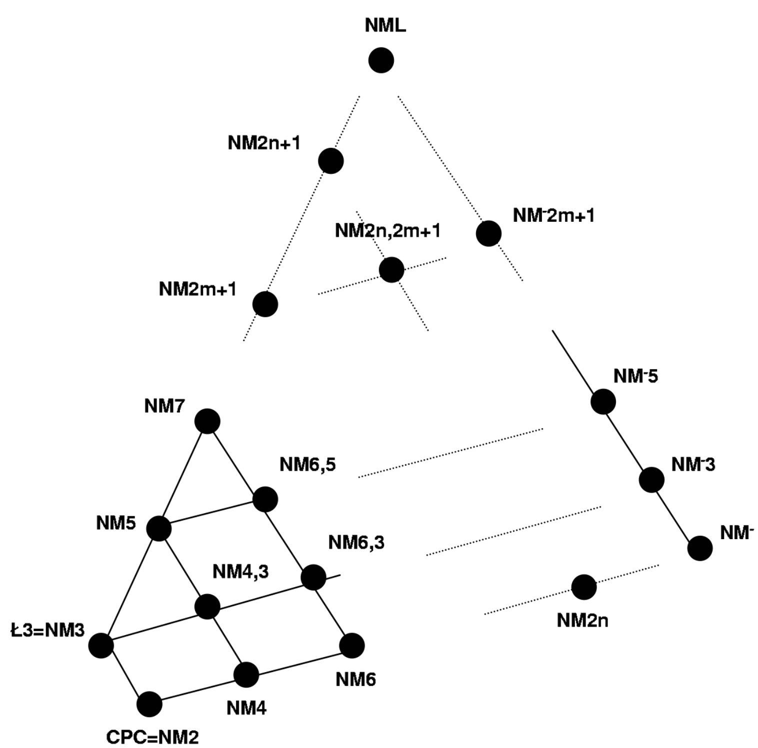
Since **NML** is algebraizable by the class of all NM-algebras, there is a dual lattice isomorphism between the lattice of all subvarieties of NM and the lattice of all axiomatic extensions of **NML**, thus:

Theorem 1.71. *All proper consistent axiomatic extensions of* ***NML*** *are:*

For every natural numbers $n, m > 0$

1. ***NM*** $^-$ = ***NML*** *plus A13*
2. ***NM2m+1*** = ***NML*** *plus* $A12_m$
3. ***NM2n*** = ***NML*** *plus* $A12_n$ *and A13*
4. ***NM*** $^-$ ***2m+1*** = ***NML*** *plus* $A13 \vee A12_m$
5. ***NM2n,2m+1*** = ***NML*** *plus* $(A12_n \wedge A13) \vee A12_m$ *with* $n > m$

Therefore, from this and the inclusion relations between subvarieties of NM given by Proposition 1.66, we obtain that the dual lattice of axiomatic extensions of the Nilpotent minimum logic will be of the form [15, Figure 2]:



Moreover, we can present some results about structural completeness and active structural completeness and introduce an axiomatization of all admissible quasiequations, given any axiomatic extension of **NML**:

Theorem 1.72. [16, Theorem 2.1 and Corollary 2.2] ***NML*** *is not structurally complete, in fact, given any variety of NM-algebras* $\mathbb{K}$ *satisfying* $\mathbb{V}(\boldsymbol{A}_3) \subseteq \mathbb{K}$*, the logic algebraized by* $\mathbb{K}$ *is not SC.*

Theorem 1.73. [16, Theorem 2.4] ***NM⁻*** *is hereditarily structurally complete.*

Theorem 1.74. [16, Theorem 2.9] ***NML*** *is hereditarily active structurally complete.*

Theorem 1.75. [16, Theorem 2.9] *Let* $\mathbb{K}$ *be a variety of NM-algebras. Then, the quasiequation* $\neg x \approx x \Rightarrow \overline{0} \approx \overline{1}$ *axiomatizes all (passive) admissible quasiequations. Thus,* $\mathbb{Q}(\boldsymbol{F}_{\mathbb{K}}(\omega))$ *is axiomatized by the quasiequation* $\neg x \approx x \Rightarrow \overline{0} \approx \overline{1}$.

Using the above statements about (almost) structural completeness on axiomatic extensions of **NML**, in [16] have been proven some results about subquasivarieties of NM, with the aim to define the lattice $L_{\mathcal{Q}}(NM)$:

Proposition 1.76. [16, Proposition 3.1] *Let* $\mathbb{M}$ *be a variety of NM-algebras and* $\mathbb{K}$ *be an* $\mathbb{M}$*-quasivariety. Then* $\mathbb{K}$ *is a proper* $\mathbb{M}$*-quasivariety iff there is* $\boldsymbol{A}_{2n+1} \in \mathbb{M} \setminus \mathbb{K}$ *for some* $n > 0$.

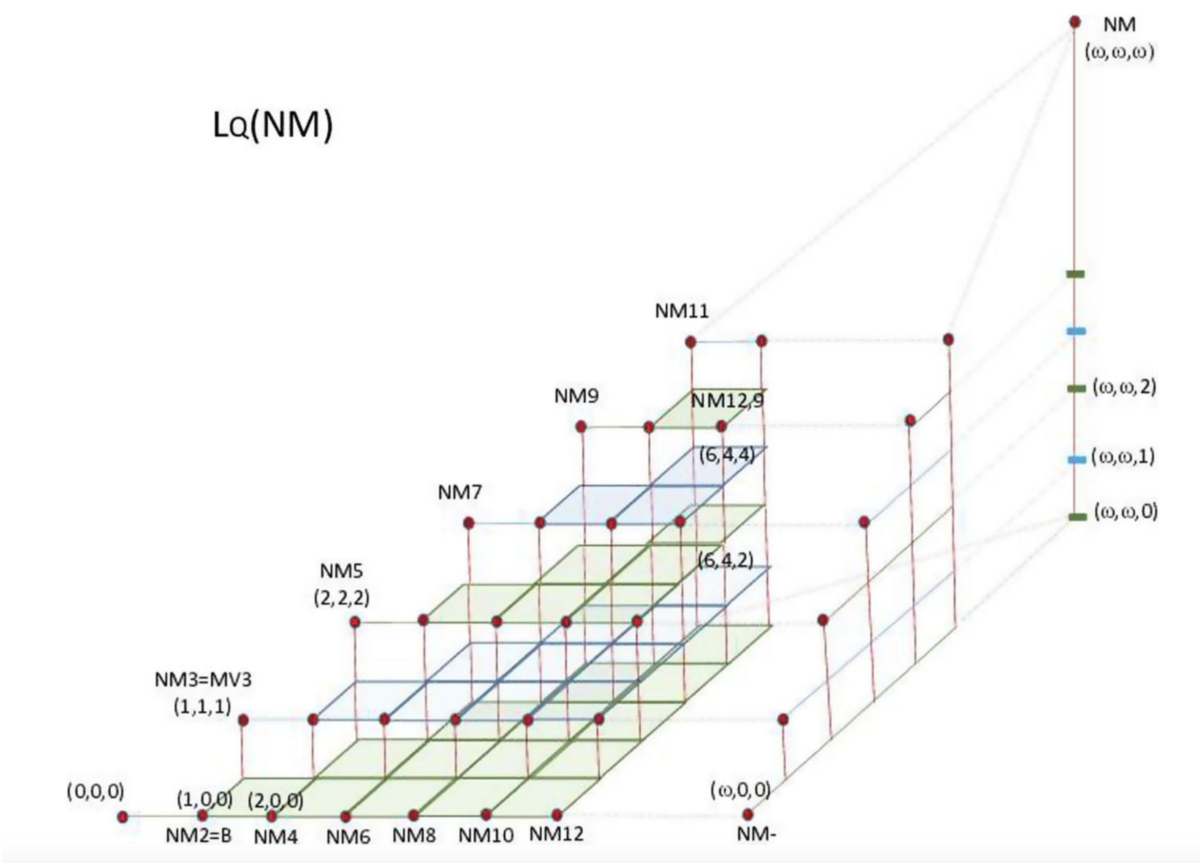
Theorem 1.77. [16, Theorem 3.3] *Let* $\mathbb{M}$ *be an NM-variety. If* $\mathbb{K}$ *is a proper* $\mathbb{M}$*-quasivariety and* $k = max\{n \in \mathbb{N} : \boldsymbol{A}_{2n+1} \in \mathbb{K}\}$*, then*

$$\mathbb{K} = \mathbb{Q}(\{\boldsymbol{A}_{2n} : \boldsymbol{A}_{2n} \in \mathbb{M}\} \cup \{\boldsymbol{A}_2 \times \boldsymbol{A}_{2m+1} : \boldsymbol{A}_{2m+1} \in \mathbb{M}\} \cup \{\boldsymbol{A}_{2k+1}\}).$$

Corollary 1.78. [16, Corollary 3.4] *Let* $\mathbb{K}$ *be a nontrivial quasivariety of NM-algebras, then*

$$\mathbb{K} = \mathbb{Q}(\{\boldsymbol{A}_{2n} : \boldsymbol{A}_{2n} \in \mathbb{V}(\mathbb{K})\} \cup \{\boldsymbol{A}_2 \times \boldsymbol{A}_{2m+1} : \boldsymbol{A}_{2m+1} \in \mathbb{V}(\mathbb{K})\} \cup \{\boldsymbol{A}_{2k+1} : \boldsymbol{A}_{2k+1} \in \mathbb{K}\}).$$

In fact, the lattice $L_{\mathcal{Q}}(NM)$ is depicted in [16, page 805] and we know that the lattice of finitary extensions of **NML** will be dually isomorphic:



Lq(NM)
NM
(ω,ω,ω)
NM11
NM9
NM12,9
(6,4,4)
NM7
(ω,ω,2)
(ω,ω,1)
(ω,ω,0)
NM5
(2,2,2)
(6,4,2)
NM3=MV3
(1,1,1)
(0,0,0)
(1,0,0)
(2,0,0)
(ω,0,0)
NM2=B
NM4
NM6
NM8
NM10
NM12
NM-

2 The Rational Nilpotent Minimum Logic

We will first study the expansion of the Nilpotent minimum logic obtained by adding rational constants to the language and how the addition of rational constants affects the lattices of axiomatic and finitary extensions and the structural completeness properties.
We will obtain analogous results to the ones achieved for the rational Gödel logic in [18].

We recall that, as it has been mentioned in Section 1.2, the rational Nilpotent minimum logic (**RNML**, for short) is defined from **NML** by adding the bookkeeping axioms of **[0,1]**. Moreover, the class of rational NM-algebras (denoted by RNM) will be its equivalent algebraic semantics and we have, for every set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathbf{RNML}} \varphi \text{ if and only if } \tau[\Gamma] \vDash_{RNM} \tau(\varphi)$$

where $\tau := \{x \approx 1\}$.

In this case, the importance of the algebra $\mathbf{[0,1]}^{\mathbf{Q}}$ is witnessed by the fact that, as has been proven in [13, Theorem 14], the following equality holds: RNM $= \mathbb{V}(\mathbf{[0,1]}^{\mathbf{Q}})$. On the other hand, RNM doesn't coincide with the quasivariety generated by $\mathbf{[0,1]}^{\mathbf{Q}}$ (see [13, Section 4]).

2.1 The lattices of axiomatic and finitary extensions

In this section, we will see, similarly to how it has been done in [18, Section 7] for the case of Gödel logic, how the addition of rational constants to **NML** affects the structure of the lattice of (axiomatic) extensions of **RNML**.

Given a real $r \in (\frac{1}{2}, 1]$, let $\mathbf{Q}_r$ be the rational NM-algebra with universe

$$\{0\} \cup ((1-r, r) \cap \mathbb{Q}) \cup \{1\}.$$

The order relation of $\mathbf{Q}_r$ is the natural order in $\mathbb{Q}$, therefore $\mathbf{Q}_r$ is a chain. We define the negation to be $\neg a := 1 - a$ for any $a \in Q_r$, since its interpretation is not established by the fact that $\mathbf{Q}_r$ is a chain. Thus, by Remark 1.60, this settles the interpretation of the lattice connectives and also of the implication (for all $a, b \in Q_r$, we get $a \to b$ to be 1 if $a \leq b$ and $max\{\neg a, b\}$ otherwise). The interpretation of $*$ is also fixed (for all $a, b \in Q_r$, we get $a * b$ to be 0 if $b \leq \neg a$ and $min\{a, b\}$ otherwise).
Finally, given a rational $q \in [0,1]$, the interpretation of $\mathbf{c}_q$ in $\mathbf{Q}_r$ is q if $q \in Q_r$, 0 if $q \leq 1 - r$ and 1 in the other cases. This way, it is clear that the bookkeeping axioms hold.

Remark 2.1. Notice that, in the case $r = 1$, we obtain $\mathbf{Q}_r = \mathbf{[0,1]}^{\mathbf{Q}} \cap \mathbf{Q}$, where the last algebra is the one mentioned in Definition 1.54.

Now, we fix a denumerable set $\{t_n : n \in \omega\}$ disjoint from $[0,1]$ and such that, for every $n \in \omega$, $\neg t_n \notin [0,1]$. Given a rational $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and some ordinal $\gamma \in \omega + 1$, let $\mathbf{Q}_p^\gamma$ be the rational NM-algebra with universe

$$\{\neg t_n : n \in \gamma\} \cup \{0\} \cup ([1-p, p] \cap \mathbb{Q}) \cup \{1\} \cup \{t_n : n \in \gamma\}.$$

We define the order relation of $\mathbf{Q}_p^\gamma$ by keeping the usual order in $\{0\} \cup ([1-p, p] \cap \mathbb{Q}) \cup \{1\}$ and inserting the totally ordered set $\{t_n : n \in \gamma\}$ between p and 1 and the set $\{\neg t_n : n \in \gamma\}$ (which is also totally ordered) between 0 an $1 - p$.
That is:

$$0 < \neg t_{n-1} < \cdots < \neg t_1 < \neg t_0 < 1 - p$$
$$p < t_0 < t_1 < \cdots < t_{n-1} < 1.$$

Therefore, $\mathbf{Q}_p^\gamma$ is a chain and, since the interpretation of the negation is not established by this fact, we define it such that: for any $a \in Q_p^\gamma \cap [0,1]$, $\neg a := 1 - a$ and, for the elements that are disjoint from $[0,1]$, $\neg(t_i) := \neg t_i$, $\neg(\neg t_i) := t_i$. This settles the interpretation of the operations as in the previous case for $\mathbf{Q}_r$: by Remark 1.60, we have fixed the interpretation of the lattice connectives (given by the order) and also the definition of $*$ and the implication. Finally, given a rational $q \in [0,1]$, the interpretation of $\mathbf{c}_q$ in $\mathbf{Q}_p^\gamma$ is q if $q \in Q_p^\gamma$, 0 if $q < 1 - p$ and 1 otherwise.

We introduce and prove some results about rational NM-chains that will be useful:

Proposition 2.2. *For every nontrivial rational NM-chain $\boldsymbol{A}$, there are $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega{+}1$ such that $\mathbb{ISP}_U(\boldsymbol{A}) = \mathbb{ISP}_U(\boldsymbol{Q}_r)$ or $\mathbb{ISP}_U(\boldsymbol{A}) = \mathbb{ISP}_U(\boldsymbol{Q}_p^\gamma)$. Moreover,*

1. *$\mathbb{ISP}_U(\boldsymbol{Q}_r)$ is axiomatized relative to the class of RNM chains by the sentences*

$$\boldsymbol{c}_{q'} \not\approx 1 \text{ for all } q' \in [\tfrac{1}{2}, r) \cap \mathbb{Q} \quad \text{and} \quad \boldsymbol{c}_q \approx 1 \text{ for all } q \in [r, 1] \cap \mathbb{Q};$$

2. *$\mathbb{ISP}_U(\boldsymbol{Q}_p^\omega)$ is axiomatized relative to the class of RNM chains by the sentences*

$$\boldsymbol{c}_p \not\approx 1 \quad \text{and} \quad \boldsymbol{c}_q \approx 1 \text{ for all } q \in (p, 1] \cap \mathbb{Q};$$

3. *$\mathbb{ISP}_U(\boldsymbol{Q}_p^n)$ is axiomatized relative to the class of RNM chains by the sentences*

$$\boldsymbol{c}_p \not\approx 1,\ \boldsymbol{c}_q \approx 1 \text{ for all } q \in (p, 1] \cap \mathbb{Q} \quad \text{and}$$
$$\forall x_0 \ldots x_{n+2} \Big(\bigvee\nolimits_{0 \le i < j \le n+2} (\boldsymbol{c}_p \vee x_i) \leftrightarrow (\boldsymbol{c}_p \vee x_j) \Big) \approx 1.$$

Proof. Let $\mathbf{A}$ be a rational NM-chain, we want to define an algebra $\mathbf{S_A}$ that has the same universal theory as $\mathbf{A}$ (we will see that this algebra can be $\mathbf{Q}_r$ or $\mathbf{Q}_p^\gamma$ for r, p, γ as desired). In order to do this, we will start by defining $\mathbf{S_A}$ as an algebra that embeds into $\mathbf{A}$.

Let $\mathbf{C}$ be the zero-generated subalgebra of $\mathbf{A}$ (its universe is comprised by the interpretations of the elements of $\mathscr{C}$). Since we assume $\mathbf{A}$ to be nontrivial, we either have $\mathbf{C} = \mathbf{Q}_r$ for some real $r \in (\frac{1}{2}, 1]$ or $\mathbf{C} = \mathbf{Q}_p^0$ for some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$.

- If $\mathbf{C} = \mathbf{Q}_r$ for some $r \in (\frac{1}{2}, 1]$ then, we take $\mathbf{S_A} := \mathbf{C}$.
- Otherwise, if $\mathbf{C} = \mathbf{Q}_p^0$ for some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$, we analyse $\downarrow (C \setminus \{1\})$ (which is the downset of $(C \setminus \{1\})$ in $\mathbf{A}$, i.e. for any $b \in A$ if there exists some $a \in (C \setminus \{1\})$ such that $b \le a$ then, $b \in \downarrow (C \setminus \{1\})$).

– If $\omega \leq |A \setminus \downarrow (C \setminus \{1\})|$ then, we take $\mathbf{S_A} := \mathbf{Q}_p^\omega$.
– If $|A \setminus \downarrow (C \setminus \{1\})| = n+1 \in \omega$ then, we take $\mathbf{S_A} := \mathbf{Q}_p^n$.

In any case, it is clear that $\mathbf{S_A}$ embeds into $\mathbf{A}$, thus, $\mathbf{S_A} \in \mathbb{IS}(\mathbf{A})$ and from this we obtain $\mathbf{S_A} \in \mathbb{ISP}_U(\mathbf{A})$.

Therefore, in order to see that $\mathbf{S_A}$ and $\mathbf{A}$ have the same universal theory it is enough to show that $\mathbf{A}$ validates the universal theory of $\mathbf{S_A}$. This last statement holds if $\mathbf{A}$ partially embeds into $\mathbf{S_A}$ thus, that is what we will prove next.

To prove what we want, it suffices to see that every finite partial subalgebra of $\mathbf{A}$ is embeddable into $\mathbf{S_A}$. Therefore, we consider $\mathbf{B}$ to be an arbitrary finite partial subalgebra of $\mathbf{A}$.
The elements of $\mathbf{B}$ can be classified depending on whether they are the interpretation of some constant (we will denote these elements: $\mathbf{c}_{q_1}^{\mathbf{A}}, \ldots, \mathbf{c}_{q_m}^{\mathbf{A}}$) or they are not (we will name these elements: $a_1, \ldots, a_n$).
We can assume, without losing generality, that, for every element of $\mathbf{B}$, its negation is also contained in $\mathbf{B}$ (in case this didn't hold we can just add the elements) and we can also suppose that we have

$$0 = \mathbf{c}_{q_1}^{\mathbf{A}} < \mathbf{c}_{q_2}^{\mathbf{A}} < \ldots < \mathbf{c}_{\frac{1}{2}}^{\mathbf{A}} < \ldots < \mathbf{c}_{q_m}^{\mathbf{A}} = 1.$$

Since we have taken $\mathbf{A}$ to be a chain, $[\mathbf{c}_{q_1}^{\mathbf{A}}, \mathbf{c}_{q_2}^{\mathbf{A}}), \ldots [\mathbf{c}_{q_{m-1}}^{\mathbf{A}}, \mathbf{c}_{q_m}^{\mathbf{A}})$ is a partition of $A \setminus \{1\}$.

Now, for every $i \leq m-1$, let $a_{i_1} < \ldots < a_{i_k}$ be the elements of $\{a_1, \ldots, a_n\}$ in the i-th component $[\mathbf{c}_{q_i}^{\mathbf{A}}, \mathbf{c}_{q_{i+1}}^{\mathbf{A}})$, we will argue it holds the fact that we can choose some $b_{i_1}, \ldots, b_{i_k} \in S_A$ such that

$$\mathbf{c}_{q_i}^{\mathbf{S_A}} < b_{i_1} < \ldots < b_{i_k} < \mathbf{c}_{q_{i+1}}^{\mathbf{S_A}}.$$

It suffices to show that the statement above holds for $[\mathbf{c}_{q_i}^{\mathbf{A}}, \mathbf{c}_{q_{i+1}}^{\mathbf{A}})$ with $q_i \geq \frac{1}{2}$. That is because: in NM-chains (thus, also in rational NM-chains) the negation is dually order preserving; we have assumed that $\mathbf{B}$ is closed under the negation; and because by the bookkeeping axioms we know $\mathbf{c}_{\frac{1}{2}}^{\mathbf{A}}$ is the negation fixpoint.
That is, if for some $[\mathbf{c}_{q_i}^{\mathbf{A}}, \mathbf{c}_{q_{i+1}}^{\mathbf{A}})$ with $q_i \geq \frac{1}{2}$ such that there are k elements of $\mathbf{B}$ laying inside and satisfying

$$\mathbf{c}_{q_i}^{\mathbf{A}} < a_{i_1} < \ldots < a_{i_k} < \mathbf{c}_{q_{i+1}}^{\mathbf{A}}$$

we have $\mathbf{c}_{q_i}^{\mathbf{S_A}} < b_{i_1} < \ldots < b_{i_k} < \mathbf{c}_{q_{i+1}}^{\mathbf{S_A}}$, for some $b_{i_1}, \ldots b_{i_k} \in S_A$, then we also have

$$\mathbf{c}_{\neg q_{i+1}}^{\mathbf{S_A}} < \neg b_{i_k} < \ldots < \neg b_{i_1} < \mathbf{c}_{\neg q_i}^{\mathbf{S_A}}$$

. Where the interval $[\mathbf{c}_{\neg q_{i+1}}^{\mathbf{A}}, \mathbf{c}_{\neg q_i}^{\mathbf{A}})$ belongs to the partition we have mentioned before (because of the closure of $\mathbf{B}$ under the negation) and we also have exactly k elements from $\mathbf{B}$ in this interval (which are the negations of the previous a_{i_j}).

Now, let's argue the existence of those b_{i_j} for rationals $q_i \geq \frac{1}{2}$:

- If $i \neq m-1$: Then, we have the interval $[\mathbf{c}^{\mathbf{A}}_{q_i}, \mathbf{c}^{\mathbf{A}}_{q_{i+1}})$ where $\mathbf{c}^{\mathbf{A}}_{q_{i+1}} \neq 1$ and, therefore, $\mathbf{c}^{\mathbf{S_A}}_{q_{i+1}} \neq 1$. Since the rationals are dense in the reals, by the definition we have given of $\mathbf{S_A}$ it is clear that $[\mathbf{c}^{\mathbf{S_A}}_{q_i}, \mathbf{c}^{\mathbf{S_A}}_{q_{i+1}})$ is an infinite set. Thus, we can find such elements b_{i_j}.

- If $i = m-1$: Then, we are considering the interval $[\mathbf{c}^{\mathbf{A}}_{q_i}, 1)$. By construction of $\mathbf{S_A}$ it is clear the existence of those b_{i_j} in $[\mathbf{c}^{\mathbf{S_A}}_{q_i}, 1)$:
  - If $\mathbf{S_A} = \mathbf{Q}_r$: it holds because of the density of $\mathbb{Q}$ in $\mathbb{R}$.
  - If $\mathbf{S_A} = \mathbf{Q}^{\omega}_p$: we have $\{t_n : n \in \omega\} \subseteq [\mathbf{c}^{\mathbf{S_A}}_{q_i}, 1)$ thus, the interval is infinite and there exist some elements we can choose to be our b_{i_j}s.
  - If $\mathbf{S_A} = \mathbf{Q}^{n}_p$: then, $|A \setminus \downarrow (C \setminus \{1\})| = n+1$ therefore, there can be at most n elements in $[\mathbf{c}^{\mathbf{A}}_{q_i}, 1)$. We know $\{t_{n'} : n' \in n\} \subseteq [\mathbf{c}^{\mathbf{S_A}}_{q_i}, 1)$ so it is clear the existence of some elements b_{i_j} satisfying what we want.

Hence, we consider a map $h : \mathbf{B} \longrightarrow \mathbf{S_A}$ and we let $h(a_{i_j}) := b_{i_j}$ (for those $a_{i_j} > \mathbf{c}^{\mathbf{A}}_{\frac{1}{2}}$) and $h(\neg a_{i_j}) := \neg b_{i_j}$ for the rest of elements of $\{a_1, \dots, a_n\}$. Moreover, we define $h(\mathbf{c}^{\mathbf{A}}_q) = \mathbf{c}^{\mathbf{A}}_q = \mathbf{c}^{\mathbf{S_A}}_q$ for every $q \in \{q_1, \dots, q_m\}$.
This completes the definition of h which is clearly an homomorphism since in NM-chains the behaviour of $\rightarrow$ and $*$ is completely determined by the order and the negation and we have: for any $a, b \in B$, $a \leq b \Rightarrow h(a) \leq h(b)$ and $h(\neg a) = \neg h(a)$.

By the definition we have given, $h : \mathbf{B} \longrightarrow \mathbf{S_A}$ is an embedding which is what we needed in order to prove that $\mathbf{A}$ and $\mathbf{S_A}$ have the same universal theory. Therefore, there are either $r \in (\frac{1}{2}, 1]$ or $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$ such that $\mathbb{ISP}_U(\mathbf{A}) = \mathbb{ISP}_U(\mathbf{Q}_r)$ or $\mathbb{ISP}_U(\mathbf{A}) = \mathbb{ISP}_U(\mathbf{Q}^{\gamma}_p)$.

Now, we will prove what remains to be seen:

1. Let $\mathbf{A}$ be a rational NM-chain validating the sentences in the statement. Then, the zero-generated subalgebra of $\mathbf{A}$ is $\mathbf{Q}_r$. Thus, by the previous construction we have given of $\mathbf{S_A}$ we know $\mathbf{S_A} = \mathbf{Q}_r$. Consequently, $\mathbf{A}$ and $\mathbf{Q}_r$ have the same universal theory and, in particular, $\mathbf{A} \in \mathbb{ISP}_U(\mathbf{Q}_r)$.

2. Let $\mathbf{A}$ be a rational NM-chain validating the sentences in the statement. Then, the zero-generated subalgebra of $\mathbf{A}$ is $\mathbf{Q}^0_p$. Thus, by the previous construction we have given of $\mathbf{S_A}$ we know $\mathbf{S_A} \in \{\mathbf{Q}^{\gamma}_p : \gamma \in \omega + 1\}$ (thus, $\mathbf{S_A} \subseteq \mathbf{Q}^{\omega}_p$) and, since $\mathbf{A}$ and $\mathbf{S_A}$ have the same universal theory, $\mathbf{A} \in \mathbb{ISP}_U(\mathbf{S_A})$.
   Therefore,
   $$\mathbf{A} \in \mathbb{ISP}_U(\mathbf{S_A}) \subseteq \mathbb{ISP}_U\mathbb{S}(\mathbf{Q}^{\omega}_p) = \mathbb{ISP}_U(\mathbf{Q}^{\omega}_p).$$

3. Let $\mathbf{A}$ be a rational NM-chain validating the sentences in the statement. By arguing as in case 2. we obtain that $\mathbf{S_A} \in \{\mathbf{Q}^{\gamma}_p : \gamma \in \omega + 1\}$ where $\mathbf{A}$ and $\mathbf{S_A}$ have the same universal theory.
   Moreover, since $\mathbf{A}$ validates
   $$\forall x_0 \dots x_{n+2} \Big(\bigvee_{0 \leq i < j \leq n+2} (\mathbf{c}_p \vee x_i) \leftrightarrow (\mathbf{c}_p \vee x_j) \Big) \approx 1,$$

it is clear that $|A\setminus \downarrow (C\setminus\{1\})| = m+1$ for some $m \le n$ where C is the universe of the zero-generated subalgebra of $\mathbf{A}$.
Therefore, $\mathbf{S_A} = \mathbf{Q}_p^m$ for some $m \le n$.
Hence,

$$\mathbf{A} \in \mathbb{ISP}_U(\mathbf{S_A}) = \mathbb{ISP}_U(\mathbf{Q}_p^m) \subseteq \mathbb{ISP}_U\mathbb{S}(\mathbf{Q}_p^n) = \mathbb{ISP}_U(\mathbf{Q}_p^n).$$

□

Corollary 2.3. *Every variety of rational NM-algebras is generated by a set of algebras of the form $\boldsymbol{Q}_r$, where $r \in (\frac{1}{2}, 1]$, or $\boldsymbol{Q}_p^\gamma$, where $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$.*

Proof. Recall that every variety is generated by its subdirectly irreducible members. As every subdirectly irreducible rational NM-algebra is a chain, the result follows from Proposition 2.2. The fact that every subdirectly irreducible element of RNM is a chain is given by Proposition 1.49 (since the logic **RNML** is a core fuzzy logic).

