UNIVERSITAT DE BARCELONA
SCHOOL OF ECONOMICS

Working Papers

Col·lecció d'Economia E24/466

SEQUENTIAL CREATION OF SURPLUS AND THE SHAPLEY VALUE

Mikel Álvarez-Mozos

Inés Macho-Stadler

David Pérez-Castrillo

ISSN 1136-8365

UNIVERSITAT DE BARCELONA
SCHOOL OF ECONOMICS

Working Papers

UB Economics *Working Paper No. 466*

Title: Sequential Creation of Surplus and the Shapley Value

Abstract: We introduce the family of games with intertemporal externalities, where two disjoint sets of players play sequentially. Coalitions formed by the present cohort generate worth today. Moreover, today's partition of players exerts an externality on the future; the worth of a coalition formed by future players is influenced by this externality. We adapt the classic Shapley axioms and study their implications in our class of games. They do not suffice to single out a unique solution. We introduce two values using the interpretation of the Shapley value as the players' expected contributions to coalitions: the one-coalition externality value and the naive value. We state a relationship between these values and the Shapley value of an associated game in characteristic function form. Our main results characterize the two values by adding one additional property to the classic Shapley axioms. A property of equal treatment of contributions leads to characterizing the one-coalition externality value. A property of equal treatment of externalities characterizes the naive value.

JEL Codes: C71, D62

Keywords: Shapley value, Externalities

Authors:

Mikel Álvarez-Mozos	Inés Macho-Stadler	David Pérez-Castrillo
Universitat de Barcelona and BEAT	Universitat Autônoma de Barcelona and BSE	Universitat Autônoma de Barcelona and BSE
Email: mikel.alvarez@ub.edu	**Email**: ines.macho@uab.cat	**Email**: david.perez@uab.cat

Date: July 2024

UNIVERSITAT DE BARCELONA
SCHOOL OF ECONOMICS

Working Papers

Acknowledgements:

We thank Chenghong Luo, Roberto Serrano, Chaoran Sun, David Wettstein, and participants in the Nanjing seminar, the SING conference in Messina, the ASSET meeting in Lisbon, REES in Castelló, and SAEe in Salamanca for many comments and discussions on this paper. We also thank MINECO and Feder (PID2020-113110GB-100 and PID2021-122403NB-I00), Generalitat de Catalunya (2021 SGR 00194 and 2021 SGR 00306), Severo Ochoa program (CEX2019-000915-S), and ICREA under the ICREA Academia program for their financial support.

1 Introduction

Our choices today may directly or indirectly affect the well-being of future generations. This is especially true for decisions with long time horizons, such as the extraction of non-renewable resources, the efforts to reduce greenhouse gas (GHG) emissions, the treatment of nuclear waste disposal, the construction of long-lived infrastructures, or the investment in technical innovation.

From a normative perspective, if today's choices shape future generations' conditions, then it is necessary to discuss how we take the future players (our children, our grandchildren, and those who will follow) into account when deciding the sharing of the surplus of these decisions.

Our paper considers this inter-generational situation by defining a new family of games, which we refer to as *games with intertemporal externalities*. It proposes cooperative solutions, acknowledging that one generation may be making decisions for people who cannot speak for their interests at the time.

Take the example of global warming. This is a cooperative game with intertemporal externalities, where today's choices are represented by the coalitions formed by today's players. Today, players are aware of the future effects of their decisions on global warming; it is estimated that a significant joint effort is needed to meet global warming below the 1.5ºC and 2ºC targets by the end of the XXI century. This is an externality for the future generation. However, these efforts to reduce greenhouse gas do not seem urgent for the present generation since the consequences will be felt in the future, and today's generation may not internalize the externalities imposed on the next generation.

In a game with intertemporal externalities, there are two disjoint sets of players. Coalitions formed by the present cohort generate worth today. Moreover, the partition of today's generation exerts an externality on the future cohort. Hence, the worth of a coalition of future players depends on the externality inherited from the past generation. In such a game, a value that shares the total surplus needs to consider the two periods and the two sets of players.

We adapt the classic Shapley axioms to games with intertemporal externalities and study their implications. However, they do not suffice to single out a unique solution. We introduce two values using the common interpretation of the Shapley value as the players' expected contributions to coalitions: the *one-coalition externality value* and the *naive value*. We show the relationship between these values and the Shapley value of an associated game in characteristic function form.

Our main results characterize the two values by adding one additional property to the classic Shapley axioms. We show that a property of equal treatment of contributions leads to characterizing the one-coalition externality value. In contrast, a property of equal treatment of externalities characterizes the naive value.

The games with intertemporal externalities differ from other cooperative games. However, they share similarities with the "games with externalities," also called "partition function form games" (Thrall and Lucas, 1963). In this class of games, there is a unique set of players, and the worth of each coalition depends on the organization of the outside players. Recent literature studies extensions of the Shapley value for this class of games (see, e.g., Myerson, 1977; Macho-Stadler, Pérez-Castrillo, and Wettstein, 2007; De Clippel and Serrano, 2008; and McQuillin, 2009; Alonso-Meijide et al., 2019).[1] However, the family of games with intertemporal externalities is not included and does not include the family of games with externalities.

The rest of the paper is organized as follows. Section 2 introduces the family of games with intertemporal externalities. Section 3 adapts the Shapley axioms and describes the structure of any value that satisfies them. Section 4 intuitively introduces the one-coalition externality and the naive values and states their relationship with the Shapley value of an associated game in characteristic function form. Sections 5 and 6 axiomatically characterize the two values, respectively. Section 7 discusses the prescription of the values for games with intertemporal additive externalities. Section 8 discusses the relationship between values for games with intertemporal externalities and values for partition function form games. Section 9 concludes.

2 Framework

We introduce a new family of games called "games with intertemporal externalities." A game with intertemporal externalities is played by two disjoint sets of players, N_1 and N_2, with $N_1 \cap N_2 = \emptyset$. We think of players in N_1 interacting at period $t = 1$, whereas players in N_2 interact at $t = 2$.[2] We denote generic players of N_1 by i, i', generic players of N_2 by j, j', and generic players of $N_1 \cup N_2$ by h, h'.

A coalition S_1 of N_1 is a group of players of that set, that is, a non-empty subset of N_1, $S_1 \subseteq N_1$. If a coalition S_1 forms, the players obtain jointly a surplus of $v_1(S_1) \in \mathbb{R}$. The worth $v_1(S_1)$ only depends on the coalition S_1 and not on how the other players in $N_1 \setminus S_1$ or N_2 are organized.

A coalition S_2 of N_2 is a non-empty subset of N_2, $S_2 \subseteq N_2$. Contrary to what happens at $t = 1$, the worth obtained by a coalition of N_2 depends not only on the identity of

[1] For reviews of the literature on values for games with externalities, see Kóczy (2018) and Macho-Stadler, Pérez-Castrillo, and Wettstein (2019).

[2] There are other environments with two sets of players where our model applies. For instance, the two groups of players may live at two completely separate locations along a river.

the players in the coalition but also on the past organization of the players in N_1; that is, there are intertemporal externalities between $t = 1$ and $t = 2$. To formally express these externalities, denote by $\mathcal{P}(M)$ the set of partitions of a finite set M. Then, if the coalition S_2 forms and the players in N_1 were organized according to the partition $P_1 \in \mathcal{P}(N_1)$, the coalition S_2 generates a surplus $v_2(S_2; P_1) \in \mathbb{R}$.

The utility is transferable among all the players; that is, the cooperative game is a transferable utility (TU) game. In our two-period interpretation of the model, being a TU game requires the existence of a perfect credit market that allows transferring money at zero interest rate (or at zero cost) in any direction between $t = 1$ and $t = 2$.

Therefore, a *game with intertemporal externalities*, or simply a *game*, is a pair (N, v) with $N = (N_1, N_2)$ and $v = (v_1, v_2)$, where $v_1 : 2^{N_1} \to \mathbb{R}$ and $v_2 : 2^{N_2} \times \mathcal{P}(N_1) \to \mathbb{R}$, with $v_1(\emptyset) = 0$ and $v_2(\emptyset; P_1) = 0$ for any $P_1 \in \mathcal{P}(N_1)$. We denote the set of all games by $\mathcal{G}$.

We look for proposals for the division of the surplus created in games with intertemporal externalities. A *value* is a mapping $\Phi : \mathcal{G} \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ that satisfies

$$\sum_{h \in N_1 \cup N_2} \Phi_h(N, v) = v_1(N_1) + v_2(N_2; \{N_1\}).$$

Note that we have in mind environments where it is efficient that the grand coalition forms in both periods. Hence, our definition of a value entails *efficiency*.

3 The "basic" axioms

In this section, we introduce some reasonable requirements to impose on a value by extending those characterizing the Shapley value in TU games without externalities. These are the axioms of linearity, anonymity, and "dummy" player. We also analyze the implications of these axioms on the characteristics of a value.

We first define the operations of *addition* and *multiplication by a scalar*, and the notions of *permutation of a game* and *dummy player*.

Definition 1. *(a) The addition of two games (N, v) and (N, v') is the game $(N, v + v')$ defined by $v + v' = (v_1 + v'_1, v_2 + v'_2)$, where $(v_1 + v'_1)(S_1) \equiv v_1(S_1) + v'_1(S_1)$ for all $S_1 \subseteq N_1$ and $(v_2 + v'_2)(S_2; P_1) \equiv v_2(S_2; P_1) + v'_2(S_2; P_1)$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.*

(b) Given a game (N, v) and a scalar $\lambda \in \mathbb{R}$, the game $(N, \lambda v)$ is defined by $\lambda v = (\lambda v_1, \lambda v_2)$, where $(\lambda v_1)(S_1) \equiv \lambda v_1(S_1)$ for all $S_1 \subseteq N_1$ and $(\lambda v_2)(S_2; P_1) \equiv \lambda v_2(S_2; P_1)$ for all $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

The permutation of a game uses the notion of a permutation of $N = (N_1, N_2)$. Given that N is composed of two disjoint sets, a permutation of N consists of a permutation of each set. That is, a *permutation of* $N = (N_1, N_2)$ is a pair $\sigma = (\sigma_1, \sigma_2)$, where σ_1 is a permutation of N_1 and σ_2 is a permutation of N_2.

Definition 2. *Let* $(N, v) \in \mathcal{G}$ *and* σ *be a permutation of* N*. The permuted game* $(N, \sigma v)$ *is defined by* $\sigma v = (\sigma v_1, \sigma v_2)$*, where* $\sigma v_1(S_1) \equiv v_1(\sigma_1(S_1))$ *for all* $S_1 \subseteq N_1$*, and* $\sigma v_2(S_2; P_1) \equiv v_2(\sigma_2(S_2); \sigma_1(P_1))$ *for all* $S_2 \subseteq N_2$ *and* $P_1 \in \mathcal{P}(N_1)$*.*

To define a dummy player, notice that a player in N_1 may influence the surplus generated at both periods. On the other hand, a player in N_2 only affects the surplus generated at $t = 2$, although her influence may depend on the organization of the players at $t = 1$. This is why the definition of a dummy player differs for the players in N_1 and N_2.

