Inconsistency lemmas and the inconsistency by cases property

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Contents

Introduction

The deduction-detachment theorem is commonly regarded as one of the central metalogical properties of classical logic: to prove that an implication holds between propositions it suffices to provide a proof of the conclusion on the basis of the assumption of the antecedent. Formally, this means that for every set $\Gamma \cup \{\varphi, \psi\}$ of formulas,

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff } \Gamma \vdash \varphi \to \psi,$$

where $\vdash$ is the consequence relation naturally associated with classical propositional logic.

Another notable feature of classical propositional logic, which emerges as a corollary of the deduction theorem, is that it admits proofs by contradiction. Formally, this means that for any set $\Gamma \cup \{\alpha\}$ of formulas,

$$\Gamma \cup \{\neg\alpha\} \text{ is inconsistent iff } \Gamma \vdash \alpha,$$

where $\vdash$ is the consequence relation naturally associated with classical propositional logic. While it is well known that intuitionistic propositional logic rejects proofs by contradiction, it still satisfies the following weaker principle: for every set $\Gamma \cup \{\alpha\}$ of formulas

$$\Gamma \cup \{\alpha\} \text{ is inconsistent iff } \Gamma \vdash \neg\alpha,$$

where $\vdash$ is the consequence relation naturally associated with intuitionistic propositional logic.

The study of deduction-detachment theorems has played a major role in shaping the evolution of abstract algebraic logic since the early eighties [5, 13] (see also [4]). Traditionally, algebraic logic has focused on the algebraic investigation of particular classes of algebras of logic, regardless of whether they could be connected to some known logic by means of the Lindenbaum-Tarski method. When such a connection could be established, there was interest in exploring the relationhip between various syntactical properties of the logic and the algebraic properties of the associated class of algebras, leading to the formulation of what are known as *bridge theorems*.

The term *bridge theorem* (see [1], pp. 133-135 and pp. 186-188) refers to results that establish a connection between two different fields, logic and algebra, by associating a purely algebraic interpretation to metalogical properties. These theorems allow us to use a better known toolbox, that of algebra, to adress logical problems, and then translate the solution back to the original logical setting. In particular, this methodology allows us to cope with an ever-increasing forest of new logics with a more uniform approach.

Abstract algebraic logic allows us to study such connections in a general context. Within this framework, a logic is determined by its formal consequence relation $\vdash$, considered as a consequence operation on the algebra of formulas. A (propositional) *logic* in some algebraic language is any pair $\langle \boldsymbol{Fm}; \vdash \rangle$ where $\boldsymbol{Fm}$ is the algebra of formulas of the appropriate type with a denumerable set of variables and $\vdash \subseteq P(Fm) \times Fm$ is a relation that satisfies that for all sets of formulas Γ, Δ and all formulas φ, ψ

(i) if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$ (identity),

(ii) if $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for all $\psi \in \Gamma$, then $\Delta \vdash \varphi$ (cut),

(iii) if $\Gamma \vdash \varphi$ and h is a substitution, then $h[\Gamma] \vdash h(\varphi)$ (substitution invariance)

where a substitution h is just an endomorphism of $\boldsymbol{Fm}$. A logic $\vdash$ is *finitary* if whenever we have $\Gamma \vdash \varphi$, there is a finite subset Δ of Γ in such a way that $\Delta \vdash \varphi$.

One class of logics that holds a prominent position within abstract algebraic logic is the class of protoalgebraic logics [2, 15], i.e., logics possessing a set of formulas that globally expresses logical equivalence. Protoalgebraicity is the weakest non-trivial property of logics which makes them amenable to most of the standard methods of algebra. Their role in algebraic logic consists in providing a framework suitable for the formulation of bridge theorems.

It is a well-known fact that a finitary protoalgebraic logic has a deduction-detachment theorem – briefly a DDT – if and only if the semilattice of compact deductive filters of every algebra of the corresponding type is dually Brouwerian (see, e.g., [13]). The bridge theorem has algebraic consequences, which in turn have logical applications crossing back over the bridge. For instance, any finitary protoalgebraic logic satisfying a DDT is filter-distributive, and the logical counterpart of filter-distributivity is the so-called *proof by cases property*, a metalogical property which has been extensively studied in [10, 12, 15, 16].

In contrast, the theory of inconsistency lemmas, or ILs, for short, has not been systematically investigated so far, with a few exceptions (see, e.g., [9, 26, 27, 28, 30]). An abstract account of the inconsistency lemma was first given by Raftery in [30]. Accordingly, a logic $\vdash$ is said to have an *inconsistency lemma* – briefly an IL – if, for $n \in \mathbb{N}^+$, there exists a finite set of formulas $\Psi_n(x_1, \ldots, x_n)$ such that for every $\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \subseteq Fm$,

$$\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \text{ is inconsistent in } \vdash \iff \Gamma \vdash \Psi_n(\varphi_1, \ldots, \varphi_n).$$

This definition encompasses the familiar inconsistency lemma of intuitionistic propositional logic, because

$$\Gamma \cup \{\alpha\} \text{ is inconsistent} \iff \Gamma \vdash \neg\alpha.$$

amounts to the conjunction, over all $n \in \mathbb{Z}^+$, of the claims

$$\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \text{ is inconsistent} \iff \Gamma \vdash \neg(\alpha_1 \wedge \ldots \wedge \alpha_n).$$

Raftery proved in [30, Thm. 3.7] that for a finitary protoalgebraic logic to have a (global) IL amounts to the demand that the join semilattice of compact deductive filters in each algebra of the corresponding type should be dually pseudo-complemented. This result, in particular implies, on lattice-theoretic grounds that a protoalgebraic logic with a DDT and a greatest compact theory has an inconsistency lemma.

Subsequently, Lávička and Přenosil [26, 28] introduced and studied the local and parametrized local versions in a similar fashion to the hierarchy of DDTs. Following the terminology introduced in [26], a logic $\vdash$ is said to have a *local inconsistency lemma* – briefly a LIL – if for every positive integer n, there exists a family Ψ_n of finite sets of formulas $I(x_1, \ldots, x_n)$ such that for every $\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \subseteq Fm$,

$$\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \text{ is inconsistent in } \vdash \iff \Gamma \vdash I(\varphi_1, \ldots, \varphi_n) \text{ for some } I \in \Psi_n.$$

The corresponding algebraic counterpart is the *maximal consistent filter extension property*, or MCFEP, for short, which a logic $\vdash$ is said to have if for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and every submatrix $\langle \boldsymbol{B}, G\rangle$ of $\langle \boldsymbol{A}, F\rangle$, for every maximal $\vdash$-filter H containing G there is a $\vdash$-filter H' containing F such that $H = H' \cap B$. This result established in [26] for protonegational logics * translates in the framework of finitary protoalgebraic logics as the following theorem:

Theorem. *[26, Thm. 6.35] Let $\vdash$ be a finitary protoalgebraic logic. The following are equivalent:*

(i) *$\vdash$ has the LIL;*

(ii) *$\vdash$ has the MCFEP and for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated;*

(iii) *The MCFEP holds in the algebra of formulas and $\vdash$ posseses a finite inconsistent set of formulas.*

Considering these advances, as the theory of the deduction-detachment theorems is rather satisfying, it is natural to ask whether a similar theory can be developed for inconsistency lemmas. Specifically, one may wonder what is necessary for a LIL to reduce to an IL. The original motivation for this work was to adress that question and obtain a theory for ILs parallel to the existing theory for DDTs.

Chapter 2 investigates the algebraic counterparts of the global and local inconsistency lemmas, previously addressed in [30, Thm. 3.7] and [26, Thm. 6.35], respectively. Here, we adapt the proof for the local version to the framework of finitary protoalgebraic logics. Moreover, as a first step to determine what is necessary for a LIL to reduce to an IL, we introduce the new notion of *(first-order) definable maximal consistent filters* – briefly DMCF. This property is reminiscent of the *(first-order) definability of principal deductive filters*, introduced by Czelakowski (see [15], pp. 132-134).

A logic $\vdash$ is said to have DMCF if for each $n \in \mathbb{Z}^+$ there exists a formula $\delta_n(x_1, \ldots, x_n)$ in the language of the first-order predicate logic (with equality), whose only non-logical symbols are the operation symbols of $\vdash$ and a unary predicate $P(x)$, such that for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and elements $a_1, \ldots, a_n \in A$,

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\}) \iff \langle \boldsymbol{A}, F\rangle \vDash \delta_n(a_1, \ldots, a_n).$$

*The class of protonegational logics is introduced in [26] as a weakening of protoalgebraicity, restricting some of its defining conditions to maximal consistent theories. Particular examples are the negation fragments of protoalgebraic logics.

In this case, for a finitary protoalgebraic logic $\vdash$ with a LIL and DMCF, we prove that any family Ψ_n witnessing the LIL must include a finite subset of sets of formulas for each $n \in \mathbb{N}^+$ such that the resulting family also witnesses the LIL for $\vdash$. However, the question of whether Ψ_n can be taken to be a singleton for every n, and obtain a global IL, is more involved.

To answer this question, in Chapter 3 we introduce a new metalogical property that arises as a consequence of the IL: a logic $\vdash$ has the *inconsistency by cases property*, or ICP, for short, when for every positive integers n, m, there exists a parameterized set $\nabla_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m, \vec{z})$ of formulas such that for any set $\Gamma \cup \{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m\}$ of formulas, $\vec{\varphi} \vdash \vec{\varphi}\nabla_{n,m}\vec{\psi}$ and $\vec{\psi} \vdash \vec{\varphi}\nabla_{n,m}\vec{\psi}$, and whenever $\Gamma \cup \{\vec{\varphi}\}$ and $\Gamma \cup \{\vec{\psi}\}$ are inconsistent in $\vdash$, then $\Gamma \cup \{\vec{\varphi}\nabla_{n,m}\vec{\psi}\}$ is inconsistent in $\vdash$, where, $\vec{\varphi}\nabla_{n,m}\vec{\psi}$ is defined as $\bigcup\{\nabla_{n,m}(\vec{\varphi}, \vec{\psi}, \vec{\gamma}) : \vec{\gamma} \in Fm\}$.

It turns out that, in parallel to the connection between the proof by cases property and filter-distributivity, the corresponding bridge theorem arises between the ICP and the notion of 1-distributivity. Recall that a lattice $\boldsymbol{A}$ with 1 is said to be *1-distributive* if whenever $a \vee b = 1$ and $a \vee c = 1$, then $a \vee (b \wedge c) = 1$ for all elements $a, b, c \in A$. We obtain the following new result:

Theorem. *Let $\vdash$ be a finitary protoalgebraic logic. The following are equivalent:*

(i) *$\vdash$ has the ICP and possesses a finite inconsistent set of formulas;*

(ii) *For every algebra $\boldsymbol{A}$, the lattice of $\vdash$-filters of $\boldsymbol{A}$ is 1-distributive;*

(iii) *The lattice of theories of $\vdash$ is 1-distributive.*

While a syntactical proof already shows that if a logic has an inconsistency lemma, then it has the inconsistency by cases property, the above theorem allows us to view this implication in the algebraic setting. Since every dually pseudo-complemented join semilattice with 1 is 1-distributive and any algebraic lattice is isomorphic to the lattice of ideals of the join semilattice of its compact elements, crossing back over the bridge to the syntactical setting, this implies that any finitary protoalgebraic logic with an IL has the ICP. Moreover, we prove that for a finitary protoalgebraic logic having a LIL witnessed by Ψ_n, the demand for the family to be directed for each $n \in \mathbb{N}^+$ amounts to the 1-distributivity of the logic.

Consequently, we are finally able to address the original question motivating this work: what is necessary for an LIL to reduce to an IL? The crucial result of the thesis – Theorem 3.10 – states that a finitary protoalgebraic logic has an IL if and only if it has the MCFEP, for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated, it has DMCF and it is filter-1-distributive.

CHAPTER 1

Preliminaries

This preliminary chapter provides a comprehensive overview of the fundamental concepts and results relevant to our work. While some facts are presented without proofs, we direct the reader to our primary sources in the literature. Section 1.1 contains a brief introduction to lattices, and to the close connection between complete lattices and closure operators. In Section 1.2 we introduce the standard notion of logic in abstract algebraic logic and the class of protoalgebraic logics.

1.1 Lattices and closure operators

We refer to [8] and [20] for an introduction to orders and lattices. We start with the definition of poset.

There are two standard ways of defining lattices: one puts them on the same (algebraic) footing as groups or rings, and the other, based on the notion of order, offers geometric insight. In order to introduce both definitions of a lattice we need to present the notion of a partial order on a set.

A binary relation $\leqslant$ on a set X is said to be a *partial order* when it is reflexive, transitive and antisymmetric. In this case, the pair $\mathbb{X} = \langle X, \leqslant \rangle$ is said to be a *partially ordered set*, a *poset*, for short.

Given a poset $\mathbb{X}$ and $x, y, z \in X$, we write $x < y$ when both $x \leqslant y$ and $x \neq y$, or equivalently $x \leqslant y$ and $y \nleqslant x$. Furthermore, we will write $x \leqslant y, z$ as an abbreviation for $x \leqslant y$ and $x \leqslant z$. A similar reading will apply to expressions of the form $x, y \leqslant z$.

Two elements x and y of a poset $\mathbb{X}$ are said to be *comparable* when either $x \leqslant y$ or $y \leqslant x$. Accordingly, we say that $\mathbb{X}$ is a *chain* when every two elements of X are comparable. By extension, a subset $Y \subseteq X$ is said to be a *chain* in $\mathbb{X}$ when the poset $(Y, \leqslant^{\mathbb{Y}})$ is a chain, where $\leqslant^{\mathbb{Y}}$ is the restriction of the order $\leqslant^{\mathbb{X}}$ to Y.

Let $\mathbb{X}$ be a poset and let $Y \subseteq X$. An *upper bound of Y in X* is any element $x \in X$ such that $y \leqslant x$, for all $y \in Y$. An element x is called the *least upper bound* (or *supremum*) of Y in $\mathbb{X}$ if x is an upper bound of Y in $\mathbb{X}$, and $x \leqslant y$ for every upper bound y of Y. Similarly, we can define what it means for an element x of $\mathbb{X}$ to be a *lower bound* and *greatest lower bound* of Y. An element x of $\mathbb{X}$ is said to be a *lower bound of Y in X* when

$x \leqslant y$, for all $y \in Y$. An element x is called the *greatest lower bound* (or *infimum*) of Y in $\mathbb{X}$ if x is a lower bound of Y in $\mathbb{X}$, and $y \leqslant x$ for every lower bound y of Y.

Let $\mathbb{X}$ be a poset and $Y \subseteq X$. An element $x \in X$ is said to be a *meet of* Y when x is the greatest lower bound of Y in $\mathbb{X}$. An element $x \in X$ is said to be a *join of* Y when x is the least upper bound of Y in $\mathbb{X}$. Given a poset $\mathbb{X}$ and $Y \subseteq X$, when the meet (resp. join) of Y exists, it is unique. Accordingly, if the meet of Y in $\mathbb{X}$ exists, we denote it by $\bigwedge Y$, and if the join of Y exists, we denote it by $\bigvee Y$. When $Y = \{x, y\}$, we will write $x \wedge y$ and $x \vee y$ instead of $\bigwedge Y$ and $\bigvee Y$.

Now we can define lattices order-theoretically: A nonempty poset $\mathbb{X}$ is said to be

(i) a *meet semilattice* when the meet of $\{x, y\}$ exists for every pair of elements $x, y \in X$;

(ii) a *join semilattice* when the join of $\{x, y\}$ exists for every pair of elements $x, y \in X$;

(iii) a *lattice* when it is both a meet semilattice and a join semilattice.

A lattice is said to be *complete* when $\bigwedge Y$ and $\bigvee Y$ exist for every $Y \subseteq X$.

Lattices and semilattices can also be viewed as algebraic structures, as we proceed to explain. To this end, we need to take a detour in universal algebra.

An *operation* of arity n on a set A is a function $f : A^n \to A$. A *type* is a map *rho* : $\mathcal{F} \to \mathbb{N}$, where $\mathcal{F}$ is a set of function symbols. In this case $\rho(f)$ is said to be the *arity* of the function symbol f, for every $f \in \mathcal{A}$. Function symbols of arity 0 are called *constants*. An *algebra* of type ρ is a pair $\boldsymbol{A} = \langle A, \mathcal{F} \rangle$ where A is a nonempty set and $\mathcal{F} = \{f^{\boldsymbol{A}} : f \in \mathcal{F}\}$ is a set of operations on A whose arity is determined by ρ in the sense that each $f^{\boldsymbol{A}}$ has arity $\rho(f)$. The set A is called the *universe* of $\boldsymbol{A}$.

Algebras of the same type are called similar and can be compared by means of maps that preserve their structure. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two similar algebras. Then $\boldsymbol{B}$ is said to be a subalgebra of $\boldsymbol{A}$ if $B \subseteq A$ and $f^{\boldsymbol{B}}$ is the restriction of $f^{\boldsymbol{A}}$ to A, for every $f \in \mathcal{F}$. In this case, we write $\boldsymbol{A} \leqslant \boldsymbol{B}$.

Given similar algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f : A \to B$ such that, for every n-ary operation g of the common type and $a_1, \ldots, a_n \in A$,

$$f(g^{\boldsymbol{A}}(a_1, \ldots, a_n)) = g^{\boldsymbol{B}}(f(a_1), ..., f(a_n)).$$

An *endomorphism* of an algebra $\boldsymbol{A}$ is a homomorphism whose domain and codomain is A. The set of endomorphisms of an algebra $\boldsymbol{A}$ will be denoted by $\mathrm{End}(\boldsymbol{A})$. An injective homomorphism is called an *embedding*, and a surjective embedding is called an *isomorphism*.

Given a type $\rho : \mathcal{F} \to \mathbb{N}$ and a set of variables X disjoint from $\mathcal{F}$, the set of *terms of type ρ over X* is the least set $T_\rho(X)$ such that

(i) $X \subseteq T_\rho(X)$;

(ii) if $c \in \mathcal{F}$ is a constant, then $c \in T_\rho(X)$;

(iii) if $\varphi_1, \ldots, \varphi_n \in T_\rho(X)$ and $f \in \mathcal{F}$ is n-ary, then $f(\varphi_1, \ldots, \varphi_n) \in T_\rho(X)$.

In the context of logic, the term algebra $T_\rho(X)$ is often called the algebra of formulas (of type ρ) and its elements are referred to as formulas. Therefore, given an algebraic language, we will denote the set of its formulas built up from a denumerable set of variables by Fm and the corresponding algebra of formulas by $\boldsymbol{Fm}$. Generic elements of X will be denoted by $x, y, z \ldots$. Moreover, the endomorphisms of $\boldsymbol{Fm}$ will be called *substitutions*.

Given a formula $\varphi \in Fm$, we write $\varphi(x_1, \ldots, x_n)$ to indicate that the variables occuring in φ are among $x_1, \ldots, x_n$. For each positive integer n, we define

$$Fm(n) := \{\varphi \in Fm : \text{the variables occurring in } \varphi \text{ are among } x_1, \ldots, x_n\}.$$

Furthermore, given $\varphi \in Fm(n)$ and an algebra $\boldsymbol{A}$ with elements $a_1, \ldots, a_n \in A$, then $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$ denotes $h(\varphi)$, where $h : \boldsymbol{Fm} \to \boldsymbol{A}$ is any homomorphism such that $h(x_i) = a_i$ for $i \in \{1, \ldots, n\}$. If $\Xi \subseteq Fm(n)$, then $\Xi^{\boldsymbol{A}}(a_1, \ldots, a_n)$ abbreviates $\{\xi^{\boldsymbol{A}}(a_1, \ldots, a_n) : \xi \in \Xi\}$.

An *equation of type ρ over X* is an expression of the form $\varphi \approx \psi$, where $\varphi, \psi \in Fm$. A *quasi-equation of type ρ over X* is an expression Φ of the form

$$(\varphi_1 \approx \psi_1 \,\&\, \ldots \,\&\, \varphi_n \approx \psi_n) \Longrightarrow \varepsilon \approx \delta,$$

where $\{\varphi_1 \approx \psi_1, \ldots, \varphi_n \approx \psi_n, \varepsilon \approx \delta\}$ is a set of equations of type ρ over X.

