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Higher Structural Reflection and Very Large Cardinals

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Contents

Abstract

One line of research in set theory aims at deriving large cardinal axioms from strengthened forms of reflection principles. This research is often motivated by the foundational goal of *justifying* the large cardinal axioms. The most comprehensive attempt in this direction is the program of *structural reflection* (SR), initiated by Joan Bagaria, whose ultimate goal is to formulate all large cardinal axioms as instances of a single, general structural reflection principle that is conceptually compelling.

The basic version of SR already gives the hierarchy of large cardinals from supercompact cardinals, through $C^{(n)}$-extendible cardinals, up to Vopěnka's Principle. A stronger version of SR, the *exact structural reflection principle* (ESR), is studied by Bagaria and Philipp Lücke, which gives almost huge cardinals, and beyond. However, ESR differs in form from the basic version of SR, rather than being direct generalization of the same principle.

In this thesis we formulate the *level by level* version and the *capturing* version of SR (CSR). CSR is a direct generalization of the basic version of SR. We introduce and study the *m-supercompact cardinals*, the $C^{(n)}$*-m-fold extendible cardinals*, and the capturing version of VP, and show that the pattern of correspondence between large cardinals and the basic version of SR also extends to the higher realm. We also apply our results to answer several open questions concerning ESR. Finally, we note that CSR, when generalized to its ω-version, leads to inconsistency.

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Chapter 0

Foundational Background

In this chapter, we discuss issues in the foundations of mathematics that motivate the work on the relation between large cardinals and reflection principles, and we place the thesis work in context.

0.1 Gödel's program and large cardinals

The Zermelo-Frankel set theory with the axiom of choice (ZFC) is the standard foundation of all of mathematics. As far as we know, all mathematical statements can be suitably translated to the language of ZFC, and all mathematical theorems are provable from the axioms of ZFC and its extensions.

Due to Gödel's incompleteness theorem, however, we have known that ZFC is incomplete. Indeed, numerous fundamental questions from a wide range of mathematical fields were shown, typically by using the powerful techniques of *inner models* and *forcing*, to be unsettled by the axioms of ZFC. These include famous examples from cardinal arithmetic, such as the *continuum hypothesis*, from combinatorics and order theory, such as *Suslin's hypothesis*, from group theory, such as *Whitehead's problem*, from topology, such as the *normal Moore space conjecture*, from measure theory, such as the *Borel conjecture*, and from descriptive set theory, such as *projective determinacy.*

A research program central to modern set theory known as *Gödel's Program* (see [35] for a discussion), initially suggested by Gödel in [19], seeks to find well-justified axioms in addition to ZFC to settle the important questions left undecided by ZFC.

Among the proposed candidates, the most important ones are the *strong axioms of infinity*, also known as the *large cardinal axioms.* These are the assertions that declare the existence of infinite cardinals with rich and complicated properties that make them extraordinarily large. These large cardinals are the *higher infinite*, as referred to in the title of [21], which is a classic text on large cardinals.

The large cardinal axioms have remarkable consequences, settling many independent questions, including the Marin-Steel result of projective determinacy (see [26]), Woodin's proof of $AD^{L(\mathbb{R})}$ (see [44]), and Solovay's result on singular cardinal arithmetic (see [33]). Moreover, numerous applications to various areas of mathematics were found, such as applications to topology (see [9] and [16]), to category theory (see [3] and [43]), and to group theory (see [8] and [14]).

The enormous success suggests that the large cardinal axioms definitely satisfy the criterion of settling many independent problems. The question, then, is whether they are well-justified principles of set theory. Indeed, the power and success may themselves be taken to provide such a justification, as suggested in the often-quoted passage by Gödel in [19, p.261]:

> There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory.

This kind of justification is referred to by Gödel as *extrinsic justification*. On the other hand, it is arguably more desirable, and at least more in line with the traditional notion of mathematical axioms, if large cardinals axioms enjoy some form of *intrinsic justification*, namely as principles that are derivable directly from the conception of the universe of sets, so that they can be seen as merely "unfold the content of the concept of set" ([19]). Indeed, some authors may go so far as to claim that intrinsic justification is the only viable form of justification (see for example [37]). Gödel himself also appears to take intrinsic justification as the more fundamental form of justification.

0.2 Reflection principles

It is in the context described above that *reflection principles* enter the picture. The simplest form of reflection principle, known as the *Levy-Montague* reflection principle, asserts that for any formula $\phi(x_0, \ldots, x_{k-1})$ in the language of set theory, if it is true in the set-theoretic universe V, then it is already true in some initial segment V_α of the universe V. Namely, for any sets $a_0, \ldots, a_{k-1}$, we have

$$V \models \text{“}\phi(a_0, \ldots, a_{k-1})\text{”} \to \exists\alpha\ V_\alpha \models \text{“}\phi(a_0, \ldots, a_{k-1})\text{”}.$$

This form of reflection is already provable in ZF, and in fact is claimed to capture the essence of ZF, as a variant of this form of reflection, known as the principle of *complete reflection*, was shown by Levy in [23] to be, modulo the axioms of extensionality, comprehension, and regularity, *equivalent to* the rest of ZF.

More importantly, the reflection principle is seen to explicate the idea that the universe V is extremely rich and complicated, such that no single formula is able to define it, so that any formula that holds in V already holds at some initial segment V_α. Indeed, this idea of the *undefinability of the universe*, which can already be traced back to Cantor, is seen by many set theorists as being part of the conception of V as being as rich as possible, such that any axiom that is derivable from such an idea can be seen as having intrinsic justification. Gödel even explicitly claimed (see [41]) that reflection principle is the "central principle" in formulating axioms of set theory, while other principles are "only heuristic principles". In particular, reflection principle should be the only source of axioms of infinity, as quoted in [41, p.285]:

> Generally I believe that, in the last analysis, every axiom of infinity should be derivable from the (extremely plausible) principle that V is undefinable, where definability is to be taken in [a] more and more generalized and idealized sense.

Thus a natural idea is to strengthen the reflection principle by taking (un)definability to *more and more generalized and idealized sense*, and attempt to derive large cardinal axioms from the resulting strengthened reflection principles.

An intuitive approach is to formulate reflection principles in higher-order logic. Thus informally, we may assert that for any higher-order formula ϕ, possibly with higher-order parameters, that if V satisfies ϕ, so does some initial segment V_α. Now by claiming that this higher-order reflective property of V *itself* holds at some initial segment V_α, we are in fact claiming that V_α is an *indescribable cardinal*, which is precisely defined as having these higher-order reflective properties with respect to lower initial segments V_β. This higher-order approach has been taken by a number of authors, including for example William Tait (see for example [36]), and was studied, with a largely negative appraisal, in an influential discussion by Peter Koellner in [22]. Without going into the precise details, we can already summarise the serious difficulties faced by this approach (for details and more discussions see [22] and section 1 of [2]).

The first difficulty is conceptually straightforward. On the standard iterative conception of the universe of sets as consisting of sets that are iteratively generated in stages, *and nothing else*, it simply does not make sense to ask whether V satisfies some higher-order formulas, as this requires the resources of the *power class*, or the power class of power class, of V, which on this conception simply do not exist. The second difficulty is that, as shown by Koellner, even we allow talks of higher-order reflection principles, these principles, when precisely formulated, fall into the *dichotomy* of being either too weak (weaker than the least *Erdös cardinal,* $\kappa(\omega)$), or outright inconsistent. Since $\kappa(\omega)$ is not strong enough to deliver the results mentioned in section 1, including the projective determinacy, and is in fact compatible with the axiom $V = L$, it is not sufficient, from a modern point of view, for a strong foundation of mathematics. Koellner thus concluded with the challenge to "formulate a strong reflection principle which is intrinsically justified on the iterative conception of set and which breaks the $\kappa(\omega)$ barrier".

0.3 The structural reflection program

If the only way to interpret the undefinability of V in a "more and more generalized sense" is to go higher-order in the style of Tait, then perhaps we are led to a skeptical attitude to the prospect of intrinsic justification for large cardinals. However, various other formulations of reflection principles have been proposed by many authors, for examples Reinhardt (see [30]), Marshall (see [29]), Welch (see [42]), Roberts (see [31]), and Mccallum (see [27]).

Arguably the most systematic proposal so far is that of Joan Bagaria's (see [2]). Bagaria proposed another interpretation of the underfinability of V, partly inspired by the following quote of Gödel ([41]):

> The universe of sets cannot be uniquely characterized (i.e., distinguished from all its initial segments) by any internal structural property of the membership relation in it which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number.

Bagaria's idea is that, instead of reflecting the *theory* of V, we may take what is reflected to be the *structural content* of V, namely the class of structures that satisfy, as the above quote suggests, some *structural property.* More precisely, we may assert that, for any "structural property" ϕ, there must be some initial segment V_α, such that V cannot be distinguished from V_α in terms of its storage of structures that satisfy ϕ, in the sense that for any structure B with $\phi(B)$ holds, there is already some structure A in V_α, with $\phi(A)$ holds, that is *very similar* to B. The notion of structural similarity here is explicated using the idea that A is isomorphic to some elementary substructure of B, i.e., there is an elementary embedding j from A to B. Thus we are led to the following formulation of reflection principle (first formulated in [1]):

SR: (*Structural Reflection*) For every definable, in the first-order language of set theory, possibly with parameters, class $\mathcal{C}$ of relational structures of the same type there exists an ordinal α that *reflects* C, i.e., for every B in $\mathcal{C}$ there exist A in $\mathcal{C} \cap V_\alpha$ and an elementary embedding from A into B.

It turns out that the above principle, taken as a schema, is *equivalent* to the very strong large cardinal axiom of *Vopěnka's Principle* (VP). Moreover, Π_1-SR, which is the above SR restricted to Π_1-definable classes in the Levy Hierarchy, is equivalent to the existence of a proper class of *supercompact cardinals*, while Π_{n+1}-SR, for $n \geq 1$, is equivalent to the existence of a proper class of $C^{(n)}$*-extendible cardinals*.

Thus if we agree that SR follows from the conception of the undefinability of V, then arguably we can conclude that VP is intrinsically justified. In particular, since VP is very strong, it is much more than sufficient to secure the examples of desirable consequences we mentioned in section 1, such as projective determinacy.

In fact Bagaria has much greater ambition: he initiated the research program of finding different forms of structural reflection principles that characterize *all* large cardinals ([2]):

> Each of these results should be regarded as a small step towards the ultimate objective of showing that all large cardinals are in fact different manifestations of a single general reflection principle.

Moreover, this program comes with the goal of even providing a *definition* of a large cardinal ([11]):

> As a consequence, they may, also, fill up an outstanding, and somewhat embarrassing, definitional void in the theory of large cardinals, i.e., the definition of 'large cardinal' itself.

Indeed, Bagaria and many collaborators have contributed to this research program, and successfully obtained many results. These include, but are not limited to, the work with Väänänen (see [12]), Gitman-Schindler (see [4]), Wilson (see [13]), and Lücke (see [6], [7], see also [24]).

In these results (surveyed in [2]), various formulations of structural reflection were proposed, and were shown to correspond in some way or another to large cardinals in different regions of the large cardinal hierarchy. For instance, the principle SR^- corresponds to small large cardinals (inaccessible, Mahlo, weakly compact); PSR (product SR) corresponds to large cardinals from strong to Woodin cardinals; GSR (Generic SR) and Strong GSR correspond to large cardinals from almost remarkable to virtually extendible cardinals; WSR (Weak SR) corresponds to large cardinals between strongly unfoldable and subtle; while ESR (Exact SR) corresponds to large cardinals beyond Vopěnka's Principle, up to the level of I1 embeddings.

Note that whether Bagaria's ambition is completely fulfilled depends on detailed evaluation of at least three questions. The question of *Universality* is whether it is true that all large cardinal axioms can indeed be formulated in terms of structural reflection. The question of *Unification* is whether these various forms of structural reflection can really be claimed to be "different manifestations of a single general reflection principle". The question of *Intrinsicality* is whether these principles can really be claimed to be justified based on the conception of the universe of sets. For a discussion of some of these issues see [11].

0.4 Higher reaches of structural reflection

I tend to agree that SR is, or at least *a priori* appears to be, a reasonable and very elegant formulation of the idea that V is undefinable. But regardless of whether one agrees with the justifiability of SR or not, still one of the most interesting questions about the whole program of SR is how high can one go in climbing the large cardinal hierarchy through SR, namely: what are the strongest structural reflection principles that are intrinsically justified? This is so because, from the point of view of justification, if a stronger principle is justified, then the weaker principles derivable from it are, *a fortiori*, also justified.

As hinted in the previous section, the strongest principles on offer so far are the principles of *exact structural reflection* (ESR), formulated in [6]. The exact structural reflection principle for some class of structures $\mathcal{C}$ asserts that there are some cardinals $\kappa < \lambda$ such that, for any structure $B \in \mathcal{C}$ of rank λ, there is some structure $A \in \mathcal{C}$ of rank κ with an elementary embedding $j : A \to B$. It is shown in [6] that ESR restricted to Π_1-definable classes implies many almost huge cardinals, making it stronger than Vopěnka's Principle. In [6] the *sequential* forms of ESR were also considered.

Definition 0.4.1. Let $0 < \eta \leq \omega$ and let L be a first-order language containing unary predicate symbols $\vec{P} = \langle \dot{P}_i : i < \eta \rangle$.

1. Given a sequence $\vec{\mu} = \langle \mu_i : i < \eta \rangle$ of cardinals with supremum μ, an L-structure A has *type* $\vec{\mu}$ *(with respect to* $\vec{P}$*)* if the universe of A has rank μ and $rk(\dot{P}_i^A) = \mu_i$ for all $i < \eta$.
2. Given a class $\mathcal{C}$ of L-structures, the *sequential ESR for* $\mathcal{C}$ *of length* $(\eta + 1)$ asserts that there is a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i : i < 1 + \eta \rangle$ of cardinals, such that for every structure B in $\mathcal{C}$ of type $\langle \lambda_{i+1} : i < \eta \rangle$, there exists an elementary embedding of a structure $A \in \mathcal{C}$ of type $\langle \lambda_i : i < \eta \rangle$ into B.

It was shown that the sequential ESR imply even stronger large cardinals, with sequential ESR of length ω implying rank-into-rank embeddings.

It is certainly true that ESR and its sequential forms are natural and interesting reflection principles in their own right. However, it seems that it is less clear, compared to the case of SR, if it is reasonable to consider ESR to be intrinsically justified. The rationale behind SR is that V is complicated enough so that the structural content of V for any definable class $\mathcal{C}$ must reflects to some V_α. But it is less clear why there should be two cardinals $\kappa < \lambda$ such that the class $\mathcal{C}$ at precisely the level λ should reflects to $\mathcal{C}$ at precisely the level κ. Much less clear still is the justifiability for sequential forms of ESR, where it is asserted that there are many cardinals such that the class $\mathcal{C}$ is required to reflect in precisely the way as prescribed in the above definition. Indeed, this was suggested by Bagaria himself and Claudio Ternullo ([11, section 5]):

> The only SRPs which would, thus, be prone to the objection are those yielding large cardinals stronger than VP; hence, by this argument's lights, it would only be ESR (again, see section 4), and the corresponding large-cardinal notions, that would be lacking intrinsic evidence.

Furthermore, in other forms of SR, the *least* relevant large cardinals can be characterized as the least cardinals that witness the relevant forms of SR. For example, in the basic form of SR, the least supercompact cardinal is the least κ that reflects all Π_1-definable, without parameters, class $\mathcal{C}$ of structures, while the least $C^{(n)}$-extendible cardinal is the least κ that reflects all Π_{n+1}-definable, without parameters, class $\mathcal{C}$ of structures, for $n \geq 1$. This phenomenon continues to hold in other cases, and it is conjectured that this is a universal phenomenon ([11, section 4]):

> **Conjecture 2** A cardinal κ is the least cardinal satisfying some large-cardinal notion iff κ is the least cardinal satisfying some Structural Reflection Principle that implies (in some inner model) κ is (weakly) inaccessible.

In the case of ESR, the relevant large cardinals proposed are the *weakly parametrically n-exact* cardinals and the *parametrically n-exact cardinals* (for precise definitions see [6] or see Chapter 6 of the thesis). Indeed it was shown that, for $n \geq 1$, the least κ that is *weakly parametrically n-exact for some* λ coincides with the least κ that satisfies Π_n-ESR(κ, λ) for some λ, and that the least κ that is *parametrically n-exact for some* λ coincides with the least κ that satisfies Σ_{n+1}-ESR(κ, λ) for some λ.

However, the global form of this characterization was an open question, namely if the least κ that is weakly parametrically n-exact for a proper class of λ coincides with the least κ that satisfies Π_n-ESR(κ, λ) for a proper class of λ, and if the least κ that is parametrically n-exact for a proper class of λ coincides with the least κ that satisfies Σ_{n+1}-ESR(κ, λ) for a proper class of λ. In Chapter 6 we show that, assuming stronger large cardinal, that this global characterization provably fails. Thus not only (at least this instance of) the above conjecture does not hold, it contradicts the existence of large cardinals, the very objects that the program of SR aims to secure.

Finally let us note that, the same nice pattern of structural reflection phenomenon occurs repeatedly in different regions of the large cardinal hierarchy. In particular, it was shown in [7] that the same pattern of SR that holds between a supercompact and a Vopěnka cardinal, and between a strong and a Woodin cardinal, also holds between a strongly unfoldable and a subtle cardinal. This was expressed in [7] by the following equation:

$$\frac{\textit{Vopěnka}}{\textit{Supercompact}} = \frac{\textit{Woodin}}{\textit{Strong}} = \frac{\textit{Subtle}}{\textit{Strongly unfoldable}}$$

The same pattern of structural reflection does not, however, appear at the level of ESR and exact cardinals.

0.5 Outline of the thesis

Given all the discussions in the previous sections, the present thesis is thus motivated by the question of whether there exists an ideal way to extend the structural reflection program to higher levels. More precisely, if there are natural structural reflection principles and large cardinal notions that are

1. as well-justified as the basic form of SR,
2. stronger than the basic form of SR,
3. satisfy **Conjecture 2**,
4. exhibit the same pattern as in the equation above.

It turns out that the answer is positive. Recall that the idea of the basic form of SR is that for any class $\mathcal{C}$, there exists some V_α that reflects $\mathcal{C}$, in the sense that for any $A \in \mathcal{C}$ there is some $B \in \mathcal{C} \cap V_\alpha$ that is *very similar to* A. The notion of structural similarity is explicated using elementary embeddability. Indeed, Bagaria writes ([2]):

> Since, in general, A may be much larger than any B in V_α , the closest resemblance of B to A is attained in the case B is isomorphic to an elementary substructure of A, i.e., B can be elementarily embedded into A.

The thought is that, short of isomorphism, the most we can ask for structural similarity is elementary embeddability.

The way we strengthen SR is thus to search for a stronger notion of structural similarity that can hold between two non-isomorphic structures, a notion that we call *capturing.*

Let us outline the content of each chapter.

In Chapter 1, we introduce our structural reflection principles. We first introduce the *level by level structural reflection* (LSR). Then we introduce the notion of capturing, and argue that it is a natural notion of structural similarity. Then we introduce the principle of *capturing structural reflection* (CSR), and argue that it is well-justified, precisely because it uses the same conceptual resources as SR, the only difference being that it employs a stronger notion of structural similarity. We also introduce the corresponding Vopěnka's principles, the LVP and CVP. We note that LSR and CSR are essentially equivalent.

In Chapter 2, we introduce the large cardinal notions that correspond to CSR. We introduce the hierarchies of *m-supercompact cardinals*, the *$C^{(n)}$-m-hyperhuge cardinals*, and the *$C^{(n)}$-m-fold extendible cardinals.* The cardinals are natural higher analogues of supercompact cardinals and $C^{(n)}$-extendible cardinals. They are formulated in model-theoretic terms, but we give combinatorial characterizations of these cardinals, using the idea of *m-supercompact measure.* We deduce basic properties of these cardinals, and determine their consistency strength. We show that for $m \geq 1$, the $C^{(n)}$-m-hyperhuge cardinals are equivalent to the $C^{(n)}$-$(m+1)$-fold extendible cardinals. This answers an open question of Sato Kentaro asked in [32].

In Chapter 3, we show that LSR and CSR, which are natural extensions of SR, correspond to 2-supercompact cardinals, $C^{(n)}$-2-fold extendible cardinals, LVP and CVP, which are natural extensions of supercompactness, $C^{(n)}$-extendibility, and VP, in exactly the same pattern we found in the correspondence between SR, VP, supercompact cardinals, and $C^{(n)}$-extendible cardinals.

In Chapter 4, we introduce a further strengthening of capturing that we call *δ-capturing*, which is formulated in a game-theoretic way. In particular, 0-capuring is the same as elementary embeddability. Based on this notion, we formulate the principles of δ-CSR and δ-CVP. We show that, for finite δ, we again see the same pattern of correspondence between δ-CSR, δ-CVP, $(\delta+1)$-supercompactness, and $C^{(n)}$-$(\delta+1)$-fold extendibility. This further supports the idea that δ-CSR is the "correct" way to strengthen SR. We further show that for infinite δ, δ-CSR is provably false in ZF.

In Chapter 5, we give a characterization and clarification of the relevant large cardinals using the general notion of Σ_n-m-supercompact cardinals, which encompasses other notions. We generalize the work of Bagaria-Goldberg in [5], which gave $C^{(n)}$-extendible cardinals an ultrafilter characterization, by showing that the Σ_n-m-supercompact cardinals can be characterized by the notion of *n-reflecting m-supercompact measures.*

In Chapter 6, we answer several open questions about ESR and exact cardinals asked by Bagaria-Lücke in [6]. The answers turn out to be quite unexpected. Firstly, the strength of ESR and exact cardinals turns out to be lower than expected, and occupies a region of the large cardinal hierarchy that was previously unexplored. Secondly, as remarked in the previous section, we show that the instance of **Conjecture 2** for the global version of ESR fails. In particular, assuming large cardinals, we show that for all $n \geq 1$, the least κ such that Σ_{n+1}-ESR(κ, λ) holds for a proper class of λ is less than the least κ that is parametrically n-exact for a proper class of λ. This is proved by application of the theory developed in previous chapters. Our results also generalize to the sequential forms of ESR.

In the concluding Chapter 7 we go back to the foundational considerations presented in this chapter. We note that, on the one hand, our results fulfill the expectations of finding natural extensions of SR to higher levels, while on the other hand, our results also constitute a challenge for the SR program as currently formulated, namely the problem of *extendibility to inconsistency*.

Chapter 1

Higher Structural Reflection

In this chapter we present and motivate two ways of formulating natural structural reflection principles of higher strength than *Vopěnka's Principle.* The first way is to employ the idea of *level by level* reflection, which is more in the spirit of the idea of *exactness* considered in [6]. The second way is to use the model-theoretic notion of *capturing*, which is generally applicable to all mathematical structures, that we will introduce below.

Recall that the original structural reflection principle (see [1] and [2]) is formulated as follows.

> SR: (*Structural Reflection*) For every definable, in the first-order language of set theory, possibly with parameters, class $\mathcal{C}$ of relational structures of the same type there exists an ordinal δ that *reflects* $\mathcal{C}$, i.e., for every B in $\mathcal{C}$ there exists A in $\mathcal{C} \cap V_\delta$ and an elementary embedding from A into B.

SR is conceptually very compelling, motivated by the idea that for any definable class $\mathcal{C}$ of structures, there is some V_δ such that the class $V_\delta \cap \mathcal{C}$ is *as rich as* $\mathcal{C}$, in the sense that for any $B \in \mathcal{C}$ there is some $A \in V_\delta \cap \mathcal{C}$ that is structurally very similar to B, namely elementarily embeddable into B.

1.1 Level by Level

A natural idea to strengthen SR is to assert that for any class $\mathcal{C}$ of structures, there is some δ that not only reflects $\mathcal{C}$, but does so in an exact, level by level way.

> LSR: (*Level by Level Structural Reflection*) For every definable, in the first-order language of set theory, possibly with parameters, class $\mathcal{C}$ of relational structures of the same type there exists an ordinal δ that *reflects $\mathcal{C}$ level by level*, i.e., for any ordinal β there is some ordinal $\alpha < \delta$ such that for every B in $\mathcal{C}$ of rank β there exists A in $\mathcal{C}$ of rank α and an elementary embedding from A into B.

LSR is motivated by the idea that there is a stronger analogy between the structural richness of $\mathcal{C}$ and that of $V_\delta \cap \mathcal{C}$, to the extent that for any β, the elements of $\mathcal{C}$ of rank β can all be matched by elements of $\mathcal{C}$ of rank α, for some $\alpha < \delta$.

Of course, due to the undefinability of definability, LSR cannot be formulated in ZFC. Thus what we really work with are the following versions of LSR, formulated for any natural number n:

$\mathbf{\Sigma_n}$-LSR: ($\mathbf{\Sigma_n}$-*Level by Level Structural Reflection*) For every Σ_n-definable, with parameters, class $\mathcal{C}$ of relational structures of the same type there is an ordinal α that *reflects* $\mathcal{C}$ *level by level.*

$\mathbf{\Pi_n}$-LSR may be similarly formulated. Furthermore, we consider the *lightface*, i.e., parameter-free versions of LSR. Namely,

Σ_n-LSR: (Σ_n-*Level by Level Structural Reflection*) For every Σ_n-definable, without parameters, class $\mathcal{C}$ of relational structures of the same type there exists an ordinal α that *reflects* $\mathcal{C}$ *level by level.*

Similarly for the lightface Π_n-LSR.

Definition 1.1.1. For a natural number n and an ordinal α, $\mathrm{LSR}(\alpha, \Sigma_n)$ holds if for any Σ_n-definable, without parameters, class $\mathcal{C}$ of relational structures of the same type, α reflects $\mathcal{C}$ level by level. Similarly for $\mathrm{LSR}(\alpha, \Pi_n)$.

Similarly, $\mathrm{LSR}(\alpha, \mathbf{\Sigma_n})$ holds if for any Σ_n-definable, with parameters in V_α, class $\mathcal{C}$ of relational structures of the same type, α reflects $\mathcal{C}$ level by level. Similarly for $\mathrm{LSR}(\alpha, \mathbf{\Pi_n})$.

We first state two easy propositions.

Proposition 1.1.2. *If* Γ *is either* Σ_n *or* Π_n*, then the following are equivalent:*

1. Γ-LSR.
2. *There is some* α *such that* $\mathrm{LSR}(\alpha, \Gamma)$ *holds.*

Proposition 1.1.3. *If* $\mathbf{\Gamma}$ *is either* $\mathbf{\Sigma_n}$ *or* $\mathbf{\Pi_n}$*, then the following are equivalent:*

1. $\mathbf{\Gamma}$-LSR.
2. *There is a proper class of* α *such that* $\mathrm{LSR}(\alpha, \mathbf{\Gamma})$ holds.

Using the same idea of level by level reflection, a natural strengthening of Vopěnka's Principle can be formulated.

Definition 1.1.4. For any natural number n, $\mathrm{LVP}(\Sigma_n)$ holds if for any Σ_n-definable, without parameters, proper class $\mathcal{C}$ of relational structures of the same type, there exist $\alpha \neq \beta$ with $\mathcal{C} \cap (V_{\alpha+1} \setminus V_\alpha) \neq \phi$ and $\mathcal{C} \cap (V_{\beta+1} \setminus V_\beta) \neq \phi$ such that for any $B \in \mathcal{C}$ of rank β there is some $A \in \mathcal{C}$ of rank α and an elementary embedding $j : A \to B$. Similarly for $\mathrm{LVP}(\Pi_n)$.

Similarly, $\mathrm{LVP}(\mathbf{\Sigma_n})$ and $\mathrm{LVP}(\mathbf{\Pi_n})$ are the corresponding assertions with parameters allowed in the definition of $\mathcal{C}$.

The idea behind the original VP is that if we have a proper class $\mathcal{C}$ of structures, then $\mathcal{C}$ must repeat itself: there is bound to be two structures in $\mathcal{C}$ with one similar to the other. The idea behind LVP is that if we have a proper class $\mathcal{C}$ of structures, then $\mathcal{C}$ must repeat itself in a strong way: there is bound to be two levels α, β of the cumulative hierarchy, where for any structure in $\mathcal{C}$ of level β, there is already some structure in $\mathcal{C}$ of level α similar to it.

1.2 Capturing

When there is an elementary embedding e from some structure A to another structure B in the same language, an equivalent statement is that there is an isomorphism between A and an elementary substructure of B. Thus there is a sense in which the structural properties of A are similar to those of B, or that A is able to capture a lot of structural information of B. Indeed the fact that elementary embedding expresses a notion of structural similarity or capturing is one of the reasons SR can be seen as a plausible principle of set theory.

A closer inspection of the concept of elementary embedding suggests, in our opinion, a natural way of strengthening it. If $e : A \to B$ is an elementary embedding and $b \in rang(B)$, then $a = e^{-1}(b)$ can be seen as a sort of *counterpart* of b, an "A's version of b", where a plays the same role in A as b plays in B, with respect to all definable structural properties. We may say in this case that A *captures* b with a.

From this perspective, if $e : A \to B$ is elementary, then any $b \in ran(e)$ is captured by $a = e^{-1}(b) \in A$. Usually we do not know what elements of B are in the range of e. In other words, we do not know which elements of B admit counterparts in A. Thus we may define the following notion:

Definition 1.2.1. For structures A and B of the same type and $b \in B$, we say A *captures* b if there is an elementary embedding $e : A \to B$ with $b \in ran(e)$.

We say A *captures* B if every element of B is captured by A.

Thus from the perspective above, to say that A captures B is to say that for any $b \in B$, there is some counterpart of b in A that plays the same structural role. Another way to see it is that for any $b \in B$, A is isomorphic to an elementary substructure of B that contains b. This gives a stronger expression of the idea that the structure of B as a whole is captured by, or reflected in, the structure of A.

Note that the notion of capturing is a general model-theoretic notion, applicable to all mathematical structures. Although the primary interest in this thesis is to abstractly study strong structural reflection principles using the notion of capturing, let us try to give some basic and concrete examples.