For every partition $P \in \mathcal{P}(N_1)$, and player $i \in N_1$, we define $P^{-i} \equiv \{S_1 \setminus \{i\} : S_1 \in P\} \cup \{\{i\}\}$. Then:

Definition 3. *(a) Player* $i \in N_1$ *is a dummy player in the game* (N, v) *if*

$$\begin{aligned} &v_1(S_1) = v_1(S_1 \backslash \{i\}) && \text{for all } S_1 \subseteq N_1 \text{ and} \\ &v_2(S_2; P_1) = v_2\left(S_2; P_1^{-i}\right) && \text{for all } S_2 \subseteq N_2 \text{ and all } P_1 \in \mathcal{P}(N_1). \end{aligned}$$

(b) Player $j \in N_2$ *is a dummy player in the game* (N, v) *if* $v_2(S_2; P_1) = v_2(S_2 \backslash \{j\}; P_1)$ *for all* $S_2 \subseteq N_2$ *and* $P_1 \in \mathcal{P}(N_1)$*.*

Note that there are two requirements for a player in the first period to be a dummy. It should be a classic dummy in the game of period 1, and it should not generate any externality in the coalitions of the second period when she leaves a coalition of the first period to remain singleton. The second requirement is in the same spirit as in the definition of a dummy player in games with externalities (for instance, Bolger, 1989; Macho-Stadler, Pérez-Castrillo, and Wettstein, 2007).

We now adapt the three original Shapley (1953b) value axioms to our environment:

Axiom 1. *Linearity: A value* Φ *is linear if*

1.1. $\Phi(N, v + v') = \Phi(N, v) + \Phi(N, v')$ *for any* $(N, v), (N, v') \in \mathcal{G}$*, and*

1.2. $\Phi(N, \lambda v) = \lambda \Phi(N, v)$ *for any* $\lambda \in \mathbb{R}$ *and* $(N, v) \in \mathcal{G}$*.*

Axiom 2. *Anonymity: A value* Φ *satisfies anonymity if for any game* $(N, v) \in \mathcal{G}$ *and permutation* σ *of* N*,*

$$\begin{aligned} &\Phi_i(N, \sigma v) = \Phi_{\sigma_1(i)}(N, v) \text{ for all } i \in N_1 \text{ and} \\ &\Phi_j(N, \sigma v) = \Phi_{\sigma_2(j)}(N, v) \text{ for all } j \in N_2. \end{aligned}$$

Axiom 3. *Dummy player: A value* Φ *satisfies the dummy player axiom if, for any game* $(N, v) \in \mathcal{G}$*,* $\Phi_h(N, v) = 0$ *if* $h \in N_1 \cup N_2$ *is a dummy player in the game* (N, v)*.*

The classic properties of linearity, anonymity, and dummy player in which our axioms are inspired characterize a unique value (Shapley, 1953b) in the set of games in characteristic function form, which we will refer to as *CFF games*. Let us denote by $\mathcal{G}^{CFF}$ the set of CFF games and $(M, \hat{w}) \in \mathcal{G}^{CFF}$ a CFF game, i.e., M is the set of players and $\hat{w} : 2^M \to \mathbb{R}$ is the characteristic function.[3] The Shapley value Sh of a player $h \in M$ can be written as

$$Sh_h(M, \hat{w}) = \sum_{S \subseteq M} \beta_h(M, S)\, \hat{w}(S) = \sum_{S \subseteq M, S \ni h} \beta_h(M, S)\left(\hat{w}(S) - \hat{w}(S \backslash \{h\})\right),$$

where the *Shapley coefficients*, $\beta_h(M, S)$, are defined for every $S \subseteq M$ by,[4]

$$\beta_h(M, S) = \begin{cases} \frac{(|S|-1)!(|M|-|S|)!)}{|M|!} & \text{if } h \in S \\ \frac{-(|S|!(|M|-|S|-1)!)}{|M|!} & \text{if } h \in M \backslash S. \end{cases} \tag{1}$$

Note that if $N_1 = \emptyset$ or $N_2 = \emptyset$, then the game with intertemporal externalities (N, v) is essentially a CFF game where the set of players is either N_2 or N_1, respectively. Therefore, any value that satisfies axioms 1 to 3 proposes the Shapley value for those games.

Moreover, consider a game (N, v) where both sets, N_1 and N_2, are non-empty, but there are no intertemporal externalities. That is, suppose that the surplus generated by any coalition of N_2 does not depend on the organization of the players in $t = 1$. Denote by $(N_1, \hat{v}_1)$ the CFF game where $\hat{v}_1(S_1) = v_1(S_1)$ for all $S_1 \in N_1$.[5] Also, for a game without externalities, denote $\hat{v}_2(S_2) \equiv v_2(S_2; P_1)$ for any $S_2 \in N_2$ and $P_1 \in \mathcal{P}(N_1)$. Then, for the game (N, v), a value satisfying the three axioms allocates the Shapley value of $(N_1, \hat{v}_1)$ to the players of N_1 and the Shapley value of $(N_2, \hat{v}_2)$ to the players of N_2. We state and prove this result in Proposition 1.

Proposition 1. *Take a value Φ satisfying linearity, anonymity, and the dummy player axiom. Also, consider a game (N, v) without externalities, that is, $v_2(S_2; P_1) = v_2(S_2; Q_1)$ for all $S_2 \subseteq N_2$ and $P_1, Q_1 \in \mathcal{P}(N_1)$. Then,*

$$\Phi_i(N, v) = Sh_i(N_1, \hat{v}_1) \text{ for all } i \in N_1 \text{ and}$$
$$\Phi_j(N, v) = Sh_j(N_2, \hat{v}_2) \text{ for all } j \in N_2.$$

[3] We will use characters with "hat," as $\hat{w}$, to easily identify when we refer to the characteristic function of a CFF game instead of a worth function in a game with intertemporal externalities.

[4] We denote $|M|$ the number of players in M, for any finite set M.

[5] Note that we use v_1 to refer to the first component of the vector v in the game with intertemporal externalities (N, v); whereas $\hat{v}_1$ is the characteristic function of the CFF game without externalities $(N_1, \hat{v}_1)$.

Proof. Define the games (N, v^a), $\left(N, v^b\right) \in \mathcal{G}$ as follows:

$$v_1^a(S_1) = v_1(S_1) \text{ for all } S_1 \subseteq N_1,$$
$$v_2^a(S_2; P_1) = 0 \text{ for all } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1),$$
$$v_1^b(S_1) = 0 \text{ for all } S_1 \subseteq N_1,$$
$$v_2^b(S_2; P_1) = v_2(S_2; P_1) \text{ for all } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1).$$

Note that $(N, v) = \left(N, v^a + v^b\right)$. Then, by linearity, $\Phi_h(N, v) = \Phi_h(N, v^a) + \Phi_h(N, v^b)$ for all $h \in N_1 \cup N_2$.

All the players in N_2 are dummy players in (N, v^a). Then, by the dummy player axiom $\Phi_j(N, v^a) = 0$ for every $j \in N_2$. Moreover, (N, v^a) is essentially a CFF game among the players in N_1 with a characteristic function $\hat{v}_1$, which is equal to the function v_1^a. Then, we can follow the same steps as in the original proof by Shapley (1953b) and conclude that $\Phi_i(N, v^a) = Sh_i(N_1, \hat{v}_1)$ for every $i \in N_1$.

Similarly, all the players in N_1 are dummy players in $\left(N, v^b\right)$: A player $i \in N_1$ does not generate any value in v_1^b, and her position in the partition formed at $t = 1$ does not affect the surplus of any coalition $S_2 \subseteq N_2$, given that $v_2\left(S_2; P_1\right) = v_2\left(S_2; Q_1\right)$ for all $P_1, Q_1 \in \mathcal{P}(N_1)$. Hence, by the dummy player property, $\Phi_i\left(N, v^b\right) = 0$, for every $i \in N_1$. Then, $\left(N, v^b\right)$ is essentially a CFF game among the players in N_2 with characteristic function $\hat{v}_2$. The classic characterization of the Shapley value implies that $\Phi_j\left(N, v^b\right) = Sh_j\left(N_2, \hat{v}_2\right)$, for every $j \in N_2$. □

Proposition 2 goes a step forward. It shows that because no externalities affect the function v_1, the worth generated at $t = 1$ should always be split only among the players in N_1, and the sharing should be done according to the Shapley value. On the other hand, the function v_2 receives the influence of players in N_1 and N_2; hence, all the players may share the surplus obtained at $t = 2$.

Proposition 2. *Take a value Φ satisfying linearity, anonymity, and the dummy player axiom. Then for every $(N, v) \in \mathcal{G}$ there exists a function f satisfying*

$$\sum_{h \in N_1 \cup N_2} f_h(N_1, N_2, v_2) = v_2(N_2; \{N_1\})$$

such that,

$$\Phi_i(N, v) = Sh_i(N_1, \hat{v}_1) + f_i(N_1, N_2, v_2) \textit{ for all } i \in N_1 \textit{ and}$$
$$\Phi_j(N, v) = f_j(N_1, N_2, v_2) \textit{ for all } j \in N_2.$$

Proof. We define the games (N, v^a) and $\left(N, v^b\right)$ as in the proof of Proposition 1. The players in N_2 are dummy players in (N, v^a) hence, $\Phi_j(N, v^a) = 0$ for every $j \in N_2$. By the same argument as in the previous proof, $\Phi_i(N, v^a) = Sh_i(N_1, \hat{v}_1)$ for every $i \in N_1$.

On the other hand, (N, v^b) is a game where players in N_1 do not generate value in $t = 1$, but they exert externalities in $t = 2$. The value obtained by the players in the game (N, v^b) can depend on the sets N_1 and N_2 and on the function v_2, but not on v_1. That is, $\Phi_h(N, v^b)$ corresponds to a function $f_h(N_1, N_2, v_2)$, for every $h \in N_1 \cup N_2$.

The linearity of the value implies $\Phi_h(N, v) = \Phi_h(N, v^a) + \Phi_h(N, v^b)$ for all $h \in N_1 \cup N_2$, which leads to the expressions of $\Phi_h(N, v)$ stated in the proposition.

Finally, $\sum_{h \in N_1 \cup N_2} f_h(N_1, N_2, v_2) = \sum_{h \in N_1 \cup N_2} \Phi_h(N, v^b) = v_1^b(N_1) + v_2^b(N_2; \{N_1\}) = v_2(N_2; \{N_1\})$ by the efficiency of Φ. □

Proposition 2 provides the structure of the payoffs received by the players according to a value that satisfies the basic axioms of linearity, anonymity, and dummy player. However, contrary to what happens in the set of CFF games, the three axioms do not characterize a unique value in the set of games with intertemporal externalities. The following sections first introduce and then characterize two values that satisfy the basic axioms together with additional properties.

4 The players' expected contribution for two random arrival processes

A common interpretation of the Shapley value of a player in a CFF game $(M, \hat{w}) \in \mathcal{G}^{CFF}$ is that it corresponds to her expected contribution to coalitions, where the distribution of coalitions arises in a particular way. Specifically, suppose the players enter a room in some order and that all $|M|!$ orderings of the players in M are equally likely. Then $Sh_h(M, \hat{w})$ is the player h' expected contribution as she enters the room.

In the following two subsections, we propose two "natural" ways players can enter the room in a game with intertemporal externalities; each leads to a value on $\mathcal{G}$.

4.1 All orderings are feasible

We first consider a situation where, to compute the expected contribution of a player, we assume that the players can "arrive" in any order. Hence, we consider orders that intersperse players in N_1 and N_2. Given the temporal dimension of our games, one could view these orders as thought experiments of how players in N_2 may perceive what happened in period 1, that is, what would have happened if the grand coalition of period 1, N_1, had not formed.

