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AN EGALITARIAN APPROACH FOR THE ADJUDICATION OF CONFLICTING CLAIMS

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Title: An egalitarian approach for the adjudication of conflicting claims

Abstract: This paper addresses the challenge of adjudicating conflicting claims by introducing a related reference point depending on the claims of agents. Egalitarian principles underpin many allocation rules, with the constrained equal awards (CEA) rule standing out as a notable example. Various other significant rules are examined through an egalitarian lens. The traditional model of adjudicating claims is enriched by incorporating an external baseline point. This study demonstrates that the CEA rule, alongside the CEL rule, the weighted CEA rule, the weighted CEL rule, the family of reverse Talmudic rules, and the entire class of CIC rules adhere to the same egalitarian criterion when a reference point is specified. However, not all allocation rules follow this criterion of equality; for instance, the Talmudic rule does not conform to it. As a second result, all the egalitarian rules are characterized by a properly defined quadratic optimization problem, as it was already known for the CEA rule..

JEL Codes: C71, C78, D63

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1 Introduction

The process of adjudicating conflicting claims arises when there's a need to allocate a finite resource among competing agents, and the available quantity falls short of meeting all demands (Aumann and Maschler, 1985). To navigate this challenge, various rules have been put forward, drawing primarily from two enduring principles of practical application: egalitarianism and proportionality.

Egalitarianism advocates for an equal distribution of resources among claimants, regardless of their individual circumstances or contributions. This principle prioritizes fairness and aims to minimize disparities by ensuring that each agent receives an equitable share of the available resources. On the other hand, proportionality suggests that the allocation of resources should be proportionate to certain factors, such as need, merit, or contribution. Unlike egalitarianism, proportionality considers the varying circumstances and characteristics of the claimants, allocating resources in accordance with their relative entitlements or worthiness.

The choice between these principles and the specific rules derived from them depends on the context, values, and goals of the adjudication process. In practice, a combination of egalitarian and proportional approaches may be employed to achieve a balanced and fair distribution of the limited resource. This nuanced approach acknowledges both the importance of equitable treatment and the recognition of individual differences among claimants. Thomson (2019) provides an extensive review encompassing all the literature pertaining to these topics.

Many allocation rules are based on an egalitarian criterion. This principle advocates for the equal distribution of resources among all involved agents, prioritizing fairness and equality. It ensures that everyone receives an equitable share, irrespective of factors such as need or contribution. Egalitarian approaches aim to minimize disparities and promote social justice. The most notable and purely egalitarian rule is the Constrained Equal Awards rule. It's distinguished, among other properties, by its aim to minimize the variance of payoffs assigned to agents compared to the "go Dutch" distribution, also known as the pure egalitarian distribution, which involves dividing the estate equally among all agents.Many other significant rules have been analyzed. Among them, we select the Constrained Equal Losses (CEL), the Talmudic rules and their extensions (T), the Proportional Rule (P), and the CIC rules.

The fundamental model for analyzing the problem of adjudicating conflicting claims is enriched by introducing a vector in the space of payoffs that represents a reference or baseline point. This is pertinent for conducting evaluations of the final allocation or proposal. By employing this reference point, we attempt to equalize the agents' perception of how they are treated. Agents' perception not only considers the absolute final outcome but also how this outcome compares with a certain reference point (Kahneman and Tversky, 1979).

Reference points in bankruptcy problems and related issues are not a new concept. Herrero (1998) incorporates the reference point as a function of the agents' claims. Pulido et al. (2002, 2008) examine bankruptcy problems incorporating reference points. Hougaard et al. (2012, 2013a, 2013b)) introduce the concept of baselines in this context, which can be seen as equivalent to reference points. Timoner and Izquierdo (2016) use ex-ante conditions to address claims problems that can be embedded in our context.

More recently, Gallice (2019) has provided justification for the utilization of reference points and has left to the reader the opportunity to introduce new solutions or rules in our context by appropriately considering reference points.

This paper illustrates that well-known rules, including CEA, CEL, Proportional, weighted CEA, weighted CEL, reverse Talmudic rule, and the broader family of CIC rules, conform to the same egalitarian criterion for a given reference point. Consequently, we characterize all these solutions by a unified relationship: the egalitarian solution. As a second result, we also characterize all of these rules through a minimization problem of a weighted sum of quadratic gaps within a well-defined and natural domain. This outcome not only offers a practical method for solving such problems but also serves as a natural extension of Schummer and Thomson's (1997) result concerning the variance of the CEA rule. The final section and the concluding remarks suggest avenues for future research.

The paper is structured as follows. Section 2 presents basic definitions and notations. Section 3 introduces the concept of egalitarian rule from a reference point. Section 4 explores the specific case where the reference point adheres to certain order-preservation properties.

Section 5 expands the model by incorporating external weights. Section 6 discusses the specific rule "concede and divide" demonstrating how non-covered rules within our model can be approached through its weighted extension. Finally, in Section 7 we provide concluding remarks.

2 Notations and definitions

Let $N = \{1, 2, \ldots, n\}$ be a set of agents. We denote by $\mathbb{R}^N$ the set of n-dimensional vectors indexed by N, i.e. if $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$, then x_i is the component relative to agent $i \in N$. Furthermore, for all $T \subseteq N$, we write $x(T) = \sum_{i \in T} x_i$, with $x(\varnothing) = 0$. On the other hand, given a real value $b \in \mathbb{R}$ we write $(b)_+ = \max\{0, b\}$.

A conflicting claims problem is a pair (E, c) where $E \geq 0$ is the amount of resource that should be distributed among a set of agents $N = \{1, 2, \ldots, n\}$, and a vector of claims $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^N_+$, where $c_i > 0$ is the claim of agent $i \in N$ over E. It is supposed $E \leq \sum_{i=1}^n c_i$. We denote the set of all claims problems on N as $\mathcal{C}^N$.

Given a claims problem (E, c), an allocation $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$ is a payoff vector that assigns to each agent $i \in N$ its share of E such that $0 \leq x_i \leq c_i$ (claim boundedness property), for all $i \in N$, and $\sum_{i \in N} x_i = E$ (efficiency property). We define the *set of admissible allocations* $D(c)$ as the set of allocations satisfying claim boundedness, i.e.

$$D(c) = \{x \in \mathbb{R}^N \mid 0 \leq x_i \leq c_i, \text{ for all } i \in N\},$$

and the *set of efficient allocations*, $H(E)$, as

$$H(E) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = E\}.$$

An *allocation rule* f assigns to each claims problem (E, c) a feasible and efficient allocation $x \in D(c) \cap H(E)$. Five of the main allocation rules are the following. The *constrained equal awards rule* (CEA) assigns the same amount λ to each agent, but not exceeding its claim, i.e.

$$f_i^{CEA}(E, c) = \min\{\lambda, c_i\}, \text{ for all } i \in N.$$

The *constrained equal losses rule* (CEL) withdraw the same amount λ from the claim of each agent, so as no agent can receive a negative amount, i.e. **??**

$$f_i^{CEL} = \max\{0,\, c_i - \lambda\}, \text{ for all } i \in N.$$

The family of *θ-reverse Talmudic* rules apply the CEL rule according to a fraction θ of the claims, $0 \leq \theta \leq 1$, if E is smaller than a fraction θ of the sum of claims, and otherwise, it distributes the excess according to the CEA rule, i.e.

$$f_i^{RT(\theta)}(E,c) = \begin{cases} \max\{0, \theta \cdot c_i - \lambda\} & \text{if } E \leq \theta \cdot c(N) \\ \theta \cdot c_i + \min\{\lambda, (1-\theta) \cdot c_i\} & \text{if } E > \theta \cdot c(N). \end{cases}$$

The *proportional rule* (P) assigns the same fraction λ of its claim to each agent $i \in N$, i.e.

$$f_i^P(E,c) = \lambda \cdot c_i, \text{ for all } i \in N.$$

The *Talmudic rule* apply the CEA rule according to half of the claims, if E is smaller than half of the sum of claims, and otherwise, it distributes the excess according to the CEL rule, i.e.

$$f_i^T(E,c) = \begin{cases} \min\{\lambda, \frac{1}{2}c_i\} & \text{if } E \leq \frac{1}{2}\, c(N) \\ \max\{\frac{1}{2}c_i, c_i - \lambda\} & \text{if } E > \frac{1}{2}\, c(N) \end{cases}$$

where, in all cases $\lambda \in \mathbb{R}_+$ such that the efficiency constraint is satisfied, i.e., $\sum_{i \in N} f_i(E,c) = E$.

3 Egalitarian rules from a reference

In this section, we define a family of rules based on a reference system. From the vector of claims, each agent is assigned a reference value, which will be used as an initial value for evaluating its gains or losses.