□

Theorem 2.4. *The following hold.*

1. *Every nontrivial variety $\mathbb{K}$ of rational NM-algebras is of the form $\mathbb{V}(\boldsymbol{Q}_r)$ for some $r \in (\frac{1}{2}, 1]$ or $\mathbb{V}(\boldsymbol{Q}_p^\gamma)$ for some $\gamma \in \omega + 1$ and $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$. Furthermore, $\mathbb{V}(\boldsymbol{Q}_r)$ is axiomatized by the equations $\{\boldsymbol{c}_q \approx 1 : q \in [r, 1] \cap \mathbb{Q}\}$ and $\mathbb{V}(\boldsymbol{Q}_p^\gamma)$ is axiomatized by the equations:*
   - *$\{\boldsymbol{c}_q \approx 1 : q \in (p, 1] \cap \mathbb{Q}\}$ and*
     $$\Big(\bigvee\nolimits_{0\le i<j\le n+2} (\boldsymbol{c}_p \vee x_i) \leftrightarrow (\boldsymbol{c}_p \vee x_j)\Big) \approx 1$$
     if $\gamma = n \in \omega$.
   - *$\{\boldsymbol{c}_q \approx 1 : q \in (p, 1] \cap \mathbb{Q}\}$, otherwise.*
2. *For all $r_1, r_2 \in (\frac{1}{2}, 1]$, $p_1, p_2 \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma_1, \gamma_2 \in \omega + 1$,*
   $\mathbb{V}(\boldsymbol{Q}_{r_1}) \subseteq \mathbb{V}(\boldsymbol{Q}_{r_2})$ if and only if $r_1 \le r_2$,
   $\mathbb{V}(\boldsymbol{Q}_{r_1}) \subseteq \mathbb{V}(\boldsymbol{Q}_{p_1}^{\gamma_1})$ if and only if $r_1 \le p_1$,
   $\mathbb{V}(\boldsymbol{Q}_{p_1}^{\gamma_1}) \subseteq \mathbb{V}(\boldsymbol{Q}_{r_1})$ if and only if $p_1 < r_1$,
   $\mathbb{V}(\boldsymbol{Q}_{p_1}^{\gamma_1}) \subseteq \mathbb{V}(\boldsymbol{Q}_{p_2}^{\gamma_2})$ if and only if either $p_1 < p_2$ or ($p_1 = p_2$ and $\gamma_1 \le \gamma_2$).
3. *The lattice of subvarieties of RNM (which we denote by $L_{\mathcal{V}}$(RNM)) is an uncountable chain isomorphic to the poset obtained by adding a new bottom element to the Dedekind-MacNeille completion of the lexicographic order of $[\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\omega + 1$.*

Proof. 2. Consider $r_1, r_2 \in (\frac{1}{2}, 1]$, $p_1, p_2 \in [\frac{1}{2}, 1) \cap \mathbb{Q}$, and $\gamma_1, \gamma_2 \in \omega + 1$. We need to prove that:

- $\mathbb{V}(\mathbf{Q}_{r_1}) \subseteq \mathbb{V}(\mathbf{Q}_{r_2})$ if and only if $r_1 \leq r_2$.

  $\Rightarrow$) To prove this implication we reason by contraposition: we assume that $r_2 < r_1$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $r_2 \leq q < r_1$. Hence, $\mathbf{c}_q \approx 1$ holds in $\mathbf{Q}_{r_2}$ but fails in $\mathbf{Q}_{r_1}$ (by the definition we have given of this algebras). Therefore, $\mathbf{Q}_{r_1} \notin \mathbb{V}(\mathbf{Q}_{r_2})$ and, consequently, $\mathbb{V}(\mathbf{Q}_{r_1}) \nsubseteq \mathbb{V}(\mathbf{Q}_{r_2})$.
  $\Leftarrow$) Now, we prove the other direction. If $r_1 \leq r_2$, then $\mathbf{Q}_{r_1} \in \mathbb{H}(\mathbf{Q}_{r_2}) \subseteq \mathbb{V}(\mathbf{Q}_{r_2})$.

- $\mathbb{V}(\mathbf{Q}_{r_1}) \subseteq \mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1})$ if and only if $r_1 \leq p_1$.

  $\Rightarrow$)To prove this implication we reason by contraposition: we assume that $p_1 < r_1$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $p_1 < q < r_1$. Hence, $\mathbf{c}_q \approx 1$ holds in $\mathbf{Q}_{p_1}^{\gamma_1}$ but fails in $\mathbf{Q}_{r_1}$ (by the definition we have given of this algebras). Therefore, $\mathbf{Q}_{r_1} \notin \mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1})$ and, consequently, $\mathbb{V}(\mathbf{Q}_{r_1}) \nsubseteq \mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1})$.
  $\Leftarrow$) Now, we prove the other direction. If $r_1 \leq p_1$, then $\mathbf{Q}_{r_1} \in \mathbb{H}(\mathbf{Q}_{p_1}^{\gamma_1}) \subseteq \mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1})$.

- $\mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1}) \subseteq \mathbb{V}(\mathbf{Q}_{r_1})$ if and only if $p_1 < r_1$.

  $\Rightarrow$) To prove this implication we reason by contraposition: we assume that $r_1 \leq p_1$. Hence, $\mathbf{c}_{p_1} \approx 1$ holds in $\mathbf{Q}_{r_1}$ but fails in $\mathbf{Q}_{p_1}^{\gamma_1}$ (by the definition we have given of this algebras). Therefore, $\mathbf{Q}_{p_1}^{\gamma_1} \notin \mathbb{V}(\mathbf{Q}_{r_1})$ and, consequently, $\mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1}) \nsubseteq \mathbb{V}(\mathbf{Q}_{r_1})$.
  $\Leftarrow$) Now, we prove the other direction. If $p_1 < r_1$, then every partial subalgebra of $\mathbf{Q}_{p_1}^{\gamma_1}$ embeds into some member of $\{\mathbf{Q}_q : q \in (p_1, r_1] \cap \mathbb{Q}\}$. This last statement implies that $\mathbf{Q}_{p_1}^{\gamma_1}$ validates the universal theory of $\{\mathbf{Q}_q : q \in (p_1, r_1] \cap \mathbb{Q}\}$ (i.e. $\mathbf{Q}_{p_1}^{\gamma_1} \in \mathbb{ISP}_U(\{\mathbf{Q}_q : q \in (p_1, r_1] \cap \mathbb{Q}\})$.
  We know $\{\mathbf{Q}_q : q \in (p_1, r_1] \cap \mathbb{Q}\} \subseteq \mathbb{H}(\mathbf{Q}_{r_1}) \subseteq \mathbb{V}(\mathbf{Q}_{r_1})$. Therefore, by what we have previously showed about the membership of $\mathbf{Q}_{p_1}^{\gamma_1}$, we obtain $\mathbf{Q}_{p_1}^{\gamma_1} \in \mathbb{V}(\mathbf{Q}_{r_1})$.

- $\mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1}) \subseteq \mathbb{V}(\mathbf{Q}_{p_2}^{\gamma_2})$ if and only if either $p_1 < p_2$ or ($p_1 = p_2$ and $\gamma_1 \leq \gamma_2$).

  $\Rightarrow$) To prove this implication we reason by contraposition: we assume that either $p_2 < p_1$ or ($p_1 = p_2$ and $\gamma_2 < \gamma_1$).
  - If $p_2 < p_1$: Then, by density, there exists a rational q such that $p_2 < q < p_1$. Hence, $\mathbf{c}_q \approx 1$ holds in $\mathbf{Q}_{p_2}^{\gamma_2}$ but fails in $\mathbf{Q}_{p_1}^{\gamma_1}$ (by the definition we have given of this algebras). Therefore, $\mathbf{Q}_{p_1}^{\gamma_1} \notin \mathbb{V}(\mathbf{Q}_{p_2}^{\gamma_2})$ and, consequently, $\mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1}) \nsubseteq \mathbb{V}(\mathbf{Q}_{p_2}^{\gamma_2})$.
  - If $p_1 = p_2$ ($= q \in [\frac{1}{2}, 1) \cap \mathbb{Q}$) and $\gamma_2 < \gamma_1$: Since $\gamma_2 < \gamma_1$, necessarily $\gamma_1 > 0$, moreover, from $\gamma_2 < \gamma_1 \in \omega + 1$ it follows that $\gamma_2 = n$ for some $n \in \omega$.
    Given the fact that $q < 1$ and $\gamma_2 = n$, it is clear that the interval $[\mathbf{c}_q, 1]$ in $\mathbf{Q}_q^{\gamma_2}$ has $n + 2$ elements, hence:
    $$\mathbf{Q}_q^{\gamma_2} = \mathbf{Q}_{p_2}^{\gamma_2} \models \bigvee_{0 \leq i < j \leq n+2} (\mathbf{c}_q \vee x_i) \leftrightarrow (\mathbf{c}_q \vee x_j) \approx 1.$$
    On the other hand, since $\gamma_1 > \gamma_2 = n$, the interval $[\mathbf{c}_q, 1]$ in $\mathbf{Q}_q^{\gamma_1} = \mathbf{Q}_{p_1}^{\gamma_1}$ has more than $n + 2$ elements. Thus, the above equation fails in $\mathbf{Q}_{p_1}^{\gamma_1}$ and, conse-

quently, $\mathbf{Q}_{p_1}^{\gamma_1} \notin \mathbb{V}(\mathbf{Q}_{p_2}^{\gamma_2})$.

$\Leftarrow$) Now, we prove the other direction:

- If $p_1 < p_2$: Then, we have $\mathbb{V}(\mathbf{Q}_{p_1}^{\gamma_1}) \subseteq \mathbb{V}(\mathbf{Q}_{p_2}) \subseteq \mathbb{V}(\mathbf{Q}_{p_2}^{\gamma_2})$ by the previous items we have proved (the third and the second, respectively).
- If $p_1 = p_2$ and $\gamma_1 \leq \gamma_2$: Then,

$$\mathbf{Q}_{p_1}^{\gamma_1} \in \mathbb{S}(\mathbf{Q}_{p_2}^{\gamma_2}) \subseteq \mathbb{V}(\mathbf{Q}_{p_2}^{\gamma_2}).$$

1. Let $\mathbb{K}$ be a nontrivial variety of rational NM-algebras, first we will show that $\mathbb{K}$ is of the form $\mathbb{V}(\mathbf{Q}_r)$ for some $r \in (\frac{1}{2}, 1]$ or $\mathbb{V}(\mathbf{Q}_p^{\gamma})$ for some $\gamma \in \omega + 1$ and $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$. By Corollary 2.3, we know $\mathbb{K}$ is generated by a set of algebras $\{A_i : i \in I\} \neq \varnothing$ of the form $\mathbf{Q}_r$, where $r \in (\frac{1}{2}, 1]$, or $\mathbf{Q}_p^{\gamma}$, where $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$.

   Thus, we want to define some algebra $\mathbf{S}$ of the previous types such that $\mathbb{K} = \mathbb{V}(\mathbf{S})$. Let $s = sup\{r \in [\frac{1}{2}, 1] : \mathbf{Q}_r \in \{A_i : i \in I\}$ or $\mathbf{Q}_r^{\gamma} \in \{A_i : i \in I\}$ for some $\gamma \in \omega + 1\}$.

   - If there exists $\gamma \in \omega + 1$ such that $\mathbf{Q}_s^{\gamma} \in \{A_i : i \in I\}$, then we take $\mathbf{S} := \mathbf{Q}_s^{\delta}$ where $\delta = sup\{\gamma \in \omega + 1 : \mathbf{Q}_s^{\gamma} \in \{A_i : i \in I\}\}$.
   - Otherwise, we take $\mathbf{S} := \mathbf{Q}_s$.

   By the inclusions we have proved in item 2 and the definition we have given of $\mathbf{S}$, it is clear that $\mathbb{K} \subseteq \mathbb{V}(\mathbf{S})$. Now, we show the other inclusion also holds:

   - If $\mathbf{S} \in \{A_i : i \in I\}$, then trivially $\mathbb{V}(\mathbf{S}) \subseteq \mathbb{K}$.
   - Otherwise, either $\mathbf{S} = \mathbf{Q}_s$ or $\mathbf{S} = \mathbf{Q}_s^{\omega}$. In both cases, every finite partial subalgebra of $\mathbf{S}$ embeds into some member of $\{A_i : i \in I\}$, therefore, $\mathbf{S}$ validates the universal theory of $\{A_i : i \in I\}$. Hence, $\mathbf{S} \in \mathbb{I}\mathbb{S}\mathbb{P}_U\{A_i : i \in I\} \subseteq \mathbb{K}$ since $\mathbb{K}$ is a variety generated by $\{A_i : i \in I\}$ and, consequently, $\mathbb{V}(\mathbf{S}) \subseteq \mathbb{K}$.

   We have proved that $\mathbb{K} = \mathbb{V}(\mathbf{S})$ and, thus, every variety of rational NM-algebras is generated by an algebra of the form $\mathbf{Q}_r$ or $\mathbf{Q}_p^{\gamma}$.

   Now, we will show that varieties of the form $\mathbb{V}(\mathbf{Q}_p^{\gamma})$ are axiomatized by the equations: $\{\mathbf{c}_q \approx 1 : q \in (p, 1] \cap \mathbb{Q}\}$ and $\left(\bigvee_{0 \leq i < j \leq n+2} (\mathbf{c}_p \vee x_i) \leftrightarrow (\mathbf{c}_p \vee x_j)\right) \approx 1$, if $\gamma = n \in \omega$; $\{\mathbf{c}_q \approx 1 : q \in (p, 1] \cap \mathbb{Q}\}$, otherwise.

   Let Σ be the set of equations given by the statement, it holds that $\mathbf{Q}_p^{\gamma} \vDash \Sigma$. On the other hand, let's consider a rational NM-algebra $\mathbf{A} \notin \mathbb{V}(\mathbf{Q}_p^{\gamma})$. By what we have proven previously we obtain that $\mathbb{V}(\mathbf{A})$ is generated by some algebra of the form $\mathbf{Q}_r$ for some $r \in (\frac{1}{2}, 1]$ or $\mathbf{Q}_{p'}^{\delta}$ for some $p' \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\delta \in \omega + 1$:

   - If $\mathbb{V}(\mathbf{A}) = \mathbb{V}(\mathbf{Q}_r)$: Then, since we are assuming that $\mathbf{A} \notin \mathbb{V}(\mathbf{Q}_p^{\gamma})$, it holds $\mathbb{V}(\mathbf{Q}_r) \nsubseteq \mathbb{V}(\mathbf{Q}_p^{\gamma})$. Now, by item 2 of this theorem, this implies that $p < r$. Therefore, by density of $\mathbb{Q}$ in $\mathbb{R}$, there is a rational q such that $p < q < r$. Hence, $\mathbf{c}_q^{\mathbf{Q}_r} \neq 1$ and, consequently, $\mathbf{Q}_r \nvDash \Sigma$, which means $\mathbf{A} \nvDash \Sigma$.

- If $\mathbb{V}(\mathbf{A}) = \mathbb{V}(\mathbf{Q}_{p'}^{\delta})$: Then, since we are assuming that $\mathbf{A} \notin \mathbb{V}(\mathbf{Q}_p^{\gamma})$, it holds $\mathbb{V}(\mathbf{Q}_{p'}^{\delta}) \nsubseteq \mathbb{V}(\mathbf{Q}_p^{\gamma})$. Now, by item 2, this implies that either $p < p'$ or ($p = p'$ and $\gamma < \delta$).
  – If $p < p'$, analogously to the previous case, $\mathbf{c}_{p'}^{\mathbf{Q}_{p'}^{\delta}} \neq 1$ and, consequently, $\mathbf{Q}_{p'}^{\delta} \not\models \Sigma$ which means $\mathbf{A} \not\models \Sigma$.
  – If $p = p'$ and $\gamma < \delta$, then $\gamma = n \in \omega$ and

$$\mathbf{Q}_{p'}^{\delta} \not\models \Big(\bigvee_{0 \leq i < j \leq n+2} (\mathbf{c}_p \vee x_i) \leftrightarrow (\mathbf{c}_p \vee x_j) \Big) \approx 1.$$

  Hence, $\mathbf{A} \not\models \Sigma$.

Therefore, we can conclude that Σ axiomatizes $\mathbb{V}(\mathbf{Q}_p^{\gamma})$.

Finally, we will show that varieties of the form $\mathbb{V}(\mathbf{Q}_r)$ are axiomatized by the equations: $\{\mathbf{c}_q \approx 1 : q \in [r, 1] \cap \mathbb{Q}\}$.
Let Σ be the set of equations given by the statement, it holds that $\mathbf{Q}_r \models \Sigma$.

On the other hand, we consider some rational NM-algebra $\mathbf{A}$ such that $\mathbf{A} \notin \mathbb{V}(\mathbf{Q}_r)$ and arguing as in the previous case: either $\mathbb{V}(\mathbf{A}) = \mathbb{V}(\mathbf{Q}_{r'})$ or $\mathbb{V}(\mathbf{A}) = \mathbb{V}(\mathbf{Q}_p^{\gamma})$ for some $r \in (\frac{1}{2}, 1], p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$.

- If $\mathbb{V}(\mathbf{A}) = \mathbb{V}(\mathbf{Q}_{r'})$ since $\mathbf{A} \notin \mathbb{V}(\mathbf{Q}_r)$, by item 2 of this theorem, $r < r'$ where there exists some rational q such that $r < q < r'$. Hence, $\mathbf{c}_q^{\mathbf{Q}_{r'}} \neq 1$ and $\mathbf{Q}_{r'} \not\models \Sigma$, which implies $\mathbf{A} \not\models \Sigma$.
- If $\mathbb{V}(\mathbf{A}) = \mathbb{V}(\mathbf{Q}_p^{\gamma})$ and $\mathbf{A} \notin \mathbb{V}(\mathbf{Q}_r)$ then, by item 2, $r \leq p < 1$. Hence, $\mathbf{c}_p^{\mathbf{Q}_p^{\gamma}} \neq 1$ and $\mathbf{Q}_p^{\gamma} \not\models \Sigma$, which implies $\mathbf{A} \not\models \Sigma$.

Therefore, we can conclude that Σ axiomatizes $\mathbb{V}(\mathbf{Q}_r)$.

3. Let $L_{\mathcal{V}}(RNM)^-$ be the poset of nontrivial varieties of rational NM-algebras, then $L_{\mathcal{V}}(RNM)^-$ is indeed a complete lattice (that is clear since: given a poset $\mathbb{X}$, if $\bigvee Y$ exists for all $Y \subseteq X$, then $\mathbb{X}$ is complete).
Moreover, $L_{\mathcal{V}}(RNM)$ is obtained by adding a new bottom element to $L_{\mathcal{V}}(RNM)^-$ hence, by 1. and 2., it is clear that $L_{\mathcal{V}}(RNM)$ is an uncountable chain and, in order to conclude the proof, it suffices to show that $L_{\mathcal{V}}(RNM)^-$ is isomorphic to the Dedekind-MacNeille completion of the poset $\mathbb{X}$ obtained by endowing $([\frac{1}{2}, 1) \cap \mathbb{Q}) \times (\omega + 1)$ with the lexicographic order:

$$\langle q_1, \gamma_1 \rangle \leq \langle q_2, \gamma_2 \rangle \text{ iff } q_1 < q_2 \text{ or } (q_1 = q_2 \text{ and } \gamma_1 \leq \gamma_2) \quad \text{for any } q_1, q_2 \in [\tfrac{1}{2}, 1) \cap \mathbb{Q}, \; \gamma_1, \gamma_2 \in \omega + 1.$$

To see what we have left to prove, we can consider the map $f : \mathbb{X} \longrightarrow L_{\mathcal{V}}(RNM)^-$ defined by:

$$f(\langle q,\gamma\rangle) := \begin{cases} \mathbb{V}(\mathbf{Q}^{\gamma}_{\frac{1}{2}}) & \text{if } q = \frac{1}{2}, \\ \mathbb{V}(\mathbf{Q}_q) & \text{if } q \neq \frac{1}{2} \text{ and } \gamma = 0, \\ \mathbb{V}(\mathbf{Q}^{n}_q) & \text{if } q \neq \frac{1}{2} \text{ and } \gamma = n+1, \\ \mathbb{V}(\mathbf{Q}^{\omega}_q) & \text{if } q \neq \frac{1}{2} \text{ and } \gamma = \omega. \end{cases}$$

Then, by the previous items we have proven, it is clear that it is an order-embedding: it is injective by 1. and, by 2., we know that for any $q_1, q_2 \in [\frac{1}{2}, 1) \cap \mathbb{Q}$, $\gamma_1, \gamma_2 \in \omega + 1$ we have $\langle q_1, \gamma_1\rangle \leq \langle q_2, \gamma_2\rangle$ iff $f(\langle q_1, \gamma_1\rangle) \subseteq f(\langle q_2, \gamma_2\rangle)$.

Moreover, $f[\mathbb{X}]$ is both join-dense and meet-dense in the complete lattice $L_{\mathcal{V}}(RNM)^-$, that is, every element of $L_{\mathcal{V}}(RNM)^-$ is a join (and also a meet) of some subset of $f[\mathbb{X}]$. This is clear since, for a subset D of a complete lattice $\mathbb{Y}$, D being join-dense in $\mathbb{Y}$ is equivalent to the fact that, for every $x, y \in Y$, $x \nleq y \Leftrightarrow$ there exists $d \in D$ such that $d \leq x$ and $d \nleq y$. Dual statements also characterize meet density.

Therefore, since up to isomorphism the Dedekind-MacNeille completion of a poset $\mathbb{Y}$ is the only completion in which $\mathbb{Y}$ is both join-dense and meet-dense (see [2, Proposition 1] and also [6]), we can conclude (by what we have recently proved) that $L_{\mathcal{V}}(RNM)^-$ is isomorphic to the Dedekind-MacNeille completion of $\mathbb{X}$. Hence, we have seen what we wanted and the proof of item 3. is concluded.

□

Remark 2.5. The axiomatization given in item 1. of Theorem 2.4 can be simplified for varieties of the form $\mathbb{V}(\mathbf{Q}_q)$ with $q \in \mathbb{Q} \cap (\frac{1}{2}, 1]$, as these can be axiomatized by the equation $\mathbf{c}_q \approx 1$. On the other hand, varieties of the form $\mathbb{V}(\mathbf{Q}_r)$ with $r \in (\frac{1}{2}, 1] \setminus \mathbb{Q}$ do not admit a finite axiomatization (that's because $\mathbb{Q}$ is dense in $\mathbb{R}$: there always exists a rational in between any two given real numbers).

Since there is a dual isomorphism between the lattice of subvarieties of RNM and the lattice of axiomatic extensions of **RNML**, Theorem 2.4 provides a full description of the last one, which is presented in the following result:

Corollary 2.6. *Every consistent axiomatic extension of* ***RNML*** *is of the form*

$\boldsymbol{RNML}_r := \boldsymbol{RNML} + \{\mathbf{c}_q : q \in [r, 1] \cap \mathbb{Q}\}$ *for some* $r \in (\frac{1}{2}, 1]$,

$\boldsymbol{RNML}^{\omega}_p := \boldsymbol{RNML} + \{\mathbf{c}_q : q \in (p, 1] \cap \mathbb{Q}\}$ *for some rational* $p \in [\frac{1}{2}, 1)$, *or*

$\boldsymbol{RNML}^{n}_p := \boldsymbol{RNML}^{\omega}_p + \bigvee_{0 \leq i < j \leq n+2} (\boldsymbol{c}_p \vee x_i) \leftrightarrow (\boldsymbol{c}_p \vee x_j)$ *for some rational* $p \in [\frac{1}{2}, 1)$ *and* $n \in \omega$.

Moreover, the lattice of axiomatic extensions of ***RNML*** *is an uncountable chain dually isomorphic to the poset obtained by adding a new bottom element to the Dedekind-MacNeille completion of the lexicographic order of* $[\frac{1}{2}, 1) \cap \mathbb{Q}$ *and* $\omega + 1$.

The structure of the lattice of arbitrary extensions of **RNML** is still unknown as in the case of **RG** (see [18, Section 7]). However, it is easy to see that it has uncountable chains and antichains, the first statement is directly obtained by Corollary 2.6. The claim that the lattice of arbitrary extensions of **RNML** has uncountable antichains is easily proven from the fact that $\{\mathbb{Q}(\mathbf{Q}_r) : r \in (\frac{1}{2}, 1]\} \cup \{\mathbb{Q}(\mathbf{Q}_p^0) : p \in [\frac{1}{2}, 1) \cap \mathbb{Q}\}$ is a set of minimal quasivarieties:

- Given some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$, we observe that, if $\mathbf{A} \in \mathbb{Q}(\mathbf{Q}_p^0)$ is nontrivial, then it validates the quasiequations of the form $\mathbf{c}_q \approx 1 \Rightarrow 0 \approx 1$, for $q \in [\frac{1}{2}, p] \cap \mathbb{Q}$.
  Thus, $\mathbf{c}_q^{\mathbf{A}} < 1$ for all $q \in [\frac{1}{2}, p] \cap \mathbb{Q}$ and, consequently, $0 < \mathbf{c}_q^{\mathbf{A}}$ for any $q \in [1-p, \frac{1}{2}] \cap \mathbb{Q}$. Hence, $\mathbf{Q}_p^0$ is embeddable into $\mathbf{A}$ and this implies that $\mathbb{Q}(\mathbf{Q}_p^0) \subseteq \mathbb{Q}(\mathbf{A})$, i.e. that $\mathbb{Q}(\mathbf{Q}_p^0)$ is a minimal quasivariety.

- Analogously, given some $r \in (\frac{1}{2}, 1]$, we can see that, if $\mathbf{A} \in \mathbb{Q}(\mathbf{Q}_r)$ is nontrivial, then $\mathbf{Q}_r$ is embeddable into $\mathbf{A}$. This implies that $\mathbb{Q}(\mathbf{Q}_r) \subseteq \mathbb{Q}(\mathbf{A})$, therefore $\mathbb{Q}(\mathbf{Q}_r)$ is a minimal quasivariety.

2.2 Structural completeness in RNML

Now, we move on to studying some structural completeness results for the rational Nilpotent minimum logic: as it has been done for the rational Gödel logic in [18, Section 8], we will obtain a full characterization of structural completeness and its variants in extensions of **RNML**. The following result characterizes PSC extensions of **RNML**:

Theorem 2.7. *The following are equivalent for an extension $\vdash$ of **RNML**:*

1. *$\vdash$ is PSC;*
2. *$\vdash$ is algebraized by a quasivariety with the JEP;*
3. *$\vdash$ is algebraized by a quasivariety whose nontrivial members have isomorphic zero-generated subalgebras.*

Proof. 1. $\Rightarrow$ 2. : Is a direct consequence of Theorem 1.30.

2. $\Rightarrow$ 3. : Is easily seen by definition of what it means for a quasivariety to have the JEP.

3. $\Rightarrow$ 1. : Consider $\mathbb{K}$ to be the quasivariety algebraizing $\vdash$, then, by Theorem 1.29, in order to see that $\vdash$ is PSC it suffices to show that every two nontrivial members of $\mathbb{K}$ (the equivalent algebraic semantics of $\vdash$) validate the same positive existential sentences.
To prove what we want, we consider two nontrivial algebras $\mathbf{A}, \mathbf{B} \in \mathbb{K}$ and, by assumption, their zero-generated subalgebras will coincide (let's denote this algebra by $\mathbf{C}$).

By Lemma 1.41 (taking $I = C \setminus \{1\}$), the set $\mathcal{F}$ of prime filters F of $\mathbf{A}$ such that $F \cap C = \{1\}$ is nonempty. Thus, we are considering a nonempty partially ordered set satisfying that every chain has an upper bound (its union) in the set, therefore, by applying Zorn's Lemma we obtain a maximal $F \in \mathcal{F}$.

Since F is prime, by Theorem 1.40, $\mathbf{A}/F$ is a chain and, by construction of F, the zero-generated subalgebra of $\mathbf{A}/F$ is isomorphic to $\mathbf{C}$. Hence, we can assume without loss of generality that $\mathbf{C}$ is a subalgebra of $\mathbf{A}/F$.
The algebra $\mathbf{A}$ is assumed to be nontrivial, therefore either $\mathbf{C} = \mathbf{Q}_r$ for some $r \in (\frac{1}{2}, 1]$ or $\mathbf{C} = \mathbf{Q}_p^0$ for some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$.