Then Φ is *valid* in an algebra $\boldsymbol{A}$ of type ρ when so is its universal closure $\forall \vec{x}\, \Phi$, i.e., for every $\vec{a} \in A$,

$$\text{if } \varphi_1^{\boldsymbol{A}}(\vec{a}) = \psi_1^{\boldsymbol{A}}(\vec{a}), \ldots, \varphi_n^{\boldsymbol{A}}(\vec{a}) = \psi_n^{\boldsymbol{A}}(\vec{a}), \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}),$$

in which case we say that $\boldsymbol{A}$ *validates* Φ. Alternatively, we say that $\boldsymbol{A}$ *satisfies* Φ or that the quasi-equation Φ *holds* in $\boldsymbol{A}$ and write $\boldsymbol{A} \vDash \Phi$. Observe that the antecedent of the above expression can be empty, in which case we have an equation.

Let K be a class of similar algebras and $\Psi \cup \{\varepsilon \approx \delta\}$ a set of equations in variables X. We define

$$\Psi \vDash_{\mathsf{K}} \varepsilon \approx \delta \Longleftrightarrow \text{for every algebra } \boldsymbol{A} \in \mathsf{K} \text{ and every } \vec{a} \in A,$$
$$\text{if } \varphi^{\boldsymbol{A}}(\vec{a}) = \psi^{\boldsymbol{A}}(\vec{a}) \text{ for all } \varphi \approx \psi \in \Psi, \text{ then } \varepsilon^{\boldsymbol{A}}(\vec{a}) = \delta^{\boldsymbol{A}}(\vec{a}).$$

The relation $\vDash_{\mathsf{K}}$ is known as the *equational consequence relative to* K.

A class of similar algebras K is called a *variety* if it is closed under subalgebras, homomorphic images, and direct products.

Birkhoff's Theorem 1.1. *[8, Thm. II.11.9] A class of similar algebras is a variety if and only if it can be axiomatized by a set of equations.*

A class of similar algebras K is called a *quasivariety* if it is closed under isomorphisms, subalgebras, direct products, and ultraproducts.

Maltsev's Theorem 1.2. *[8, Thm. V.2.25] A class of similar algebras is a quasivariety if and only if it can be axiomatized by a set of quasi-equations.*

Lattices and semilattices admit a definition as algebras which satisfy certain equations.

A *semilattice* is an algebra $\boldsymbol{A} = \langle A; \star \rangle$ such that $\star$ is a binary idempotent, commutative and associative operation, i.e., $\boldsymbol{A}$ satisfies the following equations:

(i) $x \star x \approx x$;

(ii) $x \star y \approx y \star x$;

(iii) $x \star (y \star z) \approx (x \star y) \star z$.

From a purely formal perspective, meet and join semilattices are essentially the same objects, differing only in the symbol which represents their unique binary operation, $\wedge$ and $\vee$, respectively. We shall give these two classes of algebras a different order theoretic interpretation.

A *lattice* is an algebra $\boldsymbol{A} = \langle A, \vee, \wedge \rangle$ such that $\langle A, \wedge \rangle$ is a meet semilattice and $\langle A, \vee \rangle$ is a join semilattice, that is, $\wedge$ and $\vee$ are binary idempotent, commutative and associative operations, and moreover $\boldsymbol{A}$ satisfies the equations:

$$x \wedge (y \vee x) \approx x \text{ and } x \vee (y \wedge x) \approx x.$$

We will now describe the equivalence of the order-theoretic and algebraic definitions of a lattice, and describe the translation between these definitions. By "equivalent" we mean the following: if $\boldsymbol{A}$ is a lattice according to one definition, then it is possible to construct, in a simple and uniform manner, a lattice on the same underlying set according to the other definition, and the two constructions (converting from one definition to the other) are inverses.

First we describe the construction from the algebraic definition to the order-theoretic one. Every semilattice $\boldsymbol{A}$ can be associated with two partial orders on A, namely the meet order $\leqslant_m$ and the join order $\leqslant_j$, defined respectively by the following rules:

$$a \leqslant_m b \iff a \star b = a \quad \text{and} \quad a \leqslant_j b \iff a \star b = b.$$

Accordingly, we say that $\boldsymbol{A}$ is a *meet semilattice* (respectively, *join semilattice*) when we prioritize the meet order (respectively, join order). It follows that $\langle A, \leqslant_m \rangle$ is a poset that forms a meet semilattice, and $\langle A, \leqslant_j \rangle$ is a poset that forms a join semilattice. Therefore, we have described a translation from the algebraic definition of meet and join semilattices to their order-theoretic definition.

In the converse direction, consider a poset $\mathbb{X} = \langle X, \leqslant \rangle$ which is a meet semilattice. We can define a binary operation $\wedge : X \times X \to X$ by taking $x \wedge y$ to be the meet of the set $\{x, y\}$ in $\mathbb{X}$. It then follows that the algebra $\langle X, \wedge \rangle$ is a meet semilattice in the sense of its algebraic definition. Similarly, if we consider a poset $\mathbb{X} = \langle X, \leqslant \rangle$ which is a join semilattice and define a binary operation $\vee : X \times X \to X$ taking $x \vee y$ to be the join of the set $\{x, y\}$ in $\mathbb{X}$. It then follows that the algebra $\langle X, \vee \rangle$ is a join semilattice in the sense of its algebraic definition.

From now on we shall treat lattices and semilattices both as posets and algebras without further notice.

Let $\boldsymbol{A}$ be a lattice. A *filter* F is a nonempty subset of A such that, for all $x, y \in A$, the following conditions hold:

(i) F is upward closed, i.e., if $x \in F$ and $x \leqslant y$, then $y \in F$;

(ii) If $x \in F$ and $y \in F$, then $x \wedge y \in F$.

Dually, an *ideal* I of $\boldsymbol{A}$ is a nonempty subset of A such that, for all $x, y \in A$, the following conditions hold:

(i) F is downward closed, i.e., if $x \in F$ and $y \leqslant x$, then $y \in F$;

(ii) If $x \in F$ and $y \in F$, then $x \vee y \in F$.

A filter F (resp. an ideal I) is said to be *proper* if $F \neq A$ (resp. $I \neq A$). Observe that if F is a filter on a lattice $\boldsymbol{A}$, then $\uparrow x$ is a filter of A and $\downarrow x$ is an ideal of $\boldsymbol{A}$, for every $x \in A$, where

$$\uparrow x := \{y \in A \, : \, x \leqslant y\};$$
$$\downarrow x := \{y \in A \, : \, y \leqslant x\}.$$

They are called *principal filters* and *principal ideals*, respectively. For a lattice $\boldsymbol{A}$, let us denote by $\mathcal{F}i(\boldsymbol{A})$ the set of filters of $\boldsymbol{A}$, ordered by inclusion. A filter F of $\boldsymbol{A}$ is said to be *maximal* if it is maximal in $\mathcal{F}i(\boldsymbol{A})$.

We now introduce another key concept: closure operators, which allow the general study of logic using the tools of universal algebra. In fact, we can define the abstract notion of logic as a particular kind of closure operator on the algebra of formulas of a given type, as outlined in the introduction.

A *closure operator* on a set X is a map $C : \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies the following conditions for every $Y, Z \subseteq X$:

(i) Extensivity: $Y \subseteq C(Y)$;

(ii) Idempotence: $C(C(Y)) = C(Y)$;

(iii) Monotonicity: if $Y \subseteq Z$, then $C(Y) \subseteq C(Z)$.

We say that a set $Y \subseteq X$ is *closed* when $Y = C(Y)$. When no confusion shall occur, given a closure operator C on X and $x_1, \dots, x_n \in X$, we shall write $C(x_1, \dots, x_n)$ as a shorthand for $C(\{x_1, \dots, x_n\})$.

A *closure system* on a set X is a family $S \subseteq P(X)$ closed under arbitrary intersections, that is, if for every $\{Y_i : i \in I\} \subseteq \mathcal{P}(X)$,

$$\text{if } Y_i \in S \text{ for every } i \in I, \text{ then } \bigcap_{i \in I} Y_i \in S.$$

The intersection of the empty family of subsets of X is understood here as $\cap\varnothing := X$. Consequently, the set X belongs to every closure system on X.

It is well known that the collection of closed sets of a closure operator C on X form a closure system on X and that a closure system S on X gives rise to a closure operator on X, defining

$$C_S(Y) := \bigcap\{Z \in S \, : \, Y \subseteq Z\} \text{ for every } Y \subseteq X.$$

These transformations are indeed inverse to one another. Therefore definitions and results established for closure operators transfer naturally to closure systems and viceversa.

A *consequence relation* on a set X is a relation $\vdash \subseteq \mathcal{P}(X) \times X$ satisfying the following conditions:

(i) Reflexivity: if $Y \subseteq X$ and $y \in Y$, then $Y \vdash y$;

(ii) Transitivity: for all $Y, Z \subseteq X$ and $x \in X$, if $Y \vdash z$ for all $z \in Z$ and $Z \vdash x$, then $Y \vdash x$.

If C is a closure operator on X, the relation $\vdash_C := \{\langle Y, x\rangle \in \mathcal{P}(X) \times X : x \in C(Y)\}$ is a consequence relation on X. Conversely, if $\vdash$ is a consequence relation on X, the map $C_\vdash : \mathcal{P}(X) \to \mathcal{P}(X)$ defined as $C_\vdash(Y) := \{x \in X : Y \vdash x\}$ is a closure operator on X. These transformations are inverse to each other, so the notions of a consequence relation and closure operator are coextensive.

Every closure system can be viewed as a poset ordered under the inclusion relation. In fact, this poset turns out to be a complete lattice.

Proposition 1.3. *[8, Thm. I.5.2] Let S be the closure system associated with a closure operator C on a set X. Then $\langle S, \subseteq\rangle$ is a complete lattice in which meets are intersections and joins are closures of unions. That is, for every $\{Y_i : i \in I\} \subseteq S$,*

$$\bigwedge_{i\in I} Y_i = \bigcap_{i\in I} C(Y_i) \text{ and } \bigvee_{i\in I} Y_i = C(\bigcup_{i\in I} Y_i)$$

In view of Proposition 1.3, we will often treat closure systems S as complete lattices and write S as a shorthand for $\langle S, \subseteq\rangle$. Interestingly enough, not only is every closure system a complete lattice, but (up to isomorphism) every complete lattice arises in this way. Consequently, the lattices arising from closure operators provide typical examples of complete lattices.

Theorem 1.4. *[8, Thm. I.5.3] Every complete lattice is isomorphic to a closure system.*

Given a closure operator C on a set X, a closed set Y of C is said to be *finitely generated* when there exists a finite $Z \subseteq X$ such that $Y = C(Z)$. Furthermore, C is said to be *finitary* when for every $Y \cup \{x\} \subseteq X$ such that $x \in C(Y)$ there exists a finite $Z \subseteq Y$ such that $x \in C(Z)$. Consequently, a closure operator is finitary precisely when each of its closed sets is the union of all the finitely generated closed sets contained into it.

Since the notions of a closure operator, closure system and consequence relation are coextensive, it is natural to wonder how to characterize finitary closure operators in terms of the structure of closure systems and consequence relations associated to them.

Given a poset $\mathbb{X}$, a nonempty set $Y \subseteq X$ is said to be *upward directed* in $\mathbb{X}$ when for every $x, y \in Y$ there exists $z \in Y$ such that $x, y \leqslant z$. A closure system S is said to be *inductive* when $\bigcup Y \in S$ for every family $Y \subseteq S$ that is upward directed in $\langle S, \subseteq\rangle$. A consequence relation $\vdash$ is said to be *finitary* when for every $Y \cup \{x\} \subseteq X$, if $Y \vdash x$, then there exists a finite $Z \subseteq Y$ such that $Z \vdash z$.

The following theorem states the connection between these concepts:

Theorem 1.5. *The following conditions are equivalent for a closure operator C on a set X:*

(i) *C is finitary;*

(ii) *The closure system associated with C is inductive;*

(iii) *The consequence relation associated with C is finitary.*

The finitely generated closed sets of a finitary closure operator can be described in purely lattice theoretic terms. To this end, we introduce the following notion: an element x of a complete lattice $\mathbb{X}$ is said to be *compact* if for every $Y \subseteq X$, if $x \leqslant \bigvee Y$, there exists a finite $Z \subseteq Y$ such that $x \leqslant \bigvee Z$. As a consequence of the definition, we have that compact elements are closed under binary joins.

Proposition 1.6. *The set $Comp(X)$ of compact elements of a complete lattice X is closed under binary joins in X and contains the least element of X.*

Recall from Theorem 1.4 that every complete lattice can be represented as a closure system. Therefore, we can consider the compact elements of a closure system. For a finitary closure operator, the finitely generated closed sets and the compact elements of the associated closure system coincide.

Proposition 1.7. *Let C be a finitary closure operator on a set X and S the associated closure system. A closed set $Y \subseteq X$ is finitely generated iff it is a compact element of $\langle S, \subseteq \rangle$.*

Another natural question that may arise from the fact that every complete lattice can be represented as a closure system, is which are the complete lattices isomorphic to the inductive closure systems. Before we describe these lattices, let us first introduce a useful auxiliary notion.

A subset D of a complete lattice X is said to be *join dense* (resp. *meet dense*) when each element of X is a join (resp. a meet) of some subset of D. That is, for each $x \in X$ there exists some $Y \subseteq D$ such that $x = \bigvee Y$ (resp. $x = \bigwedge Y$). The notion of join density admits equivalent formulations. Here, we present one such formulation that is particularly relevant in what follows: a subset D of a complete lattice $\mathbb{X}$ is join dense in $\mathbb{X}$ iff for every $x, y \in X$,

$$x \nleqslant y \iff \text{there exists } d \in D \text{ such that } d \leqslant x \text{ and } d \nleqslant y.$$

A complete lattice X is said to be *algebraic* when the set $Comp(X)$ of compact elements of X is join dense in X, i.e, when each element of X is a join of compact elements of X.

Let $\boldsymbol{A} = \langle A, \vee \rangle$ be a join semilattice with least element 0. An ideal of $\boldsymbol{A}$ is a downset containing 0 such that for every $a, b \in A$, if $a, b \in I$, then $a \vee b \in I$. The set of ideals of A will be denoted by $\mathcal{I}d(\boldsymbol{A})$. Equivalently, the ideals of A are the upward directed downsets of $\langle A, \leqslant_j \rangle$.

Proposition 1.8. *If $\boldsymbol{A}$ is a join semilattice with a least element, then $Id(\boldsymbol{A})$ is an inductive closure system on A.*

Let X be a complete lattice. According to Proposition 1.6, $\langle Comp(X), \vee \rangle$ is a join semilattice, with its least element being the least element of X. Consequently, $\mathcal{I}d(\langle Comp(X), \vee \rangle)$ is an inductive closure system by Proposition 1.8. Therefore, we proceed to formulate a representation theorem that relates algebraic lattices and inductive closure systems as follows:

Theorem 1.9. *Every inductive closure system is an algebraic lattice. Conversely, if $\boldsymbol{A}$ is an algebraic lattice, then $\mathcal{I}d(\langle Comp(X), \vee \rangle)$ is an inductive closure system isomorphic to $\boldsymbol{A}$.*

1.2 Logic

This section follows closely [29]. For a general background on abstract algebraic logic and for the concept of logic we work with we refer the reader to [21]. For the concept of protoalgebraic logic and related topics see [2, 15].

A *propositional logic* $\vdash$ (from now on, simply a *logic*) is a consequence relation on the set Fm of formulas of some algebraic language that, moreover, is *substitution invariant* in the sense that, for every substitution σ and every $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

This condition is also referred to as *structurality*. Among other standard abbreviations, given $\Gamma \cup \Sigma \subseteq Fm$, we write $\Gamma \vdash \Sigma$ when $\Gamma \vdash \varphi$, for all $\varphi \in \Sigma$ and, we write $\Gamma \dashv\vdash \Sigma$ whenever $\Gamma \vdash \Sigma$ and $\Sigma \vdash \Gamma$.

Every logic $\vdash$ can be associated with a closure operator $\mathrm{Cn}_{\vdash} \colon \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ defined, for every $\Gamma \subseteq Fm$, as follows:

$$\mathrm{Cn}_{\vdash}(\Gamma) := \{\varphi \in Fm \colon \Gamma \vdash \varphi\}.$$

Moreover, a set of formulas Γ is said to be a *theory* of $\vdash$ if $\Gamma = \mathrm{Cn}_{\vdash}(\Gamma)$. When ordered under the inclusion relation, the set of theories of $\vdash$ forms a closure system and, therefore, a lattice that we denote by $\mathcal{T}h(\vdash)$.

A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm$. A rule $\Gamma \rhd \varphi$ is said to be *valid* in a logic $\vdash$ when $\Gamma \vdash \varphi$. Given an algebra $\boldsymbol{A}$ in the same language as $\vdash$, a set $F \subseteq A$ is said to be a *deductive filter* of $\vdash$ (or $\vdash$*-filter*) on $\boldsymbol{A}$ when it is closed under the interpretation of the rules valid in $\vdash$, that is, when, for every $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash \varphi$ and every homomorphism $f \colon \boldsymbol{Fm} \to \boldsymbol{A}$,

$$\text{if } f[\Gamma] \subseteq F, \text{ then } f(\varphi) \in F.$$

A *(logical) matrix* is a pair $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an algebra and $F \subseteq A$. A matrix $\langle \boldsymbol{A}, F\rangle$ is *trivial* if $F = A$. Given two matrices $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, G\rangle$, we say that $\langle \boldsymbol{B}, G\rangle$ is a *submatrix* of $\langle \boldsymbol{A}, F\rangle$ if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $G = F \cap B$. In that case, we will use the notation $\langle \boldsymbol{A}, F\rangle \leqslant \langle \boldsymbol{B}, G\rangle$.

A matrix $\langle \boldsymbol{A}, F\rangle$ is said to be a *model* of a logic $\vdash$ when F is a deductive filter of $\vdash$ on $\boldsymbol{A}$. The class of models of $\vdash$ will be denoted by $\mathsf{Mod}(\vdash)$.

Remark 1.10. The class of models of a logic $\vdash$ is closed under submatrices, i.e., if $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and $\langle \boldsymbol{B}, G\rangle \leqslant \langle \boldsymbol{A}, F\rangle$, then $\langle \boldsymbol{B}, G\rangle \in \mathsf{Mod}(\vdash)$. Indeed, suppose that $\Gamma \vdash \varphi$ and consider a homomorphism $f \colon \boldsymbol{Fm} \to \boldsymbol{A}$ such that $f[\Gamma] \subseteq F$. Then $f(\varphi) \in F$ because $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$ and, therefore, F is a $\vdash$-filter on $\boldsymbol{A}$. Hence $f(\varphi) \in G$ because $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $G = F \cap B$.

A *strict homomorphism* from a matrix $\langle \boldsymbol{A}, F\rangle$ to a matrix $\langle \boldsymbol{B}, G\rangle$ is a homomorphism $f : \boldsymbol{A} \to \boldsymbol{B}$ such that, for every $a \in A$,

$$a \in F \iff f(a) \in G.$$

When ordered under the inclusion relation, the set of deductive filters of $\vdash$ on $\boldsymbol{A}$ forms a complete lattice, which we denote by $\mathcal{F}i_{\vdash}(\boldsymbol{A})$. If $\vdash$ is finitary, this lattice is algebraic, so its compact elements are just the finitely generated $\vdash$-filters. Furthermore, we denote the closure operator of deductive filter generation on $\boldsymbol{A}$ by $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash} \colon \mathcal{P}(A) \to \mathcal{P}(A)$. Then the join operation $+^{\boldsymbol{A}}$ of the lattice $\mathcal{F}i_{\vdash}(\boldsymbol{A})$ can be described, for every $F, G \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$, as

$$F +^{\boldsymbol{A}} G = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup G).$$

Proposition 1.11. *[29, Prop. 2.7] The deductive filters of a logic $\vdash$ on $\boldsymbol{Fm}$ coincide with the theories of $\vdash$. Consequently, $\mathcal{T}h(\vdash) = \mathcal{F}i_{\vdash}(\boldsymbol{Fm})$ and $\mathrm{Cn}_{\vdash}(-) = \mathrm{Fg}^{\boldsymbol{Fm}}_{\vdash}(-)$.*

A congruence θ on an algebra $\boldsymbol{A}$ is said to be *compatible* with a subset F of A provided that F is the union of θ-classes, i.e., when, for every $a, b \in A$,

$$\text{if } \langle a, b \rangle \in \theta \text{ and } a \in F, \text{ then } b \in F.$$

Proposition 1.12. *[29, Prop. 2.10] Deductive filters are closed under inverse images of homomorphisms, in the sense that if $f \colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism, $\vdash$ a logic, and $G \in \mathsf{Fi}_{\vdash}(\boldsymbol{B})$, then $f^{-1}[G] \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$. Furthermore, if f is surjective and $\mathsf{ker}(h)$ is compatible with $F \in \mathsf{Fi}_{\vdash}(\boldsymbol{A})$, then $h[F] \in \mathsf{Fi}_{\vdash}(\boldsymbol{B})$.*

Here, as usual, $\mathsf{ker}(h) := \{(a, a') \in A \times A : h(a) = h(a')\}$.