A reference system α associates to each claims vector $c \in \mathbb{R}^n_+$ a reference vector $\alpha(c) \in D(c)$. Given an allocation $x = (x_1, x_2, \ldots, x_n) \in D(c) \cap H(E)$, a claims vector c and a reference system α, the difference $x_i - \alpha_i(c)$ represents the *net payoff for agent* i from its reference. If $x_i - \alpha_i(c) > 0$, the agent i receives an amount larger than its reference, perceiving the payoff as as gain. Conversely, if $x_i - \alpha_i(c)$ is negative the payoff is interpreted as a loss. Based on this, a concept of egalitarian rule can be defined as follows.

Definition 1. *On the domain of claims problems $\mathcal{C}^N$, an allocation rule f is considered egalitarian with respect to a reference system α if, for any claims problem (E, c), the rule assigns an allocation $f(E, c) = x$ that satisfies the following condition:*

> *for all pair of agents i and j, if the net payoff for agent i is strictly smaller than the net payoff for agent j, then either agent i receives its full claim or agent j does not receive anything, i.e.*

$$\text{if } x_i - a_i < x_j - a_j \text{ then either } x_i = c_i \text{ or } x_j = 0. \tag{1}$$

Condition (1) stipulates that if there is a gap between the net payoffs of two arbitrary agents i and j, i.e. $x_i - a_i < x_j - a_j$, then no transfer of payoff from agent j to agent i can be carried out. This is because either agent i has already received the full claim or agent j has not received anything.

Next proposition shows the existence and uniqueness of an egalitarian rule for each reference system.

Theorem 1. *On the domain of claims problems $\mathcal{C}^N$ and given a reference system α, there is a unique allocation rule f that is egalitarian with respect to α. We denote it by f^α and it assigns to each claims problem (E, c) the payoff vector $x \in D(c)$ defined as*

$$x_i = f_i^\alpha(E, c) = \min\{(\lambda + a_i)_+, c_i\}, \text{ for all } i \in N,$$

where $\alpha(c) = a$, $\lambda \in \mathbb{R}$ and $\sum_{i \in N} \min\{(\lambda + a_i)_+, c_i\} = E$.

Proof. First notice that, by definition, $0 \leq f_i^\alpha(E, c) \leq c_i$, for all $i \in N$. Moreover, for each problem the value of λ does exist since $F(\lambda) = \sum_{i \in N} \min\{(\lambda + a_i)_+, c_i\}$ is a continuous function and $F(-\max_{k \in N} c_k) = 0$, $F(\max_{k \in N} c_k) = \sum_{i \in N} c_i \geq E$.

Next we show that the rule f^α satisfies condition (1). Given a problem (E, c) and $\alpha(c) =$

a, suppose that, for some $i, j \in N$, $f_i^\alpha(E,c) - a_i < f_j^\alpha(E,c) - a_j$, or equivalently

$$\min\{(\lambda + a_i)_+, c_i\} - a_i < \min\{(\lambda + a_j)_+, c_j\} - a_j. \tag{2}$$

If $(\lambda + a_i)_+ \geq c_i$ then, by definition, $f_i^\alpha(E,c) = c_i$ and thus condition (1) is satisfied.

If $(\lambda + a_i)_+ < c_i$, let us show that $f_j^\alpha(E,c) = 0$. To this aim, suppose to the contrary that $f_j^\alpha(E,c) > 0$. Then, by definition $\min\{(\lambda+a_j)_+, c_j\} > 0$, which implies that $\lambda + a_j > 0$. Hence,

$$\begin{aligned}
\lambda &\leq \max\{\lambda, -a_i\} = (\lambda + a_i)_+ - a_i = \min\{(\lambda + a_i)_+, c_i\} - a_i \\
&< \min\{(\lambda + a_j)_+, c_j\} - a_j = \min\{(\lambda + a_j)_+ - a_j, c_j - a_j\} \\
&\leq (\lambda + a_j)_+ - a_j = \lambda + a_j - a_j = \lambda,
\end{aligned}$$

where the second equality follows from the hypothesis of the case and the strict inequality follows by (2). Hence, we get a contradiction, and thus we conclude that $f_j^\alpha(E,c) = 0$. Therefore, condition (1) is also satisfied in this case.

Finally, we prove the uniqueness of the rule. Suppose there were two different rules f and g satisfying condition (1). Hence, there would be a claims problem (E,c) such that $x = f(E,c) \neq g(E,c) = y$. By the efficiency condition, $x(N) = y(N)$, we have $x_i < y_i$ and $x_j > y_j$, for some $i, j \in N$. Thus $x_i - a_i < y_i - a_i$ and $x_j - a_j > y_j - a_j$. Then, we analyze two cases. If $y_i - a_i \leq y_j - a_j$, it would follow that $x_i - a_i < x_j - a_j$ and thus, by (1), either $x_i = c_i$ or $x_j = 0$. However, if $x_i = c_i$, since $x_i < y_i$, we would get $c_i < y_i$ which contradicts $y \in D(c)$; if $x_j = 0$, then by hypothesis $x_j = 0 > y_j$, which contradicts $y \in D(c)$. In case $y_i - a_i > y_j - a_j$, we use a similar argument to get a contradiction. We conclude the rule satisfying condition (1) is unique.

□

Let us remark that in the definition of the egalitarian rule f^α, the parameter λ can be either a positive or a negative number. If it is positive, $\lambda > 0$, it represents the equal positive excess that any agent i receives from its reference value $\alpha_i(c) = a_i$ constrained to not receiving more than its claim. If it is negative, $\lambda < 0$, the parameter represents the equal loss that any agent i incurs from the reference value $\alpha_i(c) = a_i$ constrained to receiving a positive amount.

Next proposition shows that some basic classical rules are in fact egalitarian rules.

Proposition 1. *The constrained equal awards rule, the constrained equal losses rule and the family of reverse Talmudic rules are all egalitarian rules on* $\mathcal{C}^N$

Proof. For the case of the CEA rule the proof is straightforward taking the reference system $\alpha(c) = 0$ for each $c \in \mathbb{R}^N_+$.

For the CEL rule, take $\alpha(c) = c$. Indeed, in this case $f_i^\alpha(E, c) = \min\{(\lambda + c_i)_+, c_i\}$. where $\sum_{i\in N} \min\{(\lambda+c_i)_+, c_i\} = E$. Notice that when $E < \sum_{i\in N} c_i$ the value of λ must be negative, $\lambda < 0$. Otherwise, if $\lambda \geq 0$, we would have $E = \sum_{i\in N} \min\{(\lambda + c_i)_+, c_i\} = \sum_{i\in N} c_i > E$, resulting a contradiction. Moreover, for $E = \sum_{i\in N} c_i$, then we can take $\lambda = 0$. Hence, in any case it holds $\lambda \leq 0$. Thus, we can rewrite the formula for the egalitarian rule as follows:

$$f_i^\alpha(E, c) = \min\{(\lambda + c_i)_+, c_i\} = (\lambda + c_i)_+ = (c_i - \lambda')_+ = f^{CEL}(E, c).$$

where $\lambda' = -\lambda \geq 0$.

For the family of reverse Talmudic rules, and for any $0 \leq \theta \leq 1$ just take $\alpha(c) = \theta \cdot c$. Indeed, if $\theta = 0$ or $\theta = 1$ then we refer to the proofs of CEA rule and the CEL rule above.

If $0 < \theta < 1$ and $E \leq \sum_{i\in N} \theta \cdot c_i$, then the value of λ that makes $\sum_{i\in N} \min\{(\lambda + \theta \cdot c_i)_+, c_i\} = E$, is non-positive, i.e. $\lambda \leq 0$. Otherwise, if $\lambda > 0$ we would have

$$E = \sum_{i\in N} \min\{(\lambda + \theta \cdot c_i)_+, c_i\} = \sum_{i\in N} \min\{\lambda, (1 - \theta) \cdot c_i\} + \sum_{i\in N} \theta \cdot c_i > \sum_{i\in N} \theta \cdot c_i,$$

getting a contradiction. Therefore, $\lambda \leq 0$, and thus

$$\begin{aligned} f_i^{RT(\theta)}(E, c) &= \min\{(\lambda + \theta \cdot c_i)_+, c_i\} = (\lambda + \theta \cdot c_i)_+ = \max\{\theta \cdot c_i - \lambda', 0\} \\ &= f_i^{CEL}(E, \theta \cdot c), \end{aligned}$$

where $\lambda' = -\lambda$.