Moreover, we have taken F to be maximal on $\mathcal{F}$ thus, for any $a \in A/F$ strictly larger than all the elements of $C \setminus \{1\}$, $a = 1$ must hold (otherwise a wouldn't belong to F nor $C \setminus \{1\}$ and F wouldn't be maximal).
Therefore, we have seen that $\mathbf{A}/F$ is a rational NM-chain satisfying that $|(A/F) \setminus \downarrow (C \setminus \{1\})| = 1$ and, by Proposition 2.2, either $\mathbb{ISP}_U(\mathbf{A}/F) = \mathbb{ISP}_u(\mathbf{Q}_r)$ for some $r \in (\frac{1}{2}, 1]$ or $\mathbb{ISP}_U(\mathbf{A}/F) = \mathbb{ISP}_u(\mathbf{Q}_p^0)$ for some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$. Thus, $\mathbf{A}/F \in \mathbb{ISP}_U(\mathbf{C})$.

By assumption we know $\mathbf{C}$ is a subalgebra of $\mathbf{B}$ (the zero-generated one) then, we obtain $\mathbf{A}/F \in \mathbb{ISP}_U(\mathbf{B})$. Hence, there exists some embedding $f_A : \mathbf{A}/F \hookrightarrow \mathbf{B}_U$ where $\mathbf{B}_U$ is an ultrapower of $\mathbf{B}$.
We can consider $g_A : \mathbf{A} \longrightarrow \mathbf{A}/F$ to be the canonical surjection and, then, the composition $f_A \circ g_A$ will be a homomorphism from $\mathbf{A}$ to $\mathbf{B}_U$.

Finally, since positive existential sentences are preserved by homomorphic images, extensions and ultraroots, it has been showed what we wanted: every positive existential sentence that is true in $\mathbf{A}$ will also be true in $\mathbf{B}$. □

We present and prove the following result, which will be helpful to characterize structural completenes, active structural completeness and hereditary structural completeness later on:

Proposition 2.8. *Let $\mathbb{K}$ be a nontrivial sub-quasivariety of RNM and let $\boldsymbol{F}_{\mathbb{K}}(\omega)$ be its denumerably generated free algebra. Then, there are $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$ such that $\mathbb{Q}(\boldsymbol{F}_{\mathbb{K}}(\omega)) = \mathbb{Q}(\boldsymbol{Q}_r)$ or $\mathbb{Q}(\boldsymbol{F}_{\mathbb{K}}(\omega)) = \mathbb{Q}(\boldsymbol{Q}_p^\gamma)$.*

Proof. We consider $\mathbb{K}$ to be as mentioned in the statement. Now, we know that $\mathbf{F}_{\mathbb{K}}(\omega) = \mathbf{F}_{\mathbb{V}(\mathbb{K})}(\omega)$ where $\mathbb{V}(\mathbf{F}_{\mathbb{V}(\mathbb{K})}(\omega)) = \mathbb{V}(\mathbb{K})$ is a variety of RNM (the last equality is given by [7, Theorem 10.12 and Lemma 11.8]). Thus, by Theorem 2.4, there exist some $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$ such that $\mathbf{F}_{\mathbb{K}}(\omega)$ will be the denumerably generated free algebra of $\mathbb{V}(\mathbf{Q}_r)$ or $\mathbb{V}(\mathbf{Q}_p^\gamma)$.
This last statement implies that $\mathbf{F}_{\mathbb{K}}(\omega)$ will also be the denumerably generated free algebra of $\mathbb{Q}(\mathbf{Q}_r)$ or $\mathbb{Q}(\mathbf{Q}_p^\gamma)$. Hence, $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega)) \subseteq \mathbb{Q}(\mathbf{Q}_r)$ or $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega)) \subseteq \mathbb{Q}(\mathbf{Q}_p^\gamma)$.

We have left to see the other inclusion holds for both cases:

- If $\mathbf{F}_{\mathbb{K}}(\omega) = \mathbf{F}_{\mathbb{V}(\mathbf{Q}_r)}(\omega)$: then $\mathbf{Q}_r$ is the zero-generated subalgebra of $\mathbf{F}_{\mathbb{K}}(\omega)$. Therefore, $\mathbb{Q}(\mathbf{Q}_r) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$.
- If $\mathbf{F}_{\mathbb{K}}(\omega) = \mathbf{F}_{\mathbb{V}(\mathbf{Q}_p^0)}(\omega)$: analogously to the previous case we obtain $\mathbb{Q}(\mathbf{Q}_p^0) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$.
- Finally, we consider the case where $\mathbf{F}_{\mathbb{K}}(\omega) = \mathbf{F}_{\mathbb{V}(\mathbf{Q}_p^\gamma)}(\omega)$ with $\gamma > 0$: since every finite partial subalgebra of $\mathbf{Q}_p^\gamma$ embeds into $\{\mathbf{Q}_p^n : n \in \omega \text{ and } 1 \leq n \leq \gamma\}$ it is clear that $\mathbf{Q}_p^\gamma \in \mathbb{ISP}_U(\{\mathbf{Q}_p^n : n \in \omega \text{ and } 1 \leq n \leq \gamma\})$.
  Thus, if we show that each $\mathbf{Q}_p^n$ (for $n \in \omega$ and $1 \leq n \leq \gamma$) is embeddable into $\mathbf{F}_{\mathbb{K}}(\omega)$ we obtain $\mathbf{Q}_p^\gamma \in \mathbb{ISP}_U\mathbb{S}(\mathbf{F}_{\mathbb{K}}(\omega)) = \mathbb{ISP}_U(\mathbf{F}_{\mathbb{K}}(\omega)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$ and, hence, what we wanted.

  For every $1 \leq n \leq \gamma$ such that $n \in \omega$, the algebra $\mathbf{Q}_p^n$ is the chain consisting of the interval $[1-p, p] \cap \mathbb{Q}$ in between the $n+1$ element chains:

$$0 < \neg t_{n-1} < \cdots < \neg t_1 < \neg t_0$$
$$t_0 < t_1 < \cdots < t_{n-1} < 1.$$

Then, $\mathbf{Q}_p^n$ is embeddable into $\mathbf{F}_{\mathbb{K}}(\omega)$ via the map that is the identity on $\{0\} \cup ([1-p, p] \cap \mathbb{Q}) \cup \{1\}$ and that sends t_i to (the equivalence class of) the formula φ_i and $\neg t_i$ to its negation, where:

$$\varphi_0 := \mathbf{c}_p \vee x_1 \qquad \text{and} \qquad \varphi_{j+1} := x_{j+1} \vee (x_{j+1} \to \varphi_j).$$

That is, we have an embedding:

$$h : \mathbf{Q}_p^{n\Delta} \hookrightarrow \mathbf{F}_{\mathbb{K}(\omega)}$$
$$a \mapsto h(a) := \begin{cases} a & \text{if } a \in \{0\} \cup ([1-p, p] \cap \mathbb{Q}) \cup \{1\}, \\ \overline{\varphi_i} & \text{if } a = t_i, \\ \neg\overline{\varphi_i} & \text{if } a = \neg t_i. \end{cases}$$

which is what we needed in order to conclude the proof.

□

The other variants of structural completeness turn out to be equivalent among extensions of **RNML**:

Theorem 2.9. *The following are equivalent for an extension* $\vdash$ *of* ***RNML****:*

1. $\vdash$ *is HSC;*
2. $\vdash$ *is SC;*
3. $\vdash$ *is ASC;*
4. $\vdash$ *is algebraized by a quasivariety* $\mathbb{K}$ *generated by a chain* $\boldsymbol{A}$*.*

Furthermore, in condition 4., $\boldsymbol{A}$ *can be chosen either trivial or of the form* $\boldsymbol{Q}_r$ *or* $\boldsymbol{Q}_p^\gamma$*, where* $r \in (\frac{1}{2}, 1]$*,* $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ *and* $\gamma \in \omega + 1$*.*

Proof. 1. $\Rightarrow$ 2. : Is straightforward by definition of hereditary structural completeness.

2. $\Rightarrow$ 3. : Is also trivial by the explanation of what it means to be active structural complete.

3. $\Rightarrow$ 2. : Let $\mathbb{K}$ be a quasivariety algebraizing $\vdash$ and $\mathbf{F}_{\mathbb{K}}(\omega)$ be its denumerably generated free algebra.
Assume that $\vdash$ is not SC with the aim to arrive to a contradiction. From this last statement and the supposition that $\vdash$ is ASC we obtain that $\vdash$ must not be PSC.
Therefore, by Theorem 2.7, there is a zero-generated algebra $\mathbf{C} \in \mathbb{K}$ different from the zero-generated subalgebra $\mathbf{F}_{\mathbb{K}}(0)$ of $\mathbf{F}_{\mathbb{K}}(\omega)$. Then, since $\mathbf{C}$ is distinct from $\mathbf{F}_{\mathbb{K}}(0)$ and, for every $\mathbf{A} \in \mathbb{K}$, there is a set X such that $\mathbf{A} \in \mathbb{H}(\mathbf{F}_{\mathbb{K}}(\overline{X}))$, there must be some $q \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ such that $\mathbf{c}_q \approx 1$ holds in $\mathbf{C}$, but not in $\mathbf{F}_{\mathbb{K}}(0)$ (nor in $\mathbf{F}_{\mathbb{K}}(\omega)$).

Moreover, by Proposition 2.8, there are $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$ such that $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega)) = \mathbb{Q}(\mathbf{Q}_r)$ or $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega)) = \mathbb{Q}(\mathbf{Q}_p^\gamma)$. Then, we can take $\mathbf{A} \in \{\mathbf{Q}_r, \mathbf{Q}_p^\gamma\}$ such that $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega)) = \mathbb{Q}(\mathbf{A})$.
In particular $\mathbf{A} \nvDash \mathbf{c}_q \approx 1$ and, since by definition it is a chain,

$$\mathbf{A} \vDash x \vee \mathbf{c}_q \approx 1 \Longrightarrow x \approx 1.$$

Given that $\vdash$ is ASC, by Theorem 1.29, it holds $\mathbf{C} \times \mathbf{F}_{\mathbb{K}}(0) \in \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega)) = \mathbb{Q}(\mathbf{A})$, therefore,

$$\mathbf{C} \times \mathbf{F}_{\mathbb{K}}(0) \vDash x \vee \mathbf{c}_q \approx 1 \Longrightarrow x \approx 1.$$

But this last statement is false: if we consider the assignment $x \mapsto \langle 0, 1\rangle$ we obtain $\langle 0, 1\rangle \vee \langle 1, \dots\rangle \approx 1$ but $x \not\approx 1$. Hence, by assuming $\vdash$ isn't PSC we have arrived to a contradiction and this concludes our proof that $\vdash$ is structurally complete.

2. $\Rightarrow$ 4. : We assume $\vdash$ is SC and, by Theorem 1.29, this implies that $\vdash$ is algebraized by a quasivariety $\mathbb{K}$ that is generated by $\mathbf{F}_{\mathbb{K}}(\omega)$ (that is, $\mathbb{K} = \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$). Moreover, by Proposition 2.8, we obtain what we wanted: either $\mathbb{K}$ is trivial or of the form $\mathbb{Q}(\mathbf{Q}_r)$ for some $r \in (\frac{1}{2}, 1]$ or $\mathbb{Q}(\mathbf{Q}_p^\gamma)$ for some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$.

4. $\Rightarrow$ 1. : We presume that 4. holds and we distinguish the following cases:

- If $\vdash$ is algebraized by a trivial quasivariety $\mathbb{K}$: then $\vdash$ is clearly HSC.
- If $\vdash$ is algebraized by a quasivariety $\mathbb{K} = \mathbb{Q}(\mathbf{A})$ where $\mathbf{A}$ is a nontrivial chain: then, by Proposition 2.2, either $\mathbf{A} = \mathbf{Q}_r$ for some $r \in (\frac{1}{2}, 1]$ or $\mathbf{A} = \mathbf{Q}_p^\gamma$ for some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$.
  We can deduce this last statement because, by Proposition 2.2, we obtain $\mathbb{ISP}_U(\mathbf{A}) = \mathbb{ISP}_U(\mathbf{Q}_r)$ or $\mathbb{ISP}_U(\mathbf{A}) = \mathbb{ISP}_U(\mathbf{Q}_p^\gamma)$, hence $\mathbb{PISP}_U(\mathbf{A}) = \mathbb{PISP}_U(\mathbf{Q}_r)$ or $\mathbb{PISP}_U(\mathbf{A}) = \mathbb{PISP}_U(\mathbf{Q}_p^\gamma)$ and, for any class of algebras $\mathbb{K}'$, $\mathbb{PI}(\mathbb{K}') = \mathbb{IP}(\mathbb{K}')$ and $\mathbb{PS}(\mathbb{K}') \subseteq \mathbb{ISP}(\mathbb{K}')$.

  Thus, we have two different subcases:

  - First, we suppose that $\mathbf{A} = \mathbf{Q}_r$. Then, it is clear that the zero-generated subalgebra of every nontrivial member of $\mathbb{Q}(\mathbf{Q}_r)$ will be $\mathbf{Q}_r$. Thus, the quasivariety $\mathbb{Q}(\mathbf{Q}_r)$ is minimal (there is no subquasivariety different than the trivial one) and, therefore, by Theorem 1.29, we obtain that $\vdash$ is HSC.
  - Finally, we consider the case where $\mathbf{A} = \mathbf{Q}_p^\gamma$. Since we know that every extension of $\vdash$ will be algebraized by a subquasivariety of $\mathbb{K}$ then, by definition of HSC and Theorem 1.29, to prove what we want it suffices to see that every subquasivariety of $\mathbb{Q}(\mathbf{Q}_p^\gamma)$ is generated as a quasivariety by its denumerably generated free algebra.

    The previous statement holds for $\mathbb{Q}(\mathbf{Q}_p^\gamma)$: by the proof that has been given of Proposition 2.8 it is obtained that $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega)) = \mathbb{Q}(\mathbf{Q}_p^\gamma) = \mathbb{K}$.

    Moreover, if we consider a proper subquasivariety $\mathbb{K}'$ of $\mathbb{Q}(\mathbf{Q}_p^\gamma)$, we can assume $\mathbb{K}'$ to be nontrivial since otherwise it is easily seen what we want to show. Then, $\mathbf{Q}_p^0 \in \mathbb{K}'$ (by its nontriviality) and $\mathbf{Q}_p^\gamma \notin \mathbb{K}'$ (since it is proper). Consequently, there is some $n \in \omega$ such that, for all $m \in \omega + 1$,

    $$\mathbf{Q}_p^m \in \mathbb{K}' \qquad \text{if and only if} \qquad m \leq n.$$

    This together with Proposition 2.8 implies that $\mathbb{Q}(\mathbf{F}_{\mathbb{K}'}(\omega)) = \mathbb{Q}(\mathbf{Q}_p^n)$: that's because $\mathbb{Q}(\mathbf{F}_{\mathbb{K}'}(\omega)) \subseteq \mathbb{K}'$ thus, for any $m > n$, $\mathbf{Q}_p^m \notin \mathbb{Q}(\mathbf{F}_{\mathbb{K}'}(\omega))$.

From the equality $\mathbb{Q}(\mathbf{F}_{\mathbb{K}'}(\omega)) = \mathbb{Q}(\mathbf{Q}_p^n)$, in particular, we obtain:

$$\mathbb{K}' \vDash \left(\bigvee_{0 \leq i < j \leq n+2} (\mathbf{c}_p \vee x_i) \leftrightarrow (\mathbf{c}_p \vee x_j) \right) \approx 1.$$

Recall that we want to show $\mathbb{Q}(\mathbf{F}_{\mathbb{K}'}(\omega)) = \mathbb{K}'$ where, clearly, the inclusion from left to right holds. Thus, in order to prove what is left to see, we let $\mathbf{B} \in \mathbb{K}'$ and we will show that $\mathbf{B} \in \mathbb{Q}(\mathbf{F}_{\mathbb{K}'}(\omega))$.

Now, since $\mathbb{K}'$ is a quasivariety of $\mathbb{Q}(\mathbf{Q}_p^\gamma)$, we know $\mathbf{B} \in \mathbb{Q}(\mathbf{Q}_p^\gamma)$ and, by the Subdirect Decomposition Theorem (see [7, Theorem 8.6]), $\mathbf{B}$ is a subdirect product of algebras that are relatively subdirectly irreducible in $\mathbb{Q}(\mathbf{Q}_p^\gamma)$ (let's denote them by $\mathbf{C}_i$).
It is easy to see that $\mathbb{Q}(\mathbf{Q}_p^\gamma) = \mathbb{ISPP}_{\mathrm{U}}(\mathbf{Q}_p^\gamma) = \mathbb{IP}_{\mathrm{SD}}\mathbb{SP}_{\mathrm{U}}(\mathbf{Q}_p^\gamma)$. In particular, this implies that $\mathbf{C}_i \in \mathbb{IP}_{\mathrm{SD}}\mathbb{SP}_{\mathrm{U}}(\mathbf{Q}_p^\gamma)$ and, since we are assuming that they are relatively subdirectly irreducible in $\mathbb{Q}(\mathbf{Q}_p^\gamma)$, $\mathbf{C}_i \in \mathbb{ISP}_{\mathrm{U}}(\mathbf{Q}_p^\gamma)$, therefore, they are chains.

Hence, we have that $\mathbf{B}$ is a subdirect product of the chains $\{\mathbf{C}_i : i \in I\}$ in $\mathbb{Q}(\mathbf{Q}_p^\gamma)$. Furthermore, $\mathbf{B}$ validates the equation previously exhibited and so do all the $\mathbf{C}_i$.
From this, the fact that $\mathbf{C}_i \in \mathbb{Q}(\mathbf{Q}_p^\gamma)$ (for every $i \in I$) and Proposition 2.2, we obtain:

$$\{\mathbf{C}_i : i \in I\} \subseteq \mathbb{ISP}_U(\{\mathbf{Q}_p^0, \dots, \mathbf{Q}_p^n\}) \subseteq \mathbb{ISP}_U(\mathbf{Q}_p^n).$$

Thus, $\mathbf{B}$ is a subdirect product of members of $\mathbb{Q}(\mathbf{Q}_p^n)$ and, since we had shown $\mathbb{Q}(\mathbf{F}_{\mathbb{K}'}(\omega)) = \mathbb{Q}(\mathbf{Q}_p^n)$, we conclude $\mathbf{B} \in \mathbb{Q}(\mathbf{F}_{\mathbb{K}'}(\omega))$.
Therefore, $\mathbb{K}'$ is generated as a quasivariety by $\mathbf{F}_{\mathbb{K}'}(\omega)$, as we wanted to prove.

In all the cases we have distinguished we have shown to be true that $\vdash$ is HSC.

□

Next, we will introduce a result that presents bases for the admissible rules on all the axiomatic extensions of **RNML**.

Theorem 2.10. *The following hold for all $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$:*

1. *A base for the admissible rules of the logic $\boldsymbol{RNML}_r$ is given by the rules of the form $\boldsymbol{c}_q \vee z \rhd z$, for all $q \in [\frac{1}{2}, r) \cap \mathbb{Q}$;*
2. *A base for the admissible rules of $\boldsymbol{RNML}_p^\gamma$ is given by the rule $\boldsymbol{c}_p \vee z \rhd z$.*

Proof. By Theorem 2.9, we obtain that the structural completion of $\mathbf{RNML}_r$ is algebraized by $\mathbb{Q}(\mathbf{Q}_r)$ and that of $\mathbf{RNML}_p^\gamma$ by $\mathbb{Q}(\mathbf{Q}_p^\gamma)$. Thus, in order to obtain a base for the admissible rules of $\mathbf{RNML}_r$ and $\mathbf{RNML}_p^\gamma$, it is enough to find an axiomatization of $\mathbb{Q}(\mathbf{Q}_r)$ and $\mathbb{Q}(\mathbf{Q}_p^\gamma)$ relative to $\mathbb{V}(\mathbf{Q}_r)$ and $\mathbb{V}(\mathbf{Q}_p^\gamma)$, respectively (since, by Theorem 2.4, they algebraize $\mathbf{RNML}_r$ and $\mathbf{RNML}_p^\gamma$).

Proposition 2.2 and Theorem 2.4 tell us that the universal class of $\mathbf{Q}_r$ is axiomatized relative to $\mathbb{V}(\mathbf{Q}_r)$ by $\mathbf{c}_q \not\approx 1$ for all $q \in [\frac{1}{2}, r) \cap \mathbb{Q}$. Then, we let T be the set of all terms $\mathbf{c}_q$ with $q \in [\frac{1}{2}, r) \cap \mathbb{Q}$ and we consider some arbitrary $\mathbf{A} \in \mathbb{V}(\mathbf{Q}_r)$:

- If $\mathbf{A}$ validates all the quasiequations $t \vee z \approx 1 \Longrightarrow z \approx 1$ for $t \in T$, then, for every $t \in T$, $\mathbf{A}$ validates $t \not\approx 1$.
  Hence, by what we have mentioned about the axiomatization of the universal class of $\mathbf{Q}_r$ relative to $\mathbb{V}(\mathbf{Q}_r)$, $\mathbf{A} \in \mathbb{ISP}_U(\mathbf{Q}_r) \subseteq \mathbb{Q}(\mathbf{Q}_r)$.

- On the other hand, if $\mathbf{A} \in \mathbb{Q}(\mathbf{Q}_r)$, then $\mathbf{A}$ validates all quasiequations of the form $t \vee z \approx 1 \Longrightarrow z \approx 1$ since they all hold in $\mathbf{Q}_r$.

Therefore, it is clear that $\mathbb{Q}(\mathbf{Q}_r)$ is axiomatized relative to $\mathbb{V}(\mathbf{Q}_r)$ by $\mathbf{c}_q \vee z \approx 1 \Longrightarrow z \approx 1$, for all $q \in [\frac{1}{2}, r) \cap \mathbb{Q}$.

Analogously, we obtain that $\mathbb{Q}(\mathbf{Q}_p^\gamma)$ is axiomatized relative to $\mathbb{V}(\mathbf{Q}_p^\gamma)$ by $\mathbf{c}_p \vee z \approx 1 \Longrightarrow z \approx 1$.
□

2.3 Comparative analysis of the results

The addition of rational constants to the language of **NML** changes the lattice of axiomatic extensions. The lattice of the new logic is totally ordered and has uncountable elements while, on the other hand, we have a countable number of axiomatic extensions for **NML**. Moreover, logics of the type **NM2n+1** and **NM2(n+1)** can't be compared given any $n > 0$.

The change produced in the ordering of the lattice is due to the fact that now, every RNM-chain satisfies $\mathbf{c}_{\frac{1}{2}} \to 0 \approx \mathbf{c}_{\neg\frac{1}{2}}$. In other words, $\neg\mathbf{c}_{\frac{1}{2}} \approx \mathbf{c}_{\frac{1}{2}}$, consequently, every RNM-chain contains the negation fixpoint. Thus, while studying the lattice of axiomatic extensions of **RNML**, we obtain something totally ordered, as is the case for the sublattice of $\mathrm{L}_{\mathcal{V}}(NM)$ that contains all the subvarieties generated by chains with negation fixpoint.

The lattice of finitary extensions of **RNML** will also be different from the one we obtain for **NML** since the latter doesn't have neither uncountable chains nor unncountable antichains. In fact, it has a countable number of elements.

Lastly, we have seen that we can give a bases for the admissible rules on all axiomatic extensions, just as what happens in the case of **NML** (see Theorem 1.75).
But even so, the addition of rational constants to the language does change some structural completeness results. Previously, for **NML** we had hereditary active structural completeness and we had also seen that every logic algebraized by some variety of NM-algebras containing $\mathbb{V}(\mathbf{A}_3)$ was not structural complete. Thus, for **NML** we have some extensions that are ASC but not SC, in contrast to the results we obtain for **RNML**, where the notions of ASC, SC and HSC are equivalent for any extension.

3 The logic $\mathbf{NML}_\Delta$

Clearly, **NML** is a core fuzzy logic (see Definition 1.32) and, in this section, we will study the Δ-core fuzzy logic $\mathbf{NML}_\Delta$, in particular, how the addition of the Δ connective impacts the lattices of axiomatic and finitary extensions and the results about structural completeness.

By Proposition 1.49, we know that $\mathbf{NML}_\Delta$ is complete with respect to the class of NM_Δ-algebras described in Definition 1.43 (which we will denote by NM_Δ from now on), moreover, it is complete with respect to the class of NM_Δ-chains.
We have previously mentioned in Definition 1.59 that the standard NM-algebra is unique up to isomorphism. Hence, there will be a unique standard NM_Δ-algebra which we will denote by $\mathbf{[0,1]}_\Delta = \langle [0,1]; *, \to, \wedge, \vee, \neg, \Delta, 0, 1\rangle$ where Δ is as presented in (1.1) and the rest of operations as described in Definition 1.59.
Thus, by Proposition 1.50 and standard completeness of **NML**, for every set of formulas $\Gamma \cup \{\varphi\}$ we have:

$$\Gamma \vdash_{\mathbf{NML}_\Delta} \varphi \text{ if and only if there exists a finite } \Lambda \subseteq \Gamma \text{ such that } \tau[\Lambda] \vDash_{\mathbf{[0,1]}_\Delta} \tau(\varphi)$$

where $\tau := \{x \approx 1\}$.

Therefore, $\mathbf{NML}_\Delta$ is algebraized by the variety $NM_\Delta = \mathbb{V}(\mathrm{NM}_\Delta\text{-chains}) = \mathbb{V}(\mathbf{[0,1]}_\Delta)$. In fact, as in the case of **NML**, it also holds $NM_\Delta = \mathbb{Q}(\mathbf{[0,1]}_\Delta)$, since every countable NM_Δ-chain will be embeddable into $\mathbf{[0,1]}_\Delta$.

We will present some particular properties of the variety NM_Δ that will be useful later on to study the lattice of finitary extensions of the logic $\mathbf{NML}_\Delta$. But first, we introduce some concepts:

Definition 3.1. *Given a set A, the quaternary discriminator on A is defined by*

$$d^A(x,y,z,w) = \begin{cases} z & \text{if } x = y; \\ w & \text{otherwise.} \end{cases}$$

Definition 3.2. *A variety $\mathscr{V}$ is called a discriminator variety if it is generated by a class $\mathbb{K}$ for which there exists a term N such that $N^{\boldsymbol{A}} = d^{\boldsymbol{A}}$ for every $\boldsymbol{A} \in \mathbb{K}$.*

Remark 3.3. NM_Δ is a discriminator variety. That is because, as we have seen, it is generated by a class of algebras that are linearly ordered. Thus, the connective Δ will be defined in those algebras as in (1.1) and the operations $*, \to$ as in Remark 1.60.
Therefore, the term

$$N(x,y,z,w) := (\Delta(x \to y \wedge y \to x) * z) \vee (\neg\Delta(x \to y \wedge y \to x) * w)$$

will satisfy that, for every algebra $\mathbf{A}$ in the generator class of NM_Δ, $N^{\mathbf{A}} = d^{\mathbf{A}}$.

Hence, we will be able to use the following definitions and results about discriminator varieties:

Definition 3.4. *Given $\mathscr{V}$ a discriminator variety, we denote by $\mathscr{B}$ the class of algebras $\{A \in \mathscr{V} :\ A$ is simple and has no trivial subalgebra $\}$.*

In our case, where $\mathscr{V} = NM_\Delta$, $\mathscr{B}$ will be the class of all nontrivial NM_Δ-chains. This is due to Proposition 1.49 given that the notions of simple and subdirectly irreducible are equivalent because we are in a discriminator variety (check [23, page 240]).
In fact, since $\mathscr{B}$ is a universal class, it will be closed under the formation of ultraproducts.

Theorem 3.5. [4, Theorem 1]

a) $L_{\mathcal{Q}}(\mathscr{V}) = L_{\mathcal{V}}(\mathscr{V})$ *if and only if either* $\mathscr{B} = \varnothing$ *or* $L_{\mathcal{V}}(\mathscr{V}) \cong \mathbb{2}$.

b) The following are equivalent:

1. $L_{\mathcal{Q}}(\mathscr{V})$ *is modular*
2. $L_{\mathcal{Q}}(\mathscr{V})$ *is distributive*
3. $V_1 \subseteq V_2$ *or* $V_2 \subseteq V_1$, *for every* $V_1, V_2 \in L_{\mathcal{V}}(\mathscr{V}), V_1 \subseteq \mathbb{V}(\mathscr{B})$.