Example 1.2.2. Consider three structures in the same language. The first one is $\mathfrak{N}_0 = (\mathbb{N}, 0, S)$, where S is the successor function, which looks like

$$0 \to S(0) \to SS(0) \to SSS(0) \to \dots,$$

and the second one, $\mathfrak{N}_1$, which includes $\mathfrak{N}_0$ and an additional, disjoint and nonstandard "$\mathbb{Z}$-chain", which looks like

$$\begin{aligned}
&0 \to S(0) \to SS(0) \to SSS(0) \to \dots \\
&\dots \to S^{-2}(a) \to S^{-1}(a) \to a \to S(a) \to SS(a) \to \dots,
\end{aligned}$$

finally $\mathfrak{N}_2$ includes $\mathfrak{N}_1$, and a further additional disjoint $\mathbb{Z}$-chain, which looks like

$$\begin{aligned}
&0 \to S(0) \to SS(0) \to SSS(0) \to \dots \\
&\dots \to S^{-2}(a) \to S^{-1}(a) \to a \to S(a) \to SS(a) \to \dots, \\
&\dots \to S^{-2}(b) \to S^{-1}(b) \to b \to S(b) \to SS(b) \to \dots,
\end{aligned}$$

where $a \neq b$.

Let $T = Th(\mathfrak{N}_0)$, T is axiomatized (see [15], section 3.1) by the set of sentences that consists of $\forall x S(x) \neq 0$, $\forall x, y(S(x) = S(y) \to x = y)$, $\forall y(y \neq 0 \to \exists x y = S(x))$, and the schema $\forall x S^n(x) \neq x$, for all natural numbers n. Thus all three structures model T. Moreover, T admits quantifier elimination (again see [15]), and is thus model complete, so any embedding between models of T is elementary.

Thus $\mathfrak{N}_0$ is elementarily embeddable into $\mathfrak{N}_1$, and $\mathfrak{N}_1$ is elementarily embeddable into $\mathfrak{N}_2$, simply because $\mathfrak{N}_0 \prec \mathfrak{N}_1 \prec \mathfrak{N}_2$, yet all three structures are non-isomorphic. However, there seems to be a sense in which $\mathfrak{N}_1$ resembles $\mathfrak{N}_2$ more than $\mathfrak{N}_0$ resembles $\mathfrak{N}_2$, as it is $\mathfrak{N}_1$ rather than $\mathfrak{N}_0$ that looks more like $\mathfrak{N}_2$. Although the notion of capturing is not designed for this example, this difference can indeed be expressed by the notion of capturing. $\mathfrak{N}_0$ does not capture either $\mathfrak{N}_1$ or $\mathfrak{N}_2$, in particular $\mathfrak{N}_0$ does not capture a, since any elementary embedding from $\mathfrak{N}_0$ to $\mathfrak{N}_1$ or $\mathfrak{N}_2$ would fix the standard naturals. On the other hand, $\mathfrak{N}_1$ does capture $\mathfrak{N}_2$: for any $y \in \mathfrak{N}_2$, if y is standard or $S^m(a)$ for some integer m, then let e be the identity function, and if y is $S^m(b)$ for some integer m, then let $e : \mathfrak{N}_1 \to \mathfrak{N}_2$ be the function that fixes the standard part, and send $S^k(a)$ to $S^k(b)$ for all integers k. It follows that e is an embedding, and thus elementary.

It seems that the notion of capturing is a natural strengthening of elementary embeddability, and is meaningful in mathematical contexts other than set theory, thus we believe that this notion, or some variants of it, will perhaps have other applications. In Chapter 4 we will introduce a further strengthening of capturing.

In this thesis we will mainly use capturing to study strong structural reflection principles in set theory. Recall that the idea of SR is that for any class $\mathcal{C}$ of structures, there is some α such that for any $B \in \mathcal{C}$, there is some $A \in \mathcal{C} \cap V_\alpha$ such that A is structurally similar to B, where similarity is explicated by the notion of elementarary embeddability. Given that capturing is, as we have suggested, a strengthened notion of structural similarity, it is natural to consider the following strengthening of SR.

CSR (*Capturing Structural Reflection*) For every definable, in the first-order language of set theory, possibly with parameters, class $\mathcal{C}$ of relational structures of the same type there exists an ordinal δ that *capture-reflects* $\mathcal{C}$, i.e., for any $B \in \mathcal{C}$ there exists $A \in \mathcal{C} \cap V_\delta$ that captures B.

Of course, the correct formulation in ZFC is the following, for any natural number n:

$\mathbf{\Sigma_n}$-CSR: ($\mathbf{\Sigma_n}$-*Capturing Structural Reflection*) For every Σ_n-definable, with parameters, class $\mathcal{C}$ of relational structures of the same type there is an ordinal α that *capture-reflects* $\mathcal{C}$. Similarly for $\mathbf{\Pi_n}$-CSR. Moreover, Σ_n-CSR and Π_n-CSR are the corresponding lightface versions, i.e., with parameters disallowed in the definition of $\mathcal{C}$.

Definition 1.2.3. For any natural number n, $\mathrm{CSR}(\alpha, \Sigma_n)$ holds if for any Σ_n-definable, without parameters, class $\mathcal{C}$ of relational structures of the same type, α capture-reflects $\mathcal{C}$.

Similarly, $\mathrm{CSR}(\alpha, \mathbf{\Sigma_n})$ holds if for any Σ_n-definable, with parameters in V_α, class $\mathcal{C}$ of relational structures of the same type, α capture-reflects $\mathcal{C}$. Also similarly for $\mathrm{CSR}(\alpha, \mathbf{\Pi_n})$.

Given the above conceptual analysis, it seems plausible to think that if SR can be seen as intrinsically plausible, then this plausibility also extends to CSR. In Chapter 3 and 4, we will further pursue the idea that CSR is the "correct" strengthening of SR by showing that the pattern of correspondence between SR, supercompact cardinals, and $C^{(n)}$-extendible cardinals is exactly the same as the pattern of correspondence between (higher analogues of) CSR, and higher analogues of supercompact cardinals and $C^{(n)}$-extendible cardinals.

The corresponding stengthening of VP can also be formulated, with the idea being that when one has too many structures, one of them may be *very* similar to one of the others, to the extent that the latter is captured by the former:

Definition 1.2.4. For any natural number n, $\mathrm{CVP}(\Sigma_n)$ holds if for any Σ_n-definable, without parameters, proper class $\mathcal{C}$ of relational structures of the same type, there exist $A, B \in \mathcal{C}$ with $A \neq B$ and A captures B. Similarly for $\mathrm{CVP}(\Pi_n)$.

Similarly, $\mathrm{CVP}(\mathbf{\Sigma_n})$ and $\mathrm{CVP}(\mathbf{\Pi_n})$ are the corresponding assertions with parameters allowed in the definition of $\mathcal{C}$.

1.3 Convergence

A pleasing fact about LSR and CSR is that, as we will see, these two natural ways of strengthening SR actually converge.

Theorem 1.3.1. *For any $n \geq 1$ and any ordinal α, the following are equivalent:*

1. $\mathrm{LSR}(\alpha, \mathbf{\Pi_n})$ *holds.*

2. $\mathrm{CSR}(\alpha, \mathbf{\Pi_n})$ *holds.*

Similarly, $\mathrm{LSR}(\alpha, \Pi_n)$ *and* $\mathrm{CSR}(\alpha, \Pi_n)$ *are equivalent.*

The above theorem is a consequence of results we will prove later. Similarly, we have the convergence of their VP version:

Theorem 1.3.2. *For any $n \geq 1$, the following are equivalent:*

1. $\mathrm{LVP}(\mathbf{\Pi_n})$ *holds.*

2. $\mathrm{CVP}(\mathbf{\Pi_n})$ *holds.*

Similarly, $\mathrm{LVP}(\Pi_n)$ *and* $\mathrm{CVP}(\Pi_n)$ *are equivalent.*

One direction of Theorem 1.3.1 follows from the following proposition. The other direction requires more work, and will be shown in later chapters.

Proposition 1.3.3. *For any natural number n and ordinal α, if* $\mathrm{CSR}(\alpha, \mathbf{\Pi_n})$ *holds, then*

1. $\mathrm{CSR}(\alpha, \mathbf{\Sigma_{n+1}})$ *holds, and*

2. $\mathrm{LSR}(\alpha, \mathbf{\Sigma_{n+1}})$ *holds.*

Similarly if $\mathrm{CSR}(\alpha, \Pi_n)$ *holds, then both* $\mathrm{CSR}(\alpha, \Sigma_{n+1})$ *and* $\mathrm{LSR}(\alpha, \Sigma_{n+1})$ hold.

Proof. Given some natural number n and ordinal α, suppose $\mathrm{CSR}(\alpha, \mathbf{\Pi_n})$ holds.

(1): To show that $\mathrm{CSR}(\alpha, \mathbf{\Sigma_{n+1}})$ holds, let $\mathcal{C} = \{B : \phi(B, b_0, \ldots, b_{k-1})\}$ be a class of structures of the same type, where $\phi(x, y_0, \ldots, y_{k-1})$ is Σ_{n+1} and $b_0, \ldots, b_{k-1} \in V_\alpha$. Given some $B \in \mathcal{C}$, we must find some $A \in \mathcal{C} \cap V_\alpha$ that captures B.

Define $\mathcal{C}^*$ to be the class of structures of the form $(M, \in, X, b_0, \ldots, b_{k-1})$, where M is a transitive set that is Σ_n-correct, namely $M \prec_{\Sigma_n} V$, and X is any set in M. Note that $\mathcal{C}^*$ is Π_n-definable with the parameters $b_0, \ldots, b_{k-1}$.

Now consider the structure $(V_\lambda, \in, B, b_0, \ldots, b_{k-1})$, for λ large enough and $\lambda \in C^{(n+1)}$. Clearly we have $(V_\lambda, \in, B, b_0, \ldots, b_{k-1}) \in \mathcal{C}^*$. By $\mathrm{CSR}(\alpha, \mathbf{\Pi_n})$ there is some structure

$(M, \in, A, b_0, \ldots, b_{k-1}) \in \mathcal{C}^*$ that captures $(V_\lambda, \in, B, b_0, \ldots, b_{k-1})$. Thus for any $b \in B$ there is some elementary embedding

$$j : (M, \in, A, b_0, \ldots, b_{k-1}) \to (V_\lambda, \in, B, b_0, \ldots, b_{k-1})$$

with $b \in ran(j)$. By the elementarity of j, we have $j^{-1}(b) \in A$ and $type(B) = type(A)$. Moreover, for any formula $\psi(z_0, \ldots, z_{l-1})$ in the language of A and B and for any $c_0, \ldots, c_{l-1} \in A$ we have

$$\begin{aligned} A \models \psi(c_0, \ldots, c_{l-1}) &\leftrightarrow (M, \in, A, b_0, \ldots, b_{k-1}) \models \text{“}A \models \psi(c_0, \ldots, c_{l-1})\text{”} \\ &\leftrightarrow (V_\lambda, \in, B, b_0, \ldots, b_{k-1}) \models \text{“}B \models \psi(j(c_0), \ldots, j(c_{l-1}))\text{”} \\ &\leftrightarrow B \models \psi(j(c_0), \ldots, j(c_{l-1})), \end{aligned}$$

so $j \restriction A : A \to B$ is elementary. Lastly, by elementarity we have

$$(M, \in, A, b_0, \ldots, b_{k-1}) \models \text{“}\phi(A, b_0, \ldots, b_{k-1})\text{”}.$$

Thus by the upward-absoluteness of Σ_{n+1}-formulas for Σ_n-correct structures we have that $\phi(A, b_0, \ldots, b_{k-1})$ holds and so $A \in \mathcal{C}$, hence $A \in \mathcal{C} \cap V_\alpha$ captures B, as desired.

(2): To show that LSR$(\alpha, \mathbf{\Sigma_{n+1}})$ holds, again let $\mathcal{C} = \{B : \phi(B, b_0, \ldots, b_{k-1})\}$ be a class of structures of the same type, where $\phi(x, y_0, \ldots, y_{k-1})$ is Σ_{n+1} and $b_0, \ldots, b_{k-1} \in V_\alpha$. Given some β, we must find some $\gamma < \alpha$ such that for any $B \in \mathcal{C}$ of rank β there is some $A \in \mathcal{C}$ of rank γ that is elementary embeddable into B. Now define $\mathcal{C}^*$ to be the class of structures of the form $X = (M, \in, \delta, b_0, \ldots, b_{k-1})$, where M is transitive, δ is an ordinal, and M is Σ_n-correct. Again $\mathcal{C}^*$ is Π_n-definable with the parameters $b_0, \ldots, b_{k-1}$. Let $(V_\lambda, \in, \beta, b_0, \ldots, b_{k-1})$ be such that $\lambda \in C^{(n+1)}$ and λ large enough. Thus $(V_\lambda, \in, \beta, b_0, \ldots, b_{k-1}) \in \mathcal{C}^*$.

It follows from CSR$(\alpha, \mathbf{\Pi_n})$ that there exists some $(M, \in, \gamma, b_0, \ldots, b_{k-1}) \in \mathcal{C}^*$ that captures $(V_\lambda, \in, \beta, b_0, \ldots, b_{k-1})$. We claim that the γ here is as desired. To see this, note that for any structure $B \in \mathcal{C}$ of rank β, we have $B \in V_\lambda$, so there is some elementary embedding

$$j : (M, \in, \gamma, b_0, \ldots, b_{k-1}) \to (V_\lambda, \in, \beta, b_0, \ldots, b_{k-1}),$$

with some $A \in M$ such that $j(A) = B$. Thus since

$$(V_\lambda, \in, \beta, b_0, \ldots, b_{k-1}) \models \text{“}\phi(B, b_0, \ldots, b_{k-1}) \wedge rank(B) = \beta\text{”},$$

we have by elementarity that

$$(M, \in, \gamma, b_0, \ldots, b_{k-1}) \models \text{“}\phi(A, b_0, \ldots, b_{k-1}) \wedge rank(B) = \gamma\text{”},$$

thus $\phi(A, b_0, \ldots, b_{k-1})$ holds by upward-absoluteness, and $A \in \mathcal{C}$ is of rank γ. Also as before we have $j \restriction A : A \to B$ is elementary, as desired.

To conclude, note that it is clear that similar proofs work for the case CSR(α, Π_n), without considering parameters. $\square$

Chapter 2

Very Large Cardinals

In this chapter we study some very large cardinal notions, which are higher analogues of supercompactness and extendibility. In the next chapter we will show that these large cardinals correspond to the higher structural reflection principles we proposed in the previous chapter in exactly the same pattern that supercompact and extendible cardinals correspond to the original SR. In section 1.3 we answer a question of Sato Kentaro in [32].

2.1 Higher supercompactness

The notion of supercompact cardinals is a central notion in the theory of large cardinals, having numerous applications. Magidor gave a characterization of supercompactness in terms of elementary embeddings between the V_α's:

Theorem 2.1.1. (Magidor [25]) *A cardinal κ is supercompact if and only if for any $\lambda > \kappa$ and $y \in V_\lambda$ there is some $\bar{\lambda} < \kappa$, $x \in V_{\bar{\lambda}}$ and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(x) = y$ and $j(crit(j)) = \kappa$.*

We introduce here direct strengthenings of the above formulation, as follows:

Definition 2.1.2. For a natural number $m \geq 1$ and cardinal κ, κ is *m-supercompact* if for any $\lambda > \kappa$ and $y \in V_\lambda$, there is some $\bar{\lambda} < \kappa$ and $x \in V_{\bar{\lambda}}$ with some elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(x) = y$ and $j^m(crit(j)) = \kappa$.

In particular, 1-supercompactness is just supercompactness.

Remark Given the usual definition of supercompact cardinals in terms of some elementary embedding $j : V \to M$, a more intuitive way of strengthening supercompactness may be to assert that for any $\lambda \geq \kappa$, κ is the critical point of $j : V \to M$ with $\lambda < j(\kappa)$ and ${}^{j^m(\lambda)}M \subseteq M$. This is the notion of *m-hyperhugeness*, which will be discussed in the next section.

However, it is the notion of m-supercompactness that in many respects resembles supercompactness more, including its relation to higher structural reflection principles and higher analogues of Löwenheim-Skolem-Tarski theorem for higher-order logics, its characterization in terms of *reflective measures* (more on this in chapter 4), and more. On the other hand, we will show that, answering a question of Sato Kentaro, m-hyperhuge cardinals are actually higher analogues of *extendibible cardinals*, and they inherit many properties of extendible cardinals. These points will become clearer later. Thus arguably it is the Magidor characterization that better captures (some aspects of) the conceptual essence of supercompactness*.

The formulation of m-supercompactness is essentially model-theoretic in nature. However, we now show that this notion can be given purely combinatorial characterization in terms of ultrafilters.

Definition 2.1.3. For a natural number $m \geq 1$, cardinal κ, and sequences $\kappa_0, \dots, \kappa_{m-1} = \kappa$ and $\lambda_0, \dots, \lambda_{m-1} = \lambda \geq \kappa$, we say $\mathcal{U}$ is an *m-supercompact measure for* (κ, λ) if $\mathcal{U}$ is a κ_0-complete normal ultrafilter over $\mathcal{P}(\lambda)$ such that

1. $\{x \in \mathcal{P}(\lambda_{m-1}) : ot(x \cap \kappa_{i+1}) = \kappa_i\} \in \mathcal{U}$ for any $0 \leq i \leq m-2$,
2. $\{x \in \mathcal{P}(\lambda_{m-1}) : ot(x \cap \lambda_{i+1}) = \lambda_i\} \in \mathcal{U}$ for any $0 \leq i \leq m-2$, and
3. $\{x \in \mathcal{P}(\lambda_{m-1}) : ot(x \cap \lambda_0) < \kappa_0\} \in \mathcal{U}$.

We will sometimes mention $\kappa_0, \dots, \kappa_{m-1}$ and $\lambda_0, \dots, \lambda_{m-1}$ as the *target sequences*.

Note that if $m = 1$, the first two clauses of the above definition are trivial. Moreover, there is some 1-supercompact measure for (κ, λ) if and only if there is some normal measure on $\mathcal{P}_\kappa \lambda$.

A feature of the above definition is that, unlike most ultrafilter characterizations of "m-fold" versions of large cardinals, for instance the ultrafilter characterization for the m-huge cardinals, where the large cardinal κ is the *first* entry in the target sequence, here the large cardinal κ is characterized as the *last* entry of the target sequences.

Theorem 2.1.4. *For natural number $m \geq 1$ and cardinal κ, κ is m-supercompact if and only if for any $\lambda \geq \kappa$ there is an m-supercompact measure $\mathcal{U}$ for (κ, λ).*

Proof. To prove the forward direction, suppose that $m \geq 1$ is a natural number and κ is m-supercompact. First define the class function $F : Ord \to Ord$ by

> $F(\alpha) =$ the least $\beta \geq \alpha$ such that there is no m-supercompact measure $\mathcal{U}$ for (α, β), provided that there is any such β. Otherwise let $F(\alpha) = 0$.

Suppose for contradiction that our conclusion fails, namely there is some $\beta \geq \kappa$, such that there is no m-supercompact measure for (κ, β), thus $F(\kappa) \geq \kappa$. Now take λ to be some limit ordinal greater than $F(\kappa)$, and let $\bar{\lambda}$ be such that there is some elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $crit(j) = \mu$ and $j^m(\mu) = \kappa$. Since whether a set $\mathcal{U}$ is an m-supercompactness measure for $(\kappa, F(\kappa))$ can be checked in V_λ, it follows that

$$V_\lambda \models \text{“}\exists \beta \geq \kappa (F(\kappa) = \beta)\text{"}.$$

By the elementarity of j we have

$$V_{\bar{\lambda}} \models \text{“}\exists \beta \geq j^{m-1}(\mu)(F(j^{m-1}(\mu)) = \beta)\text{"}.$$

Note that the quoted statement is also true in V, and thus also in V_λ, since the relevant normal ultrafilters are all in $V_{\bar{\lambda}}$. But note that since we are supposing that $\bar{\lambda} < \kappa = j^m(\mu)$, we have that

$$V_\lambda \models \text{“}\exists \beta (j^{m-1}(\mu) \leq \beta < j^m(\mu) \wedge F(j^{m-1}(\mu)) = \beta)\text{"},$$

*Note that the similar name *m-fold supercompactness* is used in [32] to refer to what we call $(m-1)$-hyperhugeness here. For reasons explained above, we reserve the word "supercompact" for the notion of m-supercompactness. Given that 2-fold supercompact cardinals are also called hyperhuge cardinals by Toshimichi Usuba in [40], we use the terminology of $(m-1)$-hyperhugeness for m-fold supercompact cardinals.

thus by elementarity again,

$$V_{\bar{\lambda}} \models \text{"}\exists\beta(j^{m-2}(\mu) \leq \beta < j^{m-1}(\mu) \wedge F(j^{m-2}(\mu)) = \beta)\text{"}$$

which is true in V and in V_λ. Iterating this argument, we will eventually get

$$V_{\bar{\lambda}} \models \text{"}\exists\beta(\mu \leq \beta < j(\mu) \wedge F(\mu) = \beta)\text{"}.$$

Let $\beta = F(\mu)$, we have

$$j^{m-1}(\beta) = j^{m-1}(F(\mu)) = F(j^{m-1}(\mu)),$$

and note that we have seen that $F(j^{m-1}(\mu)) < \bar{\lambda}$. But now it makes sense to define

$$\mathcal{U} = \{X \subseteq \mathcal{P}(j^{m-1}(\beta)) : j\text{``}j^{m-1}(\beta) \in j(X)\}.$$

One can verify that $\mathcal{U}$ is a μ-complete normal ultrafilter over $\mathcal{P}(j^{m-1}(\beta))$. Moreover, we have, for any $0 \leq i \leq (m-2)$, that

1. $ot(j\text{``}j^{m-1}(\beta) \cap j^{i+2}(\mu)) = j^{i+1}(\mu)$ implies $\{x \in \mathcal{P}(j^{m-1}(\beta)) : ot(x \cap j^{i+1}(\mu)) = j^i(\mu)\} \in \mathcal{U}$.
2. $ot(j\text{``}j^{m-1}(\beta) \cap j^{i+2}(\beta)) = j^{i+1}(\beta)$ implies $\{x \in \mathcal{P}(j^{m-1}(\beta)) : ot(x \cap j^{i+1}(\beta)) = j^i(\beta)\} \in \mathcal{U}$.
3. $ot(j\text{``}j^{m-1}(\beta) \cap j(\beta)) = \beta < j(\mu)$ implies $\{x \in \mathcal{P}(j^{m-1}(\beta)) : ot(x \cap \beta) < \mu\} \in \mathcal{U}$.

It follows that $\mathcal{U}$ is an m-supercompactness measure for $(j^{m-1}(\mu), j^{m-1}(\beta))$ (with target sequences $\mu, j(\mu), \ldots, j^{m-1}(\mu)$ and $\beta, j(\beta), \ldots, j^{m-1}(\beta)$), contradicting the fact that $F(j^{m-1}(\mu)) = j^{m-1}(\beta)$.

For the converse direction, given any $\eta \geq \kappa$, take $\lambda > \eta$ to be a $\beth$-fixed point, so that $|V_\lambda| = \lambda$. Suppose for contradiction that we have some $y \in V_\lambda$ so that

> there is no $\delta < \kappa$ with some $x \in V_\delta$ and an elementary embedding $k : V_\delta \to V_\lambda$ such that $k^m(crit(k)) = \kappa$ and $k(x) = y$.

Now take $\kappa_0, \ldots, \kappa_{m-1} = \kappa$ and $\lambda_0, \ldots, \lambda_{m-1} = \lambda$ such that there is an m-supercompactness measure $\mathcal{U}$ over $\mathcal{P}(\lambda)$ for (κ, λ).

Take the ultrapower embedding $j : V \to Ult(V, \mathcal{U}) \cong M$ given by $\mathcal{U}$. Thus $[id]_{\mathcal{U}} = j\text{``}\lambda$, so ${}^\lambda M \subseteq M$, and we have for any $0 \leq i \leq m-2$

1. $\{x \in \mathcal{P}(\lambda_{m-1}) : ot(x \cap \kappa_{i+1}) = \kappa_i\} \in \mathcal{U}$ implies $j(\kappa_i) = \kappa_{i+1}$,
2. $\{x \in \mathcal{P}(\lambda_{m-1}) : ot(x \cap \lambda_{i+1}) = \lambda_i\} \in \mathcal{U}$ implies $j(\lambda_i) = \lambda_{i+1}$, and
3. $\{x \in \mathcal{P}(\lambda_{m-1}) : ot(x \cap \lambda_0) < \kappa_0\} \in \mathcal{U}$ implies $\lambda_0 < j(\kappa_0)$.

It also follows from the above by elementarity that $\lambda_i < j(\kappa_i)$ for any $0 \leq i \leq m-1$. Moreover we have $crit(j) = \kappa_0$. Also by the elementarity of j we have

> $M \models$ "there is no $\delta < j(\kappa)$ with some $x \in V_\delta$ and an elementary embedding $k : V_\delta \to V_{j(\lambda)}$ such that $k^m(crit(k)) = j(\kappa)$ and $k(x) = j(y)$".

Now since $|V_\lambda| = \lambda$, an argument by induction will show that $V_\lambda = (V_\lambda)^M$. Moreover, since $|j{\restriction}_{V_\lambda}| = |V_\lambda| = \lambda$, we have $k = j{\restriction}_{V_\lambda} \in M$. However, note that we have

1. $k : V_\lambda \to (V_{j(\lambda)})^M$ is elementary,

2. $\lambda < j(\kappa)$,

3. $k^m(crit(k)) = j^m(crit(j)) = j^m(\kappa_0) = j(\kappa_{m-1}) = j(\kappa)$, and

4. $y \in V_\lambda$ with $k(y) = j(y)$,

thus k is exactly the embedding that is supposed to be missing in M, a contradiction. This completes the proof. $\square$

Note that in the forward direction of the proof of the above theorem, we did not consider y and x as in the definition of m-supercompactness, thus we have the following:

Corollary. *For a natural number $m \geq 1$, a cardinal κ is m-supercompact if for any $\lambda > \kappa$ there is $\bar{\lambda} < \kappa$ and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j^m(crit(j)) = \kappa$.*

Note that the assertion "there is an m-supercompact measure for (α, β)" is a Δ_2-statement with parameters α and β, since it is equivalent to both

$$\forall X(X = V_{\beta+5} \to X \models \text{“there is an } m\text{-supercompact measure for } (\alpha,\beta)\text{", and}$$

$$\exists X(X = V_{\beta+5} \wedge X \models \text{“there is an } m\text{-supercompact measure for } (\alpha,\beta)\text{".}$$

It follows by Theorem 2.1.4 that "α is m-supercompact" is a Π_2-statement.

Now we show that the m-supercompact cardinals form a proper hierarchy in a strong sense, both in terms of relative consistency strength and the relative sizes of least instances.

Let us first recall the notions of $C^{(n)}$-cardinals. $C^{(n)}$ is defined to be the class of ordinals α such that $V_\alpha \prec_{\Sigma_n} V$, namely V_α is a Σ_n-elementary substructure of V. $C^{(n)}$ is a club class for each natural number n, and the statement “$\alpha \in C^{(n)}$” is Π_n-definable. Moreover, the class $C^{(1)}$ consists of precisely the $\beth$-fixed point, namely uncountable cardinals κ such that $V_\kappa = H_\kappa$ For more details see [1].

Theorem 2.1.5. *For natural numbers $1 \leq n < m$, if κ is an m-supercompact cardinal, then*

1. κ is n-supercompact,

2. $\kappa \in C^{(2)}$,

3. $V_\kappa \models$ ZFC+"there is a proper class of n-supercompact cardinals", and

4. κ is a limit of n-supercompact cardinals.

Proof. For (1), suppose $1 \leq n < m$ are natural numbers and κ is m-supercompact. Let $\lambda = \beth_\lambda > \kappa$, and by Theorem 2.1.4, let $\mathcal{U}$ be an m-supercompact measure for (κ, λ), with target sequences $\kappa_0, \ldots, \kappa_{m-1} = \kappa$ and $\lambda_0, \ldots, \lambda_{m-1} = \lambda$. Let $j : V \to Ult(V,\mathcal{U}) \cong M$ be the ultrapower embedding given by $\mathcal{U}$, we then have $crit(j) = \kappa_0$, $j(\kappa_i) = \kappa_{i+1}$ and $j(\lambda_i) = \lambda_{i+1}$ for $0 \leq i \leq m-2$, and ${}^\lambda M \subseteq M$.

For any $\delta = \delta_0 < j(\kappa_0)$, we have $j^{n-1}(\delta_0) < j^n(\kappa_0) = \kappa_n \leq \kappa_{m-1} < \lambda$. Let $\delta_i = j^i(\delta_0)$ for $0 \leq i \leq n-1$. Now define the ultrafilter

$$\mathcal{U}^* = \{X \subseteq \mathcal{P}(\delta_{n-1}) : j``\delta_{n-1} \in j(X)\},$$

one can check that $\mathcal{U}^*$ is an n-supercompact measure for $(\kappa_{n-1}, \delta_{n-1})$, with target sequences $\kappa_0, \ldots, \kappa_{n-1}$ and $\delta_0, \ldots, \delta_{n-1}$. Now since $\delta_{n-1} < \lambda = \beth_\lambda$ and ${}^\lambda M \subseteq M$, we have $\mathcal{U}^* \in M$, so

$$M \models \text{"there is an } n\text{-supercompact measure for } (\kappa_{n-1}, \delta_{n-1})\text{"},$$

so by elementarity we have an n-supercompact measure for $(\kappa_{n-2}, \delta_{n-2})$, which is again true in M. By applying elementarity $(n-1)$-times, we get that there is an n-supercompact measure for (κ_0, δ_0). Now since $\delta_0 < \kappa_1$ is arbitrarily chosen, we have that for any $\delta < \kappa_1$, there is an n-supercompact measure for (κ_0, δ). This can be checked in V_{κ_1}, which implies that

$$V_{\kappa_1} \models \text{"}\kappa_0 \text{ is } n\text{-supercompact"}.$$

By applying elementarity we have

$$(V_{j(\kappa_{m-1})})^M \models \text{"}\kappa_{m-1} \text{ is } n\text{-supercompact"}.$$

Now note that we have $(V_\lambda)^M = V_\lambda$. It follows that it is true in V that κ_{m-1} is at least δ-n-supercompact for any $\delta < \lambda$. But the λ from the beginning is arbitrarily chosen, thus $\kappa_{m-1} = \kappa$ is n-supercompact, which proves (1).

It follows from (1) that in particular κ is supercompact, thus $\kappa \in C^{(2)}$, which proves (2).