If $0 < \theta < 1$ and $E \geq \sum_{i\in N} \theta \cdot c_i$, then it holds that the λ value that makes

$$\sum_{i\in N} \min\{(\lambda + \theta \cdot c_i)_+, c_i\} = E,$$

is non-negative, i.e. $\lambda \geq 0$. Otherwise, if $\lambda < 0$ we would have

$$\begin{aligned} E &= \sum_{i\in N} \min\{(\lambda+\theta\cdot c_i)_+, c_i\} = \sum_{i\in N} (\lambda+\theta\cdot c_i)_+ = \sum_{i\in N} max\{\lambda+\theta\cdot c_i, 0\} \\ &= \sum_{i\in N} max\{\lambda, -\theta\cdot c_i\} + \sum_{i\in N} \theta\cdot c_i < \sum_{i\in N} \theta\cdot c_i. \end{aligned}$$

Therefore, $\lambda \geq 0$, and thus

$$\begin{aligned} f_i^{RT(\theta)}(E,c) &= \min\{(\lambda+\theta\cdot c_i)_+, c_i\} = \min\{\lambda+\theta\cdot c_i, c_i\} \\ &= \theta\cdot c_i + \min\{\lambda, (1-\theta)\cdot c_i)\} = \theta\cdot c_i + f_i^{CEA}(E,(1-\theta)\cdot c). \end{aligned}$$

□

The reference system α can be interpreted as agents' payoff expectations before knowing the available amount to be distributed. Some examples are those analyzed in Proposition 1. Other instances of a reference system are:

- ⋆ *Smallest best* α^{SB}. Every agent expects to receive at least as much as the next smallest claim after its own claim. Supposing $c_1 \leq c_2 \leq \ldots \leq c_n$, then

$$\alpha_1^{SB}(c) = 0 \text{ and } \alpha_i^{SB}(c) = c_{i-1}, \text{ for } i = 2,\ldots,n.$$

- ⋆ *Average of smaller claims* α^{Av}. Every agent expects to receive at least the average of smaller claims. Supposing $c_1 \leq c_2 \leq \ldots \leq c_n$, then

$$\alpha_1^{Av}(c) = 0 \text{ and } \alpha_i^{Av}(c) = \frac{1}{i-1}\sum_{k<i} c_k, \text{ for } i = 2,\ldots,n.$$

- ⋆ *Truncated claim* α^{TC}. Every agent expects to receive its claim if it is small enough compared to the average of claims; otherwise, it will consider the average claim as the reference. For $i = 1,2,\ldots,n$

$$\alpha_i^{TC}(c) = \min\{c_i, \frac{1}{n}c(N)\}.$$

- ⋆ *Serial sharing* α^{S}. The smallest claimant expects to receive an equal part of its claim, $\frac{c_1}{n}$. The second smallest claimant expects to receive the previous amount plus the equal

REFERENCE SYSTEMS

90AGENT	90CLAIM	90α^{CEA}	90α^{CEL}	90$\alpha^{RT(\frac{1}{2})}$	90α^{SB}	90α^{Av}	90α^{TC}	90α^{S}
1	2000	0	2000	1000	0	0	2000	666.66
2	3000	0	3000	1500	2000	2000	3000	1166.66
3	4000	0	4000	2000	3000	2500	3000	2166.66

Table 1: Reference systems for the three-agent problem with $c = (2000, 3000, 4000)$.

part, shared with the $n-1$ remaining agents, of the difference between c_2 and c_1, $\frac{c_2-c_1}{n-1}$, and so on. Supposing $c_1 \leq c_2 \leq \ldots \leq c_n$, then

$$\alpha_i^S(c) = \sum_{k=1}^{i} \frac{c_k - c_{k-1}}{n-k+1}, \text{ for } i = 1, \ldots, n,$$

where we take $c_0 = 0$.

In all cases the reader can check that $0 \leq \alpha_i(c) \leq c_i$, for all $i \in N$.

Table 2 shows a numerical application of the reference systems introduced above for c = (2000, 3000, 4000). Each of these systems defines an egalitarian rule. For instance, consider the truncated claim system $\alpha^{TC} = (2000, 3000, 3000)$. If $E = 1000$ the egalitarian allocation is

$$f^{\alpha^{TC}}(E, c) = (0,\ 500, 500).$$

The corresponding net payoffs $x_i - \alpha_i^{TC}(c)$ are

$$(x_1 - \alpha_1^{TC}(c), x_2 - \alpha_2^{TC}(c), x_3 - \alpha_3^{TC}(c)) = (-2000, -2500, -2500).$$

Observe that the loss of agent 1 from its reference, -2000, is larger than the one of agent 2, -2500. When this happens, egalitarian condition (1) requires, as a consequence, that either agent 1 should not receive anything, or agent 2 should receive its full claim. In this case agent 1 is assigned a zero payoff.

If now we take the same reference system but $E = 8500$, then we have

$$f^{\alpha^{TC}}(E, c) = (2000,\ 3000, 3500)$$

ALLOCATIONS FOR $E = 1000$ AND CLAIMS $c = (2000, 3000, 4000)$

	CEA	CEL	RT($\frac{1}{2}$)	SB	Av	TC	S
1	$333\frac{1}{3}$ (0.3)	0 (-2)	0 (-1)	0 (0)	0 (0)	0 (-2)	0 (-0.6)
2	$333\frac{1}{3}$ (0.3)	0 (-3)	250 (-1.25)	0 (-2)	250 (-1.75)	500 (-2.5)	0 (-1.16)
3	$333\frac{1}{3}$ (0.3)	1000 (-3)	750 (-1.25)	1000 (-2)	750 (-1.75)	500 (-2.5)	1000 (-1.16)
$\sum$	1000	1000	1000	1000	1000	1000	1000

ALLOCATIONS FOR $E = 8500$ AND CLAIMS $c = (2000, 3000, 4000)$

	CEA	CEL	RT($\frac{1}{2}$)	SB	Av	TC	S
1	2000 (2)	$1833\frac{1}{3}$ (-0.1)	2000 (1)	1500 (1.5)	1500 (1.5)	2000 (0)	2000 (1.33)
2	3000 (3)	$2833\frac{1}{3}$ (-0.1)	3000 (1.5)	3000 (1)	3000 (1)	3000 (0)	2750 (1.58)
3	3500 (3.5)	$3833\frac{1}{3}$ (-0.1)	3500 (1.5)	4000 (1)	4000 (1.5)	3500 (0.5)	3750 (1.58)
$\sum$	8500	8500	8500	8500	8500	8500	8500

Table 2: Egalitarian allocations corresponding to different reference systems and claims vector $c = (2000, 3000, 4000)$. In the first table $E = 1000$ and in the second one $E = 8500$. In brackets we indicate the agents' net payoffs in thousands of units and rounded to one decimal place.

with net payoffs $(0, 0, 500)$. Observe that the net payoff of agent 1 is now smaller than that of agent 3, but in this case agent 1 receives its full claim to fulfill condition (1). Similarly for agents 2 and 3, and for agents 1 and 2.

In Table 2 you can find egalitarian allocations with respect to different reference systems when $E = 1000$ and $E = 8500$, respectively. After each agents' payoff, in brackets, it is shown its net payoff with respect to the corresponding reference value. In all cases you can easily verify that condition (1) is met.

4 Order-preserving reference systems

An egalitarian rule is based on its associated reference system. For any claims vector c it recommends a reference point $\alpha(c) = a \in \mathbb{R}^N$. This reference point is only required to be feasible, i.e. $0 \leq a_i \leq c_i$, for all $i \in N$. However, in the literature some equity criteria has been introduced to refine the feasible set. Indeed, we can frame the selection of a reference point by using two order preservation principles:

- *Order preservation in awards.* The larger an agent's claim, the greater its gain.
- *Order preservation in losses.* The larger an agent's claim, the greater its loss (from that claim).

This way we define the order preserving set as follows

$$\mathcal{O}(c) = \left\{ x \in D(c) \;\middle|\; \text{for all } i, j \in N, \text{if } c_i \leq c_j \text{ then } \begin{array}{l} x_i \leq x_j \text{ and} \\ c_i - x_i \leq c_j - x_j \end{array} \right\}.$$

All the reference systems discussed in the previous section recommend reference points in the order-preserving, except for the *smallest best* system and the average of smaller claims system. Indeed, take the reference point proposed by this system in Table 1, i.e. $\alpha^{SB}(2000, 3000, 4000) = (0, 2000, 3000)$. This vector respects order-preservation in awards, but not in losses since

$$c_1 - \alpha_1^{SB} = 2000 - 0 > c_2 - \alpha_2^{SB} = 3000 - 2000 = 1000.$$

Next proposition shows that, only if the reference point is in the order-preserving set, the egalitarian rule proposes allocations in the order-preserving set (see Figure 1).