Moreover, if the language has a constant, then the above conditions are equivalent to

4. $L_{\mathcal{V}}(\mathscr{V})$ *is a chain or* $L_{\mathcal{Q}}(\mathscr{V}) = L_{\mathcal{V}}(\mathscr{V})$.

c) $L_{\mathcal{Q}}(\mathscr{V})$ *is a chain if and only if some of the following conditions hold*

1. $L_{\mathcal{Q}}(\mathscr{V}) = L_{\mathcal{V}}(\mathscr{V})$ *is a chain*
2. $L_{\mathcal{V}}(\mathscr{V}) = \{\mathbb{V}(\varnothing), \mathbb{V}(\mathscr{B}), \mathscr{V}\} \cong \mathbb{3}$ *(and, hence,* $L_{\mathcal{Q}}(\mathscr{V}) \cong \mathbb{4}$*).*

d) $L_{\mathcal{Q}}(\mathscr{V})$ *is a Boolean lattice iff* $L_{\mathcal{Q}}(\mathscr{V}) = L_{\mathcal{V}}(\mathscr{V})$ *is a Boolean lattice.*

e) The pair $(L_{\mathcal{V}}(\mathscr{V}), \mathbb{V}(\mathscr{B}))$ *determines* $L_{\mathcal{Q}}(\mathscr{V})$, *i.e., if* $\mathscr{V}'$ *is a discriminator variety such that* $\mathscr{B}'$ *is closed under the formation of ultraproducts and such that* $L_{\mathcal{V}}(\mathscr{V}')$ *is isomorphic to* $L_{\mathcal{V}}(\mathscr{V})$ *via an isomorphism which carries* $\mathbb{V}(\mathscr{B}')$ *in* $\mathbb{V}(\mathscr{B})$, *then* $L_{\mathcal{Q}}(\mathscr{V}') \cong L_{\mathcal{Q}}(\mathscr{V})$.

Where $L_{\mathcal{V}}(\mathscr{V})$ (respectively $L_{\mathcal{Q}}(\mathscr{V})$) denote the lattice of subvarieties (respectively subquasivarieties) of $\mathscr{V}$.

3.1 The lattices of axiomatic and finitary extensions

First, we want to study the lattice of axiomatic extensions of $\mathbf{NML}_\Delta$, equivalently, since they are dually isomorphic, we will analize the lattice of proper subvarieties of NM_Δ.
By Remark 1.67 we know that, for each $n \in \mathbb{N} \setminus \{0\}$, there is only one NM-chain $\mathbf{A}_n$ with exactly n elements, up to isomorphism. Therefore, since every NM_Δ-chain is an expansion of an NM-chain (seen in Definition 1.43) and Δ is uniquely defined in NM_Δ-chains (by Proposition 1.47), we obtain that, for each $n \in \mathbb{N} \setminus \{0\}$, there is (up to isomorphism) only one NM_Δ-chain with exactly n elements which we will denote by $\mathbf{A}_n^\Delta$.
Moreover, we have already seen that there is just one standard NM_Δ-algebra $\mathbf{[0,1]}_\Delta$ up to isomorphism.

Remark 3.6. As in the case of NM-algebras, any finitely generated subalgebra of a nontrivial NM_Δ-chain is finite, that means, it will be isomorphic to $\mathbf{A}_{2n}^\Delta$ or $\mathbf{A}_{2n+1}^\Delta$ for some $n > 0$.

The following result is a direct consequence of Proposition 1.66:

Proposition 3.7.

- $\boldsymbol{A}_{2m+1}^{\Delta}$ *is a subalgebra of* $\boldsymbol{A}_{2n+1}^{\Delta}$ *iff* $m \leq n$,
- $\boldsymbol{A}_{2m}^{\Delta}$ *is a subalgebra of* $\boldsymbol{A}_{2n+1}^{\Delta}$ *iff* $0 < m \leq n$,
- $\boldsymbol{A}_{2m}^{\Delta}$ *is a subalgebra of* $\boldsymbol{A}_{2n}^{\Delta}$ *iff* $0 < m \leq n$,
- $\boldsymbol{A}_{2n+1}^{\Delta}$ *is not a subalgebra of* $\boldsymbol{A}_{2m}^{\Delta}$ *for any* $n, m > 0$,
- $\boldsymbol{A}_{m}^{\Delta}$ *is embeddable into* $\boldsymbol{[0,1]}_{\Delta}$ *for every* $m > 1$,
- $\boldsymbol{A}_{2m}^{\Delta}$ *is embeddable into* $\boldsymbol{[0,1]}_{\Delta}^{-}$ *for every* $m > 0$.

In fact, by Definitions 1.43 and 1.69, we know that, just as stated for the case of NM-algebras, an NM_{Δ}-chain satisfies $S_n(x_0, \dots, x_n) \approx 1$ if and only if it has less than $2n+2$ elements. Furthermore, a nontrivial NM_{Δ}-chain satisfies $BP(x) \approx 1$ if and only if it does not contain the negation fixpoint.

Moreover, we had previously mentioned $NM_{\Delta} = \mathbb{V}(\mathrm{NM}_{\Delta}\text{-chains})$ and it has been seen that NM_{Δ}-chains have the same properties as the ones presented for NM-chains. Therefore, we will be able to characterize, classify and axiomatize all axiomatic extensions of $\mathbf{NML}_{\Delta}$ (or, equivalently, all proper subvarieties of NM_{Δ}) analogously as in the case of **NML**:

Theorem 3.8. (The proof is analogous to the one of [15, Theorem 1]) *A variety of* NM_{Δ}*-algebras is a proper subvariety of* NM_{Δ} *if and only if it does not contain some* $\boldsymbol{A}_{k}^{\Delta}$ *with* $1 < k$.

Corollary 3.9. $NM_{\Delta} = \mathbb{V}(\{\boldsymbol{A}_{n}^{\Delta} : n \in \omega, 0 < n\}) = \mathbb{V}(\{\boldsymbol{A}_{2n+1}^{\Delta} : n \in \omega\}) = \mathbb{Q}(\{\boldsymbol{A}_{n}^{\Delta} : n \in \omega, 0 < n\})$.

Corollary 3.10. *Let* $\boldsymbol{A}$ *be an infinite* NM_{Δ}*-chain containing the negation fixpoint. Then* $\mathbb{V}(\boldsymbol{A}) = NM_{\Delta} = \mathbb{Q}(\boldsymbol{A})$.

Since any finitely generated NM_{Δ}-chain is finite and every NM_{Δ}-algebra is a subdirect product of NM_{Δ}-chains, it is easy to see that:

Proposition 3.11. *Every variety of* NM_{Δ}*-algebras is generated by its finite* NM_{Δ}*-chains.*

Proof. Let W be a variety of NM_{Δ}-algebras, we want to see that $W = \mathbb{V}(W_{\text{finite chains}})$. The inclusion from right to left is clear, hence, it is enough to show $W \subseteq \mathbb{V}(W_{\text{finite chains}})$.

In order to see what we want, first we will prove that, given W and some other variety of NM_{Δ}-algebras $W' \subseteq W$:

W' is a proper subvariety $\Leftrightarrow$ W' doesn't contain some finite NM_{Δ}-chain of W.

$\Leftarrow$) Holds trivially.

$\Rightarrow$) Since W' is a proper subvariety, there is some equation $\epsilon \approx \delta$ with m variables (let's denote them $\{x_0, \dots, x_{m-1}\}$) satisfied in W' and not in W.

Hence, there is some nontrivial algebra $\mathbf{A} \in W$ such that $\mathbf{A} \notin W'$. That is, there exist some elements $a_0, \ldots, a_{m-1} \in A$ such that $\epsilon(a_0, \ldots, a_{m-1}) \neq \delta(a_0, \ldots, a_{m-1})$.
By Proposition 1.49 we know every NM_Δ-algebra is a subdirect product of NM_Δ-chains, thus, we can assume, without losing generality, that $\mathbf{A}$ is an NM_Δ-chain.
We take $\mathbf{B}$ to be the subalgebra of $\mathbf{A}$ generated by $\{a_0, \ldots, a_{m-1}\}$. Since every finitely generated NM_Δ-chain is finite, $\mathbf{B}$ is a finite NM_Δ-chain such that $\mathbf{B} \in \mathbb{S}(\mathbf{A}) \subseteq W$ and $\mathbf{B} \not\models \epsilon \approx \delta$. Therefore, $\mathbf{B} \notin W'$.
Thus, it has been seen that W' doesn't contain some finite NM_Δ-chain.

Now, we can finally argue that $W = \mathbb{V}(W_{\text{finite chains}})$. That is because $\mathbb{V}(W_{\text{finite chains}})$ contains every finite NM_Δ-chain of W, hence, by what we have proven, it must not be a proper subvariety. ☐

From the above results and completeness for $\mathbf{NML}_\Delta$ (Proposition 1.49) we obtain:

Proposition 3.12. *$\boldsymbol{NML}_\Delta$ is decidable. Moreover, so will be every axiomatic extension.*

The proper subvarieties of NM_Δ are characterized and axiomatized just as stated in Theorem 1.70 for the case of NM but with the respective algebras $\mathbf{A}^\Delta_{2n}$, $\mathbf{A}^\Delta_{2m+1}$, $[\mathbf{0,1}]^-_\Delta$. The proof is analogous to the one in [15, Theorem 3]:

Theorem 3.13. *Every proper nontrivial subvariety of NM_Δ is of one of the following types:*

1. $\mathbb{V}([\boldsymbol{0,1}]^-_\Delta) = \mathbb{V}(\{\boldsymbol{A}^\Delta_{2k} : k \in \omega\}) = \mathbb{Q}(\{\boldsymbol{A}^\Delta_{2k} : k \in \omega\})$,
2. $\mathbb{V}(\boldsymbol{A}^\Delta_{2m+1}) = \mathbb{Q}(\boldsymbol{A}^\Delta_{2m+1})$ *for some* $m > 0$,
3. $\mathbb{V}(\boldsymbol{A}^\Delta_{2n}) = \mathbb{Q}(\boldsymbol{A}^\Delta_{2n})$ *for some* $n > 0$,
4. $\mathbb{V}([\boldsymbol{0,1}]^-_\Delta, \boldsymbol{A}^\Delta_{2m+1}) = \mathbb{V}(\{\boldsymbol{A}^\Delta_{2k} \colon k \in \omega\} \cup \{\boldsymbol{A}^\Delta_{2m+1}\}) = \mathbb{Q}(\{\boldsymbol{A}^\Delta_{2k} \colon k \in \omega\} \cup \{\boldsymbol{A}^\Delta_{2m+1}\})$ *for some* $m > 0$,
5. $\mathbb{V}(\boldsymbol{A}^\Delta_{2n}, \boldsymbol{A}^\Delta_{2m+1}) = \mathbb{Q}(\boldsymbol{A}^\Delta_{2n}, \boldsymbol{A}^\Delta_{2m+1})$ *for some* $m, n \in \omega$ *such that* $0 < m < n$.

Moreover, if Σ is any set of equations axiomatizing NM_Δ, then

1. $\mathbb{V}([\boldsymbol{0,1}]^-_\Delta)$ *is axiomatized by Σ plus the equation* $BP(x) \approx \overline{1}$,
2. $\mathbb{V}(\boldsymbol{A}^\Delta_{2m+1})$ *is axiomatized by Σ plus the equation* $S_m(x_0, \ldots, x_m) \approx \overline{1}$,
3. $\mathbb{V}(\boldsymbol{A}^\Delta_{2n})$ *is axiomatized by Σ plus the equations* $S_n(x_0, \ldots, x_n) \approx \overline{1}$ *and* $BP(x) \approx \overline{1}$,
4. $\mathbb{V}([\boldsymbol{0,1}]^-_\Delta, \boldsymbol{A}^\Delta_{2m+1})$ *is axiomatized by Σ plus the equation* $BP(x) \vee S_m(x_0, \ldots, x_m) \approx \overline{1}$,
5. $\mathbb{V}(\boldsymbol{A}^\Delta_{2n}, \boldsymbol{A}^\Delta_{2m+1})$ *with* $m < n$ *is axiomatized by Σ plus* $(BP(x) \wedge S_n(x_0, \ldots, x_n)) \vee S_m(x_0, \ldots, x_m) \approx \overline{1}$.

Furthermore, from Proposition 3.7 we can easily obtain relations among NM_Δ varieties:

Proposition 3.14.

- $\mathbb{V}(\boldsymbol{A}^\Delta_{2n+1}) \subseteq \mathbb{V}(\boldsymbol{A}^\Delta_{2m+1})$ *for every* $n \leq m$,

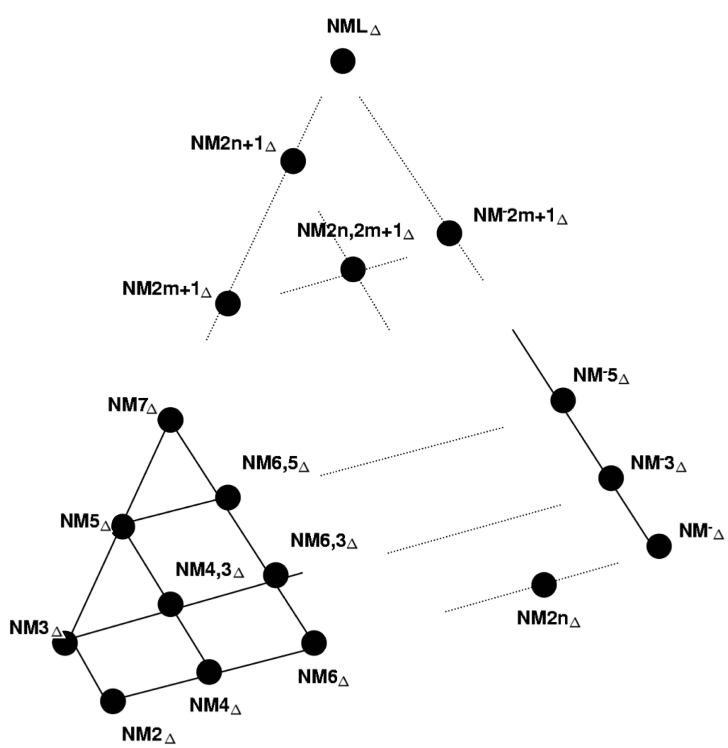
- $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2n}) \subseteq \mathbb{V}(\boldsymbol{A}^{\Delta}_{2m+1})$ *for every* $0 < n \leq m$,
- $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2n}) \subseteq \mathbb{V}(\boldsymbol{A}^{\Delta}_{2m})$ *for every* $0 < n \leq m$,
- $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2n}) \subseteq \mathbb{V}(\boldsymbol{[0,1]}^{-}_{\Delta})$ *for every* $n > 0$,
- $\mathbb{V}(\boldsymbol{[0,1]}^{-}_{\Delta}) \cap \mathbb{V}(\boldsymbol{A}^{\Delta}_{2m+1}) = \mathbb{V}(\boldsymbol{A}^{\Delta}_{2m})$ *for every* $m > 0$,
- $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2n}) \cap \mathbb{V}(\boldsymbol{A}^{\Delta}_{2m+1}) = \mathbb{V}(\boldsymbol{A}^{\Delta}_{2\,min\{n,m\}})$ *for every* $n, m > 0$.

We know there is a lattice isomorphism between the lattice of all subvarieties of NM_{Δ} and the lattice of all axiomatic extensions of $\mathbf{NML}_{\Delta}$. Thus, from Theorem 3.13, we have:

Theorem 3.15. *All proper consistent axiomatic extensions of* $\boldsymbol{NML}_{\Delta}$ *are:*
For every natural numbers $n, m > 0$

1. $\boldsymbol{NM}^{-}_{\Delta} = \boldsymbol{NML}_{\Delta}$ *plus A13,*
2. $\boldsymbol{NM2m{+}1}_{\Delta} = \boldsymbol{NML}_{\Delta}$ *plus* $A12_m$,
3. $\boldsymbol{NM2n}_{\Delta} = \boldsymbol{NML}_{\Delta}$ *plus* $A12_n$ *and A13,*
4. $\boldsymbol{NM^{-}2m{+}1}_{\Delta} = \boldsymbol{NML}_{\Delta}$ *plus* $A13 \vee A12_m$,
5. $\boldsymbol{NM2n,2m{+}1}_{\Delta} = \boldsymbol{NML}_{\Delta}$ *plus* $(A12_n \wedge A13) \vee A12_m$ *with* $n > m$.

In fact, the relations between NM_{Δ}-varieties stated in Proposition 3.14 can also be translated to relations among extensions of the logic we are studying. Hence, we obtain the following lattice of axiomatic extensions of $\mathbf{NML}_{\Delta}$ (with the dual order):



Remark 3.16. This lattice happens to be equal to the one obtained for the case of the logic **NML**.

Now, we will proceed to examine the lattice of finitary extensions of the logic.
In order to do so, we will use the isomorphism between the lattice of all subquasivarieties of NM_Δ and the lattice of all finitary extensions of the logic, and we will focus on studying $L_{\mathcal{Q}}(\mathrm{NM}_\Delta)$.

We recall Remark 3.3 and use Theorem 3.5:
It is clear that $\mathbf{A}_2^\Delta \in \mathscr{B} = \{\mathbf{A} \in NM_\Delta : \ \mathbf{A}$ is simple and has no trivial subalgebra$\} \neq \varnothing$. Furthermore, we have previously studied $L_{\mathcal{V}}(NM_\Delta)$ and we know it is not isomorphic to $\mathbb{2}$, hence, by Theorem 3.5 a), we obtain that $L_{\mathcal{Q}}(NM_\Delta) \neq L_{\mathcal{V}}(NM_\Delta)$.

Actually, the language of NM_Δ contains a constant (e.g. the 0-ary function 1), therefore, by the inequality we have proven and Theorem 3.5 b) 4., we deduce that $L_{\mathcal{Q}}(NM_\Delta)$ is not modular nor distributive and, by d) and c), $L_{\mathcal{Q}}(NM_\Delta)$ is not a Boolean lattice nor a chain.

In fact, if we were able to find a locally finite discriminator variety whose lattice of all subvarieties was isomorphic to $L_{\mathcal{V}}(NM_\Delta)$ then, by Theorem 3.5 e), we could know how the lattice of subquasivarieties $L_{\mathcal{Q}}(NM_\Delta)$ would look (see Remark 3.33 for an example).
The local finiteness of the variety would assure us that the requirement that $\mathscr{B}'$ is closed under the formation of ultraproducts is satisfied (see [4, Lemma 13]).

Remark 3.17. NM_Δ is a locally finite variety. That is, every finitely generated subalgebra of some NM_Δ-algebra is, in fact, finite. Therefore, every subquasivariety of NM_Δ will also be locally finite.

In order to get more information about $L_{\mathcal{Q}}(NM_\Delta)$, we will present some results about locally finite NM_Δ-quasivarieties that will let us obtain all quasivarieties of NM_Δ-algebras, establish inclusion relations between them and characterize and axiomatize its generators.

Theorem 3.18. *Every finite NM_Δ-algebra is isomorphic to a direct product of finite simple NM_Δ-algebras.*

Proof. Given some finite NM_Δ-algebra $\mathbf{A}$, by [7, Theorem 7.10] we obtain that $\mathbf{A}$ is isomorphic to a direct product of directly indecomposable algebras.
Moreover, since NM_Δ is a discriminator variety, from [23, page 240] we deduce that directly indecomposable algebras are simple. In particular, we will have a direct product of finite simple algebras since, otherwise, $\mathbf{A}$ wouldn't be finite. □

Remark 3.19. Thus, by the last result and Proposition 1.49, every finite NM_Δ-algebra is isomorphic to a direct product of finite NM_Δ-chains.

Definition 3.20. *A critical algebra is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras.*

The interest of critical algebras is given by the following theorem:

Theorem 3.21. [17, Theorem 2.3] *Every locally finite quasivariety is generated by its critical algebras.*

Moreover, we will be able to characterize all NM_Δ-critical algebras, similarly to how it has been done in [17], by means of the following lemma:

Lemma 3.22. *If $\boldsymbol{A}^\Delta_{n_0} \times \cdots \times \boldsymbol{A}^\Delta_{n_{l-1}}$ is embeddable into $\prod_{j\in J} \boldsymbol{A}^\Delta_{m_j}$ where the set $\{m_j : j \in J\}$ is finite, then*

1. *For every $i < l$, there exists $j \in J$ such that $n_i \leq m_j$ and n_i odd $\Rightarrow m_j$ odd.*
2. *For every $j \in J$, there exists some $i < l$ such that $n_i \leq m_j$ and m_j even $\Rightarrow n_i$ even.*

Proof. 1. Given $\mathbf{A}^\Delta_{n_0} \times \cdots \times \mathbf{A}^\Delta_{n_{l-1}}$ embeddable into $\prod_{j\in J} \mathbf{A}^\Delta_{m_j}$, then

$$\mathbf{A}^\Delta_{n_0} \times \cdots \times \mathbf{A}^\Delta_{n_{l-1}} \in \mathbb{V}(\textstyle\prod_{j\in J} \mathbf{A}^\Delta_{m_j}) = \mathbb{V}(\{\mathbf{A}^\Delta_{m_j} : j \in J\}).$$

Hence, for every $i < l$, $\mathbf{A}^\Delta_{n_i} \in \mathbb{V}(\{\mathbf{A}^\Delta_{m_j} : j \in J\})$. Given

$$M(x, y, z) = (\Delta(x \to y \wedge y \to x) * y) \vee (\neg\Delta(x \to y \wedge y \to x) * z),$$

it is clear, by [7, Theorem 12.3] that $\mathbb{V}(\{\mathbf{A}^\Delta_{m_j} : j \in J\})$ is a congruence-distributive variety. Thus, from this and the fact that the set $\{m_j : j \in J\}$ is finite we can apply [7, Corollary 6.10] which states that the class of subdirectly irreducible algebras of $\mathbb{V}(\{\mathbf{A}^\Delta_{m_j} : j \in J\})$ will be in $\mathbb{HS}(\{\mathbf{A}^\Delta_{m_j} : j \in J\})$.

Moreover, by Proposition 3.7,

$$\mathbb{HS}(\{\mathbf{A}^\Delta_{m_j} : j \in J\}) = \mathbb{H}(\{\mathbf{A}^\Delta_{n} : \exists j \in J \text{ such that } n \leq m_j \text{ and } n \text{ odd} \Rightarrow m_j \text{ odd}\})$$

where a homomorphic image of a finite NM_Δ-chain will also be a finite NM_Δ-chain (whose cardinality is smaller or equal).

Furthermore, given $0 < k < n$, we can't have an exhaustive homomorphism $h : \mathbf{A}^\Delta_{2n} \twoheadrightarrow \mathbf{A}^\Delta_{2k+1}$. That's because, in that case, there would be some $a \in A^\Delta_{2n}$ such that $h(a) = h(\neg a)$ would be the negation fixpoint (denoted by 0 in these algebra). Then, if we consider $b = max\{a, \neg a\}$ and $c = min\{a, \neg a\}$, we have $b \to c = c$. Hence, $h(b \to c) = 0$ while $h(b) \to h(c) = 0 \to 0 = k$. We had taken $k \neq 0$, therefore, such homomorphism doesn't exist.
Moreover, for any $\mathbf{A}^\Delta_{2k}$, it is clear that this algebra cannot be a homomorphic image of any algebra that has a negation fixpoint.

Thus, $\mathbb{H}(\mathbf{A}^\Delta_n)$ is the set of $\mathbf{A}^\Delta_k$ such that $k \leq n$ and k odd $\Leftrightarrow n$ odd.

Therefore, the class of irreducible members of $\mathbb{V}(\{\mathbf{A}^\Delta_{m_j} : j \in J\})$ will be contained in $\mathbb{I}(\{\mathbf{A}^\Delta_n : \exists j \in J$ such that $n \leq m_j$ and n odd $\Rightarrow m_j$ odd$\})$.
By Proposition 1.49, we know NM_Δ-chains are subdirectly irreducible members of NM_Δ, hence, for every $i < l$, $\mathbf{A}^\Delta_{n_i}$ belongs to $\mathbb{V}(\{\mathbf{A}^\Delta_{m_j} : j \in J\})_{SI}$.

Then, we have proven that, for every $i < l$, there exists some $j \in J$ such that $n_i \leq m_j$ and n_i odd $\Rightarrow m_j$ odd.

2. Given some $\mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}$ embeddable into $\prod_{j\in J}\mathbf{A}^{\Delta}_{m_j}$, we can consider γ to be such an embedding and, for every $j \in J$, we can take the natural projection $\pi_j : \prod_{j\in J} \mathbf{A}^{\Delta}_{m_j} \twoheadrightarrow \mathbf{A}^{\Delta}_{m_j}$.
   Then, for every $j \in J$, $\gamma_j = \pi_j \circ \gamma : \mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}} \to \mathbf{A}^{\Delta}_{m_j}$ is an homomorphism and, by the Homomorphism Theorem [7, Theorem 6.12],

$$\mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}/\mathrm{Ker}(\gamma_j) \cong \gamma_j(\mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}) \subseteq \mathbf{A}^{\Delta}_{m_j}.$$

   From this, since in a discriminator variety the concepts of simple, subdirectly irreducible and directly indecomposable are equivalent ([23, page 240]), we obtain that $\mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}/\mathrm{Ker}(\gamma_j)$ is simple.
   By definition of simple algebra, this last statement is equivalent to $\mathrm{Ker}(\gamma_j)$ being a maximal congruence relation of $\mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}$.

   Now, we consider [8, Lemma 2.3] and remark that this result also holds for NM_{Δ}-algebras since the proof is general. Thus, from this, the maximality of $\mathrm{Ker}(\gamma_j)$ and the fact that the $\mathbf{A}^{\Delta}_{n_i}$'s are simple (that is, $\mathrm{Con}(\mathbf{A}^{\Delta}_{n_i})= \{{\mathbf{A}^{\Delta}_{n_i}}^2, \mathrm{Id}_{\mathbf{A}^{\Delta}_{n_i}}\}$) we obtain that there is some $i < l$ such that:

$$\mathrm{Ker}(\gamma_j)= {\mathbf{A}^{\Delta}_{n_0}}^2 \times \cdots \times {\mathbf{A}^{\Delta}_{n_{i-1}}}^2 \times \mathrm{Id}_{\mathbf{A}^{\Delta}_{n_i}} \times {\mathbf{A}^{\Delta}_{n_{i+1}}}^2 \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}.$$

   Therefore, for every $j \in J$, there exists $i < l$ such that:

$$\mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}/\mathrm{Ker}(\gamma_j) \cong \mathbf{A}^{\Delta}_{n_i} \subseteq \mathbf{A}^{\Delta}_{m_j}.$$

   Finally, by Proposition 3.7, we have proven what we wanted: for every $j \in J$, there exists $i < l$ such that $n_i \leq m_j$ and m_j even $\Rightarrow n_i$ even.

□

Now, we can give a characterization of all critical NM_{Δ}-algebras:

Theorem 3.23. *An NM_{Δ}-algebra $\boldsymbol{A}$ is critical if and only if $\boldsymbol{A}$ is isomorphic to a finite NM_{Δ}-algebra $\boldsymbol{A}^{\Delta}_{n_0} \times \cdots \times \boldsymbol{A}^{\Delta}_{n_{l-1}}$ satisfying the following conditions:*

1. *For every $i, j < l$, $i \neq j$ implies $n_i \neq n_j$.*
2. *If there exists some n_j with $j < l$ such that for some $i \neq j$ with $i < l$ it holds $n_i \leq n_j$ and n_i odd $\Rightarrow n_j$ odd, then n_j is unique. That is, for any $s, r < l$ such that $s \neq r$, $n_s \leq n_r$ and n_s odd $\Rightarrow n_r$ odd it must hold $n_r = n_j$.*

Proof. $\Leftarrow$) First, we assume that $\mathbf{A} = \mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}$ satisfies conditions 1. and 2. and we will show that $\mathbf{A}$ is critical. Before doing that, we will prove the following claim:

Claim: *Every proper subalgebra of $\boldsymbol{A}$ is embeddable into a subalgebra of $\boldsymbol{A}$ of the form $\boldsymbol{A}^{\Delta}_{d_0} \times \cdots \times \boldsymbol{A}^{\Delta}_{d_{l-1}}$* where $d_i \leq n_i$ and d_i odd $\Rightarrow n_i$ odd for every $i < l$ and there exists some $j < l$ such that $d_j \neq n_j$.

Proof. Let $\mathbf{B}$ be a proper subalgebra of $\mathbf{A}$. Since $\mathbf{A}$ is finite, so will be $\mathbf{B}$ and then, by Remark 3.19, we obtain that $\mathbf{B}$ is isomorphic to an NM_Δ-algebra $\mathbf{A}^\Delta_{r_0} \times \cdots \times \mathbf{A}^\Delta_{r_{k-1}}$.

Now, for every $i < l$, we can consider the natural projection

$$\pi_i : \mathbf{A} = \mathbf{A}^\Delta_{n_0} \times \cdots \times \mathbf{A}^\Delta_{n_{l-1}} \twoheadrightarrow \mathbf{A}^\Delta_{n_i}$$

and we can take γ_i to be $\pi_i \upharpoonright \mathbf{B}$. Then, $\mathbf{B}$ will be embeddable into $\gamma_0(\mathbf{B}) \times \cdots \times \gamma_{l-1}(\mathbf{B})$ (not necessarily exhaustively).
Moreover, since for every $i < l$ the inclusion $\gamma_i(\mathbf{B}) \subseteq \mathbf{A}^\Delta_{n_i}$ holds, by Proposition 3.7, we have $\gamma_i(\mathbf{B}) = \mathbf{A}^\Delta_{d_i}$ for some $d_i \leq n_i$ such that d_i odd $\Rightarrow n_i$ odd.

Thus, if there is some $j < l$ such that $d_j \neq n_j$ we have already seen what we wanted. Therefore, let's assume the opposite: for every $i < l$, $\gamma_i(\mathbf{B}) = \mathbf{A}^\Delta_{n_i}$.