For (3), we continue to work in the same setting. Define

$$\mathcal{U}' = \{X \subseteq \kappa_0 : \kappa_0 \in j(X)\},$$

note that the fact that $V_{\kappa_1} \models$ "κ_0 is n-supercompact" also implies that

$$\{\mu < \kappa_0 : V_{\kappa_0} \models \text{"}\mu \text{ is } n\text{-supercompact"}\} \in \mathcal{U}',$$

which gives

$$V_{\kappa_0} \models \text{"there is a proper class of } n\text{-supercompact cardinals"}.$$

By applying elementarity $(m-1)$-times and the fact that $V_{\kappa_{m-1}} = (V_{\kappa_{m-1}})^M$, it follows that

$$V_{\kappa_{m-1}} \models \text{"there is a proper class of } n\text{-supercompact cardinals"},$$

which gives (3).

Since $\kappa \in C^{(2)}$ and "μ is n-supercompact" is Π_2, it follows that it is true in V that κ is a limit of n-supercompact cardinals, which proves (4). $\square$

It follows that clause (2) of the above theorem is optimal. In particular, the least m-supercompact cardinal κ cannot be in $C^{(3)}$, otherwise V_κ satisfies the Σ_3-statement "there is an m-supercompact cardinal", which gives a real m-supercompact cardinal below κ, contradicting its minimality.

Thus the least m-supercompact cardinal κ is less than any cardinal in $C^{(3)}$. On the other hand, the least extendible cardinal, if it exists, is in $C^{(3)}$, so is greater than κ. On the other hand, the consistency strength of m-supercompact cardinals, for $m \geq 2$, is much greater than the strength of extendible cardinals, as shown in the next theorem.

To compare the strength of m-supercompact cardinals with familiar large cardinals, recall that κ is *m-huge with target sequence* $\kappa = \lambda_0, \ldots, \lambda_m$ if there is an elementary embedding $j : V \to M$ for some transitive M with $crit(j) = \kappa$, $j^i(\kappa) = \lambda_i$ for $0 \leq i \leq m$, and ${}^{\lambda_m}M \subseteq M$. κ is *m-superhuge* if κ is huge with a proper class of target sequence $\lambda_0, \ldots \lambda_m$. hugeness is characterized in terms of ultrafilters:

Proposition 2.1.6. (Theorem 24.8 in [21]) *For a natural number $m \geq 1$, a cardinal κ is m-huge if and only if $\kappa > \omega$ and there is a κ-complete normal ultrafilter over some $\mathcal{P}(\lambda)$ with $\kappa = \lambda_0 < \lambda_1 < \dots \lambda_m = \lambda$ so that for each $i < m$,*

$$\{x \in \mathcal{P}(\lambda) : ot(x \cap \lambda_{i+1}) = \lambda_i\} \in \mathcal{U}.$$

Note that the proof of the above proposition shows that the ultrafilter $\mathcal{U}$ can also be assumed to be on $\mathcal{P}_{\lambda_{m-1}}\lambda$.

We may strengthen hugeness by saying κ is $C^{(n)}$-m-huge, for some natural numbers $m \geq 1$ and n, if κ is m-huge with target sequence $\kappa = \lambda_0, \dots, \lambda_m$ such that $\lambda_1 \in C^{(n)}$. Similarly, κ is $C^{(n)}$-m-superhuge if κ is $C^{(n)}$-m-huge with a proper class of target sequence $\kappa = \lambda_0, \dots, \lambda_m$.

Theorem 2.1.7. *If $m \geq 2$ and κ is m-supercompact, then for any natural number n, V_κ is a model of ZFC plus "there is a proper class of $C^{(n)}$-$(m-1)$-superhuge cardinals".*

Proof. Suppose $m \geq 2$ and κ is m-supercompact, take some $\bar{\kappa} < \kappa$ and some elementary embedding $j : V_{\bar{\kappa}+1} \to V_{\kappa+1}$ with $j^m(crit(j)) = \kappa$. Let $crit(j) = \mu$, then we have $j^{m-1}(\mu) = \bar{\kappa}$. Note that since $j^{m-1}(\mu)$ is inaccessible, we have $\mathcal{P}(\mathcal{P}_{j^{m-2}(\mu)}j^{m-1}(\mu)) \subseteq V_{\bar{\kappa}+1}$. So we may define

$$\mathcal{U} = \{X \subseteq \mathcal{P}_{j^{m-2}(\mu)}j^{m-1}(\mu) : j``j^{m-1}(\mu) \in j(X)\},$$

it follows from standard verification that $\mathcal{U}$ is a $\mu = crit(j)$-complete normal ultrafilter over $\mathcal{P}_{j^{m-2}(\mu)}j^{m-1}(\mu)$. Moreover the fact that $ot(j``j^{m-1}(\mu) \cap j^{i+1}(\mu)) = j^i(\mu)$ for any $0 \leq i \leq (m-1)$ implies that

$$\{x \in \mathcal{P}_{j^{m-2}(\mu)}j^{m-1}(\mu) : ot(x \cap j^{i+1}(\mu)) = j^i(\mu)\} \in \mathcal{U}$$

for any $0 \leq i \leq (m-1)$, which shows that μ is $(m-1)$-huge with target sequence $\mu, j(\mu), \dots j^{m-1}(\mu)$. Define the normal ultrafilter on μ induced by j, namely

$$\mathcal{U}^* = \{X \subseteq \mu : \mu \in j(X)\},$$

since $\mathcal{U} \in V_{\kappa+1}$, it holds in $V_{\kappa+1}$ that μ is $(m-1)$-huge with target sequence $\mu, j(\mu), \dots j^{m-1}(\mu)$, so we have

$$T_0 \coloneqq \{\lambda < \mu : V_{\bar{\kappa}+1} \models \text{“}\lambda \text{ is } (m-1)\text{-huge with target sequence } \lambda, \mu, \dots j^{m-2}(\mu)\text{"}\} \in \mathcal{U}^*,$$

but the quoted statement above is true also in $V_{\kappa+1}$, so we get that for any $\lambda \in T_0$, since $j(\lambda) = \lambda$,

$$T_1 \coloneqq \{\delta < \mu : V_{\bar{\kappa}+1} \models \text{“}\lambda \text{ is } (m-1)\text{-huge with target sequence } \lambda, \delta, \mu, \dots j^{m-3}(\mu)\text{"}\} \in \mathcal{U}^*.$$

By repeatedly applying this argument we will get $T_0, \dots, T_{m-1} \in \mathcal{U}^*$ such that given any $\lambda_0, \dots, \lambda_{m-1}$ with $\lambda_i \in T_i$ for $0 \leq i \leq (m-1)$, we have λ_0 is $(m-1)$-huge with target sequences $\lambda_0, \dots, \lambda_{m-1}$.

Now since V_μ is a model of ZFC, we have for nay natural number n,

$$C^{(n)}_\mu = \{\alpha < \mu : V_\mu \models \text{“}\alpha \text{ is in } C^{(n)}\text{"}\}$$

is a club set in μ. Since $\mathcal{U}^*$ is a normal measure, we have $C^{(n)}_\mu \in \mathcal{U}^*$. Thus for any $0 \leq i \leq (m-1)$ we have $C^{(n)}_\mu \cap T_i \in \mathcal{U}^*$. Thus for any $\lambda_0 \in T_0$ and $\lambda_0 < \alpha < \mu$ we may pick $\lambda_i \in C_n^\mu \cap T_i$ all greater than α for $1 \leq i \leq (m-1)$. It follows that λ_0 is $C^{(n)}$-$(m-1)$-superhuge in V_μ, thus we have

$$V_\mu \models \text{"there is a proper class of } C^{(n)}\text{-}m\text{-superhuge cardinals"}.$$

Finally, by applying elementarity m-times, we have

$$V_\kappa \models \text{"there is a proper class of } C^{(n)}\text{-}m\text{-superhuge cardinals"},$$

as desired. □

Corollary. *If κ is 2-supercompact, then for any natural number n, V_κ is a model of ZFC plus "there is a proper class of $C^{(n)}$-superhuge cardinals".*

An upper bound of the strength of 2-supercompact cardinals is a hyperhuge cardinal, to which we now turn our attention.

2.2 Hyperhugeness

In this section we study the *m-hyperhuge cardinals*, together with their $C^{(n)}$-versions. As mentioned in section 2.1, m-hyperhuge cardinals were first introduced and known as $(m+1)$-fold supercompact cardinals in [32].

Hyperhuge cardinals have important applications, as they are first used by Toshimichi Usuba in [40] to prove his seminal result in *set-theoretic geology* that *the mantle is a ground of V*[*].

Definition 2.2.1. For a natural number m, a cardinal κ is λ-m-hyperhuge if there is an elementary embedding $j : V \to M$, M transitive, with $crit(j) = \kappa$, $\lambda < j(\kappa)$, and ${}^{j^m(\lambda)}M \subseteq M$. κ is *m-hyperhuge* if κ is λ-m-hyperhuge for all (equivalently, for a proper class of) λ.

κ is *λ-$C^{(n)}$-m-hyperhuge*, for a natural number n, if κ is λ-m-hyperhuge as above, and additionally $j(\kappa) \in C^{(n)}$. κ is *$C^{(n)}$-m-hyperhuge* if κ is λ-$C^{(n)}$-m-hyperhuge for all (equivalently, for a proper class of) λ.

Note that we allow $m = 0$, in which case m-hyperhuge cardinals are just supercompact cardinals. 1-hyperhuge cardinal will simply be called *hyperhuge cardinals.*

Here we state a lemma which is simple but will be used repeatedly.

Lemma 2.2.2. *If λ is an ordinal and M is a transitive model of ZFC, ${}^\delta M \subseteq M$ for any $\delta < \lambda$, and $M \models$ "$|V_\lambda| = \lambda$", then $V_\lambda = (V_\lambda)^M$, thus $|V_\lambda| = \lambda$.*

Proof. By induction on $\alpha < \lambda$ we show that $V_\alpha = (V_\alpha)^M$. Suppose this holds for α, then since $M \models$ "$|V_\alpha| < \lambda$", it is true in V that $|V_\alpha| < \lambda$, thus if $X \subseteq V_\alpha$ then $|X| < \lambda$, so $X \in M$. It follows that $V_{\alpha+1} = (V_{\alpha+1})^M$. The limit case is immediate. □

Proposition 2.2.3. *For $m \geq 1$, if κ is m-hyperhuge, then κ is $C^{(1)}$-m-hyperhuge.*

Proof. Take any $\lambda \geq \kappa$, and let $j : V \to M$ witnesses that κ is λ-m-hyperhuge. Since $\kappa \in C^{(1)}$, by elementarity we have $(j(\kappa) \in C^{(1)})^M$, namely $(|V_{j(\kappa)}| = j(\kappa))^M$. By Lemma 2.2.2 and ${}^{j(\lambda)}M \subseteq M$, we have $(V_{j(\kappa)})^M = V_{j(\kappa)}$ and $|V_{j(\kappa)}| = j(\kappa)$, thus indeed $j(\kappa) \in C^{(1)}$. □

[*]An inner model W is a *ground* of V if V is a set-forcing extension of W, and the *mantle* is the intersection of all grounds of V. For more see [17] and [40].

The above proposition does not hold when $m = 0$, in which case $C^{(n)}$-0-hyperhuge cardinals are also known as the *$C^{(n)}$-supercompact cardinals*, first introduced by Bagaria in [1]. In [20], Yair Hayut, Menachem Magidor and Alejandro Poveda studied the *Identity Crisis* of $C^{(n)}$-supercompact cardinals. They showed (assuming stronger large cardinal hypotheses) that, on the one hand, it is consistent that the first supercompact cardinal is less than the first $C^{(1)}$-supercompact cardinal, while on the other hand, it is consistent that the first strongly compact cardinal is the first $C^{(n)}$-supercompact cardinal, simultaneously for all n.

Interestingly, m-hyperhugeness can also be combinatorially characterized in terms of m-supercompact measures: m-supercompact cardinals are the *last* entries of the target sequences of m-supercompact measures, while m-hyperhuge cardinals are the *first* entries of the target sequences of $(m+1)$-supercompact measures.

Theorem 2.2.4. *For a natural number m and cardinals $\kappa < \lambda$, κ is λ-m-hyperhuge if and only if there are $\kappa_m < \lambda_m$ and an $(m+1)$-supercompact measure $\mathcal{U}$ for (κ_m, λ_m) with target sequences $\kappa = \kappa_0, \ldots, \kappa_m$ and $\lambda = \lambda_0, \ldots, \lambda_m$.*

Proof. For the forward direction, suppose $\kappa < \lambda$ and κ is λ-m-hyperhuge, witnessed by $j : V \to M$ with $crit(j) = \kappa$, $\lambda < j(\kappa)$ and ${}^{j^m(\lambda)}M \subseteq M$. Define

$$\mathcal{U} = \{X \subseteq \mathcal{P}(j^m(\lambda)) : j``j^m(\lambda) \in j(X)\},$$

it is not hard to verify that $\mathcal{U}$ is an $(m+1)$-supercompactness measure for $(j^m(\kappa), j^m(\lambda))$ with target sequences $\kappa, \ldots, j^m(\kappa)$ and $\lambda, \ldots, j^m(\lambda)$.

Conversely, suppose there are $\kappa_m < \lambda_m$ and an $(m+1)$-supercompact measure for (κ_m, λ_m) with target sequences $\kappa = \kappa_0, \ldots, \kappa_m$ and $\lambda = \lambda_0, \ldots, \lambda_m$. Then let $j : V \to Ult(V, \mathcal{U}) \cong M$ be the ultrapower embedding given by $\mathcal{U}$. It follows that $crit(j) = \kappa_0 = \kappa$, $j(\kappa) > \lambda_0 = \lambda$, $j^m(\lambda) = \lambda_m$ and ${}^{j^m(\lambda)}M \subseteq M$, witnessing that κ is λ-m-hyperhuge. $\square$

In light of the above characterization, we will also use the following terminology:

Definition 2.2.5. If $\kappa \leq \lambda$ and $\mathcal{U}$ is a $(m+1)$-supercompact measure for some (κ_m, λ_m) with target sequences $\kappa = \kappa_0, \ldots, \kappa_m$ and $\lambda = \lambda_0, \ldots, \lambda_m$, we will also say that $\mathcal{U}$ is an *m-hyperhuge measure for* (κ, λ) (with the same target sequence $\kappa = \kappa_0, \ldots, \kappa_m$ and $\lambda = \lambda_0, \ldots, \lambda_m$).

Corollary. *Given natural numbers $m \geq 1$, n, and cardinal κ. For $\lambda \geq \kappa$, κ is λ-$C^{(n)}$-m-hyperhuge if and only if there are $\kappa_m < \lambda_m$ and an m-hyperhuge measure $\mathcal{U}$ for (κ, λ) with target sequences $\kappa = \kappa_0, \ldots, \kappa_m$, $\lambda = \lambda_0, \ldots, \lambda_m$, and $\kappa_1 \in C^{(n)}$. We will sometimes call such a measure $\mathcal{U}$ a $C^{(n)}$-m-hyperhuge measure for (κ, λ).*

If $m = 0$, then the $C^{(n)}$-supercompact cardinals (i.e., $C^{(n)}$-0-hyperhuge cardinals) can be characterized via the existence of certain *long extenders*. It follows that for $n \geq 1$ and $\lambda \geq \kappa$, "κ is λ-$C^{(n)}$-supercompact" is Σ_{n+1}-expressible, and thus "κ is $C^{(n)}$-supercompact" is Π_{n+2}. For details see section 5 of [1].

Note that unlike "there is an m-supercompact measure for (κ, λ) with target sequences $\kappa_0, \ldots, \kappa_{m-1} = \kappa$ and $\lambda_0, \ldots, \lambda_{m-1} = \lambda$", which we have seen is Δ_2, "there is an m-hyperhuge measure for (κ, λ) with target sequences $\kappa = \kappa_0, \ldots, \kappa_m$ and $\lambda_0, \ldots, \lambda_m$" is Σ_2, but not Π_2.

Thus it follows from the above corollary that for $n \geq 1$ and m, "κ is λ-$C^{(n)}$-m-hyperhuge" is Σ_{n+1}, and "κ is $C^{(n)}$-m-hyperhuge" is Π_{n+2}. In particular "κ is m-hyperhuge" is Π_3.

As in the case of m-supercompact cardinals, the $C^{(n)}$-m-hyperhuge cardinals form a proper hierarchy in a strong sense, where usually if we decrease either m or n, we get a notion both weaker in consistency strength and smaller in size of the least instance.

Proposition 2.2.6. *For natural numbers $m, n \geq 1$, if κ is a $C^{(n)}$-m-hyperhuge cardinal, then the following hold:*

1. *κ is $C^{(n')}$-m'-hyperhuge for any $m' \leq m$ and $n' \leq n$,*

2. *$\kappa \in C^{(n+2)}$,*

3. *$V_\kappa \models$ ZFC+"there is a proper class of $C^{(n)}$-m'-hyperhuge cardinals", and κ is a limit of $C^{(n)}$-m'-hyperhuge cardinals, for any $m' < m$, and*

4. *If $n \geq 2$, $V_\kappa \models$ ZFC+"there is a proper class of $C^{(n')}$-m-hyperhuge cardinals", and κ is a limit of $C^{(n')}$-m-hyperhuge cardinals, for any $n' < n$.*

Proof. (1) follows simply by definition.

For (2), note that if $\lambda > \kappa$ and $|V_\lambda| = \lambda$, then if $j : V \to M$ witnesses the $C^{(n)}$-m-hyperhugeness of κ, with $\lambda < j(\kappa) \in C^{(n)}$, it follows that $(V_{j(\lambda)})^M = V_{j(\lambda)}$, thus $j{\restriction}V_\lambda : V_\lambda \to V_{j(\lambda)}$ is elementary and witness that κ is λ-$C^{(n)}$-extendible. Thus κ is $C^{(n)}$-extendible, and is in $C^{(n+2)}$ by Proposition 3.4 of [1].

For (3), note that κ is clearly almost m-huge, so (3) follows from Theorem 2.2.9 below.

For (4), note that we require $n \geq 2$, because as Proposition 2.2.3 shows, m-hyperhugeness, $C^{(0)}$-m-hyperhugeness and $C^{(1)}$-m-hyperhugeness are all the same. Now to prove (4), if $n \geq 2$ and $n' < n$, then as remarked before, "κ is $C^{(n')}$-m-hyperhuge" is $\Pi_{n'+2}$-expressible, so "there exists some $C^{(n')}$-m-hyperhuge cardinal" is $\Sigma_{n'+3}$, therefore also Σ_{n+2}, so by (2) this is true in V_κ. Now given $\alpha < \kappa$ such that $V_\kappa \models$ "α is $C^{(n')}$-m-hyperhuge", the statement "there exists some $C^{(n')}$-m-hyperhuge cardinal greater than α" is Σ_{n+2} with the parameter α, and is true in V witnessed by κ itself, so the statement is again true in V_κ. It follows that there are unboundedly many $\alpha < \kappa$ such that $V_\kappa \models$ "α is $C^{(n')}$-m-hyperhuge", and α is $C^{(n')}$-m-hyperhuge in V, as desired. □

In particular, hyperhuge cardinals are in $C^{(3)}$. Given that "there exists a $C^{(n)}$-m-hyperhuge cardinal" is Σ_{n+3}, it follows that clause (2) of the previous theorem is optimal.

Proposition 2.2.7. *For natural numbers $m, n \geq 1$, if there is an $C^{(n+2)}$-m-hyperhuge cardinal, then there is a proper class of $C^{(n)}$-m-hyperhuge cardinals.*

Proof. Given such $m, n \geq 1$ and some $C^{(n+2)}$-m-hyperhuge κ, by Proposition 2.2.6, it is true in V_κ that there is a proper class of $C^{(n)}$-m-hyperhuge cardinals. Note that "there is a proper class of $C^{(n)}$-m-hyperhuge cardinals" is Π_{n+4}-expressible, so it holds in V given that $\kappa \in C^{(n+4)}$. □

Let us observe that, if $m, n \geq 1$, then a $C^{(n+1)}$-m-hyperhuge cardinal κ does not imply that there is some $C^{(n)}$-m-hyperhuge cardinal $\lambda > \kappa$, for if it does, then the $C^{(n+1)}$-m-hyperhugeness of κ would reflects down to V_λ, given that $\lambda \in C^{(n+2)}$ and "κ is $C^{(n+1)}$-m-hyperhuge" is Π_{n+3}, hence we would be proving the consistency of a $C^{(n+1)}$-m-hyperhuge cardinal.

The next proposition shows that, in terms of consistency strength, m-hyperhuge cardinals are strictly stronger than $(m+1)$-supercompact cardinals.

Proposition 2.2.8. *For a natural number $m \geq 1$, if κ is m-hyperhuge, then*

1. *κ is $(m+1)$-supercompact and a limit of $(m+1)$-supercompact cardinals.*

2. *V_κ is a model of ZFC plus "there is a proper class of $(m+1)$-supercompact cardinals".*

Proof. For any $\delta > \kappa$, let $\lambda > \delta$ with $|V_\lambda| = \lambda$, and let $\mathcal{U}$ be an m-hyperhuge measure for (κ, λ), with target sequences $\kappa = \kappa_0, \ldots, \kappa_m$ and $\lambda = \lambda_0, \ldots, \lambda_m$. Let $j : V \to Ult(V, \mathcal{U}) \cong M$ be the ultrapower embedding. We have $crit(j) = \kappa$, $\lambda < j(\kappa)$ and ${}^{j^m(\lambda)}M \subseteq M$. By applying elementarity and Lemma 2.2.2 repeatedly, it follows that $|V_{\lambda_i}| = \lambda_i$ for any $0 \leq i \leq m$, and $V_{j^m(\lambda)} = (V_{j^m(\lambda)})^M$.

Note that since $j^m(\delta) < j^m(\lambda)$, letting

$$\mathcal{U}^* = \{X \subseteq \mathcal{P}(j^m(\delta)) : j``j^m(\delta) \in j(X)\},$$

it follows that $\mathcal{U}^*$ is an $(m+1)$-supercompact measure for $(j^m(\kappa), j^m(\delta))$, and $\mathcal{U}^*$ is in $V_{j^m(\lambda)} = (V_{j^m(\lambda)})^M$. By elementarity, there is an $(m+1)$-supercompact measure for $(j^{m-1}(\kappa), j^{m-1}(\delta))$ in $V_{j^{m-1}(\lambda)} \subseteq M$. By applying elementarity $(m-1)$ more times, there is an $(m+1)$-supercompact measure for (κ, δ). Since $\delta > \kappa$ is arbitrary, it follows from Theorem 2.1.4 that κ is $(m+1)$-supercompact. Moreover, since "there is an $(m+1)$-supercompact cardinal" is Σ_3, we have by Proposition 2.2.6 that there is some $(m+1)$-supercompact cardinal $\alpha < \kappa$. Again "there is some $(m+1)$-supercompact cardinal greater than α" is Σ_3 with the parameter α, so it is reflected in V_κ. Thus there are unboundedly many $\alpha < \kappa$ that are $(m+1)$-supercompact in V_κ, and also in V. Thus (1) and (2) follow. □

Now we give an upper bound on the strength of $C^{(n)}$-m-hyperhuge cardinals. We say a cardinal κ is *almost m-huge*, for $m \geq 1$, if there is an elementary embedding $j : V \to M$ with $crit(j) = \kappa$ and ${}^\lambda M \subseteq M$ for any $\lambda < j^m(\kappa)$.

Theorem 2.2.9. *For $m \geq 1$, if κ is almost m-huge, then for any natural number n, V_κ is a model of ZFC plus "there is a proper class of $C^{(n)}$-$(m-1)$-hyperhuge cardinals".*

Proof. If $m = 1$, then $C^{(n)}$-$(m-1)$-hyperhuge cardinals are the $C^{(n)}$-supercompact cardinals, and our conclusion follows from Theorem 2.21 of [38].

Now suppose $m \geq 2$ and κ is almost m-huge, witnessed by some elementary embedding $j : V \to M$ with $crit(j) = \kappa$ and ${}^\lambda M \subseteq M$ for any $\lambda < j^m(\kappa)$.

For any $\lambda < j(\kappa)$, we have $j^{m-1}(\lambda) < j^m(\kappa)$, so by the closure property of M we have $j``j^{m-1}(\lambda) \in M$, so it makes sense to define

$$\mathcal{U} = \{X \subseteq \mathcal{P}(j^{m-1}(\lambda)) : j``j^{m-1}(\lambda) \in j(X)\}$$

which is an m-supercompact measure for $(j^{m-1}(\kappa), j^{m-1}(\lambda))$, namely an $(m-1)$-hyperhuge measure for (κ, λ), with target sequences $\kappa, \ldots, j^{m-1}(\kappa)$ and $\lambda, \ldots, j^{m-1}(\lambda)$. It follows that $\mathcal{U} \in V_{j^m(\kappa)} = (V_{j^m(\kappa)})^M$, by Lemma 2.2.2. So we have

$M \models$ "$\forall \lambda < j(\kappa)$ there is an $(m-1)$-hyperhuge measure for (κ, λ) in $V_{j^m(\kappa)}$ with some target sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of length $(m-1)$ with $\alpha_0 = \kappa$ and $\alpha_1 = j(\kappa)$",

Let $\mathcal{U}^*$ be the induced normal ultrafilter on κ by j, namely $\mathcal{U}^* = \{X \subseteq \kappa : \kappa \in j(X)\}$. By elementarity, we have

$D = \{\mu < \kappa : \forall \lambda < \kappa$ there is an $(m-1)$-hyperhuge measure for (μ, λ) in $V_{j^{m-1}(\kappa)}$ with some target sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of length $(m-1)$ with $\alpha_0 = \mu$ and $\alpha_1 = \kappa\} \in \mathcal{U}^*$,

Since all the measures mentioned above are in M, we have for any $\mu \in D$ that

$D_\mu = \{\delta < \kappa : \forall \lambda < \delta$ there is an $(m-1)$-hyperhuge measure for (μ, λ) in $V_{j^{m-2}(\kappa)}$ with some target sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of length $(m-1)$ with $\alpha_0 = \mu$ and $\alpha_1 = \delta\} \in \mathcal{U}^*$.

By letting $C^{(n)}_\kappa = \{\alpha < \kappa : V_\kappa \models$ "$\alpha \in C^{(n)}$"$\}$, we get that $D_\mu \cap C^{(n)}_\kappa \in \mathcal{U}^*$. Now it follows by applying elementarity $(m-2)$ more times that we will get that for any $\mu \in D$ and $\mu < \beta < \kappa$, there is some $\beta < \delta \in C^{(n)}_\kappa$ such that there is an $(m-1)$-hyperhuge measure for (μ, δ) in V_κ with some target sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of length $(m-1)$ with $\alpha_0 = \mu$ and $\alpha_1 = \delta$. It follows that $V_\kappa \models$ "there is a proper class of $C^{(n)}$-$(m-1)$-hyperhuge cardinals", as desired. $\square$

Corollary. *For any natural number n and cardinal κ, if κ is almost 2-huge, then V_κ is a model of ZFC plus "there is a proper class of $C^{(n)}$-hyperhuge cardinals".*

In particular, so far we have shown that in terms of consistency strength,

$$\begin{aligned}&\text{supercompact} < \text{almost huge} < C^{(n)}\text{-superhuge}\\ &< \text{2-supercompact} < \text{hyperhuge} < \text{almost 2-huge}.\end{aligned}$$

2.3 Higher extendibility

In this section we consider m-fold extendible cardinals (which were first introduced in [32]) and their $C^{(n)}$ versions. Our definition of m-fold extendibility is slightly different but equivalent to the definition in [32].

Definition 2.3.1. For a natural number $m \geq 1$, a cardinal κ is *η-m-fold extendible* if there is some elementary embedding $j : V_{j^{m-1}(\kappa+\eta)} \to V_\delta$ for some δ, with $crit(j) = \kappa$ and $\kappa + \eta < j(\kappa)$.

κ is *m-fold extendible* if it is η-m-fold extendible for all (equivalently, a proper class of) η.

Moreover, for a natural number n, κ is *η-$C^{(n)}$-m-fold extendible* if it is η-m-fold extendible as above, but additionally we have $j(\kappa) \in C^{(n)}$. κ is *$C^{(n)}$-m-fold extendible* if it is η-$C^{(n)}$-m-fold extendible for all η.

Note that the 1-fold extendible cardinals are just extendible cardinals. Moreover, it is not difficult to see that m-fold extendible cardinals, $C^{(0)}$-m-fold extendible cardinals, and $C^{(1)}$-m-fold extendible cardinals are the same.

Both m-hyperhuge cardinals and m-fold extendible cardinals, without the $C^{(n)}$ variants, were studied by Sato in [32], but the precise relation between the two was unknown. Sato asked the following question*:

Question 2.3.2. Are the statements "there is an m-hyperhuge cardinal" and "there is an $(m+1)$-fold extendible cardinal" equiconsistent, for $m \geq 1$?

Based on the analysis of the reflective properties of the $C^{(n)}$-m-hyperhuge cardinals in the previous section, it follows that the answer is positive, and more: it turns out that the two notions, even their $C^{(n)}$ versions, are not only equiconsistent, but provably equivalent.

Theorem 2.3.3. *For natural numbers $m, n \geq 1$, a cardinal κ is $C^{(n)}$-m-hyperhuge if and only if κ is $C^{(n)}$-$(m+1)$-fold extendible.*

Proof. For the forward direction, suppose κ is $C^{(n)}$-m-hyperhuge, but not $C^{(n)}$-$(m+1)$-fold extendible. Let $\lambda = |V_\lambda|$ be such that κ is not λ-$C^{(n)}$-$(m+1)$-fold extendible, and let $\mathcal{U}$ be an m-hyperhuge measure for (κ, λ), with target sequences $\kappa = \kappa_0, \ldots, \kappa_m$ and $\lambda = \lambda_0, \ldots, \lambda_m$, with $\kappa_1 \in C^{(n)}$.

*Phrased in our terms, where we use "m-hyperhuge" instead of "$(m+1)$-fold supercompact".

Let $j : V \to Ult(V,\mathcal{U}) \cong M$ be the ultrapower embedding, we have $crit(j) = \kappa$, $\lambda < j(\kappa) = \kappa_1 \in C^{(n)}$, and ${}^{j^m(\lambda)}M \subseteq M$. By elementarity, $M \models$ "$|V_{j(\lambda)}| = j(\lambda)$", so by Lemma 2.2.2 $|V_{j(\lambda)}| = j(\lambda)$, thus again $M \models$ "$|V_{j^2(\lambda)}| = j^2(\lambda)$", which is true in V if $m \geq 2$. By repeating the same argument we have $(V_{j^m(\lambda)})^M = V_{j^m(\lambda)}$ and $|V_{j^m(\lambda)}| = j^m(\lambda)$.