Proposition 2. *Given a claims vector $c \in \mathbb{R}^N_{++}$ and a reference system α, the following two conditions are equivalent:*

1. *$f^\alpha(E, c) \in \mathcal{O}(c)$ for all $0 \leq E \leq c(N)$,*

2. *$\alpha(c) \in \mathcal{O}(c)$.*

Proof.

$1 \Rightarrow 2$) Taking $E = \sum_{i \in N} \alpha_i(c)$, then by definition of egalitarian rule we have $f^\alpha(E, c) = \alpha(c) \in \mathcal{O}(c)$, since we are assuming, by hypothesis, $f^\alpha(E, c) \in \mathcal{O}(c)$, for all $0 \leq E \leq c(N)$.

$2 \Rightarrow 1$) Let i and j be two agents such that $c_i \leq c_j$. Since $a = \alpha(c) \in \mathcal{O}(c)$, then $a_i \leq a_j$ and $c_i - a_i \leq c_j - a_j$. Hence, it is straightforward that

$$f_i^\alpha(E, c) = \min\{(\lambda + a_i)_+, c_i\} \leq \min\{(\lambda + a_j)_+, c_j\} = f_j^\alpha(E, c).$$

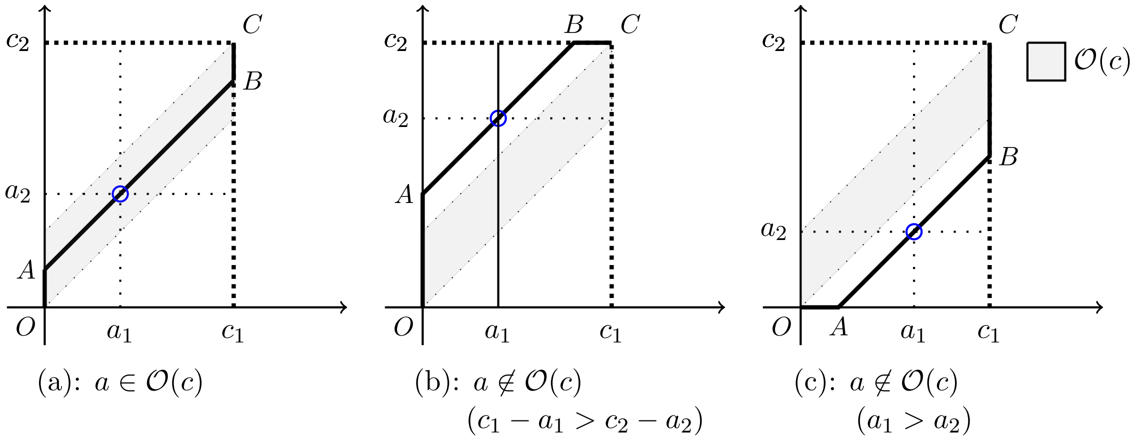

Figure 1: Two-person problem $(E, (c_1, c_2))$ and reference point $\alpha(c) = (a_1, a_2)$, where $c_1 < c_2$. The path of awards (OABC) of the egalitarian rule for three different reference points. In (a), the reference point is in the order-preserving set $\mathcal{O}(c)$ and thus, by Proposition 2, the path of awards is in the order-preserving set. In (b) and (c) the reference point is above or below the order-preserving set $\mathcal{O}(c)$ (shadow grey band) and thus the path of awards is not included in the order-preserving set.

Moreover, if $\lambda + a_j \leq 0$, then $\lambda + a_i \leq 0$, and thus $f_i^\alpha(E, c) = f_j^\alpha(E, c) = 0$. Hence

$$c_i - f_i^\alpha(E, c) = c_i \leq c_j \leq c_j - f_j^\alpha(E, c).$$

If $\lambda + a_j > 0$, then

$$\begin{aligned} c_i - f_i^\alpha(E, c) &= c_i - \min\{(\lambda + a_i)_+, c_i\} = \max\{c_i - (\lambda + a_i)_+, 0\} \\ &\leq \max\{c_i - \lambda - a_i, 0\} \leq \max\{c_j - \lambda - a_j, 0\} \\ &= c_j + \max\{-\lambda - a_j, -c_j\} = c_j - \min\{\lambda + a_j, c_j\} \\ &= c_j - \min\{(\lambda + a_j)_+, c_j\} = c_j - f_j^\alpha(E, c), \end{aligned}$$

where the second inequality comes from $c_i - a_i \leq c_j - a_j$. □

When the reference system always proposes points within the order-preserving set, we say the reference system is order-preserving, and the corresponding rule is an **order-preserving egalitarian rule**. Next, we show that the entire family of CIC rules consists of order-preserving egalitarian rules.

Thomson (2008) define CIC rules with the aim of generalizing and including some of the allocation rules. Next we show that CIC rules are a particular case of egalitarian rules. To formally define CIC rules, let $\mathcal{H} = (F^k, G^k)_{k=1}^{n-1}$, where $n = |N|$, be a list of pairs of real-valued functions defined on $\mathbb{R}^n$, such that for each $c \in \mathbb{R}_+^N$, the sequence $(F^k(c))_{k=1}^{n-1}$ is nowhere decreasing, the sequence $(G^k(c))_{k=1}^{n-1}$ is nowhere increasing such that

$$0 \leq F^1(c) \leq F^2(c) \leq \ldots \leq F^{n-1}(c) \leq G^{n-1}(c) \leq \ldots \leq G^1(c) \leq c(N),$$

and the following CIC relations hold: suppose (w.l.o.g.) that $c_1 \leq c_2 \leq \ldots \leq c_n$, then

$$\begin{aligned} \frac{-F^{n-1}(c)}{n} + \frac{G^{n-1}(c)}{n} &= c_1; \\ &\vdots \\ c_{k-1} + \frac{F^{n-k+1}(c) - F^{n-k}(c)}{n-k+1} + \frac{G^{n-k}(c) - G^{n-k+1}(c)}{n-k+1} &= c_k, \text{ for } k = 2, \ldots, n-1; \\ &\vdots \\ c_{n-1} + \frac{F^1(c)}{1} + \frac{c(N) - G^1(c)}{1} &= c_n. \end{aligned} \tag{3}$$

Then, the *CIC* rule, $f_{\mathcal{H}}^{CIC}$, assigns to any claims problem (E, c) a feasible allocation $x = f_{\mathcal{H}}^{CIC}(E, c) \in D(c) \cap H(E)$ defined as follows: take $F^0(c) := 0$ and $G^0(c) := c(N)$, then

- If $0 < E \leq F^1(c)$, then

$$x_n = E \text{ and } x_i = 0 \text{ for all } i < n. \tag{4}$$

- If $F^{r-1}(c) < E \leq F^r(c)$, for some $r = 2, \ldots, n-1$, we have[1]

$$\begin{aligned} x_i &= \left[\sum_{k=n-i+1}^{r-1} \frac{F^k(c) - F^{k-1}(c)}{k} \right] + \frac{E - F^{r-1}(c)}{r}, && \text{for } i \geq n+1-r \\ x_i &= 0, && \text{for } i < n+1-r. \end{aligned} \tag{5}$$

- If $F^{n-1}(c) < E \leq G^{n-1}(c)$, then

[1] A sumatory over an empty set of indices is equal to 0.

$$x_i = \left[\sum_{k=n-i+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k}\right] + \frac{E - F^{n-1}(c)}{n}, \text{ for } i = 1, \ldots, n. \tag{6}$$

- If $G^r(c) < E \leq G^{r-1}(c)$, for $r = 2, \ldots, n-1$, then

$$\begin{aligned} x_i &= \left[\sum_{k=n-i+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k}\right] + c_1 + \left[\sum_{k=r+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k}\right], && \text{for } i \geq n+1-r \\ &+ \frac{E - G^r(c)}{r} && \\ x_i &= c_i, && \text{for } i < n+1-r. \end{aligned} \tag{7}$$

- If $G^1(c) < E \leq c(N)$, then

$$x_i = c_i, \text{ for } i = 1, \ldots, n-1 \text{ and } x_n = E - \sum_{i=1}^{n-1} c_i. \tag{8}$$

Next theorem reinterprets CIC rules as order-preserving egalitarian rules.

Theorem 2. *Any CIC rule is an order-preserving egalitarian rule.*

Proof. Let $f_{\mathcal{H}}^{CIC}$ be the CIC rule corresponding to the set of functions $\mathcal{H} = (F^k, G^k)_{k=1}^{n-1}$ satisfying the CIC relations. We first show that $f_{\mathcal{H}}^{CIC}$ is an egalitarian rule. To this aim we shall prove that, for any claims problems (E, c),

$$f_{\mathcal{H}}^{CIC}(E, c) = f^{\alpha}(E, c)$$

for a properly defined reference system α. For shake of simplicity and without loss of generality, let us suppose that

$$c_1 \leq c_2 \leq \ldots \leq c_n.$$

Then, define the reference system α as follows:

$$\begin{aligned} \alpha_1(c) &= 0 \\ \alpha_i(c) &= \sum_{k=n-i+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k}, \quad \text{for } i = 2, \dots, n, \end{aligned} \tag{9}$$

with $F^0(c) = 0$. Notice that, by CIC relations, $\alpha(c) \in D(c)$. Next, we check that $x = f_{\mathcal{H}}^{CIC}(E, c)$ satisfies the egalitarian condition (1) relative to the reference point $a = \alpha(c)$ defined in (9).