From Propositions 1.11 and 1.12 we deduce:

Corollary 1.13. *[29, Cor. 2.11] If Γ is a theory of a logic $\vdash$ and σ a substitution, then $\sigma^{-1}[\Gamma]$ is also a theory of $\vdash$.*

Protoalgebraic logics were introduced in [15] and [13, 14] and constitute the core of abstract algebraic logic. Their theory is enshrined in the monograph [15].

Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$, the *Leibniz congruence* of F on $\boldsymbol{A}$, in symbols $\boldsymbol{\Omega}^{\boldsymbol{A}} F$, is the largest congruence of $\boldsymbol{A}$ compatible with F. The Leibniz congruence always exists (see, e.g., [29, Prop. 2.13]).

A logic $\vdash$ is said to be *protoalgebraic* if there exists a set $\Delta(x, y, \vec{z})$ of formulas such that, for every model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$ and $a, b \in A$,

$$\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F \iff \Delta^{\boldsymbol{A}}(a, b, \vec{c}) \subseteq F, \text{ for every } \vec{c} \in A.$$

In this case, we say that Δ is a set of *equivalence formulas* for $\vdash$.

Protoalgebraic logics admit an even simpler syntactic description which, however, does not guarantee that $\Delta(x, y)$ is a set of equivalence formulas.

Theorem 1.14. *[29, Thm. 3.8] A logic $\vdash$ is protoalgebraic iff there exists a set $\Delta(x, y)$ of formulas such that*

$$\varnothing \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Remark 1.15. As a consequence of Theorem 1.14, every logic possessing an implication $\to$ such that $\varnothing \vdash x \to x$ and $x, x \to y \vdash y$ is protoalgebraic, as witnessed by the set $\delta(x, y) = \{x \to y\}$. Because of this, most familiar logics are protoalgebraic.

Numerous additional characterizations of protoalgebraicity are known. One noteworthy example is the Correspondence Theorem, which will be used repeatedly in what follows. The Correspondence Theorem of universal algebra [8, Thm. II.6.20] states that if $f : \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then the congruence lattice $Con(\boldsymbol{B})$ is isomorphic to the sublattice of $Con(\boldsymbol{A})$ consisting of the congruences of $\boldsymbol{A}$ that extend $\mathsf{ker}(f)$. The existence of a similar isomorphism characterizes protoalgebraic logics, as we proceed to explain.

Given a logic , an algebra $\boldsymbol{A}$, and a subset $F \subseteq A$, we denote the sublattice of $\mathcal{F}i(\boldsymbol{A})$ consisting of the deductive filters extending F by $\mathcal{F}i(\boldsymbol{A})^F$. When $\boldsymbol{A}$ is $\boldsymbol{Fm}$, we will write $\mathcal{T}h(\vdash)^F$ instead of $\mathcal{F}i(\boldsymbol{A})^F$.

A logic $\vdash$ is said to have the *correspondence property* when for every strict surjective homomorphism $f : \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ between models of $\vdash$, the direct image map

$$f[-] : \mathcal{F}i_{\vdash}(\boldsymbol{A})^F \to \mathcal{F}i_{\vdash}(\boldsymbol{B})^G$$

is a well-defined lattice isomorphism.

Theorem 1.16. *[29, Thm. 3.21] A logic $\vdash$ is protoalgebraic iff it has the correspondence property.*

The following proposition, which will be usefull in what follows, relies on the equivalence between protoalgebraicity and the correspondence property.

Proposition 1.17. *Let $\vdash$ be a protoalgebraic logic. For all $\Gamma \cup \{\alpha, \beta\} \subseteq \boldsymbol{Fm}$,*

$$\Gamma, \alpha \vdash \beta \iff \textit{there is } \Gamma' \in Th(\vdash) \textit{ satisfying } \Gamma', x \vdash y \textit{ and } \Gamma \vdash \sigma\Gamma' \textit{ for a substitution } \sigma \textit{ such that } \sigma x = \alpha \textit{ and } \sigma y = \beta.$$

Proof. Let $\Gamma \cup \{\alpha, \beta\} \subseteq \boldsymbol{Fm}$. The right-to-left implication follows from substitution-invariance regardless of the protoalgebraicity of $\vdash$. To prove the converse implication, let σ be a substitution such that $\sigma x = \alpha$ and $\sigma y = \beta$, and suppose that $\Gamma, \alpha \vdash \beta$. Define $\Gamma' := \sigma^{-1}[\mathrm{Cn}_{\vdash}(\Gamma)]$. It is clear that $\Gamma \vdash \sigma\Gamma'$. Moreover observe that σ is a strict surjective homomorphism between the models $\langle \boldsymbol{Fm}, \Gamma'\rangle$ and $\langle \boldsymbol{Fm}, \mathrm{Cn}_{\vdash}(\Gamma)\rangle$ of $\vdash$. Then, since $\vdash$ is protoalgebraic, it has the correspondence property, which implies that the direct map

$$\sigma[-] : \mathcal{T}h(\vdash)^{\Gamma'} \to \mathcal{T}h(\vdash)^{\mathrm{Cn}_{\vdash}(\Gamma)}$$

is a well-defined order isomorphism.

Therefore $\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Gamma')]$ is a $\vdash$-theory extending Γ because $\mathrm{Cn}_{\vdash}(\{x\} \cup \Gamma') \in \mathcal{T}h(\vdash)^{\Gamma'}$ and $\sigma[-]$ is well-defined. Furthermore, $\sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Gamma')]$ also contains α because $\sigma x = \alpha$. Hence $\mathrm{Cn}_{\vdash}(\{\alpha\} \cup \Gamma) \subseteq \sigma[\mathrm{Cn}_{\vdash}(\{x\} \cup \Gamma')]$, and since the correspondence property guarantees that $\sigma^{-1}\sigma = id$, we get

$$\sigma^{-1}[\mathrm{Cn}_{\vdash}(\{\alpha\} \cup \Gamma)] \subseteq \mathrm{Cn}_{\vdash}(\{x\} \cup \Gamma').$$

Together with the assumption that $\beta \in \mathrm{Cn}_{\vdash}(\{\alpha\} \cup \Gamma)$ and $\sigma y = \beta$ we obtain that $y \in \mathrm{Cn}_{\vdash}(\{x\} \cup \Gamma')$. Thus $\Gamma', x \vdash y$. ⊠

Theorem 1.18. *A logic $\vdash$ is protoalgebraic iff the following is true whenever F and G are $\vdash$-filters of an algebra $\boldsymbol{A}$, and θ is a congruence of $\boldsymbol{A}$: if $F \subseteq G$ and θ is compatible with F, then θ is compatible with G.*

Characterizing the process of filter generation in arbitrary algebras is not a simple task. In the context of finitary logics, the operation of deductive filter generation can be described as follows. Let $\vdash$ be a finitary logic, $\mathcal{A}$ an algebra, and $X \subseteq A$. For every $n \in \mathbb{N}$ we define a set X_n as follows:

$$X_0 := X;$$
$$X_{m+1} := X_m \cup \{a \in A : \text{ there are a finite set of formulas } \Gamma \cup \{\varphi\} \text{ and a homomorphism } h : \boldsymbol{Fm} \to \boldsymbol{A} \text{ such that } \Gamma \vdash \varphi \text{ and } h[\Gamma] \subseteq X_m \text{ and } h(\varphi) = a\}.$$

Then

$$\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(X) = \bigcup_{n \in \mathbb{N}} X_n.$$

Yet, in the protoalgebraic case, the following result is useful.

Lemma 1.19. *[6, Thm. 3.1] Let $\vdash$ be a finitary protoalgebraic logic and let $\boldsymbol{A}$ be an algebra with $X \cup \{a\} \subseteq A$. Then $a \in \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(X)$ iff there exists a finite set of formulas $\Gamma \cup \{\varphi\}$ and a homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $\Gamma \vdash \varphi$ and $h[\Gamma] \subseteq X \cup \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\varnothing)$ and $h(\varphi) = a$.*

To conclude this chapter, we present the class of algebraizable logics, introduced by Blok and Pigozzi [3]. The connection between these logics and their algebraic counterpart goes beyond a mere completeness theorem; it requires a more robust connection as we shall see.

Given a set $\Delta(x, y)$ of formulas and a set Ψ of equations, we shall abbreviate

$$\Delta[\Psi] := \bigcup\{\Delta(\varphi, \psi) : \varphi \approx \psi \in \Psi\}.$$

A finitary logic $\vdash$ is said to be *algebraizable* if there exist a finite set $\tau(x)$ of equations, a finite set $\Delta(x, y)$ of formulas, and a quasivariety K such that

(i) $\Gamma \vdash \varphi \iff \tau[\Gamma] \vDash_{\mathsf{K}} \tau(\varphi)$;

(ii) $\Psi \vDash_{\mathsf{K}} \varepsilon \approx \delta \iff \Delta[\Psi] \vdash \Delta(\varepsilon, \delta)$;

(iii) $\varphi \vdash \Delta[\tau(\varphi)]$ and $\Delta[\tau(\varphi)] \vdash \varphi$;

(iv) $\varepsilon \approx \delta \vDash_{\mathsf{K}} \tau[\Delta(\varepsilon, \delta)]$ and $\tau[\Delta(\varepsilon, \delta)] \vDash_{\mathsf{K}} \varepsilon \approx \delta$.

for every set $\Gamma \cup \{\varphi\}$ and every set $\Psi \cup \{\varepsilon \simeq \delta\}$ of equations.

In this case, K is said to be an *equivalent algebraic semantics* for $\vdash$. In addition, we say that τ, Δ and K witness the algebraization of the logic $\vdash$. The first condition in the definition states that K is a τ-algebraic semantics for $\vdash$, i.e., $\vdash$ can be interpreted into K by means of the set $\tau(x)$ of equations that allows to translate sets Γ of formulas into sets $\tau[\Gamma]$ of equations. Condition (ii) states that this interpretation can be reversed, in the sense that K can also be interpreted into by means of the set $\Delta(x, y)$ of formulas that allows to translate sets ψ of equations into sets $\Delta[\Psi]$ of formulas. Lastly, Conditions (iii) and (iv) require that these two interpretations are inverses of each other up to provability equivalence. Because of this, the definition of an algebraizable logic essentially states that the consequence relations $\vdash$ and $\vDash_{\mathsf{K}}$ are equivalent, as witnessed by the translations $\tau(x)$ and $\Delta(x, y)$.

In view of Proposition 1.12, given a logic $\vdash$ and an algebra $\boldsymbol{A}$, the lattice $\mathcal{F}i_{\vdash}(\boldsymbol{A})$ of deductive filters of $\vdash$ on $\boldsymbol{A}$ can be expanded with the unary operation $\{\sigma^{-1} : \sigma \in \text{End}(\boldsymbol{A})\}$. Similarly, given a quasivariety K, the lattice $Con_{\mathsf{K}}(\boldsymbol{A})$ of K-congruences of $\boldsymbol{A}$ can also be expanded with the unary operations $\{\sigma^{-1} : \sigma \in \text{End}(\boldsymbol{A})\}$. Thus, we define

$$\mathcal{F}i_{\vdash}(\boldsymbol{A})^{+} := \langle \mathcal{F}i_{\vdash}(\boldsymbol{A}); \wedge, \vee, \{\sigma^{-1} : \sigma \in \text{End}(\boldsymbol{A})\}\rangle$$
$$Con_{\mathsf{K}}(\boldsymbol{A})^{+} := \langle Con_{\mathsf{K}}(\boldsymbol{A}); \wedge, \vee, \{\sigma^{-1} : \sigma \in \text{End}(\boldsymbol{A})\}\rangle$$

The above structures can be viewed as algebras whose type consists of two binary operations, $\wedge$ and $\vee$, along with a set of unary operations $\{\sigma^{-1} : \sigma \in \text{End}(\boldsymbol{A})\}$. Accordingly, an isomorphism from $\mathcal{F}i_{\vdash}(\boldsymbol{A})^{+}$ to $Con_{\mathsf{K}}(\boldsymbol{A})^{+}$ is an isomorphism between the underlying lattice structures $h : \mathcal{F}i_{\vdash}(\boldsymbol{A}) \to Con_{\mathsf{K}}(\boldsymbol{A})$ that additionally commutes with inverse endomorphisms, in the sense that

$$h(\sigma^{-1}[F]) = \sigma^{-1}[h(F)], \text{ for every } \sigma \in \text{End}(\boldsymbol{A}).$$

Theorem 1.20. *The following conditions are equivalent for a finitary logic $\vdash$ and a quasivariety K:*

(i) *$\vdash$ is algebraizable with equivalent algebraic semantics K;*

(ii) *$\mathcal{T}h(\vdash)^{+} \cong Con_{\mathsf{K}}(\boldsymbol{Fm})^{+}$;*

(iii) *$\mathcal{F}i_{\vdash}(\boldsymbol{A})^{+} \cong Con_{\mathsf{K}}(\boldsymbol{A})^{+}$, for every algebra $\boldsymbol{A}$ of the suitable type.*

In view of the implication $(i) \Rightarrow (ii)$ in the Isomorphism Theorem, every algebraizable logic induces an isomorphism between lattices of deductive filters and of K-congruences.

CHAPTER 2

Inconsistency lemmas

In this chapter we formulate an abstract account of the global and local inconsistency lemma, and investigate their algebraic counterparts, previously presented in [30, Thm. 3.7] and [26, Thm. 6.35], respectively. Moreover, at the end of the chapter we introduce a novel notion, that of *(first order) definable maximal consistent filters*. The main contribution of this chapter – Theorem 2.15 – establishes that for a finitary protoalgebraic logic $\vdash$ with a local inconsistency lemma and definable maximal consistent filters, any family Ψ_n witnessing the LIL must include a finite subset of sets of formulas for each $n \in \mathbb{Z}^+$ such that the resulting family also witnesses the LIL for $\vdash$. This represents a significant improvement towards understanding what is necessary for a LIL to reduce to an IL.

Given a logic $\vdash$, a set Ξ of formulas is said to be *inconsistent in* $\vdash$ if $\Xi \vdash \alpha$ for all $\alpha \in Fm$.

Remark 2.1. Notice that if a logic $\vdash$ possesses a finite and inconsistent set of formulas Ξ, then $\Xi \dashv\vdash h[\Xi]$ for every substitution h. Indeed, consider an arbitrary substitution h and let $x_1, \ldots, x_n$ be the variables occuring in the formulas of Ξ. Let also $y_1, \ldots, y_n$ be variables different from $x_1, \ldots, x_n$. Since Ξ is inconsistent in $\vdash$, we have that $\Xi \vdash \xi(y_1, \ldots, y_n)$ for each $\xi(y_1, \ldots, y_n) \in \Xi$. Now, as Ξ is finite we can define another substitution h' such that it agrees with h on the variables $x_1, \ldots, x_n$ and $h'(y_i) = x_i$ for $i \in \{1, \ldots, n\}$, while it sends any other variable to x_1. Therefore, $h'[\Xi] \vdash h'(\xi(y_1, \ldots, y_n))$ which implies that $h[\Xi] \vdash \xi(x_1, \ldots, x_n)$. Hence, $h[\Xi] \vdash \Xi$. Moreover, notice that it follows from the inconsistency of Ξ that $\Xi \vdash h[\Xi]$.

Using this fact we can prove the equivalence of the following conditions, which will be useful in the proof of the forthcoming results.

Proposition 2.2. *The following statements are equivalent for a logic* $\vdash$*:*

(i) $\vdash$ *possesses some finite inconsistent set of formulas.*

(ii) *There is some finite set of formulas* $\Xi(x)$ *such that* $A = \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(\Xi^{\boldsymbol{A}}(a))$ *for every algebra* $\boldsymbol{A}$ *and every* $a \in A$*.*

(iii) $\vdash$ *has a greatest compact theory.*

Proof. (i)⇒(ii): First suppose that Ξ is a finite set of formulas inconsistent in $\vdash$. Without loss of generality we may assume that Ξ consists of formulas in just one variable x. Otherwise, we can take the substitution h that maps every variable to x. Then, Remark 2.1 guarantees that $h[\Xi] \vdash \Xi$, making $h[\Xi]$ a set of formulas in variable x only, finite, and inconsistent in $\vdash$. Therefore, if we consider an arbitrary algebra $\boldsymbol{A}$ and $a \in A$, the inconsistency of $\Xi(x)$ implies that $b \in \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi(a))$ for every $b \in A$, so it must be the case that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$.

(ii)⇒(iii): Assume that there is some finite set of formulas $\Xi(x)$ such that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$ for every algebra $\boldsymbol{A}$ and every $a \in A$. Then, in particular for $\boldsymbol{A} = \boldsymbol{Fm}$, the statement amounts to $Fm = \mathrm{Cn}_{\vdash}(\Xi(x))$. Thus, Fm is a compact theory, and, therefore, $\vdash$ has a greatest compact theory.

(iii)⇒(i): Assume that $\vdash$ has a greatest compact theory, say Γ. This means that there is some finite set Ξ of formulas such that $\Gamma = \mathrm{Cn}_{\vdash}(\Xi)$ and $\mathrm{Cn}_{\vdash}(\Delta) \subseteq \Gamma$ for every finite $\Delta \subseteq Fm$. Since $Fm = \bigcup\{\mathrm{Cn}_{\vdash}(\Delta) : \Delta \subseteq Fm \text{ and } \Delta \text{ is finite}\}$, it must be the case that $Fm = \mathrm{Cn}_{\vdash}(\Xi)$. Therefore, Ξ is finite and inconsistent in $\vdash$. ⊠

In the case of classical and intuitionistic propositional logic, the theory Fm is compact. Among the finite sets that generate the theory Fm we find $\Xi = \{x, \neg x\}$ and also $\Xi = \{\bot\}$.

A logic $\vdash$ is said to have an *inconsistency lemma* – briefly an IL – if, for every positive integer n, there exists a finite set of formulas $\Psi_n(x_1, \ldots, x_n)$ such that for every $\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \subseteq Fm$,

$$\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \text{ is inconsistent in } \vdash \iff \Gamma \vdash \Psi_n(\alpha_1, \ldots, \alpha_n).$$

Following the terminology introduced in [30], the family $\{\Psi_n : n \in \mathbb{Z}^+\}$ is called an *elementary IL-sequence*. Observe that the convention $n \in \mathbb{Z}^+$ is significant. Without it, the constant-free formulation of classical logic would lack an inconsistency lemma. Indeed, if we take $0 \in \mathbb{N}$, the set Ψ_0 cannot exist since we do not have any constant in the language.

Moreover, the definition of an inconsistency lemma could be streamlined in the case of logics with a binary connective $\wedge$ such that $x, y \dashv\vdash x \wedge y$. However, not all logics with an IL have such connective. For example, the family

$$\{\{x_1 \to (x_2 \to (\ldots \to (x_n \to \bot)\ldots))\} : n \in \mathbb{Z}^+\}$$

witnesses an IL for the $\to, \bot$ fragment of intuitionistic logic.

A logic $\vdash$ is said to have a *local inconsistency lemma* – briefly a LIL – if for every positive integer n, there exists a family Ψ_n, possibly infinite, of finite sets of formulas $I(x_1, \ldots, x_n)$ such that for every $\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \subseteq Fm$,

$$\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \text{ is inconsistent in } \vdash \iff \Gamma \vdash I(\alpha_1, \ldots, \alpha_n) \text{ for some } I \in \Psi_n.$$

If the family Ψ_n witnessing the LIL consists of just one finite set of formulas $I(x_1, \ldots, x_n)$ for each $n \in \mathbb{Z}^+$, then the LIL reduces to the IL.

Remark 2.3. Let $\vdash$ be a finitary logic with the LIL witnessed by the family $\{\Psi_n : n \in \mathbb{Z}^+\}$. Then $\vdash$ possesses some finite inconsistent set of formulas, or equivalently, there is some finite set of formulas $\Xi(x)$ such that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$ for every algebra $\boldsymbol{A}$ and every $a \in A$. Indeed, if $\{\Psi_n : n \in \mathbb{Z}^+\}$ witnesses the LIL for $\vdash$, then it follows from the definition of the LIL that $I(x_1, \ldots, x_n) \cup \{x_1, \ldots, x_n\}$ is finite and inconsistent in $\vdash$ for every $I \in \Psi_n$ because $I(x_1, \ldots, x_n) \vdash I(x_1, \ldots, x_n)$. The same reasoning is valid for every $I(x) \in \Psi_1$, so we may assume that there is some finite set of formulas $\Xi(x)$ inconsistent in $\vdash$. This is because, if we consider an arbitrary algebra $\boldsymbol{A}$ and $a \in A$, the inconsistency of $\Xi(x)$ implies that $b \in \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$ for every $b \in A$. Therefore, it must be the case that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$. Conversely, if we assume that there is some finite set of formulas $\Xi(x)$ such that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$ for every algebra $\boldsymbol{A}$ and every $a \in A$, in particular for $\boldsymbol{A} = \boldsymbol{Fm}$, the statement amounts to $Fm = \mathrm{Cn}_{\vdash}(\Xi(x))$ for some finite set of formulas $\Xi(x)$. Thus, $\Xi(x)$ is finite and inconsistent in $\vdash$.