Then, by the Homomorphism Theorem [7, Theorem 6.12], for all $i < l$, $\mathbf{B}/\mathrm{Ker}(\gamma_i) \cong \mathbf{A}^\Delta_{n_i}$ and, since $\mathbf{A}^\Delta_{n_i}$ is simple then, by definition, $\mathrm{Ker}(\gamma_i)$ is a maximal congruence relation of $\mathbf{B}$.
Thus, by [8, Lemma 2.3] (which we have previously mentioned that also holds for NM_Δ-algebras), by the fact that we had seen $\mathbf{B} \cong \mathbf{A}^\Delta_{r_0} \times \cdots \times \mathbf{A}^\Delta_{r_{k-1}}$ for simple $\mathbf{A}^\Delta_{r_i}$'s and by the maximality of $\mathrm{Ker}(\gamma_i)$ we obtain that there is some $j < k$ such that

$$\mathrm{Ker}(\gamma_i)= {\mathbf{A}^\Delta_{r_0}}^2 \times \cdots \times {\mathbf{A}^\Delta_{r_{j-1}}}^2 \times \mathrm{Id}_{\mathbf{A}^\Delta_{r_j}} \times {\mathbf{A}^\Delta_{r_{j+1}}}^2 \times \cdots \times \mathbf{A}^\Delta_{r_{k-1}}.$$

Hence, for every $i < l$, there exists some $j < r$ such that $\mathbf{B}/\mathrm{Ker}(\gamma_i) \cong \mathbf{A}^\Delta_{r_j} = \mathbf{A}^\Delta_{n_i}$.

By condition 1. we know that $i_1 \neq i_2$ implies $n_{i_1} \neq n_{i_2}$. Therefore, $l \leq k$, because otherwise there would exist some $i_1, i_2 < l$ and some $j < r$ such that $i_1 \neq i_2$ and $\mathbf{A}^\Delta_{n_{i_1}} = \mathbf{A}^\Delta_{r_j} = \mathbf{A}^\Delta_{n_{i_2}}$, contradicting our assumption.

Moreover, for each $i_1, i_2 < l$ such that $i_1 \neq i_2$ the corresponging $j_1, j_2 < r$ must also be different. Otherwise, $\mathbf{A}^\Delta_{n_{i_1}} = \mathbf{A}^\Delta_{r_{j_1}} = \mathbf{A}^\Delta_{r_{j_2}} = \mathbf{A}^\Delta_{n_{i_2}}$ holds, which again contradicts condition 1.

Then, since $l \leq k$ and we obtain

$$\mathbf{B} \cong \mathbf{A}^\Delta_{r_0} \times \cdots \times \mathbf{A}^\Delta_{r_{k-1}} = \mathbf{A}^\Delta_{n_0} \times \cdots \times \mathbf{A}^\Delta_{n_{l-1}} \times \mathbf{A}^\Delta_{m_l} \times \cdots \times \mathbf{A}^\Delta_{m_{r-1}} =$$
$$\mathbf{A} \times \mathbf{A}^\Delta_{m_l} \times \cdots \times \mathbf{A}^\Delta_{m_{r-1}}$$

where $m_l, \ldots, m_{r-1} \in \{r_0, \ldots, r_{k-1}\}$. But this implies $|\mathbf{A}| \leq |\mathbf{B}|$, which contradicts the fact that $\mathbf{B}$ is a proper subalgebra of $\mathbf{A}$.

Thus, it never occurs the case where there is no $j < l$ such that $d_j \neq n_j$ and the claim has been proven. ☐

Now, we continue with the proof of the theorem. Given the algebra $\mathbf{A} = \mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}$ satisfying conditions 1. and 2., we assume $\mathbf{A} \in \mathbb{Q}(\{\mathbf{B} \subsetneq \mathbf{A}\})$ with the goal of arriving to a contradiction.
In that case, it will be clear that $\mathbf{A}$ is a critical algebra since these are finite algebras not belonging to the quasivariety generated by all its proper subalgebras and we already know $\mathbf{A}$ is finite.

Since $\mathbf{A} \in \mathbb{Q}(\{\mathbf{B} \subsetneq \mathbf{A}\})$, then, by the claim,

$$\mathbf{A} \in \mathbb{ISPP}_U(\{\mathbf{A}^{\Delta}_{d_0} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1}} \colon \forall i\ d_i \leq n_i,\ d_i \text{ odd} \Rightarrow n_i \text{ odd};\ \exists j\ d_j \neq n_j\}).$$

Moreover, given that $\{\mathbf{A}^{\Delta}_{d_0} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1}} \colon \forall i\ d_i \leq n_i,\ d_i \text{ odd} \Rightarrow n_i \text{ odd};\ \exists j\ d_j \neq n_j\}$ is a finite set of finite NM_{Δ}-algebras, by [7, Lemma 6.5], we have: $\mathbf{A} \in \mathbb{ISP}(\{\mathbf{A}^{\Delta}_{d_0} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1}} \colon \forall i\ d_i \leq n_i,\ d_i \text{ odd} \Rightarrow n_i \text{ odd};\ \exists j\ d_j \neq n_j\})$.

Thus, $\mathbf{A} = \mathbf{A}^{\Delta}_{n_0} \times \cdots \times \mathbf{A}^{\Delta}_{n_{l-1}}$ is embeddable into $\prod_{k<m}(\mathbf{A}^{\Delta}_{d_{0,k}} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1,k}})^{\alpha_k}$ where

$$\{\mathbf{A}^{\Delta}_{d_{0,k}} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1,k}} \colon k < m\} \subseteq \{\mathbf{A}^{\Delta}_{d_0} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1}} \colon \forall i\ d_i \leq n_i,\ d_i \text{ odd} \Rightarrow n_i \text{ odd};\ \exists j\ d_j \neq n_j\}.$$

Since the set $\{d_{t,k} : t < l, k < m\}$ is finite, we can apply Lemma 3.22. First, we consider two possible cases:

- There exist $i, j < l$ such that $i \neq j$, $n_i \leq n_j$ and n_i odd $\Rightarrow n_j$ odd:
  Then, by the fact that $\mathbf{A}$ satisfies condition 2., n_j must be unique. By Lemma 3.22 1., there exists some $\mathbf{A}^{\Delta}_{d_{t,k}}$ such that $n_j \leq d_{t,k}$ and n_j odd $\Rightarrow d_{t,k}$ odd. In other words, there exists some

  $$\mathbf{A}^{\Delta}_{d_{0,k}} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1,k}} \in \{\mathbf{A}^{\Delta}_{d_0} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1}} \colon \forall i\ d_i \leq n_i,\ d_i \text{ odd} \Rightarrow n_i \text{ odd};\ \exists j\ d_j \neq n_j\}$$

  such that $n_j \leq d_{t,k}$ and n_j odd $\Rightarrow d_{t,k}$ odd for some $t < l$. Combining this with the fact that n_j is unique and, by definition, $d_{t,k} \leq n_t$ and $d_{t,k}$ odd $\Rightarrow n_t$ odd, we obtain $n_j = d_{t,k} = n_t$.
  Thus, by condition 1., $j = t$.

  Since we know

  $$\mathbf{A}^{\Delta}_{d_{0,k}} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1,k}} \in \{\mathbf{A}^{\Delta}_{d_0} \times \cdots \times \mathbf{A}^{\Delta}_{d_{l-1}} \colon \forall i\ d_i \leq n_i,\ d_i \text{ odd} \Rightarrow n_i \text{ odd};\ \exists j\ d_j \neq n_j\},$$

  there exists some $r < l$ such that $d_{r,k} \leq n_r$, $d_{r,k}$ odd $\Rightarrow n_r$ odd and $d_{r,k} \neq n_r$ (hence, it must hold $r \neq j$).
  Now, by Lemma 3.22 2., there exists some $s < l$ such that $n_s \leq d_{r,k}$ and n_s odd $\Rightarrow d_{r,k}$ odd. Then, combining this with the properties given by how we have defined r, we obtain $\mathbf{n_s \leq n_r}$ **and** $\mathbf{n_s}$ **odd** $\Rightarrow \mathbf{n_r}$ **odd** (with $\mathbf{s \neq r}$, since n_s is strictly less than n_r).
  Furthermore, in this case, we had considered the existence of some $\mathbf{i \neq j}$ such that $\mathbf{n_i \leq n_j}$ **and** $\mathbf{n_i}$ **odd** $\Rightarrow \mathbf{n_j}$ **odd**.
  We have seen $\mathbf{r \neq j}$ and this leads us to a contradiction with respect to condition 2.

- For all $i, j < l$ such that $i \neq j$ either $n_i > n_j$ or n_i odd $\not\Rightarrow n_j$ odd:
  Then, the argument of the previous case follows analogously by taking any n_j with $j < l$.

Therefore, we have arrived to a contradiction in both cases, as we wanted. Then, $\mathbf{A}$ is finite and $\mathbf{A} \notin \mathbb{Q}(\{\mathbf{B} \subsetneq \mathbf{A}\})$ which means that $\mathbf{A}$ is critical.

$\Rightarrow$) We proceed to prove the converse, let $\mathbf{A}$ be a critical NM_Δ-algebra. Then, by definition, $\mathbf{A}$ is finite and, by Theorem 3.18, we can suppose, without loss of generality, that

$$\mathbf{A} = \mathbf{A}_{n_0}^{\Delta\ m_0} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta\ \ m_{r-1}}$$

for some $n_0, \ldots, n_{r-1}, m_0, \ldots, m_{r-1} \in \omega$ and $n_i \neq n_j$ when $i \neq j$. We assume not all m_i's are equal to 1 in order to arrive to a contradiction.

Let $m = max\{m_0, \ldots, m_{r-1}\}$, then we consider the map

$$\beta : \mathbf{A}_{n_0}^{\Delta\ m_0} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta\ \ m_{r-1}} \to \mathbf{A}_{n_0}^{\Delta\ m} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta\ \ m}$$

such that, for every $k < r$,

$$\beta(b(0), \ldots, b(r-1))\ (k) = (b(k)(1), \ldots, b(k)(m_k), \overbrace{b(k)(1), \ldots, b(k)(1)}^{m-m_k}).$$

This function gives an embedding from $\mathbf{A}$ into $\mathbf{A}_{n_0}^{\Delta\ m} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta\ \ m} \cong (\mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta})^m$. Thus, $\mathbf{A} \in \mathbb{Q}(\mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta})$.

On the other hand, the map

$$\alpha : \mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta} \longrightarrow \mathbf{B} \subsetneq \mathbf{A}_{n_0}^{\Delta\ m_0} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta\ \ m_{r-1}}$$
$$(a(0), \ldots, a(r-1)) \longmapsto \alpha(a) = \Big(\overbrace{a(0), \ldots, a(0)}^{m_0}, \ldots, \overbrace{a(r-1), \ldots, a(r-1)}^{m_{r-1}} \Big)$$

defines an isomorphism from $\mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta}$ into a proper subalgebra of $\mathbf{A}$ (because there is some m_i different than 1).
This leads us to a contradiction since, then, $\mathbf{A} \in \mathbb{Q}(\mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta})$ implies $\mathbf{A}$ is not critical.
Thus, $m_0 = \cdots = m_{r-1} = 1$ and $\mathbf{A} = \mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta}$ with $n_i \neq n_j$ if $i \neq j$, that is, satisfying condition 1.

Now, we just have left to see condition 2. holds. We will assume it fails and, once more, arrive to a contradiction.
We suppose there exist $i \neq j$ and $k \neq s$ such that $n_i \leq n_j$, $n_k \leq n_s$, n_i odd $\Rightarrow n_j$ odd, n_k odd $\Rightarrow n_s$ odd and $j \neq s$.

Since $i \neq j$, $n_i \leq n_j$ and n_i odd $\Rightarrow n_j$ odd, by Proposition 3.7, we have that the map

$$\mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{j-1}}^{\Delta} \times \mathbf{A}_{n_{j+1}}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta} \longrightarrow \mathbf{B} \subsetneq \mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta}$$
$$(a(0), \ldots, a(j-1), a(j+1), \ldots, a(r-1)) \longmapsto (a(0), \ldots, a(j-1), a(i), a(j+1), \ldots, a(r-1))$$

is a isomorphism between $\mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{j-1}}^{\Delta} \times \mathbf{A}_{n_{j+1}}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta}$ and some proper subalgebra of $\mathbf{A}$.

Analogously, $\mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{s-1}}^{\Delta} \times \mathbf{A}_{n_{s+1}}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta}$ is also isomorphic to a proper subalgebra of $\mathbf{A}$.

Finally, we observe that we have an embedding:

$$\delta : \mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta} \hookrightarrow {\mathbf{A}_{n_0}^{\Delta}}^2 \times \cdots \times {\mathbf{A}_{n_{j-1}}^{\Delta}}^2 \times \mathbf{A}_{n_j}^{\Delta} \times {\mathbf{A}_{n_{j+1}}^{\Delta}}^2 \times \cdots \times {\mathbf{A}_{n_{s-1}}^{\Delta}}^2 \times \mathbf{A}_{n_s}^{\Delta} \times {\mathbf{A}_{n_{s+1}}^{\Delta}}^2 \times \cdots \times {\mathbf{A}_{n_{r-1}}^{\Delta}}^2$$

$$(a(0), \ldots, a(r-1)) \longmapsto \delta(a(0), \ldots, a(r-1))\ (i) = \begin{cases} (a(i), a(i)) & \text{if } i \neq j, i \neq r, \\ a(i) & \text{otherwise .} \end{cases}$$

Therefore, $\mathbf{A} \in \mathbb{Q}(\mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{j-1}}^{\Delta} \times \mathbf{A}_{n_{j+1}}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta},\ \mathbf{A}_{n_0}^{\Delta} \times \cdots \times \mathbf{A}_{n_{s-1}}^{\Delta} \times \mathbf{A}_{n_{s+1}}^{\Delta} \times \cdots \times \mathbf{A}_{n_{r-1}}^{\Delta})$ which implies that $\mathbf{A}$ belongs to the quasivariety generated by all its proper subalgebras, contradicting the fact that $\mathbf{A}$ is critical.
Thus, it is clear that condition 2. also holds and we have showed what we wanted to prove. □

Corollary 3.24. *Every critical NM_{Δ}-algebra is of one of the following types:*

1. $\boldsymbol{A}_n^{\Delta}$ *for some* $n > 0$
2. $\boldsymbol{A}_{2n}^{\Delta} \times \boldsymbol{A}_{2m}^{\Delta}$ *for some* $0 < n < m$
3. $\boldsymbol{A}_{2n}^{\Delta} \times \boldsymbol{A}_{2m+1}^{\Delta}$ *for some* $n, m > 0$
4. $\boldsymbol{A}_{2n+1}^{\Delta} \times \boldsymbol{A}_{2m+1}^{\Delta}$ *for some* $0 < n < m$
5. $\boldsymbol{A}_{2k+1}^{\Delta} \times \boldsymbol{A}_{2n}^{\Delta} \times \boldsymbol{A}_{2m}^{\Delta}$ *for some* $0 < k < n < m$
6. $\boldsymbol{A}_{2k+1}^{\Delta} \times \boldsymbol{A}_{2n}^{\Delta} \times \boldsymbol{A}_{2m+1}^{\Delta}$ *for some* $0 < k < n, m$

Moreover, we can try to give an axiomatization of quasivarieties generated just by one critical NM_{Δ}-algebra. In order to do so, we present the following formula and give some results about it:

Definition 3.25. *For every $n > 0$ we consider the formula*

$$SL_n(x_1, \ldots, x_n) := \Delta\big(\bigvee_{i<n} (x_i \to x_{i+1})\big)$$

Theorem 3.26. *A direct product of finite NM_{Δ}-chains $\boldsymbol{A} = \prod_{i=1}^{m} \boldsymbol{A}_{k_i}^{\Delta}$ satisfies:*

1. $\neg SL_n(x_1, \ldots, x_n) \approx \overline{1} \Rightarrow y \approx \overline{1}$ *iff there exists some* $\boldsymbol{A}_{k_i}^{\Delta}$ *with* $k_i < n$.
2. $SL_n(x_1, \ldots, x_n) \approx \overline{1}$ *iff for every* $\boldsymbol{A}_{k_i}^{\Delta}$ *holds* $k_i < n$*, that is, every* $\boldsymbol{A}_{k_i}^{\Delta}$ *has strictly less than n elements.*
3. $\Delta(x \leftrightarrow \neg x) \vee \neg SL_n(x_1, \ldots, x_n) \approx \overline{1} \Rightarrow y \approx \overline{1}$ *iff there exists some* $\boldsymbol{A}_{k_i}^{\Delta}$ *with* k_i *even such that* $k_i < n$.

4. $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_n(x_1, \ldots, x_n) \approx \overline{1}$ *iff every* $\boldsymbol{A}_{k_i}^{\Delta}$ *satisfies that* k_i *is even and* $k_i < n$.

5. $\Delta(x \leftrightarrow \neg x) \approx \overline{1} \Rightarrow y \approx \overline{1}$ *iff there exists some* $\boldsymbol{A}_{k_i}^{\Delta}$ *with* k_i *even.*

Proof. 1. $\Rightarrow$) We will assume, by contraposition, that every $i \leq m$ satisfies $k_i \geq n$. Then, our goal is to prove that $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$ doesn't satisfy $\neg SL_n(x_1, \ldots, x_n) \approx \overline{1} \Rightarrow y \approx \overline{1}$. That is, we want to find some $\overline{a}_1, \ldots, \overline{a}_n \in \mathbf{A}$ such that $\neg SL_n(\overline{a}_1, \ldots, \overline{a}_n) \approx \overline{1}$.

Since for every $i \leq m$, $k_i \geq n$ holds, we can choose $\overline{a}_1, \ldots, \overline{a}_n$ such that, for any $j < n$, $a_j(i) > a_{j+1}(i)$. Hence, it is clear that $\neg SL_n(\overline{a}_1, \ldots, \overline{a}_n) \approx \overline{1}$ which is what we wanted to see.

$\Leftarrow$) Given some algebra $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$ such that there exists some $\mathbf{A}_{k_i}^{\Delta}$ satisfying $k_i < n$, we will prove that this $\mathbf{A}_{k_i}^{\Delta}$ doesn't satisfy $\neg SL_n(x_1, \ldots, x_n) \approx \overline{1}$ for any evaluation, therefore, neither does $\mathbf{A}$. Hence, it always holds the quasiequation.

For any $a_1, \ldots, a_n \in \mathbf{A}_{k_i}^{\Delta}$, since $k_i < n$, there exists some $j < n$ such that $a_j \leq a_{j+1}$. Thus, $a_j \rightarrow a_{j+1} = \overline{1}$ and $\neg\Delta\big(\bigvee_{j<n}(a_j \rightarrow a_{j+1})\big) = \overline{0}$, which is what we wanted to show.

2. $\Rightarrow$) Given some algebra $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$ it is clear that it satisfies $SL_n(x_1, \ldots, x_n) \approx \overline{1}$ if and only if every $\mathbf{A}_{k_i}^{\Delta}$ does.

We will assume, by contraposition, there exists some $i \leq m$ such that $k_i \geq n$ and, then, we will prove that $\mathbf{A}_{k_i}^{\Delta}$ doesn't satisfy $SL_n(x_1, \ldots, x_n) \approx \overline{1}$, therefore, neither does $\mathbf{A}$.

Since $\mathbf{A}_{k_i}^{\Delta}$ has at least n elements, there exist some $a_1, \cdots, a_n \in A_{k_i}^{\Delta}$ such that $a_j > a_{j+1}$ for every $j < n$. Then, for all j, $a_j \rightarrow a_{j+1} = \neg a_j \vee a_{j+1} < \overline{1}$ since $a_{j+1} \neq \overline{1}$ and $a_j \neq \overline{0}$. Thus, $\Delta\big(\bigvee_{j<n}(a_j \rightarrow a_{j+1})\big) = \overline{0}$.

Therefore, it is clear that $SL_n(x_1, \ldots, x_n) \approx \overline{1}$ is not satisfied.

$\Leftarrow$) Given some algebra $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$ such that, for every $\mathbf{A}_{k_i}^{\Delta}$, $k_i < n$ holds, we will prove that, for all $i \leq m$, $\mathbf{A}_{k_i}^{\Delta}$ satisfies $SL_n(x_1, \ldots, x_n) \approx \overline{1}$, therefore, so does $\mathbf{A}$.

Given any $i \leq m$, we take some arbitrary $a_1, \ldots, a_n \in A_{k_i}^{\Delta}$. Since we are assuming $|\mathbf{A}_{k_i}^{\Delta}| < n$, there exists some $j < n$ such that $a_j \leq a_{j+1}$ then, $a_j \rightarrow a_{j+1} = \overline{1}$. Thus, $SL_n(x_1, \ldots, x_n) \approx \overline{1}$ is satisfied.

3. $\Rightarrow$) We will assume, by contraposition, that every $i \leq m$ satisfies that k_i is odd or $k_i \geq n$. Then, our goal is to prove that $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$ doesn't satisfy $\Delta(x \leftrightarrow \neg x) \vee \neg SL_n(x_1, \ldots, x_n) \approx \overline{1} \Rightarrow y \approx \overline{1}$. That is, we want to find some $\overline{a}, \overline{a}_1, \ldots, \overline{a}_n \in \mathbf{A}$ such that $\Delta(\overline{a} \leftrightarrow \neg\overline{a}) \vee \neg SL_n(\overline{a}_1, \ldots, \overline{a}_n) \approx \overline{1}$.

We choose $\overline{a}, \overline{a}_1, \ldots, \overline{a}_n$ such that, for any $i \leq m$:

- If k_i is odd: we take $a(i)$ to be the negation fixpoint of $\mathbf{A}_{k_i}^{\Delta}$ and consider $a_1(i), \dots, a_n(i)$ to be any elements of the algebra.
Thus, $\Delta(a(i) \leftrightarrow \neg a(i)) = \overline{1}$.
- If $k_i \geq n$: we take $a_1(i), \dots, a_n(i)$ to be elements of $\mathbf{A}_{k_i}^{\Delta}$ such that, for any $j < n$, $a_j(i) > a_{j+1}(i)$ and consider $a(i)$ to be any element of the algebra.
Then, $\neg SL_n(a_1(i), \dots, a_n(i)) = \overline{1}$.

Hence, it is clear that $\Delta(\overline{a} \leftrightarrow \neg\overline{a}) \vee \neg SL_n(\overline{a}_1, \dots, \overline{a}_n) \approx \overline{1}$ which is what we wanted to see.

$\Leftarrow$) Let $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$ be an arbitrary algebra with some $\mathbf{A}_{k_i}^{\Delta}$ such that k_i is even and $k_i < n$, we will prove that this $\mathbf{A}_{k_i}^{\Delta}$ doesn't satisfy $\Delta(x \leftrightarrow \neg x) \vee \neg SL_n(x_1, \dots, x_n) \approx \overline{1}$ for any evaluation, therefore, neither does $\mathbf{A}$. Hence, it always holds the quasiequation.

For any $a, a_1, \dots, a_n \in \mathbf{A}_{k_i}^{\Delta}$, since k_i is even, a will never be a negation fixpoint, that is, $\Delta(a \leftrightarrow \neg a) = \overline{0}$.
Moreover, we also know $k_i < n$, therefore there exists some $j < n$ such that $a_j \leq a_{j+1}$. This means that $a_j \to a_{j+1} = \overline{1}$ and $\neg\Delta\big(\bigvee_{j<n}(a_j \to a_{j+1})\big) = \overline{0}$.

Then, we have seen $\Delta(a \leftrightarrow \neg a) \vee \neg SL_n(a_1, \dots, a_n) = \overline{0}$ for any $a, a_1, \dots, a_n \in \mathbf{A}_{k_i}^{\Delta}$, which is what we wanted to show.

4. $\Rightarrow$) Given some algebra $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$, it is clear that it satisfies $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_n(x_1, \dots, x_n) \approx \overline{1}$ if and only if every $\mathbf{A}_{k_i}^{\Delta}$ does.

   We will assume, by contraposition, that there exists some $i \leq m$ such that either $k_i \geq n$ or k_i is odd and we will prove that, then, $\mathbf{A}_{k_i}^{\Delta}$ doesn't satisfy $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_n(x_1, \dots, x_n) \approx \overline{1}$, therefore, neither does $\mathbf{A}$.

   - If $k_i \geq n$: then, by 2., $SL_n(x_1, \dots, x_n) \approx \overline{1}$ is not satisfied, hence, neither is $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_n(x_1, \dots, x_n) \approx \overline{1}$.
   - If k_i is odd: we can consider $a \in \mathbf{A}_{k_i}^{\Delta}$ to be the negation fixpoint. Then, $\neg\Delta(a \leftrightarrow \neg a) = \overline{0}$

   Therefore, it is clear that $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_n(x_1, \dots, x_n) \approx \overline{1}$ is not satisfied in $\mathbf{A}$.

   $\Leftarrow$) Given some algebra $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$ such that, for every $\mathbf{A}_{k_i}^{\Delta}$, k_i is even and $k_i < n$, we will prove that, for all $i \leq m$, $\mathbf{A}_{k_i}^{\Delta}$ satisfies $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_n(x_1, \dots, x_n) \approx \overline{1}$, therefore, so does $\mathbf{A}$.

   Given any $i \leq m$, we take $a, a_1, \dots, a_n \in A_{k_i}^{\Delta}$. Since k_i is even, a cannot be a negation fixpoint, thus, $\neg\Delta(a \leftrightarrow \neg a) = \overline{1}$. Moreover, by 2., it is clear that $SL_n(a_1, \dots, a_n) = \overline{1}$ holds. Therefore, $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_n(x_1, \dots, x_n) \approx \overline{1}$ is satisfied in $\mathbf{A}$.

5. $\Rightarrow$) We will assume, by contraposition, that k_i is odd for every $i \leq m$. Then, our goal is to prove that $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}_{k_i}^{\Delta}$ doesn't satisfy $\Delta(x \leftrightarrow \neg x) \approx \overline{1} \Rightarrow y \approx \overline{1}$. That is, we

want to find some $\overline{a} \in \mathbf{A}$ such that $\Delta(\overline{a} \leftrightarrow \neg\overline{a}) \approx \overline{1}$.

Since k_i is odd for every $i \leq m$, we can choose $a(i)$ to be the negation fixpoint of $\mathbf{A}^{\Delta}_{k_i}$. Hence, it is clear that $\Delta(\overline{a} \leftrightarrow \neg\overline{a}) \approx \overline{1}$ which is what we wanted to see.

$\Leftarrow$) Given some algebra $\mathbf{A} = \prod_{i=1}^{m} \mathbf{A}^{\Delta}_{k_i}$ such that there exists some $\mathbf{A}^{\Delta}_{k_i}$ satisfying k_i is even, it is clear that this $\mathbf{A}^{\Delta}_{k_i}$ doesn't satisfy $\Delta(x \leftrightarrow \neg x) \approx \overline{1}$ for any evaluation since there is no negation fixpoint. Therefore, neither does $\mathbf{A}$ and it always holds the quasiequation.

□

Remark 3.27. We can express the axiomatization of varieties of NM_{Δ}-algebras presented in Theorem 3.13 in terms of $\neg\Delta(x \leftrightarrow \neg x)$ and $SL_n(x_1, \ldots, x_n)$. That is because in NM_{Δ} $BP(x) \approx \overline{1}$ is satisfied if and only if $\neg\Delta(x \leftrightarrow \neg x) \approx \overline{1}$ is (see Definition 1.69) and, analogously, $S_n(x_0, \ldots, x_n) \approx \overline{1}$ holds if and only if $SL_{2(n+1)}(x_1, \ldots, x_{2(n+1)}) \approx \overline{1}$ also does.

In fact, when some chain without negation fixpoint is considered, we can substitute $S_n(x_0, \ldots, x_n) \approx \overline{1}$ by $SL_{2n+1}(x_1, \ldots, x_{2n+1}) \approx \overline{1}$ since it is clear that it will have an even number of elements.