If $n = 1$ the statement trivially holds. From now on, we assume $n \geq 2$. If $E = 0$ then $x = 0$ and (1) is trivially satisfied. If $0 < E \leq F^1(c)$, then, since $x_n = E$ and $x_i = 0$ for $i = 1, \dots, n-1$ (see(4)),

$$x_1 - a_1 = 0,\ x_i - a_i = -\sum_{k=n-i+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k}, \text{ for } i = 2, \dots, n-1$$

$$\text{and } x_n - a_n = E - \sum_{k=1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} = (E - F^1(c)) - \sum_{k=2}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k},$$

where $a_i = \alpha_i(c)$, for $i = 1, \dots, n$ (see (9)). Notice that

$$0 = x_1 - a_1 \geq x_2 - a_2 \geq \dots \geq x_n - a_n,$$

where the last inequality follows from $E \leq F^1(c)$. Since $x_i = 0$, for all $i \in N \setminus \{n\}$, condition (1) is satisfied.

If $F^{r-1}(c) < E \leq F^r(c)$, for some $r = 2, \dots, n-1$, we have, by (5),

$$\begin{aligned} x_i - a_i &= -\sum_{k=r}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} + \frac{E - F^{r-1}(c)}{r}, &&\text{for } i \geq n+1-r \\ x_i - a_i &= -\sum_{k=n-i+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k}, &&\text{for } i < n+1-r. \end{aligned}$$

We discuss three cases. If $i, j \geq n+1-r$, then $x_i - a_i = x_j - a_j$ and (1) holds. If $i, j < n+1-r$, then $x_i = x_j = 0$, and then condition (1) is trivially satisfied. If $j < n+1-r \leq i$, then we claim $x_i - a_i \leq x_j - a_j$. Indeed,

$$\begin{aligned}
x_i - a_i &= -\sum_{k=r}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} + \frac{E - F^{r-1}(c)}{r} \\
&= -\sum_{k=r}^{n-j} \frac{F^k(c) - F^{k-1}(c)}{k} - \sum_{k=n-j+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} + \frac{E - F^{r-1}(c)}{r} \\
&= -\sum_{k=r+1}^{n-j} \frac{F^k(c) - F^{k-1}(c)}{k} - \frac{F^r(c) - F^{r-1}(c)}{r} + (x_j - a_j) + \frac{E - F^{r-1}(c)}{r} \\
&= x_j - a_j - \sum_{k=r+1}^{n-j} \frac{F^k(c) - F^{k-1}(c)}{r} - \frac{F^r(c) - E}{r} \leq x_j - a_j.
\end{aligned}$$

If $x_i - a_i = x_j - a_j$ condition (1) holds. If $x_i - a_i < x_j - a_j$, then, since $x_j = 0$, condition (1) is also satisfied.

If $F^{n-1}(c) < E \leq G^{n-1}(c)$, then, by (6), we have

$$x_i - a_i = \frac{E - F^{n-1}(c)}{n}, \text{ for } i = 1, \ldots, n.$$

Thus, (1) holds.

If $G^r(c) < E \leq G^{r-1}(c)$, for $r = 2, \ldots, n-1$, then, by (7), we have

$$\begin{aligned}
x_i - a_i &= c_1 + \sum_{k=r+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k} + \frac{E - G^r(c)}{r}, \quad \text{for } i \geq n+1-r \\
x_i - a_i &= c_1 + \sum_{k=n-i+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k}, \qquad \text{for } i < n+1-r,
\end{aligned}$$

since, by CIC relations,

$$c_i = c_1 + \sum_{k=n-i+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} + \sum_{k=n-i+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k}, \text{ for all } i = 1, 2, \ldots n.$$

We discuss three cases. If $i, j < n+1-r$, then, by (7), $x_i = c_i$ and $x_j = c_j$ and thus (1) is satisfied. If $i, j \geq n+1-r$, then $x_i - a_i = x_j - a_j$ and (1) holds. If $j < n+1-r \leq i$, then $x_j - a_j \leq x_i - a_i$, since $j < n+1-r$ and thus $r+1 \leq n-j+1$. In this case, if $x_i - a_i = x_j - a_j$ then (1) holds; if not, $x_j - a_j < x_i - a_i$, but $x_j = c_j$ and thus condition (1) is also satisfied.

Finally, if $G^1(c) < E \leq c(N)$, then, by (8) and the CIC relations (3),

$$\begin{aligned} x_i - a_i &= c_1 + \sum_{k=n-i+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k}, && \text{for } i = 1, 2, \ldots, n-1 \\ x_n - a_n &= c_1 + \sum_{k=2}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k} + E - G^1(c). \end{aligned}$$

Clearly $x_n - a_n \geq x_{n-1} - a_{n-1} \geq \ldots \geq x_1 - a_1$. However (1) is satisfied since $x_i = c_i$, for all $i = 1, \ldots, n-1$. Thus, we conclude $f_{\mathcal{H}}^{CIC}$ is an egalitarian rule

To check that any CIC rule is order-preserving, and due to Proposition 2, it is enough to show that $a = \alpha(c) \in \mathcal{O}(c)$ (see (9)). Indeed, without loss of generality let us suppose $c_1 \leq c_2 \leq \ldots \leq c_n$. Then, since $\{F^k(c)\}_{k=1,\ldots,n-1}$ is a non-decreasing sequence of values, it easily follows from the CIC relations that if $c_i \leq c_j$, then $a_i \leq a_j$ and if $c_i = c_j$, then $a_i = a_j$. Moreover, take a pair of agents $i, j \in N$ such that $c_i \leq c_j$. In case $c_i = c_j$, we know that $a_i = a_j$, and thus $c_i - a_i = c_j - a_j$. Otherwise, $c_i < c_j$ and so, by the CIC relations, see (3), we have

$$\begin{aligned} c_i - a_i &= \sum_{k=n-i+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} + c_1 + \sum_{k=n-i+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k} \\ &\quad - \sum_{k=n-i+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} \\ &= c_1 + \sum_{k=n-i+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k} \leq c_1 + \sum_{k=n-j+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k} \\ &= \sum_{k=n-j+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} + c_1 + \sum_{k=n-j+1}^{n-1} \frac{G^{k-1}(c) - G^k(c)}{k} \\ &\quad - \sum_{k=n-j+1}^{n-1} \frac{F^k(c) - F^{k-1}(c)}{k} \\ &= c_j - a_j. \end{aligned}$$

Hence, we conclude $a \in \mathcal{O}(c)$.

$\square$

5 Weighted claims problems

In this section we extend the domain of claims problems by introducing weights that qualify the importance of agents. We represent a system of weights by any positive vector $w = (\omega_i)_{i \in N}$, $\omega_i > 0$, for all $i \in N$. A weighted claims problem will be then defined by the triplet (E, c, ω). On this extended domain of claims problems, an egalitarian rule with respect to a reference system α is defined as follows.

Definition 2. *On the domain of weighted claims problems (E, c, ω), the egalitarian rule relative to a reference system α, f^α, is defined as:*

$$f_i^\alpha(E, c, \omega) = \min\{(\lambda \omega_i + a_i)_+, c_i\}, \textit{ for all } i \in N,$$

where $a = \alpha(c)$, $\lambda \in \mathbb{R}$ and $\sum_{i=1}^n f_i^\alpha(E, c, \omega) = E$.

Notice that if $\omega_1 = \ldots = \omega_n$ then we recover the expression of an egalitarian rule expressed in Theorem 1. The rule defined above is well-defined and assigns a unique allocation[2] for any weighted claims problems. The proof is left to the reader and follows the arguments of the proof of Theorem 1 .

Now, we can reinterpret the classical proportional rule as an egalitarian allocation taking as a reference system $\alpha(c) = 0$ and weighting vector equal to the claims vector, $\omega = c$, i.e.

$$f^P(E, c) = f^{\alpha=0}(E, c, \omega = c).$$

Similarly, the weighted CEA rule, CEA^ω and the weighted CEL rule, CEL^ω,(see Thomson (2019), p. 84) fit in this model. That is,

$$CEA^\omega(E, c) = f^{\alpha=0}(E, c, \omega)$$

and

[2]The allocation is unique for any weighted claims problem since the expression $\min\{(\lambda\omega_i + a_i)_+, c_i\}$ is increasing with respect to λ.

$$CEL^{\omega}(E,c) = f^{\alpha=c}(E,c,(\frac{1}{\omega_1},\ldots,\frac{1}{\omega_n})).$$

Analogously to the non-weighted problem, the rule of Definition 2 is the unique that satisfies the following egalitarian condition.