2.1 The transfer of inconsistency lemmas

Observe that if the family $\{\Psi_n : n \in \mathbb{Z}^+\}$ witnesses the LIL for a logic $\vdash$, then it is the case that for every $\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \subseteq Fm$,

$$Fm = \mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, \ldots, \alpha_n\}) \iff I(\alpha_1, \ldots, \alpha_n) \subseteq \mathrm{Cn}_{\vdash}(\Gamma) \text{ for some } I \in \Psi_n,$$

or equivalently, since the deductive filters of the algebra of formulas are the $\vdash$-theories, that for any $\vdash$-filter F of the algebra of formulas $\boldsymbol{Fm}$ and any $\alpha_1, \ldots, \alpha_n \in Fm$,

$$Fm = F + \mathrm{Fg}_{\vdash}(\{\alpha_1, \ldots, \alpha_n\}) \iff I(\alpha_1, \ldots, \alpha_n) \subseteq F \text{ for some } I \in \Psi_n.$$

The next result extends this equivalence from $\boldsymbol{Fm}$ to arbitrary algebras. It is the first example we present of the so-called transfer theorems: theorems which transfer a given property of a logic $\vdash$ (understood as the closure operator/system over the set of formulas) to the analogous property of the closure operator/system of all $\vdash$-filters over any algebra.

While this result can be found already in [30, Thm. 3.6], it was originally presented for the particular case of a global inconsistency lemma. Here, we adapt it for the local version of the inconsistency lemma. Its proof relies on the protoalgebraicity of $\vdash$. A different proof is provided in [26, Thm. 6.32] for the transfer of the LIL in the more general framework of compact protonegational logics.

Theorem 2.4. *Let $\vdash$ be a finitary protoalgebraic logic with the LIL witnessed by the family $\{\Psi_n : n \in \mathbb{Z}^+\}$. Then for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and elements $a_1, \ldots, a_n \in A$,*

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\}) \iff I^{\boldsymbol{A}}(a_1, \ldots, a_n) \subseteq F \text{ for some } I \in \Psi_n.$$

Proof. Let $\langle \boldsymbol{A}, F\rangle$ be a model of $\vdash$ and $a_1, \ldots, a_n \in A$. First observe that it follows from the LIL that $I(x_1, \ldots, x_n) \cup \{x_1, \ldots, x_n\}$ is finite and inconsistent in $\vdash$ for every $I \in \Psi_n$ because $I(x_1, \ldots, x_n) \vdash I(x_1, \ldots, x_n)$. Since infinitely many variables are at our disposal, this implies that $I(x_1, \ldots, x_n) \cup \{x_1, \ldots, x_n\} \vdash y$ for some variable y different from $x_1, \ldots, x_n$. Therefore, if we assume that $I^{\boldsymbol{A}}(a_1, \ldots, a_n) \subseteq F$ for some $I \in \Psi_n$, then $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\})$.

It remains to prove the forward implication in the statement. To this end, let $a \in A$ and observe that the assumption that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\})$ amounts to

$\Xi^{A}(a) \subseteq \mathrm{Fg}^{A}_{\vdash}(F \cup \{a_1, \ldots, a_n\})$ for some finite inconsistent set of formulas $\Xi(x)$ and $a \in A$. The existence of such a set is guaranteed by the LIL, as seen in Remark 2.3. It then follows from the assumption that $\vdash$ is finitary and protoalgebraic together with Lemma 1.19 that for each $\xi \in \Xi$ there exists a finite set of formulas $\Gamma_\xi \cup \{\varphi_\xi\}$ and a homomorphism $h_\xi : \boldsymbol{Fm} \to \boldsymbol{A}$ such that

$$\Gamma_\xi \vdash \varphi_\xi \text{ and } h_\xi[\Gamma_\xi] \subseteq F \cup \{a_1, \ldots, a_n\} \text{ and } h_\xi(\varphi_\xi) = \xi^{A}(a).$$

Then, by substitution-invariance, we may assume that for distinct $\xi, \xi' \in \Xi$ the sets of variables occurring in $\Gamma_\xi \cup \{\varphi_\xi\}$ and $\Gamma_{\xi'} \cup \{\varphi_{\xi'}\}$ are pairwise disjoint and that all these variables are among $y_1, y_2, y_3, \ldots$ Consequently, we can consider a homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that for every $\xi \in \Xi$ it acts as h_ξ on the variables that occur in $\Gamma_\xi \cup \{\varphi_\xi\}$, and $h(x_i) = a_i$ for $i \in \{1, \ldots, n\}$ and $h(x_{n+1}) = a$. Therefore we get

$$\bigcup_{\xi \in \Xi} \Gamma_\xi \vdash \{\varphi_\xi : \xi \in \Xi\} \text{ and } h[\bigcup_{\xi \in \Xi} \Gamma_\xi] \subseteq F \cup \{a_1, \ldots, a_n\}.$$

Moreover, the protoalgebraicity of $\vdash$ amounts to the existence of a set $\Delta(x, y)$ of formulas such that $\varnothing \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Hence, from $x, \Delta(x, y) \vdash y$ we get that

$$\Delta(\varphi_\xi, \xi(x_{n+1})), \varphi_\xi \vdash \xi(x_{n+1}) \text{ for each } \xi \in \Xi,$$

and it follows from $\varnothing \vdash \Delta(x, x)$ that for every $\delta \in \Delta$ and $\xi \in \Xi$,

$$h(\delta(\varphi_\xi, \xi(x_{n+1})) = \delta^{A}(h(\varphi_\xi), h(\xi(x_{n+1}))) = \delta^{A}(\xi^{A}(a), \xi^{A}(a)) \in F.$$

So, if

$$\Gamma := \bigcup_{\xi \in \Xi} \Gamma_\xi \cup \bigcup \Delta(\varphi_\xi, \xi(x_{n+1})) \cup \{x_1, \ldots, x_n\},$$

then $\Gamma \vdash \Xi(x_{n+1})$ and $h[\Gamma] \subseteq F \cup \{a_1, \ldots, a_n\}$.

Observe that we can write $\Gamma = \Gamma' \cup \Pi_1 \cup \ldots \cup \Pi_n$, say, where $h[\Gamma'] \subseteq F$ and $x_i \in \Pi_i$ and $h[\Pi_i] = \{a_i\}$ for $i \in \{1, \ldots, n\}$. We may assume without loss of generality that $\Pi_i = \{x_i\}$. Indeed, $h[\Gamma'] \subseteq F$ remains true if for every $i \in \{1, \ldots, n\}$ we add to Γ' the formulas $\delta(x_i, \alpha)$ for every $\delta \in \Delta$ and $x_i \neq \alpha \in \Pi_i$, since $h(\delta(x_i, \alpha)) = \delta(h(\alpha), h(\alpha)) \in F$ because $\varnothing \vdash \Delta(x, x)$. Furthermore, for each $i \in \{1, \ldots, n\}$ and each $\alpha \in \Pi_i$, we also have that $\Delta(x_i, \alpha), x_i \vdash \alpha$ (because $x, \Delta(x, y) \vdash y$) and $h(x_i) = h(\alpha)$. Thus, by extending Γ' in this way, we obtain a set Γ^* such that $h[\Gamma^*] \subseteq F$ and $\Gamma^* \cup \{x_1, \ldots, x_n\} \vdash \Xi(x_{n+1})$.

In other words, we may assume that $\Gamma = \{\gamma_1, \ldots, \gamma_m, x_1, \ldots, x_n\}$ where h sends $\gamma_1, \ldots, \gamma_m$ into F and $x_1, \ldots, x_n$ into $\{a_1, \ldots, a_n\}$. Then, since Γ is inconsistent in $\vdash$ because $\Gamma \vdash \Xi(x_{n+1})$, the LIL implies that $\gamma_1, \ldots, \gamma_m \vdash I(x_1, \ldots, x_n)$ for some $I \in \Psi_n$. Together with the fact that $h[\{\gamma_1, \ldots, \gamma_m\}] \subseteq F$, this yields that

$$I^{A}(a_1, \ldots, a_n) = h[I(x_1, \ldots, x_n)] \subseteq F.$$

⊠

As a corollary we obtain the analogous result for the inconsistency lemma.

Theorem 2.5. *[30, Thm. 3.6] Let $\vdash$ be a finitary protoalgebraic logic with the IL witnessed by the family $\{\Psi_n : n \in \mathbb{Z}^+\}$. Then for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and elements $a_1, \ldots, a_n \in A$,*

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\}) \iff \Psi^{\boldsymbol{A}}_n(a_1, \ldots, a_n) \subseteq F.$$

The transfer of inconsistency lemmas, Theorems 2.4 and 2.5 allow us to prove semantic characterizations for both local and global inconsistency lemmas.

2.2 The algebraic counterpart of the global inconsistency lemma

We shall first focus on the algebraic counterpart of the global inconsistency lemma. We will prove that a finitary protoalgebraic logic has an inconsistency lemma if and only if its join semilattice of compact theories is *dually pseudo-complemented*.

A join semilattice $\boldsymbol{A}$ is said to be *dually pseudo-complemented* if it has a greatest element 1 and for each $a \in A$, there exists an element $a^* \in A$ such that for every $b \in A$,

$$a \vee b = 1 \iff a^* \leqslant b.$$

It follows from the definition that a join semilattice $\boldsymbol{A}$ is dually pseudo-complemented if it has a maximum element 1 and for each $a \in A$ there exists the smallest $b \in A$ such that $a \vee b = 1$, in which case b is taken to be a^*. Consequently, every dually pseudo-complemented semilattice has a minimum element, namely $0 := 1^*$.

Clearly, the compact $\vdash$-filters of an algebra $\boldsymbol{A}$ form a join semilattice with minimum element 0 under the operation $+^{\boldsymbol{A}}$, and the inclusion as the join order. Therefore, the connection between an IL and dual pseudo-complements, suggested by Theorem 2.5, is as follows.

Theorem 2.6. *[30, Thm. 3.7] Let $\vdash$ be a finitary protoalgebraic logic. Then the following conditions are equivalent.*

(i) *$\vdash$ has an inconsistency lemma.*

(ii) *For every algebra $\boldsymbol{A}$, the compact $\vdash$-filters of $\boldsymbol{A}$ form a dually pseudo-complemented semilattice with respect to $+^{\boldsymbol{A}}$.*

(iii) *The join semilattice of compact $\vdash$-theories is dually pseudo-complemented.*

Proof. (i)⇒(ii): Observe that A is compact by Remark 2.3. Therefore, A is the maximum element of the semilattice of compact deductive filters of $\boldsymbol{A}$. Furthermore, recall from Proposition 1.7 that the deductive filters of the form $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \ldots, a_n\})$ for arbitrary elements $a_1, \ldots, a_n$ of an algebra $\boldsymbol{A}$ are exactly the compact elements of the lattice of $\vdash$-filters on $\boldsymbol{A}$. Therefore, we will prove that for any $n \in \mathbb{Z}^+$ and any elements $a_1, \ldots, a_n$ of an algebra $\boldsymbol{A}$, we have

$$(\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \ldots, a_n\}))^* = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Psi^{\boldsymbol{A}}_n(a_1, \ldots, a_n)).$$

To this end, let $\boldsymbol{A}$ be an algebra and $a_1, \ldots, a_n \in A$. By assumption there exists a family $\{\Psi_n : n \in \mathbb{Z}^+\}$ witnessing the IL for $\vdash$. From Theorem 2.5 it follows that for any $\vdash$-filter F of $\boldsymbol{A}$

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\}) \iff \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}\Psi^{\boldsymbol{A}}_n(a_1, \ldots, a_n) \subseteq F.$$

In particular, this equivalence implies that

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \dots, a_n\}) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}\Psi^{\boldsymbol{A}}_n(a_1, \dots, a_n).$$

Additionally, observe that under the given assumption of Ψ_n being finite, we have that $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}\Psi^{\boldsymbol{A}}_n(a_1, \dots, a_n)$ is compact. Consequently, $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}\Psi^{\boldsymbol{A}}_n(a_1, \dots, a_n)$ is equal to the dual pseudocomplement of $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \dots, a_n\})$.

(ii)⇒(iii): It is clear that if for every algebra $\boldsymbol{A}$, the compact $\vdash$-filters of $\boldsymbol{A}$ form a dually pseudo-complemented semilattice with respect to $+^{\boldsymbol{A}}$, then for the particular case of $\boldsymbol{A}$ being the algebra of formulas $\boldsymbol{Fm}$, the join semilattice of compact $\vdash$-theories is dually pseudo-complemented.

(iii)⇒(i): Let $n \in \mathbb{Z}^+$ and consider a finite set $\Psi'_n \subseteq Fm$ such that

$$\mathrm{Cn}_{\vdash}\Psi'_n = (\mathrm{Cn}_{\vdash}(x_1, \dots, x_n))^*$$

in the semilattice of compact $\vdash$-theories. This choice is enabled by the assumption that the join semilattice of compact $\vdash$-theories is dually pseudo-complemented. We can also deduce from this assumption that Fm is a compact $\vdash$-theory, or alternatively, that there is some finite $\Xi(x)$ inconsistent in $\vdash$ in view of Proposition 2.2.

Consequently, for any $\vdash$-theory Γ, regardless whether it is compact or not, the following equivalence holds:

$$Fm = \Gamma + \mathrm{Cn}_{\vdash}(x_1, \dots, x_n) \iff \mathrm{Cn}_{\vdash}\Psi'_n \subseteq \Gamma, \tag{2.1}$$

This is due to the compactness of Fm and the fact that Γ is a join of compact elements of the lattice of all $\vdash$-theories, because this lattice is algebraic.

We define $\Psi_n := g[\Psi'_n]$, where g is a substitution that fixes $x_1, \dots, x_n$ and sends all other variables to x_1. Then Ψ_n is a finite subset of Fm where only variables $x_1, \dots, x_n$ occur. Moreover, given $\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \subseteq Fm$, let h be a surjective substitution that sends x_i to α_i for $i \in \{1, \dots, n\}$ and that sends to α_1 all other variables occurring in Ψ'_n. This substitution exists because Ψ'_n is finite. Then $h[\Psi'_n] = \Psi_n(\alpha_1, \dots, \alpha_n)$ and $\Gamma = h[h^{-1}[\Gamma]]$ because h is surjective.

We need to show that $\Gamma \cup \{\alpha_1, \dots, \alpha_n\}$ is inconsistent in $\vdash$ if and only if $\Gamma \vdash \Psi_n(\alpha_1, \dots, \alpha_n)$. Proposition 1.12 implies that $h^{-1}[\mathrm{Cn}_{\vdash}\Gamma]$ and $h^{-1}[\mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, \dots, \alpha_n\})]$ are $\vdash$-theories. Moreover, $\mathsf{ker}(h)$ is compatible with $h^{-1}[\mathrm{Cn}_{\vdash}\Gamma]$. This is because if $\langle \alpha, \beta \rangle \in \mathsf{ker}(h)$, i.e. $h(\alpha) = h(\beta)$, and $\alpha \in h^{-1}[\mathrm{Cn}_{\vdash}\Gamma]$, then $h(\alpha) = h(\beta) \in \mathrm{Cn}_{\vdash}\Gamma$, so $\beta \in h^{-1}[\mathrm{Cn}_{\vdash}\Gamma]$. Now, the protoalgebraicity of $\vdash$ together with Theorem 1.18 imply that $\mathsf{ker}(h)$ is also compatible with the larger theory

$$Y := h^{-1}[\mathrm{Cn}_{\vdash}\Gamma] + \mathrm{Cn}_{\vdash}(x_1, \dots, x_n) = \mathrm{Cn}_{\vdash}(h^{-1}[\mathrm{Cn}_{\vdash}\Gamma] \cup \{x_1, \dots, x_n\}).$$

Therefore, $h[Y]$ is a theory, by Proposition 1.12. It follows that $h[Y]$ extends $\mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, \dots, \alpha_n\})$, because it contains

$$h[h^{-1}[\Gamma] \cup \{x_1, \dots, x_n\}] = \Gamma \cup \{\alpha_1, \dots, \alpha_n\}.$$

On the other hand, it is clear that $Y \subseteq h^{-1}[\mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, \dots, \alpha_n\})]$ because $h(x_i) = \alpha_i$ for each $i \leqslant n$. Hence

$$h[Y] = \mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, \dots, \alpha_n\}).$$

Finally, we can finish the proof of the theorem with the following equivalences.

$$\begin{aligned}
\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \text{ is inconsistent in } \vdash &\iff \Gamma, \alpha_1, \ldots, \alpha_n \vdash \Xi \\
&\iff \Gamma, \alpha_1, \ldots, \alpha_n \vdash h[\Xi] \\
&\iff h[\Xi] \subseteq h[Y] \\
&\iff \Xi \subseteq Y \\
&\iff Fm = h^{-1}[\mathrm{Cn}_{\vdash}\Gamma] + \mathrm{Cn}_{\vdash}(x_1, \ldots, x_n) \\
&\iff \Psi'_n \subseteq h^{-1}[\mathrm{Cn}_{\vdash}\Gamma] \\
&\iff h[\Psi'_n] \subseteq \mathrm{Cn}_{\vdash}\Gamma \\
&\iff \Gamma \vdash h[\Psi'_n] \\
&\iff \Gamma \vdash \Psi_n(\alpha_1, \ldots, \alpha_n).
\end{aligned}$$

The first and second equivalences follow from the fact that Ξ is inconsistent in $\vdash$ together with Remark 2.1, that is, $\Xi \dashv\vdash h[\Xi]$. The third is a consequence of $h[Y] = \mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1, \ldots, \alpha_n\})$. The forth holds because, as mentioned above, $\mathsf{ker}(h)$ is compatible with Y. The fifth one by the inconsistency of Ξ and the definition of Y. The sixth by Condition (2.1). The last three are staightforward. ⊠

Given that for a finite algebra $\boldsymbol{A}$, its compact $\vdash$-filters and its $\vdash$ filters coincide, the previous theorem together with the fact that the $\vdash$-filters of $\boldsymbol{A}$ form a lattice, yields the following result.

Corollary 2.7. *[30, Cor. 3.8] If a finitary protoalgebraic logic $\vdash$ has an inconsistency lemma, then every finite algebra has a dually pseudo-complemented lattice of $\vdash$-filters.*

It is worth recalling at this point the parallelism with the theory of the deduction-detachment theorem. A logic $\vdash$ is said to have a *deduction-detachment theorem* – briefly a DDT – if there exists a set of formulas $I(x, y)$ such that for every $\Gamma \cup \{\alpha, \beta\} \subseteq Fm$,

$$\Gamma \cup \{\alpha\} \vdash \beta \iff \Gamma \vdash I(\alpha, \beta).$$

A finitary protoalgebraic logic has a deduction-detachment theorem –briefly a DDT– if and only if the join semilattice of compact deductive filters of every algebra of the corresponding type is dually Brouwerian (see, e.g., [13, 25]). Moreover, a dually Brouwerian join semilattice is dually pseudo-complemented if it has a greatest element. Crossing back over the bridge, the next result follows from Theorem 2.6.

Corollary 2.8. *[30, Thm. 3.9] If a logic with a greatest compact theory has a deduction-detachment theorem, then it has an inconsistency lemma.*

Given a quasivariety K and an algebra $\boldsymbol{A}$ of the same type, the K-*congruences* of $\boldsymbol{A}$ are the congruences θ such that $\boldsymbol{A}/\theta \in \mathsf{K}$. They form an algebraic closure system over $\mathsf{K} \times \mathsf{K}$, and therefore, an algebraic lattice, ordered by inclusion. For every quasivariety K, the K–congruence lattices of $\boldsymbol{A}$ and $\boldsymbol{A}/Cg^{\boldsymbol{A}}_{\mathsf{K}}\varnothing$ are isomorphic, where $Cg^{\boldsymbol{A}}_{\mathsf{K}}Y$ denotes the least K–congruence of $\boldsymbol{A}$ containing a subset Y of $A \times A$. Since $\boldsymbol{A}/Cg^{\boldsymbol{A}}_{\mathsf{K}}\varnothing \in \mathsf{K}$, the following conclusion can be drawn from Theorem 2.6.