Thus, we can now proceed to give an axiomatization of the quasivarieties generated by a single critical algebra:

Corollary 3.28. *Given Σ any set of equations axiomatizing NM_{Δ}:*

1. $\mathbb{Q}(\boldsymbol{A}^{\Delta}_{n}) = \mathbb{V}(\boldsymbol{A}^{\Delta}_{n})$*, by Theorem 3.13, is axiomatized by* Σ *plus the equation* $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_{n+1}(x_1 \ldots, x_{n+1}) \approx \overline{1}$ *if* n *is even and, otherwise, by* Σ *and* $SL_{n+1}(x_1, \ldots, x_{n+1}) \approx \overline{1}$.
2. $\mathbb{Q}(\boldsymbol{A}^{\Delta}_{2n} \times \boldsymbol{A}^{\Delta}_{2m})$ *with* $0 < n < m$ *is axiomatized by* Σ *plus:*
   - $\neg\Delta(x \leftrightarrow \neg x) \wedge SL_{2m+1}(x_1 \ldots, x_{2m+1}) \approx \overline{1}$*, which axiomatizes* $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2m})$*, and*
   - $\neg SL_{2n+1}(x_1, \ldots, x_{2n+1}) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
3. $\mathbb{Q}(\boldsymbol{A}^{\Delta}_{2n} \times \boldsymbol{A}^{\Delta}_{2m+1})$ *satisfying* $n \leq m$ *is axiomatized by* Σ *plus:*
   - $SL_{2(m+1)}(x_1, \ldots, x_{2(m+1)}) \approx \overline{1}$*, which axiomatizes* $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2m+1})$*, and*
   - $\Delta(x \leftrightarrow \neg x) \vee \neg SL_{2n+1}(x_1, \ldots, x_{2n+1}) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
4. $\mathbb{Q}(\boldsymbol{A}^{\Delta}_{2n} \times \boldsymbol{A}^{\Delta}_{2m+1})$ *satisfying* $n > m$ *is axiomatized by* Σ *plus:*
   - $\big(\neg\Delta(x \leftrightarrow \neg x) \wedge SL_{2n+1}(x_1, \ldots, x_{2n+1})\big) \vee SL_{2(m+1)}(x_1, \ldots, x_{2(m+1)}) \approx \overline{1}$*, which axiomatizes* $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2n}, \boldsymbol{A}^{\Delta}_{2m+1})$*,*
   - $\neg SL_{2(m+1)}(x_1, \ldots, x_{2(m+1)}) \approx \overline{1} \Rightarrow y \approx \overline{1}$ *and*
   - $\Delta(x \leftrightarrow \neg x) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
5. $\mathbb{Q}(\boldsymbol{A}^{\Delta}_{2n+1} \times \boldsymbol{A}^{\Delta}_{2m+1})$ *with* $0 < n < m$ *is axiomatized by* Σ *plus:*
   - $SL_{2(m+1)}(x_1, \ldots, x_{2(m+1)}) \approx \overline{1}$*, which axiomatizes* $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2m+1})$*, and*

- $\neg SL_{2(n+1)}(x_1, \ldots, x_{2(n+1)}) \approx \overline{1} \Rightarrow y \approx \overline{1}$.

6. $\mathbb{Q}(\boldsymbol{A}^{\Delta}_{2k+1} \times \boldsymbol{A}^{\Delta}_{2n} \times \boldsymbol{A}^{\Delta}_{2m})$ *with* $0 < k < n < m$ *is axiomatized by* Σ *plus:*
   - $\big(\neg\Delta(x \leftrightarrow \neg x) \wedge SL_{2m+1}(x_1, \ldots, x_{2m+1})\big) \vee SL_{2(k+1)}(x_1, \ldots, x_{2(k+1)}) \approx \overline{1}$, *which axiomatizes* $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2m}, \boldsymbol{A}^{\Delta}_{2k+1})$,
   - $\neg SL_{2(k+1)}(x_1, \ldots, x_{2(k+1)}) \approx \overline{1} \Rightarrow y \approx \overline{1}$ *and*
   - $\Delta(x \leftrightarrow \neg x) \vee \neg SL_{2n+1}(x_1, \ldots, x_{2n+1}) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
7. $\mathbb{Q}(\boldsymbol{A}^{\Delta}_{2k+1} \times \boldsymbol{A}^{\Delta}_{2n} \times \boldsymbol{A}^{\Delta}_{2m+1})$ *with* $0 < k < n \leq m$ *is axiomatized by* Σ *plus:*
   - $SL_{2(m+1)}(x_1, \ldots, x_{2(m+1)}) \approx \overline{1}$, *which axiomatizes* $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2m+1})$,
   - $\neg SL_{2(k+1)}(x_1, \ldots, x_{2(k+1)}) \approx \overline{1} \Rightarrow y \approx \overline{1}$ *and*
   - $\Delta(x \leftrightarrow \neg x) \vee \neg SL_{2n+1}(x_1, \ldots, x_{2n+1}) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
8. $\mathbb{Q}(\boldsymbol{A}^{\Delta}_{2k+1} \times \boldsymbol{A}^{\Delta}_{2n} \times \boldsymbol{A}^{\Delta}_{2m+1})$ *with* $0 < k < m < n$ *is axiomatized by* Σ *plus:*
   - $\big(\neg\Delta(x \leftrightarrow \neg x) \wedge SL_{2n+1}(x_1, \ldots, x_{2n+1})\big) \vee SL_{2(m+1)}(x_1, \ldots, x_{2(m+1)}) \approx \overline{1}$, *which axiomatizes* $\mathbb{V}(\boldsymbol{A}^{\Delta}_{2n}, \boldsymbol{A}^{\Delta}_{2m+1})$,
   - $\neg SL_{2(k+1)}(x_1, \ldots, x_{2(k+1)}) \approx \overline{1} \Rightarrow y \approx \overline{1}$ *and*
   - $\Delta(x \leftrightarrow \neg x) \approx \overline{1} \Rightarrow y \approx \overline{1}$.

Now that we have studied the class of critical NM_{Δ}-algebras, we give a result that allows us to further classify and distinguish quasivarieties of NM_{Δ}-algebras in terms of their generators:

Lemma 3.29. *Let* $\{\boldsymbol{A}^{\Delta}_{n_{i_1}} \times \cdots \times \boldsymbol{A}^{\Delta}_{n_{i_{l(i)}}} : i \in I\}$ *and* $\{\boldsymbol{A}^{\Delta}_{m_{j_1}} \times \cdots \times \boldsymbol{A}^{\Delta}_{m_{j_{l(j)}}} : j \in J\}$ *two finite families of critical* NM_{Δ}*-algebras, then*

$$\mathbb{Q}(\{\boldsymbol{A}^{\Delta}_{n_{i_1}} \times \cdots \times \boldsymbol{A}^{\Delta}_{n_{i_{l(i)}}} : i \in I\}) \subseteq \mathbb{Q}(\{\boldsymbol{A}^{\Delta}_{m_{j_1}} \times \cdots \times \boldsymbol{A}^{\Delta}_{m_{j_{l(j)}}} : j \in J\})$$

if and only if, for every $i \in I$*, there exists a non-empty subset* $H \subseteq J$ *such that:*

1. *For any* $1 \leq k \leq l(i)$ *there are* $j \in H$ *and* $1 \leq r \leq l(j)$ *such that* $n_{i_k} \leq m_{j_r}$ *and* n_{i_k} *odd* $\Rightarrow m_{j_r}$ *odd.*
2. *For any* $j \in H$ *and* $1 \leq r \leq l(j)$ *there exists some* $1 \leq k \leq l(i)$ *such that* $n_{i_k} \leq m_{j_r}$ *and* n_{i_k} *odd* $\Rightarrow m_{j_r}$ *odd.*

Proof. $\Rightarrow$) Assume $\mathbb{Q}(\{\mathbf{A}^{\Delta}_{n_{i_1}} \times \cdots \times \mathbf{A}^{\Delta}_{n_{i_{l(i)}}} : i \in I\}) \subseteq \mathbb{Q}(\{\mathbf{A}^{\Delta}_{m_{j_1}} \times \cdots \times \mathbf{A}^{\Delta}_{m_{j_{l(j)}}} : j \in J\})$ for some finite families of critical NM_{Δ}-algebras.
Then, by [7, Lemma 6.5], it is clear that, for every $i \in I$, there exists some $\varnothing \neq H \subseteq J$ such that $\mathbf{A}^{\Delta}_{n_{i_1}} \times \cdots \times \mathbf{A}^{\Delta}_{n_{i_{l(i)}}}$ is embeddable into $\prod_{j \in H}(\mathbf{A}^{\Delta}_{m_{j_1}} \times \cdots \times \mathbf{A}^{\Delta}_{m_{j_{l(j)}}})^{\alpha_j}$. Therefore, since the set $\bigcup_{j \in H}\{m_{j_r} : r \leq j(l)\}$ is finite, we can apply Lemma 3.22, and 1. and 2. follow from Lemma 3.22 conditions 1. and 2., respectively.

$\Leftarrow$) To prove the converse we show that, for every $i \in I$,

$$\mathbf{A}^{\Delta}_{n_{i_1}} \times \cdots \times \mathbf{A}^{\Delta}_{n_{i_{l(i)}}} \in \mathbb{ISP}(\{\mathbf{A}^{\Delta}_{m_{j_1}} \times \cdots \times \mathbf{A}^{\Delta}_{m_{j_{l(j)}}} : j \in H\})$$

where H is the subset of J defined in the hypothesis.

Given any $i \in I$, by condition 1., for every $1 \leq k \leq l(i)$ we can choose $j \in H$ (which we will name j_k) and $1 \leq r_k \leq l(j_k)$ such that $n_{i_k} \leq m_{j_{k r_k}}$ and n_{i_k} odd $\Rightarrow m_{j_{k r_k}}$ odd.

Then, the following map will be an embedding:

$$\beta : \mathbf{A}^{\Delta}_{n_{i_1}} \times \cdots \times \mathbf{A}^{\Delta}_{n_{i_{l(i)}}} \to \prod_{1 \leq k \leq l(i)} \mathbf{A}^{\Delta}_{m_{j_{k1}}} \times \cdots \times \mathbf{A}^{\Delta}_{m_{j_{k l(j_k)}}}$$

$$\overline{a} \longmapsto \beta(\overline{a})\ (k)(r) = \begin{cases} \overline{a}(k) & \text{if } r = r_k, \\ \overline{a}(l) & \text{otherwise .} \end{cases}$$

where l satisfies $1 \leq l \leq l(i)$ and is such that $n_{i_l} \leq m_{j_{kr}}$ and n_{i_l} odd $\Rightarrow m_{j_{kr}}$ odd. This l exists by condition 2.
Therefore, it has been showed what we needed in order to conclude the proof. □

With all the previous given results, now we have more information about the lattice of subquasivarieties $L_{\mathcal{Q}}(NM_{\Delta})$. In fact, by Remark 3.17, Theorem 3.21 and Corollary 3.24, we can determine all the quasivarieties of NM_{Δ}-algebras.

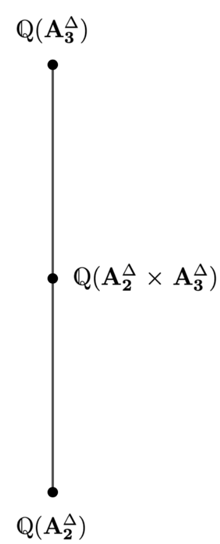
Even so, it will be difficult to fully determine the lattice $L_{\mathcal{Q}}(NM_{\Delta})$ due to its dimension, but we can study the lattices of quasivarieties contained in some varieties of NM_{Δ}-algebras:

Example 3.30. The lattice of all quasivarieties contained in $\mathbb{V}(\mathbf{A}^{\Delta}_3)$.

By the characterization given in Theorem 3.23, it is clear that the critical NM_{Δ}-algebras contained in $\mathbb{V}(\mathbf{A}^{\Delta}_3)$ are

$$\mathbb{I}(\{\mathbf{A}^{\Delta}_2,\ \mathbf{A}^{\Delta}_3,\ \mathbf{A}^{\Delta}_2 \times \mathbf{A}^{\Delta}_3\}).$$

Thus, by Lemma 3.29, the lattice of quasivarieties contained in $\mathbb{V}(\mathbf{A}^{\Delta}_3)$ is the following:



Example 3.31. The lattice of all quasivarieties contained in $\mathbb{V}(\mathbf{A}_4^\Delta)$.

The class of critical NM_Δ-algebras contained in the variety, by Theorem 3.23, is given by

$$\mathbb{I}(\{\mathbf{A}_2^\Delta,\ \mathbf{A}_4^\Delta,\ \mathbf{A}_2^\Delta\times\mathbf{A}_4^\Delta\}).$$

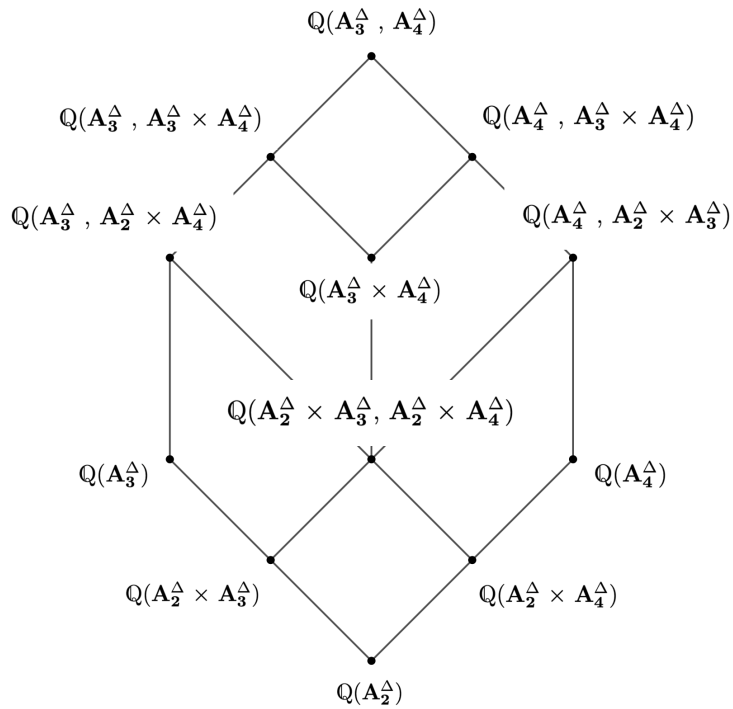
Therefore, the lattice of all the subquasivarieties will be similar to the one in the previous example.

Example 3.32. The lattice of all quasivarieties contained in $\mathbb{V}(\mathbf{A}_3^\Delta,\ \mathbf{A}_4^\Delta)$.

By Theorem 3.23, the critical algebras contained in the variety are the ones in

$$\mathbb{I}(\{\mathbf{A}_2^\Delta,\ \mathbf{A}_3^\Delta,\ \mathbf{A}_4^\Delta,\ \mathbf{A}_2^\Delta\times\mathbf{A}_3^\Delta,\ \mathbf{A}_2^\Delta\times\mathbf{A}_4^\Delta,\ \mathbf{A}_3^\Delta\times\mathbf{A}_4^\Delta\}).$$

Thus, by Theorem 3.21, we can determine all subquasivarieties and, by Lemma 3.29, we obtain the following lattice:

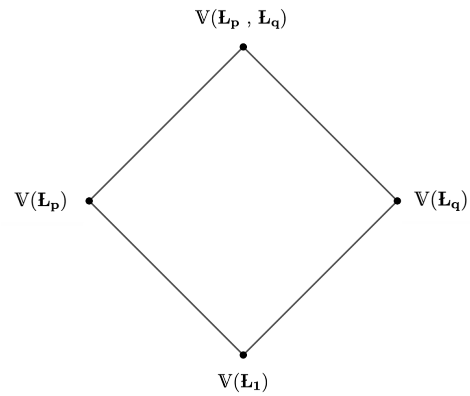


Where the presented quasivarieties that are generated by a single algebra have already been axiomatized in Corollary 3.28. Moreover, in some result introduced later on (see Proposition 3.35) there is also an axiomatization of $\mathbb{Q}(\mathbf{A}_2^\Delta\times\mathbf{A}_3^\Delta,\ \mathbf{A}_2^\Delta\times\mathbf{A}_4^\Delta)$ and the axiomatization of $\mathbb{Q}(\mathbf{A}_3^\Delta,\ \mathbf{A}_4^\Delta) = \mathbb{V}(\mathbf{A}_3^\Delta,\ \mathbf{A}_4^\Delta)$ is clear, by Theorem 3.13.

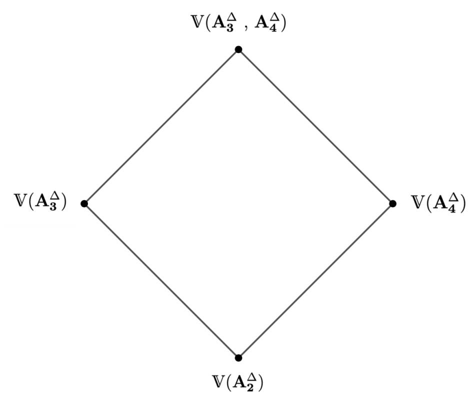
Furthermore, we could give an axiomatization of some of the remaining quasivarieties of the lattice.
We know all of them are contained in $\mathbb{V}(\mathbf{A}_3^\Delta,\ \mathbf{A}_4^\Delta)$ thus, let Σ be a set of constituted by $\big(\neg\Delta(x \leftrightarrow \neg x) \wedge SL_5(x_1, \dots, x_5)\big) \vee SL_4(x_1, \dots, x_4) \approx \overline{1}$ plus any set of equations axiomatizing NM_Δ, then:

- $\mathbb{Q}(\mathbf{A}_3^\Delta, \mathbf{A}_3^\Delta \times \mathbf{A}_4^\Delta)$ is axiomatized by Σ plus $\neg SL_4(x_1, \ldots, x_4) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
- $\mathbb{Q}(\mathbf{A}_4^\Delta, \mathbf{A}_3^\Delta \times \mathbf{A}_4^\Delta)$ is axiomatized by Σ plus $\Delta(x \leftrightarrow \neg x) \approx \overline{1} \Rightarrow y \approx \overline{1}$.

Remark 3.33. In fact, we could have obtained the previous lattice by applying Theorem 3.5 e). That is because $\mathbb{V}(\text{Ł}_p, \text{Ł}_q)$ is also a locally finite discriminator variety (see [17, Lemma 2.6]) whose lattice of subvarieties is the following:



Since the lattice of subvarieties of $\mathbb{V}(\mathbf{A}_3^\Delta, \mathbf{A}_4^\Delta)$ is analogous:



then, by [17, Figure 1] and Theorem 3.5 we would have obtained, in a different way, the same lattice of subquasivarieties as the one presented in the previous example.

Remark 3.34. From Theorem 3.5 we can also deduce that, the sublattice of $L_{\mathcal{Q}}(NM_\Delta)$ that consists of the quasivarieties generated by chains $\mathbf{A}_{k_i}^\Delta$ with k_i even, will be distributive and modular. This is obtained from section b) applying 3. to $L_{\mathcal{V}}(NM_\Delta^-)$ where NM_Δ^- denotes $\mathbb{V}(\mathbf{[0,1]}_\Delta^-)$.

3.2 Structural completeness in $\mathbf{NML}_\Delta$

Once we have studied both the lattices of subvarieties and subquasivarieties, we proceed to study some structural completeness properties about the logic $\mathbf{NML}_\Delta$

First, we can identify which are the least quasivarieties of $L_{\mathcal{Q}}(NM_\Delta)$:

Proposition 3.35. *For any quasivariety* $\mathbb{K}$ *of* NM_Δ*-algebras that generates* $\mathbb{V}(\boldsymbol{A}^\Delta_{2n},\ \boldsymbol{A}^\Delta_{2m+1})$ *for some* $n > 0$, $n, m \in \omega + 1$, *we have:*

$$\mathbb{Q}(\boldsymbol{A}^\Delta_2 \times \boldsymbol{A}^\Delta_{2n},\ \boldsymbol{A}^\Delta_2 \times \boldsymbol{A}^\Delta_{2m+1}) \subseteq \mathbb{K}.$$

In fact, given Σ *to be any set of equations axiomatizing* NM_Δ*, the quasivariety* $\mathbb{Q}(\boldsymbol{A}^\Delta_2 \times \boldsymbol{A}^\Delta_{2n},$ $\boldsymbol{A}^\Delta_2 \times \boldsymbol{A}^\Delta_{2m+1})$ *is axiomatized by:*

- *If* $m < n$, *with* $n \in \omega$: Σ *plus the equation* $\big(\neg\Delta(x \leftrightarrow \neg x) \wedge SL_{2n+1}(x_1, \dots, x_{2n+1})\big) \vee SL_{2(m+1)}(x_1, \dots, x_{2(m+1)}) \approx \overline{1}$, *which axiomatizes* $\mathbb{V}(\boldsymbol{A}^\Delta_{2n},\ \boldsymbol{A}^\Delta_{2m+1})$, *and also* $\neg SL_3(x_1, \dots, x_3) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
- *If* $m < n$, *with* $n = \omega$: Σ *plus* $\neg\Delta(x \leftrightarrow \neg x) \vee SL_{2(m+1)}(x_1, \dots, x_{2(m+1)}) \approx \overline{1}$, *which axiomatizes* $\mathbb{V}(\boldsymbol{[0,1]}^-_\Delta,\ \boldsymbol{A}^\Delta_{2m+1})$, *and* $\neg SL_3(x_1, \dots, x_3) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
- *If* $n \leq m$, *such that* $m \in \omega$: Σ *plus* $SL_{2(m+1)}(x_1, \dots, x_{2(m+1)}) \approx \overline{1}$, *which axiomatizes* $\mathbb{V}(\ \boldsymbol{A}^\Delta_{2m+1})$, *and* $\neg SL_3(x_1, \dots, x_3) \approx \overline{1} \Rightarrow y \approx \overline{1}$.
- *If* $n \leq m$, *with* $m = \omega$: Σ *plus* $\neg SL_3(x_1, \dots, x_3) \approx \overline{1} \Rightarrow y \approx \overline{1}$.

Proof. Let $\mathbb{K}$ be a quasivariety that generates $\mathbb{V}(\mathbf{A}^\Delta_{2n},\ \mathbf{A}^\Delta_{2m+1})$ for some $n, m \in \omega + 1$, $n > 0$. By Theorem 3.21, $\mathbb{K}$ will be generated by its critical algebras, that is, $\mathbb{K} = \mathbb{Q}(\{\mathbf{C}_i : i \in I\})$. We proceed to show that there exist critical algebras $\mathbf{C}_i$ and $\mathbf{C}_j$, for some $i, j \in I$, such that $\mathbf{A}^\Delta_{2n}$ and $\mathbf{A}^\Delta_{2m+1}$ are, respectively, one of its components:

- If $n, m \in \omega$: We know $\mathbb{V}(\mathbb{K}) = \mathbb{V}(\mathbf{A}^\Delta_{2n}, \mathbf{A}^\Delta_{2m+1})$ and, in our way to a contradiction, we can assume that there is no critical algebra in $\mathbb{K}$ such that $\mathbf{A}^\Delta_{2n}$ is one of its components. Then, it is clear that every component without negation point of any critical algebra of $\mathbb{K}$ has to have less than $2n$ elements. Let

  $$k = max\{r \in \omega : \exists i \in I \text{ such that } \mathbf{A}^\Delta_{2r} \text{ is a component of } \mathbf{C}_i\}.$$

  We would have that $\mathbb{V}(\mathbb{K}) \subseteq \mathbb{V}(\mathbf{A}^\Delta_{2k}, \mathbf{A}^\Delta_{2m+1}) \subsetneq \mathbb{V}(\mathbf{A}^\Delta_{2n}, \mathbf{A}^\Delta_{2m+1})$, contradicting our assumption.

  By arguing similarly, we obtain that there also exists some critical algebra $\mathbf{C}_j$ such that $\mathbf{A}^\Delta_{2m+1}$ is one of its components.

- If $n = \omega$, $m \in \omega$: We know $\mathbb{V}(\mathbb{K}) = \mathbb{V}(\mathbf{[0,1]}^-_\Delta, \mathbf{A}^\Delta_{2m+1})$ and we can prove, analogously to the previous case, that there must exist some critical algebra $\mathbf{C}_j$ in $\mathbb{K}$ such that $\mathbf{A}^\Delta_{2m+1}$ is one of its components.

  On the other hand, we can assume there exists some $k \in \omega$ satisfying that

  $$k = max\{r \in \omega : \exists i \in I \text{ such that } \mathbf{A}^\Delta_{2r} \text{ is a component of } \mathbf{C}_i\}.$$

  Then, we would obtain that $\mathbb{V}(\mathbb{K}) \subseteq \mathbb{V}(\mathbf{A}^\Delta_{2k}, \mathbf{A}^\Delta_{2m+1})$ which would be a contradiction. Thus, there is no upper bound on the cardinal of the chains without negation fixpoint satisfying that they are a component of one of the critical algebras of $\mathbb{K}$. Hence, this is also valid for $\mathbf{[0,1]}^-_\Delta$ (which, in this case, is our algebra $\mathbf{A}^\Delta_{2n}$).

- If $m = \omega$: Arguing analogously we obtain the same as in the previous cases.

Thus, there must exist some $\mathbf{C}_i$ satisfying that $\mathbf{A}_{2n}^{\Delta}$ is one of its components and $\mathbf{A}_{2m+1}^{\Delta}$ will also satisfy the same with respect to some $\mathbf{C}_j$.
Therefore, $\mathbf{A}_2^{\Delta} \times \mathbf{A}_{2n}^{\Delta} \in \mathbb{IS}(\mathbf{C}_i) \subseteq \mathbb{Q}(\mathbf{C}_i) \subseteq \mathbb{K}$ and $\mathbf{A}_2^{\Delta} \times \mathbf{A}_{2m+1}^{\Delta} \in \mathbb{IS}(\mathbf{C}_j) \subseteq \mathbb{Q}(\mathbf{C}_j) \subseteq \mathbb{K}$ which implies that the inclusion we wanted to show holds. □

Theorem 3.36. *The logic* $\boldsymbol{NML_{\Delta}}$ *is not structurally complete.*

Proof. We know the logic $\mathbf{NML}_{\Delta}$ is algebraized by the class of NM_{Δ}-algebras. Moreover, by Theorem 3.13, every variety $\mathbb{K}$ of NM$_{\Delta}$-algebras is of the form $\mathbb{V}(\mathbf{A}_{2n}^{\Delta}, \mathbf{A}_{2m+1}^{\Delta})$ for some $n, m \in \omega + 1$, $n > 0$.

By definition of structural completion, $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$ will generate $\mathbb{K}$ as a variety, thus, by Proposition 3.35, $\mathbb{Q}(\mathbf{A}_2^{\Delta} \times \mathbf{A}_{2n}^{\Delta}, \mathbf{A}_2^{\Delta} \times \mathbf{A}_{2m+1}^{\Delta}) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$. In fact, both quasivarieties generate $\mathbb{K}$, therefore, since $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$ is structurally complete, they will be equal.

Then, if $n \neq 1$ or $m \neq 0$, $\mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega)) = \mathbb{Q}(\mathbf{A}_2^{\Delta} \times \mathbf{A}_{2n}^{\Delta}, \mathbf{A}_2^{\Delta} \times \mathbf{A}_{2m+1}^{\Delta}) \subsetneq \mathbb{V}(\mathbf{A}_{2n}^{\Delta}, \mathbf{A}_{2m+1}^{\Delta})$ because $\mathbf{A}_{2n}^{\Delta} \notin \mathbb{Q}(\mathbf{A}_2^{\Delta} \times \mathbf{A}_{2n}^{\Delta}, \mathbf{A}_2^{\Delta} \times \mathbf{A}_{2m+1}^{\Delta})$. Hence, by Theorem 1.29, it is clear that any logic algebraized by some $\mathbb{V}(\mathbf{A}_{2n}^{\Delta}, \mathbf{A}_{2m+1}^{\Delta})$ with $n \neq 1$ or $m \neq 0$ is not structurally complete.
□

Corollary 3.37. *Let* $\mathbb{M}$ *be a variety of* NM_{Δ}*-algebras such that* $\mathbb{V}(\boldsymbol{A}_2^{\Delta}) \subsetneq \mathbb{M}$. *Then, the logic algebraized by* $\mathbb{M}$ *is not structurally complete.*

Remark 3.38. The previous Corollary tells us that there is no nontrivial variety of NM$_{\Delta}$-algebras that is structurally complete. Therefore, $\mathbf{NM}^{-}_{\Delta}$ won't be SC, unlike what happens in the case of $\mathbf{NM}^{-}$ (see Theorem 1.73).

Theorem 3.39. $\boldsymbol{NML_{\Delta}}$ *is hereditarily active structurally complete.*

Proof. We will prove this from an algebraic perspective. Let $\mathbb{K}$ be a subvariety of NM$_{\Delta}$. Using the characterization of ASC given in Theorem 1.29, it will be enough to show that, for any $\mathbf{B} \in \mathbb{K}_{SI}$, $\mathbf{A}_2^{\Delta} \times \mathbf{B} \in \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$ in order to prove that $\mathbb{K}$ is ASC.

Given any $\mathbf{B} \in \mathbb{K}_{SI}$, by Proposition 1.49, $\mathbf{B}$ is a NM_{Δ}-chain of $\mathbb{K}$. Moreover, by Theorem 3.13, we know $\mathbb{K}$ is of the form $\mathbb{V}(\mathbf{A}_{2n}^{\Delta}, \mathbf{A}_{2m+1}^{\Delta})$ for some $n, m \in \omega + 1$ such that $n > 0$ and, from Proposition 3.35, we obtain the inclusion $\mathbb{Q}(\mathbf{A}_2^{\Delta} \times \mathbf{A}_{2n}^{\Delta}, \mathbf{A}_2^{\Delta} \times \mathbf{A}_{2m+1}^{\Delta}) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$.