Take a game $(N, v) \in \mathcal{G}$. An *ordering* of $N_1 \cup N_2$ is an injective mapping $\omega : N_1 \cup N_2 \to \{1, \ldots, |N_1| + |N_2|\}$. Let $\Omega(N_1 \cup N_2)$ denote the set of orderings of $N_1 \cup N_2$. The set of players present at a given step k (that is, the set of predecessors together with the player who arrives at k), with $k \in \{1, \ldots, |N_1| + |N_2|\}$, is $\omega^{-1}(\{1, \ldots, k\})$. We divide this set in two:

$$B_1^\omega(k) = \omega^{-1}(\{1, \ldots, k\}) \cap N_1,$$
$$B_2^\omega(k) = \omega^{-1}(\{1, \ldots, k\}) \cap N_2,$$

and we define $B_1^\omega(0) = B_2^\omega(0) = \emptyset$. That is, $B_1^\omega(k)$ (respectively, $B_2^\omega(k)$) is the set of players who have arrived at step k who belong to N_1 (respectively, N_2).

We compute the contribution of a player given an ordering ω. Take the player who arrives in the k^{th} step, that is, player $\omega^{-1}(k)$. If she belongs to N_1, then she contributes to the surplus obtained according to v_1 since the worth of the coalition $B_1^\omega(k)$ may be different from that of $B_1^\omega(k-1)$ due to the addition of $\omega^{-1}(k)$. Hence, the first contribution of player $\omega^{-1}(k)$ is $v_1(B_1^\omega(k)) - v_1(B_1^\omega(k-1))$. Moreover, player $\omega^{-1}(k)$ may also contribute by changing the externality that players in N_1 exert over the coalition of N_2 formed at this step, that is, $B_2^\omega(k)$ (that coincides with $B_2^\omega(k-1)$). In this logic, we assume that the players in N_1 who have not arrived yet, that is, those in $N_1 \setminus B_1^\omega(k)$, remain singletons. Hence, the contribution of player $\omega^{-1}(k)$ to the worth generated by the players in N_2 is

$$v_2(B_2^\omega(k); \{B_1^\omega(k)\} \cup \{\{i\} : i \in N_1 \setminus B_1^\omega(k)\}) \\ - v_2(B_2^\omega(k); \{B_1^\omega(k-1)\} \cup \{\{i\} : i \in N_1 \setminus B_1^\omega(k-1)\}).$$

If the player $\omega^{-1}(k)$ is in N_2, she may only change the surplus generated by the function v_2. This contribution depends on the set of players in N_1 who have already arrived. Following the same logic as before, the contribution of $\omega^{-1}(k)$, in this case, is

$$v_2(B_2^\omega(k); \{B_1^\omega(k)\} \cup \{\{i\} : i \in N_1 \setminus B_1^\omega(k)\}) \\ - v_2(B_2^\omega(k-1); \{B_1^\omega(k)\} \cup \{\{i\} : i \in N_1 \setminus B_1^\omega(k)\}).$$

Therefore, using that $B_2^\omega(k) = B_2^\omega(k-1)$ if $\omega^{-1}(k) \in N_1$ and $B_1^\omega(k) = B_1^\omega(k-1)$ if $\omega^{-1}(k) \in N_2$, we can write the contribution to (N, v) of the player who arrives at step $k \in \{1, \ldots, |N_1| + |N_2|\}$ of ω as:

$$\begin{aligned} m_k^\omega(N, v) &= v_1(B_1^\omega(k)) - v_1(B_1^\omega(k-1)) \\ &+ v_2(B_2^\omega(k); \{B_1^\omega(k)\} \cup \{\{i\} : i \in N_1 \setminus B_1^\omega(k)\}) \\ &- v_2(B_2^\omega(k-1); \{B_1^\omega(k-1)\} \cup \{\{i\} : i \in N_1 \setminus B_1^\omega(k-1)\}). \end{aligned}$$

The *one-coalition externality value* Φ^{1c} allocates to every player $h \in N_1 \cup N_2$ in the game (N, v) her expected contribution to the game when all the orderings have the same probability, that is,[6]

$$\Phi_h^{1c}(N,v) = \frac{1}{(|N_1|+|N_2|)!} \sum_{\omega \in \Omega(N_1 \cup N_2)} m_{\omega(h)}^{\omega}(N,v). \tag{2}$$

Note that Φ^{1c} is a well-defined value because, for each order, the contributions of all the players in $N_1 \cup N_2$ add up to $v_1(N_1) + v_2(N_2; \{N_1\})$: for any $\omega \in \Omega(N_1 \cup N_2)$,

$$\begin{aligned} \sum_{k=1}^{|N_1|+|N_2|} m_k^{\omega}(N,v) &= v_1(N_1) + v_2\left(N_2; \{N_1\}\right) - v_1(\emptyset) - v_2(\emptyset; \{\{i\} : i \in N_1\}) \\ &= v_1(N_1) + v_2\left(N_2; \{N_1\}\right). \end{aligned}$$

We now relate the one-coalition externality value of a game with intertemporal externalities to the Shapley value of an associated CFF game. For any game $(N,v) \in \mathcal{G}$, define the *associated game* $(N_1 \cup N_2, \hat{v}) \in \mathcal{G}^{CFF}$ as follows:

$$\hat{v}(S) \equiv v_1\left(S \cap N_1\right) + v_2\left(S \cap N_2; \{S \cap N_1\} \cup \{\{i\} : i \in N_1 \setminus S\}\right),$$

for every $S \subseteq N_1 \cup N_2$. Proposition 3 states that the one-coalition externality value of (N,v) and the Shapley value of $(N_1 \cup N_2, \hat{v})$ coincide.

Proposition 3. *For any game with intertemporal externalities $(N,v) \in \mathcal{G}$,*

$$\Phi^{1c}(N,v) = Sh(N_1 \cup N_2, \hat{v}). \tag{3}$$

Proof. The set of the orderings that allow computing the Shapley value of the game $(N_1 \cup N_2, \hat{v})$ is the same set that we have used to define the one-coalition externality value of (N,v). Moreover, it is immediate to check that, for any order, a player's contribution in both games is the same. Hence, the two values coincide. □

4.2 Players in N_1 go first

The existence of intertemporal externalities suggests that we may want only to consider orderings where the players in N_1 go before the players in N_2; we call them "constrained orderings." For a game $(N,v) \in \mathcal{G}$, a *constrained ordering* of $N_1 \cup N_2$ is an injective mapping $\theta : N_1 \cup N_2 \to \{1, \ldots, |N_1| + |N_2|\}$ such that $\theta(i) < \theta(j)$, for all $i \in N_1$ and $j \in N_2$. We denote by $\Theta(N_1 \cup N_2)$ the set of constrained orderings of $N_1 \cup N_2$. As above, $B_1^{\theta}(k)$ and $B_2^{\theta}(k)$ are the sets of players who have arrived at step k and belong to N_1 and N_2, respectively.

[6] We call it the one-coalition externality value because it only considers the externalities exerted when, at most, one coalition of N_1 is formed.

We compute a player's contribution given a constrained ordering θ. When a player $j \in N_2$ arrives, all the players in N_1 are already in the room; hence, N_1 has been formed. Therefore, the order of arrival does not change the externality that the players in N_1 generate on the worth of the coalitions in N_2. Thus, the contribution in (N, v) of the player who arrives at step $k \in \{1, \dots, |N_1| + |N_2|\}$ of θ is:

$$m_k^\theta(N, v) = \begin{cases} v_1(B_1^\theta(k)) - v_1(B_1^\theta(k-1)) & \text{if } \theta^{-1}(k) \in N_1 \\ v_2(B_2^\theta(k); \{N_1\}) - v_2(B_2^\theta(k-1); \{N_1\}) & \text{if } \theta^{-1}(k) \in N_2. \end{cases}$$

We define the *naive value* $\varPhi^n$ as the players' expected contribution to constrained orderings when the probability of these orderings is the same. Considering the number of constrained orderings is $|N_1|!\,|N_2|!$, we have:

$$\varPhi_h^n(N, v) = \frac{1}{|N_1|!\,|N_2|!} \sum_{\theta \in \varTheta(N_1 \cup N_2)} m_{\theta(h)}^\theta(N, v),$$

for any $h \in N_1 \cup N_2$. The function $\varPhi^n$ is well-defined since it is efficient.

There are similarities between the constrained orderings that we have used in the construction of the naive value and the "ordered partitions" used in the definition of the *weighted Shapley value* by Kalai and Samet (1987).[7] To see the relationship, we first recall one way to compute the weighted Shapley value for the simple "weight systems" where all the players' weights are the same, but the set of players M is partitioned in a non-singleton ordered set $\varSigma = (S_1, ..., S_m)$.[8] Consider $(M, \hat{w}) \in \mathcal{G}^{CFF}$. The weighted Shapley value with the system $\varSigma$ of player h, which we denote $Sh_h^\varSigma(M, \hat{w})$, corresponds to h's expected marginal contribution to $\hat{w}$ when the only feasible orderings of M have all the players of S_t precede those of S_{t+1} for $t = 1, ..., m-1$ and all the feasible orderings have the same probability.

The following proposition states that the naive value corresponds to the weighted Shapley value for the ordered partition $\varSigma = (N_1, N_2)$ of $N_1 \cup N_2$ of the CFF game $(N_1 \cup N_2, \hat{v})$, which we have used to characterize the one-coalition externality value.

Proposition 4. *Let $\varSigma = (N_1, N_2)$. For any game with intertemporal externalities $(N, v) \in \mathcal{G}$,*

$$\varPhi^n(N, v) = Sh^\varSigma(N_1 \cup N_2, \hat{v}). \tag{4}$$

Proof. Consider $\varSigma = (N_1, N_2)$. Then $Sh^\varSigma(N_1 \cup N_2, \hat{v})$ is the vector of the players' expected marginal contributions to $\hat{v}$ when the feasible orderings of $N_1 \cup N_2$ are those where the players

[7] The family of weighted Shapley value was introduced by Shapley (1953a).

[8] We use the notation $\varSigma$ instead of P to indicate that the partition is ordered.

of N_1 precede those of N_2, and all the feasible orderings have the same probability. This is the same set of orderings that we call constrained orderings.

All the predecessors of a player $i \in N_1$, if any, belong to N_1. Hence, given the definition of $\hat{v}$, the marginal contribution in a feasible ordering of a player $i \in N_1$ to a coalition $S \subseteq N_1 \setminus \{i\}$ is just

$$\hat{v}(S \cup \{i\}) - \hat{v}(S) = v_1 (S \cup \{i\}) - v_1 (S) \,.$$

On the other hand, the set of predecessors of any player $i \in N_2$ includes N_1. Therefore, the marginal contribution in a feasible ordering of a player $j \in N_2$ to a coalition $S = N_1 \cup S_2$, with $S_2 \subseteq N_2 \setminus \{j\}$, is

$$\hat{v}(S \cup \{j\}) - \hat{v}(S) = v_2 (S_2 \cup \{j\}; \{N_1\}) - v_2 (S_2; \{N_1\}) \,.$$

The two previous marginal contributions are the same as those in the definition of the naive value; hence, (4) holds. □

An easy implication of Proposition 4 is that we can relate the naive value of a game $(N, v) \in \mathcal{G}$ with the Shapley value of two CFF games, the first involving the players of N_1 and the second involving the players in N_2:

$$\Phi^n_h(N, v) = \begin{cases} Sh_h(N_1, \hat{v}_1) & \text{if } h \in N_1 \\ Sh_h\left(N_2, \hat{v}_2^{N_1}\right) & \text{if } h \in N_2, \end{cases} \tag{5}$$

where

$$\hat{v}_2^{N_1}(S_2) = v_2(S_2; \{N_1\}), \tag{6}$$

for every $S_2 \subseteq N_2$.