Theorem 2.9. *[30, Thm. 3.7] Let K be a quasivariety that algebraizes a finitary logic $\vdash$. Then the following conditions are equivalent.*

(i) $\vdash$ *has an inconsistency lemma.*

(ii) *For every algebra $\boldsymbol{A}$, the join semilattice of compact* K*–congruences of $\boldsymbol{A}$ is dually pseudo-complemented.*

(iii) *For every $\boldsymbol{A} \in \mathsf{K}$, the join semilattice of compact* K*–congruences of $\boldsymbol{A}$ is dually pseudo-complemented.*

Observe that if moreover K is a variety, then $\vdash$ has an inconsistency lemma iff every algebra in K has a dually pseudo-complemented join semilattice of compact congruences.

2.3 The algebraic counterpart of the local inconsistency lemma

We shall now focus on the algebraic counterpart of the local inconsistency lemma.

Given a logic $\vdash$, a $\vdash$-theory is said to be *maximal consistent* if it is maximal non-trivial in $\mathcal{T}h(\vdash)$. Similarly, given an algebra $\boldsymbol{A}$, a deductive filter of $\vdash$ on $\boldsymbol{A}$ is *maximal consistent* if it is maximal non-trivial in $\mathcal{F}i(\vdash)$, i.e., if F is maximal and $F \neq A$. We denote the collection of maximal consistent filters as $\mathcal{M}ax\mathcal{F}i_{\vdash}(\boldsymbol{A})$. Following the terminology introduced in [26], these filters are called *simple*. The motivation to call these theories and filters simple comes from universal algebra, where an algebra is called simple provided $Con(A)$ has two elements; however, in this thesis, we opt for the self-explanatory term "maximal consistent".

A logic $\vdash$ has the *maximal consistent filter extension property* (MCFEP, for short) if for every model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$ and every submatrix $\langle \boldsymbol{B}, G \rangle$ of $\langle \boldsymbol{A}, F \rangle$ and every $H \in \mathcal{M}ax\mathcal{F}i_{\vdash}(\boldsymbol{B})$ with $G \subseteq H$ there is $H' \in Fi_{\vdash}(\boldsymbol{A})$ with $F \subseteq H'$ such that $H = H' \cap B$. Note that in the definition we do not require H' to be maximal.

Lemma 2.10. *[7, Lem. 36] If a logic $\vdash$ has a finite inconsistent set of formulas, then the class of its non-trivial models is closed under submatrices.*

Proof. Let $\Xi(x)$ be a finite set of formulas inconsistent in $\vdash$. Observe that since $\Xi(x)$ is finite, we may assume that Ξ is in variable x only. Let $\langle \boldsymbol{B}, G \rangle$ be a submatrix of a non-trivial model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$. As the class of arbitrary models of $\vdash$ is closed under submatrices, then $\langle \boldsymbol{B}, G \rangle$ is also a model of $\vdash$. Suppose towards a contradiction that the matrix $\langle \boldsymbol{B}, G \rangle$ is trivial, that is, $G = B$. Since $\langle \boldsymbol{A}, F \rangle$ is non-trivial by assumption, there is some element $a \in A \setminus F$. Consider a homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h(x) = b$ for some $b \in B$ and $h(y) = a$. Then we have that

$$h[\Xi(x)] = \Xi^{\boldsymbol{A}}(h(x)) = \Xi^{\boldsymbol{A}}(b) = \Xi^{\boldsymbol{B}}(b) \subseteq B = G \subseteq F.$$

The first equality above follows from the fact that h is a homomorphism, the second from the definition of h, and the third equality and last inclusion follow from the assumption that $\langle \boldsymbol{B}, G \rangle$ is a submatrix of $\langle \boldsymbol{A}, F \rangle$, which by definition implies that $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and that $B = B \cap F$. The inclusion $\Xi^{\boldsymbol{B}}(b) \subseteq B$ is straightforward, while the equality $B = G$ holds by the assumption that $\langle \boldsymbol{B}, G \rangle$ is trivial.

Therefore, it follows from $h[\Xi(x)] \subseteq F$ together with the fact that $\Xi(x) \vdash y$ that $a = h(y) \in F$, which is a contradiction. ⊠

Lemma 2.11. *If a finitary logic $\vdash$ has a finite inconsistent set of formulas, then every non-trivial $\vdash$-filter is contained in some maximal non-trivial $\vdash$-filter.*

Proof. Let $\boldsymbol{A}$ be an algebra and F a non trivial $\vdash$-filter on A. Clearly, every maximal element in the poset $\mathbb{X}$ with universe

$$\{F^+ \in \mathcal{F}i_{\vdash}(\boldsymbol{A}) \,:\, F \subseteq F^+ \text{ and } F^+ \neq A\}$$

ordered by the inclusion relation is a maximal non-trivial $\vdash$-filter on $\boldsymbol{A}$ extending F.

We will use Zorn's Lemma to establish the existence of such a maximal element. By assumption, F is non-trivial, which implies that $F \in \mathbb{X}$, hence $\mathbb{X}$ is non-empty, so it suffices to prove that every chain in $\mathbb{X}$ has an upperbound in $\mathbb{X}$. Accordingly, consider a chain $\{F_i \,:\, i \in I\}$ in $\mathbb{X}$. First observe that $\mathcal{F}i_{\vdash}(\boldsymbol{A})$ is an inductive closure system. This is a consequence of Theorem 1.5 together with the assumption that $\vdash$ is a finitary consequence relation. Therefore, $\cup_{i\in I}F_i$ is a $\vdash$-filter, and it contains F. Moreover, it is non-trivial. If it was not the case, since $\vdash$ contains a finite inconsistent set of formulas Ξ, we could consider a homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that there would exist $n \in \mathbb{N}$ such that $h[\Xi] = \Xi^{\boldsymbol{A}} \subseteq \cup_{i\leqslant n}F_i$. But then, since $\{F_i \,:\, i \in I\}$ is a chain, there would exist F_n such that $F_i \subseteq F_n$ for $i \leqslant n$, and, therefore, $\Xi^{\boldsymbol{A}} \subseteq F_n$. Consequently, F_n would be a trivial filter, i.e., $F_n = Fm$, contradicting the fact that for every $i \in I$, the filter F_i is non-trivial.

Therefore, the existence of a finite set of formulas inconsistent in $\vdash$ ensures that $\mathbb{X}$ is closed under unions of chains and we can apply Zorn's Lemma to deduce that such a maximal element always exists. ⊠

The following result is already established in [26] for the class of protonegational logics. This class of logics is presented as a weakening of protoalgebraicity, restricting some of its defining conditions to maximal consistent theories. Particular examples are the negation fragments of protoalgebraic logics. As we shall not need to employ this notion, we omit its definition, which can be found in [26, pp. 108]. In the framework of finitary protoalgebraic logics, the result translates as the following theorem:

Theorem 2.12. *[26, Thm. 6.35] Let $\vdash$ be a finitary protoalgebraic logic. The following are equivalent:*

(i) *$\vdash$ enjoys the LIL;*

(ii) *$\vdash$ enjoys the MCFEP and for every algebra $\boldsymbol{A}$ the deductive filter A is compact;*

(iii) *The MCFEP holds in the algebra of formulas and $\vdash$ possesses a finite inconsistent set of formulas.*

Proof. (i)⇒(ii): Suppose that $\vdash$ has the LIL witnessed by $\{\Psi_n \,:\, n \in \mathbb{Z}^+\}$. It follows from remark 2.3 that for every algebra $\boldsymbol{A}$, the deductive filter A is compact. More precisely, $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$ for some finite inconsistent set of formulas $\Xi(x)$ and for all $a \in A$.

Therefore, we turn to prove that $\vdash$ enjoys the MCFEP. Accordingly, let $\langle \boldsymbol{A}, F\rangle$ be a model of $\vdash$, $\langle \boldsymbol{B}, G\rangle$ a submatrix of $\langle \boldsymbol{A}, F\rangle$ and $H \in MaxFi_{\vdash}(\boldsymbol{B})$ with $G \subseteq H$. Define

$$H' = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup H).$$

It follows from the definition of H' that $F \subseteq H'$ and that $H \subseteq H' \cap B$. Hence, we shall prove the reverse inclusion $H' \cap B \subseteq H$. We claim that to prove $H' \cap B \subseteq H$, it suffices to show that $A \neq H'$.

To this end, consider $b \in H' \cap B$. We need to prove that if $A \neq H'$, then it must be the case that $b \in H$. First, the maximality of H implies that $b \in H$ iff $\mathrm{Fg}_{\vdash}^{\boldsymbol{B}}(H \cup \{b\}) \neq B$. Hence, by the transfer of LIL (Theorem 2.4) we get that the latter is equivalent to $I^{\boldsymbol{B}}(b) \nsubseteq H$ for every $I \in \Psi_1$. Therefore, we obtain

$$b \in H \iff I^{\boldsymbol{B}}(b) \nsubseteq H \text{ for every } I \in \Psi_1.$$

Now, observe that, by assumption, $\langle \boldsymbol{B}, G\rangle$ is a submatrix of $\langle \boldsymbol{A}, F\rangle$. This implies by the definition of submatrix that $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$, so $I^{\boldsymbol{A}}(b) = I^{\boldsymbol{B}}(b)$. Consequently, the right-hand side of the equivalence can be rewritten as $I^{\boldsymbol{A}}(b) \nsubseteq H$ for every $I \in \Psi_1$. But it is clear that, if $I^{\boldsymbol{A}}(b) \nsubseteq \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(F \cup H)$, then $I^{\boldsymbol{A}}(b) \nsubseteq H$. Therefore, we have

$$b \in H \Longleftarrow I^{\boldsymbol{A}}(b) \nsubseteq \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(F \cup H) = H' \text{ for every } I \in \Psi_1.$$

Again, by Theorem 2.4, the right-hand side amounts to $A \neq \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(H' \cup \{b\})$. But this is is equivalent to $A \neq H'$ by the assumption that $b \in H'$. Hence, we get

$$b \in H \Longleftarrow A \neq H'.$$

Therefore it suffices to show that $A \neq H'$. By way of contradiction suppose that it is not the case, i.e., $A = \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(F \cup H)$. Then, since $A = \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(\Xi^{\boldsymbol{A}}(a))$ for some finite inconsistent set of formulas $\Xi(x)$ and $a \in A$, and $\mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(-)$ is finitary, there are elements $f_1, \ldots, f_n \in F$ and $h_1, \ldots, h_m \in H$ such that

$$\mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(\Xi^{\boldsymbol{A}}(a)) = \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(f_1, \ldots, f_n, h_1, \ldots, h_m),$$

but then it must be the case that

$$A = \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(f_1, \ldots, f_n, h_1, \ldots, h_m).$$

It then follows from Theorem 2.4 that there is $I \in \Psi_m$ such that

$$I^{\boldsymbol{A}}(h_1, \ldots, h_m) \subseteq \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(f_1, \ldots, f_n) \subseteq F.$$

Since $h_1, \ldots, h_n \in H \subseteq B$ and $G = F \cap B$, we obtain

$$I^{\boldsymbol{B}}(h_1, \ldots, h_m) = I^{\boldsymbol{A}}(h_1, \ldots, h_m) \subseteq F \cap B = G.$$

One more time, the transfer of LIL and $G \cup \{h_1, \ldots, h_n\} \subseteq H$ imply that that

$$B = \mathrm{Fg}_{\vdash}^{\boldsymbol{B}}(G \cup \{h_1, \ldots, h_m\}) \subseteq H.$$

In other words, we conclude that H is trivial, i.e., $H = B$, which contradicts the assumption that $H \in \mathit{MaxFi}_{\vdash}(\boldsymbol{B})$. Hence $H = H' \cap B$, as desired.

(ii)⇒(iii): If $\vdash$ enjoys the MCFEP and for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated, then for the particular case of $\boldsymbol{A}$ being the algebra of formulas $\boldsymbol{Fm}$, the MCFEP holds for $\boldsymbol{A} = \boldsymbol{Fm}$, and $Fm = \mathrm{Cn}_{\vdash}(\Xi)$ for some finite set Ξ, i.e., Ξ is a finite inconsistent set in $\vdash$.

(iii)⇒(i): Define an inconsistency sequence as

$$\Psi_n = \{\Sigma \cap Fm(\{x_1, \dots, x_n\}) : \{x_1, \dots, x_n\} \cup \Sigma \text{ is inconsistent in } \vdash \text{ and } \Sigma \in \mathcal{T}h(\vdash)\}.$$

First we prove that for any $I \in \Psi_n$ we have that $\{x_1, \dots, x_n\} \cup I$ is inconsistent in $\vdash$. In other words, for every $\Sigma \in \mathcal{T}h(\vdash)$, if $\{x_1, \dots, x_n\} \cup \Sigma$ is inconsistent in $\vdash$, then $\{x_1, \dots, x_n\} \cup (\Sigma \cap Fm(\{x_1, \dots, x_n\}))$ is still inconsistent in $\vdash$.

Suppose, with a view to contradiction, that this is not the case. Then for some $\Sigma \in Th(\vdash)$ such that $\{x_1, \dots, x_n\} \cup \Sigma$ is inconsistent in $\vdash$, we have that

$$\mathrm{Cn}_{\vdash}[\{x_1, \dots, x_n\} \cup (\Sigma \cap Fm(\{x_1, \dots, x_n\}))] \neq Fm.$$

We define then two matrices

$$\langle \boldsymbol{A}, F \rangle := \langle \boldsymbol{Fm}, \Sigma \rangle,$$
$$\langle \boldsymbol{B}, G \rangle := \langle \boldsymbol{Fm}(\{x_1, \dots, x_n\}), \Sigma \cap Fm(\{x_1, \dots, x_n\}) \rangle,$$

both being models of $\vdash$. This is because Σ is a $\vdash$-theory, i.e., F is a filter of $\vdash$ on $\boldsymbol{A} = \boldsymbol{Fm}$ by Proposition 1.11. Moreover, $\langle \boldsymbol{B}, G \rangle$ is a submatrix of $\langle \boldsymbol{A}, F \rangle$ since $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $G = F \cap A$. Therefore, $\langle \boldsymbol{B}, G \rangle$ is also a model of $\vdash$ by Remark 1.10.

Let

$$J = Fm(\{x_1, \dots, x_n\}) \cap \mathrm{Cn}_{\vdash}[\{x_1, \dots, x_n\} \cup (\Sigma \cap Fm(\{x_1, \dots, x_n\}))].$$

Observe that J is a non-trivial $\vdash$-filter on $\boldsymbol{B}$. This is a consequence of Remark 1.11, and Lemma 2.10 together with the assumption that $\vdash$ possesses a finite inconsistent set of formulas, since $\langle \boldsymbol{B}, J \rangle$ is a submatrix of

$$\langle \boldsymbol{Fm}, \mathrm{Cn}_{\vdash}[\{x_1, \dots, x_n\} \cup (\Sigma \cap Fm(\{x_1, \dots, x_n\}))] \rangle,$$

which is non-trivial model of $\vdash$.

In virtue of Lemma 2.11 there exists a maximal non-trivial filter H such that $J \subseteq H \subseteq Fm(\{x_1, \dots, x_n\})$. We also have that $G \subseteq H$ because $G \subseteq J$. It then follows from the MCFEP that there exists a $\vdash$-filter H' on $\boldsymbol{A}$ such that $F \subseteq H'$ and $H = H' \cap B$.

Since $F \subseteq H'$, we get that $\Sigma \subseteq H'$. Moreover, we have $\{x_1, \dots, x_n\} \subseteq H'$ because $\{x_1, \dots, x_n\} \subseteq J \subseteq H$ and $H = H' \cap B$. We also have that H' is non-trivial, i.e., $H' \neq Fm$ because $H \neq Fm(\{x_1, \dots, x_n\})$ and $H = H' \cap B = H' \cap Fm(\{x_1, \dots, x_n\})$. As a result, $\mathrm{Cn}_{\vdash}[\{x_1, \dots, x_n\} \cup \Sigma] \subseteq H' \neq Fm$, which implies that $\{x_1, \dots, x_n\} \cup \Sigma$ is consistent — contradiction. Thus, for every $I \in \Psi_n$ we have that $\{x_1, \dots, x_n\} \cup I$ is inconsistent in $\vdash$.

In particular, we get one direction of the LIL, that is, for every $\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \subseteq Fm$,

$$\Gamma \vdash I(\alpha_1, \dots, \alpha_n) \text{ for some } I \in \Psi_n \text{ implies that } \Gamma \cup \{\alpha_1, \dots, \alpha_n\} \text{ is inconsistent in } \vdash.$$

For the converse direction of the LIL, suppose that $\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \subseteq Fm$ is inconsistent in $\vdash$ and take a surjective substitution σ such that $\sigma(x_i) = \alpha_i$ for every

$i \in \{1, \ldots, n\}$. Since $\vdash$ is protoalgebraic, Proposition 1.17, together with the assumption that $\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \subseteq Fm$ is inconsistent in $\vdash$, implies that $\sigma^{-1}[\mathrm{Cn}_{\vdash}(\Gamma)] \cup \{x_1, \ldots, x_n\}$ is inconsistent in $\vdash$.

Let $I = \sigma^{-1}[\mathrm{Cn}_{\vdash}(\Gamma)] \cap Fm(\{x_1, \ldots, x_n\})$. Clearly $I \in \Psi_n$ by the definition of Ψ_n. Moreover, from the assumption that σ is surjective, we get that $\sigma(\sigma^{-1}(\Gamma)) = \Gamma$. Together with the fact that $\sigma^{-1}(\Gamma) \vdash I$, by substitution invariance, we conclude that $\Gamma \vdash \sigma(I(x_1, \ldots, x_n)) = I(\alpha_1, \ldots, \alpha_n)$. In conclusion,

$$\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \text{ is inconsistent in } \vdash \text{ implies that } \Gamma \vdash I(\alpha_1, \ldots, \alpha_n) \text{ for some } I \in \Psi_n.$$

⊠

Let K be a quasivariety. We say that K has the *maximal relative congruence extension property* (MRCEP) if, for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{K}$, if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $\theta \in Con_{\mathsf{K}}\boldsymbol{B}$ maximal, then there is some $\theta' \in Con_{\mathsf{K}}(\boldsymbol{A})$ such that $\theta' \cap B^2 = \theta$. If K is a variety, we say that it has the *maximal congruence extension property* (MCEP).

Theorem 2.13. *Let $\vdash$ be a finitary algebraizable logic with equivalent algebraic semantics K. Then $\vdash$ has a local inconsistenct lemma if and only if K has the MRCEP and for every algebra $\boldsymbol{A}$ the total congruence is compact.*

At this point in the discussion, it is worth presenting the bridge theorem that connects the local deduction theorem and the so-called *filter extension property*. We begin by formally defining the notions of *local deduction-detachment theorem* and *filter extension property*.

A logic $\vdash$ is said to have a *local deduction-detachment theorem* – briefly a LDDT – if there exists a family Ψ of sets of formulas $I(x, y)$ such that for every $\Gamma \cup \{\alpha, \beta\} \subseteq Fm$,

$$\Gamma \cup \{\alpha\} \vdash \beta \iff \Gamma \vdash I(\alpha, \beta) \text{ for some } I \in \Psi.$$

A logic $\vdash$ has the *filter extension property*, or FEP, for short) if for every model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$ and every submatrix $\langle \boldsymbol{B}, G \rangle$ of $\langle \boldsymbol{A}, F \rangle$ and every $H \in Fi_{\vdash}(\boldsymbol{B})$ with $G \subseteq H$ there is $H' \in Fi_{\vdash}(\boldsymbol{A})$ with $F \subseteq H'$ such that $H = H' \cap B$.

In parallelism to Theorem 2.12, the corresponding bridge theorem arises between the LDDT and the FEP: a finitary protoalgebraic logic has the LDDT if and only if it has the FEP (see, [14, Thm. II.1]).

2.4 First-order definability of maximal consistent filters

To conclude this chapter, we introduce a new notion reminiscent of the *(first-order) definability of principal deductive filters* introduced by Czelakowski (see [15], pp. 132-134).

Recall that a logic $\vdash$ is a consequence relation on the set Fm of formulas of some algebraic language L. A logical matrix $\langle \boldsymbol{A}, F \rangle$ can be viewed as a first order structure in the algebraic language L extended with a unary predicate symbol $P(x)$. More precisely, a matrix can be regarded as an algebra equipped with the interpretation of $P(x)$ given by the designated set $F \subseteq A$. An intuitive reading of logical matrices suggests that the set of designated elements F represents truth inside the set of truth-values A, so $P(x)$ can be understood as a truth predicate and F as the truth set of $\langle \boldsymbol{A}, F \rangle$.