Theorem 3. *Let f be an allocation rule defined on the domain of weighted claims problems and α be an arbitrary reference system. The following statements are equivalent:*

1. *The allocation rule f is egalitarian relative to α, $f = f^{\alpha}$.*
2. *For all weighted claims problem (E,c,ω) and for all i, $j \in N$,*

$$\textit{if } \frac{x_i - \alpha_i(c)}{\omega_i} < \frac{x_j - \alpha_j(c)}{\omega_j} \textit{ then either } x_i = c_i \textit{ or } x_j = 0, \tag{10}$$

   where $x = f(E,c,\omega)$.

The proof is analogous to the one of Theorem 1 and it is left to the reader.

Thomson (2019, Theorem 13.13) characterizes the constrained equal awards rule as the rule that minimizes the variance with respect to the pure egalitarian allocation. The variance can be reinterpreted as the squared of the distance to the equal division point. Focusing on this distance interpretation, next theorem characterizes egalitarian rules on the domain of weighted problems as the ones that minimize the weighted squared of the distance with respect to the reference point. If we apply this new theorem to the case of equal weights, we obtain new characterizations of the classical egalitarian rules as those that minimize the squared distance with respect to the corresponding reference systems.

Theorem 4. *Let (E,c,ω) be a weighted claims problem. Then, the egalitarian rule relative to a reference system α is characterized by*

$$f^{\alpha}(E,c,\omega) = \underset{x\in\mathcal{D}(c)\cap H(E)}{arg\min} \sum_{i\in N} \frac{1}{\omega_i}(x_i - \alpha_i(c))^2.$$

Proof. Let $a = \alpha(c)$ and x^* be the unique[3] solution of the minimization problem

$$\underset{x \in D(c) \cap H(E)}{\arg\min} \sum_{i \in N} \frac{1}{\omega_i}(x_i - a_i)^2.$$

Suppose on the contrary that $x^* \neq f^\alpha(E, c, \omega)$ and thus, by Theorem 3, x^* does not satisfy (10). Hence, there exist at least two agents $i, j \in N$, such that $\frac{1}{\omega_i}(x_i^* - a_i) < \frac{1}{\omega_j}(x_j^* - a_j)$, but $x_i^* < c_i$ and $x_j^* > 0$. Next, we define $x' \in \mathbb{R}_+^N$ as follows:

$$x_i' = x_i^* + \varepsilon,\ x_j' = x_j^* - \varepsilon,\ \text{and } x_k' = x_k^* \text{ else,}$$

where $0 < \varepsilon < \min \left\{ c_i - x_i^*,\ x_j^*,\ 2 \cdot \dfrac{\dfrac{x_j^* - a_j}{\omega_j} - \dfrac{x_i^* - a_i}{\omega_i}}{\dfrac{1}{\omega_j} + \dfrac{1}{\omega_i}} \right\}$.

Let us remark that, by definition of ε, $x' \in D(c) \cap H(E)$. However,

$$\begin{aligned}
\sum_{k \in N} \frac{(x_k' - a_k)^2}{\omega_k} &= \sum_{k \in N \setminus \{i,j\}} \frac{(x_k^* - a_k)^2}{\omega_k} + \frac{1}{\omega_i}(x_i^* + \varepsilon - a_i)^2 + \frac{1}{\omega_j}(x_j^* - \varepsilon - a_j)^2 \\
&= \sum_{k \in N} \frac{(x_k^* - a_k)^2}{\omega_k} + 2\varepsilon \left(\frac{x_i^* - a_i}{\omega_i} - \frac{x_j^* - a_j}{\omega_j} \right) + \frac{1}{\omega_i}\varepsilon^2 + \frac{1}{\omega_j}\varepsilon^2 \\
&= \sum_{k \in N} \frac{(x_k^* - a_k)^2}{\omega_k} + \varepsilon \left(2 \left(\frac{x_i^* - a_i}{\omega_i} - \frac{x_j^* - a_j}{\omega_j} \right) + \varepsilon \left(\frac{1}{\omega_i} + \frac{1}{\omega_j} \right) \right) \\
&< \sum_{k \in N} \frac{(x_k^* - a_k)^2}{\omega_k}
\end{aligned}$$

where the strict inequality follows from $\varepsilon < 2 \cdot \dfrac{\dfrac{x_j^* - a_j}{\omega_j} - \dfrac{x_i^* - a_i}{\omega_i}}{\dfrac{1}{\omega_j} + \dfrac{1}{\omega_i}}$. As a consequence, we have reached a contradiction with the fact that $x^* = \underset{x \in D(c) \cap H(E)}{\arg\min} \sum_{i \in N} \dfrac{1}{\omega_i}(x_i - a_i)^2$. Thus, we conclude that x^* satisfies (10) and $x^* = f^\alpha(E, c, \omega)$. □

[3]Notice that we minimize a strictly convex function over a non-empty compact and convex domain.

6 The concede and divide rule

In previous sections we introduced weighted claims problems and demonstrated that egalitarian rules select allocations that minimizes the weighted squared distance from the reference point. Among this family of rules, the *concede and divide* rule (see Thomson, 2019) cannot be interpreted this way.

Let us recall the formula of the *concede and divide* rule (CD). Let $N = \{1, 2\}$, and define

$$m_1 = (E - c_2)_+ \text{ and } m_2 = (E - c_1)_+$$

then

$$CD_1(E, c) = m_1 + \frac{1}{2}(E - m_1 - m_2) \quad \text{and} \quad CD_2(E, c) = m_2 + \frac{1}{2}(E - m_1 - m_2)$$

If $c_1 \leq c_2$, the formula is

$$CD(E,(c_1,c_2)) = \begin{cases} (\frac{E}{2}, \frac{E}{2}) & \text{if } 0 \leq E \leq c_1 \\ (\frac{c_1}{2}, E - \frac{1}{2}c_1) & \text{if } c_1 \leq E \leq c_2 \\ (\frac{E}{2} + \frac{c_1 - c_2}{2}, \frac{E}{2} + \frac{c_2 - c_1}{2}) & \text{if } c_2 \leq E \leq c_1 + c_2. \end{cases}$$

Proposition 3. *The concede and divide rule is not an egalitarian rule.*

Proof. Suppose on the contrary there would exist a reference system $\alpha(c)$ such that for any claims problem $(E, c = (c_1, c_2))$

$$CD(E, c) = f^{\alpha}(E, c), \tag{11}$$

for all $0 \leq E \leq c_1 + c_2$. Next, we will show why this is not possible.

Indeed, taking $0 < E < c_1 < c_2$ we know, by the definition of the CD rule, that $CD_1(E, c) = CD_2(E, c) = \frac{E}{2}$. Hence, by (11), we have

$$\min\{(\lambda + \alpha_1(c))_+, c_1\} = \min\{(\lambda + \alpha_2(c))_+, c_2\} = \frac{E}{2}.$$

Since $\frac{E}{2} > 0$, we can deduce that $\lambda + \alpha_1(c) > 0$ and $\lambda + \alpha_2(c) > 0$, and thus

$$\min\{\lambda + \alpha_1(c),\, c_1\} = \min\{\lambda + \alpha_2(c),\, c_2\} = \frac{E}{2}. \tag{12}$$

We claim that $\alpha_1(c) = \alpha_2(c)$. Otherwise, suppose $\alpha_1(c) \neq \alpha_2(c)$. Taking (12) into account, and since $E < c_1 < c_1 + c_2$, we can analyze two cases:

$$\text{case 1)}\ \min\{\lambda + \alpha_1(c),\, c_1\} = \lambda + \alpha_1(c) \text{ and } \min\{\lambda + \alpha_2(c),\, c_2\} = c_2$$

or

$$\text{case 2)}\ \min\{\lambda + \alpha_1(c),\, c_1\} = c_1 \text{ and } \min\{\lambda + \alpha_2(c),\, c_2\} = \lambda + \alpha_2(c)$$

In the first case, since $\lambda + \alpha_1(c) > 0$, we have

$$E = \min\{\lambda + \alpha_1(c),\, c_1\} + \min\{\lambda + \alpha_2(c),\, c_2\} = \lambda + \alpha_1(c) + c_2 > c_2 \geq c_1,$$

in contradiction with $E < c_1$.