Now note that the statement "κ is not λ-$C^{(n)}$-$(m+1)$-fold extendible" is Π_{n+1} with parameters κ and λ, since it is equivalent to

$$\neg\exists\delta, j(j : V_{j^m(\lambda)} \to V_\delta \text{ is elementary, } crit(j) = \kappa, \text{ and } \lambda < j(\kappa) \in C^{(n)}).$$

Since $j(\kappa) \in C^{(n)}$, the above statement is reflected in $V_{j(\kappa)} = (V_{j(\kappa)})^M$. By Proposition 2.2.6, $\kappa \in C^{(n+1)}$, so by elementarity $M \models$ "$j(\kappa) \in C^{(n+1)}$", thus

$$M \models \text{"}\kappa \text{ is not } \lambda\text{-}C^{(n)}\text{-}(m+1)\text{-fold extendible"}.$$

However, we have $j{\restriction}V_{j^m(\lambda)} : V_{j^m(\lambda)} \to (V_{j^{m+1}(\lambda)})^M \in M$, which is after all the required embedding in M to witness the λ-$C^{(n)}$-$(m+1)$-fold extendibility of κ. Contradiction.

For the converse direction, suppose κ is $C^{(n)}$-$(m+1)$-fold extendible and $\lambda = \kappa + \lambda$. Let $\eta > \lambda$, and let $j : V_{j^m(\eta)} \to V_\delta$, with $crit(j) = \kappa$ and $\eta < j(\kappa) \in C^{(n)}$.

Since $\lambda < \eta$, $j^m(\lambda) < j^m(\eta)$, we may define

$$\mathcal{U} = \{X \subseteq \mathcal{P}(j^m(\lambda)) : j``j^m(\lambda) \in j(X)\}$$

and one can verify that this is a $C^{(n)}$-m-hyperhuge measure for (κ, λ), with target sequences $\kappa, \ldots, j^m(\kappa)$ and $\lambda, \ldots, j^m(\lambda)$. Since λ can be chosen to be arbitrarily large, κ is $C^{(n)}$-m-hyperhuge. □

Note that the above is also true if $n = 0$. Also note that if $m = 0$, the converse direction of the above theorem says that $C^{(n)}$-extendible cardinals are $C^{(n)}$-supercompact, which also holds, by Corollary 2.29 of [38]. Also, the forward direction for $m = 0$ is provably false. Poveda showed in [28] that for any n, any $C^{(n)}$-extendible cardinal is a limit of $C^{(n)}$-supercompact cardinals, which also implies that consistencywise, $C^{(n)}$-extendible cardinals are stronger than $C^{(n)}$-supercompact cardinals.

It follows from the above theorem that our analysis of properties and consistency strength of $C^{(n)}$-m-hyperhuge cardinals in the previous section carry over directly to $C^{(n)}$-$(m+1)$-fold extendible cardinals.

Chapter 3

Correspondence

As remarked in Chapter 1, the research program of structural reflection (for a survey see [2]) has shown that the same pattern of correspondence holds between different forms of structural reflection principles and large cardinal axioms of various levels of consistency strength, as emphasized in [7]. However, although the region between supercompactness and Vopěnka's Principle is the first region in which such a pattern of correspondence was discovered (initially presented in [1]), subsequent developments have centered on weaker principles. Although stronger forms of structural reflection principles were proposed (see [6]), the same pattern of correspondence between them and large cardinals no longer holds.

In this chapter we show that the higher structural reflection principles introduced in chapter 1, namely LSR and CSR, which we have argued are natural and motivated principles, correspond to the very large cardinals we introduced in chapter 2 in exactly the same pattern desired, advancing the pattern of correspondence to large cardinals of much greater strength than before. In particular, Π_1-LSR and Π_1-CSR correspond to 2-supercompact cardinals as Π_1-SR corresponds to supercompact cardinals, and Π_{n+1}-LSR and Π_{n+1}-CSR correspond to $C^{(n)}$-2-fold extendible cardinals as Π_{n+1}-SR corresponds to $C^{(n)}$-extendible cardinals, for all $n \geq 1$.

3.1 The Π_1 case

LVP(Π_1) is very strong, implying the existence of some 2-supercompact cardinal, thus by Theorem 2.1.7 has stronger strength than a proper class of $C^{(n)}$-superhuge cardinals, for all n.

Theorem 3.1.1.

1. *If* LVP(Π_1) *holds, then there exists a* 2*-supercompact cardinal.*

2. *If* LVP($\boldsymbol{\Pi_1}$) *holds, then there exists a proper class of* 2*-supercompact cardinals.*

Proof. For (1): suppose LVP(Π_1) holds, but there are no 2-supercompact cardinals. Then define the function $F : Ord \to Ord$ as in Theorem 2.1.4, namely $F(\alpha)$ = the least $\beta > \alpha$ such that there is no 2-supercompact measure $\mathcal{U}$ for (α, β), provided that there is any such β. Otherwise let $F(\alpha) = 0$. Define a class $\mathcal{D}$ of ordinals such that $\gamma \in \mathcal{D}$ if and only if α is closed under F, namely

$$\forall \alpha < \gamma \exists \beta < \gamma F(\alpha) = \beta.$$

Thus under our assumption we have $\alpha < F(\alpha)$ for any ordinal α.

Now define a class $\mathcal{C}$ of structures of the form $(V_\alpha, \in, \alpha, \delta, f)$ such that $\delta < \alpha \in C^{(1)}$, α is a limit point of $\mathcal{D}$, and f is a function with domain ω and range unbounded in α. We claim that $\mathcal{C}$ is Π_1-definable without parameters. To see this, note that $X \in \mathcal{C}$ if and only if X is of the form $(X_0, X_1, X_2, X_3, X_4)$ such that

(i) X_2, X_3 are ordinals and $X_3 \in X_2$,

(ii) $X_2 \in C^{(1)}$,

(iii) $X_0 = V_{X_2}$,

(iv) $X_1 = \in \restriction X_0$,

(v) $\exists x \in X_0(x = \omega \wedge dom(f) = x \wedge \forall y \in X_2 \exists z \in x(y < f(z)))$, and

(vi) $\forall x \in X_2 \exists y \in X_2(x \in y \wedge X_0 \models \text{“}y \in \mathcal{D}\text{”})$.

The point is that for any ordinal $\eta < \alpha$, $\eta \in \mathcal{D}$ if and only if $V_\alpha \models \text{“}\eta \in \mathcal{D}\text{”}$, since the relevant normal ultrafilters are all in V_α. The role of f is to ensure that α is of countable cofinality.

Claim 3.1.2. $\mathcal{C}$ is a proper class.

Proof. For any ordinal ξ, we describe a function G as follows: let $\xi_0 = \xi$, and given ξ_i, let $\xi_i^0 \geq \sup_{\alpha \leq \xi_i} F(\alpha)$, and let ξ_i^1 be the least element of $C^{(1)}$ above ξ_i. Then set $\xi_{i+1} = max\{\xi_i^0, \xi_i^1\}$. Let $G(\xi) = \sup_{i\in\omega} \xi_i \in \mathcal{D}$. Let $\alpha = \{G^i(\xi) : i \in \omega\}$. Then we have $cf(\alpha) = \omega$ and there is some structure $X \in \mathcal{C}$ of rank $\alpha > \xi$. Since ξ is arbitrary, this implies that $\mathcal{C}$ forms a proper class. $\square$

Thus by LVP(Π_1), there are $\alpha \neq \beta$ with $\mathcal{C} \cap (V_{\alpha+1} \setminus V_\alpha) \neq \phi$ and $\mathcal{C} \cap (V_{\beta+1} \setminus V_\beta) \neq \phi$ such that for any $B \in \mathcal{C}$ of rank β there is some $A \in \mathcal{C}$ of rank α and an elementary embedding $j : A \to B$. Thus we have $\alpha < \beta$. Now consider the structure $(V_\beta, \in, \beta, \alpha, f) \in \mathcal{C}$, where f is any cofinal function from ω to β. It follows that there is some $(V_\alpha, \in, \alpha, \delta, g) \in \mathcal{C}$ with an elementary embedding

$$j : (V_\alpha, \in, \alpha, \delta, g) \to (V_\beta, \in, \beta, \alpha, f).$$

Note that since $\delta < \alpha = j(\delta)$, we have $crit(j) \leq \delta$. But $crit(j)$ cannot be δ, since we have

$$(V_\beta, \in, \beta, \alpha, f) \models cf(\alpha) = \omega,$$

so by elementarity

$$(V_\alpha, \in, \alpha, \delta, g) \models cf(\delta) = \omega,$$

but $crit(j)$ is inaccessible. Thus let $crit(j) = \mu$, we have $\mu < \delta$.

Claim 3.1.3. $\{\delta, \mu, j(\mu)\} \subseteq lim(\mathcal{D})$.

Proof. To see that $\delta \in lim(\mathcal{D})$, note that since $\alpha \in lim(\mathcal{D})$, we have $(V_\beta, \in, \beta, \alpha, f) \models$ “$\alpha \in lim(\mathcal{D})$”, so by elementarity we have $(V_\alpha, \in, \alpha, \delta, g) \models$ “$\delta \in lim(\mathcal{D})$”, which is true in V.

To see that $\mu \in lim(\mathcal{D})$, suppose not. Then since $\mathcal{D}$ is a club class, there must be some $\eta < \mu$ such that there is no element in $\mathcal{D}$ strictly between η and μ. But then note that since $\mathcal{D}$ is definable in both $(V_\alpha, \in, \alpha, \delta, g)$ and $(V_\beta, \in, \beta, \alpha, f)$, and the least element $\xi \in \mathcal{D}$ above μ is below δ, we have by elementarity that $j(\xi)$ is the least element of $\mathcal{D}$ above $j(\eta) = \eta$, which is ξ, but then we have $j \restriction V_{\xi+2} : V_{\xi+2} \to V_{\xi+2}$ is a nontrivial elementary embedding, contradicting the *Kunen Inconsistency.*

Lastly, since $(V_\alpha, \in, \alpha, \delta, g) \models$ “$\mu \in lim(\mathcal{D})$”, we have $j(\mu) \in lim(\mathcal{D})$ by elementarity. $\square$

Now let $\eta = min\{\delta, j(\mu)\}$. Since $\delta, j(\mu) \in \mathcal{D}$, it follows that there is some $\lambda < \eta$ such that $F(\mu) = \lambda$, namely there is no 2-supercompact measure $\mathcal{U}$ for (μ, λ). But since $\lambda < \delta$, we have $j(\lambda) < j(\delta) = \alpha$. Moreover we have $\lambda < j(\mu)$, so if we define

$$\mathcal{U} = \{X \subseteq \mathcal{P}(j(\lambda)) : j``j(\lambda) \in j(X)\},$$

then one can verify that $\mathcal{U}$ is a 2-supercompact measure $\mathcal{U}$ for $(j(\mu), j(\lambda))$, with target sequences $\mu, j(\mu)$ and $\lambda, j(\lambda)$. Since $\mathcal{U} \in V_\beta$, by the elementarity of j we have

$$(V_\alpha, \in, \alpha, \delta, g) \models \text{“There is a 2-supercompact measure } \mathcal{U} \text{ for } (\mu, \lambda)\text{”}$$

which is true in V, but this contradicts the definition of F. Thus there must indeed exist some 2-supercompact cardinal.

(2): Suppose that the class of 2-supercompact cardinals is bounded in Ord, then let η be such that there is no 2-supercompact cardinal above η. Now argue similarly as in (1): define $\mathcal{D}$ to be the class of limit ordinals γ such that

$$\eta < \gamma \wedge \forall \alpha < \gamma[(\eta < \alpha) \to (\exists \beta < \gamma F(\alpha) = \beta)].$$

Then proceed as before, and define the class $\mathcal{C}$ of structures of the form $(V_\alpha, \in, \alpha, \delta, \eta, f)$ such that $\eta < \delta < \alpha$, $\alpha \in C^{(1)} \cap lim(\mathcal{D})$, and f is a function with domain ω and range unbounded in α. Again $\mathcal{C}$ is Π_1-definable with the parameter η, and can be shown to be a proper class as before. Thus by LVP($\mathbf{\Pi_{n+1}}$) there are $\alpha < \beta$ and $(V_\alpha, \in, \alpha, \delta, \eta, g), (V_\beta, \in, \beta, \alpha, \eta, f) \in \mathcal{C}$ with some elementary embedding

$$j : (V_\alpha, \in, \alpha, \delta, \eta, g) \to (V_\beta, \in, \beta, \alpha, \eta, f).$$

Arguing as in (1), and noting the fact that $j(\eta) = \eta$, we have $crit(j) = \mu > \eta$. Similarly we have some $\lambda < min\{j(\mu), \delta\}$ such that $F(\mu) = \lambda$ but there is some 2-supercompact measure $\mathcal{U}$ for (μ, λ), contradiction. $\square$

Conversely, a 2-supercompact cardinal κ implies that CSR($\kappa, \mathbf{\Sigma_2}$) holds.

Theorem 3.1.4. *If κ is 2-supercompact, then* CSR($\kappa, \mathbf{\Sigma_2}$) *holds.*

Proof. Suppose that κ is 2-supercompact but CSR($\kappa, \mathbf{\Sigma_2}$) fails. Let $\lambda > \kappa$ be such that V_λ is sufficiently correct (for example $\lambda \in C^{(5)}$ is more than enough) so that $V_\lambda \models$ “CSR($\kappa, \mathbf{\Sigma_2}$) fails”.

By Theorem 2.1.4, let $\mathcal{U}$ be a 2-supercompact measure for (κ, λ), with some target sequences $\bar{\kappa}, \kappa$ and $\bar{\lambda}, \lambda$. Let $j : V \to Ult(V, \mathcal{U}) \cong M$ be the conrresponding ultrapower embedding. We then have $crit(j) = \bar{\kappa}$, $\bar{\lambda} < j(\bar{\kappa}) = \kappa$, $j(\bar{\lambda}) = \lambda$, and ${}^\lambda M \subseteq M$. Since $|V_\lambda| = \lambda$ we have $V_\lambda = (V_\lambda)^M$, so

$$M \models \text{“}V_\lambda \models \text{“CSR}(\kappa, \mathbf{\Sigma_2}) \text{ fails”}\text{”}$$

hence by the elementarity of j we have $V_{\bar{\lambda}} \models$“CSR($\bar{\kappa}, \mathbf{\Sigma_2}$) fails”. So there are $b_0, \ldots, b_{k-1} \in V_{\bar{\kappa}}$ and some Σ_2-formula ϕ such that

$$V_{\bar{\lambda}} \models \text{“}\phi(X, b_0, \ldots, b_{k-1}) \text{ defines a class of structures of the same type”},$$

(though it does not immediately follow that the quoted statement is true in V), yet there is some $B \in V_{\bar{\lambda}}$ such that

$V_{\bar{\lambda}} \models$ “$\phi(B, b_0, \ldots, b_{k-1})$ holds and
there is no $A \in V_{\bar{\kappa}}$ with $\phi(A, b_0, \ldots, b_{k-1})$ holds that captures B”.

By elementarity and the fact that $j(b_i) = b_i$ for all $0 \leq i \leq (k-1)$, it is also true in V_λ that $\phi(X, b_0, \ldots, b_{k-1})$ defines a class of structures of the same type. Moreover, since $M \models$ "$\lambda \in C^{(1)}$", we have $\bar{\lambda} \in C^{(1)}$ by elementarity, so by the upward-absoluteness of Σ_2-formula, we have $\phi(B, b_0, \ldots, b_{k-1})$ holds in V, and also in V_λ. By elementarity we have

$$V_\lambda \models \text{“}\phi(j(B), b_0, \ldots, b_{k-1}) \text{ holds and}$$
$$\text{there is no } A \in V_\kappa \text{ with } \phi(A, b_0, \ldots, b_{k-1}) \text{ holds that captures } j(B)\text{”}$$

and it follows that B and $j(B)$ are of the same type. Since $B \in V_{\bar{\lambda}} \subseteq V_\kappa$, it follows that B does not capture $j(B)$, so there is some $y \in j(B)$ such that there is no elementary embedding $k : B \to j(B)$ with $y \in ran(k)$. By elementarity we have

$$M \models \text{“There is no elementary embedding } k : j(B) \to j^2(B) \text{ with } j(y) \in ran(k)\text{”}$$

but note that since $|j(B)| < |V_\lambda| = \lambda$ and ${}^\lambda M \subseteq M$, we have $k = j \restriction j(B) : j(B) \to j^2(B)$ is elementary, and is in M. But we have $k(y) = j(y)$, so $y \in ran(k)$ so k is the elementary embedding that is supposed to be missing in M, a contradiction. Thus CSR($\kappa, \mathbf{\Sigma_2}$) must hold. $\square$

Next we show that in general CVP implies LVP:

Proposition 3.1.5. *For any natural number $n \geq 1$, if* CVP($\mathbf{\Pi_n}$) *holds, then* LVP($\mathbf{\Pi_n}$) *holds. Similarly if* CVP(Π_n) *holds, then* LVP(Π_n) *holds.*

Proof. Suppose CVP($\mathbf{\Pi_n}$) holds. To show that LVP($\mathbf{\Pi_n}$) holds, given a proper class of structures $\mathcal{C} = \{B : \phi(B, b_0, \ldots, b_{k-1})\}$, where $\phi(x, y_0, \ldots, y_{k-1})$ is Π_n, we must find $\alpha \neq \beta$ with $\mathcal{C} \cap (V_{\alpha+1} \setminus V_\alpha) \neq \phi$ and $\mathcal{C} \cap (V_{\beta+1} \setminus V_\beta) \neq \phi$ for any $B \in \mathcal{C}$ of rank β there is some $A \in \mathcal{C}$ of rank α and an elementary embedding $j : A \to B$. Define the class $\mathcal{C}^*$ of structures of the form $(V_\delta, \in, \delta, \alpha, b_0, \ldots, b_{k-1})$, where $\delta > \alpha$ are ordinals, δ is the least element of $C^{(n)}$ above α, and $\mathcal{C} \cap (V_{\alpha+1} \setminus V_\alpha) \neq \phi$. $\mathcal{C}^*$ is Π_n-definable with the parameters $b_0, \ldots, b_{k-1}$, since $X \in \mathcal{C}^*$ if and only if $X = (X_0, X_1, X_2, X_3, X_4, \ldots, X_{k+3})$, where

(i) X_2, X_3 are ordinals, with $X_3 < X_2$,

(ii) $X_0 = V_{X_2}$,

(iii) $X_1 = \in \restriction X_0$,

(iv) $X_{4+i} = b_i$ for $0 \leq i \leq k-1$,

(v) $X_2 \in C^{(n)} \wedge \forall \beta \in X_2 (X_3 < \beta \to X_0 \models \text{“}\beta \notin C^{(n)}\text{”})$, and

(vi) $X_0 \models$ "$\exists B(rank(B) = X_3 \wedge \phi(B, X_4, \ldots, X_{k+3})$ holds".

Since $\mathcal{C}$ is a proper class, it is clear that $\mathcal{C}^*$ is also a proper class, thus by CVP($\mathbf{\Pi_n}$) there are $X \neq Y \in \mathcal{C}^*$ such that X captures Y, where X is of the form $(V_\delta, \in, \delta, \alpha, b_0, \ldots, b_{k-1})$, and Y is of the form $(V_\mu, \in, \mu, \beta, b_0, \ldots, b_{k-1})$. It follows that $\alpha \neq \beta$, otherwise we have $\delta = \mu$ by the definition of $\mathcal{C}^*$, which implies that $X = Y$. But now for any $B \in \mathcal{C}$ of rank β, we have $B \in V_\mu$, and so there is an elementary embedding

$$j : (V_\delta, \in, \delta, \alpha, b_0, \ldots, b_{k-1}) \to (V_\mu, \in, \mu, \beta, b_0, \ldots, b_{k-1})$$

with $B \in ran(j)$. But then letting $j(A) = B$, we have

$$(V_\mu, \in, \mu, \beta, b_0, \ldots, b_{k-1}) \models \text{“}rank(B) = \beta \wedge \phi(B, b_0, \ldots, b_{k-1}) \text{ holds”}$$

so by elementarity

$$(V_\delta, \in, \delta, \alpha, b_0, \ldots, b_{k-1}) \models \text{“}rank(A) = \alpha \wedge \phi(A, b_0, \ldots, b_{k-1}) \text{ holds”}$$

which implies that $A \in \mathcal{C}$. Moreover, $j \restriction A : A \to B$ is elementary. It follows that α, β are as desired, so LVP($\mathbf{\Pi_n}$) holds.

That CVP(Π_n) implies LVP(Π_n) can be proved similarly, without considering parameters. □

Corollary. *The following statements are equivalent:*

1. LVP(Π_1).
2. LSR($\kappa, \mathbf{\Sigma_2}$) *holds for some* κ.
3. CVP(Π_1).
4. CSR($\kappa, \mathbf{\Sigma_2}$) *holds for some* κ.
5. *There exists a* 2*-supercompact cardinal.*

Proof. (1) implies (5) by Theorem 3.1.1. (5) implies (4) by Theorem 3.1.4. (4) implies (3) and (2) implies (1) straightforwardly by definitions. (3) implies (1) by Proposition 3.1.5, which implies (5), which then implies (4), which implies (2) by Proposition 1.3.3. This closes the cycle. □

Similarly we obtain the boldface version of the equivalence.

Corollary. *The following statements are equivalent:*

1. LVP($\mathbf{\Pi_1}$).
2. LSR($\kappa, \mathbf{\Sigma_2}$) *holds for a proper class of* κ.
3. CVP($\mathbf{\Pi_1}$).
4. CSR($\kappa, \mathbf{\Sigma_2}$) *holds for a proper class of* κ.
5. *There exists a proper class of* 2*-supercompact cardinals.*

LSR can also be used to characterize the least 2-supercompact cardinal:

Theorem 3.1.6. *If* κ *is the least ordinal such that* LSR(κ, Π_1) *holds, then* κ *is* 2*-supercompact.*

Proof. Suppose κ is the least ordinal such that LSR(κ, Π_1) holds, but is not 2-supercompact. Then there is no 2-supercompact cardinals $\leq \kappa$, otherwise we have, by Theorem 3.1.4, some $\alpha < \kappa$ such that CSR($\alpha, \mathbf{\Pi_1}$) holds, which by Proposition 1.3.3 implies LSR($\alpha, \mathbf{\Pi_1}$) holds, contradicting the minimality of κ.

Thus $D := \{\alpha : \exists \lambda \leq \kappa$ such that there is a 2-supercompact measure for $(\lambda, \alpha)\}$ is a bounded class of ordinals, so we can take some $\beta > max\{\kappa, \sup D\}$ where V_β is sufficiently correct (say $\beta \in C^{(4)}$) and $cf(\beta) = \omega$. Now define the class $\mathcal{C}$ of structures of the form $(V_\gamma, \in, \gamma, \xi, \eta, f)$, where $\eta < \xi < \gamma$, $\gamma \in C^{(1)}$, and $f : \omega \to \gamma$ is cofinal. It is clear that $\mathcal{C}$ is Π_1-definable without parameters.

Now by LSR(κ, Π_1), there is some α such that for any $B \in \mathcal{C}$ of rank β there is some $A \in \mathcal{C}$ of rank α elementarily embeddable into B. So consider some $(V_\beta, \in, \beta, \kappa, \alpha, f) \in \mathcal{C}$, which exists since $cf(\beta) = \omega$. Thus we get some $(V_\alpha, \in \alpha, \lambda, \eta, g) \in \mathcal{C}$ with elementary embedding

$$j : (V_\alpha, \in \alpha, \lambda, \eta, g) \to (V_\beta, \in, \beta, \kappa, \alpha, f).$$

As before, we have $crit(j) \leq \eta$, which implies $crit(j) < \eta$ given that $cf(\alpha) = \omega$. Let $crit(j) = \mu < \eta$.

Claim 3.1.7. $j(\mu) \leq \lambda$.

Proof. Suppose for contradiction that $j(\mu) > \lambda$. Since we are supposing that κ is the least ordinal with LSR(κ, Π_1) holds and that V_β is sufficiently correct, this property of κ is reflected in V_β, thus by elementarity

$$(V_\alpha, \in \alpha, \lambda, \eta, g) \models \text{“}\lambda \text{ is the least ordinal such that LSR}(\lambda, \Pi_1) \text{ holds”},$$

now since we have $j(\mu) > \lambda$, we have

$$(V_\beta, \in, \beta, \kappa, \alpha f) \models \text{“}V_\alpha \models \text{“}\exists \xi < j(\mu) \text{ LSR}(\xi, \Pi_1) \text{ holds””},$$

which gives

$$(V_\alpha, \in \alpha, \lambda, \eta, g) \models \text{“}V_\eta \models \text{“}\exists \xi < \mu \text{ LSR}(\xi, \Pi_1) \text{ holds””}.$$

Let $\xi < \mu$ be such a witness. We have, by elementarity, that

$$(V_\alpha, \in \alpha, \lambda, \eta, g) \models \text{“LSR}(j(\xi), \Pi_1) \text{ holds”},$$

and finally

$$(V_\beta, \in, \beta, \kappa, \alpha, f) \models \text{“LSR}(j^2(\xi), \Pi_1) \text{ holds”}.$$

Thus, by the correctness of V_β, it is true in V that $j^2(\xi) = \xi < \kappa$ is such that LSR(ξ, Π_1) holds, contradicting the minimality of κ. □

Claim 3.1.8. If $\gamma = min\{\eta, j(\mu)\}$, then there is a 2-supercompact measure for (μ, ξ) for any ξ with $\mu \leq \xi < \gamma$.

Proof. Let ξ be such that $\mu \leq \xi < \gamma$. Since $\xi < \eta$, we have $j(\xi) < \alpha$. So we define

$$\mathcal{U} = \{X \subseteq \mathcal{P}(j(\xi)) : j\text{“}j(\xi) \in j(X)\}$$

and one can verify that $\mathcal{U}$ is a 2-supercompact measure for $(j(\mu), j(\xi))$ with target sequences $\mu, j(\mu)$ and $\xi, j(\xi)$. Since $\mathcal{U} \in V_\beta$, we have by elementarity that

$$(V_\alpha, \in, \alpha, \lambda, \eta, g) \models \text{“There is a 2-supercompact measure } \mathcal{U} \text{ for } (\mu, \xi)\text{”}$$

which is true in V. □

Claim 3.1.9. There is some natural number i such that $\alpha < j^i(\mu)$.

Proof. Suppose not, then we have $\mu_\omega = \sup_{i \in \omega} j^i(\mu) < \alpha$, which implies $\mu_\omega < \lambda$, thus $j : V_{\mu_\omega+2} : V_{\mu_\omega+2} \to V_{\mu_\omega+2}$ contradicts the *Kunen Inconsistency*. □

Now we distinguish between two cases, both giving rise to a contradiction.

Case 0: $\gamma = \eta$.

In this case, observe that we have $V_\eta \models$ “μ is 2-supercompact”, so we have

$$V_\alpha \models \text{“}j(\mu) \text{ is 2-supercompact”}$$

and thus $V_\beta \models$ "$j^2(\mu)$ is 2-supercompact", which implies that it is true in V that $j^2(\mu)$ is 2-supercompact, since $V_\beta \prec_{\Sigma_2} V$. But since $j(\mu) \leq \lambda$, we have $j^2(\mu) \leq \kappa$, which is a contradiction.
Case 1: $\gamma = j(\mu)$.

In this case we have $V_{j(\mu)} \models$ "μ is 2-supercompact". Let $i \geq 2$ be the least with $j^i(\mu) \geq \alpha$. Note that we have $V_\mu \prec V_{j(\mu)}$, so by elementarity we have $V_{j^k(\mu)} \prec V_{j^{k+1}(\mu)}$ for any $0 \leq k < i$. It follows that $V_{j^i(\mu)} \models$ "μ is 2-supercompact". Since for any $\xi < \alpha$, the witnessing 2-supercompact measure for (μ, ξ) is in V_α, we have $V_\alpha \models$ "μ is 2-supercompact", implying $V_\beta \models$ "$j(\mu)$ is 2-supercompact", which also holds in V. But since $j(\mu) < \kappa$, this is a contradiction.

Thus it follows that κ must be 2-supercompact, which concludes the proof. □

Corollary. *The following are equivalent for every cardinal κ:*

1. *κ is the least 2-supercompact cardinal.*
2. *κ is the least ordinal such that $\mathrm{CSR}(\kappa, \boldsymbol{\Sigma_2})$ holds.*
3. *κ is the least ordinal such that $\mathrm{CSR}(\kappa, \Pi_1)$ holds.*
4. *κ is the least ordinal such that $\mathrm{LSR}(\kappa, \boldsymbol{\Sigma_2})$ holds.*
5. *κ is the least ordinal such that $\mathrm{LSR}(\kappa, \Pi_1)$ holds.*

Proof. The three properties of κ, $\mathrm{CSR}(\kappa, \boldsymbol{\Sigma_2})$, $\mathrm{CSR}(\kappa, \Pi_1)$ and $\mathrm{LSR}(\kappa, \boldsymbol{\Sigma_2})$, all imply that $\mathrm{LSR}(\kappa, \Pi_1)$ holds, where $\mathrm{CSR}(\kappa, \Pi_1)$ implies $\mathrm{LSR}(\kappa, \Pi_1)$ by Proposition 1.3.3. But the least ordinal κ such that $\mathrm{LSR}(\kappa, \Pi_1)$ holds is 2-supercompact by Theorem 3.1.6, which implies that all of the three properties above hold at κ. It follows that (1)-(5) are all equivalent. □

We also have a boldface version of the above corollary:

Corollary. *The following are equivalent for every cardinal κ:*

1. *κ is either 2-supercompact cardinal or a limit of 2-supercompact cardinals.*
2. $\mathrm{CSR}(\kappa, \boldsymbol{\Pi_1})$.
3. $\mathrm{LSR}(\kappa, \boldsymbol{\Pi_1})$.