Equation (5) highlights that, according to the naive value, players in N_1 only receive the value they create at the first period. They do not enjoy or suffer the consequences of the externality generated in the second period by forming the grand coalition in the first period.

In this section, we have introduced the values Φ^{1c} and Φ^n for the set of games with intertemporal externalities $\mathcal{G}$. Each value is obtained as the expected contribution to coalitions for a particular arrival process. We have also shown that they correspond to the Shapley value and a weighted Shapley value, respectively, of the associated CFF game $(N_1 \cup N_2, \hat{v})$. In the following two sections, we propose new properties to complement the basic axioms described in Section 3 to characterize Φ^{1c} and Φ^n.

5 Characterization of the one-coalition externality value

This section characterizes the one-coalition externality value by adding an equal treatment property to the basic axioms. To present this axiom, we first define the notion of *equally relevant* players. As we discuss after the definition, two players must satisfy a demanding condition to be equally relevant.

Definition 4. *(a) Players $i, i' \in N_1$ are equally relevant in (N, v) if*

$$v_1(S_1) - v_1(S_1 \setminus \{i\}) = v_1(S_1) - v_1(S_1 \setminus \{i'\}) \qquad \text{for every } S_1 \subseteq N_1, \text{ and}$$
$$v_2(S_2; P_1) - v_2(S_2; P_1^{-i}) = v_2(S_2; P_1) - v_2(S_2; P_1^{-i'}) \text{ for every } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1).$$

(b) Players $j, j' \in N_2$ are equally relevant in (N, v) if

$$v_2(S_2; P_1) - v_2(S_2 \setminus \{j\}; P_1) = v_2(S_2; P_1) - v_2(S_2 \setminus \{j'\}; P_1) \text{ for every } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1).$$

(c) Players $i \in N_1$ and $j \in N_2$ are equally relevant in (N, v) if

$$v_1(S_1) = v_1(S_1 \setminus \{i\}) \text{ for all } S_1 \subseteq N_1, \text{ and}$$
$$v_2(S_2; P_1) - v_2(S_2 \setminus \{j\}; P_1) = v_2(S_2; P_1) - v_2(S_2; P_1^{-i}) \text{ for every } S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1).$$

We provide an intuition of Definition 4. Consider two players in N_1. To be equally relevant, the two players must contribute equally to *any* coalition of N_1 (not only to those coalitions containing both players, as in the classic definition of "symmetric players"). Moreover, the effect in the worth of any coalition of N_2 of having either of the two players isolated from the coalition structure of N_1 is also the same. Hence, considering two players in N_1 equally relevant requires satisfying a demanding condition. The condition for two players in N_2 to be equally relevant is in the same spirit as the first part of the condition for players in N_1. There is no requirement in terms of externalities since they do not create any. Finally, we also propose a definition of equally relevant players for players who belong to different periods, namely one agent in N_1 and one agent in N_2. In this case, we require that the player in N_1 does not have an effect on v_1 (since the player in N_2 cannot have any effect) and that the contribution of the player in N_2 be the same as the externality effect of the isolation of the player in N_1.

The following axiom we propose, which we call "equal treatment of contributions," requires that two equally relevant players obtain the same payoff in the value. As discussed above, being equally relevant is a very demanding condition; hence, the axiom is weak. In fact, for equally relevant players in N_2, the property is implied by the axiom of anonymity. We state this fact in Remark 1.

Remark 1. *Consider two equally relevant players $j, j' \in N_2$. Then, anonymity implies $\Phi_j(N, v) = \Phi_{j'}(N, v)$, for any $(N, v) \in \mathcal{G}$. Indeed, let Φ be an anonymous value and $\sigma = (\sigma_1, \sigma_2)$ a permutation of N, with σ_1 the identity on N_1 and σ_2 the permutation on N_2 such that $\sigma_2(j) = j'$, $\sigma_2(j') = j$, and $\sigma_2(j'') = j''$, for every $j'' \in N_2 \setminus \{j, j'\}$. Then, we can see that $\sigma v = v$ because j and j' are equally relevant and by anonymity $\Phi_j(N, v) = \Phi_{j'}(N, v)$.*

We now introduce the axiom of equal treatment of contributions.

Axiom 4. *Equal treatment of contributions: A value Φ satisfies equal treatment of contributions if, for any game $(N, v) \in \mathcal{G}$, $\Phi_h(N, v) = \Phi'_h(N, v)$, for any equally relevant players $h, h' \in N_1 \cup N_2$.*

Theorem 1 states the characterization of the one-coalition externality value using the axiom of equal treatment of contributions.

Theorem 1. *The one-coalition externality value Φ^{1c} is the only value satisfying the axioms of linearity, anonymity, dummy player, and equal treatment of contributions.*

Proof. We first show that Φ^{1c} satisfies all the properties. We use Proposition 3 and Shapley's original axioms for CFF games.

The linearity of Φ^{1c} follows from (a) the associated CFF game of the sum of two games is the sum of the two corresponding associated CFF games, (b) the associated CFF game of the product of a game and a scalar is the product of the corresponding associated CFF game and the scalar, and (c) the linearity of the Shapley value.

Similarly, the anonymity of Φ^{1c} follows from the fact that the associated CFF game of a permuted game is a permuted game of the associated CFF game and the anonymity of the Shapley value.

For the dummy player property, let $i \in N_1$ be a dummy player in (N, v). Then, for every $S \subseteq N_1 \cup N_2$,

$$\begin{aligned}\hat{v}(S) =& v_1(S \cap N_1) + v_2\left(S \cap N_2; \{S \cap N_1\} \cup \{\{i'\} : i' \in N_1 \setminus S\}\right)\\ =& v_1((S \setminus \{i\}) \cap N_1) + v_2\left(S \cap N_2; \{(S \setminus \{i\}) \cap N_1\} \cup \{\{i'\} : i' \in N_1 \setminus (S \setminus \{i\})\}\right)\\ =& v_1((S \setminus \{i\}) \cap N_1) + v_2\left((S \setminus \{i\}) \cap N_2; \{(S \setminus \{i\}) \cap N_1\} \cup \{\{i'\} : i' \in N_1 \setminus (S \setminus \{i\})\}\right)\\ =& \hat{v}(S \setminus \{i\}),\end{aligned}$$

where the first and last equalities follow the definition of $\hat{v}$, the second equality holds because $i \in N_1$ is a dummy player, and the third equality holds because $(S \setminus \{i\}) \cap N_2 = S \cap N_2$.

Similarly, if $j \in N_2$ is a dummy player in (N, v) then, for every $S \subseteq N_1 \cup N_2$,

$$\begin{aligned}\hat{v}(S) =& v_1(S \cap N_1) + v_2\left(S \cap N_2; \{S \cap N_1\} \cup \{\{i\} : i \in N_1 \setminus S\}\right) \\ =& v_1(S \cap N_1) + v_2\left((S \setminus \{j\}) \cap N_2; \{S \cap N_1\} \cup \{\{i\} : i \in N_1 \setminus S\}\right) \\ =& v_1((S \setminus \{j\}) \cap N_1) + v_2\left((S \setminus \{j\}) \cap N_2; \{(S \setminus \{j\}) \cap N_1\} \cup \{\{i\} : i \in N_1 \setminus (S \setminus \{j\})\}\right) \\ =& \hat{v}(S \setminus \{j\}),\end{aligned}$$

where the second equality holds because $j \in N_2$ is a dummy player and the third because $(S \setminus \{j\}) \cap N_1 = S \cap N_1$.

Given that $\hat{v}(S) = \hat{v}(S \setminus \{h\})$ for every $S \subseteq N_1 \cup N_2$ if $h \in N_1 \cup N_2$ is a dummy player, the dummy player property of Φ^{1c} follows from the homonymous property of the Shapley value for CFF games.

We now prove that Φ^{1c} satisfies equal treatment of contributions. Let $(N, v) \in \mathcal{G}$ and $i, i' \in N_1$ be equally relevant players in (N, v). We show that the two players obtain the same payoff in Φ^{1c} by proving that they are symmetric in the associated game $(N_1 \cup N_2, \hat{v})$. Consider any $S \subseteq N_1 \cup N_2$ such that $i, i' \in S$. Then,

$$\begin{aligned}\hat{v}(S \setminus \{i\}) =& v_1\left((S \setminus \{i\}) \cap N_1\right) + v_2\left((S \setminus \{i\}) \cap N_2; \{(S \setminus \{i\}) \cap N_1\} \cup \{\{l\} : l \in N_1 \setminus (S \setminus \{i\})\}\right) \\ =& v_1\left((S \cap N_1) \setminus \{i\}\right) + v_2\left(S \cap N_2; \{(S \cap N_1) \setminus \{i\}\} \cup \{\{l\} : l \in N_1 \setminus S, \{i\}\}\right) \\ =& v_1\left((S \cap N_1) \setminus \{i\}\right) + v_2\left(S \cap N_2; P_1^{-i}\right),\end{aligned}$$

where $P_1 \equiv \{S \cap N_1\} \cup \{\{l\} : l \in N_1 \setminus S\}$. A similar equation holds for $\hat{v}(S \setminus \{i'\})$. Since i and i' are equally relevant players, $v_1\left((S \cap N_1) \setminus \{i\}\right) = v_1\left((S \cap N_1) \setminus \{i'\}\right)$ and $v_2\left(S \cap N_2; P_1^{-i}\right) = v_2\left(S \cap N_2; P_1^{-i'}\right)$; hence, $\hat{v}(S \setminus \{i\}) = \hat{v}(S \setminus \{i'\})$, and the players are symmetric, as we wanted to prove.

We do not need to show that the property holds for two equally relevant players $j, j' \in N_2$ because of Remark 1.

Finally, we show that if $i \in N_1$ and $j \in N_2$ are equally relevant in (N, v), then they are symmetric players in $(N_1 \cup N_2, \hat{v})$. Following the same steps as above, we can check that $\hat{v}(S \setminus \{j\}) = v_1\left(S \cap N_1\right) + v_2\left((S \cap N_2) \setminus \{j\}; P_1\right)$. Then, for any $S \subseteq N_1 \cup N_2$ such that $i, j \in S$, we have $\hat{v}(S \setminus \{i\}) = \hat{v}(S \setminus \{j\})$ because $v_1\left((S \cap N_1) \setminus \{i\}\right) = v_1\left(S \cap N_1\right)$ and $v_2\left(S \cap N_2; P_1^{-i}\right) = v_2\left((S \cap N_2) \setminus \{j\}; P_1\right)$. Therefore, i and j obtain the same payoff in $Sh(N_1 \cup N_2, \hat{v})$, and, hence, in $\Phi^{1c}(N, v)$.