A logic $\vdash$ is said to have *(first order) definable maximal consistent filters* –briefly a DMCF – if for each $n \in \mathbb{Z}^+$ there exists a formula $\delta_n(x_1, \ldots, x_n)$ in the language of the first-order predicate logic (without equality), whose only non-logical symbols are the operation symbols of $\vdash$ and a unary predicate $P(x)$, such that for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and elements $a_1, \ldots, a_n \in A$,

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\}) \iff \langle \boldsymbol{A}, F\rangle \vDash \delta_n(a_1, \ldots, a_n).$$

A class of similar algebras is *elementary* if it is the model class of a set of first-order sentences.

Lemma 2.14. *The class of models of a finitary logic is elementary.*

Proof. For every finite set of formulas $\Gamma \cup \{\varphi\}$ such that $\Gamma \vdash \varphi$ we consider the sentence

$$\forall \vec{x}(\bigwedge_{\gamma \in \Gamma} P(\gamma) \Rightarrow P(\varphi)).$$

To prove that the set of all these sentences axiomatizes $\mathsf{Mod}(\vdash)$, it suffices to consider a matrix $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and prove that it is a model of $\vdash$ if and only if it satisfies these axioms. It is clear that if $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash$, then it satisfies these axioms. Conversely, assume that $\langle \boldsymbol{A}, F\rangle$ satisfies the axioms. Consider $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash \varphi$ and $\vec{a} \in A$ such that $\Gamma^{\boldsymbol{A}}(\vec{a}) \subseteq F$. Since $\vdash$ is finitary we may assume that Γ is finite. Now $\forall \vec{x}(\bigwedge_{\gamma \in \Gamma} P(\gamma) \Rightarrow P(\varphi))$ is an axiom and $\Gamma^{\boldsymbol{A}}(\vec{a}) \subseteq F$, so that $\varphi^{\boldsymbol{A}}(\vec{a}) \in F$. Therefore, $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash$. ⊠

Theorem 2.15. *Let $\vdash$ be a finitary protoalgebraic logic with the LIL witnessed by the family $\{\Psi_n : n \in \mathbb{Z}^+\}$. Then the following are equivalent:*

(i) *For every $n \in \mathbb{Z}^+$ there exists a finite $\Psi'_n \subseteq \Psi$ such that $\{\Psi'_n : n \in \mathbb{Z}^+\}$ also witnesses the LIL for $\vdash$;*

(ii) *$\vdash$ has definable maximal consistent filters.*

Proof. (i)⇒(ii): Let $\langle \boldsymbol{A}, F\rangle$ be a model of $\vdash$ and $a_1, \ldots, a_n \in A$. It follows from the semantic version of the LIL, Theorem 2.4, that

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\}) \iff I^{\boldsymbol{A}}(a_1, \ldots, a_n) \subseteq F \text{ for some } I \in \Psi_n.$$

However, by assumption there exists a finite $\Psi'_n \subseteq \Psi$ such that $\{\Psi'_n : n \in \mathbb{Z}^+\}$ also witnesses the LIL for $\vdash$. Suppose that Ψ'_n is the finite family of finite sets of formulas $\{I_1(x_1, \ldots, x_n), \ldots, I_m(x_1, \ldots, x_n)\}$. Then the above display amounts to

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\}) \iff I_i^{\boldsymbol{A}}(a_1, \ldots, a_n) \subseteq F \text{ for some } i \in \{1, \ldots, m\}.$$

Observe that $I_i^{\boldsymbol{A}}(a_1, \ldots, a_n) \subseteq F$ for some $i \in \{1, \ldots, m\}$ if and only if

$$\langle \boldsymbol{A}, F\rangle \vDash \bigvee_{i \leqslant m} \bigwedge_{\gamma \in I_i} P(\gamma(a_1, \ldots, a_n)).$$

Therefore, we can define a first-order formula $\delta_n(x_1, \ldots, x_n)$ which witnesses the definability of maximal consistent filters for $\vdash$ as follows:

$$\delta_n(x_1, \ldots, x_n) := \bigvee_{i \leqslant m} \bigwedge_{\gamma \in I_i} P(\gamma(x_1, \ldots, x_n)).$$

(ii)⇒(i): Let $\{\Psi_n : n \in \mathbb{Z}^+\}$ be the family witnessing the LIL for $\vdash$, and let $\{\delta_n(x_1,\dots,x_n) : n \in \mathbb{Z}^+\}$ be the family of first-order formulas witnessing the definability of maximal consistent filters for $\vdash$. Since $\vdash$ is finitary, Lemma 2.14 guarantees that the class of models of $\vdash$ is axiomatized by some set Σ of sentences. We first prove that for every $n \in \mathbb{Z}^+$

$$\Sigma \cup \{\bigvee_{\gamma \in I} \neg P(\gamma(x_1,\dots,x_n)) : I \in \Psi_n\} \vdash_{FOL} \neg\delta_n(x_1,\dots,x_n),$$

where, $\vdash_{FOL}$ denotes first-order entailment. To this end, consider a model $\langle \boldsymbol{A}, F\rangle$ of Σ and $a_1,\dots,a_n \in A$ such that

$$\langle \boldsymbol{A}, F\rangle \vDash \bigvee_{\gamma \in I} \neg P(\gamma(a_1,\dots,a_n)) \text{ for every } I \in \Psi_n.$$

Then, $I^{\boldsymbol{A}}(a_1,\dots,a_n) \nsubseteq F$ for every $I \in \Psi_n$, which, together with the semantic version of the LIL, Theorem 2.4, implies that $A \neq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1,\dots,a_n\})$. As a consequence, $\langle \boldsymbol{A}, F\rangle \vDash \neg\delta_n(a_1,\dots,a_n)$ because $\vdash$ has DMCF witnessed by the family $\{\delta_n(x_1,\dots,x_n) : n \in \mathbb{Z}^+\}$.

It then follows from the compactness theorem of first-order logic that there is a finite family $\Psi'_n = \{I_1(x_1,\dots,x_n),\dots,I_m(x_1,\dots,x_n)\} \subseteq \Psi_n$ such that

$$\Sigma \cup \{\bigvee_{\gamma \in I_i} \neg P(\gamma(x_1,\dots,x_n)) : i \leqslant m\} \vdash_{FOL} \neg\delta_n(x_1,\dots,x_n). \tag{2.2}$$

Now, we shall prove that Ψ'_n also witnesses the LIL for $\vdash$. We show that for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and elements $a_1,\dots,a_n \in A$

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1,\dots,a_n\}) \iff I^{\boldsymbol{A}}_i(a_1,\dots,a_n) \subseteq F \text{ for some } i \leqslant m.$$

The right-to-left implication holds because $\{\Psi_n : n \in \mathbb{Z}^+\}$ witnesses the LIL for $\vdash$ and $\Psi'_n = \{I_1(x_1,\dots,x_n),\dots,I_m(x_1,\dots,x_n)\} \subseteq \Psi_n$. To prove the converse implication we reason by contraposition. Assume that $I^{\boldsymbol{A}}_i(a_1,\dots,a_n) \nsubseteq F$ for every $i \leqslant m$. Then we have that

$$\langle \boldsymbol{A}, F\rangle \vDash \bigvee_{\gamma \in I_1} \neg P(\gamma(a_1,\dots,a_n)) \wedge \dots \wedge \bigvee_{\gamma \in I_m} \neg P(\gamma(a_1,\dots,a_n)).$$

Since $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\vdash)$, then $\langle \boldsymbol{A}, F\rangle \vDash \Sigma$. Therefore, by the display 2.2 we conclude that $\langle \boldsymbol{A}, F\rangle \vDash \neg\delta_n(a_1,\dots,a_n)$, which amounts to $A \neq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1,\dots,a_n\})$ because $\{\delta_n(x_1,\dots,x_n) : n \in \mathbb{Z}^+\}$ witnesses the definability of maximal consistent filters.

Since Proposition 1.11 establishes that the deductive filters of the algebra of formulas are the $\vdash$-theories, in particular we have that for every $\Gamma \cup \{\alpha_1,\dots,\alpha_n\} \subseteq Fm$,

$$Fm = \mathrm{Cn}_{\vdash}(\Gamma \cup \{\alpha_1,\dots,\alpha_n\}) \iff I(\alpha_1,\dots,\alpha_n) \subseteq \mathrm{Cn}_{\vdash}(\Gamma) \text{ for some } I \in \Psi'_n,$$

which is equivalent to the statement that for every $\Gamma \cup \{\alpha_1,\dots,\alpha_n\} \subseteq Fm$,

$$\Gamma \cup \{\alpha_1,\dots,\alpha_n\} \text{ is inconsistent in } \vdash \iff \Gamma \vdash I(\alpha_1,\dots,\alpha_n) \text{ for some } I \in \Psi'_n.$$

Hence, we conclude that $\Psi'_n = \{I_1(x_1,\dots,x_n),\dots,I_m(x_1,\dots,x_n)\}$ also witnesses the LIL for $\vdash$.

⊠

Following the parallelism with the theory for deduction-detachment theorem, we present at this point the notion of *(first-order) definable principal filters* (see [15], pp. 132-134), which a logic $\vdash$ is said to have if there exists a formula $\delta(x, y)$ in the language of the first-order predicate logic (with equality), whose only non-logical symbols are the operation symbols of $\vdash$ and a unary predicate $P(x)$, such that for every model $\langle \boldsymbol{A}, F \rangle$ of $\vdash$ and elements $a, b \in A$,

$$b \in \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a\}) \iff \langle \boldsymbol{A}, F \rangle \vDash \delta(a, b).$$

If a finitary protoalgebraic logic $\vdash$ has a LDDT witnessed by some family of formulas Ψ, then there exists a finite $\Psi' \subseteq \Psi$ such that Ψ'_n also witnesses the LDDT for $\vdash$ if and only if $\vdash$ has definable principal filters (see [15, Thm. 2.2.1]).

CHAPTER 3

The inconsistency by cases property

In this final chapter we present the crucial result of the thesis – Theorem 3.10 – which states that a finitary protoalgebraic logic has an IL if and only if it satisfies the following conditions: (1) $\vdash$ has the MCFEP; (2) for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated; (3) $\vdash$ has DMCF; and (4) $\vdash$ is filter-1-distributive. To this end, we introduce the so-called *inconsistency by cases property*. Theorem 3.4 provides, in the framework of finitary protoalgebraic logics, a characterization of this novel property in terms of an algebraic property of its lattice of deductive filters on any algebra of the suitable type, namely filter-1-distributivity.

Let $\nabla_{n,m}(x_1,\ldots,x_n,y_1,\ldots,y_m,\vec{z})$ be a set of formulas in variables $x_1,\ldots,x_n$, $y_1,\ldots,y_m$ and possible parameters $\vec{z}$. We define

$$\alpha_1,\ldots,\alpha_n\nabla_{n,m}\beta_1,\ldots,\beta_m \text{ as } \bigcup\{\nabla_{n,m}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m,\vec{\gamma}) : \vec{\gamma}\in Fm\}.$$

We start by defining a useful notion. A parameterized set $\nabla_{n,m}(x_1,\ldots,x_n,y_1,\ldots,y_m,\vec{z})$ of formulas is a *parameterized protodisjunction* (or just a *protodisjunction* if $\nabla_{n,m}$ has no parameters) in $\vdash$ whenever $\vec{\alpha}\vdash\vec{\alpha}\nabla_{n,m}\vec{\beta}$ and $\vec{\beta}\vdash\vec{\alpha}\nabla_{n,m}\vec{\beta}$.

We observe that the notion of protodisjunction is not interesting on its own since, essentially, any theorem (or set of theorems) in variables $x_1,\ldots,x_n,y_1,\ldots,y_m$ of a given logic would be a protodisjunction in this logic; we introduce it primarily as a tool for simplifying the presentation of upcoming definitions and results.

A more characterizing property of disjunction is given by the so-called *proof by cases* of classical disjunction, which has already been explored for arbitrary logics in the literature (see [10, 12, 15, 16]). We say that a logic $\vdash$ has the *proof by cases property* if for every n, m there exists a parameterized protodisjunction $\nabla_{n,m}(x_1,\ldots,x_n,y_1,\ldots,y_m,\vec{z})$ such that for any set $\Gamma,\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m,\chi$ of formulas, if $\Gamma,\vec{\alpha}\vdash\chi$ and $\Gamma,\vec{\beta}\vdash\chi$, then $\Gamma,\vec{\alpha}\nabla_{n,m}\vec{\beta}\vdash\chi$.

In a similar fashion, we introduce another relevant property: we say that a logic $\vdash$ enjoys the *inconsistency by cases property (ICP)* if for every positive integers n, m, there exists a parameterized protodisjunction $\nabla_{n,m}(x_1,\ldots,x_n,y_1,\ldots,y_m,\vec{z})$ such that for any set $\Gamma,\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m$ of formulas, if $\Gamma\cup\{\vec{\alpha}\}$ and $\Gamma\cup\{\vec{\beta}\}$ are inconsistent in $\vdash$, then $\Gamma\cup\{\vec{\alpha}\nabla_{n,m}\vec{\beta}\}$ is inconsistent in $\vdash$.

Proposition 3.1. *If a logic has an inconsistency lemma, then it has the inconsistency by cases property witnessed by protodisjunctions* $\nabla_{n,m}$ *without parameters.*

Proof. Let $\vdash$ be a logic with the IL witnessed by the family $\{\Psi_n : n \in \mathbb{Z}^+\}$ and define

$$\nabla_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \Psi_k(\Psi_n(x_1, \ldots, x_n), \Psi_m(y_1, \ldots, y_m)),$$

where $k = |\Psi_n \cup \Psi_m|$. First we shall prove that $\nabla_{n,m}$ is a protodisjunction. To this end, consider $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in Fm$. Since $\Psi_n(\alpha_1, \ldots, \alpha_n) \vdash \Psi_n(\alpha_1, \ldots, \alpha_n)$, it follows from the IL that $\{\alpha_1, \ldots, \alpha_n\} \cup \Psi_n(\alpha_1, \ldots, \alpha_n)$ is inconsistent in $\vdash$. Therefore, $\{\alpha_1, \ldots, \alpha_n\} \cup \Psi_n(\alpha_1, \ldots, \alpha_n) \cup \Psi_m(\beta_1, \ldots, \beta_m)$ is also inconsistent in $\vdash$. Together with the IL this implies that

$$\alpha_1, \ldots, \alpha_n \vdash \Psi_k(\Psi_n(\alpha_1, \ldots, \alpha_n), \Psi_m(\beta_1, \ldots, \beta_m)).$$

A symmetrical argument yields that

$$\beta_1, \ldots, \beta_m \vdash \Psi_k(\Psi_n(\alpha_1, \ldots, \alpha_n), \Psi_m(\beta_1, \ldots, \beta_m)).$$

Hence, $\vec{\alpha} \vdash \vec{\alpha} \nabla_{n,m} \vec{\beta}$ and $\vec{\beta} \vdash \vec{\alpha} \nabla_{n,m} \vec{\beta}$. Thus $\nabla_{n,m}$ is a protodisjunction.

We turn to prove that $\nabla_{n,m}$ witnesses the ICP for $\vdash$, i.e., that for any set of formulas $\Gamma, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$, if $\Gamma \cup \{\vec{\alpha}\}$ and $\Gamma \cup \{\vec{\beta}\}$ are inconsistent in $\vdash$, then $\Gamma \cup \Psi_k(\Psi_n(\vec{\alpha}), \Psi_m(\vec{\beta}))$ is inconsistent in $\vdash$. In view of the IL, this amounts to showing that if $\Gamma \vdash \Psi_n(\vec{\alpha})$ and $\Gamma \vdash \Psi_m(\vec{\beta})$, then $\Gamma \vdash \Psi_l(\Psi_k(\Psi_n(\vec{\alpha}), \Psi_m(\vec{\beta})))$, where $l = |\Psi_k(\Psi_n(\vec{\alpha}), \Psi_m(\vec{\beta}))|$.

Observe that, by the transitivity of $\vdash$, it suffices to prove that

$$\Psi_n(\vec{\alpha}) \cup \Psi_m(\vec{\beta}) \vdash \Psi_l(\Psi_k(\Psi_n(\vec{\alpha}), \Psi_m(\vec{\beta}))),$$

which, considering the IL translates to

$$\Psi_n(\vec{\alpha}) \cup \Psi_m(\vec{\beta}) \cup \Psi_k(\Psi_n(\vec{\alpha}), \Psi_m(\vec{\beta})) \text{ is inconsistent in } \vdash .$$

Considering the IL again, this is true since $\Psi_k(\Psi_n(\vec{\alpha}), \Psi_m(\vec{\beta})) \vdash \Psi_k(\Psi_n(\vec{\alpha}), \Psi_m(\vec{\beta}))$. ⊠

We now characterize the inconsistency by cases property. In parallel with the connection between the proof by cases property and filter-distributivity (see [10, 12, 15, 16]), the corresponding bridge theorem links the ICP to filter-1-distributivity. First we must introduce some notions from lattice theory.

A join semilattice $\boldsymbol{A}$ with maximum element 1 is called *1-distributive* if for all $a, b, c \in A$ with $a \vee b = 1$ and $a \vee c = 1$, then $a \vee d = 1$ for some element $d \leqslant b, c$. Similarly, a lattice $\boldsymbol{A}$ with maximum element 1 is said to be *1-distributive* if whenever $a \vee b = 1$ and $a \vee c = 1$, then $a \vee (b \wedge c) = 1$ for all elements $a, b, c \in A$. This notion is sometimes referred to as *join-semidistributivity at 1*.

A logic $\vdash$ is *filter-1-distributive* if for each algebra $\boldsymbol{A}$ the lattice of $\vdash$-filters $\mathcal{F}i_{\vdash}(\boldsymbol{A})$ is 1-distributive. We omit the prefix 'filter-' whenever the corresponding property holds for the algebra of formulas $\boldsymbol{Fm}$.

Proposition 3.2. *Let $\boldsymbol{A}$ be a semilattice with minimum* 0 *and maximum* 1*. Then $\boldsymbol{A}$ is* 1*-distributive if and only if the lattice of ideals $\mathcal{I}d(\boldsymbol{A})$ is* 1*-distributive.*

Proof. Suppose that $\boldsymbol{A}$ is a 1-distributive semilatice. Let $I, J, K \in \mathcal{I}d(\boldsymbol{A})$ such that $I + J = A$ and $I + K = A$. We shall prove that $I + (J \cap K) = A$. From $1 \in I + J$ we obtain that $a' \vee b = 1$ for some $a' \in I$ and $b \in J$. From $1 \in I + K$ we obtain that $a'' \vee c = 1$ for some $a'' \in I$ and $c \in K$. Let $a = a' \vee a''$, then $a \vee b = 1$ and $a \vee c = 1$. Then it follows from the 1-distributivity of $\boldsymbol{A}$ that $a \vee (d) = 1$ for some element $d \in A$ such that $d \leqslant b, c$. Therefore, $d \in J \cap K$ because I and J are downsets and, by assumption, $b \in J$ and $c \in K$. Consequently, $1 \in I + (J \cap K)$ which implies that $I + (J \cap K) = A$. Thus, $\mathcal{I}d(\boldsymbol{A})$ is 1-distributive.

Conversely, suppose that $\mathcal{I}d(\boldsymbol{A})$ is 1-distributive and let $a, b, c \in A$ such that $a \vee b = 1$ and $a \vee c = 1$. Then, $\downarrow a + \downarrow b = A$ and $\downarrow a + \downarrow c = A$, which together with the 1-distributivity of $\mathcal{I}d(\boldsymbol{A})$ implies that $\downarrow a + (\downarrow b \cap \downarrow c) = A$. Therefore, it must be the case that $a \vee d = 1$ for some element $d \in A$ with $d \leqslant b, c$. Hence, $\boldsymbol{A}$ is 1-distributive. ⊠

Since every algebraic lattice is isomorphic to the lattice of ideals of the join semilattice of its compact elements, by Theorem 1.9, we get:

Corollary 3.3. *Let $\boldsymbol{A}$ be an algebraic lattice whose maximum element is compact. Then $\boldsymbol{A}$ is* 1*-distributive if and only if the join semilattice of its compact elements is* 1*-distributive.*

Theorem 3.4. *Let $\vdash$ be a finitary protoalgebraic logic. The following are equivalent*

(i) *$\vdash$ enjoys the ICP and $\vdash$ possesses a finite inconsistent set of formulas.*

(ii) *$\vdash$ is filter-*1*-distributive.*

(iii) *The lattice $\mathcal{T}h(\vdash)$ is* 1*-distributive.*

Proof. (i)⇒(ii): Let $\boldsymbol{A}$ be an algebra over the same language of $\vdash$. Since the logic is finitary, Theorem 1.5 and Theorem 1.9 imply that the lattice of its deductive filters, $\mathcal{F}i_{\vdash}(\boldsymbol{A})$ is algebraic. It also follows from the representation theorem of algebraic lattices, Theorem 1.9, that $\mathcal{F}i_{\vdash}(\boldsymbol{A})$ is isomorphic to the lattice of ideals of the join semilattice of compact deductive filters of $\boldsymbol{A}$. In symbols, $\mathcal{F}i_{\vdash}(\boldsymbol{A}) \cong \mathcal{I}d(Comp\mathcal{F}i_{\vdash}(\boldsymbol{A}))$. Furthermore, the finitary property also implies that the compact elements are just the finitely generated $\vdash$–filters of $\boldsymbol{A}$ by Proposition 1.7. Therefore, in view of Corollary 3.3, it suffices to show that the join semilattice of finitely generated $\vdash$–filters of $\boldsymbol{A}$ is a 1-distributive join semilattice.