Proof. Observe that if κ is a limit of ordinals α such that $\mathrm{CSR}(\alpha, \boldsymbol{\Pi_1})$ holds, then we also have $\mathrm{CSR}(\kappa, \boldsymbol{\Pi_1})$ holds. It follows that (1) implies (2). (2) implies (3) by Proposition 1.3.3. To see that (3) implies (1), suppose for a contradiction that (3) holds but there is some $\nu < \kappa$ such that for any γ with $\nu \leq \gamma \leq \kappa$, γ is not 2-supercompact. Now we can argue similarly as in the proof of Theorem 3.1.6 as follows. In the proof, define instead that $D = \{\alpha : \exists\lambda(\eta \leq \lambda \leq \kappa)$ such that there is some 2-supercompact measure for $(\lambda, \alpha)\}$. Let $\mathcal{C}$ of the structures of the form $(V_\gamma, \in, \gamma, \xi, \eta, \nu, f)$, where the conditions except for ν are the same as in the proof. $\mathcal{C}$ is then a Π_1-definable class with the parameter $\nu < \kappa$. Thus we may proceed similarly, and we are ensured that the critical point μ as in the proof is greater than ν, which is fixed by j, thus as in both Case 0 and Case 1, we find some ordinal in the interval $[\nu, \kappa]$ to be 2-supercompact, contradicting the assumption. □

3.2 The Π_n case

In this section we consider the general Π_n versions of LSR and CSR, for $n \geq 2$.

Theorem 3.2.1. *For any natural number $n \geq 1$,*

1. *If* $\mathrm{LVP}(\Pi_{n+1})$ *holds, then there exists a $C^{(n)}$-2-fold extendible cardinal.*

2. *If* $\mathrm{LVP}(\boldsymbol{\Pi_{n+1}})$ *holds, then there exists a proper class of $C^{(n)}$-2-fold extendible cardinals.*

Proof. (1): suppose $\mathrm{LVP}(\Pi_{n+1})$ holds, but there are no $C^{(n)}$-2-fold extendible cardinals. Define a class $\mathcal{D}$ of ordinals such that $\alpha \in \mathcal{D}$ if and only if

(i) $\alpha \in C^{(n+1)}$, and

(ii) For any $\beta < \alpha$, there is some γ with $\beta \leq \gamma < \alpha$ such that β is not γ-$C^{(n)}$-2-fold extendible.

We claim that $\mathcal{D}$ is a Π_{n+1}-definable, without parameters, class of ordinals. To see this, first note that clause (i) is Π_{n+1}-definable, and note that clause (ii) is equivalent to

$$\alpha \in C^{(n+1)} \wedge \forall \beta \in \alpha \exists \gamma \in \alpha(\beta \leq \gamma \wedge \forall X(X = V_\alpha \to X \models \text{“}\beta \text{ is not } \gamma\text{-}C^{(n)}\text{-2-fold extendible”}))$$

which is a Π_{n+1}-statement. The reason of the equivalence is that “β is γ-$C^{(n)}$-2-fold extendible” is a Σ_{n+1}-statement, since it is equivalent to

$$\exists \delta \exists j(j : V_{j(\beta+\gamma)} \to V_\delta \text{ is elementary} \wedge crit(j) = \beta \wedge j(\beta) > \beta + \gamma \wedge j(\beta) \in C^{(n)})$$

so it holds in V if and only if it holds in V_α, which is Σ_{n+1}-correct.

Now define a class $\mathcal{C}$ of structures of the form $(V_\alpha, \in, \alpha, \delta)$ such that $\delta < \alpha$, α is a limit point of $\mathcal{D}$ with $cf(\alpha) = \omega$. $\mathcal{C}$ is Π_{n+1}-definable, as $X \in \mathcal{C}$ if and only if X is of the form (X_0, X_1, X_2, X_3) such that

(i) X_2, X_3 are ordinals and $X_3 \in X_2$,

(ii) $X_0 = V_{X_2}$,

(iii) $X_1 = \in \restriction X_0$,

(iv) $\forall x \in X_2 \exists y \in X_2(x \in y \wedge y \in \mathcal{D})$.

(v) $cf(X_2) = \omega$,

and we have seen that $y \in \mathcal{D}$ is Π_{n+1}-definable.

Claim 3.2.2. $\mathcal{C}$ is a proper class

Proof. To see this, note that since we are supposing that there are no $C^{(n)}$-2-fold extendible cardinals, so if $\alpha \in C^{(n+2)}$, then for any $\beta < \alpha$, the Σ_{n+2}-statement “β is not $C^{(n)}$-m-fold extendible” reflects down to V_α, so indeed there is some $\gamma \geq \beta$ less than α such that β is not γ-$C^{(n)}$-2-fold extendible. Thus in fact we have $C^{(n+2)} \cap \{\alpha : cf(\alpha) = \omega\} \subseteq \mathcal{D}$, so $\mathcal{D}$ is a proper class, implying that $\mathcal{C}$ is also a proper class. $\square$

Now by LVP(Π_{n+1}), there are $\alpha \neq \beta$ with $\mathcal{C} \cap (V_{\alpha+1} \setminus V_\alpha) \neq \phi$ and $\mathcal{C} \cap (V_{\beta+1} \setminus V_\beta) \neq \phi$ such that for any $B \in \mathcal{C}$ of rank β there is some $A \in \mathcal{C}$ of rank α and an elementary embedding $j : A \to B$. Thus we have $\alpha < \beta$. Now consider the structure $(V_\beta, \in, \beta, \alpha) \in \mathcal{C}$. It follows that there is some $(V_\alpha, \in, \alpha, \delta) \in \mathcal{C}$ with an elementary embedding

$$j : (V_\alpha, \in, \alpha, \delta) \to (V_\beta, \in, \beta, \alpha).$$

Note that since $\delta < \alpha = j(\delta)$, we have $crit(j) \leq \delta$, which implies, since $cf(\delta) = \omega$ by elementarity, that $crit(j) = \mu < \delta$.

Claim 3.2.3. $\{\delta, \mu, j(\mu)\} \subseteq \mathcal{D}$.

Proof. First note that $\mathcal{D}$ is a closed and unbounded class definable in both V_α and V_β, since $\alpha, \beta \in C^{(n+1)}$.

Since we have $(V_\beta, \in, \beta, \alpha) \models$ "$\alpha \in \mathcal{D}$" and $j(\delta) = \alpha$, by elementarity and the correctness of V_α we have $\delta \in \mathcal{D}$.

Now suppose $\mu \notin \mathcal{D}$, then there is some $\lambda < \mu$ such that there is no $\lambda' \in \mathcal{D}$ with $\lambda \leq \lambda' < \mu$. However since $\alpha \in lim(\mathcal{D})$, the least $\eta \in \mathcal{D}$ above λ is less than α. Thus by elementarity we have

$$j(\eta) = j(\text{the least ordinal in } \mathcal{D} \text{ above } \lambda) = \text{the least ordinal in } \mathcal{D} \text{ above } j(\lambda)$$

which is again η, but then $j \restriction V_{\eta+2} : V_{\eta+2} \to V_{\eta+2}$ would contradict the *Kunen Inconsistency*.

Lastly $j(\mu) \in \mathcal{D}$ follows again from elementarity and the fact that $\alpha, \beta \in C^{(n+1)}$. □

Now let $\eta = min\{\delta, j(\mu)\}$, since $\delta, j(\mu) \in \mathcal{D}$, it follows that there is some $\lambda = \mu + \lambda < \eta$ such that μ is not λ-$C^{(n)}$-2-fold extendible. But since $\lambda < \delta$, we have $j(\lambda) < \alpha$. Moreover we have $\lambda < j(\mu)$, and $j(\mu) \in \mathcal{D} \subseteq C^{(n)}$. Thus in fact $j \restriction V_{j(\lambda)} : V_{j(\lambda)} \to V_{j^2(\lambda)}$ is elementary and witnesses that μ is λ-$C^{(n)}$-2-fold extendible, which is a contradiction.

(2): Suppose that the class of $C^{(n)}$-2-fold extendible cardinals are bounded in Ord, then let η be such that there is no $C^{(n)}$-2-fold extendible cardinals above η. Now argue similarly as in (1): define $\mathcal{D}$ to be the class of ordinals such that $\alpha \in \mathcal{D}$ if and only if

(i) $\alpha \in C^{(n+1)}$,

(ii) $\alpha > \eta$, and

(iii) For any $\beta < \alpha$ such that $\eta < \beta$ there is some γ with $\beta \leq \gamma < \alpha$ such that β is not γ-$C^{(n)}$-2-fold extendible,

which is Π_{n+1}-definable with the parameter η. Then proceed as before, and define the class $\mathcal{C}$ of structures of the form $(V_\alpha, \in, \alpha, \delta, \eta)$ such that $\eta < \delta < \alpha$, α is a limit point of $\mathcal{D}$ with $cf(\alpha) = \omega$. Again $\mathcal{C}$ is Π_{n+1}-definable with the parameter η, and can be shown to be a proper class as before. Thus by LVP($\mathbf{\Pi_{n+1}}$) there are $\alpha < \beta$ and $(V_\alpha, \in, \alpha, \delta, \eta), (V_\beta, \in, \beta, \alpha, \eta) \in \mathcal{C}$ with some elementary embedding

$$j : (V_\alpha, \in, \alpha, \delta, \eta) \to (V_\beta, \in, \beta, \alpha, \eta).$$

Now argue as before, noting that $j(\eta) = \eta$ implies $crit(j) = \mu > \eta$. It follows that μ has some degree of $C^{(n)}$-2-fold extendibility that leads to a contradiction. □

Theorem 3.2.4. *For any natural number n and cardinal κ, if κ is $C^{(n)}$-hyperhuge, then* CSR($\kappa, \mathbf{\Sigma_{n+2}}$) *holds. In particular, if κ is hyperhuge, then* CSR($\kappa, \mathbf{\Sigma_3}$) *holds.*

Proof. Given a natural number n, suppose that κ is $C^{(n)}$-hyperhuge. Let $\mathcal{C} = \{B : \phi(B, b_0, \ldots, b_{k-1})\}$ be a class of structures of the same type, where $\phi(x, y_0, \ldots, y_{k-1})$ is Σ_{n+2} and $b_0, \ldots, b_{k-1} \in V_\kappa$. For any $B \in \mathcal{C}$, we must find $A \in V_\kappa \cap \mathcal{C}$ that captures B. We may assume $rk(B) > \kappa$, otherwise the conclusion trivially holds.

Suppose for contradiction that there is no $A \in \mathcal{C} \cap V_\kappa$ that covers B. Now since $\phi(x, y_0, \ldots, y_{k-1})$ is of the form $\exists z \psi(z, x, y_0, \ldots, y_{k-1})$, where $\psi(z, x, y_0, \ldots, y_{k-1})$ is Π_{n+1}, there is some a such that $\psi(a, B, b_0, \ldots, b_{k-1})$ holds in V.

Let $\lambda > max\{rank(a), rank(B)\}$ be such that $\lambda \in C^{(1)}$, and by Corollary 2.2, let $\mathcal{U}$ be a hyperhuge measure for (κ, λ) with target sequences κ, κ' and λ, λ' with $\kappa' \in C^{(n)}$. Let $j : V \to Ult(V, \mathcal{U}) \cong M$ be the ultrapower embedding, we then have $crit(j) = \kappa, j(\kappa) = \kappa' > \lambda, j(\lambda) = \lambda'$, and ${}^{j(\lambda)}M \subseteq M$.

Since $|V_\lambda| = \lambda$ we have $(V_{j(\lambda)})^M = V_{j(\lambda)}$. Moreover, by elementarity we have $|V_{j(\lambda)}| = j(\lambda)$ is true in M, thus also in V. Now since $rank(B) < \lambda < j(\kappa)$, we have $B \in V_{j(\kappa)} \subseteq M$.

Claim 3.2.5. $M \models$ "$\phi(B, b_0, \ldots, b_{k-1})$ holds."

Proof. Note that since $rank(a), rank(B) < j(\kappa) \in C^{(n)}$ and $\psi(a, B, b_0, \ldots, b_{k-1})$ is a Π_{n+1}-statement, it reflect down to $V_{j(\kappa)}$, thus we have

$$V_{j(\kappa)} \models \text{“}\psi(a, B, b_0, \ldots, b_{k-1}) \text{ holds”}.$$

Furthermore, by Proposition 2.2.6 we have $\kappa \in C^{(n+1)}$ (in fact $\kappa \in C^{(n+2)}$), so by elementarity we have that

$$M \models \text{“}j(\kappa) \in C^{(n+1)}\text{”},$$

thus $V_{j(\kappa)}$ is Π_{n+1}-correct in M, which implies that

$$M \models \text{“}\phi(B, b_0, \ldots, b_{k-1})\text{”},$$

as desired. $\square$

Now since we are supposing that there is no $A \in \mathcal{C} \cap V_\kappa$ that captures B, by elementarity,

$$M \models \text{“there is no } A \in V_{j(\kappa)} \text{ with } \phi(A, b_0, \ldots, b_{k-1}) \text{ holds such that } A \text{ captures } j(B)\text{”}.$$

By the above claim and the definition of capturing, it follows that

$$\begin{aligned} M \models \text{“there is some } b \in j(B) \text{ such that} \\ \text{there is no elementary embedding } k : B \to j(B) \text{ with } b \in ran(k)\text{”}. \end{aligned}$$

Let b be as above, it follows by the closure property of M that it is true in V that there is no elementary embedding $k : B \to j(B)$ with $b \in ran(k)$. Thus again by elementarity we have

$$M \models \text{“there is no elementary embedding } k : j(B) \to j^2(B) \text{ with } j(b) \in ran(k)\text{”}.$$

However, since $rank(j(B)) < j(\lambda)$, we have $|j \restriction j(B)| = |j(B)| < |V_{j(\lambda)}| = j(\lambda)$, so $k = j \restriction j(B) : j(B) \to j^2(B)$ is in M. But k is an elementary embedding with $k(b) = j(b)$ in its range, which is what M is supposed to be missing, contradiction.

The last part of the theorem then follows from the fact that every hyperhuge cardinal is $C^{(1)}$-hyperhuge. $\square$

Corollary. *The following are equivalent for every natural number $n \geq 1$:*

1. LVP(Π_{n+1}).

2. LSR($\kappa, \mathbf{\Sigma_{n+2}}$) *holds for some* κ.

3. CVP(Π_{n+1}).

4. CSR($\kappa, \mathbf{\Sigma_{n+2}}$) *holds for some* κ.

5. *There exists a* $C^{(n)}$*-hyperhuge cardinal.*

6. *There exists a* $C^{(n)}$*-2-fold extendible cardinal.*

Proof. That (1) implies (6) is Theorem 3.2.1. (6) implies (5) by Theorem 2.3.3, and (5) implies (4) by Theorem 3.2.4. That (4) implies (3) is straightforward from the definitions. To see that (3) implies (2), first note that (3) implies (1) by Proposition 3.1.5, then (1) implies (4), which implies (2) by Proposition 1.3.3. Lastly, that (2) implies (1) is also straightforward from the definitions. □

Corollary. *The following statements are equivalent:*

1. LVP(Π_2).

2. LSR($\alpha, \mathbf{\Sigma_3}$) *holds for some* α.

3. CVP(Π_2).

4. CSR($\alpha, \mathbf{\Sigma_3}$) *holds for some* α.

5. *There exists a hyperhuge cardinal.*

6. *There exists a 2-fold extendible cardinal.*

Corollary. *The following are equivalent for every natural number* $n \geq 1$*:*

1. LVP($\mathbf{\Pi_{n+1}}$).

2. LSR($\kappa, \mathbf{\Sigma_{n+2}}$) *holds for a proper class of* κ.

3. CVP($\mathbf{\Pi_{n+1}}$).

4. CSR($\kappa, \mathbf{\Sigma_{n+2}}$) *holds for a proper class of* κ.

5. *There exists a proper class of* $C^{(n)}$*-hyperhuge cardinals.*

6. *There exists a proper class of* $C^{(n)}$*-2-fold extendibles cardinals.*

Theorem 3.2.6. *If* $n \geq 1$ *and* κ *is the least ordinal with* LSR(κ, Π_{n+1}) *holds, then* κ *is* $C^{(n)}$*-2-fold extendible.*

Proof. Suppose κ is the least ordinal with LSR(κ, Π_{n+1}) holds but is not $C^{(n)}$-2-fold extendible, then there are no $C^{(n)}$-2-fold extendible cardinals $\leq \kappa$, since if $\eta < \kappa$ is $C^{(n)}$-2-fold extendible, then by Theorem 2.3.3, η is $C^{(n)}$-hyperhuge, so by Theorem 3.2.4, CSR(η, Π_{n+1}) holds, so by Proposition 1.3.3, LSR(η, Π_{n+1}) holds, contradicting the minimality of κ.

Take some $\beta > \kappa$ so that V_β is sufficiently correct, say $\beta \in C^{(n+4)}$, and $cf(\beta) = \omega$. Let $\mathcal{C}$ be the Π_{n+1}-definable class of structures of the form $(V_\alpha, \in, \alpha, \xi, \eta)$ where $\eta < \xi < \alpha \in lim(C^{(n+1)})$ and $cf(\alpha) = \omega$.

By LSR(κ, Π_{n+1}), there is some α such that for any $B \in \mathcal{C}$ of rank β there is some $A \in \mathcal{C}$ of rank α elementarily embeddable into B. Let $(V_\beta, \in, \beta, \kappa, \alpha) \in \mathcal{C}$ and let $(V_\alpha, \in \alpha, \xi, \eta) \in \mathcal{C}$ with elementary embedding $j : (V_\alpha, \in \alpha, \xi, \eta) \to (V_\beta, \in, \beta, \kappa, \alpha)$. As before, we have $\mu = crit(j) \leq \eta$, implying $\mu < \eta$.

Claim 3.2.7. $j(\mu) \leq \xi$.

Proof. Similarly as in Theorem 3.1.6, suppose $j(\mu) > \xi$. By the correctness of V_β and the elementarity of j, we get

$$(V_\alpha, \in \alpha, \xi, \eta) \models \text{“}\xi \text{ is the least ordinal with } \mathrm{LSR}(\xi, \Pi_{n+1}) \text{ holds”}$$

which implies that

$$(V_\beta, \in, \beta, \kappa, \alpha) \models \text{“}V_\alpha \models \text{“}\exists \nu < j(\mu)\ \mathrm{LSR}(\nu, \Pi_n) \text{ holds”}\text{”}$$

which gives

$$(V_\alpha, \in \alpha, \xi, \eta) \models \text{“}V_\eta \models \text{“}\exists \nu < \mu\ \mathrm{LSR}(\nu, \Pi_n) \text{ holds”}\text{”}.$$

Let $\nu < \mu$ be such a witness, we have by applying elementarity twice that

$$(V_\beta, \in, \beta, \kappa, \alpha) \models \text{“}\mathrm{LSR}(j^2(\nu), \Pi_n) \text{ holds”}$$

thus it is true in V that $j^2(\nu) = \nu$ and $\mathrm{LSR}(\nu, \Pi_n)$ holds, contradicting the minimality of κ. □

Claim 3.2.8. If $\gamma = min\{\eta, j(\mu)\}$, then μ is ν-$C^{(n)}$-2-fold extendible for any $\nu < \gamma$.

Proof. First note that since $C^{(n)}$ is a club class, similar arguments as in Theorem 3.2.1 show that μ and $j(\mu)$ are both in $C^{(n)}$. For any $\nu < \gamma$, we have $j(\mu + \nu) < \alpha$, so $j \restriction V_{j(\mu+\nu)} : V_{j(\mu+\nu)} \to V_{j^2(\mu+\nu)}$ witnesses that μ is ν-$C^{(n)}$-2-fold extendible. □

Now we distinguish between two cases, both give rise to a contradiction.
Case 0: $\gamma = \eta$.

Note that since $V_\beta \models$ “$\alpha \in C^{(n+1)}$” and $j(\eta) = \alpha$, we have by elementarity that it is true in V_α and hence in V that $\eta \in C^{(n+1)}$. For any $\nu < \eta$, the assertion “μ is ν-$C^{(n)}$-2-fold extendible” is Σ_{n+1} with parameters μ and ν, so it reflects in V_η. It follows that we have $V_\eta \models$ “μ is $C^{(n)}$-2-fold extendible”, so $V_\alpha \models$ “$j(\mu)$ is $C^{(n)}$-2-fold extendible”, and thus $V_\beta \models$ “$j^2(\mu)$ is $C^{(n)}$-2-fold extendible”, which is true in V. But since $j(\mu) \leq \xi$, we have $j^2(\mu) \leq \kappa$, which is a contradiction.
Case 1: $\gamma = j(\mu)$.

In this case we can argue as in Case 0 that $V_{j(\mu)} \models$ “μ is $C^{(n)}$-2-fold extendible”. There is some least natural number $i \geq 2$ with $j^i(\mu) \geq \alpha$. Again noting that $V_{j^k(\mu)} \prec V_{j^{k+1}(\mu)}$ for any $0 \leq k < i$, we have $V_{j^i(\mu)} \models$ “μ is $C^{(n)}$-2-fold extendible”. It follows that $V_\alpha \models$ “μ is $C^{(n)}$-2-fold extendible”, thus it is true in V_β and also in V that $j(\mu)$ is $C^{(n)}$-2-fold extendible, which is a contradiction since $j(\mu) < \kappa$. □

Corollary. *The following are equivalent for every natural number $n \geq 1$ and cardinal κ:*

1. *κ is the least $C^{(n)}$-2-fold extendible cardinal.*
2. *κ is the least $C^{(n)}$-hyperhuge cardinal.*
3. *κ is the least ordinal such that $\mathrm{CSR}(\kappa, \mathbf{\Sigma_{n+2}})$ holds.*
4. *κ is the least ordinal such that $\mathrm{CSR}(\kappa, \Pi_{n+1})$ holds.*
5. *κ is the least ordinal such that $\mathrm{LSR}(\kappa, \mathbf{\Sigma_{n+2}})$ holds.*
6. *κ is the least ordinal such that $\mathrm{LSR}(\kappa, \Pi_{n+1})$ holds.*

Proof. Among the six properties mentioned, $\mathrm{LSR}(\kappa, \Pi_{n+1})$ is implied by all of the other five. But by Theorem 3.2.6, the least κ such that $\mathrm{LSR}(\kappa, \Pi_{n+1})$ holds has all the other five properties, thus (1)-(6) are all equivalent. □

Similarly as in the Π_1 case, we have the boldface version of the above corollary:

Corollary. *The following are equivalent for every natural number $n \geq 1$ and cardinal κ:*

1. *κ is either a $C^{(n)}$-2-fold extendible cardinal or a limit of $C^{(n)}$-2-fold extendible cardinals.*
2. *κ is either a $C^{(n)}$-hyperhuge cardinal or a limit of $C^{(n)}$-hyperhuge cardinals*
3. $\mathrm{CSR}(\kappa, \mathbf{\Pi_{n+1}})$.
4. $\mathrm{LSR}(\kappa, \mathbf{\Pi_{n+1}})$.

Proof. (1) implies (2) Theorem 2.3.3. (2) implies (3) by Theorem 3.2.4 plus the fact that the class of α such that $\mathrm{CSR}(\alpha, \mathbf{\Pi_{n+1}})$ holds is closed under limits. (3) implies (4) by Proposition 1.3.3.

To see that (4) implies (1), suppose for contradiction that (4) holds but there is some $\nu < \kappa$ such that for any γ with $\nu \leq \gamma \leq \kappa$, γ is not $C^{(n)}$-2-fold extendible. Now we can argue similarly as in the proof of Theorem 3.2.6 as follows. In the proof, let $\mathcal{C}$ instead be the structures of the form $(V_\alpha, \in, \alpha, \xi, \eta, \nu)$, where $\nu < \eta$ and the rest of the conditions are the same as in the proof. $\mathcal{C}$ is then a Π_{n+1}-definable class with the parameter $\nu < \kappa$. Thus we may proceed similarly, and we are ensured that the critical point μ as in the proof is greater than ν, which is fixed by j, thus as in both Case 0 and Case 1, we find some ordinal in the interval $[\nu, \kappa]$ to be $C^{(n)}$-2-fold extendible, contradicting the assumption. □

Chapter 4

Still Higher Reflection

In Chapter 3 we extended the pattern of structural reflection phenomena to the level of 2-supercompact cardinals, $C^{(n)}$-hyperhuge cardinals and $C^{(n)}$-2-fold extendible cardinals. This immediately gives rise to the question:

Question 4.0.1. Are there natural structural reflection principles that correspond, in the same pattern, to m-supercompact cardinals, $C^{(n)}$-m-hyperhuge cardinals and $C^{(n)}$-m-fold extendible cardinals in general, for every natural number m?

In this chapter we give a positive answer to the above question. We introduce natural game-theoretic extensions of the notion of capturing, and use them to formulate the principle δ-*CSR*, which are extensions of CSR, that correspond to the relevant large cardinals in the desired pattern, thus extending the same pattern of structural reflection phenomena to what is close to the upper limit of the large cardinal hierarchy, up to the region just below rank into rank axioms. Moreover, the original principle SR and the principle CSR in previous chapters are the special cases of 0-CSR and 1-CSR, respectively, so the results and proofs in this chapter also cover those special cases.

4.1 The capturing game

Recall that we say A captures B if for any $b \in B$ there is an elementary embedding $e : A \to B$ with $b \in ran(e)$, namely there is some $a \in A$ with $e(a) = b$. Conceptually, a can be seen as a counterpart of b in A.

Imagine that some person P_A wants to demonstrate that the structural properties of B are captured by A, so that when given any $b \in B$ by person P_B, P_A is able to show some $a \in A$, and promises the existence of an elementary embedding $e : A \to B$ with $e(a) = b$. What if P_B is still unsatisfied, and propose *another* $b' \in B$, demanding P_A to find *another* $a' \in A$, such that there is an elementary $e : A \to B$ with not only $e(a) = b$, but also $e(a') = b'$, to demonstrate how well A captures B? If P_A succeeds in showing such a', what if P_B, still unsatisfied, proposes still another b'', and so on, for many, even infinitely many steps?

Let us describe a game. Given two structures A and B of the same type and some ordinal δ, the *δ-capturing game of length δ on A and B*, denoted by $Cap_\delta(A, B)$, is described as follows. There are two players, P_B and P_A. At stage α, where $\alpha < \delta$, P_B chooses some element $b_\alpha \in B$, and then P_A chooses some element $a_\alpha \in A$. After δ many steps, two sequences of elements $\vec{a} = (a_\alpha : \alpha < \delta)$ and $\vec{b} = (b_\alpha : \alpha < \delta)$ have been chosen, and the pair $(\vec{a}, \vec{b})$ is the *play*. We say the play $(\vec{a}, \vec{b})$ *is a win for player P_A*, or *P_A wins the play*, if there is some elementary embedding $e : A \to B$ with $e(a_\alpha) = b_\alpha$ for all $\alpha < \delta$. Otherwise, we say $(\vec{a}, \vec{b})$ *is a win for player* P_B, or P_B *wins the play.* Note that in the case $\delta = 0$, there is only one play, namely (ϕ, ϕ), on the game $Cap_\delta(A, B)$, and P_A wins if there simply exists some elementary embedding $e : A \to B$.

A *strategy* σ for the player P_A on the game $Cap_\delta(A, B)$ is a function that, given partial plays at stages $\alpha < \delta$, when it is the turn for P_A to move, outputs a choice a_α for P_A. Namely, given a pair of sequences $(\vec{a}, \vec{b})$, where $\vec{a} = (a_\beta : \beta < \alpha)$ and $\vec{b} = (b_\beta : \beta \leq \alpha)$, where $\alpha < \delta$, such that $\{a_\beta : \beta < \alpha\} \subseteq A$ and $\{b_\beta : \beta \leq \alpha\} \subseteq B$, we have $\sigma((\vec{a}, \vec{b}))$ is an element of A, to be chosen as a_α. A *winning strategy* σ for the player P_A in the game $Cap_\delta(A, B)$ is a strategy such that, whenever a play is played by P_A according to the strategy σ, P_A wins the play. In other words, for any play $(\vec{a}, \vec{b})$ on $Cap_\delta(A, B)$, if for any $\alpha < \delta$ we have $\sigma((\vec{a}\restriction\alpha, \vec{b}\restriction(\alpha + 1))) = a_\alpha$, then P_A wins the play $(\vec{a}, \vec{b})$. A strategy for P_B is similarly defined.

With the notion of the capturing game, we can formulate the intuitive idea that the better P_A is able to repeatedly answer P_B's challenge, the better A can be said to capture the structure of B.

Definition 4.1.1. For all structures A and B of the same type and some ordinal δ, we say *A δ-captures B* if the player P_A has a winning strategy in the game $Cap_\delta(A, B)$.

Thus A 0-captures B if and only if there is some elementary embedding $e : A \to B$. This is the case where P_B is very permissive, not challenging P_A at all, and is satisfied with any elementary embedding.

Note that if $|A| = \kappa < \lambda = |B|$, then trivially P_B has a winning strategy for the game $Cap_\lambda(A, B)$, simply by enumerating all elements of B. Also if $|A| = |B| = \kappa$, then if $A \cong B$ then P_A has a winning strategy for the game $Cap_\kappa(A, B)$, by choosing according to some isomorphism. Otherwise P_B has a winning strategy, by enumerating the elements of B.

We also give here some basic examples to illustrate the notion of δ-capturing by extending Example 1.2.2:

Example 4.1.2. Recall that the model $\mathfrak{N}_0 = (\mathbb{N}, 0, S)$ is the standard natural numbers equipped with a distinguished element 0 and the successor function S. $\mathfrak{N}_1$ is $\mathfrak{N}_0$ plus an additional disjoint "$\mathbb{Z}$-chain", and $\mathfrak{N}_2$ is $\mathfrak{N}_1$ plus yet another disjoint $\mathbb{Z}$-chain. In general we may let $\mathfrak{N}_\kappa$ be the model which includes $\mathfrak{N}_0$ and κ-many disjoint $\mathbb{Z}$-chains, for any cardinal number κ, finite or infinite. For $\alpha < \kappa < \lambda$, we may without loss of generality assume that the αth $\mathbb{Z}$-chain in $\mathfrak{N}_\kappa$ and the αth $\mathbb{Z}$-chain in $\mathfrak{N}_\lambda$ are identical, since any two $\mathbb{Z}$ chains are isomorphic. Thus let $\mathbb{Z}_\alpha$ denote the αth $\mathbb{Z}$-chain in $\mathfrak{N}_\kappa$, for any $\kappa > \alpha$. Moreover we may assume that for any ordinal α, $\mathbb{Z}_\alpha$ is of the form

$$\cdots \to S^{-2}(o_\alpha) \to S^{-1}(o_\alpha) \to o_\alpha \to S(o_\alpha) \to S(S(o_\alpha)) \to \dots$$

for some o_α.