- If $n, m \in \omega$: Since $\mathbf{B}$ is a NM$_{\Delta}$-chain of $\mathbb{K} = \mathbb{V}(\mathbf{A}_{2n}^{\Delta}, \mathbf{A}_{2m+1}^{\Delta})$, either $\mathbf{B}$ has no fixpoint and has strictly less than $2n + 1$ elements or $|\mathbf{B}| < 2(m + 1)$. In both cases it is clear that $\mathbf{A}_2^{\Delta} \times \mathbf{B} \in \mathbb{Q}(\mathbf{A}_2^{\Delta} \times \mathbf{A}_{2n}^{\Delta}, \mathbf{A}_2^{\Delta} \times \mathbf{A}_{2m+1}^{\Delta}) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{K}}(\omega))$.

- If $m = \omega$: Then, $\mathbb{K} = \mathbb{V}([\mathbf{0{,}1}]_{\Delta}) = \mathbb{V}(\{\mathbf{A}_k^{\Delta} : k \in \omega\})$ and, since $\mathbf{A}_2^{\Delta} \times [\mathbf{0{,}1}]_{\Delta}$ partially embeds into $\{\mathbf{A}_2^{\Delta} \times \mathbf{A}_k^{\Delta} : k \in \omega\}$, $\mathbb{Q}(\mathbf{A}_2^{\Delta} \times [\mathbf{0{,}1}]_{\Delta}) = \mathbb{Q}(\{\mathbf{A}_2^{\Delta} \times \mathbf{A}_k^{\Delta} : k \in \omega\})$.

  By [17, Theorem 2.8], we know $\mathbf{B}$ (which is a chain) will be embeddable into an ultraproduct of its finitely generated subalgebras. Since NM$_{\Delta}$ is a locally finite variety, every finitely generated algebra is finite thus, $\mathbf{B} \in \mathbb{ISP}_U(\{\mathbf{A}_k : k \in I\})$. Therefore,

$$\mathbf{A}_2^\Delta \times \mathbf{B} \hookrightarrow \mathbf{A}_2^\Delta \times \prod_{k \in I} \mathbf{A}_k^\Delta / F \cong \prod_{k \in I} \left(\mathbf{A}_2^\Delta \times \mathbf{A}_k^\Delta\right)/F$$

Hence, $\mathbf{A}_2^\Delta \times \mathbf{B} \in \mathbb{ISPP}_U(\{\mathbf{A}_2^\Delta \times \mathbf{A}_k^\Delta : k \in \omega\}) \subseteq \mathbb{Q}(\mathbf{F}_\mathbb{K}(\omega))$.

- Otherwise, if $m \in \omega$ and $n = \omega$: We know $\mathbb{K} = \mathbb{V}([\mathbf{0{,}1}]_\Delta^-, \mathbf{A}_{2m+1}^\Delta)$ and, since $\mathbf{A}_2^\Delta \times [\mathbf{0{,}1}]_\Delta^-$ partially embeds into $\{\mathbf{A}_2^\Delta \times \mathbf{A}_{2k}^\Delta : k \in \omega\}$, $\mathbb{Q}(\mathbf{A}_2^\Delta \times [\mathbf{0{,}1}]_\Delta^-, \mathbf{A}_2^\Delta \times \mathbf{A}_{2m+1}^\Delta) = \mathbb{Q}(\{\mathbf{A}_2^\Delta \times \mathbf{A}_{2k}^\Delta : k \in \omega\} \cup \{\mathbf{A}_2^\Delta \times \mathbf{A}_{2m+1}^\Delta\})$.

  If $\mathbf{B}$ is a chain of $\mathbb{K}$ with negation fixpoint it is clear that it will have less than $2m+2$ elements, thus, $\mathbf{A}_2^\Delta \times \mathbf{B} \in \mathbb{Q}(\mathbf{A}_2^\Delta \times \mathbf{A}_{2n}^\Delta, \mathbf{A}_2^\Delta \times \mathbf{A}_{2m+1}^\Delta) \subseteq \mathbb{Q}(\mathbf{F}_\mathbb{K}(\omega))$.

  On the other hand, if $\mathbf{B}$ is a chain without negation fixpoint, by arguing as in the previous case, it is also shown what we wanted.

□

Finally, from the previous results, an axiomatization of all passive admissible rules is directly obtained:

Corollary 3.40. *Let $\mathbb{K}$ be a variety of NM_Δ-algebras. Then, $\mathbb{Q}(\boldsymbol{F}_\mathbb{K}(\omega))$ is axiomatized by the quasiequation $\neg SL_3(x_1, \ldots, x_3) \approx \overline{1} \Rightarrow \overline{0} \approx \overline{1}$. Thus, the quasiequation $\neg SL_3(x_1, \ldots, x_3) \approx \overline{1} \Rightarrow \overline{0} \approx \overline{1}$ axiomatizes all (passive) admissible quasiequations.*

3.3 Comparative analysis of the results

On one hand, it has been showed that adding the Δ connective to the language doesn't change the lattice of axiomatic extensions.
On the other hand, from Example 3.31 we can deduce that the lattices $L_\mathcal{Q}(\mathbb{V}(\mathbf{A}_4^\Delta))$ and $L_\mathcal{Q}(\mathbb{V}(\mathbf{A}_4))$ are different. We know $\mathbb{V}(\mathbf{A}_4) = \mathbb{Q}(\mathbf{A}_4)$ and, from Corollary 1.78, we get that its only proper subquasivariety is $\mathbb{Q}(\mathbf{A}_2)$. Therefore, $L_\mathcal{Q}(\mathbb{V}(\mathbf{A}_4))$ has only two elements while $L_\mathcal{Q}(\mathbb{V}(\mathbf{A}_4^\Delta))$ has three and, even though the lattices of axiomatic extensions of **NML** and $\mathbf{NML}_\Delta$ are equal, we don't have the same for the finitary extensions.

With respect to (almost) structural completeness results, we have proven that $\mathbf{NML}_\Delta$ is not structurally complete, just like what happens for **NML**. Moreover, both logics are hereditarily active sructurally complete and we can axiomatize all (passive) admissible quasiequations in the two cases (see Theorem 1.75 and Corollary 3.40).
The difference between the results obtained for both of them lies in the fact that $\mathbf{NM}^-$ is HSC while $\mathbf{NM}^-_\Delta$ is not even SC (as mentioned in Remark 3.38).

4 The logic $\mathbf{RNML}_\Delta$

Now, we will focus on studying the logic $\mathbf{RNML}_\Delta$, which is obtained by the addition of rational constants to $\mathbf{NML}_\Delta$. In fact, the calculus $\vdash_{\mathbf{RNML}_\Delta}$ is defined by extending the axioms of $\mathbf{NML}_\Delta$ with the bookkeeping axioms of **[0,1]**.

Equivalently, we can also consider the logic $\mathbf{RNML}_\Delta$ to be the Δ-core fuzzy logic corresponding to the core fuzzy logic **RNML** (Proposition 1.46).

We move on to defining the equivalent algebraic semantics for the logic:

Definition 4.1. *An algebra* $\boldsymbol{A}$ *in the language of* MTL_Δ*-algebras expanded with the constants in* $\mathscr{C} = \{\boldsymbol{c}_q \colon q \in [0,1] \cap \mathbb{Q}\}$ *is said to be a rational* NM_Δ*-algebra if the* MTL_Δ*-reduct of* $\boldsymbol{A}$ *is an* NM_Δ*-algebra and* $\boldsymbol{A}$ *validates the bookkeeping axioms* $\mathscr{B}(\boldsymbol{[0,1]})$ *presented in Definition 1.52.*

We denote by RNM_Δ the variety of rational NM_Δ-algebras.

Remark 4.2. Alternatively, we could have described these algebras as in Definition 1.43 given **L** to be the logic **RNML**.

Definition 4.3. *The canonical rational* NM_Δ*-algebra can be obtained by expanding the standard* NM_Δ*-algebra* $\boldsymbol{[0,1]}_\Delta$ *with the natural interpretation of the constants in* $\mathscr{C}$ *(interpreting* $\boldsymbol{c}_q$ *as the rational q, for every* $q \in [0,1] \cap \mathbb{Q}$*). We will denote this algebra by* $\boldsymbol{[0,1]}_\Delta^{\boldsymbol{Q}}$ *and its subalgebra with universe* $[0,1] \cap \mathbb{Q}$ *by* $\boldsymbol{[0,1]}_\Delta^{\boldsymbol{Q}} \cap \boldsymbol{Q}$*.*

For readability's sake we will usually omit the superscript **Q** from here onwards.

It is clear that the variety RNM_Δ algebraizes $\mathbf{RNML}_\Delta$, that is, for every set of formulas $\Gamma \cup \{\varphi\}$, as:

$$\Gamma \vdash_{\mathbf{RNML}_\Delta} \varphi \text{ if and only if } \tau[\Gamma] \vDash_{RNM_\Delta} \tau(\varphi)$$

where $\tau := \{x \approx 1\}$.

Remark 4.4. By definition of rational NM_Δ-algebras and Proposition 1.49, RNM_Δ will also be a discriminator variety since every algebra **A** in the generator class of RNM_Δ will satisfy $N^{\mathbf{A}} = d^{\mathbf{A}}$ where N and d are as presented in Remark 3.3 and Definition 3.1.

We will consider, as in the case of **RNML**, two types of rational NM_Δ-chains:

- Given a real $r \in (\frac{1}{2}, 1]$, we take $\mathbf{Q}_r{}^\Delta$ to be the expansion of $\mathbf{Q}_r$ (introduced in Section 2) with the connective Δ defined as in (1.1).

  Remark 4.5. In the case $r = 1$, we obtain $\mathbf{Q}_r{}^\Delta = \mathbf{[0,1]}_\Delta^{\mathbf{Q}} \cap \mathbf{Q}$, where the last algebra is the one mentioned in Definition 4.3.

- Given a rational $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and some ordinal $\gamma \in \omega + 1$, let $\mathbf{Q}_p^{\gamma\Delta}$ be the rational NM_Δ-algebra obtained from expanding the algebra $\mathbf{Q}_p^\gamma$ (presented in Section 2) with a connective Δ defined as in (1.1).

4.1 The lattice of axiomatic extensions

In this section we will just focus on the axiomatic extensions of the logic, which we have already mentioned that can be seen as the Δ-core fuzzy logic corresponding to the core fuzzy logic **RNML**.
In Section 3 we have showed that the addition of the Δ connective to **NML** doesn't change the lattice of axiomatic extensions. Thus, although we don't fully study all the extensions of $\mathbf{RNML}_\Delta$, we can analyze if this is still the case when comparing the lattice of axiomatic extensions of the logic to the one of **RNML**.

We proceed similarly to how it was done in Section 2 and present some results about RNM_Δ-chains and varieties of rational NM_Δ-algebras. Since the operation Δ is uniquely defined in all RNM_Δ-chains (see Proposition 1.47), we obtain a result analogous to the one from Proposition 2.2:

Proposition 4.6. *For every nontrivial rational NM_Δ-chain $\boldsymbol{A}$, there are $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$ such that $\mathbb{ISP}_U(\boldsymbol{A}) = \mathbb{ISP}_U(\boldsymbol{Q_r}^\Delta)$ or $\mathbb{ISP}_U(\boldsymbol{A}) = \mathbb{ISP}_U(\boldsymbol{Q}_p^{\gamma\Delta})$. Moreover,*

1. $\mathbb{ISP}_U(\boldsymbol{Q_r}^\Delta)$ *is axiomatized relative to the class of RNM_Δ chains by the sentences*

$$\boldsymbol{c}_{q'} \not\approx 1 \text{ for all } q' \in [\tfrac{1}{2}, r) \cap \mathbb{Q} \quad \text{and} \quad \boldsymbol{c}_q \approx 1 \text{ for all } q \in [r, 1] \cap \mathbb{Q}.$$

   That is, $\mathbb{ISP}_U(\boldsymbol{Q_r}^\Delta)$ is the class of RNM_Δ-chains that have $\boldsymbol{Q_r}^\Delta$ as the interpretation of the constants.

2. $\mathbb{ISP}_U(\boldsymbol{Q}_p^{\omega\Delta})$ *is axiomatized relative to the class of RNM_Δ chains by the sentences*

$$\boldsymbol{c}_p \not\approx 1 \quad \text{and} \quad \boldsymbol{c}_q \approx 1 \text{ for all } q \in (p, 1] \cap \mathbb{Q}.$$

   In other words, $\mathbb{ISP}_U(\boldsymbol{Q}_p^{\omega\Delta})$ is the class of RNM_Δ-chains that have $\boldsymbol{Q}_p^{0\Delta}$ as the interpretation of the constants.

3. $\mathbb{ISP}_U(\boldsymbol{Q}_p^{n\Delta})$ *is axiomatized relative to the class of RNM_Δ chains by the sentences*

$$\boldsymbol{c}_p \not\approx 1,\ \boldsymbol{c}_q \approx 1 \text{ for all } q \in (p, 1] \cap \mathbb{Q} \quad \text{and}$$
$$\forall x_0 \dots x_{n+2} \Big(\bigvee_{0 \le i < j \le n+2} (\boldsymbol{c}_p \vee x_i) \leftrightarrow (\boldsymbol{c}_p \vee x_j) \Big) \approx 1.$$

   That is, $\mathbb{ISP}_U(\boldsymbol{Q}_p^{\omega\Delta})$ is the class of RNM_Δ-chains that have $\boldsymbol{Q}_p^{0\Delta}$ as the interpretation of the constants and contain less than $n+2$ elements above $\boldsymbol{c}_p$.

The proof is easily obtained by slightly adapting the one from Proposition 2.2.

Corollary 4.7. *Every variety of rational NM_Δ-algebras is generated by a set of algebras of the form $\boldsymbol{Q_r}^\Delta$, where $r \in (\frac{1}{2}, 1]$, or $\boldsymbol{Q}_p^{\gamma\Delta}$, where $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$.*

Proof. We know every variety is generated by its subdirectly irreducible members [7, Corollary 9.7] and, by Proposition 1.49, subdirectly irreducible rational NM_Δ-algebras are RNM_Δ-chains. Thus, what we wanted to see is derived from Proposition 4.6.
□

In the following theorem we give an axiomatization of some varieties of rational NM_Δ-algebras and we show how these are related between them, in terms inclusion:

Theorem 4.8. *The following hold.*

1. *Any variety of rational NM_Δ-algebras of the form $\mathbb{V}(\boldsymbol{Q_r}^\Delta)$, where $r \in (\frac{1}{2}, 1]$, is axiomatized by the equations $\{\neg\Delta(\boldsymbol{c}_q) \approx 1 : q \in [\frac{1}{2}, r) \cap \mathbb{Q}\}$ and $\{\boldsymbol{c}_q \approx 1 : q \in [r, 1] \cap \mathbb{Q}\}$ and varieties of the form $\mathbb{V}(\boldsymbol{Q}_p^{\gamma\Delta})$, with $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$, are axiomatized by the equations:*
   - $\neg\Delta(\boldsymbol{c}_p) \approx 1$, $\{\boldsymbol{c}_q \approx 1 : q \in (p, 1] \cap \mathbb{Q}\}$ *and*
   $$\left(\bigvee_{0 \leq i < j \leq n+2} (\boldsymbol{c}_p \vee x_i) \leftrightarrow (\boldsymbol{c}_p \vee x_j)\right) \approx 1$$
   if $\gamma = n \in \omega$.
   - $\neg\Delta(\boldsymbol{c}_p) \approx 1$ *and* $\{\boldsymbol{c}_q \approx 1 : q \in (p, 1] \cap \mathbb{Q}\}$, *otherwise.*
2. *For all $r_1, r_2 \in (\frac{1}{2}, 1]$, $p_1, p_2 \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma_1, \gamma_2 \in \omega + 1$,*

   $\mathbb{V}(\boldsymbol{Q}_{r_1}{}^\Delta) \subseteq \mathbb{V}(\boldsymbol{Q}_{r_2}{}^\Delta)$ *if and only if* $r_1 = r_2$,

   $\mathbb{V}(\boldsymbol{Q}_{r_1}{}^\Delta) \subseteq \mathbb{V}(\boldsymbol{Q}_{p_1}^{\gamma_1\Delta})$ *never holds,*

   $\mathbb{V}(\boldsymbol{Q}_{p_1}^{\gamma_1\Delta}) \subseteq \mathbb{V}(\boldsymbol{Q}_{r_1}{}^\Delta)$ *never holds,*

   $\mathbb{V}(\boldsymbol{Q}_{p_1}^{\gamma_1\Delta}) \subseteq \mathbb{V}(\boldsymbol{Q}_{p_2}^{\gamma_2\Delta})$ *if and only if* $p_1 = p_2$ *and* $\gamma_1 \leq \gamma_2$.

Proof. 2. Consider $r_1, r_2 \in (\frac{1}{2}, 1]$, $p_1, p_2 \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma_1, \gamma_2 \in \omega + 1$. Wee need to prove that:

- $\mathbb{V}(\mathbf{Q}_{r_1}{}^\Delta) \subseteq \mathbb{V}(\mathbf{Q}_{r_2}{}^\Delta)$ if and only if $r_1 = r_2$.

  $\Rightarrow$) We show this direction holds by contraposition:

  - If $r_2 < r_1$: Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number such that $r_2 \leq q < r_1$. Hence, $\mathbf{c}_q \approx 1$ holds in $\mathbf{Q}_{r_2}{}^\Delta$ but fails in $\mathbf{Q}_{r_1}{}^\Delta$ (by the definition we have given of this algebras). Therefore, $\mathbf{Q}_{r_1}{}^\Delta \notin \mathbb{V}(\mathbf{Q}_{r_2}{}^\Delta)$ and, consequently, $\mathbb{V}(\mathbf{Q}_{r_1}{}^\Delta) \nsubseteq \mathbb{V}(\mathbf{Q}_{r_2}{}^\Delta)$.
  - If $r_1 < r_2$: Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number such that $r_1 \leq q < r_2$. Hence, $\neg\Delta(\mathbf{c}_q) \approx 1$ holds in $\mathbf{Q}_{r_2}{}^\Delta$ but fails in $\mathbf{Q}_{r_1}{}^\Delta$ (by the definition we have given of this algebras). Therefore, $\mathbf{Q}_{r_1}{}^\Delta \notin \mathbb{V}(\mathbf{Q}_{r_2}{}^\Delta)$ and, consequently, $\mathbb{V}(\mathbf{Q}_{r_1}{}^\Delta) \nsubseteq \mathbb{V}(\mathbf{Q}_{r_2}{}^\Delta)$.

  In fact, this could also be proven directly by assuming $\mathbf{Q}_{r_1}{}^\Delta \in \mathbb{V}(\mathbf{Q}_{r_2}{}^\Delta)$ and seeing that, then, $r_1 = r_2$ must hold. Since $\mathbf{Q}_{r_1}{}^\Delta$ is simple and the notions of simple and subdirectly irreducible coincide for RNM_Δ-algebras, by [7, Theorem 6.8] we would have $\mathbf{Q}_{r_1}{}^\Delta \in \mathbb{HSP}_U(\mathbf{Q}_{r_2}{}^\Delta)$. Moreover, $\mathbf{Q}_{r_2}{}^\Delta$ is also simple and the class of simple algebras is closed under $\mathbb{S}$ and $\mathbb{P}_U$, hence, $\mathbf{Q}_{r_1}{}^\Delta \in \mathbb{ISP}_U(\mathbf{Q}_{r_2}{}^\Delta)$.

  Therefore, by Proposition 4.6, $\mathbf{Q}_{r_1}{}^\Delta$ would have $\mathbf{Q}_{r_2}{}^\Delta$ as the interpretation of its constants. This implies $r_1 = r_2$, which is what we wanted to see.

$\Leftarrow$) Is clear.

- $\mathbb{V}({\mathbf{Q}_{r_1}}^{\Delta}) \subseteq \mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta})$ never holds.

  We reason by cases:

  - If $p_1 < r_1$: Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number such that $p_1 < q < r_1$. Hence, $\mathbf{c}_q \approx 1$ holds in ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}$ but fails in ${\mathbf{Q}_{r_1}}^{\Delta}$ (by the definition we have given of this algebras). Therefore, ${\mathbf{Q}_{r_1}}^{\Delta} \notin \mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta})$ and, consequently, $\mathbb{V}({\mathbf{Q}_{r_1}}^{\Delta}) \nsubseteq \mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta})$.

  - If $r_1 \leq p_1$: Then, $\neg\Delta(\mathbf{c}_{p_1}) \approx 1$ holds in ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}$ but fails in ${\mathbf{Q}_{r_1}}^{\Delta}$ (by the definition we have given of this algebras). Therefore, ${\mathbf{Q}_{r_1}}^{\Delta} \notin \mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta})$ and, consequently, $\mathbb{V}({\mathbf{Q}_{r_1}}^{\Delta}) \nsubseteq \mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta})$.

- $\mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}) \subseteq \mathbb{V}({\mathbf{Q}_{r_1}}^{\Delta})$ never holds.

  We reason by cases:

  - If $r_1 \leq p_1$: Then, $\mathbf{c}_{p_1} \approx 1$ holds in ${\mathbf{Q}_{r_1}}^{\Delta}$ but fails in ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}$ (by the definition we have given of this algebras). Therefore, ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta} \notin \mathbb{V}({\mathbf{Q}_{r_1}}^{\Delta})$ and, consequently, $\mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}) \nsubseteq \mathbb{V}({\mathbf{Q}_{r_1}}^{\Delta})$.

  - If $p_1 < r_1$: Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $p_1 < q < r_1$. Hence, $\neg\Delta(\mathbf{c}_q) \approx 1$ holds in ${\mathbf{Q}_{r_1}}^{\Delta}$ but fails in ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}$ (by the definition we have given of this algebras). Therefore, ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta} \notin \mathbb{V}({\mathbf{Q}_{r_1}}^{\Delta})$ and, consequently, $\mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}) \nsubseteq \mathbb{V}({\mathbf{Q}_{r_1}}^{\Delta})$.

- $\mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}) \subseteq \mathbb{V}({\mathbf{Q}_{p_2}^{\gamma_2}}^{\Delta})$ if and only if $p_1 = p_2$ and $\gamma_1 \leq \gamma_2$.

  $\Rightarrow$) By contraposition, we will show this direction holds:

  - If $p_2 < p_1$: Then, by density, there exists a rational q such that $p_2 < q < p_1$. Hence, $\mathbf{c}_q \approx 1$ holds in ${\mathbf{Q}_{p_2}^{\gamma_2}}^{\Delta}$ but fails in ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}$ (by the definition we have given of this algebras). Therefore, ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta} \notin \mathbb{V}({\mathbf{Q}_{p_2}^{\gamma_2}}^{\Delta})$ and, consequently, $\mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}) \nsubseteq \mathbb{V}({\mathbf{Q}_{p_2}^{\gamma_2}}^{\Delta})$.

  - If $p_1 < p_2$: Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number $p_1 < q < p_2$. Hence, $\neg\Delta(\mathbf{c}_q) \approx 1$ holds in in ${\mathbf{Q}_{p_2}^{\gamma_2}}^{\Delta}$ but fails in ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}$ (by the definition we have given of this algebras). Therefore, ${\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta} \notin \mathbb{V}({\mathbf{Q}_{p_2}^{\gamma_2}}^{\Delta})$ and, consequently, $\mathbb{V}({\mathbf{Q}_{p_1}^{\gamma_1}}^{\Delta}) \nsubseteq \mathbb{V}({\mathbf{Q}_{p_2}^{\gamma_2}}^{\Delta})$.

  - If $p_1 = p_2 (= q \in [\frac{1}{2}, 1) \cap \mathbb{Q})$ and $\gamma_2 < \gamma_1$: Since $\gamma_2 < \gamma_1$, necessarily $\gamma_1 > 0$ moreover, from $\gamma_2 < \gamma_1 \in \omega + 1$ it follows that $\gamma_2 = n$ for some $n \in \omega$.
    Given the fact that $q < 1$ and $\gamma_2 = n$, it is clear that the interval $[\mathbf{c}_q, 1]$ in ${\mathbf{Q}_q^{\gamma_2}}^{\Delta}$ has $n + 2$ elements, hence:

$$ {\mathbf{Q}_q^{\gamma_2}}^{\Delta} = {\mathbf{Q}_{p_2}^{\gamma_2}}^{\Delta} \vDash \bigvee_{0 \leq i < j \leq n+2} (\mathbf{c}_q \vee x_i) \leftrightarrow (\mathbf{c}_q \vee x_j) \approx 1. $$

On the other hand, since $\gamma_1 > \gamma_2 = n$, the interval $[\mathbf{c}_q, 1]$ in $\mathbf{Q}_q^{\gamma_1\Delta} = \mathbf{Q}_{p_1}^{\gamma_1\Delta}$ has more than $n+2$ elements. Thus, the above equation fails in $\mathbf{Q}_{p_1}^{\gamma_1\Delta}$ and, consequently, $\mathbf{Q}_{p_1}^{\gamma_1\Delta} \notin \mathbb{V}(\mathbf{Q}_{p_2}^{\gamma_2\Delta})$.

$\Leftarrow$) If $p_1 = p_2$ and $\gamma_1 \leq \gamma_2$ then,

$$\mathbf{Q}_{p_1}^{\gamma_1\Delta} \in \mathbb{S}(\mathbf{Q}_{p_2}^{\gamma_2\Delta}) \subseteq \mathbb{V}(\mathbf{Q}_{p_2}^{\gamma_2\Delta}).$$

1. We will show that varieties of the form $\mathbb{V}(\mathbf{Q}_p^{\gamma\Delta})$ are axiomatized by the equations: $\neg\Delta(\mathbf{c}_p) \approx 1$, $\{\mathbf{c}_q \approx 1 : q \in (p, 1] \cap \mathbb{Q}\}$ and $\left(\bigvee_{0\leq i<j\leq n+2}(\mathbf{c}_p \vee x_i) \leftrightarrow (\mathbf{c}_p \vee x_j)\right) \approx 1$, if $\gamma = n \in \omega$; $\neg\Delta(\mathbf{c}_p) \approx 1$ and $\{\mathbf{c}_q \approx 1 : q \in (p, 1] \cap \mathbb{Q}\}$, otherwise.

Let Σ be the set of equations given by the statement, it holds that $\mathbf{Q}_p^{\gamma\Delta} \vDash \Sigma$.
On the other hand, let's consider a rational NM_Δ-algebra $\mathbf{A} \notin \mathbb{V}(\mathbf{Q}_p^{\gamma\Delta})$. By what we have proven in Corollary 4.7, we obtain that $\mathbb{V}(\mathbf{A})$ is generated by some algebras of the form $\mathbf{Q}_r{}^\Delta$ for some $r \in (\frac{1}{2}, 1]$ and/or $\mathbf{Q}_{p'}^{\delta}{}^\Delta$ for some $p' \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\delta \in \omega + 1$.

Thus, since $\mathbf{A} \notin \mathbb{V}(\mathbf{Q}_p^{\gamma\Delta})$, by item 2., there must exist some $\mathbf{Q}_r{}^\Delta$ or some $\mathbf{Q}_{p'}^{\delta}{}^\Delta$ with $p \neq p'$ or $\gamma < \delta$ among the generators of $\mathbb{V}(\mathbf{A})$.

- If we have some $\mathbf{Q}_r{}^\Delta$: Then, either $p < r$ or $r \leq p$. In the first case, by density of $\mathbb{Q}$ in $\mathbb{R}$, there is a rational q such that $p < q < r$. Hence, $\mathbf{c}_q^{\mathbf{Q}_r{}^\Delta} \neq 1$ and, consequently, $\mathbf{Q}_r{}^\Delta \nvDash \Sigma$ which means $\mathbf{A} \nvDash \Sigma$.
  On the other hand, if we have $r \leq p$, then $\mathbf{c}_p^{\mathbf{Q}_r{}^\Delta} = 1$ and, consequently, $\mathbf{Q}_r{}^\Delta \nvDash \Sigma$ which means $\mathbf{A} \nvDash \Sigma$.
- If there exists some algebra $\mathbf{Q}_{p'}^{\delta}{}^\Delta$: Then, either $p < p'$, $p' < p$ or $p = p'$ and $\gamma < \delta$.
  - If $p < p'$, analogously to the previous case, $\mathbf{c}_{p'}^{\mathbf{Q}_{p'}^{\delta}{}^\Delta} \neq 1$ and, consequently, $\mathbf{Q}_{p'}^{\delta}{}^\Delta \nvDash \Sigma$ which means $\mathbf{A} \nvDash \Sigma$.
  - If $p' < p$, by density of the $\mathbb{Q}$ in $\mathbb{R}$, there exists some rational $p' < q < p$. Thus, $\neg\Delta(\mathbf{c}_q^{\mathbf{Q}_{p'}^{\delta}{}^\Delta}) \neq 1$ and, consequently, $\mathbf{Q}_{p'}^{\delta}{}^\Delta \nvDash \Sigma$ which means $\mathbf{A} \nvDash \Sigma$.
  - If $p = p'$ and $\gamma < \delta$, then $\gamma = n \in \omega$ and

$$\mathbf{Q}_{p'}^{\delta}{}^\Delta \nvDash \Big(\bigvee_{0\leq i<j\leq n+2}(\mathbf{c}_p \vee x_i) \leftrightarrow (\mathbf{c}_p \vee x_j)\Big) \approx 1.$$

    Hence, $\mathbf{A} \nvDash \Sigma$.