To this end, first observe that it follows from the assumption that $\vdash$ possesses a finite inconsistent set of formulas, together with Proposition 2.2 that there is some finite set of formulas $\Xi(x)$ such that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$ for every $a \in A$. Therefore, the deductive filter A is finitely generated, i.e., $Comp\mathcal{F}i_{\vdash}(\boldsymbol{A})$ has a maximum element.

Moreover, consider elements $a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_k \in A$ and suppose that $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \ldots, a_n) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \ldots, b_m) = A$ and $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \ldots, a_n) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(c_1, \ldots, c_k) = A$. To prove that $Comp\mathcal{F}i_{\vdash}(\boldsymbol{A})$ is a 1-distributive join semilattice, we need to establish the existence of a finite subset $D \subseteq A$ satisfying the following conditions:

1. $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(D) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{b_1, \ldots, b_m\}), \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{c_1, \ldots, c_k\})$;

2. $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1,\dots,a_n\}) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(D) = A$.

To achieve this, consider the protodisjunction $\nabla_{m,k}$ given by the ICP associated with the integers m,k. Take a homomorphism $g : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $g(x_i) = b_i$ for $i \in \{1,\dots,m\}$ and $g(y_j) = c_j$ for $j \in \{1,\dots,k\}$, where $x_1,\dots,x_m,y_1,\dots,y_k$ are disjoint variables. Define

$$D := g[x_1,\dots,x_m\nabla^{\boldsymbol{A}}_{m,k}y_1,\dots,y_k] = b_1,\dots,b_m\nabla^{\boldsymbol{A}}_{m,k}c_1,\dots,c_k.$$

It follows from the ICP that $\vec{x} \vdash \vec{x}\nabla_{m,k}\vec{y}$ and $\vec{y} \vdash \vec{x}\nabla_{m,k}\vec{y}$ because $\nabla_{m,k}$ is a protodisjunction. This implies that for every $\vdash$-filter F on $\boldsymbol{A}$ and homomorphism $f : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $f[\vec{x}] \subseteq F$ or $f[\vec{y}] \subseteq F$, we have $f[\vec{x}\nabla_{m,k}\vec{y}] \subseteq F$. Since $g[\vec{x}] \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{b_1,\dots,b_m\})$ and $g[\vec{y}] \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{c_1,\dots,c_k\})$, then $g[\vec{x}\nabla_{m,k}\vec{y}] \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{b_1,\dots,b_m\}), \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{c_1,\dots,c_k\})$. Hence, taking D as above, the first condition is met. Observe that the set D need not be finite. However, the assumption that $\vdash$ is finitary implies that the consequence operation of $\vdash$-filter generation on $\boldsymbol{A}$ is finitary. So $a \in \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(D)$ implies that $a \in \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(D_0)$ for some finite $D_0 \subseteq D$.

Thus, we focus on proving the second condition, i.e.,

$$\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1,\dots,a_n\}) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1,\dots,b_m\nabla^{\boldsymbol{A}}c_1,\dots,c_k) = A.$$

Recall from Proposition 2.2 that the assumption that there is a finite set of inconsistent formulas in $\vdash$ implies that A is finitely generated, i.e., $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$ for some finite inconsistent set of formulas $\Xi(x)$ and any $a \in A$. Therefore, the condition

$$\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1,\dots,a_n\}) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{b_1,\dots,b_m\}) = A$$

translates to

$$\Xi^{\boldsymbol{A}}(a) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1,\dots,a_n\}) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{b_1,\dots,b_m\}).$$

It then follows from the fact that $\vdash$ is finitary and protoalgebraic together with Lemma 1.19 that for each $\xi \in \Xi$ there exists a finite set of formulas $\Gamma^b_\xi \cup \{\varphi^b_\xi\}$ and a homomorphism $h^b_\xi : \boldsymbol{Fm} \to \boldsymbol{A}$ such that

$$\Gamma^b_\xi \vdash \varphi^b_\xi \text{ and } h^b_\xi[\Gamma^b_\xi] \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1,\dots,a_n\}) \cup \{b_1,\dots,b_m\} \text{ and } h^b_\xi(\varphi^b_\xi) = \xi^{\boldsymbol{A}}(a).$$

Then, by substitution-invariance and since infinitely many variables are at our disposal, we may assume that for distinct $\xi,\xi' \in \Xi$, the sets of variables occurring in $\Gamma^b_\xi \cup \{\varphi^b_\xi\}$ and $\Gamma^b_{\xi'} \cup \{\varphi^b_{\xi'}\}$ are pairwise disjoint and that all these variables are among $y_1,y_2,y_3,\dots$, and distinct from $x_1,\dots,x_{m+1}$. Consequently, we can consider a homomorphism $h^b : \boldsymbol{Fm} \to \boldsymbol{A}$ such that for every $\xi \in \Xi$ it acts as h^b_ξ on the variables that occur in $\Gamma^b_\xi \cup \{\varphi^b_\xi\}$, and we set $h^b(x_i) = b_i$ for $i \in \{1,\dots,m\}$ and $h^b(x_{m+1}) = a$. Therefore we get

$$\bigcup_{\xi\in\Xi} \Gamma^b_\xi \vdash \{\varphi^b_\xi : \xi \in \Xi\} \text{ and } h^b[\bigcup_{\xi\in\Xi} \Gamma^b_\xi] \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1,\dots,a_n\}) \cup \{b_1,\dots,b_m\}.$$

Moreover, recall from Theorem 1.14 that the protoalgebraicity of $\vdash$ amounts to the existence of a set $\Delta(x,y)$ of formulas such that $\varnothing \vdash \Delta(x,x)$ and $x, \Delta(x,y) \vdash y$. Hence, from $x, \Delta(x,y) \vdash y$ we get that

$$\Delta(\varphi_\xi^b, \xi(x_{m+1})), \varphi_\xi^b \vdash \xi(x_{m+1}),$$

and it follows from $\varnothing \vdash \Delta(x,x)$ that for every $\delta \in \Delta$,

$$h^b(\delta(\varphi_\xi^b, \xi(x_{m+1})) = \delta^{\boldsymbol{A}}(h^b(\varphi_\xi^b), h^b(\xi(x_{n+1}))) = \delta^{\boldsymbol{A}}(\xi^{\boldsymbol{A}}(a), \xi^{\boldsymbol{A}}(a)) \in \mathrm{Fg}_\vdash^{\boldsymbol{A}}(\{a_1, \dots, a_n\}).$$

So, if we consider

$$\Gamma^b := \bigcup_{\xi \in \Xi} \Gamma_\xi^b \cup \bigcup_{\xi \in \Xi} \Delta(\varphi_\xi^b, \xi(x_{m+1})) \cup \{x_1, \dots, x_m\},$$

then $\Gamma^b \vdash \Xi(x_{m+1})$ and $h^b[\Gamma^b] \subseteq \mathrm{Fg}_\vdash^{\boldsymbol{A}}(\{a_1, \dots, a_n\}) \cup \{b_1, \dots, b_m\}$.

Similar reasoning applies to the assumption that

$$\mathrm{Fg}_\vdash^{\boldsymbol{A}}(a_1, \dots, a_n) +^{\boldsymbol{A}} \mathrm{Fg}_\vdash^{\boldsymbol{A}}(c_1, \dots, c_k) = A.$$

In this case, there exists a finite set of formulas Γ^c and a homomorphism $h^c : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $\Gamma^c \vdash \Xi(x_{m+1})$ and $h^c[\Gamma^c] \subseteq \mathrm{Fg}_\vdash^{\boldsymbol{A}}(a_1, \dots, a_n) \cup \{c_1, \dots, c_k\}$ and $h^c(x_{m+1}) = a$.

Again by substitution-invariance and with infinitely many variables at our disposal, we may assume that the sets of variables occurring in Γ^b and Γ^c are pairwise disjoint and that all these variables are different from x_{m+1}. Consequently, we can consider a homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that it acts as h^b on the variables that occur in Γ^b, as h^c on the variables that occur in Γ^c, and $h(x_{m+1}) = a$.

Observe that we can write $\Gamma^b = \Pi^b \cup \Sigma^b$ where $\Sigma^b := \{x_1, \dots, x_m\}$ and

$$\Pi^b := \bigcup_{\xi \in \Xi} \Delta(\varphi_\xi^b, \xi(x_{m+1})) \cup \bigcup \{\Delta(x_i, \alpha) \, : \, i \leqslant m \text{ and } \alpha \in \bigcup_{\xi \in \Xi} \Gamma_\xi^b \text{ and } h^b(\alpha) = b_i\}.$$

Notice first that $h[\Sigma^b] \subseteq \{b_1, \dots, b_m\}$ because $h^b(x_i) = b_i$ for $i \in \{1, \dots, m\}$. Also notice first that $h[\Pi^b] \subseteq \mathrm{Fg}_\vdash^{\boldsymbol{A}}(a_1, \dots, a_n)$. This is because for every $i \in \{1, \dots, m\}$,

$$h^b[\Delta(x_i, \alpha)] = \Delta^{\boldsymbol{A}}(h^b(x_i), h^b(\alpha)) = \Delta^{\boldsymbol{A}}(b_i, b_i),$$
$$h^b[\Delta(\varphi_\xi^b, \xi(x_{m+1}))] = \Delta^{\boldsymbol{A}}(h(\varphi_\xi^b), h(\xi(x_{m+1}))) = \Delta^{\boldsymbol{A}}(\xi^{\boldsymbol{A}}(a), \xi^{\boldsymbol{A}}(a)),$$

and $\varnothing \vdash \Delta(x,x)$ by Theorem 1.14.

Finally, notice that $\Pi^b \cup \Sigma^b \vdash \Xi(x_{m+1})$. This is because $\Delta(x,y), x \vdash y$ by Theorem 1.14, so we have that $\Delta(x_i, \alpha), x_i \vdash \alpha$. Hence $\Pi^b \cup \Sigma^b \vdash \alpha$ for every $\alpha \in \bigcup_{\xi \in \Xi} \Gamma_\xi^b$. Moreover $\bigcup_{\xi \in \Xi} \Delta(\varphi_\xi^b, \xi(x_{m+1})) \subseteq \Pi^b$ and $\{x_1, \dots, x_m\} \subseteq \Sigma^b$. Hence

$$\Gamma^b = \bigcup_{\xi \in \Xi} \Gamma_\xi^b \cup \bigcup_{\xi \in \Xi} \Delta(\varphi_\xi^b, \xi(x_{m+1})) \cup \{x_1, \dots, x_m\} \subseteq \Pi^b \cup \Sigma^b,$$

which together with $\Gamma^b \vdash \Xi(x_{m+1})$, implies that $\Pi^b \cup \Sigma^b \vdash \Xi(x_{m+1})$.

Furthermore, following a similar strategy, we can write $\Gamma^c = \Pi^c \cup \Sigma^c$ where $\Sigma^c := \{y_1, \ldots, y_k\}$ and

$$\Pi^c := \bigcup_{\xi \in \Xi} \Delta(\varphi^c_\xi, \xi(x_{m+1})) \cup \bigcup \{\Delta(x_j, \alpha) \,:\, j \leqslant k \text{ and } \alpha \in \bigcup_{\xi \in \Xi} \Gamma^c_\xi \text{ and } h^c(\alpha) = c_j\}.$$

With this arrangement, by an analogous reasoning, we observe that $h[\Sigma^c] \subseteq \{c_1, \ldots, c_k\}$, $h[\Pi^c] \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \ldots, a_n)$, and $\Pi^c \cup \Sigma^c \vdash \Xi(x_{m+1})$

Now, letting $\Pi = \Pi^b \cup \Pi^c$, we have that $\Pi \cup \Sigma^b$ and $\Pi \cup \Sigma^c$ are inconsistent in $\vdash$. The inconsistency by cases property guarantees that $\Pi \cup \Sigma^b \nabla_{m,k} \Sigma^c$ is inconsistent in $\vdash$, or stated equivalently that $\Pi \cup \Sigma^b \nabla_{m,k} \Sigma^c \vdash \Xi(x_{m+1})$. Together with the fact that $h[\Pi] \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \ldots, a_n\})$, and $h[\Sigma^b \nabla_{m,k} \Sigma^c] \subseteq b_1, \ldots, b_m \nabla^{\boldsymbol{A}}_{m,k} c_1, \ldots, c_k$, and $h[\Xi(x_{m+1})] = \Xi^{\boldsymbol{A}}(a)$, this implies that

$$\Xi^{\boldsymbol{A}}(a) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \ldots, a_n\}) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \ldots, b_m \nabla^{\boldsymbol{A}}_{m,k} c_1, \ldots, c_k).$$

Given that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$, we conclude that

$$\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \ldots, a_n\}) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \ldots, b_m \nabla^{\boldsymbol{A}}_{m,k} c_1, \ldots, c_k) = A.$$

By the definition of D, this is $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\{a_1, \ldots, a_n\}) +^{\boldsymbol{A}} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(D) = A$. Although D is not necessarily finite, the assumption that $\vdash$ is finitary implies that the operation of $\vdash$-filter generation is also finitary, so we may assume that D is finite.

(ii)⇒(iii): If $\vdash$ is filter-1-distributive, then for the particular case of $\boldsymbol{A}$ being the algebra of formulas $\boldsymbol{Fm}$, the lattice $\mathcal{T}h(\vdash)$ is 1-distributive.

(iii)⇒(i): Define

$$\nabla_{n,m}(x_1, \ldots, x_n, y_1, \ldots, y_m, \vec{z}) = \mathrm{Cn}_{\vdash}(x_1, \ldots, x_n) \cap \mathrm{Cn}_{\vdash}(y_1, \ldots, y_m).$$

Given a set Γ and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ of formulas, let σ be a surjective substitution such that $\sigma x_i = \alpha_i$ for $i \in \{1, \ldots, n\}$ and $\sigma y_j = \beta_j$ for $j \in \{1, \ldots, m\}$. If we knew that $\mathrm{Cn}_{\vdash}(\Gamma) + \mathrm{Cn}_{\vdash}[\vec{\alpha} \nabla_{n,m} \vec{\beta}] = Fm \iff \mathrm{Cn}_{\vdash}(\Gamma) + [\mathrm{Cn}_{\vdash}(\vec{\alpha}) \cap \mathrm{Cn}_{\vdash}(\vec{\beta})] = Fm$, then we could write the following chain of equivalences demonstrating that $\vdash$ enjoys the ICP:

$$\begin{aligned} \mathrm{Cn}_{\vdash}(\Gamma \cup \{\vec{\alpha} \nabla_{n,m} \vec{\beta}\}) = Fm &\iff \mathrm{Cn}_{\vdash}(\Gamma) + \mathrm{Cn}_{\vdash}[\vec{\alpha} \nabla_{n,m} \vec{\beta}] = Fm \\ &\iff \mathrm{Cn}_{\vdash}(\Gamma) + [\mathrm{Cn}_{\vdash}(\vec{\alpha}) \cap \mathrm{Cn}_{\vdash}(\vec{\beta})] = Fm \\ &\iff [\mathrm{Cn}_{\vdash}(\Gamma) + \mathrm{Cn}_{\vdash}(\vec{\alpha})] \cap [\mathrm{Cn}_{\vdash}(\Gamma) + \mathrm{Cn}_{\vdash}(\vec{\beta})] = Fm \\ &\iff \mathrm{Cn}_{\vdash}(\Gamma \cup \{\vec{\alpha}\}) \cap \mathrm{Cn}_{\vdash}(\Gamma \cup \{\vec{\beta}\}) = Fm \end{aligned}$$

The first and last equivalences follow from the definition of $+$ in $\mathcal{T}h(\vdash)$, while the third is a consequence of the 1-distributivity of $\mathcal{T}h(\vdash)$. The rest of the proof is dedicated to proving the second equivalence, namely,

$$\mathrm{Cn}_{\vdash}(\Gamma) + \mathrm{Cn}_{\vdash}[\vec{\alpha} \nabla_{n,m} \vec{\beta}] = Fm \iff \mathrm{Cn}_{\vdash}(\Gamma) + [\mathrm{Cn}_{\vdash}(\vec{\alpha}) \cap \mathrm{Cn}_{\vdash}(\vec{\beta})] = Fm.$$

To this end, we establish the following two claims:

Claim 3.5. *$\Sigma + \mathrm{Cn}_{\vdash}\sigma\Gamma = \sigma[\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\Gamma]$ for each theory Σ and set of formulas Γ.*

Proof. Recall that the deductive filters of a logic $\vdash$ on the algebra of formulas coincide with the theories of $\vdash$, i.e, $\mathcal{T}h(\vdash) = \mathcal{F}i_{\vdash}(\boldsymbol{Fm})$ and $\mathrm{Cn}_{\vdash}(-) = \mathrm{Fg}_{\vdash}^{\boldsymbol{Fm}}(-)$. It then follows that σ is a strict surjective homomorphism beetween the models $\langle \boldsymbol{Fm}, \sigma^{-1}\Sigma\rangle$ and $\langle \boldsymbol{Fm}, \Sigma\rangle$ of $\vdash$, where Σ is a $\vdash$-theory. Since $\vdash$ is protoalgebraic, it has the correspondence property by Theorem 1.16, which implies that the direct image map

$$\sigma[-] : \mathcal{T}h(\vdash)^{\sigma^{-1}\Sigma} \to \mathcal{T}h(\vdash)^{\Sigma}$$

is a well-defined lattice isomorphism.

Therefore, for every set of formulas $\Gamma \subseteq Fm$ we have that $\sigma[\mathrm{Cn}_{\vdash}(\Gamma \cup \sigma^{-1}[\Sigma])]$ is a $\vdash$-theory because it is the image under σ of $\mathrm{Cn}_{\vdash}(\Gamma \cup \sigma^{-1}[\Sigma])$ which is a theory of $\vdash$ extending $\sigma^{-1}[\Sigma]$. Together with the fact that $\Sigma \cup \sigma[\Gamma] \subseteq \sigma[\mathrm{Cn}_{\vdash}(\Gamma \cup \sigma^{-1}[\Sigma])]$, because σ is surjective, this implies that $\mathrm{Cn}_{\vdash}(\Sigma \cup \sigma[\Gamma]) \subseteq \sigma[\mathrm{Cn}_{\vdash}(\sigma^{-1}[\Sigma] \cup \Gamma)]$. Then, the inclusion $\Sigma + \mathrm{Cn}(\sigma[\Gamma]) \subseteq \sigma[\mathrm{Cn}_{\vdash}(\Gamma) + \sigma^{-1}[\Sigma]]$ follows from the definition of the join operator in $\mathcal{T}h(\vdash)$ and the fact that $\sigma^{-1}\Sigma$ and Σ are $\vdash$-theories.