Claim 4.1.3. *If $m < n$ are natural numbers, then $\mathfrak{N}_m$ m-captures $\mathfrak{N}_n$, but $\mathfrak{N}_m$ does not $(m + 1)$-capture $\mathfrak{N}_n$.*

Proof. To see that $\mathfrak{N}_m$ m-captures $\mathfrak{N}_n$, let us describe a strategy for P_A: for the first pick b_0 of P_B, if b_0 is some standard natural number k, then let $a_0 = k$ as well. If b_0 is of the form $S^k(o_j)$ for some natural number $j < n$ and some integer k, then let a_0 be $S^k(o_0)$.

In general, at each stage $i < m$, there are three possibilities for b_i.

If b_i is some standard natural number k, then let $a_i = k$.

If b_i is from some $\mathbb{Z}$-chain already picked before, namely b_i is of the form $S^k(o_j)$ for some natural number $j < n$ and some integer k, such that there is some b_s for some $s < i$

such that $b_s = S^l(o_j)$ for some integer l, then a_s must come from some $\mathbb{Z}_t$ for some $t < m$ (if in the previous stages P_A has played according to our strategy), and we let b_i be $S^k(o_t)$.

If b_i is from some new $\mathbb{Z}$-chain, namely b_i is of the form $S^k(o_j)$ for some natural number $j < n$ and some integer k, such that there is no b_s for some $s < i$ such that $b_s = S^l(o_j)$ for some integer l, then we let t be the least natural number less than m such that there is no a_s that comes from $\mathbb{Z}_t$ for all $s < i$, and let a_i be $S^k(o_t)$. Such a t must exists, because we have only played for less than $i < m$ many rounds, so the available $\mathbb{Z}$-chains in $\mathfrak{N}_m$ have not been exhausted.

If P_A plays accordingly, then in the end we may define the map $e : A \to B$ as follows. There are three possibilities for elements of A. Firstly let e be the identity on $\mathbb{N}$. Secondly for any $i < m$, if there is some $j < m$ such that a_j is of the form $S^k(o_i)$ for some integer k, namely if the chain $\mathbb{Z}_i$ was invoked in the choosing process, then b_j must be of the form $S^k(o_t)$ for some $t < n$, and we define $e(S^l(o_i)) = S^l(o_t)$ for any integer l, namely e is the natural isomorphism between $\mathbb{Z}_i$ and $\mathbb{Z}_t$. Thirdly, consider

$$A' := \{i < m : \text{there is no } j < m \text{ such that } a_j \text{ is of the form } S^k(o_i) \text{ for some integer } k\},$$

namely the set of indexes of $\mathbb{Z}$-chains in $\mathfrak{N}_m$ not invoked before, then there must also be at least $|A'|$ many $\mathbb{Z}$-chains in $\mathfrak{N}_n$ not invoked before. Thus for $i \in A'$, we may let e the natural isomorphism between $\mathbb{Z}_i$ and some $\mathbb{Z}_t$ for some $t < n$, where $\mathbb{Z}_t$ was not invoked before.

It follows that e as defined is an embedding from A into B, hence also an elementary embedding from A into B, such that $e(a_i) = b_i$ for all $i < m$. Thus $\mathfrak{N}_m$ m-captures $\mathfrak{N}_n$.

To see that $\mathfrak{N}_m$ does not $(m+1)$-capture $\mathfrak{N}_n$, we describe a winning strategy for P_B: at each stage $i < (m+1)$, P_B simply chooses some elements from $\mathbb{Z}_i$, no matter what P_A chooses. In the end, if there exists some $e : A \to B$ with $e(a_i) = b_i$ for all $i < (m+1)$, then since $\mathfrak{N}_m$ only has m $\mathbb{Z}$-chains, there must be some $j, s < (m+1)$ such that b_j and b_s come from, by our strategy, different $\mathbb{Z}$-chains, while a_j and a_s come from the same $\mathbb{Z}$-chain, say $a_j = S^k(o_t)$ and $a_s = S^l(o_t)$ for some natural number $t < m$ and some integers k and l. Without loss of generality assume $k < l$, we have $\mathfrak{N}_m \models$ "a_s can be reached from a_j in $(l-k)$ steps", but it fails in $\mathfrak{N}_n$ that b_s can be reached from b_j in $(l-k)$ steps, thus e is not elementary. Thus $\mathfrak{N}_m$ does not $(m+1)$-capture $\mathfrak{N}_n$. □

In fact the above argument can be generalized:

Claim 4.1.4. *For any finite or infinite cardinals $\kappa < \lambda$ and any ordinal $\alpha < \kappa^+$, $\mathfrak{N}_\kappa$ α-captures $\mathfrak{N}_\lambda$, but $\mathfrak{N}_\kappa$ does not κ^+-capture $\mathfrak{N}_\lambda$.*

Proof. This is proved by modifying the proof for the previous claim. To see that $\mathfrak{N}_\kappa$ α-captures $\mathfrak{N}_\lambda$, we first order the $\mathbb{Z}$-chains in $\mathfrak{N}_\kappa$ in order-type α. Now we describe a strategy for P_A as follows. At each stage $\beta < \alpha$, if b_β is standard, let $a_\beta = b_\beta$. If b_β is from some chain Z_γ such that there is already some $\eta < \beta$ with $b_\eta \in \mathbb{Z}_\gamma$, then a_η is from some $\mathbb{Z}_\xi$ for some $\xi < \alpha$, and we let a_β be the element in $\mathbb{Z}_\xi$ that corresponds to the position of b_β in $\mathbb{Z}_\gamma$. If b_β is from some new $\mathbb{Z}$-chain, then let $\gamma < \alpha$ be the least such that there is no a_η, for some $\eta < \beta$, that comes from $\mathbb{Z}_\gamma$. Such a γ must exists, if P_A plays according to our strategy. It follows that there is in the end an elementary embedding $e : A \to B$ that is defined in the natural way, similarly as in the previous claim.

One way to see that $\mathfrak{N}_\kappa$ does not κ^+-captures $\mathfrak{N}_\lambda$, if κ is infinite, is that P_B can employ the strategy of enumerating κ^+ elements in $\mathfrak{N}_\lambda$. Another way to see it is that P_B can use the strategy of choosing, at each stage $\beta < \kappa^+$, some element from the chain $\mathbb{Z}_\beta$, so that in the end there will be many a_β's that come from the same chain, so any map that sends a_β to b_β, for all $\beta < \kappa^+$, cannot be elementary. □

As remarked in Chapter 1, although our primary interest in this thesis is in formulating structural reflection principles, and we only consider basic examples here, it seems that the notion of δ-capturing may have applications in other mathematical contexts. In any case, there are many basic properties of the capturing games that merit further investigations.

To conclude the section, note that the following proposition is intuitively straightforward and not difficult to prove. However, it makes it easier for later arguments to go through, and we will use this proposition repeatedly.

Proposition 4.1.5. *For all natural numbers n and for all structures A, B of the same type, the following are equivalent:*

1. *A n-captures B.*

2. $\forall b_0 \in B\ \exists a_0 \in A \ldots \forall b_{n-1} \in B\ \exists a_{n-1} \in A$ *such that there is an elementary embedding $e : A \to B$ with $e(a_i) = b_i$ for $0 \leq i < n$. In particular, A 0-captures B if and only if there is some elementary embedding $e : A \to B$.*

4.2 δ-CSR

With the notion of δ-capturing, the natural step to take is to formulate CSR using the more general δ-capturing in place of capturing.

$\mathbf{\Sigma_n}$-δ-CSR: ($\mathbf{\Sigma_n}$-δ-*Capturing Structural Reflection*) For every Σ_n-definable, with parameters, class $\mathcal{C}$ of relational structures of the same type there is an ordinal α that δ-*capture-reflects* $\mathcal{C}$.,i.e., for any $B \in \mathcal{C}$ there is some $A \in \mathcal{C} \cap V_\alpha$ that δ-captures B.

Similarly for $\mathbf{\Pi_n}$-δ-CSR. Moreover, Σ_n-δ-CSR and Π_n-δ-CSR are the corresponding lightface versions, i.e., with parameters disallowed in the definition of $\mathcal{C}$.

Definition 4.2.1. For every natural number n, δ-CSR(α, Σ_n) holds if for every Σ_n-definable, without parameters, class $\mathcal{C}$ of relational structures of the same type, α δ-capture-reflects $\mathcal{C}$. Similarly for δ-CSR(α, Π_n).

δ-CSR$(\alpha, \mathbf{\Sigma_n})$ holds if for any Σ_n-definable, with parameters in V_α, class $\mathcal{C}$ of relational structures of the same type, α δ-capture-reflects $\mathcal{C}$. Also similarly for δ-CSR$(\alpha, \mathbf{\Pi_n})$.

Thus the original SR is just 0-SR, and CSR is 1-SR. Now recall that a part of the supposed justification for the original SR comes from the idea of taking the notion of elementary embeddability to express the informal notion of structural similarity, and we argued in Chapter 1 that, since capturing is a stronger notion of structural similarity, any justification for SR seems to carry over to CSR. Now given that δ-capturing seems to be a yet stronger notion of structural similarity, which naturally extends the notion of capturing, it would seem that the justification that supports SR and CSR would also support δ-CSR as well.

Moreover, in the next section we show that there are indeed reasons to think that δ-capturing is the natural way to extend capturing, by showing that m-CSR, for all natural number m, corresponds to $(m+1)$-supercompact cardinals and $C^{(n)}$-$(m+1)$-fold extendible cardinals in completely the same way CSR corresponds to 2-supercompact cardinals and $C^{(n)}$-2-fold extendible cardinals.

We can also formulate the Vopěnka's Principle version:

Definition 4.2.2. For any natural number n, δ-CVP(Σ_n) holds if for any Σ_n-definable, without parameters, proper class $\mathcal{C}$ of relational structures of the same type, there exist $A, B \in \mathcal{C}$ with $A \neq B$ and A δ-captures B. Similarly for δ-CVP(Π_n).

Similarly, δ-CVP($\mathbf{\Sigma_n}$) and δ-CVP($\mathbf{\Pi_n}$) are the corresponding assertions with parameters allowed in the definition of $\mathcal{C}$.

Note that VP is just 0-CVP and CVP is 1-CVP.

Let us remark that, using similar ideas behind the formulation of δ-CSR, we can in fact formulate strengthenings of level by level structural reflection principles that relate to δ-CSR just as LSR relate to CSR, but it seems to me that the CSR versions are the more natural and general versions, so we focus on the CSR versions here.

Let us first summarize the strength of δ-CSR. On the one hand, the results in the next secion implies that the strength of the axiom schema m-CSR, for finite $m \geq 1$, is strictly between an m-huge cardinal and an $(m+1)$-huge cardinal, so the whole hierarchy of m-CSR, for all finite m, occupies the region of large cardinal hierarchy that is essentially cofinal below an I3 embedding. On the other hand, the principle Σ_0-δ-CSR, for any infinite δ, is outright inconsistent.

Theorem 4.2.3. *For any ordinal α, ω-CSR(α, Σ_0) fails.*

Proof. Suppose that there is some ordinal α such that ω-CSR(α, Σ_0) holds. Consider the Σ_0-definable class $\mathcal{C}$ of structures of the form $(M, \in)$, where M is transitive. Let $\beta > \alpha$, and consider $B = (V_\beta, \in) \in \mathcal{C}$. Then there is some $A = (M, \in) \in \mathcal{C} \cap V_\alpha$ such that A ω-captures B. Now let σ be a winning strategy for P_A in the game $Cap_\omega(A, B)$, and we define recursively the sequence $(\delta_i : i \in \omega)$ as follows. Let $\delta_0 = o(M)$, namely the least ordinal not in M. Since $\delta_0 < \alpha$, we have $\delta_0 \in V_\beta$. Given $\delta_0, \ldots, \delta_i$, consider the partial play $p := ((\delta_j : 1 \leq j \leq i), (\delta_k : 0 \leq k \leq i))$ on the game $Cap_\omega(A, B)$, we let δ_{i+1} be $\sigma(p) \in A \subseteq B$. By our construction of the sequence, there is an elementary embedding $j : A \to B$ such that $j(\delta_i) = \delta_{i-1}$ for all $1 \leq i < \omega$. But note that $\delta_1 \in M$, so $\delta_1 < o(M) = \delta_0$, so by elementarity $\delta_{i+1} < \delta_i$ for all $0 \leq i < \omega$, namely $(\delta_i : i \in \omega)$ is an infinite descending chain, which is a contradiction. $\square$

Note that the above proof actually shows that for all transitive M, N such that $o(M) < o(N)$, M does not ω-capture N. Also note that the the proof does not use the axiom of choice, instead the principle ω-CSR(α, Σ_0) directly contradicts the well-foundedness of the ordinals.

4.3 General Correspondence

In this section we show that, generalizing Bagaria's results and our previous results in Chapter 3, if δ is a finite ordinal, namely a natural number, then δ-CSR relates to $(\delta+1)$-supercompact cardinals and $C^{(n)}$-$(\delta+1)$-fold extendible cardinals in the desired pattern, which answers our Question 4.0.1.

The proofs will be briefer in this section when we use similar methods as in the proofs of previous chapters.

Theorem 4.3.1. *For every natural number m,*

1. *If m-CVP(Π_1) holds, then there exists an $(m+1)$-supercompact cardinal.*

2. *If m-CVP($\mathbf{\Pi_1}$) holds, then there exists a proper class of $(m+1)$-supercompact cardinals.*

Proof. For (1): suppose m-CVP(Π_1) holds, but there are no $(m+1)$-supercompact cardinals. Then define the function $F(\alpha)$ = the least $\beta > \alpha$ such that for any $\xi \leq \alpha$ there is no $(m+1)$-supercompact measure $\mathcal{U}$ for (ξ, β), provided that there is any such β. Otherwise $F(\alpha) = 0$. Define a class $\mathcal{D}$ of ordinals such that $\gamma \in \mathcal{D}$ if and only if for any $\alpha < \gamma$ there is some $\beta < \gamma$ with $F(\alpha) = \beta$.

Consider the Π_1-definable, without parameters, class $\mathcal{C}$ of structures of the form $(V_\alpha, \in, \alpha, \delta)$ such that $\delta < \alpha \in C^{(1)}$, α is the least limit point of $\mathcal{D}$ above δ. Since under our assumption we have $\alpha < F(\alpha)$ for any ordinal α, it is clear that $\mathcal{C}$ is a proper class. Moreover, note that if $(V_\alpha, \in, \alpha, \delta) \in \mathcal{C}$, then $cf(\alpha) = \omega$. This is because $\mathcal{D}$ is a club class, so if $\delta < \alpha$ and $cf(\alpha) \geq \omega_1$, there must be some $\delta < \alpha' < \alpha$ with $cf(\alpha') = \omega$ but $\alpha' \in lim(\mathcal{D})$.

By m-CVP(Π_1), there are $A \neq B$, both in $\mathcal{C}$, such that $A = (V_\alpha, \in, \alpha, \delta)$ m-captures $B = (V_\beta, \in, \beta, \eta)$. It follows that $\delta \neq \eta$, otherwise $\alpha = \beta$, and so $A = B$. Thus we have $\delta < \eta$, which implies that $\delta < \alpha \leq \eta < \beta$, since if $\alpha > \eta$, then the least limit point of $\mathcal{D}$ above η would not be β.

Now if $m \geq 1$, we construct recursively the sequences $(x_i : i < m)$ and $(y_i : i < m)$ in a "diagonal" way as follows. Let $y_0 = \alpha$. By Proposition 4.1.5 we have

$$\begin{aligned}&\exists x_0 \in A \forall y_1 \in B \exists x_1 \in A, \ldots, \forall y_{m-1} \in B \exists x_{m-1} \in A\\ &\exists j : A \to B \text{ is elementary and } j(x_i) = y_i \text{ for } 0 \leq i \leq (m-1).\end{aligned}$$

Choose such an x_0 as above. In general given $(x_i : 0 \leq i \leq k)$ and $(y_i : 0 \leq i \leq k)$, for $k+1 < m$, we may inductively assume that

$$\begin{aligned}&\forall y_{k+1} \in B \exists x_{k+1} \in A, \ldots, \forall y_{m-1} \in B \exists x_{m-1} \in A\\ &\exists j : A \to B \text{ is elementary and } j(x_i) = y_i \text{ for } 0 \leq i \leq (m-1)\end{aligned}$$

and we pick y_{k+1} to be precisely x_k (this makes sense since we have $x_k \in A \subseteq B$), and choose any x_{k+1} that satisfies the displayed statement above. If $m = 0$ then we skip this choosing process.

By the way we construct the sequences $(x_i : i < m)$ and $(y_i : i < m)$, there is in the end some elementary embedding $j : (V_\alpha, \in, \alpha, \delta) \to (V_\beta, \in, \beta, \eta)$ with $j(\delta) = \eta$ and $j(x_i) = y_i$ for all $0 \leq i < m$. It follows that we have $j(x_i) = y_i = x_{i-1}$ for any $i \geq 1$, and $j(x_0) = y_0 = \alpha$. By applying elementarity iteratively, this implies that all the x_i's and y_i's are all ordinals of countable cofinality, and $x_i < j(x_i)$ for any $0 \leq i < m$. Also $j(\delta) = \eta > \delta$. Thus j is nontrivial and we have that $crit(j) = \mu \leq x_{m-1}$ (in case $m > 0$ so x_{m-1} exists), and thus $crit(j) < x_{m-1}$, since μ is regular.

Claim 4.3.2. $\{\alpha, x_0, \ldots, x_{m-1}, \mu, j(\mu)\} \subseteq \mathcal{D}$.

Proof. That $\alpha \in \mathcal{D}$ is already given. Note that in general the statement "$x \in \mathcal{D}$" can be decided in V_ν, for any ordinal ν and $x \in V_\nu$, so since $\alpha = y_0 \in \mathcal{D}$, we have $(V_\beta, \in, \beta, \eta) \models$ "$\alpha \in \mathcal{D}$". By elementarity and the fact that $j(x_0) = \alpha$, we have $x_0 \in \mathcal{D}$ as well. By applying elementarity repeatedly and the fact that $j(x_i) = x_{i-1}$ for any $1 \leq i < m$, it follows that $\{\alpha, x_0, \ldots, x_{m-1}\} \subseteq \mathcal{D}$.

To see that $\mu \in \mathcal{D}$, note that since $\mathcal{D}$ is a club class, if $\mu \notin \mathcal{D}$, there is some $\lambda < \mu$ such that there is no $\lambda' \in \mathcal{D}$ with $\lambda \leq \lambda' < \mu$. However since $\alpha \in lim(\mathcal{D})$, the least $\xi \in \mathcal{D}$ above λ is less than α. Thus since λ is fixed by j and ξ is definable from λ, we have $j(\xi) = \xi$, which gives $j \restriction V_{\xi+2} : V_{\xi+2} \to V_{\xi+2}$ as a nontrivial elementary embedding, which impossible by the Kunen Inconsistency.

Lastly $j(\mu) \in \mathcal{D}$ follows from elementarity. □

Now let $\xi = min\{x_{m-1}, \alpha, j(\mu)\}$ (note that $\xi = \alpha$ is possible only if $m = 0$, so there are no $(x_i : i < m)$ and $(y_i : i < m)$ from the beginning). By the above claim there is some $\gamma < \xi$ such that there is no $(m+1)$-supercompact measure $\mathcal{U}$ for (μ, γ).

However, since $\gamma < x_{m-1}$, we have $j^m(\gamma) < j^m(x_{m-1}) = j(x_0) = y_0 = \alpha$ (where $j^0(\gamma)$ is defined to be γ). Thus define

$$\mathcal{U} = \{X \subseteq \mathcal{P}(j^m(\gamma)) : j``j^m(\gamma) \in j(X)\}$$

one can verify that $\mathcal{U}$ is an $(m+1)$-supercompact measure for $(j^m(\mu), j^m(\gamma))$ (with target sequences $\mu, j(\mu), \ldots, j^m(\mu)$ and $\gamma, j(\gamma), \ldots, j^m(\gamma)$). Since $\mathcal{U} \in V_\beta$, so by elementarity, there must be some $(m+1)$-supercompact measure for $(j^{m-1}(\mu), j^{m-1}(\gamma))$ which is again in V_β. By applying elementarity repeatedly we get some $(m+1)$-supercompact measure for (μ, γ), which is a contradiction.

(2): Suppose that the class of $(m+1)$-supercompact cardinals is bounded in Ord, then let ν be such that there is no $(m+1)$-supercompact cardinal above ν. Now argue similarly as in (1): define $\mathcal{D}$ to be the class of ordinals $\gamma > \nu$ such that any $\alpha < \gamma$ with $\nu < \alpha$ there is some $\beta < \gamma$ with $F(\alpha) = \beta$. Then proceed as before, and define the class $\mathcal{C}$ of structures of the form $(V_\alpha, \in, \alpha, \delta, \nu)$ such that $\nu < \delta < \alpha$, and α is a limit point of $\mathcal{D}$. Again $\mathcal{C}$ is Π_1-definable with the parameter η, and can be shown to be a proper class. Thus by m-CVP($\mathbf{\Pi_1}$) there is some $(V_\alpha, \in \alpha, \delta, \nu) \in \mathcal{C}$ that m-captures some $(V_\beta, \in, \beta, \eta, \nu) \in \mathcal{C}$. Arguing as before, and noting that $j(\nu) = \nu$, we have that $crit(j) = \mu > \nu$ has some degree of $(m+1)$-supercompactness that leads to a contradiction. □

Theorem 4.3.3. *For every natural number m and cardinal κ, if κ is $(m+1)$-supercompact, then m-*CSR$(\kappa, \mathbf{\Sigma_2})$ holds.

Proof. We prove the case where $m = 3$. It will be clear how the general case can be proved similarly, although the notation will become more cumbersome and less readable.

Suppose that κ is 4-supercompact but 3-CSR$(\kappa, \mathbf{\Sigma_2})$ fails. Let $\lambda > \kappa$ be such that V_λ is sufficiently correct, say $\lambda \in C^{(5)}$, so that $V_\lambda \models$ "3-CSR$(\kappa, \mathbf{\Sigma_2})$ fails".

Let $\mathcal{U}$ be a 4-supercompact measure for (κ, λ), with target sequences $(\kappa_0, \kappa_1, \kappa_2, \kappa_3 = \kappa)$ and $(\lambda_0, \lambda_1, \lambda_2, \lambda_3 = \lambda)$. Let $j : V \to Ult(V, \mathcal{U}) \cong M$ be the conrresponding ultrapower embedding. We then have $crit(j) = \kappa_0$, $j(\kappa_i) = \kappa_{i+1}$ and $j(\lambda_i) = \lambda_{i+1}$ for $0 \leq i \leq 2$, and ${}^\lambda M \subseteq M$.

Since $|V_\lambda| = \lambda$ it follows that $V_\lambda = (V_\lambda)^M$, so $M \models$ "$V_\lambda \models$ "CSR$(\kappa, \mathbf{\Sigma_2})$ fails"", so by applying elementarity 3 times we have $V_{\lambda_0} \models$"CSR$(\kappa_0, \mathbf{\Sigma_2})$ fails".

Thus there are $b_0, \ldots, b_{k-1} \in V_{\kappa_0}$ and some Σ_2-formula ϕ such that

$V_{\lambda_0} \models$ "$\phi(X, b_0, \ldots, b_{k-1})$ defines a class of structures of the same type,
and there is some B with $\phi(B, b_0, \ldots, b_{k-1})$ holds but
there is no $A \in V_{\kappa_0}$ with $\phi(A, b_0, \ldots, b_{k-1})$ holds that 3-captures B".

Let B be a witness of the above statement. By elementarity and the fact that $j(b_i) = b_i$ for all $0 \leq i \leq (k-1)$, it is also true in V_{λ_1} that $\phi(X, b_0, \ldots, b_{k-1})$ defines a class of structures of the same type. Moreover, we have $\lambda_i \in C^{(1)}$ for $0 \leq i \leq 3$, so we have $\phi(B, b_0, \ldots, b_{k-1})$ holds in V, and also in V_{λ_1}. Also,

$V_{\lambda_1} \models$ "$\phi(j(B), b_0, \ldots, b_{k-1})$ holds and
there is no $A \in V_{\kappa_1}$ with $\phi(A, b_0, \ldots, b_{k-1})$ holds that 3-captures $j(B)$".

It follows that B and $j(B)$ are of the same type. Since $B \in V_{\lambda_0} \subseteq V_{\kappa_1}$, it follows that B does not 3-capture $j(B)$, so by Proposition 4.1.5, it follows that there is some fixed $y_0 \in j(B)$ such that

$$\neg\exists x_0 \in B \forall y_1 \in j(B) \exists x_1 \in B \forall y_2 \in j(B) \exists x_2 \in B$$
$$\exists k : B \to j(B) \text{ is elementary with } k(x_i) = y_i \text{ for all } 0 \leq i < 3.$$

By elementarity we have

$$M \models \text{“}\neg\exists x_0 \in j(B) \forall y_1 \in j^2(B) \exists x_1 \in j(B) \forall y_2 \in j^2(B) \exists x_2 \in j(B)$$
$$\exists k : j(B) \to j^2(B) \text{ is elementary with } k(x_0) = j(y_0) \text{ and } k(x_i) = y_i \text{ for all } 1 \leq i < 3\text{”}$$

which is true in V by the closure property of M. Note that the reason for the “$k(x_0) = j(y_0)$” above is because y_0 is a set that is already fixed. Now note that since $y_0 \in j(B)$, it follows that there is some fixed $y_1 \in j^2(B)$ such that

$$\neg\exists x_1 \in j(B) \forall y_2 \in j^2(B) \exists x_2 \in j(B)$$
$$\exists k : j(B) \to j^2(B) \text{ is elementary with } k(y_0) = j(y_0) \text{ and } k(x_i) = y_i \text{ for all } 1 \leq i < 3$$

By elementarity we have

$$M \models \text{“ }\neg\exists x_1 \in j^2(B) \forall y_2 \in j^3(B) \exists x_2 \in j^2(B)$$
$$\exists k : j^2(B) \to j^3(B) \text{ is elementary with } k(j(y_0)) = j^2(y_0),\ k(x_1) = j(y_1) \text{ and } k(x_2) = y_2\text{”}$$

which is true in V. Again this is the result of fixing some y_1. But since $y_1 \in j^2(B)$, this implies that there is some fixed $y_2 \in j^3(B)$ such that

$$\neg\exists x_2 \in j^2(B) \exists k : j^2(B) \to j^3(B) \text{ is elementary}$$
$$\text{with } k(j(y_0)) = j^2(y_0),\ k(y_1) = j(y_1) \text{ and } k(x_2) = y_2.$$

Finally, we have by elementarity that

$$M \models \text{“}\neg\exists x_2 \in j^3(B)\ \exists k : j^3(B) \to j^4(B) \text{ is elementary}$$
$$\text{with } k(j^2(y_0)) = j^3(y_0),\ k(j(y_1)) = j^2(y_1) \text{ and } k(x_2) = j(y_2)\text{”}.$$

However, letting $k := j \restriction j^3(B) : j^3(B) \to j^4(B)$, we have k is elementary, and $|k| = |j^3(B)| < |V_{j^3(\lambda_0)}| = \lambda$. Thus $k \in M$ by ${}^\lambda M \subseteq M$. Moreover, we have $k(j^2(y_0)) = j^3(y_0)$, $k(j(y_1)) = j^2(y_1)$ and $k(y_2) = j(y_2)$. Thus k is exactly the embedding that is missing in M, contradiction. $\square$

Corollary. *The following are equivalent for every natural number m:*

1. m-CVP(Π_1).
2. m-CSR$(\kappa, \mathbf{\Sigma_2})$ holds for some κ.
3. There exists an $(m+1)$-supercompact cardinal.

Corollary. *The following are equivalent for every natural number m:*

1. m-CVP$(\mathbf{\Pi_1})$.
2. m-CSR$(\kappa, \mathbf{\Sigma_2})$ holds for a proper class of κ.

3. There exists a proper class of $(m+1)$-supercompact cardinals.

Corollary. *The following are equivalent for every natural number m and cardinal κ:*

1. *κ is either an $(m+1)$-supercompact cardinal or a limit of $(m+1)$-supercompact cardinals.*

2. *m-*CSR$(\kappa, \mathbf{\Pi_1})$ holds.

*Moreover, the least κ such that m-*CSR(κ, Π_1) holds is the least $(m+1)$-supercompact cardinal.

Proof. This can be proved by combining the methods in the proofs of Theorem 3.1.6 and Theorem 4.3.1. □

Theorem 4.3.4. *For all natural numbers $n \geq 1$ and m:*

1. *If m-*CVP(Π_{n+1}) holds, then there exists a $C^{(n)}$-$(m+1)$-fold extendible cardinal.

2. If m-CVP$(\mathbf{\Pi_{n+1}})$ holds, then there is a proper class of $C^{(n)}$-$(m+1)$-fold extendible cardinals.

Proof. (1): Suppose m-CVP(Π_{n+1}) holds, but there is no $C^{(n)}$-$(m+1)$-fold extendible cardinal. Define a class $\mathcal{D}$ of ordinals such that $\alpha \in \mathcal{D}$ if and only if

(i) $\alpha \in lim(C^{(n+1)})$, and

(ii) For any $\beta < \alpha$, there is some γ with $\beta \leq \gamma < \alpha$ such that for any $\xi \leq \beta$, ξ is not γ-$C^{(n)}$-$(m+1)$-fold extendible.

$\mathcal{D}$ is a Π_{n+1}-definable, without parameters, class of ordinals. To see this, note that clause (i) is equivalent to

$$\alpha \in C^{(n+1)} \wedge \forall X(X = V_\alpha \rightarrow X \models \text{“}\forall\beta\exists\gamma \geq \beta(\gamma \in C^{(n+1)})\text{”})$$

and clause (ii) is equivalent to the conjunction of $\alpha \in C^{(n+1)}$ and

$$\forall X(X = V_\alpha \rightarrow X \models \text{“}\forall\beta\exists\gamma \geq \beta\forall\xi \leq \beta(\xi \text{ is not } \gamma\text{-}C^{(n)}\text{-}(m+1)\text{-fold extendible)”}).$$

Now let $\mathcal{C}$ be the class of structures of the form $(V_\alpha, \in, \alpha, \delta)$ such that α is the least limit point of $\mathcal{D}$ above δ. $\mathcal{C}$ is Π_{n+1}-definable without parameters. Moreover, since we are supposing that there are no $C^{(n)}$-$(m+1)$-fold extendible cardinals, we have $C^{(n+2)} \subseteq \mathcal{D}$, thus $\mathcal{D}$ is a proper club class, which implies that $\mathcal{C}$ is a proper class. Note that if $(V_\alpha, \in, \alpha, \delta) \in \mathcal{C}$, then $cf(\alpha) = \omega$.