Therefore, we can conclude that Σ axiomatizes $\mathbb{V}(\mathbf{Q}_p^{\gamma\Delta})$.

Finally, we will show that varieties of the form $\mathbb{V}(\mathbf{Q}_r{}^\Delta)$ are axiomatized by the equations: $\{\neg\Delta(\mathbf{c}_q) \approx 1 : q \in [\frac{1}{2}, r) \cap \mathbb{Q}\}$ and $\{\mathbf{c}_q \approx 1 : q \in [r, 1] \cap \mathbb{Q}\}$.
Let Σ be the set of equations given by the statement, it holds that $\mathbf{Q}_r{}^\Delta \vDash \Sigma$ (clearly by how we have defined $\mathbf{Q}_r{}^\Delta$).

On the other hand, we consider some rational NM$_\Delta$-algebra $\mathbf{A}$ such that $\mathbf{A} \notin \mathbb{V}({\mathbf{Q}_r}^\Delta)$ and, arguing as in the previous case, by Corollary 4.7 we know $\mathbb{V}(\mathbf{A})$ will be generated by a set of algebras of the form ${\mathbf{Q}_r}^\Delta$, for some $r \in (\frac{1}{2}, 1]$, and/or ${\mathbf{Q}_p^\delta}^\Delta$, for some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\delta \in \omega + 1$. By item 2. and the fact that $\mathbf{A} \notin \mathbb{V}({\mathbf{Q}_r}^\Delta)$, there must exist some ${\mathbf{Q}_r}^\Delta$ with $r' \neq r$ or some ${\mathbf{Q}_{p'}^\delta}^\Delta$ among the generators of $\mathbb{V}(\mathbf{A})$.

- If there exists some ${\mathbf{Q}_{r'}}^\Delta$ with $r \neq r'$ then, either $r < r'$ or $r' < r$. In the first case, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some rational q such that $r < q < r'$. Hence, $\mathbf{c}_q^{{\mathbf{Q}_{r'}}^\Delta} \neq 1$ and ${\mathbf{Q}_{r'}}^\Delta \not\models \Sigma$, which implies $\mathbf{A} \not\models \Sigma$.
  In the other case, if we have $r' < r$, again by density, there exists some rational $r' < q < r$, hence, $\neg\Delta(\mathbf{c}_q^{{\mathbf{Q}_{r'}}^\Delta}) \neq 1$ and ${\mathbf{Q}_{r'}}^\Delta \not\models \Sigma$, which implies $\mathbf{A} \not\models \Sigma$.
- For the case in which we have some algebra $\mathbf{Q}_p^{\gamma\Delta}$ either $r \leq p < 1$ or $p < r$. In the first case, $\mathbf{c}_p^{\mathbf{Q}_p^{\gamma\Delta}} \neq 1$ and $\mathbf{Q}_p^{\gamma\Delta} \not\models \Sigma$, which implies $\mathbf{A} \not\models \Sigma$.
  In the second case, by density of $\mathbb{Q}$ in $\mathbb{R}$, there is some rational $p < q < r$. Thus, $\neg\Delta(\mathbf{c}_q^{\mathbf{Q}_p^{\gamma\Delta}}) \neq 1$ and $\mathbf{Q}_p^{\gamma\Delta} \not\models \Sigma$, which implies $\mathbf{A} \not\models \Sigma$.

Therefore, we can conclude that Σ axiomatizes $\mathbb{V}({\mathbf{Q}_r}^\Delta)$.

□

Remark 4.9. The axiomatization given in item 1. of Theorem 4.8 can be simplified for varieties of the form $\mathbb{V}({\mathbf{Q}_q}^\Delta)$ with $q \in \mathbb{Q} \cap (\frac{1}{2}, 1]$, as these can be axiomatized by the equations $\mathbf{c}_q \approx 1$ and $\{\neg\Delta(\mathbf{c}_q) \approx 1 : q \in [\frac{1}{2}, r) \cap \mathbb{Q}\}$. Varieties of the form $\mathbb{V}({\mathbf{Q}_r}^\Delta)$ with $r \in (\frac{1}{2}, 1] \setminus \mathbb{Q}$ do not admit a simpler axiomatization (that's because $\mathbb{Q}$ is dense in $\mathbb{R}$: there always exists a rational in between any two given real numbers).

In Corollary 4.7 we have given a characterization of all RNM$_\Delta$-varieties in terms of their generators and, in the following proposition, we prove that they allow us to fully distinguish different varieties:

Proposition 4.10. *Let V_1, V_2 be two varieties of rational NM$_\Delta$-algebras. Denote by $\mathbb{K}_1, \mathbb{K}_2$ the respective sets of rational NM$_\Delta$-chains of the form ${\mathbf{Q}_r}^\Delta$, $\mathbf{Q}_p^{\gamma\Delta}$ that generate the varieties. If $\mathbb{K}_1$ and $\mathbb{K}_2$ are finite and satisfy that, given any $\mathbf{Q}_p^{\gamma\Delta} \in \mathbb{K}_i$, we have ${\mathbf{Q}_p^\delta}^\Delta \in \mathbb{K}_i$ for all $\delta < \gamma$; then, the fact that $\mathbb{K}_1$ and $\mathbb{K}_2$ are different implies that so are V_1 and V_2.*

Proof. Assume the sets of generators are different, then, without losing generality, we can suppose that there is some algebra of the form ${\mathbf{Q}_r}^\Delta$ or $\mathbf{Q}_p^{\gamma\Delta}$ (let's denote it by $\mathbf{A}$) that is in $\mathbb{K}_1$ but doesn't belong to $\mathbb{K}_2$.
We presume $\mathbf{A} \in V_2$ in order to reach a contradiction. Then, it will be clear that $V_1 \neq V_2$.

Since RNM$_\Delta$-chains are simple, in fact, subdirectly irreducible (see Remark 4.4 and [23, page 240]), by [7, Theorem 6.8] we obtain that $\mathbf{A} \in \mathbb{HSP}_U(\mathbb{K}_2) = \cup_{\mathbf{B}\in\mathbb{K}_2}\mathbb{HSP}_U(\mathbf{B})$, where the last equality is due to the fact that $\mathbb{K}_2$ is finite. Moreover, $\mathbb{K}_2$ is a set of simple elements, hence, $\mathbf{A} \in \cup_{\mathbf{B}\in\mathbb{K}_2}\mathbb{ISP}_U(\mathbf{B})$.
Therefore, there exists some $\mathbf{B} \in \mathbb{K}_2$ such that $\mathbf{A}$ belongs to it universal class, that is, $\mathbf{A}$ satisfies the axiomatization presented in Proposition 4.6.

We distinguish two cases:

- If $\mathbf{A}$ is of the form $\mathbf{Q}_r{}^{\Delta}$ for some $r \in (\frac{1}{2}, 1]$:
  Then, by Proposition 4.6, $\mathbf{B}$ has to be $\mathbf{Q}_r{}^{\Delta}$. This contradicts our assumption that $\mathbf{A} \notin \mathbb{K}_2$.
- If $\mathbf{A}$ is of the form $\mathbf{Q}_p^{\gamma\Delta}$ for some $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and some $\gamma \in \omega + 1$:
  Then, by Proposition 4.6, $\mathbf{B}$ has to be of the form $\mathbf{Q}_p^{\delta\Delta}$ for some $\delta \geq \gamma$ but, in that case, by the definition we have given of the sets of generator,s we obtain $\mathbf{Q}_p^{\gamma\Delta} \in \mathbb{K}_2$. This implies that $\mathbf{A}$ belongs to $\mathbb{K}_2$, which contradicts our assumption.

Then, it is clear that $\mathbf{A} \notin V_2$ and, then, $V_1 \neq V_2$. □

Corollary 4.11. *There is a countable chain of rational NM$_\Delta$-varieties.*

Since there is a dual isomorphism between the lattice of subvarieties of RNM$_\Delta$ and the lattice of axiomatic extensions of $\mathbf{RNML}_\Delta$, Theorem 4.8, Corollary 4.7 and Proposition 4.10 provide some information of the last one, which is presented in the following result:

Corollary 4.12. *Some consistent axiomatic extensions of* $\boldsymbol{RNML}_\Delta$ *are of the form*

$\boldsymbol{RNML}_{r\Delta} := \boldsymbol{RNML}_\Delta + \{\neg\Delta(\boldsymbol{c}_q) : q \in [\frac{1}{2}, r) \cap \mathbb{Q}\} + \{\boldsymbol{c}_q : q \in [r, 1] \cap \mathbb{Q}\}$ *for some* $r \in (\frac{1}{2}, 1]$,

$\boldsymbol{RNML}^{\omega}_{p\Delta} := \boldsymbol{RNML}_\Delta + \neg\Delta(\boldsymbol{c}_p) + \{\boldsymbol{c}_q : q \in (p, 1] \cap \mathbb{Q}\}$ *for some rational* $p \in [\frac{1}{2}, 1)$,

$\boldsymbol{RNML}^{n}_{p\Delta} := \boldsymbol{RNML}^{\omega}_{p\Delta} + \bigvee_{0 \leq i < j \leq n+2} (\boldsymbol{c}_p \vee x_i) \leftrightarrow (\boldsymbol{c}_p \vee x_j)$ *for some rational* $p \in [\frac{1}{2}, 1)$ *and* $n \in \omega$.

Moreover, the lattice of axiomatic extensions of $\boldsymbol{RNML}_\Delta$ *has uncountable antichains and countable chains.*

4.2 Structural completeness in RNML$_\Delta$

We move on to studying the structural completeness of some varieties of RNM$_\Delta$-algebras. In order to do so, first, we present the following result:

Proposition 4.13. *Given any* $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ *and* $\gamma \in \omega + 1$, *the equalities* $\mathbb{V}(\boldsymbol{Q}_r{}^{\Delta}) = \mathbb{Q}(\boldsymbol{Q}_r{}^{\Delta})$ *and* $\mathbb{V}(\boldsymbol{Q}_p^{\gamma\Delta}) = \mathbb{Q}(\boldsymbol{Q}_p^{\gamma\Delta})$ *hold.*

Proof. Let $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$, we consider $\mathbf{A} \in \{\mathbf{Q}_r{}^{\Delta}, \mathbf{Q}_p^{\gamma\Delta}\}$. The inclusion $\mathbb{Q}(\mathbf{A}) \subseteq \mathbb{V}(\mathbf{A})$ is trivial and, to show the other one also holds, we take $\mathbf{B} \in \mathbb{V}(\mathbf{A})$ and prove that $\mathbf{B} \in \mathbb{Q}(\mathbf{A})$.

By Proposition 1.49, $\mathbf{B}$ will be representable as a subdirect product of chains in $\mathbb{V}(\mathbf{A})$. Since all RNM$_\Delta$-chains are subdirectly irreducible (see Remark 4.4 and [23, page 240]), we can apply [7, Theorem 6.8] and deduce that $\mathbf{B} \in \mathbb{ISPHSP}_U(\mathbf{A})$.
Given that $\mathbf{A}$ is simple and the class of simple algebras is closed under $\mathbb{S}$ and $\mathbb{P}_U$, we obtain $\mathbf{B} \in \mathbb{ISPISP}_U(\mathbf{A}) = \mathbb{ISPP}_U(\mathbf{A}) = \mathbb{Q}(\mathbf{A})$, which is what we wanted to see. □

From the information we have obtained about the previous varieties, we can deduce the following statement about their structural completeness:

Theorem 4.14. *For any $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$, $\mathbb{V}(\boldsymbol{Q}_r{}^{\Delta})$ and $\mathbb{V}(\boldsymbol{Q}_p^{\gamma\Delta})$ are structurally complete.*

Proof. Given any $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $\gamma \in \omega + 1$, let $\mathbf{A} \in \{\mathbf{Q}_r{}^{\Delta}, \mathbf{Q}_p^{\gamma\Delta}\}$. By Theorem 1.29, in order to see that $\mathbb{V}(\mathbf{A})$ is structurally complete it will be enough to see that it is generated as a quasivariety by $\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega)$.

On one hand, we have $\mathbb{Q}(\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega)) \subseteq \mathbb{V}(\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega)) = \mathbb{V}(\mathbf{A})$, where the last equality is given by [7, Theorem 10.12 and Lemma 11.8].

On the other hand, we have left to show the other inclusion, we proceed as we have done for Proposition 2.8 (similarly to the proof of [18, Proposition 8.3]):

- If $\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega) = \mathbf{F}_{\mathbb{V}(\mathbf{Q}_r{}^{\Delta})}(\omega)$: then $\mathbf{Q}_r{}^{\Delta}$ is the zero-generated subalgebra of $\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega)$. Therefore, $\mathbb{V}(\mathbf{Q}_r{}^{\Delta}) = \mathbb{Q}(\mathbf{Q}_r{}^{\Delta}) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega))$.
- If $\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega) = \mathbf{F}_{\mathbb{V}(\mathbf{Q}_p^{0\Delta})}(\omega)$: analogously to the previous case we obtain $\mathbb{V}(\mathbf{Q}_p^{0\Delta}) = \mathbb{Q}(\mathbf{Q}_p^{0\Delta}) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega))$.
- Finally, we consider the case where $\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega) = \mathbf{F}_{\mathbb{V}(\mathbf{Q}_p^{\gamma\Delta})}(\omega)$ with $\gamma > 0$: since every finite partial subalgebra of $\mathbf{Q}_p^{\gamma\Delta}$ embeds into $\{\mathbf{Q}_p^{n\Delta} : n \in \omega \text{ and } 1 \leq n \leq \gamma\}$ it is clear that $\mathbf{Q}_p^{\gamma\Delta} \in \mathbb{ISP}_U(\{\mathbf{Q}_p^{n\Delta} : n \in \omega \text{ and } 1 \leq n \leq \gamma\})$.
  Thus, if we show that each $\mathbf{Q}_p^{n\Delta}$ (for $n \in \omega$ and $1 \leq n \leq \gamma$) is embeddable into $\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega)$ we obtain $\mathbf{Q}_p^{\gamma\Delta} \in \mathbb{ISP}_U\mathbb{S}(\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega)) = \mathbb{ISP}_U(\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega)) \subseteq \mathbb{Q}(\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega))$ and, hence, what we wanted.

  For every $1 \leq n \leq \gamma$ such that $n \in \omega$, the algebra $\mathbf{Q}_p^{n\Delta}$ is the chain consisting of the interval $[1-p, p] \cap \mathbb{Q}$ in between the $n+1$ element chains:

  $$0 < \neg t_{n-1} < \cdots < \neg t_1 < \neg t_0$$
  $$t_0 < t_1 < \cdots < t_{n-1} < 1.$$

  As happened in Proposition 2.8, $\mathbf{Q}_p^{n\Delta}$ can be embedded into $\mathbf{F}_{\mathbb{V}(\mathbf{A})}(\omega)$ considering the map that is the identity on $\{0\} \cup ([1-p, p] \cap \mathbb{Q}) \cup \{1\}$ and that sends t_i to (the equivalence class of) the formula φ_i and $\neg t_i$ to its negation, where:

  $$\varphi_0 := \mathbf{c}_p \vee x_1 \qquad \text{and} \qquad \varphi_{j+1} := x_{j+1} \vee (x_{j+1} \to \varphi_j).$$

  That is, we have an embedding:

  $$h : \mathbf{Q}_p^{n\Delta} \hookrightarrow \mathbf{F}_{\mathbb{V}(\mathbf{A})}$$
  $$a \mapsto h(a) := \begin{cases} a & \text{if } a \in \{0\} \cup ([1-p, p] \cap \mathbb{Q}) \cup \{1\}, \\ \overline{\varphi_i} & \text{if } a = t_i, \\ \neg\overline{\varphi_i} & \text{if } a = \neg t_i. \end{cases}$$

  which is what we needed in order to conclude the proof.

□

4.3 Comparative analysis of the results

In the case of the lattice of axiomatic extensions of $\mathbf{RNML}_\Delta$ we have obtained that the addition of the Δ connective does have an impact on the results, unlike what we had for the logic $\mathbf{NML}_\Delta$. The lattice of axiomatic extensions of **RNML** is an uncountable chain, while in $\mathbf{RNML}_\Delta$ not all the axiomatic extensions are comparable between them. Moreover, we have seen that there are many more elements in the lattice of $\mathbf{RNML}_\Delta$.

On the other hand, regarding the lattice of axiomatic extensions of $\mathbf{NML}_\Delta$ (of **NML**, equivalently) we can also spot some differences: in there we don't have uncountable antichains unlike what happens for $\mathbf{RNML}_\Delta$.

For this logic, we have proven that there is an uncountable number of axiomatic extensions that are structurally complete. This differs from the results obtained for $\mathbf{NML}_\Delta$, which doesn't have structural completeness for any axiomatic extensions different than $\mathbf{NM2}_\Delta$ (see Corollary 3.37).
On the other hand, for **NML** we have a countable number of structurally complete extensions. In particular $\mathbf{NM}^{-}$ is hereditarily structurally complete (see Theorem 1.73), that is, every extension of $\mathbf{NM}^{-}$ is SC.

4.4 The addition of bookkeeping axioms for Δ to the logic $\mathbf{RNML}_\Delta$

In the previous section, the RNM_Δ-algebras we have studied don't necessarily satisfy that the interpretation of the constants constitutes a subuniverse closed under the operations.
For example, given some algebra $\mathbf{Q}_p^{0\Delta} \times \mathbf{Q}_r{}^\Delta$ with $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$ and $r \in (\frac{1}{2}, 1]$ such that $p \geq r$, we know:

$$\begin{cases} \mathbf{c}_p^{\mathbf{Q}_p^{0\Delta}} = p, \\ \mathbf{c}_p^{\mathbf{Q}_p^{0\Delta} \times \mathbf{Q}_r{}^\Delta} = (p, 1). \end{cases}$$

But then, $\Delta(\mathbf{c}_p^{\mathbf{Q}_p^{0\Delta}}) = 0$ and $\Delta(\mathbf{c}_p^{\mathbf{Q}_p^{0\Delta} \times \mathbf{Q}_r{}^\Delta}) = (0, 1)$ which is not the interpretation of any constant of $\mathbf{Q}_p^{0\Delta} \times \mathbf{Q}_r{}^\Delta$.

Therefore, now we will study the logic obtained from $\mathbf{RNML}_\Delta$ by adding some axioms assuring that the rational constants behave well with respect to the Δ connective.

Definition 4.15. *We call Δ-bookkeeping axioms of **[0,1]**$_\Delta$ the axioms in the language of **[0,1]**$_\Delta$ expanded with the constants in $\mathscr{C} = \{\boldsymbol{c}_q : q \in [0, 1] \cap \mathbb{Q}\}$ that are of the form:*

$$\Delta(\boldsymbol{c}_p) \leftrightarrow \boldsymbol{c}_{\Delta(p)},$$

for all $p \in [0, 1] \cap \mathbb{Q}$.

*We can also express them as equations of the form: $\Delta(\boldsymbol{c}_p) \approx \boldsymbol{c}_{\Delta(p)}$, where will denote by $\mathscr{B}_\Delta$(**[0,1]**$_\Delta$) the set constituted by these equations, for all $p \in [0, 1] \cap \mathbb{Q}$.*

The logic $\mathbf{CRNML}_\Delta$ (canonical rational Nilpotent minimum logic) is defined from $\mathbf{RNML}_\Delta$ by adding the Δ-bookkeeping axioms of $[\mathbf{0,1}]_\Delta$.

Now, we will present the equivalent algebraic semantics for this logic:

Definition 4.16. *A rational NM_Δ-algebra is considered to be a $CRNM_\Delta$-algebra if it validates the Δ-bookkeeping axioms $\mathscr{B}_\Delta([\mathbf{0,1}]_\Delta)$.*

We denote by CRNM_Δ the variety of rational NM_Δ-algebras satisfying $\mathscr{B}_\Delta([\mathbf{0,1}]_\Delta)$. We have decided on these names because satisfying these last equations implies that every constant has to be evaluated differently (as happens for the canonical interpretation).
In fact, the canonical CRNM_Δ-algebra will coincide with the one from RNM_Δ (see Definition 4.3).
From the viewpoint of logic, the variety CRNM_Δ algebraizes $\mathbf{CRNML}_\Delta$, that is, for every set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathbf{CRNML}_\Delta} \varphi \text{ if and only if } \tau[\Gamma] \vDash_{CRNM_\Delta} \tau(\varphi)$$

where $\tau := \{x \approx 1\}$.

Remark 4.17. Studying whether some algebra satisfies $\Delta(\mathbf{c}_p) \leftrightarrow \mathbf{c}_{\Delta(p)}$ for every $p \in [0,1] \cap \mathbb{Q}$ is the same as considering if the axioms $\{\neg\Delta(\mathbf{c}_q) \approx 1 : q \in [\frac{1}{2}, 1) \cap \mathbb{Q}\}$ and $\mathbf{c}_1 \approx 1$ hold, which are the ones axiomatizing $\mathbb{V}({\mathbf{Q}_1}^\Delta)$ (see Theorem 4.8).
Thus, from the previous RNM_Δ-chains presented in Section 4.2, it is clear that the only one satisfying the Δ-bookkeeping axioms will be ${\mathbf{Q}_1}^\Delta$.

Now, we can proceed to study the varieties of CRNM_Δ-algebras. Since all of them will be, in particular, varieties of RNM_Δ-algebras, the results proven in the previous subsection will also be valid in this case. In particular, from Corollary 4.7 and our previous Remark, we obtain:

Proposition 4.18. *Every variety of $CRNM_\Delta$-algebras is of the form $\mathbb{V}({\boldsymbol{Q}_1}^\Delta)$. Moreover, $\mathbb{V}({\boldsymbol{Q}_1}^\Delta)$ is axiomatized by equations $\{\neg\Delta(\boldsymbol{c}_q) \approx 1 : q \in [\frac{1}{2}, 1) \cap \mathbb{Q}\}$ and $\boldsymbol{c}_1 \approx 1$.*

Hence, from the dual isomorphism that exists between the lattice of subvarieties of CRNM_Δ and the lattice of axiomatic extensions of $\mathbf{CRNML}_\Delta$, we deduce:

Corollary 4.19. *The logic $\boldsymbol{CRNML}_\Delta$ has no proper consistent axiomatic extensions.*

Finally, from Theorem 4.14:

Theorem 4.20. *The logic $\boldsymbol{CRNML}_\Delta$ is structurally complete.*

5 Conclusions

In this thesis we have studied some expansions of the Nilpotent minimum logic, in particular, we have focused on the rational Nilpotent minimum logic, the logic $\mathbf{NML}_\Delta$, the rational $\mathbf{NML}_\Delta$ logic, and its further extension with additional bookkeeping axioms for the Δ connective. We have examined their lattices of axiomatic and finitary extensions, along with several results on structural completeness.

We can now summarize what we have seen during the project and explain how the expansion of the language affects, in each case, our object of study.

Regarding the lattices of axiomatic and finitary extensions, the ones corresponding to **NML** are presented in [16]. The first one has a countable number of elements, not all of which are comparable, such as **NM2n+1** and **NM2(n+1)** given any $n > 0$. However, when rational constants are added to the language, the lattice of axiomatic extensions becomes an uncountable chain, which makes the lattice different than the one we have for **NML**.
That is because every RNM-chain satisfies $\neg\mathbf{c}_{\frac{1}{2}} \approx \mathbf{c}_{\frac{1}{2}}$, in other words, contains the negation fixpoint. Therefore, we obtain a totally ordered lattice, as is the case of the sublattice of $\mathrm{L}_{\mathcal{V}}(NM)$ constituted by all the subvarieties generated by chains with negation fixpoint.

On the other hand, concerning the lattice of finitary extensions of **NML**, there is also a countable number of elements. This differs from the lattice obtained for **RNML** which contains both uncountable chains and antichains.

Next, we analyze what happens when we expand the **NML** language with the Delta Baaz connective. In this case, we have proven that the lattice of axiomatic extensions is the same as the one for **NML**.
Although adding the Δ connective does not affect axiomatic extensions, it does affect the lattice of finitary extensions. Even if we have not completely defined this lattice for $\mathbf{NML}_\Delta$, we know all its elements (thanks to the study of critical algebras) and we have graphically represented some of its sublattices. From this, we have been able to deduce that the lattices of finitary extensions of **NML** and $\mathbf{NML}_\Delta$ do not coincide, as $L_{\mathcal{Q}}(\mathbb{V}(\mathbf{A}_4^\Delta))$ and $L_{\mathcal{Q}}(\mathbb{V}(\mathbf{A}_4))$ are different (the first one has three elements while the latter only two).

Finally, we analyze the rational $\mathbf{NML}_\Delta$ logic, which is an expansion of both **RNML** and $\mathbf{NML}_\Delta$. We limited our study to its lattice of axiomatic extensions, which has countable chains and uncountable antichains, making it more similar to the lattice of finitary extensions of **RNML** (which also has this property) than to the one of axiomatic extensions. The later is an uncountable chain but, for $\mathbf{RNML}_\Delta$, we have seen that not all axiomatic extensions are comparable. Thus, they are different and, additionally, it is clear that there are many more elements in the lattice of $\mathbf{RNML}_\Delta$.
From this, we observe that although the addition of the Δ connective didn't affect the lattice of axiomatic extensions of $\mathbf{NML}_\Delta$, this is not the case for $\mathbf{RNML}_\Delta$.

We can also compare the results obtained in Section 4 with the lattice of axiomatic extensions of $\mathbf{NML}_\Delta$ (equivalently, of **NML**). Clearly, they will be different since the last lattice doesn't have uncountable antichains.

Lastly, we recall that the expansion obtained from $\mathbf{RNM}_\Delta$ by adding bookkeeping axioms for the Δ connective has no proper axiomatic extensions.

We proceed to analyze the results concerning structural completeness. In [16] it is seen that the logic **NML** is hereditarily active structurally complete but not structurally complete, and neither is any logic algebraized by a variety of NM-algebras containing $\mathbb{V}(\mathbf{A}_3)$. However, the extension **NM**$-$ is structurally complete (HSC, to be precise). Furthermore, we know we can axiomatize the admissible quasiequations (equivalently, provide a basis for admissible rules) for any axiomatic extension of **NML**.

When we add rational constants to the language, some of these results change. We still can provide a basis for the admissible rules on all axiomatic extensions, but we no longer have logics that are ASC (active structurally complete) and not SC, as was previously the case for logics algebraized by a variety $\mathbb{V}(\mathbf{A}_3) \subseteq \mathbb{K}$. This is because the concepts of hereditary structural completeness, structural completeness, and active structural completeness are equivalent for any extension of **RNML**. Additionally, a characterization of passive structural completeness is presented.

Now, we turn to analyze how adding the Delta Baaz connective to the language of **NML** influences its results on structural completeness, presented earlier. We prove that $\mathbf{NML}_\Delta$ is not structurally complete but it is hereditarily active structurally complete, similarly to what we have for **NML**. Moreover, we can also axiomatize the admissible quasiequations of every axiomatic extension of the logic. The difference here is that, in this case, no axiomatic extension other than $\mathbf{NM2}_\Delta$ is structurally complete, whereas for **NML** it is shown that $\mathbf{NM}^-$ is HSC.

Finally, in the project we study the rational $\mathbf{NML}_\Delta$ logic, which can either be seen as an expansion of **RNML** or $\mathbf{NML}_\Delta$. It has been proven in the thesis that there are several axiomatic extensions of this logic that are structurally complete (in fact, an uncountable amount). The same does not hold for the case of $\mathbf{NML}_\Delta$, where the only structurally complete axiomatic extension is the logic $\mathbf{NM2}_\Delta$. However, for **NML**, we have a countable number of axiomatic extensions that are SC (all axiomatic extensions of $\mathbf{NM}^-$, for example). We conclude by recalling that the expansion obtained from $\mathbf{RNML}_\Delta$ by adding bookkeeping axioms for the Δ connective has been proven to be structurally complete.

There is future work that could be done with the aim of providing a continuation to the results presented in the thesis:

- The finitary extensions of the $\mathbf{RNML}_\Delta$ logic could be analyzed. Further research could also include providing more information about the active structural completeness of the logic and its extensions and attempting to axiomatize all admissible quasiequations for any given axiomatic extension.
- The $\mathcal{Q}$-universality of the logic $\mathbf{RNML}_\Delta$ could be studied. The necessary conditions

to apply [1, Corollary 3.4] are not satisfied (the language is not finite, hence, RNM_Δ-algebras do not have finite type). Nevertheless, by [7, Theorem 12.2], it is easily checked that RNM_Δ is congruence-permutable, hence, it is congruence-modular (see [7, Theorem 5.10]). From this and [26, Theorem 1] it is deduced that the variety of RNM_Δ-algebras has the Fraser-Horn property.

Thus, we can consider the class of algebras of the form ${\mathbf{Q}_r}^\Delta$ or ${\mathbf{Q}_p^0}^\Delta$, given any $r \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2}, 1) \cap \mathbb{Q}$. Even though their algebras will not be finite, the class satisfies the Fraser-Horn property and contains infinitely many algebras, none of which is embeddable into any other one and each of which is hereditarily simple.
Therefore, maybe there could exist an infinite family of RNM_Δ-algebras satisfying the conditions P1)-P4) presented in [1, page 1054].

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