For the uniqueness, let Φ be a value on $\mathcal{G}$ satisfying the properties. By Proposition 1, we only need to prove that the value is uniquely determined for the games $(N, v^b) \in \mathcal{G}^b \equiv \{(N, v) \in \mathcal{G} : v_1(S_1) = 0 \text{ for all } S_1 \subseteq N_1\}$. To show it, we use a basis of the family of games $\mathcal{G}^b$. For any non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$, we define the unanimity game of $(S_2; P_1)$,

$(N, v^{(S_2;P_1)}) \in \mathcal{G}^b$, by[9]

$$v_1^{(S_2;P_1)}(T_1) \equiv 0 \text{ for all } T_1 \subseteq N_1$$
$$v_2^{(S_2;P_1)}(T_2; Q_1) \equiv \begin{cases} 1 & \text{if } S_2 \subseteq T_2 \text{ and } P_1 \preceq Q_1 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\{(N, v^{(S_2;P_1)}) : \emptyset \neq S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1)\}$ is a basis of $\mathcal{G}^b$. Clearly, $\mathcal{G}^b$ is a vector space of dimension $(2^{|N_2|} - 1)|\mathcal{P}(N_1)|$. Then, it is enough to check that the set of unanimity games is linearly independent. We do it by contradiction. Let $\{\lambda_{(S_2;P_1)} : \emptyset \neq S_2 \subseteq N_2 \text{ and } P_1 \in \mathcal{P}(N_1)\}$ be a set of scalars such that $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v^{(S_2;P_1)}$ is the null game. Suppose that not all the scalars are equal to zero. Then, we choose one of them, $\lambda_{(T_2;Q_1)} \neq 0$, such that for every $T_2' \subseteq T_2$ and $Q_1' \preceq Q_1$, $\lambda_{(T_2';Q_1')} = 0$. The worth of $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v^{(S_2;P_1)}$ evaluated in $(T_2; Q_1)$ is $\sum_{\substack{\emptyset \neq S_2 \subseteq N_2 \\ P_1 \in \mathcal{P}(N_1)}} \lambda_{(S_2;P_1)} v_2^{(S_2;P_1)}(T_2; Q_1) = \lambda_{(T_2;Q_1)} \neq 0$, which is a contradiction and proves the claim.

By linearity, we only need to show that Φ is uniquely determined for every element of the basis. Consider $(N, v^{(S_2;P_1)})$, for any non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. For convenience, we write the partition P_1 as $P_1 = \{A_1, \ldots, A_k\} \cup \{\{l\} : l \in A_{k+1}\}$, where $A_1, \ldots, A_k$ are non-singleton coalitions. That is, A_{k+1} includes all the players, if any, of the singleton coalitions of P_1. We prove that $\Phi_h(N, v^{(S_2;P_1)})$ is uniquely determined for every $h \in N_1 \cup N_2$.

First, take $j \in N_2 \setminus S_2$. It is easy to check that j is a dummy player in $v^{(S_2;P_1)}$. Then, by the dummy player property, $\Phi_j(N, v^{(S_2;P_1)}) = 0$.

Second, consider $i \in A_{k+1}$. Then i is a dummy player in $v^{(S_2;P_1)}$. Indeed, $v_1^{(S_2;P_1)}(T_1) = v_1^{(S_2;P_1)}(T_1 \setminus \{i\}) = 0$ for all $T_1 \subseteq N_1$. Moreover, $P_1 \preceq Q_1$ if and only if $P_1 \preceq Q_1^{-i}$ for every $Q_1 \in \mathcal{P}(N_1)$; hence, $v_2^{(S_2;P_1)}(T_2; Q_1) = v_2^{(S_2;P_1)}(T_2; Q_1^{-i})$ for every $T_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. Then, by the dummy player property, $\Phi_i(N, v^{(S_2;P_1)}) = 0$.

We next show that the payoffs to the agents in $S_2 \cup A_1 \cup \cdots \cup A_k$ are also uniquely determined.

Let $j \in S_2$. Then, $v_2^{(S_2;P_1)}(T_2 \setminus \{j\}; Q_1) = 0$ for every $T_2 \subseteq N_2$ and every $Q_1 \in \mathcal{P}(N_1)$. This implies that all the players in S_2 are equally relevant (see part (b) of Definition 4). Then by equal treatment of contributions, Φ allocates the same payoff to the agents in S_2.

Next, consider $i \in N_1 \setminus A_{k+1}$. Observe that $P_1 \not\preceq Q_1^{-i}$ for every $Q_1 \in \mathcal{P}(N_1)$, because player i forms a singleton coalition in Q_1^{-i} and belongs to a non-singleton coalition in P_1. Then $v_2^{(S_2;P_1)}(T_2; Q_1^{-i}) = 0$ for every $T_2 \subseteq N_2$ and every $Q_1 \in \mathcal{P}(N_1)$. Recall that, since $v^{(S_2;P_1)} \in \mathcal{G}^b$, players in N_1 do not generate value in the first period. Hence, all the players in

[9] Given $P, Q \in \mathcal{P}(M)$, we say that P is finer than Q and write $P \preceq Q$ if for every $S \in P$ there is a $T \in Q$ such that $S \subseteq T$.

$N_1 \setminus A_{k+1}$ are equally relevant (see part (b) of Definition 4). Therefore, by equal treatment of contributions, Φ allocates the same payoff to all of them.

Moreover, note that we have just seen that for every $i \in N_1 \setminus A_{k+1}$ and $j \in S_2$, $v_1^{(S_2;P_1)}(T_1) = v_1^{(S_2;P_1)}(T_1 \setminus \{i\}) = 0$ for all $T_1 \subseteq N_1$ and

$$\begin{aligned} v_2^{(S_2;P_1)}(T_2;Q_1) - v_2^{(S_2;P_1)}(T_2 \setminus \{j\};Q_1) =& v_2^{(S_2;P_1)}(T_2;Q_1) \\ =& v_2^{(S_2;P_1)}(T_2;Q_1) - v_2^{(S_2;P_1)}(T_2;Q_1^{-i}), \end{aligned}$$

for every $T_2 \subseteq N_2$ and every $Q_1 \in \mathcal{P}(N_1)$. Therefore, $i \in N_1 \setminus A_{k+1}$ and $j \in S_2$ are equally relevant players. Hence, by equal treatment of contributions, Φ allocates the same payoff to all players in $S_2 \cup A_1 \cup \cdots \cup A_k$. Efficiency implies that this payoff is unique, which proves the theorem. □

Theorem 1 provides a characterization of the one-coalition externality value based on an axiom that postulates that two equally relevant players obtain the same payoff. The definition of equal relevance takes into account not only the "direct" effect of a player in the worth of a coalition but also the "indirect" effect she may have through an externality. It gives a similar weight to both effects. In particular, if the contribution to a coalition of a player in N_2 is of the same magnitude as the externality generated by a player in N_1 (see part (c) of Definition 4), then these players must obtain the same payoff, according to the axiom of equal treatment.

6 Characterization of the naive value

To characterize the naive value, we use an axiom related to the equal treatment of the players in N_1 who generate similar externalities. We focus on the idea that if two players of N_1 have a similar role in generating externalities, their payoff should be the same. We introduce this idea in some simple games, denoted $u^{(S_2;P_1)}$. These games are part of a basis for the set of games with intertemporal externalities $\mathcal{G}$.[10]

Consider a non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. We define the game $u^{(S_2;P_1)} \equiv \left(u_1^{(S_2;P_1)}, u_2^{(S_2;P_1)}\right)$, where $u_1^{(S_2;P_1)} : 2^{N_1} \to \mathbb{R}$ and $u_2^{(S_2;P_1)} : 2^{N_2} \times \mathcal{P}(N_1) \to \mathbb{R}$, by:

$$u_1^{(S_2;P_1)}(T_1) = 0 \quad \text{for all} \quad T_1 \subseteq N_1$$

$$u_2^{(S_2;P_1)}(T_2;Q_1) = \begin{cases} 1 & \text{if } (T_2;Q_1) = (S_2;P_1) \\ 0 & \text{otherwise.} \end{cases}$$

[10] In the proof of Theorem 1, we have used for convenience a different basis, which we denoted $\{v^{(S_2;P_1)}\}_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1)}$, for the same set of games.

The set $\{u^{(S_2;P_1)}\}_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)}$ is a basis for the set of games $\mathcal{G}^b \equiv \{(N,v)\in\mathcal{G} : v_1(S_1)=0 \text{ for all } S_1\subseteq N_1\}$.[11],[12] Indeed, for any game $(N,v^b)\in\mathcal{G}^b$, we have:

$$v^b = \sum_{\emptyset\neq S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v_2^b(S_2;P_1)u^{(S_2;P_1)}. \tag{7}$$

In the game $u^{(S_2;P_1)}$, the role of all the players in N_1 is "similar": it is only when they form precisely the partition P_1 that they generate an externality on the coalition S_2. Our new axiom states that since the role of the players in N_1 in the game $u^{(S_2;P_1)}$ is similar, they should receive the same payoff in "compensation" for the externality that they generate. We call it the axiom of "equal treatment of externalities."

Axiom 5. *Equal Treatment of Externalities: A value Φ satisfies equal treatment of externalities if*

$$\Phi_i\left(N,u^{(S_2;P_1)}\right) = \Phi_{i'}\left(N,u^{(S_2;P_1)}\right) \quad \textit{for all} \;\; i,i'\in N_1, S_2\subseteq N_2, \;\textit{and}\; P_1\in\mathcal{P}(N_1). \tag{8}$$

Lemma 1 provides some information about the payoff obtained by the players in a value that satisfies equal treatment of externalities in addition to the basic axioms. It is instrumental in the proof of our following theorem.

Lemma 1. *Consider a value Φ that satisfies linearity, anonymity, dummy player, and equal treatment of externalities. Then, there exists weights $\{\gamma(S_2;P_1)\}_{\emptyset\neq S_2\subseteq N_2;P_1\in\mathcal{P}(N_1)}$ satisfying $\sum_{P_1\in\mathcal{P}(N_1)}\gamma(S_2;P_1)=1$ for all $S_2\subseteq N_2$, such that*

$$\Phi_i(N,v) = Sh_i(N_1,\hat{v}_1) + \sum_{S_2\subseteq N_2, P_1\in\mathcal{P}(N_1)} v_2(S_2;P_1)\Phi_k(N,u^{(S_2;P_1)}) \tag{9}$$

$$\Phi_j(N,v) = Sh_j(N_2,\hat{v}_2^{\gamma}) - \sum_{S_2\subseteq N_2, P_1\in\mathcal{P}(N_1), S_2\not\supseteq\{j\}} \frac{|N_1|}{|N_2\setminus S_2|} v_2(S_2;P_1)\Phi_k(N,u^{(S_2;P_1)}), \tag{10}$$

[11] In the game $u^{(S_2;P_1)}$, forming the grand coalition in both periods is not efficient unless $(S_2;P_1) = (\{N_2\},\{N_1\})$. We use these games for convenience. However, the same analysis can be done if we define a basis using the functions $w^{(S_2;P_1)}$, which are identical to $u^{(S_2;P_1)}$ except that $w^{(S_2;P_1)}(R_2;Q_1)=1$ if either $(R_2;Q_1)=(S_2;P_1)$ or $(R_2;Q_1)=(\{N_2\};\{N_1\})$.