Thus by m-CVP(Π_{n+1}), there are $A = (V_\alpha, \in, \alpha, \delta), B = (V_\beta, \in, \beta, \eta) \in \mathcal{C}$ with $A \neq B$ such that A m-captures B. It follows that $\delta \neq \eta$, otherwise $\alpha = \beta$ so $A = B$. Thus we have $\delta < \eta$, which implies that $\delta < \alpha \leq \eta < \beta$. Now if $m \geq 1$, we construct recursively the sequences $(x_i : i < m)$ and $(y_i : i < m)$ in exactly the same way as in Theorem 4.3.1, which result in sequences $(x_i : i < m)$ and $(y_i : i < m)$, such that there is some elementary embedding $j : (V_\alpha, \in, \alpha, \delta) \rightarrow (V_\beta, \in, \beta, \eta)$ with $j(x_i) = y_i = x_{i-1}$ for any $i \geq 1$, and $j(x_0) = y_0 = \alpha$. Similarly as in Theorem 4.3.1, the x_i's and y_i's are ordinals with countable cofinality, and $x_i < j(x_i)$ for any $0 \leq i < m$. Also $j(\delta) = \eta > \delta$. Thus j is nontrivial and we have that $crit(j) < x_{m-1}$.

Claim 4.3.5. $\{\alpha, x_0, \ldots, x_{m-1}, \mu, j(\mu)\} \subseteq \mathcal{D}$.

Proof. That $\alpha \in \mathcal{D}$ is already given. That $x_i \in \mathcal{D}$ for $0 \leq i < m$ is shown similarly as in Theorem 4.3.1, by noting that $\mathcal{D}$ is definable in both V_α and V_β, since $\alpha, \beta \in C^{(n+1)}$. That $\mu, j(\mu) \in \mathcal{D}$ is also shown similarly. □

Now let $\xi = min\{x_{m-1}, \alpha, j(\mu)\}$ ($\xi = \alpha$ can only happen if $m = 0$). By the above claim there is some $\gamma < \xi$ such that μ is not γ-$C^{(n)}$-$(m+1)$-fold extendible.

However, since $\mu + \gamma < x_{m-1}$, we have $j^m(\mu + \gamma) < j^m(x_{m-1}) = j(x_0) = y_0 = \alpha$. So in fact $k = j \restriction V_{j^m(\mu+\gamma)} : V_{j^m(\mu+\gamma)} \to V_{j^{m+1}(\mu+\gamma)}$ is elementary, with $crit(k) = \mu$, $k(\mu) \in \mathcal{D} \subseteq C^{(n)}$, and $\mu + \gamma < \xi \leq j(\mu) = k(\mu)$, so k witnesses that μ is γ-$C^{(n)}$-$(m+1)$-fold extendible, which is a contradiction.

(2) is shown similarly as before, noting that if ν is such that there is no $C^{(n)}$-$(m+1)$-fold extendible cardinal above ν, then we use the parameter ν in the definition of some class $\mathcal{C}$ and the critical point of $j : A \to B$, for some $A, B \in \mathcal{C}$, will be greater than ν, and has some degree of $C^{(n)}$-$(m+1)$-fold extendibility that leads to a contradiction. □

Theorem 4.3.6. *For all natural numbers $n \geq 1$ and m, and for every cardinal κ, if κ is $C^{(n)}$-$(m+1)$-fold extendible, then m-CSR$(\kappa, \mathbf{\Sigma_{n+2}})$ holds.*

Proof. We prove the case where $m = 3$, where the general case can be proved similarly. Suppose that κ is $C^{(n)}$-4-fold extendible. Let $\mathcal{C} = \{B : \phi(B, b_0, \dots, b_{k-1})\}$ be a class of structures of the same type, where $\phi(x, y_0, \dots, y_{k-1})$ is Σ_{n+2} and $b_0, \dots, b_{k-1} \in V_\kappa$. For any $B \in \mathcal{C}$, suppose for contradiction there is no $A \in \mathcal{C} \cap V_\kappa$ that 3-captures B. Since $\phi(x, y_0, \dots, y_{k-1})$ is of the form $\exists z \psi(z, x, y_0, \dots, y_{k-1})$, where $\psi(z, x, y_0, \dots, y_{k-1})$ is Π_{n+1}, there is some a such that $\psi(a, B, b_0, \dots, b_{k-1})$ holds in V. We may suppose that $rk(B) > \kappa$.

Let $\lambda > max\{rank(a), rank(B)\}$, and let $j : V_{j^3(\kappa+\lambda)} \to V_\delta$ witnesses that κ is λ-$C^{(n)}$-4-fold extendible, with $crit(j) = \kappa$, $\kappa + \lambda < j(\kappa)$, and $j(\kappa) \in C^{(n)}$. Since $\kappa \in C^{(n+2)}$ by Proposition 2.2.6, it follows that there is no $A \in V_\kappa$ such that $V_\kappa \models$ "$A \in \mathcal{C}$" and A 3-captures B.

By elementarity, there is no $A \in V_{j(\kappa)}$ such that $V_{j(\kappa)} \models$ "$A \in \mathcal{C}$" and A 3-captures $j(B)$. Now since $B, a \in V_{j(\kappa)}$, the Π_{n+1} statement $\psi(a, B, b_0, \dots, b_{k-1})$ reflects to $V_{j(\kappa)}$, thus $V_{j(\kappa)} \models$ "$\phi(B, b_0, \dots, b_{k-1})$ holds", so B does not 3-capture $j(B)$. By Proposition 4.1.5 it follows that there is some fixed $y_0 \in j(B)$ such that

$$\neg \exists x_0 \in B \forall y_1 \in j(B) \exists x_1 \in B \forall y_2 \in j(B) \exists x_2 \in B$$
$$\exists k : B \to j(B) \text{ is elementary with } k(x_i) = y_i \text{ for all } 0 \leq i < 3\text{''}.$$

Now we proceed exactly as in the proof of Theorem 4.3.3 to reach a contradiction. □

Corollary. *The following are equivalent for all natural numbers $n \geq 1$ and m:*

1. m-CVP(Π_{n+1}).
2. m-CSR$(\kappa, \mathbf{\Sigma_{n+2}})$ holds for some κ.
3. There exists a $C^{(n)}$-$(m+1)$-fold extendible cardinal.

Corollary. *The following are equivalent for all natural numbers $n \geq 1$ and m:*

1. m-CVP$(\mathbf{\Pi_{n+1}})$.
2. m-CSR$(\kappa, \mathbf{\Sigma_{n+2}})$ holds for a proper class of κ.
3. There is a proper class of $C^{(n)}$-$(m+1)$-fold extendible cardinals.

Corollary. *The following are equivalent for all natural numbers $n \geq 1$ and m, and every cardinal κ:*

1. *κ is either a $C^{(n)}$-$(m+1)$-fold extendible cardinal or a limit of $C^{(n)}$-$(m+1)$-fold extendible cardinals.*
2. *m-*CSR$(\kappa, \mathbf{\Pi_{n+1}})$ holds.

*Moreover, the least κ such that m-*CSR(κ, Π_{n+1}) holds is the least $C^{(n)}$-$(m+1)$-fold extendible cardinal.

Proof. This can be proved by combining the techniques in the proofs of Theorem 3.2.6 and Theorem 4.3.4. □

Chapter 5

Characterization

In this short chapter we give a synthesis and clarification of the relevant large cardinal notions by showing that the notion of $C^{(n)}$-m-fold extendible cardinals and $C^{(n)}$-m hyperhuge cardinals can be characterized by the more general notion of *Σ_n-m-supercompact cardinals*. This notion of large cardinals is a generalization of both the m-supercompact cardinals introduced in Chapter 3 and the Σ_n-supercompact cardinals introduced by Joan Bagaria and Alejandro Poveda in [10], which are used to characterize $C^{(n)}$-extendible cardinals.

Moreover, in [5], Joan Bagaria and Gabriel Goldberg introduced the notion of *reflecting measures*, and used them to give $C^{(n)}$-extendible cardinals a characterization in terms of ultrafilters, which was the first formulation of extendible and $C^{(n)}$-extendible cardinals in terms of measures. Generalizing this result, we show that Σ_n-m-supercompact cardinals can be characterized in terms of generalizations of reflecting measures.

5.1 The general notion

Definition 5.1.1. *Given natural numbers $m \geq 1$, n and cardinal κ, for $\lambda \in C^{(n)}$ greater than κ, we say κ is λ-Σ_n-m-supercompact if for any $y \in V_\lambda$, there is $\bar{\lambda} < \kappa$ and $x \in V_{\bar{\lambda}}$, with $\bar{\lambda} \in C^{(n)}$ and some elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(x) = y$ and $j^m(crit(j)) = \kappa$.*

κ is Σ_n-m-supercompact if κ is λ-Σ_n-m-supercompact for any $\lambda \in C^{(n)}$ greater than κ.

In the paper [1] where SR was first introduced, Bagaria used the notion of $C^{(n)+}$-*extendible cardinal* in order to study the equivalence between SR and large cardinals, where κ is $C^{(n)+}$-extendible if for any $\lambda \in C^{(n)}$ greater than κ, κ is λ-$C^{(n)}$-extendible, witnessed by some $j : V_\lambda \to V_\delta$ with the additional requirement that $\delta \in C^{(n)}$. It was not known what is the precise relation between $C^{(n)+}$-extendible cardinals and $C^{(n)}$-extendible cardinals.

Later it was shown independently by Konstantinos Tsaprounis in [39] and Victoria Gitman and Joel Hamkins in [18] that the two notions are equivalent. See also [3].

One can also define similar notions in the m-fold case:

Definition 5.1.2. *For natural numbers $m \geq 1$, n, cardinal κ and ordinal $\eta \in C^{(n)}$, we say κ is η-$C^{(n)+}$-m-fold extendible if there is some elementary embedding $j : V_{j^{m-1}(\kappa+\eta)} \to V_\delta$ with $crit(j) = \kappa$, $\kappa + \eta < j(\kappa)$, and $\{\delta, \eta, j(\eta), \ldots j^{m-1}(\eta), \kappa, j(\kappa), \ldots, j^m(\kappa)\} \subseteq C^{(n)}$.*

κ is $C^{(n)+}$-m-fold extendible if κ is η-$C^{(n)+}$-m-fold extendible for any $\eta \in C^{(n)}$.

This may seem to be a stronger definition, but it turns out that again this $C^{(n)+}$ strengthening is equivalent to the original notion:

Corollary. *For natural numbers $m \geq 1$, n and cardinal κ, the following are equivalent:*

1. κ *is* $C^{(n)}$*-m-fold extendible.*

2. κ *is* $C^{(n)+}$*-m-fold extendible.*

This will be a corollary of the following theorem, which gives a more complete picture of the relevant large cardinal notions:

Theorem 5.1.3. *For natural numbers* $m, n \geq 1$ *and cardinal* κ*, the following are equivalent:*

1. κ *is* Σ_{n+1}*-*$(m+1)$*-supercompact.*

2. κ *is* $C^{(n)}$*-m-hyperhuge.*

3. κ *is* $C^{(n)}$*-*$(m+1)$*-fold extendible.*

Moreover, we have

(i) If $m = 0$ *and* $n = 0$*, then (1) and (2) are still equivalent, but (3) is strictly stronger.*

(ii) If $m \neq 0$ *but* $n = 0$*, then (2) and (3) are still equivalent, but (1) is strictly weaker.*

(iii) If $m = 0$ *but* $n \neq 0$*, then (1) and (3) are still equivalent, but (2) is strictly weaker.*

Proof. For $m, n \geq 1$, Theorem 2.3.3 already shows that (2) and (3) are equivalent. We now show that for natural numbers $n \geq 1$ and m, (1) and (3) are equivalent.

(1) implies (3): We will actually show that (1) implies that κ is $C^{(n)+}$-$(m+1)$-fold extendible, to give the above corollary. Define an ordinal function F by letting $F(\alpha)$ be the least $\eta \in C^{(n)}$ above α such that α is not η-$C^{(n)+}$-$(m+1)$-fold extendible, if there is such an η, and $F(\alpha) = 0$ otherwise.

Suppose for a contradiction $F(\kappa) = \eta > \kappa$. Let $\lambda \in C^{(n+1)}$ be greater than η, and $\bar{\lambda} \in C^{(n+1)}$ less than κ, with some elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ and $j^{m+1}(crit(j)) = \kappa$. Let $crti(j) = \mu$.

By noting that $C^{(n)}$ is a club class and $\bar{\lambda} \in lim(C^{(n)})$, one can use similar argument as in Theorem 3.1.4 to show that $\{\mu, j(\mu), \ldots j^{m+1}(\mu)\} \subseteq C^{(n)}$, otherwise there is a violation of Kunen Inconsistency.

Note that "$F(\kappa) = \eta$" is Δ_{n+2} (this is where we need $n \geq 1$), since it is equivalent to the conjunction of the Σ_{n+1}-assertion

$$\kappa < \eta \in C^{(n)} \wedge \forall \eta' < \eta[(\kappa < \eta' \wedge \eta' \in C^{(n)}) \to \kappa \text{ is } \eta'\text{-}C^{(n)+}\text{-}(m+1)\text{-fold extendible}]$$

and the Π_{n+1}-assertion "κ is not η-$C^{(n)+}$-$(m+1)$-fold extendible". Given that $V_{\bar{\lambda}} \prec_{\Sigma_{n+1}} V$ and $V_\lambda \prec_{\Sigma_{n+1}} V$, the formula "$F(\alpha) = \beta$" is absolute between $V_{\bar{\lambda}}$, V_λ, and V.

Thus the assertion "$F(\kappa) = \eta$" reflects down to V_λ. By elementarity

$$V_{\bar{\lambda}} \models \text{"}\exists \xi > j^m(\mu) F(j^m(\mu)) = \xi\text{"},$$

which implies, by $\xi < \bar{\lambda} < \kappa$ and absoluteness, that

$$V_\lambda \models \text{"}\exists \xi (j^m(\mu) < \xi < j^{m+1}(\mu) \wedge F(j^m(\mu)) = \xi)\text{"},$$

now similarly as in Theorem 2.1.4, by applying elementarity iteratively we get some $\nu = F(\mu)$ with $\mu < \nu < j(\mu)$. Note that we have $j^m(\nu) = j^m(F(\mu)) = F(j^m(\mu)) < \bar{\lambda}$, so since $\nu \in C^{(n)}$, by elementarity we have $\{\nu, j(\nu), \ldots, j^{m+1}(\nu)\} \subseteq C^{(n)}$. Also we have seen that $\{\mu, j(\mu), \ldots, j^{m+1}(\mu)\} \in C^{(n)}$, so $j \restriction V_{j^m(\nu)} : V_{j^m(\nu)} \to V_{j^{m+1}(\nu)}$ witnesses that μ is ν-$C^{(n)+}$-$(m+1)$-fold extendible, a contradiction.

(3) implies (1): Suppose for contradiction that $\lambda > \kappa$, $\lambda \in C^{(n+1)}$, and $y \in V_\lambda$ but there is no $\bar{\lambda} < \kappa$ with $\bar{\lambda} \in C^{(n+1)}$ and $x \in V_{\bar{\lambda}}$ with some elementary $k : V_{\bar{\lambda}} \to V_\lambda$ with $k^{m+1}(crit(k)) = \kappa$ and $k(x) = y$. Let $\lambda' > \lambda$ with $\lambda' \in C^{(1)}$, and $j : V_{j^m(\lambda')} \to V_\delta$ witnesses the λ'-$C^{(n)}$-$(m+1)$-fold extendibility of κ.

Since $\kappa \in C^{(n+1)}$ by Proposition 2.2.6 and Theorem 2.3.3, we have

$$V_{j^m(\lambda')} \models \text{“There is no } \bar{\lambda} < \kappa \text{ with } V_\kappa \models \text{“}\bar{\lambda} \in C^{(n+1)}\text{” and } x \in V_{\bar{\lambda}}$$
$$\text{with some elementary } k : V_{\bar{\lambda}} \to V_\lambda \text{ with } k^{m+1}(crit(k)) = \kappa \text{ and } k(x) = y\text{”}$$

which implies that

$$V_\delta \models \text{“There is no } \bar{\lambda} < j(\kappa) \text{ with } V_{j(\kappa)} \models \text{“}\bar{\lambda} \in C^{(n+1)}\text{” and } x \in V_{\bar{\lambda}}$$
$$\text{with some elementary } k : V_{\bar{\lambda}} \to V_{j(\lambda)} \text{ with } k^{m+1}(crit(k)) = j(\kappa) \text{ and } k(x) = j(y)\text{”.}$$

Since $j(\kappa) \in C^{(n)}$, $V_{j(\kappa)} \models$ “$\lambda \in C^{(n+1)}$”. It follows that there is no elementary $k : V_\lambda \to V_{j(\lambda)}$ with $k^{m+1}(crit(k)) = j(\kappa)$ and $k(y) = j(y)$.

Now by applying elementarity m-more times, we get that there is no elementary $k : V_{j^m(\lambda)} \to V_{j^{m+1}(\lambda)}$ with $k^{m+1}(crit(k)) = j^{m+1}(\kappa)$ and $k(j^m(y)) = j^{m+1}(y)$, which contradicts the existence of $k = j \restriction V_{j^m(\lambda)}$.

Now for (i), note that if $m, n = 0$ then (1) and (2) are both asserting that κ is supercompact, by observing the fact that if $j : V_{\bar{\lambda}} \to V_\lambda$ is elementary and $\lambda \in C^{(1)}$, then $\bar{\lambda} \in C^{(1)}$. Also, (3) says that κ is extendible, which has stronger consistency strength and a larger least instance.

For (ii), note that if $m \neq 0$ but $n = 0$, then the equivalence of (2) and (3) follows from Theorem 2.3.3, the fact that m-hyperhugeness is the same as $C^{(1)}$-m-hyperhugeness, and similarly for $(m+1)$-fold extendibility and $C^{(1)}$-$(m+1)$-fold extendibility. (1) is the notion of m-supercompactness, which by Proposition 2.2.8 is weaker in strength and with a smaller least instance.

For (iii), note that if $m = 0$ but $n \neq 0$, (1) and (3) both say that κ is $C^{(n)}$-extendible, since our argument above does not assume that $m \geq 1$. Also (1) is the notion of $C^{(n)}$-supercompactness, which as remarked before, was shown in [28] to be weaker in strength and with a smaller least instance. □

As in Corollary 2.1, the above also gives the following:

Corollary. *For every natural numbers $m \geq 1$ and n, a cardinal κ is Σ_n-m-supercompact if for any $\lambda \in C^{(n)}$ greater than κ, there is $\bar{\lambda} < \kappa$, with $\bar{\lambda} \in C^{(n)}$ and some elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j^m(crit(j)) = \kappa$.*

Corollary. ([10]) *For natural number $n \geq 1$, κ is $C^{(n)}$-extendible if and only if κ is Σ_{n+1}-supercompact.*

5.2 Reflecting measures

We now define the notion of an n-reflecting m-supercompact measure that generalizes the notion of *n-reflecting measure* introduced in [5]. Using Theorem 5.1.3, we give characterizations of m-supercompact cardinals and $C^{(n)}$-m-fold extendible cardinals in terms of n-reflecting m-supercompact measures, generalizing the results in [5].

Definition 5.2.1. *Given natural numbers $m \geq 1$, n, cardinal κ and $\lambda \geq \kappa$, we say that an m-supercompact measure $\mathcal{U}$ for (κ, λ) is n-reflecting if $T^n_\lambda = \{x \in \mathcal{P}(\lambda) : ot(x) \in C^{(n)}\} \in \mathcal{U}$.*

It was shown in [5] that a cardinal $\kappa is supercompact if and only if there is a 11-reflecting normal measure on$ $reflecting 11-supercompact measure for (,)(\kappa,\lambda)) for any \lambda \geq \kappa in C(1) C^{(1)}. In fact more is true, namely that a$ $reflecting :$

Theorem 5.2.2. *For natural number $m \geq 1$,*

1. *If κ is a cardinal and $\lambda \geq \kappa$ is in $C^{(1)}$, then every m-supercompact measure $\mathcal{U}$ for (κ, λ) is 1-reflecting.*
2. *A cardinal κ is m-supercompact if and only if for every $\lambda \in C^{(1)}$ greater than or equal to κ, there is a 1-reflecting m-supercompact measure for (κ, λ).*

Proof. Note that (2) immediately follows from (1). For (1), suppose that $\kappa \leq \lambda \in C^{(1)}$ and $\mathcal{U}$ is an m-supercompact measure for (κ, λ), then let $j : V :\to Ult(V, \mathcal{U}) \cong M$ be the ultrapower embedding, with ${}^\lambda M \subseteq M$. Define $\mathcal{U}^* = \{X \subseteq \mathcal{P}(\lambda) : j``\lambda \in j(X)\}$. Then, since we have

$$j(T^1_\lambda) = \{x \in \mathcal{P}^M(j(\lambda)) : (ot(x) \in C^{(1)})^M\}$$

and also $(V_\lambda)^M = V_\lambda$ and $ot(j``\lambda) = \lambda$, it follows that $j``\lambda \in j(T^1_\lambda)$, thus $T^1_\lambda \in \mathcal{U}^*$. But also

$$X \in \mathcal{U}^* \text{ iff } j``\lambda \in j(X) \text{ iff (by normality) } [id]_\mathcal{U} \in j(X) \text{ iff } \{x \in \mathcal{P}(\lambda) : x \in X\} = X \in \mathcal{U},$$

So in fact $T^1_\lambda \in \mathcal{U} = \mathcal{U}^*$. □

More generally:

Theorem 5.2.3. *For all natural numbers $m, n \geq 1$, a cardinal κ is Σ_n-m-supercompact if and only if for every $\lambda \in C^{(n)}$ greater or equal to κ there is an n-reflecting m-supercompact measure for (κ, λ).*

Proof. For the forward direction, using similar ideas as before, let $F : Ord \to Ord$ be such that $F(\alpha)$ is the least $\beta \in C^{(n)}$ above α such that there is no n-reflecting m-supercompact measure for (α, β), if such a β exists. Otherwise $F(\alpha) = 0$.

Suppose for a contradiction that κ is Σ_n-m-supercompact but $F(\kappa) > \kappa$. Let $\lambda > F(\kappa)$ be in $C^{(n)}$ and $j : V_{\bar{\lambda}} \to V_\lambda$ is elementary, for some $\bar{\lambda} < \kappa$ in $C^{(n)}$, with $j^m(crit(j)) = \kappa$. Let $crit(j) = \mu$ and $\mu_i = j^i(\mu)$ for $0 \leq i \leq m$

It follows by arguments similar to the direction (1) to (3) of Theorem 5.1.3 that $\mu_{m-1} < F(\mu_{m-1}) < \mu_m$, which by elementarity implies that $\mu_0 < F(\mu_0) < \mu_1$ and also that $j^{m-1}(F(\mu_0)) = F(\mu_{m-1})$. Define

$$\mathcal{U} = \{X \subseteq \mathcal{P}(F(\mu_{m-1})) : j``F(\mu_{m-1}) \in j(X)\}$$

Since $F(\mu_{m-1}) = ot(j``F(\mu_{m-1})) \in C^{(n)}$, and this reflects to V_λ, it follows that

$$T^n_{F(\mu_{m-1})} = \{x \in \mathcal{P}(F(\mu_{m-1})) : ot(x) \in C^{(n)}\} = \{x \in \mathcal{P}(F(\mu_{m-1})) : V_{\bar{\lambda}} \models ``ot(x) \in C^{(n)}"\} \in \mathcal{U}$$

so $\mathcal{U}$ is an n-reflecting m-supercompact measure for $(\mu_{m-1}, F(\mu_{m-1}))$, with targets sequences $\mu_0, \ldots, \mu_{m-1}$ and $F(\mu_0), \ldots, F(\mu_{m-1})$, contradicting the definition of F.

Conversely, if $\kappa \leq \lambda \in C^{(n)}$ and $\mathcal{U}$ is an n-reflecting m-supercompact measure for (κ, λ), then let $j : V :\to Ult(V, \mathcal{U}) \cong M$ be the ultrapower embedding. Thus ${}^\lambda M \subseteq M$ and $j^{m-1}(crit(j)) = \kappa$. Then, since as in Theorem 5.2.2 we have $X \in \mathcal{U}$ if and only if $j``\lambda \in j(X)$, $T^n_\lambda \in \mathcal{U}$ and $j(T^n_\lambda) = \{x \in \mathcal{P}^M(j(\lambda)) : (ot(x) \in C^{(n)})^M\}$ implies that $(\lambda \in C^{(n)})^M$. Thus, since $k = j \restriction V_\lambda \in M$,

$M \models$ "$\exists \bar\lambda < j(\kappa)$ with $\bar\lambda \in C^{(n)}$ and an elementary $k : V_{\bar\lambda} :\to V_{j(\lambda)}$ with $k^m(crit(k)) = j(\kappa)$"

and so there is some $\bar\lambda < \kappa$ with $\bar\lambda \in C^{(n)}$ and an elementary $k : V_{\bar\lambda} \to V_\lambda$ with $k^m(crit(k)) = \kappa$. It follows by Corollary 5.1 that κ is Σ_n-m-supercompact. $\square$

Corollary. *For all natural numbers $n, m \geq 1$, a cardinal κ is $C^{(n)}$-m-fold-extendible if and only if for every $\lambda \in C^{(n+1)}$ greater than or equal to κ, there is an $(n+1)$-reflecting m-supercompact measure for (κ, λ).*

Proof. This simply follows from Theorem 5.1.3 and Theorem 5.2.3. $\square$

Corollary. ([5]) *For every natural number $n \geq 1$, a cardinal κ is $C^{(n)}$-extendible if and only if for all $\lambda \in C^{(n+1)}$ greater than or equal to κ there is an $(n+1)$-reflecting 1-supercompact measure for (κ, λ).*

Chapter 6

On Exact Structural Reflection

In this chapter we study the principles of *Exact Structural Reflection* (ESR) proposed by Joan Bagaria and Philipp Lücke in [6]. These principles are motivated by their search for structural reflection principles with higher consistency strength than *Vopěnka's Principle*. Many basic questions remained open, including the exact strength of ESR, and the pattern in which they correspond to large cardinals. In the following we will clarify these issues, in particular we answer several open questions asked in the paper.

Interestingly, the answers to these questions turn out to be somewhat unexpected. We observe some connections between ESR, exact cardinals, and our notion of Σ_n-m-supercompact cardinals. Some of the questions are answered by applications of the theory of these cardinals and their correspondence to the capturing reflection principles.

6.1 ESR and exact cardinals

We first recall some basic definitions and results, all of them from [6].

Definition 6.1.1. Given cardinals $\kappa < \lambda$ and a class $\mathcal{C}$ of structures of the same type, we say $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds if for any structure $B \in \mathcal{C}$ of rank λ, there is some $A \in \mathcal{C}$ of rank κ and an elementary embedding from A to B. Given a natural number n,

1. $\Sigma_n(P)$-$\mathrm{ESR}(\kappa, \lambda)$ holds if for any $\mathcal{C}$ that is Σ_n-definable with parameters in P, $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds. If P is empty we write Σ_n-$\mathrm{ESR}(\kappa, \lambda)$.

2. $\Sigma_n(P)$-$\mathrm{ESR}(\kappa)$ holds if $\Sigma_n(P)$-$\mathrm{ESR}(\kappa, \lambda)$ holds for some $\lambda > \kappa$.

3. $\Sigma_n(P)^{ic}$-$\mathrm{ESR}(\kappa, \lambda)$ holds if for any $\mathcal{C}$ that is Σ_n-definable with parameters in P and is closed under isomorphic copies, $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds. Similarly for $\Sigma_n(P)^{ic}$-$\mathrm{ESR}(\kappa)$.

Similarly for $\Pi_n(P)$-$\mathrm{ESR}(\kappa, \lambda)$, $\Pi_n(P)$-$\mathrm{ESR}(\kappa)$, $\Pi_n(P)^{ic}$-$\mathrm{ESR}(\kappa, \lambda)$, and $\Pi_n(P)^{ic}$-$\mathrm{ESR}(\kappa)$.

With the motivation of finding large cardinal notions that correspond to the ESR principles, Bagaria and Lücke introduced the *weakly exact cardinals* and *exact cardinals*. We say a set M is $\Pi_n(P)$-*upwards correct*, for some natural number n and class P, if for any Π_n-formula $\phi(x_0, \ldots, x_{k-1})$ and parameters $b_0, \ldots, b_{k-1} \in M \cap P$, if $M \models$ "$\phi(x_0, \ldots, x_{k-1})$" then $\phi(x_0, \ldots, x_{k-1})$ holds. The notion of $\Pi_n(P)$-*downwards correct* is similarly defined.

Definition 6.1.2. Given a natural number $n \geq 1$, a cardinal κ is *weakly n-exact for a cardinal $\lambda > \kappa$* if for any $A \in V_{\lambda+1}$, there is some transitive, $\Pi_n(V_{\kappa+1})$-upwards correct M with $V_\kappa \cup \{\kappa\} \subseteq M$, a cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$ and an elementary embedding

$j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A \in ran(j)$. If we further require that $j(crit(j)) = \kappa$ then we say κ is *weakly prarametrically n-exact for λ**.

Weakly exact cardinals correspond to ESR for Π_n-definable classes:

Theorem 6.1.3. *The following statements are equivalent for all cardinals κ and all natural numbers $n > 0$:*

1. *κ is the least regular cardinal such that Π_n^{ic}-ESR(κ) holds.*
2. *κ is the least cardinal such that Π_n-ESR(κ) holds.*
3. *κ is the least cardinal such that $\Pi_n(V_\kappa)$-ESR(κ) holds.*
4. *κ is the least cardinal that is weakly n-exact for some cardinal $\lambda > \kappa$.*
5. *κ is the least cardinal that is weakly parametrically n-exact for some cardinal $\lambda > \kappa$.*

Definition 6.1.4. Given a a natural number n, a cardinal κ is *n-exact for some cardinal $\lambda > \kappa$* if for any $A \in V_{\lambda+1}$ there is a cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\kappa$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, an elementary substructure X of $H_{\kappa'}$, with $V_\kappa \cup \{\kappa\} \subseteq X$, and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A \in ran(j)$. If we further require that $j(crit(j)) = \kappa$ then we say that κ is *parametrically n-exact for λ*.