Similarly, the same contradiction is reached for the second case

$$E = \min\{\lambda + \alpha_1(c),\, c_1\} + \min\{\lambda + \alpha_2(c),\, c_2\} = c_1 + \lambda + \alpha_2(c) > c_1.$$

Thus we conclude $\alpha_1(c) = \alpha_2(c)$.

Now take $E = \frac{c_1 + c_2}{2}$. Then, by definition of the CD rule

$$CD_1(E, c) = \frac{1}{2}c_1 \text{ and } CD_2(E, c) = \frac{1}{2}c_2.$$

By (11)

$$\min\{(\lambda + \alpha_1(c))_+,\, c_1\} = \frac{1}{2}c_1 \text{ and } \min\{(\lambda + \alpha_2(c))_+,\, c_2\} = \frac{1}{2}c_2.$$

and thus, since $c_1 > 0$ and $c_2 > 0$

$$\lambda + \alpha_1(c) = \frac{1}{2}c_1 \text{ and } \lambda + \alpha_2(c) = \frac{1}{2}c_2,$$

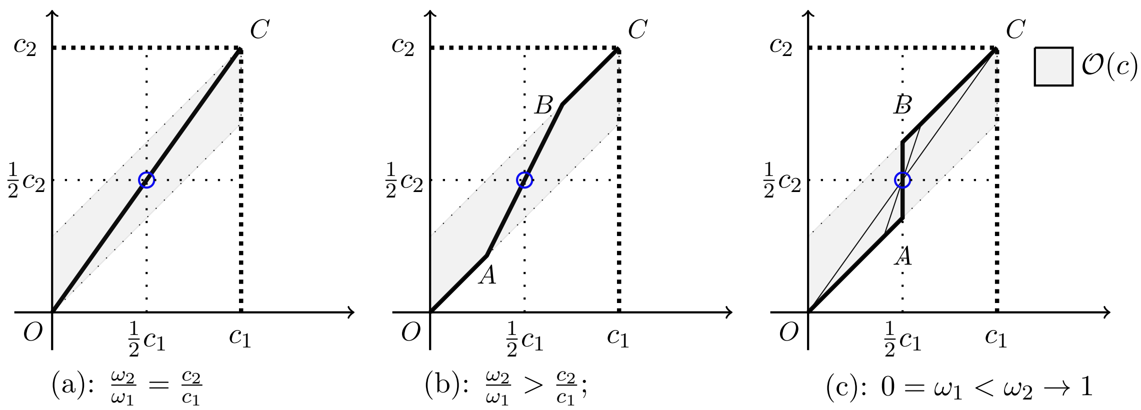

Figure 2: Two-person problem $(E, (c_1, c_2))$, $c_1 < c_2$. Minimizing the weighted squared distance in the order preserving set to the reference point $(\frac{1}{2}c_1, \frac{1}{2}c_2)$. The path of awards (OABC) for different weighting vectors. In (a), the relative weights of agents coincides the relative claims. In (b) the weight of agent 2 is strictly larger than the one of agent 1. In (c) the difference between the weights is pushed to the limit $w_1 \to 0$ and $w_2 \to 1$.

Since $c_1 > 0$, $c_2 > 0$ and $\alpha_1(c) = \alpha_2(c)$ (see the claim proved above), we get

$$0 = \lambda + \alpha_1(c) - \lambda - \alpha_2(c) = \frac{1}{2}(c_1 - c_2),$$

which implies that $c_1 = c_2$, getting a contradiction. Thus, we conclude that the *coincide and divide* rule is not an egalitarian rule. □

As a direct consequence of Proposition 3 the Talmudic rule and all ICI rules (Thomsom, 2008) are not egalitarian rules. This is because the *concede and divide* rule is a specific case of them. However, the *concede and divide* rule can still be reinterpreted as the one that minimizes the weighted squared distance with respect to the half-claims vector among allocations that are in the order-preserving set and where weights reflect some priority between agents. In Figure 2 we illustrate this with several examples.

Proposition 4. *Let $N = \{1, 2\}$ and (E, c) be a claims problem with $c_1 < c_2$. Then,*

$$CD(E,c) = \lim_{\substack{\omega = (\omega_1,\omega_2) \to (0,1) \\ \omega \in S}} \ \underset{x \in \mathcal{O}(c) \cap H(E)}{arg\min} \ \frac{1}{w_1}(x_1 - \frac{1}{2}c_1)^2 + \frac{1}{\omega_2}(x_2 - \frac{1}{2}c_2)^2,$$

where $S = \{\omega = (\omega_1, \omega_2) \in \mathbb{R}^2_{++} \mid \omega_1 + \omega_2 = 1$ and $\frac{\omega_2}{c_2} > \frac{\omega_1}{c_1}\}$.

Proof. Since by efficiency, $x_1 = E - x_2$, the function to be minimized is

$$\varphi(x_2) = \frac{1}{w_1}(E - x_2 - \frac{1}{2}c_1)^2 + \frac{1}{\omega_2}(x_2 - \frac{1}{2}c_2)^2, \tag{13}$$

subject to

$$\max\{E - c_1, \frac{E}{2}\} \leq x_2 \leq \min\{\frac{E}{2} + \frac{1}{2}(c_2 - c_1), E\}. \tag{14}$$

The unconstrained optimum ($\varphi'(x_2) = 0$, where $\varphi''(x_2) = \frac{2}{\omega_1} + \frac{2}{\omega_2} > 0$) is attained at

$$x_2^* = E\,\omega_2 - \frac{1}{2}[\omega_2\, c_1 - \omega_1\, c_2]. \tag{15}$$

For studying the constrained optimum, we focus on the case $\dfrac{\omega_2}{c_2} > \dfrac{\omega_1}{c_1}$ and thus [4]

$$\Delta = c_1\,\omega_2 - c_2\,\omega_1 > 0 \text{ and } \omega_2 > \frac{1}{2}. \tag{16}$$

Given (c_1, c_2) and (ω_1, ω_2) satisfying (16), we claim that the optimal value of x_2 depending on the value of the estate E, $x_2(E)$, is

$$x_2^*(E, c) = \begin{cases} \dfrac{E}{2} & \text{if} \quad 0 \leq E < \dfrac{\Delta}{2\omega_2 - 1} \\ E\,\omega_2 - \frac{1}{2}\Delta & \text{if } \dfrac{\Delta}{2\omega_2 - 1} \leq E \leq \dfrac{c_2 - c_1}{2\omega_2 - 1} + \dfrac{\Delta}{2\omega_2 - 1} \\ \dfrac{E}{2} + \dfrac{1}{2}(c_2 - c_1) & \text{if } \dfrac{c_2 - c_1}{2\omega_2 - 1} + \dfrac{\Delta}{2\omega_2 - 1} < E \leq c_1 + c_2, \end{cases} \tag{17}$$

where $\Delta = \omega_2\, c_1 - \omega_1\, c_2$. Indeed, first notice that

$$0 < \frac{\Delta}{2\omega_2 - 1} = \frac{\omega_2\, c_1 - \omega_1\, c_2}{2\omega_2 - 1} \leq \frac{\omega_2\, c_1 - \omega_1\, c_1}{2\omega_2 - 1} = c_1, \tag{18}$$

where the first strict inequality follows from (16), the second one from $c_1 < c_2$ and the last equality since $\omega_1 = 1 - \omega_2$.

Now we discuss three cases:

[4]Notice that condition $\dfrac{\omega_2}{c_2} > \dfrac{\omega_1}{c_1}$ implies $\omega_2 > \dfrac{c_2}{c_1}\omega_1 > \omega_1$, which encompasses the particular case $\omega_2 \to 1$ and $\omega_1 \to 0$. Moreover $\dfrac{\omega_2}{c_2} > \dfrac{\omega_1}{c_1}$ implies $\frac{\omega_2}{\omega_1} > \frac{c_2}{c_1}$ or $\frac{\omega_2}{1-\omega_2} > \frac{c_2}{c_1} > 1$ and then $\omega_2 > \frac{1}{2}$.

Case 1: $0 \leq E < \dfrac{\Delta}{2\omega_2 - 1}$.

We have $E < \dfrac{\omega_2\, c_1 - \omega_1\, c_2}{2\omega_2 - 1} = \dfrac{1}{2}\dfrac{\omega_2\, c_1 - \omega_1\, c_2}{\omega_2 - \frac{1}{2}}$, and thus $E\,\omega_2 - \frac{1}{2}E < \frac{1}{2}(\omega_2\, c_1 - \omega_1\, c_2)$.

We conclude, by (15),

$$x_2^* = E\,\omega_2 - \frac{1}{2}(\omega_2\, c_1 - \omega_1\, c_2) < \frac{1}{2}E = \max\{E - c_1,\, \frac{1}{2}E\},$$

where the last equality follows from the hypothesis of the case and (18). Since $x_2^* < \frac{1}{2}\,E$, x_2^* is not in the domain of the constrained optimization problem, see (14) and the objective function (13) is strictly increasing to the right of x_2^*. Hence, the constrained optimum is attained at $\frac{E}{2}$, as indicated in (17) .