[12] Consider the game (N,u^{S_1}), where $u^{S_1}=\left(u_1^{S_1},u_2^{S_1}\right)$ is defined by:

$$u_1^{S_1}(R_1)\equiv\begin{cases}1 & \text{if } R_1=S_1\\ 0 & \text{otherwise.}\end{cases}$$
$$u_2^{S_1}(R_2;Q_1)\equiv 0 \quad \text{for all} \quad (R_2;Q_1)\in 2^{N_2}\times\mathcal{P}(N_1).$$

Then, the set $\{(N,u^{S_1})\}_{\emptyset\neq S_1\subseteq N_1}\cup\{(N,u^{(S_2;P_1)})\}_{\emptyset\neq S_2\subseteq N_2,P_1\in\mathcal{P}(N_1)}$ constitutes a basis of the whole set of games with intertemporal externalities $\mathcal{G}$.

for any $i \in N_1$ *and* $j \in N_2$*, where* $\Phi_k(N, u^{(S_2;P_1)})$ *is the payoff obtained by any* $k \in N_1$ *in the basis game* $(N, u^{(S_2;P_1)})$ *and* $(N_2, \hat{v}_2^{\gamma})$ *is the CFF game defined by*

$$\hat{v}_2^{\gamma}(S_2) \equiv \sum_{P_1 \in \mathcal{P}(N_1)} \gamma(S_2; P_1) v_2(S_2; P_1) \tag{11}$$

for any $S_2 \subseteq N_2$*.*

Proof. The proof is in the Appendix. □

Lemma 1 states that the axiom of equal treatment of externalities, together with linearity, anonymity, and the dummy player axiom, restricts the set of values. However, it does not single out one value. To do it, we strengthen this axiom.

Equal treatment of externalities advocates that the players in N_1 should receive the same payoff in a basis game $u^{(S_2;P_1)}$ because their role in creating the externality is similar. "Strong equal treatment of externalities" requires that since the role of the players in N_1 in the games $u^{(S_2;P_1)}$ and $u^{(S_2;P_1')}$ is similar, for any $P_1, P_1' \in \mathcal{P}(N_1)$, their payoffs in these game should also be the same.

Axiom 6. *Strong Equal Treatment of Externalities: A value* Φ *satisfies strong equal treatment of externalities if*

$$\Phi_i\left(N, u^{(S_2;P_1)}\right) = \Phi_{i'}\left(N, u^{(S_2;P_1')}\right) \quad \text{for all} \ \ i, i' \in N_1, S_2 \subseteq N_2, \ \text{and} \ P_1, P_1' \in \mathcal{P}(N_1). \tag{12}$$

Theorem 2 uses Lemma 1 to characterize the naive value through our basic axioms plus the strong equal treatment of externalities axiom.

Theorem 2. *The naive value* Φ^n *is the only value satisfying the axioms of linearity, anonymity, dummy player, and strong equal treatment of externalities.*

Proof. We first show that Φ^n satisfies the four axioms. Given the characterization of Φ^n provided in Equation (5), it is immediate to check that it satisfies linearity, anonymity, and dummy player. It also satisfies strong treatment of externalities because $\Phi_i^n(N, u^{(S_2;P_1)}) = 0$ for all $i \in N_1$, nonempty $S_2 \subseteq N_2$, and $P_1 \in \mathcal{P}_1$.

Notice that Φ^n corresponds to the value identified in Lemma 1 when the weights are $\gamma^n(S_2; P_1) \equiv 0$ and $\gamma^n(S_2; \{N_1\}) \equiv 1$, for all $S_2 \subseteq N_2$ and $P_1 \neq \{N_1\}$. For these weights, $\hat{v}_2^{N_1} = \hat{v}_2^{\gamma^n}$ (see Equations (6) and (11)).

We now prove that Φ^n is the only value that satisfies all the axioms. Take Φ, satisfying the axioms. We show that $\Phi(N, v) = \Phi^n(N, v)$ for all $(N, v) \in \mathcal{G}$.

First, take $i \in N_1$. Strong equal treatment of externalities requires that, for any nonempty $S_2 \subseteq N_2$, $\Phi_i\left(N, u^{(S_2;P_1)}\right)$ is the same for all $P_1 \in \mathcal{P}(N_1)$. Equation (20) (see the Appendix) implies that $\sum_{P_1 \in \mathcal{P}(N_1)} \Phi_i\left(N, u^{(S_2;P_1)}\right) = 0$. Therefore, $\Phi_i\left(N, u^{(S_2;P_1)}\right) = 0$ for all $i \in N_1$, nonempty $S_2 \subseteq N_2$, and $P_1 \in \mathcal{P}(N_1)$. Then, using Equation (9), $\Phi_i(N,v) = Sh_i(N_1, \hat{v}_1) = \Phi_i^n(N,v)$ for any $i \in N_1$.

Take now $j \in N_2$. Equation (10), together with $\Phi_k\left(N, u^{(S_2;P_1)}\right) = 0$ for all $k \in N_1$, imply that

$$\Phi_j(N,v) = Sh_j\left(N_2, \hat{v}_2^\gamma\right), \tag{13}$$

where $\hat{v}_2^\gamma$ is defined in (11), for some weight system γ. We prove that it is necessarily the case that $\gamma = \gamma^n$ by induction on the size of the coalition S_2. If $S_2 = N_2$, efficiency requires $\gamma(N_2; P_1) = 0$, for any $P_1 \neq \{N_1\}$. Otherwise, suppose $\gamma(N_2; P_1) \neq 0$ for some $P_1 \neq \{N_1\}$, and consider the game $v = u^{(N_2;P_1)}$. For this game, $\hat{v}_2^\gamma(N_2) = \gamma(N_2; P_1)$. Therefore, $Sh(N_2, \hat{v}_2^\gamma)$ shares $\gamma(N_2; P_1) \neq 0$ among the players in N_2, whereas the efficiency of Φ requires that the sum of the players' payoff be $v_2(N_2; \{N_1\}) = 0$. Moreover, $\gamma(N_2; P_1) = 0$ for any $P_1 \neq \{N_1\}$ implies $\gamma(N_2; \{N_1\}) = 1$. Hence, $\gamma(N_2; P_1) = \gamma^n(N_2; P_1)$ for all $P_1 \in \mathcal{P}_1$.

By the induction argument, assume that $\gamma(S_2; P_1) = \gamma^n(S_2; P_1)$ for all $P_1 \in \mathcal{P}(N_1)$ holds for all $S_2 \subseteq N_2$ with $|S_2| \geq m$, for $1 < m \leq |N_2|$.

Consider $S_2 \subseteq N_2$ with $|S_2| = m-1$, $j \in N_2 \setminus S_2$, and $P_1 \in \mathcal{P}(N_1)$. Define the game (N,w) by $w \equiv u^{(S_2 \cup \{j\};P_1)} + u^{(S_2;P_1)}$. That is, the worth of the coalitions $S_2 \cup \{j\}$ and S_2 is 1 if the partition P_1 has been formed; the worth of a coalition is zero in any other case. The agent $j \in N_2$ is a dummy player in (N,w); hence, the dummy player axiom implies $\Phi_j(N,w) = 0$. Moreover, given the worth of the coalitions in w, the CFF game $(N_2, \hat{w}_2^\gamma)$ satisfies

$$\begin{aligned}
w_2^\gamma(S_2 \cup \{j\}) =& \gamma(S_2 \cup \{j\}; P_1)\\
w_2^\gamma(S_2) =& \gamma(S_2; P_1)\\
w_2^\gamma(T_2) =& 0 \text{ for all } T_2 \neq S_2, T_2 \neq S_2 \cup \{j\}.
\end{aligned}$$

The contribution of j to any coalition in the game $(N_2, \hat{w}_2^\gamma)$ is zero, except possibly to S_2. Her contribution to S_2 is $\gamma(S_2 \cup \{j\}; P_1) - \gamma(S_2; P_1)$. Then, $0 = \Phi_j(N,w) = Sh_j\left(N_2, w_2^\gamma\right)$ implies that this contribution must be zero; hence, $\gamma(S_2; P_1) = \gamma(S_2 \cup \{j\}; P_1)$ for all $P_1 \in \mathcal{P}_1$. Since $|S_2 \cup \{j\}| = m$, we use the induction argument and obtain $\gamma(S_2; P_1) = \gamma(S_2 \cup \{j\}; P_1) = \gamma^n(S_2 \cup \{j\}; P_1) = 0$ for all $P_1 \neq \{N_1\}$ and $\gamma(S_2; \{N_1\}) = \gamma(S_2 \cup \{j\}; \{N_1\}) = \gamma^n(S_2 \cup \{j\}; \{N_1\}) = 1$.

This completes the induction argument. We have shown that $\gamma = \gamma^n$; hence, Φ^n is the only value satisfying the four axioms. □

Theorem 2 identifies the naive value as the only one satisfying the basic Shapley axioms and rewarding the externalities generated by the players in N_1 in a strong symmetric way. It shows that the only way symmetric treatment of externalities (in a strong sense) is compatible with the Shapley axioms is to disregard the externalities completely.

7 Games with intertemporal additive externalities

In this section, we introduce a particular class of games in $\mathcal{G}$., which we call *games with intertemporal additive externalities*. They are games where the intertemporal externality does not vary across the coalitions that can be formed in the second period; it only depends on the partition created in the first period. In this class of games, we first illustrate the form of any sharing rules satisfying the basic axioms. Then, we highlight the differences in the distribution of the surplus between the one-coalition externality and the naive values.

Formally, a game with intertemporal additive externalities $(N, v) \in \mathcal{G}$ satisfies $v_2(\emptyset; P_1) = 0$ for any $P_1 \in \mathcal{P}(N_1)$ and, for every non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$,

$$v_2(S_2; P_1) = \hat{v}_2(S_2) + e(P_1).$$

The function $\hat{v}_2 : 2^{N_2} \setminus \emptyset \to \mathbb{R}$ provides the worth generated by any non-empty coalition of players in N_2, and the function $e : \mathcal{P}(N_1) \to \mathbb{R}$ measures the externality generated in any coalition by the partition formed among the players in N_1. We normalize the function such that $e(\{\{i\} : i \in N_1\}) = 0$. This assumption is without loss of generality as we could subtract the worth of the partition of the singletons from all the externalities and add it to $v_2(S_2)$ for all non-empty $S_2 \subseteq N_2$.

Consider a value Φ satisfying linearity, anonymity, and dummy player. We decompose any game with additive externalities $(N, v) \in \mathcal{G}$ as the sum of two games (N, v') and (N, v'') as follows. The game (N, v') satisfies $v'_1 = v_1$ and $v'_2(S_2; P_1) = \hat{v}_2(S_2)$ for any $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$. The game (N, v'') is defined by $v''_1 = 0$ and $v''_2(\emptyset; P_1) = 0$ and $v''_2(S_2; P_1) = e(P_1)$ for any non-empty $S_2 \subseteq N_2$ and $P_1 \in \mathcal{P}(N_1)$.

Note that (N, v') is a game without externalities. Then, by Proposition 1, $\Phi_i(N, v') = Sh_i(N_1, \hat{v}_1)$ for all $i \in N_1$ and $\Phi_j(N, v') = Sh_j(N_2, \hat{v}_2)$ for all $j \in N_2$. Therefore, if the externality is additive, the values that satisfy the basic axioms only differ in how they share the surplus $e(\{N_1\})$ among the players in $N_1 \cup N_2$. Moreover, by anonymity, all the players in N_2 obtain the same payoff, hence $\Phi_j(N, v'') = \Phi_{j'}(N, v'')$ for every $j, j' \in N_2$.

We now describe the precise sharing proposed by Φ^n and Φ^{1c} for the class of games $\mathcal{G}^A$.