To prove the converse inclusion, let $\xi \in \sigma[\sigma^{-1}[\Sigma] + \mathrm{Cn}_{\vdash}\Gamma]$, then $\xi = \sigma\delta$ for some $\delta \in Fm$ such that $\sigma^{-1}[\Sigma] \cup \Gamma \vdash \delta$. Thus, by the surjectivity of σ, we have that $\Sigma, \sigma\Gamma \vdash \sigma\delta$, i.e., $\Sigma, \sigma\Gamma \vdash \xi$ so that $\xi \in \Sigma + \mathrm{Cn}_{\vdash}(\sigma\Gamma)$, as required. □

Claim 3.6. *$\Sigma + (\mathrm{Cn}_{\vdash}(\sigma x_1, \ldots, \sigma x_n) \cap \mathrm{Cn}_{\vdash}(\sigma y_1, \ldots, \sigma y_m)) = Fm$ if and only if $\Leftrightarrow \Sigma + \mathrm{Cn}_{\vdash}[\sigma(\mathrm{Cn}_{\vdash}(x_1, \ldots, x_n) \cap \mathrm{Cn}_{\vdash}(y_1, \ldots, y_m))] = Fm$ for every theory Σ.*

Proof. Let Σ be a theory and recall that σ is a surjective substitution. Due to the 1-distributivity of $\mathcal{T}h(\vdash)$ together with the Claim 3.5, we can write the equivalences

$$\begin{aligned}\Sigma + (\mathrm{Cn}_{\vdash}\sigma\vec{x} \cap \mathrm{Cn}_{\vdash}\sigma\vec{y}) = Fm &\iff [\Sigma + \mathrm{Cn}_{\vdash}\sigma\vec{x}] \cap [\Sigma + \mathrm{Cn}_{\vdash}\sigma\vec{y}] = Fm \\ &\iff \sigma[\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\vec{x}] \cap \sigma[\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\vec{y}] = Fm.\end{aligned}$$

Now using the same argument as before, the protoalgebraicity of $\vdash$ implies that the direct image map

$$\sigma[-] : \mathcal{T}h(\vdash)^{\sigma^{-1}\Sigma} \to \mathcal{T}h(\vdash)^{\Sigma}$$

is a well-defined isomorphism.

Therefore both $\sigma[\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\vec{x}]$ and $\sigma[\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\vec{y}]$ are $\vdash$-theories because they are images under σ of theories extending $\sigma^{-1}\Sigma$. Moreover, it follows from the fact that $\sigma[-]$ is an isomorphism that the intersection of these two theories is the set of all formulas if and only if so does the intersection of its preimages under σ. Hence, $\sigma[\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\vec{x}] \cap \sigma[\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\vec{y}] = Fm \iff [\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\vec{x}] \cap [\sigma^{-1}\Sigma + \mathrm{Cn}_{\vdash}\vec{y}] = Fm$.

As a consequence of the 1-distributivity of $\mathcal{T}h(\vdash)$, the latter amounts to $\sigma^{-1}\Sigma + [\mathrm{Cn}_{\vdash}\vec{x} \cap \mathrm{Cn}_{\vdash}\vec{y}] = Fm$, which is equivalent to $\sigma(\sigma^{-1}\Sigma + [\mathrm{Cn}_{\vdash}\vec{x} \cap \mathrm{Cn}_{\vdash}\vec{y}]) = Fm$ because $\sigma[-]$ is a lattice-isomorphism so it preserves the top element of the lattice structure. Applying the previous claim, Claim 3.5, to the above expression, we obtain the right-hand side of the equivalence in the claim statement: $\Sigma + \mathrm{Cn}_{\vdash}[\sigma(\mathrm{Cn}_{\vdash}\vec{x} \cap \mathrm{Cn}_{\vdash}\vec{y})] = Fm$.

Therefore, tracing back through the chain of equivalences, we have established the desired result: $\Sigma + (\mathrm{Cn}_{\vdash}\sigma\vec{x} \cap \mathrm{Cn}_{\vdash}\sigma\vec{y}) = Fm \Leftrightarrow \Sigma + \mathrm{Cn}_{\vdash}[\sigma(\mathrm{Cn}_{\vdash}\vec{x} \cap \mathrm{Cn}_{\vdash}\vec{y})] = Fm$. □

Finally, we can finish the proof of the theorem by a series of equivalences

$$\begin{aligned}\mathrm{Cn}_{\vdash}(\Gamma) + [\mathrm{Cn}_{\vdash}(\vec{\alpha}) \cap \mathrm{Cn}_{\vdash}(\vec{\beta})] = Fm &\iff \mathrm{Cn}_{\vdash}(\Gamma) + [\mathrm{Cn}_{\vdash}(\sigma\vec{x}) \cap \mathrm{Cn}_{\vdash}(\sigma\vec{y})] = Fm \\ &\iff \mathrm{Cn}_{\vdash}(\Gamma) + \mathrm{Cn}_{\vdash}[\sigma(\mathrm{Cn}_{\vdash}(\vec{x}) \cap \mathrm{Cn}_{\vdash}(\vec{y}))] = Fm \\ &\iff \mathrm{Cn}_{\vdash}(\Gamma) + \mathrm{Cn}_{\vdash}[\vec{\alpha}\nabla_{n,m}\vec{\beta}] = Fm\end{aligned}$$

The first equivalence holds by the definition of σ, while the second follows from Claim 3.6. To justify the last equivalence, observe that from the definitions of $\nabla_{n,m}$ and σ it follows that $\vec{\alpha}\nabla_{n,m}\vec{\beta}$ is the union of $\mathrm{Cn}_{\vdash}(\sigma\vec{x}) \cap \mathrm{Cn}_{\vdash}(\sigma\vec{y})$ over all possible substitutions σ such that $\sigma x_i = \alpha_i$ and $\sigma y_j = \beta_j$. Consequently, $\sigma(\mathrm{Cn}_{\vdash}(\vec{x}) \cap \mathrm{Cn}_{\vdash}(\vec{y})) \subseteq \vec{\alpha}\nabla_{n,m}\vec{\beta}$, and the left-to-right implication follows trivially. We prove the reverse inclusion.

To this end, let $\delta(\vec{x},\vec{y},\vec{z}) \in \nabla_{n,m}$. It suffices to show that

$$\sigma(\mathrm{Cn}_{\vdash}(\vec{x}) \cap \mathrm{Cn}_{\vdash}(\vec{y})) \vdash \delta(\vec{\alpha},\vec{\beta},\vec{\gamma}) \text{ for every } \vec{\gamma} \subseteq Fm.$$

Observe that σ is surjective, so there is $\vec{\eta} \subseteq Fm$ such that $\sigma\vec{\eta} = \vec{\gamma}$. Since $\nabla_{n,m}$ is a protodisjunction, we have that $\vec{x} \vdash \delta(\vec{x},\vec{y},\vec{\eta})$ and $\vec{y} \vdash \delta(\vec{x},\vec{y},\vec{\eta})$. Therefore $\delta(\vec{x},\vec{y},\vec{\eta}) \in \mathrm{Cn}_{\vdash}(\vec{x}) \cap \mathrm{Cn}_{\vdash}(\vec{y})$. Hence $\delta(\vec{\alpha},\vec{\beta},\vec{\gamma}) \in \sigma[\mathrm{Cn}_{\vdash}(\vec{x}) \cap \mathrm{Cn}_{\vdash}(\vec{y})]$. ⊠

Recall that Proposition 3.1 shows that any finitary logic satisfying an inconsistency lemma also has the inconsistency by cases property. While this result is proven syntactically, a purely algebraic proof can also be provided crossing back over the bridge given by Theorem 3.4. To this end, we present the following observation:

Remark 3.7. Every dually pseudo-complemented join semilattice with 1 is a 1-distributive semilattice. Indeed, let $\boldsymbol{A}$ be a dually pseudo-complemented semilattice and consider $a, b, c \in A$ such that $a \vee b = 1$ and $a \vee c = 1$. It follows from the pseudo-complementation of $\boldsymbol{A}$ that there exists a smallest element $a^* \in A$ such that $a \vee a^* = 1$. Therefore, we get that $a^* \vee b = b$ and $a^* \vee c = c$. Therefore,$a^* \leqslant b, c$, which implies that $\boldsymbol{A}$ is a 1-distributive join semilattice.

Consequently, the bridge theorems for the IL and the ICP, Theorem 2.6 and 3.4, respectively, allow us to obtain the following weaker version of Proposition 3.1.

Corollary 3.8. *If a finitary protoalgebraic logic $\vdash$ has an inconsistency lemma, then it has the inconsistency by cases property.*

Proof. In view of the semantic characterizations of the inconsistency lemma and the inconsistency by cases property, Theorems 2.6 and 3.4, respectively, the present theorem's statement amounts to the claim that if for any algebra $\boldsymbol{A}$ over the appropriate language, if its semilattice of compact $\vdash$-filters is dually pseudo-complemented with respect to $+^{\boldsymbol{A}}$, then its lattice of $\vdash$-filters is 1-distributive.

As the $\vdash$-filter lattice of an algebra $\boldsymbol{A}$ is algebraic, then it is isomorphic to the lattice of ideals of the join semilattice of compact filters of $\boldsymbol{A}$ by Proposition 1.9. In symbols, $\mathcal{F}i_{\vdash}(\boldsymbol{A}) \simeq \mathcal{I}d(Comp\mathcal{F}i_{\vdash}(\boldsymbol{A}))$. Consequently, to establish the 1-distributivity of $\mathcal{F}i_{\vdash}(\boldsymbol{A})$, it suffices to prove the same property for the lattice of ideals of $Comp\mathcal{F}i_{\vdash}(\boldsymbol{A})$. To this end, let $\boldsymbol{A}$ be an algebra such that the compact filters of $\boldsymbol{A}$ form a dually pseudo-complemented semilattice. The proof proceeds as follows: since the semilattice

$Comp\mathcal{F}i_{\vdash}(\boldsymbol{A})$ is dually pseudo-complemented, we can deduce from Remark 3.7 that it is 1-distributive, which together with Proposition 3.2 implies that the lattice of ideals $\mathcal{I}d(Comp\mathcal{F}i_{\vdash}(\boldsymbol{A}))$ is also 1-distributive. ☒

3.1 From the local inconsistency lemma to the global inconsistency lemma

According to Proposition 3.1 (or Corollary 3.8), every finitary protoalgebraic logic $\vdash$ with an inconsistency lemma has the inconsistency by cases property, which in turn implies by Theorem 3.4 that it is filter-1-distributive. Establishing whether the logics which admit a local inconsistency lemma are filter-1-distributive is more involved. Before we answer this question, we shall consider one more property.

Consider a logic $\vdash$, a non-empty family of sets of formulas $\{I_\alpha(x,\vec{z}) : \alpha \leqslant \beta\}$ has the *refinement property* with respect to a given logic $\vdash$ if for every pair $I_{\alpha_1}, I_{\alpha_2}$ with $\alpha_1, \alpha_2 < \beta$ there is a set I_α with $\alpha < \beta$ such that $I_\alpha \subseteq \mathrm{Cn}_{\vdash}(I_{\alpha_1}) \cap \mathrm{Cn}_{\vdash}(I_{\alpha_2})$.

Theorem 3.9. *Let $\vdash$ be a finitary protoalgebraic logic with the LIL witnessed by the family $\{\Psi_n : n \in \mathbb{Z}^+\}$. The following assertions are equivalent:*

(i) *$\vdash$ is filter-1-distributive.*

(ii) *For every $n \in \mathbb{Z}^+$, the family Ψ_n has the refinement property.*

Proof. (i)⇒(ii): Observe that if we assume the filter-1-distributivity of $\vdash$, then it can be inferred, considering the algebra of formulas, that the lattice $\mathcal{T}h(\vdash)$ of theories of $\vdash$ is 1-distributive.

Let $\{\Psi_n : n \in \mathbb{Z}^+\}$ be the family of sets of formulas which witnesses the local inconsistency lemma for $\vdash$, and consider $I_1, I_2 \in \Psi_n$ and $\alpha_1, \ldots, \alpha_n \in Fm$. Since $I_1(\alpha_1,\ldots,\alpha_n) \vdash I_1(\alpha_1,\ldots,\alpha_n)$ and $I_2(\alpha_1,\ldots,\alpha_n) \vdash I_2(\alpha_1,\ldots,\alpha_n)$, the local inconsistency lemma for $\vdash$ implies that both sets of formulas $\{\alpha_1,\ldots,\alpha_n\} \cup I_1(\alpha_1,\ldots,\alpha_n)$ and $\{\alpha_1,\ldots,\alpha_n\} \cup I_2(\alpha_1,\ldots,\alpha_n)$ are inconsistent in $\vdash$. Using the definition of $+$ in $\mathcal{T}h(\vdash)$, this can be equivalently stated as $\mathrm{Cn}_{\vdash}(\{\alpha_1,\ldots,\alpha_n\}) + \mathrm{Cn}_{\vdash}(I_1(\alpha_1,\ldots,\alpha_n)) = Fm$ and $\mathrm{Cn}_{\vdash}(\{\alpha_1,\ldots,\alpha_n\}) + \mathrm{Cn}_{\vdash}(I_2(\alpha_1,\ldots,\alpha_n)) = Fm$. In view of the 1-distributivity of $\mathcal{T}h(\vdash)$, it follows that

$$\mathrm{Cn}_{\vdash}(\{\alpha_1,\ldots,\alpha_n\}) + [\mathrm{Cn}(I_1(\alpha_1,\ldots,\alpha_n)) \cap \mathrm{Cn}(I_2(\alpha_1,\ldots,\alpha_n))] = Fm.$$

Together with the LIL, this implies that there is some $I \in \Psi_n$ such that

$$\mathrm{Cn}(I_1(\alpha_1,\ldots,\alpha_n)) \cap \mathrm{Cn}(I_2(\alpha_1,\ldots,\alpha_n)) \vdash I(\alpha_1,\ldots,\alpha_n).$$

We conclude that $I_1(\alpha_1,\ldots,\alpha_n) \vdash I(\alpha_1,\ldots,\alpha_n)$ and $I_2(\alpha_1,\ldots,\alpha_n) \vdash I(\alpha_1,\ldots,\alpha_n)$ for some $I \in \Psi_n$. Thus, for every $n \in \mathbb{Z}^+$, the family Ψ_n has the refinement property.

(ii)⇒(i): In order to show that the lattice $\mathcal{F}i_{\vdash}(\boldsymbol{A})$ is 1-distributive for every algebra $\boldsymbol{A}$ of the appropiate language, recall from Proposition 1.9 that for a finitary logic $\vdash$, the $\vdash$-filter lattice of an algebra $\boldsymbol{A}$ is algebraic and, therefore, it is isomorphic to the lattice of ideals of the join semilattice of compact deductive filters of $\boldsymbol{A}$. In symbols, $\mathcal{F}i_{\vdash}(\boldsymbol{A}) \simeq \mathcal{I}d(Comp\mathcal{F}i_{\vdash}(\boldsymbol{A}))$. Furthermore, the compact elements of $\mathcal{F}i_{\vdash}(\boldsymbol{A})$ are just the finitely generated filters of $\boldsymbol{A}$. Therefore, in view of Proposition 3.2, it suffices to show that for an arbitrary algebra $\boldsymbol{A}$, the join semilattice $\mathcal{C}omp\mathcal{F}i_{\vdash}(\boldsymbol{A})$ is 1-distributive.

To this end, we observe that it follows from the assumption that $\vdash$ has a LIL, together with Remark 2.3, that there is some finite set of formulas $\Xi(x)$ such that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(\Xi^{\boldsymbol{A}}(a))$ for every $a \in A$. Therefore, the deductive filter A is finitely generated, i.e., $\mathcal{C}omp\mathcal{F}i_{\vdash}(\boldsymbol{A})$ has a maximum element.

Now, let $a_1, \dots, a_n, b_1, \dots, b_m, c_1, \dots, c_k \in A$ such that

$$\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \dots, a_n) +^{A} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \dots, b_m) = A \text{ and } \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \dots, a_n) +^{A} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(c_1, \dots, c_k) = A.$$

From the definition of $+$ in $\mathcal{T}h(\vdash)$ together with the semantic LIL, Theorem 2.4, it follows that $I_1^{\boldsymbol{A}}(a_1, \dots, a_n) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \dots, b_m)$ and $I_2^{\boldsymbol{A}}(a_1, \dots, a_n) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(c_1, \dots, c_k)$ for some $I_1, I_2 \in \Psi_n$.

Since, by assumption, for every $n \in \mathbb{Z}^+$ the family Ψ_n has the refinement property with respect to $\vdash$, there is some $I \in \Psi_n$ such that $I \subseteq \mathrm{Cn}_{\vdash}(I_1) \cap \mathrm{Cn}_{\vdash}(I_2)$. Therefore, let $h : \boldsymbol{Fm} \to \boldsymbol{A}$ be a homomorphism such that $h(x_i) = a_i$ for $i \in \{1, \dots, n\}$. From $I_1^{\boldsymbol{A}}(a_1, \dots, a_n) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \dots, b_m)$ it follows that $h(I_1(x_1, \dots, x_n)) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \dots, b_m)$ and from $I_2^{\boldsymbol{A}}(a_1, \dots, a_n) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(c_1, \dots, c_k)$ that $h(I_2(x_1, \dots, x_n)) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(c_1, \dots, c_k)$. Hence, we obtain

$$h(I(x_1, \dots, x_n)) = I^{\boldsymbol{A}}(a_1, \dots, a_n) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \dots, b_m), \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(c_1, \dots, c_k).$$

This implies that $\mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(Z) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(b_1, \dots, b_m), \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(c_1, \dots, c_k)$ for $Z = I^{\boldsymbol{A}}(a_1, \dots, a_n)$. Moreover, Z is a finite subset of A because I is guaranteed to be finite by the definition of the LIL. Finally, from $I^{\boldsymbol{A}}(a_1, \dots, a_n) \subseteq \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(Z)$, together with Theorem 2.4, we obtain that $A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(Z \cup \{a_1, \dots, a_n\})$, which is equivalent to

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(a_1, \dots, a_n) +^{A} \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(Z),$$

by the definition of the join operator in $\mathcal{T}h(\vdash)$. Thus, we have proved that $\vdash$ is filter-1-distributive. ⊠

To conclude this chapter, we shall apply the preceding results to characterize the set of logics $\vdash$ that have an inconsistency lemma. The theorem presented below is reminiscent of [15, Thm. 2.6.2], which provides a similar characterization concerning the deduction-detachment theorem.

Theorem 3.10. *A finitary protoalgebraic logic $\vdash$ has an inconsistency lemma if and only if it has the maximal consistent filter extension property, for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated, it has definable maximal consistent filters, and it is filter-1-distributive.*

Proof. Observe that if a logic $\vdash$ has an inconsistency lemma, then in particular it has a local inconsistency lemma witnessed by the family Ψ_n consisting of just one finite set of formulas $I(x_1, \dots, x_n)$ for each $n \in \mathbb{Z}^+$. The forward implication then follows directly from Theorem 2.12, Theorem 2.15, Corollary 3.8, and Theorem 3.4.

We turn to prove the converse implication. To this end, assume that $\vdash$ has the maximal consistent filter extension property and for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated, then Theorem 2.12 guarantees that $\vdash$ enjoys the LIL witnessed by a family $\{\Psi_n : n \in \mathbb{Z}^+\}$, i.e., for every $\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \subseteq Fm$,

$$\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \text{ is inconsistent in } \vdash \Longleftrightarrow \Gamma \vdash I(\alpha_1, \dots, \alpha_n) \text{ for some } I \in \Psi_n.$$

Additionally, assuming that $\vdash$ has definable maximal consistent filters, then Theorem 2.15 implies that for every $n \in \mathbb{Z}^+$, the family Ψ_n can be assumed to be finite. In other words, we can assume that there exists $m \in \mathbb{Z}^+$ such that $\Psi_n(x_1, \dots, x_n)$ is a finite family of finite sets of formulas $\{I_1(x_1, \dots, x_n), \dots, I_m(x_1, \dots, x_n)\}$. Consequently, for every $\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \subseteq Fm$,

$$\Gamma \cup \{\alpha_1, \dots, \alpha\} \text{ is inconsistent } \iff \Gamma \vdash I_1(\alpha_1, \dots, \alpha_n) \text{ or } \dots \text{ or } \Gamma \vdash I_m(\alpha_1, \dots, \alpha_n).$$

Finally, under the assumption that $\vdash$ is filter-1-distributive, we claim that for every $n \in \mathbb{Z}^+$, the family Ψ_n may be assumed to consist of just one finite set of formulas $I(x_1, \dots, x_n)$. Indeed, in view of Theorem 3.9, the filter-1-distributivity of $\vdash$ implies that for every $n \in \mathbb{Z}^+$, the family Ψ_n has the refinement property with respect to $\vdash$, i.e, for every pair $I_1, I_2 \in \Psi_n$ there is a set $I \in \psi_n$ such that $I_1 \vdash I$ and $I_2 \vdash I$. As a consequence there exists $I \in \{I_1(x_1, \dots, x_n), \dots, I_m(x_1, \dots, x_n)\}$ such that $I_i(x_1, \dots, x_n) \vdash I(x_1, \dots, x_n)$ for every $i \in \{1, \dots, m\}$. Therefore, for every $\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \subseteq Fm$, we have

$$\Gamma \cup \{\alpha_1, \dots, \alpha_n\} \text{ is inconsistent } \iff \Gamma \vdash I(\alpha_1, \dots, \alpha_n).$$

⊠

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