Case 2: $\dfrac{\Delta}{2\omega_2 - 1} \leq E \leq \dfrac{(c_2 - c_1)}{2\omega_2 - 1} + \dfrac{\Delta}{2\omega_2 - 1}$.

Similarly to the previous case we can prove that $x_2^* \geq \frac{1}{2}\,E$. Indeed, as

$$\frac{\Delta}{2\omega_2 - 1} = \frac{1}{2}\frac{\omega_2 c_1 - \omega_1 c_2}{\omega_2 - \frac{1}{2}} \leq E,$$

we obtain

$$\frac{1}{2}(\omega_2 c_1 - \omega_1 c_2) \leq E\omega_2 - \frac{E}{2},$$

or equivalently

$$\frac{E}{2} \leq x_2^* = E\omega_2 - \frac{1}{2}\Delta.$$

Furthermore,

$$\begin{aligned}
x_2^* &= E\omega_2 - \tfrac{1}{2}(\omega_2 c_1 - \omega_1 c_2) = \omega_2\left[E - \tfrac{1}{2}c_1\right] + \tfrac{1}{2}\omega_1 c_2 \\
&= E - c_1 - (1-\omega_2)\,(E - c_1) + \tfrac{1}{2}(\omega_2 c_1 + \omega_1 c_2) \\
&= E - c_1 - (\omega_1)\,(E - c_1) + \tfrac{1}{2}(\omega_2 c_1 + \omega_1 c_2) \\
&\geq E - c_1 - (\omega_1)\,c_2 + \tfrac{1}{2}(\omega_2 c_1 + \omega_1 c_2) = E - c_1 + \tfrac{1}{2}[\omega_2 c_1 - \omega_1 c_2] \\
&> E - c_1.
\end{aligned}$$

where the first inequality follows from $c_2 \geq E$, and thus $c_1 + c_2 \geq E$, and the strict inequality from $\frac{\omega_2}{c_2} > \frac{\omega_1}{c_1}$. Thus, we have proved that $\max\{E - c_1,\, \frac{E}{2}\} \leq x_2^*$.

On the other hand,

$$x_2^* = E\omega_2 - \frac{1}{2}[\omega_2 c_1 - \omega_1 c_2] < E\,\omega_2 < E,$$

where the first inequality follows for $\frac{\omega_2}{c_2} \geq \frac{\omega_1}{c_1}$.

Furthermore, since $E \leq \dfrac{(c_2 - c_1)}{2\omega_2 - 1} + \dfrac{\Delta}{2\omega_2 - 1}$,

$$E \leq \frac{1}{2}\frac{c_2 - c_1}{(\omega_2 - \frac{1}{2})} + \frac{1}{2}\frac{\Delta}{(\omega_2 - \frac{1}{2})} \text{ and thus}$$

$$E\omega_2 - \frac{1}{2}E \leq \frac{1}{2}(c_2 - c_1) + \frac{1}{2}\Delta$$

Hence,

$$x_2^* = E\omega_2 - \frac{1}{2}(\omega_2 c_1 - \omega_1 c_2) \leq \frac{1}{2}\,E + \frac{1}{2}(c_2 - c_1).$$

We conclude that for this range of E

$$\max\{E - c_1,\, \frac{E}{2}\} \leq x_2^* \leq \min\{\frac{1}{2}(c_2 - c_1) + \frac{1}{2}\,E,\, E\},$$

and $x_2^* = E\omega_2 - \frac{\Delta}{2}$ is the solution for the constrained optimization problem, as indicated in (17).

Case 3: $\dfrac{c_2-c_1}{2\omega_2-1}+\dfrac{\Delta}{2\omega_2-1}<E\leq c_1+c_2$.

Since $\dfrac{c_2-c_1}{2\omega_2-1}+\dfrac{\Delta}{2\omega_2-1}<E$, then $\dfrac{1}{2}\dfrac{c_2-c_1}{(\omega_2-\frac{1}{2})}+\dfrac{1}{2}\dfrac{\Delta}{(\omega_2-\frac{1}{2})}<E$, or equivalently

$$\frac{1}{2}(c_2-c_1)+\frac{1}{2}\Delta<E\omega_2-\frac{E}{2}<E-\frac{E}{2}=\frac{E}{2}.$$

Hence,

$$x_2^*=E\omega_2-\frac{1}{2}\Delta>\frac{1}{2}E+\frac{1}{2}(c_2-c_1)=\min\{\frac{1}{2}E+\frac{1}{2}(c_2-c_1),\,E\},$$

where the last equality come from the fact that $c_2-c_1<E$ since

$$\frac{c_2-c_1}{2}<\frac{1}{2}(c_2-c_1)+\frac{\Delta}{2}<E\omega_2-\frac{E}{2}<E-\frac{E}{2}=\frac{E}{2}.$$

Therefore $x_2^*=E\omega_2-\frac{1}{2}\Delta$ is not in the domain of the constrained optimization problem. As the objective function is strictly convex, and thus, is strictly decreasing to the left of x_2^*, the constrained optimum is attained at $\frac{E}{2}+\frac{1}{2}(c_2-c_1)$, as indicated in (17) .

Finally, taking limits in (17) we obtain

$$\lim_{\substack{(\omega_1,\omega_2)\to(0,1)\\ \omega\in S}} x_2^*(E,c)=\begin{cases}\dfrac{E}{2} & \text{if}\quad 0\leq E<c_1\\[2ex] E-\frac{1}{2}c_1 & \text{if}\quad c_1\leq E\leq c_2\\[2ex] \dfrac{E}{2}+\dfrac{1}{2}(c_2-c_1) & \text{if}\quad c_2<E\leq c_1+c_2,\end{cases}\tag{19}$$

that coincides with the payoff to agent 2 according to the concede and divide rule, finishing the proof.

□

Aumann and Maschler (1985) already stated that the extension of the *concede and divide* rule to the n-person problem is the Talmudic rule. We conjecture that the result of Proposition 4 can be extended to the n-person claims problem, and thereby proving that the Talmudic rule can be viewed as the output of a minimization problem. To this aim, the idea is to take weights that pairwise replicate the relative extreme weights used in Proposition 4. That is, suppose $c_1 \leq c_2 \leq \ldots c_n$; then, we can assign the following weights to agents, depending on some fixed value $M \geq 1$

$$\omega_1(M) = \frac{1}{\mathcal{M}}, \omega_2(M) = \frac{M}{\mathcal{M}}, \omega_3(M) = \frac{M^2}{\mathcal{M}}, \ldots, \omega_n(M) = \frac{M^{n-1}}{\mathcal{M}},$$

where $\mathcal{M} = 1 + M + M^2 + \ldots + M^{n-1}$.

We conjecture that for any problem (E, c) the Talmudic rule T assigns the allocation

$$T(E, c) = \lim_{M \to \infty} \underset{x \in \mathcal{O}(c) \cap H(E)}{arg \min} \sum_{i=1}^{n} \frac{1}{w_i(M)} (x_i - \frac{1}{2} c_i)^2.$$

7 Conclusions

In this paper, we have revealed that basic allocation rules merely select the nearest efficient point to a given reference vector. This opens the door to introduce new rules with a properly defined and sensible reference vector, as illustrated in Table 1.

The approach used in the paper allows to define new rules by selecting the closest point to a reference point, while also restricting the domain of allocations. For instance, the constrained egalitarian rule (see Thomson, 2019, p. 34) can be interpreted in this manner. This rule can be seen as one that minimizes the variance, as stated in Theorem 4, while restricting the domain to those payoff vectors below half of the claims vector and above of half of the claims vector.

However, there are other well-accepted solution concepts such as the family of Talmudic rules, the minimal overlap rule, or the largest family of ICI rules that are not covered by this approach. In Section 6, we discuss how to reinterpret one of these rules as a rule that select the closest point to a reference by employing extreme weights. This reinterpretation shows the versatility and potential of our approach in extending the applicability of traditional rules. A promising area for future research could be to introduce two reference points instead of one, as proposed by Gimenez-Gómez et al. (2016), to characterize this uncovered family of rules.

To conclude this section, it is worth noting that interesting properties of solutions can be analyzed and discovered from this perspective. For example, it would be intriguing to explore the subset of egalitarian rules that exhibit consistency. Specifically, we are interested in understanding the relationships between the reference point and the consistency property of the corresponding egalitarian solution. This exploration could yield deeper insights underlying fair allocation and contribute to the development of more robust and equitable mechanisms.

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