Consider the naive value, $\Phi = \Phi^n$. Equation (5) implies that $\Phi^n_i(N, v'') = Sh_i(N, \hat{v}''_1) = 0$ for all $i \in N_1$. Moreover, since all the players in N_2 must obtain the same payoff, $\Phi^n_j(N, v'') = \frac{e(N_1)}{|N_2|}$ for all $j \in N_2$. That is, the naive value divides equally the externality (positive or negative) generated by the formation of the grand coalition N_1 among the players in N_2.

Consider now the one-coalition externality value, $\Phi = \Phi^{1c}$. For this value, the externality generated by the formation of N_1 is shared among the players in $N_1 \cup N_2$ and not only among the players in N_2. Using Equation (2), we compute the equal value assigned by Φ^{1c} to the players in N_2:

$$\Phi_j^{1c}(N,v'') = \sum_{\substack{S_1 \subseteq N_1 \\ |S_1| \geq 2}} \frac{|S_1|!(|N_1 \cup N_2| - |S_1| - 1)!}{|N_1 \cup N_2|!} e(\{S_1\} \cup \{\{l\} : l \in N_1 \setminus S_1\}),$$

for all $j \in N_2$. On the other hand, the players in N_1 are not symmetric. Following also Equation (2), the contributions of a player in N_1 determine the value that Φ^{1c} assigns to her:

$$\Phi_i^{1c}(N,v'') = \sum_{\substack{S_1 \subseteq N_1 \\ S_1 \supseteq \{i\}}} \left(\frac{(|S_1|-1)!(|N_1|-|S_1|)!}{|N_1|!} - \frac{(|S_1|-1)!(|N_1 \cup N_2|-|S_1|)!}{|N_1 \cup N_2|!} \right) \times$$
$$\big(e(\{S_1\} \cup \{\{l\} : l \in N_1 \setminus S_1\}) - e(\{S_1 \setminus \{i\}\} \cup \{\{l\} : l \in N_1 \setminus (S_1 \setminus \{i\})\})\big),$$

for all $i \in N_1$.

To illustrate the previous results on how the externality is shared among the players in N_1 and N_2, consider two games (N, v'') with $N_1 = \{1,2,3\}$, $N_2 = \{4\}$, one with positive externalities (that is, forming larger coalitions in $t = 1$ generates a higher surplus in $t = 2$) and another with negative externalities (that is, forming larger coalitions at $t = 1$ generates lower surplus at $t = 2$), normalizing so that $e(\{1\},\{2\},\{3\}) = 0$:

P_1	$e^+(P_1)$	$e^-(P_1)$
$e(\{1\},\{2\},\{3\})$	0	0
$e(\{1,2\},\{3\})$	0	-3
$e(\{1,3\},\{2\})$	3	-3
$e(\{2,3\},\{1\})$	6	-3
$e(\{1,2,3\})$	9	-9

We denote by (N, v''^{+}) the game with positive externalities, the one whose externality function is e^+ and by (N, v''^{-}) the game with negative externalities, whose function is e^-.

In the game with positive externalities, we have:

$$\Phi^n(N, v''^{+}) = (0,0,0,9) \quad \text{and} \quad \Phi^{1c}(N, v''^{+}) = (1,2,3,3),$$

whereas in the game with negative externalities, we have:

$$\Phi^n(N, v''^{-}) = (0,0,0,-9) \quad \text{and} \quad \Phi^{1c}(N, v''^{+}) = (-2,-2,-2,-3).$$

As the example illustrates, the naive value allocates the (positive or negative) externalities generated by forming the grand coalition in the first period only to the agents in the second period. On the other hand, the one-coalition externality value is more sensitive. It divides the externalities among the players in the two periods. It rewards or punishes the agents in the first period depending on whether the externalities they create are positive (as it happens in (N, v''^{+}) or negative (as in (N, v''^{-})). In the game (N, v''^{+}), the formation of the grand coalition N_1 leads to the highest positive externality. Therefore, the players in N_1 receive a positive payoff of 6 in total. On the other hand, when they form N_1 in the game (N, v''^{-}), the externality created by the agents in $t = 1$ is the most negative. The one-coalition externality value allocates a total payoff of -6 to those agents.

We note that the previous example only describes the sharing of the surplus (or the loss) due to the externality. In most environments, the players in N_1 would both make some profits at $t = 1$ and generate externalities at $t = 2$. Consider, for instance, the game with positive externalities. We may think of situations (like the formation of coalitions in $t = 1$ to reduce emissions in $t = 2$) where the players in N_1 obtain fewer profits in the first period if they form the grand coalition (the one reducing emissions the most), but the players in the second period inherit a better environment, which leads to more profits. In this case, it may make sense to compensate these players for the cost they encounter by sharing $v_1(N_1)$ instead of the total worth obtained in some other partition, as the one-coalitional externality suggests.

8 Discussion on the relationship with values for partition function form games

Given the existence of externalities between two sets of players in a game with intertemporal externalities, there are similarities between the class of games we analyze in this paper and the set of games in partition function form (PFF games). In contrast with a CFF game, a PFF game considers that the worth of a coalition may depend on the organization of the rest of the players, that is, on the whole partition of players. As discussed in the Introduction, the literature has provided several values for PFF games that extend the Shapley value.

This section shows that a game with intertemporal externalities can be adapted into a "traditional" PFF game. Then, we discuss what values for games with intertemporal externalities can be obtained through such a procedure and their relationship to the values we introduce and analyze in this paper.

Take a game with intertemporal externalities (N, v). The most intuitive way to transform it into a PFF game $(N_1 \cup N_2, \tilde{v})$ is by defining the worth function $\tilde{v}$ as follows:

$$\tilde{v}(S, P) \equiv v_1(S \cap N_1) + v_2(S \cap N_2; P \tilde{\cap} N_1), \tag{14}$$

for any $S \subseteq N_1 \cup N_2$ and $P \in \mathcal{P}(N_1 \cup N_2)$ with $S \in P$, where we denote $P \tilde{\cap} N_1 \equiv \{R \cap N_1 \mid R \in P\} \setminus \varnothing$. That is, the game $(N_1 \cup N_2, \tilde{v})$ associates to a coalition S of $N_1 \cup N_2$ when the partition is P the sum of the worth in $t = 1$ of the players of S who are in N_1 plus the worth in $t = 2$ of the players in S who are in N_2. The worth of $S \cap N_2$ is computed taking into account that players in N_1 are organized according to the restriction of P to N_1.[13]

In the class of PFF games, several extensions of the Shapley value (Macho-Stadler et al., 2007; Pham Do and Norde, 2007; McQuillin, 2009) can be obtained through the "average approach." This approach consists of defining, for each PFF game, an "average" CFF game, where the worth of a coalition is a weighted average of the worth of the coalition for all the possible partitions that include it. Then, we obtain an extension of the Shapley value by applying this value to the resulting average CFF game. Each way of doing averages (i.e., each weight system) leads to a different extension of the Shapley value.

A weight system α is a function that associates a non-negative weight to each coalition and partition that contains it, with the condition that $\sum_{P \ni S, P \in \mathcal{P}(N_1 \cup N_2)} \alpha(S, P) = 1$, for all $S \subseteq N_1 \cup N_2$. Then, given the PFF game $(N_1 \cup N_2, \tilde{v})$ and the weight system α, the average approach constructs the CFF game $(N_1 \cup N_2, \hat{v}^{\alpha})$ as follows:

$$\hat{v}^{\alpha}(S) = \sum_{P \ni S, P \in \mathcal{P}(N_1 \cup N_2)} \alpha(S, P) \tilde{v}(S, P),$$

for any $S \subseteq N_1 \cup N_2$.

Using the two previous steps, we can go from a game $(N, v) \in \mathcal{G}$ to a PFF game $(N_1 \cup N_2, \tilde{v})$, and from this to a CFF game $(N_1 \cup N_2, \hat{v}^{\alpha})$. Given the expression for $\tilde{v}$ in (14), we obtain

$$\hat{v}^{\alpha}(S) = v_1(S \cap N_1) + \sum_{P \ni S, P \in \mathcal{P}(N_1 \cup N_2)} \alpha(S, P) v_2(S \cap N_2, P \tilde{\cap} N_1), \tag{15}$$

for any $S \subseteq N_1 \cup N_2$.

Hence, we can propose an extension of the Shapley value for games with intertemporal externalities by computing $Sh(N_1 \cup N_2, \hat{v}^{\alpha})$. Each vector of weights α that is symmetric, in the sense that it only depends on the sizes of the coalitions, and that satisfies a condition derived from the dummy player axiom (see Theorem 1 in Macho-Stadler et al., 2007) leads to an extension of the Shapley value to the class of games $\mathcal{G}$.

Pham Do and Norde (2007) and de Clippel and Serrano (2008) propose an extension, called the "externality-free" value, which corresponds to the weights $\alpha(S, P) = 1$ if $P =$

[13] The previous expression of $\tilde{v}$ is also obtained if we consider the initial game (N, v) a game with two "issues," in the sense of Diamantoudi et al. (2015). Following the approach of that paper, each period can be considered an issue in which all the players participate; only the players in N_1 generate worth in the first issue, and only those in N_2 generate worth in the second issue, although the organization of all the players in N_1 matters for that worth. Diamantoudi et al. (2015) propose a way to convert a game with several issues into a PFF game. Easy calculations show that going from (N, v) to a game with two issues and from that game to a PFF game results in $(N_1 \cup N_2, \tilde{v})$.

$\{S\} \cup \{\{h\} : h \in (N_1 \cup N_2) \setminus S\}$ and $\alpha(S, P) = 0$ otherwise. Using these weights, we obtain

$$\hat{v}^{\alpha}(S) = v_1(S \cap N_1) + v_2(S \cap N_2; \{S \cap N_1\} \cup \{\{h\} : h \in N_1 \setminus S\}) = \hat{v}(S),$$

for any $S \subseteq N_1 \cup N_2$. Therefore, according to Proposition 3, applying the previous procedure and then using the externality-free value leads to the one-coalition externality value for games with intertemporal externalities.

Can we also obtain the naive value using this procedure? Note first that, using (5) and (6), we can write the naive value as $\Phi^n(N, v) = Sh(N_1 \cup N_2, \hat{v}^n)$, where $\hat{v}^n(S) \equiv v_1(S \cap N_1) + v_2(S \cap N_2; \{N_1\})$, for every $S \subseteq N_1 \cup N_2$. Therefore, we can rephrase the question as, are there weights that lead to $\hat{v}^{\alpha}(S) = \hat{v}^n(S) = v_1(S \cap N_1) + v_2(S \cap N_2; \{N_1\})$ for any $S \subseteq N_1 \cup N_2$? The answer is negative. The reason is that, for any $S \subseteq N_1 \cup N_2$, $\hat{v}^{\alpha}(S)$ puts weights to the worth of the coalition $S \cap N_2$ when $S \cap N_1$ is an element of the partition P_1 (since $S \cap N_1 \in P \tilde{\cap} N_1$, for any (S, P) with $S \in P$), whereas $\hat{v}^n(S)$ only takes into account the worth of $S \cap N_2$ when $P_1 = N_1$. Therefore, $\hat{v}^{\alpha}(S)$ is typically different from $\hat{v}^n(S)$, for any $S \not\supseteq N_1$.

The fact that the naive value cannot be derived from an extension of the Shapley value for PFF games through the procedure previously described is not a weakness of the value. On the contrary, this fact highlights the particularities of the framework that we have analyzed. In PFF games, all the players have a priori equal possibilities to create worth and externalities with the other players. On the other hand, in a game with intertemporal externalities, the possibilities to create worth and externalities for players in N_1 are very different from those in N_2. A value for the set of games with intertemporal externalities may take into account these differences, as the naive value does.

9 Conclusion

We have introduced a new class of games whose main feature is that the organization of the coalitions formed by one set of players generates externalities on the worth obtained by the coalitions formed by another set of players. This model fits environments with intergenerational externalities, where the decisions taken (the coalitions formed) by countries or individuals at a moment in time strongly influence the surplus that countries or the future generation of individuals can obtain later. We propose a cooperative game analysis based on the original Shapley axioms for this class of games with intertemporal externalities. However, these axioms are insufficient to single out a unique value for our class of games.

We introduce two extensions of the Shapley value, the one-coalition externality value and the naive value, defined as the players' expected contribution to coalitions. Each extension is based on a different distribution probability of the arrival of the players. We characterize the one-coalition externality value using the equal treatment axiom, requiring two equally relevant players to obtain the same payoff. Similarly, we characterize the naive value with the axiom of strong equal treatment of externalities, which requires that similar externalities be rewarded or penalized in the same way.

Other values can be proposed for our class of games. For instance, one could consider the value obtained by assuming that players in the second period arrive before those in the first period. This may be seen as an imaginary backward induction made by players in the present anticipating that their descendants will form the grand coalition. This proposal is a technically interesting counterpart of the naive value as it distributes the externalities generated (on the grand coalition of the second period) among the agents in the first period. We believe that the unintuitive orderings in which it is based may make it an unappealing way to share the surplus in our setting.

Further research on games with intertemporal externalities may propose other axioms to extend the Shapley value. It may also use different approaches extending, for instance, the nucleolus. Moreover, we have deliberately focused on games where the externality is only intertemporal. In environments such as today's countries' negotiations on pollution abatement, there are not only intertemporal externalities but also inter-coalitional externalities affecting today's welfare. Finally, we can also consider non-cooperative games that would implement the cooperative solutions proposed in this paper. These analyses, while interesting, fall beyond the scope of the current paper.

Appendix

Proof of Lemma 1. We decompose the game (N, v) in the games (N, v^a) and $\left(N, v^b\right)$, as in the proof of Proposition 1. We know that $\Phi(N, v^a)$ allocates $Sh(N_1, \hat{v}_1)$ to the players in N_1 and 0 to the players of N_2. We now focus on $\Phi(N, v^b)$.

Since Φ satisfies *linearity*, then

$$\Phi_h(N, v^b) = \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1)} v^b(S_2; P_1) \Phi_h(N, u^{(S_2;P_1)}) \ \text{ for all } \ h \in N_1 \cup N_2. \tag{16}$$

The *anonymity* of Φ implies that

$$\Phi_j(N, u^{(S_2;P_1)}) = \Phi_{j'}(N, u^{(S_2;P_1)}) \ \text{ if } \ j, j' \in S_2, \text{ or } j, j' \in N_2 \setminus S_2, \tag{17}$$

and its *efficiency* implies that

$$\sum_{h \in N_1 \cup N_2} \Phi_h(N, u^{(S_2;P_1)}) = 0 \ \text{ if } \ (S_2; P_1) \neq (N_2, \{N_1\}), \tag{18}$$

$$\sum_{h \in N_1 \cup N_2} \Phi_h(N, u^{(N_2,\{N_1\})}) = 1. \tag{19}$$

Moreover, because Φ satisfies *linearity*, *anonymity*, and *dummy player*, then

$$\sum_{P_1 \in \mathcal{P}(N_1)} \Phi_h(N, u^{(S_2;P_1)}) = \begin{cases} 0 & \text{if } \ h \in N_1 \\ \beta_h(N_2, S_2) & \text{if } \ h \in N_2 \end{cases} \tag{20}$$

where $\beta_h(N_2, S_2)$ are the Shapley coefficients, see (1). Equation (20) follows from Proposition 1 because $\sum_{P_1 \in \mathcal{P}(N_1)}(N, u^{(S_2;P_1)})$ is a game without externalities; hence, the worth $\sum_{P_1 \in \mathcal{P}(N_1)}(N, u^{(S_2;P_1)})(N_2; \{N_1\})$ (which is equal to 0 unless $S_2 = N_2$, in which case the worth is 1) is shared among the players in N_2 according to their Shapley value.

Using (8) and (17), we can express Equations (18) and (19) as follows:

$$|N_1|\, \Phi_k(N, u^{(S_2;P_1)}) + |S_2|\, \Phi_j(N, u^{(S_2;P_1)}) + |N_2 \setminus S_2|\, \Phi_{j'}(N, u^{(S_2;P_1)}) = 0 \tag{21}$$

for any $k \in N_1$, $j \in S_2$, and $j' \in N_2 \setminus S_2$, and

$$|N_1|\, \Phi_k(N, u^{(N_2,\{N_1\})}) + |N_2|\, \Phi_j(N, u^{(N_2,\{N_1\})}) = 1, \tag{22}$$

for any $k \in N_1$ and $j \in N_2$.

We write Equation (21) as:

$$\Phi_{j'}(N, u^{(S_2;P_1)}) = -\frac{|S_2|}{|N_2 \setminus S_2|}\Phi_j(N, u^{(S_2;P_1)}) - \frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)}), \tag{23}$$

for any $k \in N_1$, $j \in S_2$, and $j' \in N_2 \setminus S_2$, and we notice that the Shapley coefficients satisfy the following relation:

$$|S_2|\, \beta_j(N_2, S_2) + |N_2 \setminus S_2|\, \beta_{j'}(N_2, S_2) = 0 \tag{24}$$

for all $j \in S_2$ and $j' \in N_2 \setminus S_2$.

Using (24), we substitute $|N_2 \setminus S_2|$ in Equation (23) to obtain:

$$\Phi_{j'}(N, u^{(S_2;P_1)}) = \beta_{j'}(N_2, S_2)\frac{1}{\beta_j(N_2, S_2)}\Phi_j(N, u^{(S_2;P_1)}) - \frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)}), \tag{25}$$

for any $k \in N_1$, $j \in S_2$, and $j' \in N_2 \setminus S_2$.

Define the "weights" $\gamma(S_2; P_1)$ as follows:

$$\gamma(S_2; P_1) \equiv \frac{1}{\beta_j(N_2, S_2)}\Phi_j(N, u^{(S_2;P_1)}), \tag{26}$$

where j is any player in S_2.

Notice that, using (20), $\sum_{P_1 \in \mathcal{P}(N_1)} \gamma(S_2; P_1) = \frac{1}{\beta_j(N_2,S_2)} \sum_{P_1 \in \mathcal{P}(N_1)} \Phi_j(N, u^{(S_2;P_1)}) = 1$ (where j is any player in S_2), for all $S_2 \subseteq N_2$.

Then, Equations (26) and (25) lead to

$$\Phi_j(N, u^{(S_2;P_1)}) = \beta_j(N_2, S_2)\gamma(S_2; P_1), \tag{27}$$

$$\Phi_{j'}(N, u^{(S_2;P_1)}) = \beta_{j'}(N_2, S_2)\gamma(S_2; P_1) - \frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)}), \tag{28}$$

for any $j \in S_2$, $j' \in N_2 \setminus S_2$, and $k \in N_1$.

Using (16), (27), and (28), we can express the worth of any player $j \in N_2$ in a game (N, v^b) according to a value Φ that satisfies linearity, anonymity, and equal treatment of externalities as follows:

$$\begin{aligned}
\Phi_j(N, v^b) = & \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1)} v^b(S_2; P_1)\Phi_j(N, u^{(S_2;P_1)}) \\
= & \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1), S_2 \supseteq \{j\}} v^b(S_2; P_1)\beta_j(N_2, S_2)\gamma(S_2; P_1) \\
+ & \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1), S_2 \not\supseteq \{j\}} v^b(S_2; P_1)\left(\beta_j(N_2, S_2)\gamma(S_2; P_1) - \frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)})\right) \\
= & \sum_{\emptyset \neq S_2 \subseteq N_2} \beta_j(N_2, S_2) \sum_{P_1 \in \mathcal{P}(N_1)} \gamma(S_2; P_1)v^b(S_2; P_1) \\
- & \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1), S_2 \not\supseteq \{j\}} v^b(S_2; P_1)\frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)}) \\
= & Sh_j(N_2, \hat{v}_2^\gamma) - \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1), S_2 \not\supseteq \{j\}} \frac{|N_1|}{|N_2 \setminus S_2|}\Phi_k(N, u^{(S_2;P_1)})v^b(S_2; P_1),
\end{aligned}$$

where k is any player in N_1.

Similarly, using (16), we can express the worth of any player $i \in N_1$ as follows:

$$\Phi_i(N, v^b) = \sum_{\emptyset \neq S_2 \subseteq N_2, P_1 \in \mathcal{P}(N_1)} v^b(S_2; P_1)\Phi_i(N, u^{(S_2;P_1)}).$$

Given that $\Phi_k(N, v^{(S_2;P_1)})$ is the same for every $k \in N_1$, linearity and equal treatment of externalities imply that all the players in N_1 obtain the same payoff in a game (N, v^b).

Finally, the expression in the lemma follows from the linearity of Φ and the fact that $v = v^a + v^b$. □

References

Alonso-Meijide, J. M., Álvarez-Mozos, M., Fiestras-Janeiro, M. G., Jiménez-Losada, A. (2019). "Complete null agent for games with externalities." *Expert Systems with Applications* 135, 1-11.

Bolger, E.M. (1989). “A set of axioms for a value for partition function games.” *International Journal of Game Theory* 18(1), 37-44.

De Clippel, G., Serrano, R. (2008). “Marginal contributions and externalities in the value.” *Econometrica* 76(6), 1413-1436.

Diamantoudi, E., Macho-Stadler, I., Pérez-Castrillo, D., Xue, L. (2015). “Sharing the surplus in games with externalities within and across issues.” *Economic Theory* 60(2), 315-343.

Kalai, E., Samet, D. (1987). “On weighted Shapley values.” *International Journal of Game Theory* 16, 205-222.

Kóczy, L.Á. (2018). *Partition function form games.* Theory and Decision Library C, 48, 312.

Macho-Stadler, I., Pérez-Castrillo, D., Wettstein, D. (2007). “Sharing the surplus: An extension of the Shapley value for environments with externalities.” *Journal of Economic Theory* 135(1), 339-356.

Macho-Stadler, I., Pérez-Castrillo, D., Wettstein, D. (2019). “Extensions of the Shapley value for environments with externalities.” In *Handbook of the Shapley Value* (pp. 131-155). Chapman and Hall/CRC.

McQuillin, B. (2009). “The extended and generalized Shapley value: Simultaneous consideration of coalitional externalities and coalitional structure.” *Journal of Economic Theory* 144(2), 696-721.

Myerson, R. (1977). “Values of games in partition function form.” *International Journal of Game Theory* 6, 23-31.

Pham Do, K. H., Norde, H. (2007). “The Shapley value for partition function form games.” *International Game Theory Review* 9(2), 353-360.

Shapley, L. S. (1953a). “Additive and non-additive set functions.” PhD Thesis, Department of Mathematics, Princeton University.

Shapley, L. S. (1953b). “A value for n-person games.” *Annals of Mathematics Study* 28, 307-317.

Thrall, R. M., Lucas, W. F. (1963). "N-person games in partition function form." *Naval Research Logistics Quarterly* 10(1), 281-298.