Exact cardinals correspond to ESR for Σ_{n+1}-definable classes.

Theorem 6.1.5. *The following are equivalent for all cardinals κ and all natural numbers $n > 0$:*

1. *κ is the least cardinal such that Σ_{n+1}-ESR(κ) holds.*
2. *κ is the least cardinal such that $\Sigma_{n+1}(V_\kappa)$-ESR(κ) holds.*
3. *κ is the least cardinal that is n-exact for some cardinal $\lambda > \kappa$.*
4. *κ is the least cardinal that is parametrically n-exact for some cardinal $> \kappa$.*

Many questions remained open, two of them being the exact *consistency strength* of the exact cardinals, and the *pattern of correspondence* between ESR and exact cardinals. These issues will be elaborated and discussed in the following sections.

6.2 On strength

Here are three relevant results from [6] regarding the strength of exact cardinals:

Proposition 6.2.1. If κ is either parametrically 0-exact for λ or weakly parametrically 1-exact for λ, then the set of $\mu < \kappa$ such that μ is almost huge with target κ is stationary in κ.

Proposition 6.2.2. If κ is the least huge cardinal, then κ is not 1-exact for any $\lambda > \kappa$.

Proposition 6.2.3. If κ is the critical point of some I3 embedding $j : V_\delta \to V_\delta$ and $l, m, n < \omega$, then it holds in V_δ that $j^l(\kappa)$ is parametrically n-exact for $j^{l+m+n}(\kappa)$.

*The definition here is a bit different from the version of [6] that is currently published, which uses "$\Pi_n(V_{\kappa+1})$-correct" instead of "$\Pi_n(V_{\kappa+1})$-upwards correct". The definition here is the correct one; see the forthcoming corrigendum.

One naturally wonders what is the precise consistency strength of the exact cardinals, in particular the following open question was asked in [6]:

Question 6.2.4 ([6]). Does Con(ZFC+"there is a huge cardinal") imply Con(ZFC+"Σ_2-ESR(κ) holds for some κ")?

The three results above show that, on the one hand, 0-exact cardinals and weakly 1-exact cardinals, which are among the weakest in the whole hierarchy, already imply many almost huge cardinals, and the least huge cardinal κ cannot be 1-exact for any $\lambda > \kappa$. On the other hand, the upper bound for the whole hierarchy given is the very strong axiom of an I3 embedding. These results can natually be taken to indicate that the answer to 6.2.4 might be negative, as the remarks in [6] suggest.

Perhaps surprisingly, it turns out that the answer is positive in the strongest sense: a huge cardinal is strictly stronger than the whole hierarchy of n-exact cardinals, for all n:

Theorem 6.2.5. *If κ is a huge cardinal, then for any natural number n, V_κ is a model of ZFC plus the following statement:*

> *"There is a proper class of cardinals μ such that μ is parametrically n-exact for a proper class of cardinals λ".*

Proof. Let n be a natural number. Let $\kappa = \kappa_0$ be a huge cardinal with target κ_1, and let $\mathcal{U}$ be a witnesssing normal ultrafilter over $\mathcal{P}(\kappa_1)$. Let $j : V \to Ult(V,\mathcal{U}) \cong M$ be the corresponding ultrapower embedding, with $crit(j) = \kappa_0, j(\kappa_0) = \kappa_1$, and ${}^{\kappa_1}M \subseteq M$. Furthermore let $j(\kappa_1) = \kappa_2$.

First note that we have $V_{\kappa_1} = (V_{\kappa_1})^M \subseteq (V_{\kappa_2})^M$. Also, $V_{\kappa_0} \prec V_{\kappa_1}$, so by elementarity, we have $V_{\kappa_1} \prec (V_{\kappa_2})^M$. Moreover, note that by elementarity we have $(V_{\kappa_2})^M$ is a model of ZFC.

Let α be such that $\kappa_1 < \alpha < \kappa_2$ and $(V_{\kappa_2})^M \models$ "$\alpha \in C^{(n+1)}$". This makes sense since $(V_{\kappa_2})^M \models$ ZFC implies

$$(V_{\kappa_2})^M \models \text{"}C^{(n+1)} \text{ is a club class"}.$$

Note that it follows that $M \models$ "$\alpha = \beth_\alpha > \beth_{\kappa_1}$". Now for any $B \in V_{\kappa_1+1}$, let X be an elementary substructure of $(V_\alpha)^M$ with $V_{\kappa_1} \cup \{B, \kappa_1\} \subseteq X$ and $|X| = \kappa_1$. Note that this is possible, since we have by elementarity, that $M \models$ "κ_1 is inacessible", which is true in V by the closure property of M. Thus $|V_{\kappa_1}| = \kappa_1$. Now consider the mapping $k = j{\restriction}X : X \to j(X)$. k is elementary, and we have $crit(k) = \kappa_0$, $k(\kappa_0) = j(\kappa_0) = \kappa_1$, $k(\kappa_1) = \kappa_2$, and $k(B) = j(B)$. Since ${}^{\kappa_1}M \subseteq M$, we have $k \in M$ and

> $M \models$ "$\exists\, Y$ such that $Y \prec V_\beta$ for some $\beta > \beth_{\kappa_1}$ with $V_\beta \prec_{\Sigma_{n+1}} V_{\kappa_2}$ and
> $V_{\kappa_1} \cup \{\kappa_1\} \subseteq Y$, and there is an elementary embedding $k : Y \to j(X)$ such that
> $j(B) \in ran(k)$, $k(\kappa_1) = \kappa_2$, and $k(crit(k)) = \kappa_1$".

By the elementarity of j, the following holds in V:

> "$\exists\, Y$ such that $Y \prec V_\beta$ for some $\beta > \beth_{\kappa_0}$ with $V_\beta \prec_{\Sigma_{n+1}} V_{\kappa_1}$ and
> $V_{\kappa_0} \cup \{\kappa_0\} \subseteq Y$, and there is an elementary embedding $k : Y \to X$ such that
> $B \in ran(k)$, $k(\kappa_0) = \kappa_1$, and $k(crit(k)) = \kappa_0$".

Take some witnessing Y, β and k as above. Observe that we have Y, β and k are all in $(V_{\kappa_2})^M$. Moreover we also have $V_\beta \prec_{\Sigma_{n+1}} (V_{\kappa_2})^M$. Also recall that $X \prec V_\alpha \prec_{\Sigma_{n+1}} (V_{\kappa_2})^M$.

It follows that

$$(V_{\kappa_2})^M \models \text{“}\exists\, Y \text{ such that } Y \prec V_\beta \text{ for some } \beta > \beth_{\kappa_0} \text{ with } \beta \in C^{(n)} \text{ and }$$
$$V_{\kappa_0} \cup \{\kappa_0\} \subseteq Y, \text{ and there is an elementary embedding } k : Y \to V_\alpha \text{ for some }$$
$$\alpha \in C^{(n+1)} \text{ such that } B \in ran(k),\ k(\kappa_0) = \kappa_1, \text{ and } k(crit(k)) = \kappa_0\text{”}.$$

Noting that $H_\alpha = V_\alpha$ for any $\alpha \in C^{(1)}$, that the $B \in V_{\kappa_1+1}$ is arbitrarily chosen, and that $(V_{\kappa_1+1})^{(V_{\kappa_2})^M} = V_{\kappa_1+1}$, the above implies that in fact

$$(V_{\kappa_2})^M \models \text{“}\kappa_0 \text{ is parametrically } n\text{-exact for } \kappa_1\text{”}.$$

Now define the ultrafilter $\mathcal{U}^*$ on κ_0 induced by j, namely

$$\mathcal{U}^* = \{X \in \mathcal{P}(\kappa_0) : \kappa_0 \in j(X)\},$$

it follows that

$$D = \{\mu < \kappa_0 : V_{\kappa_1} \models \text{“}\mu \text{ is parametrically } n\text{-exact for } \kappa_0\text{”}\} \in \mathcal{U}^*.$$

Take any element $\mu \in D$, since $j(\mu) = \mu$, we have

$$(V_{\kappa_1})^M = V_{\kappa_1} \models \text{“}j(\mu) \text{ is parametrically } n\text{-exact for } \kappa_0\text{”},$$

which implies that

$$D_\mu = \{\lambda < \kappa_0 : V_{\kappa_0} \models \text{“}\mu \text{ is parametrically } n\text{-exact for } \lambda\text{”}\} \in \mathcal{U}^*.$$

Combining all of the above, we have

$$V_{\kappa_0} \models \text{ZFC+“there is a proper class of cardinals } \mu \text{ such that}$$
$$\mu \text{ is parametrically } n\text{-exact for a proper class of cardinals } \lambda\text{”},$$

as desired. □

In particular, we lower the upper bound of consistency strength significantly from an I3 cardinal to a huge cardinal.

We also give now a finer analysis of the lower bound. A cardinal κ is called an A_2 cardinal (with target λ), if κ is the critical point of some elementary embedding $j : V_\alpha \to V_\beta$ with $j(\kappa) = \lambda \leq \alpha$. This notion was introduced in [34], with consistency strength strictly between an almost huge cardinal and a huge cardinal. We may strengthen this notion by considering its “super” version and $C^{(n)}$ version. For the moment let us say that κ is $C^{(n)}$-super A_2 if for any δ, κ is A_2 with target $\lambda \in C^{(n)}$ greater than δ.

Proposition 6.2.6. *If κ is either parametrically 0-exact for λ or weakly parametrically 1-exact for λ, then for any natural number n, V_κ is a model of ZFC plus “there is a proper class of $C^{(n)}$-super A_2 cardinal”.*

Proof. We show the case where κ is parametrically 0-exact. It should be clear how the other case can be proved similarly. Suppose κ is parametrically 0-exact for $\lambda > \kappa$. Since $rk(V_\lambda) = \lambda$, there is some $\kappa' > \beth_\kappa$, some $\lambda' \in C^{(1)}$ greater than λ, some $X \prec H_{\kappa'}$, with $V_\kappa \cup \{\kappa\} \subseteq X$, and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $V_\lambda \in ran(j)$ and $j(crit(j)) = \kappa$. Let $crit(j) = \nu$.

Now let $j(A) = V_\lambda$, it follows from elementarity and $V_\kappa \subseteq X$ that $A = V_\kappa$, so in fact $k = j{\restriction}V_\kappa : V_\kappa \to V_\lambda$ witnesses that $crit(k) = \nu$ is A_2 with target κ. But note that $\lambda' \in C^{(1)}$ implies that $V_{\lambda'} = H_{\lambda'}$, so $k \in H_{\lambda'}$. Thus let $\mathcal{U} = \{X \subseteq \mathcal{P}(\nu) : \nu \in j(X)\}$ be the normal ultrafilter induced by j on ν. We get that $C = \{\mu < \nu : X \models \text{“}\mu \text{ is } A_2 \text{ with target } \nu\text{”}\} \in \mathcal{U}$. Note that if $\mu \in C$, then “μ is A_2 with target ν” also holds in V and in $H_{\lambda'}$. It follows that $C_\mu = \{\delta < \nu : X \models \text{“}\mu \text{ is } A_2 \text{ with target } \delta\text{”}\} \in \mathcal{U}$, and since $C^n_\nu = \{\alpha < \nu : V_\nu \models \text{“}\alpha \in C^{(n)}\text{”}\}$ is a club set, we have $C^n_\nu \in \mathcal{U}$, thus $C_\mu \cap C^n_\nu \in \mathcal{U}$. It follows that if $\mu \in C$ then $V_\nu \models$ “μ is a $C^{(n)}$-super A_2 cardinal”, So it holds in both V_ν and, by elementarity, V_κ, that there is a proper class of $C^{(n)}$-super A_2 cardinals, as desired. □

Additionally we give some brief remarks on the strength of A_2 cardinals. We say a cardinal κ is *almost huge with target* λ if there is some elementary embedding $j : V \to M$ for some transitive M, with $crit(j) = \kappa$, $j(\kappa) = \lambda$, and ${}^\delta M \subseteq M$ for every $\delta < \lambda$. Similarly we say κ is $C^{(n)}$-super almost huge, for some natural number n, if for any δ, κ is almost huge with some target $\lambda \in C^{(n)}$ greater than δ.

Proposition 6.2.7. If κ is A_2, then V_κ is a model of ZFC+“there is a proper class of $C^{(n)}$-super almost huge cardinals”.

Proof. Suppose $j : V_\alpha \to V_\beta$ is elementary, with $crit(j) = \kappa$ and $j(\kappa) \leq \alpha$. For any $\kappa \leq \lambda < j(\kappa)$, define $\mathcal{U}_\lambda$ by $X \in \mathcal{U}_\lambda$ if and only if $X \subseteq \mathcal{P}_\kappa\lambda \wedge j\text{“}\lambda \in j(X)$. It can be verified that each $\mathcal{U}_\lambda$ is a normal ultrafilter over $\mathcal{P}_\kappa\lambda$. Moreover, by Theorem 24.11 of [21], it follows that $\langle \mathcal{U}_\lambda : \kappa \leq \lambda < j(\kappa)\rangle$ is a *coherent sequence* (please see [21] for the deails) witnessing that κ is almost huge with target $j(\kappa)$. Furthermore, the almost hugeness of κ with target $j(\kappa)$ can be correctly verified by $V_{j(\kappa)+1}$. It follows that $V_\beta \models$ “κ is almost huge with target $j(\kappa)$”.

Arguing similarly as in Proposition 6.2.6, we see that if $\mathcal{U}$ is be the ultrafilter on κ derived from j, we have $A = \{\gamma < \kappa : \gamma \text{ is almost huge with target } \kappa\} \in \mathcal{U}$, which implies $\{\mu < \kappa : \gamma \text{ is almost huge with target } \mu \text{ and } V_\kappa \models \text{“}\mu \in C^{(n)}\text{”}\} \in \mathcal{U}$ for any $\gamma \in A$, so the conclusion follows. □

Thus we have seen that, firstly, an A_2 cardinal is stronger than the whole hierarchy of $C^{(n)}$-super almost huge cardinals, secondly, a parametrically 0-exact cardinal is stronger than the hierarchy of $C^{(n)}$-super A_2 cardinals, and finally, a huge cardinal is stronger than the hierarchy of parametrically n-exact cardinals. Thus interestingly, there turn out to be rich hierarchies of large cardinals strictly between the seemingly narrow interval between almost hugeness and hugeness. Moreover, we see that large cardinal notions prompted by the study of structural reflection principle give rise to new regions of the large cardinal hierarchy not studied before.

6.3 On pattern

Note that Theorem 6.1.3 and 6.1.5 only give us the correspondence between ESR and exact cardinals in their local forms. However, usually in the study of structural reflection principles, we are more interested in the global forms of the reflection principles, thus it is important to know if the patterns of correspondence continue to hold, namely if Theorem 6.1.3 and 6.1.5 continue to be true globally. More precisely, the following open question was proposed in [6]:

Question 6.3.1 ([6]). Are the following statements equivalent for every cardinal κ and every natural number $n \geq 1$?

1. κ is the least cardinal that is weakly parametrically n-exact for a proper class of cardinals λ.

2. κ is the least cardinal such that Π_n-ESR(κ, λ) holds for a proper class of cardinals λ.

The analogous question for the Σ_{n+1} case was also open:

Question 6.3.2 ([6]). Are the following statements equivalent for every cardinal κ and every natural number $n \geq 1$?

1. κ is the least cardinal that is parametrically n-exact for a proper class of cardinals λ.

2. κ is the least cardinal such that Σ_{n+1}-ESR(κ, λ) holds for a proper class of cardinals λ.

In this section we answer the above two questions by applications of the theory of Σ_n-m-supercompact cardinals. The answers turn out to be unexpected but quite interesting: it is provable from ZFC that the answer to Question 6.3.1 is positive for $n \geq 2$, but it is provable, assuming the existence of a 2-supercompact cardinal, that the answer is negative when $n = 1$. In fact more is true: the assertion that a 2-supercompact cardinal exists *is equivalent* to the assertion that the answer is negative when $n = 1$.

Moreover, this pattern also holds for Question 6.3.2: for any $n \geq 1$, the assertion that a Σ_n-2-supercompact cardinal exists is equivalent to the assertion that the answer to the question is negative.

We also show that weakened versions of the two questions have positive answers.

Proposition 6.3.3. *For every natural number $n \geq 1$, if κ is Σ_n-2-supercompact, then κ is a limit of cardinals $\mu < \kappa$ such that Σ_{n+1}-ESR(μ, λ) holds for a proper class of cardinals λ.*

Proof. Suppose κ is Σ_n-2-supercompact, then CSR$(\kappa, \mathbf{\Pi_n})$ holds. In the case $n = 1$ this follows from Theorem 3.1.4. In the case $n \geq 2$, first note that by Theorem 5.1.3, κ is $C^{(n-1)}$-hyperhuge, and so by Theorem 3.2.4 CSR$(\kappa, \mathbf{\Pi_n})$ holds.

Now for any $\alpha < \kappa$, consider the Π_n-definable, with parameter α, class $\mathcal{C}$ of structures of the form $(V_\gamma, \in, \gamma, \lambda, \alpha)$, where $\gamma \in C^{(n)}$ and $\lambda < \gamma$.

For any ordinal $\lambda \geq \kappa$, we can take some $\gamma \in C^{(n+1)}$ greater than λ, and consider the structure $(V_\gamma, \in, \gamma, \lambda, \alpha) \in \mathcal{C}$. By CSR$(\kappa, \mathbf{\Pi_n})$, there is some $(V_\xi, \in, \xi, \mu, \alpha) \in \mathcal{C}$ that captures $(V_\gamma, \in, \gamma, \lambda, \alpha)$.

Claim 6.3.4. Σ_{n+1}-ESR(μ, λ) holds.

Proof. For any class $\mathcal{C}^*$ of structures that is Σ_{n+1}-definable without parameters, take $B \in \mathcal{C}^*$ of rank λ, by capturing there is some elementary embedding

$$j : (V_\xi, \in, \xi, \mu, \alpha) \to (V_\gamma, \in, \gamma, \lambda, \alpha)$$

and some A with $j(A) = B$. Since $\gamma \in C^{(n+1)}$ we have $V_\gamma \models$ "$rk(B) = \lambda \wedge B \in \mathcal{C}^*$", so by elementarity $V_\xi \models$ "$rk(A) = \mu \wedge A \in \mathcal{C}^*$". The point here is that the definition of $\mathcal{C}^*$ does not have parameter, thus we can conclude directly from $\xi \in C^{(n)}$ and upward absoluteness that $A \in \mathcal{C}^*$ (if $\mathcal{C}^*$ is defined with some parameter b then it can happen that $j^{-1}(b) \neq b$ so it does not follow that $A \in \mathcal{C}^*$). Moreover, $j \restriction A : A \to B$ is elementary. Thus Σ_{n+1}-ESR(μ, λ) holds. □

Note that $\mu < j(\mu) = \lambda$ implies $\alpha < \mu$, thus what we have shown is this: for any $\alpha < \kappa$ and $\lambda \geq \kappa$, there is some μ between α and κ with Σ_{n+1}-ESR(μ, λ) holds. Thus there must be some fixed μ between α and κ with Σ_{n+1}-ESR(μ, λ) holds for a proper class of λ, as desired. □

Lemma 6.3.5. *For every natural number n and cardinals $\kappa < \lambda < \delta$, if $\Pi_n(P)$-ESR(κ, λ) holds and $\Pi_n(P)$-ESR(λ, δ) holds, then $\Pi_n(P)$-ESR(κ, δ) holds.*

Proof. Let $\mathcal{C}$ be a Π_n-definable, with parameters in P, class of structures of the same type. If $C \in \mathcal{C}$ is of rank δ, then we have some $B \in \mathcal{C}$ of rank λ with $j : B \to C$ elementary. Thus if $A \in \mathcal{C}$ is of rank κ and $k : A \to B$ is elementary, then $j \circ k : A \to C$ is elementary. □

Lemma 6.3.6 ([6], Corollary 5.5*). *The following statements are equivalent for all natural numbers $n > 0$ and all cardinals $\kappa < \lambda$,*

1. Π_n-ESR(κ, λ) *holds.*
2. *For all $A_0, \dots, A_{k-1} \in V_{\lambda+1}$ and $\lambda < \lambda' \in C^{(n)}$ there exists a transitive, $\Pi_n(V_{\kappa+1})$-upwards correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A_0, \dots, A_{k-1} \in ran(j)$.*

Lemma 6.3.7. *For every natural number $n \geq 1$, if κ is the least cardinal such that Π_n-ESR(κ, λ) holds for a proper class of cardinals λ, but κ is not weakly parametrically n-exact for a proper class of cardinals λ, then there is some cardinal $\xi < \kappa$, such that Π_n-ESR(ξ, ν) holds for some Σ_n-2-supercompact cardinal ν.*

Proof. Suppose our assumptions hold, then for any $\alpha < \kappa$, the β's such that Π_n-ESR(α, β) holds are bounded, so there is some $\delta \in C^{(n+1)}$ large enough such that for any $\alpha < \kappa$, Π_n-ESR(α, β) fails for any $\beta \geq \delta$. Fix such a δ.

By assumption we can take some $\lambda > \delta$ such that Π_n-ESR(κ, λ) holds but κ is not weakly parametrically n-exact for λ. It follows by definition that there must be some $A \in V_{\lambda+1}$ such that there is no transitive, $\Pi_n(V_{\kappa+1})$-upwards correct M with $V_\kappa \cup \{\kappa\} \subseteq M$, with some cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $A \in ran(j)$, and $j(crit(j)) = \kappa$. Let A be such a set.

Now let $\lambda' \in C^{(n)}$ be greater than λ. By Lemma 6.3.6, there is some M that is transitive, $\Pi_n(V_{\kappa+1})$-upwards correct, with $V_\kappa \cup \{\kappa\} \subseteq M$, and there is an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A, \kappa, \delta \in ran(j)$. It follows from $A \in ran(j)$ that $j(crit(j)) \neq \kappa$, so $j(crit(j)) < \kappa$. Let $crit(j) = \mu$ and $j(\xi) = \delta$. Note that since $j(\mu) < \kappa$ there must be some natural number $i \geq 2$ such that $j^i(\mu) > \kappa$, otherwise there is a violation of the Kunen Inconsistency.

Claim 6.3.8. Π_n-ESR$(j(\mu), j^i(\mu))$ holds.

Proof. We first show that for any k with $1 \leq k \leq (i-1)$ we have Π_n-ESR$(j^k(\mu), j^{k+1}(\mu))$ holds. Then the claim follows from Lemma 6.3.5. Now suppose $1 \leq k \leq (i-1)$ but Π_n-ESR$(j^k(\mu), j^{k+1}(\mu))$ fails. Then let $\mathcal{C}$ be a Π_n-definable, without parameters, class of structures that is a counterexample. Thus it is true in $V_{\lambda'} = H_{\lambda'} \prec_{\Sigma_n} V$ that there is some $Y \in \mathcal{C}$ of rank $j^{k+1}(\mu)$ with no $X \in \mathcal{C}$ of rank $j^k(\mu)$ and an elementary embedding $k : X \to Y$. By elementarity, it is true in M that there is some $Y \in \mathcal{C}$ of rank $j^k(\mu)$ with no $X \in \mathcal{C}$ of rank $j^{k-1}(\mu)$ and an elementary embedding $k : X \to Y$. Let Y be such a witness. By elementarity again, it is true in $V_{\lambda'}$ that there is no X of rank $j^k(\mu)$ with an elementary embedding $k : X \to j(Y)$.

But note that since M is $\Pi_n(V_{\kappa+1})$-upward absolute, $Y \in V_{\kappa+1}$ (as $j^k(\mu) \leq \kappa$), and $\mathcal{C}$ is Π_n-definable without parameters, we have $Y \in \mathcal{C}$ holds in V, and in $V_{\lambda'}$, thus $k = j{\restriction}Y : Y \to j(Y) \in V_{\lambda'}$ is elementary, which is a contradiction. □

*This is a variant of Corollary 5.5, but inspecting the proof of Corollary 5.5 shows that this can also be proven.

Since $j(\mu) < \kappa$ and Π_n-ESR$(j(\mu), j^i(\mu))$ holds, it follows that $j^i(\mu) < \delta$, since we are supposing that for no $\alpha < \kappa$ and $\beta \geq \delta$ can Π_n-ESR(α, β) hold. Now suppose our conclusion fails, then in particular, we have $j^i(\mu)$ is not Σ_n-2-supercompact, since $j(\mu) < \kappa$ and Π_n-ESR$(j(\mu), j^i(\mu))$ holds. Thus in fact $j^i(\mu)$ must fail to be Σ_n-i-supercomact, given that $i \geq 2$, and Σ_n-i-supercompactness implies Σ_n-2-supercompactness. Now note that "$j^i(\mu)$ is not Σ_n-i-supercompact" is a Σ_{n+1}-statement, since it is equivalent to

$$\exists \eta > j^i(\mu), X(\eta \in C^{(n)} \wedge X = V_{\eta+1} \wedge \text{there is no } k, \nu \in X \text{ such that}$$
$$\nu \in C^{(n)},\ k : V_\nu \to V_\eta \text{ is elementary and } k^i(crit(k)) = j^i(\mu)).$$

Thus, since $\delta \in C^{(n+1)}$, the above statement holds in V_δ, so by elementarity and $j(\xi) = \delta$ we have

$V_\xi \models$ "there exists $\eta \in C^{(n)}$ greater than $j^{i-1}(\mu)$ such that there is no $\nu \in C^{(n)}$ less than $j^{i-1}(\mu)$ with an elementary embedding $k : V_\nu \to V_\eta$ with $k^i(crit(k)) = j^{i-1}(\mu)$".

By elementarity,

$V_\delta \models$ "there is no $\nu \in C^{(n)}$ less than $j^i(\mu)$ such that there is an elementary embedding $k : V_\nu \to V_{j(\eta)}$ with $k^i(crit(k)) = j^i(\mu)$".

But note that $\eta < \xi < \kappa < j^i(\mu)$, and since $\lambda' \in C^{(n)}$, we have $V_{\lambda'} \models$ "$\delta \in C^{(n)}$", thus $M \models$ "$\xi \in C^{(n)}$". But since $\delta < \lambda$, by elementarity $\xi < \kappa$, so by $\Pi_n(V_{\kappa+1})$-upward absoluteness, $\xi \in C^{(n)}$, thus $\eta \in C^{(n)}$. It follows that $j{\restriction}V_\eta : V_\eta \to V_{j(\eta)}$ is the required elementary embedding k, which is a contradiction. ☐

Proposition 6.3.9. *For every natural number $n \geq 1$, if κ is Σ_{n+1}-2-supercompact, then κ is weakly parametrically $(n+1)$-exact for a proper class of cardinals λ.*

Proof. Suppose κ is Σ_{n+1}-2-supercompact, then by Theorem 5.1.3, κ is $C^{(n)}$-2-fold extendible, which implies by Corollary 5.1 that κ is $C^{(n)+}$-2-fold extendible. For any ordinal $\alpha \in C^{(n+1)}$ greater than κ, take $\delta \in C^{(n)}$ greater than α. By assumption, we have some elementary embedding $j : V_{j(\delta)} \to V_\eta$ for some η, with $crit(j) = \kappa$, $\delta < j(\kappa)$, and $\{\delta, j(\delta), j(\kappa), j^2(\kappa), \eta\} \subseteq C^{(n)}$.

Claim 6.3.10. κ is weakly parametrically $(n+1)$-exact for $j(\kappa)$.

Proof. For any $A \in V_{j(\kappa)+1}$, consider $k = j{\restriction}V_{j(\alpha)} : V_{j(\alpha)} \to V_{j^2(\alpha)} \in V_\eta$. Thus

$$V_\eta \models \text{"There is some elementary embedding } k : V_{j(\alpha)} \to V_{j^2(\alpha)}$$
$$\text{with } k(j(\kappa)) = j^2(\kappa),\ k(crit(k)) = j(\kappa), \text{ and } j(A) \in ran(k)\text{"},$$

so by elementarity,

$$V_{j(\delta)} \models \text{"There is some elementary embedding } k : V_\alpha \to V_{j(\alpha)}$$
$$\text{with } k(\kappa) = j(\kappa),\ k(crit(k)) = \kappa, \text{ and } A \in ran(k)\text{"}.$$

Now since $V_{j(\delta)} \models$ "$\alpha \in C^{(n)}$", we have $V_\eta \models$ "$j(\alpha) \in C^{(n)}$", which implies that $j(\alpha) \in C^{(n)}$. Also note that we have V_α is $\Pi_{n+1}(V_{\kappa+1})$-upwards correct, $V_\kappa \cup \{\kappa\} \subseteq V_\alpha$, and since $j(\alpha) \in C^{(1)}$, $j(\alpha) > \beth_{j(\kappa)}$ and $H_{j(\alpha)} = V_{j(\alpha)}$. Thus since A is arbitrary, we can conclude that κ is weakly parametrically $(n+1)$-exact for $j(\kappa)$, as desired. ☐

Finally since α can be chosen to be arbitrarily large, κ is weakly parametrically $(n+1)$-exact for a proper class of cardinals, as desired. $\square$

Lemma 6.3.11 ([6], Proposition 5.3)**.** *For every natural number $n \geq 1$, if κ is weakly parametrically n-exact for λ, then $\Pi_n(V_\kappa)$-ESR(κ, λ) holds.*

Combining all the previous results, now we can give a positive answer to Question 6.3.1 in the cases when $n \geq 2$:

Theorem 6.3.12. *For every natural number $n \geq 2$ and cardinal κ, the following are equivalent:*

1. *κ is the least cardinal that is weakly parametrically n-exact for a proper class of cardinals λ.*

2. *κ is the least cardinal such that* Π_n-ESR(κ, λ) *holds for a proper class of cardinals λ.*

Proof. By Lemma 6.3.11, it is sufficient to show that the least κ such that Π_n-ESR(κ, λ) holds for a proper class of cardinals λ is weakly parametrically n-exact for a proper class of cardinals λ. Suppose for contradiction that κ is a counterexample. Now by Lemma 6.3.7, there is some cardinal $\xi < \kappa$ such that Π_n-ESR(ξ, ν) holds for some Σ_n-2-supercompact cardinal ν. By Proposition 6.3.9, ν is weakly parametrically $(n-1)$-exact for a proper class of cardinals λ. By Lemma 6.3.11, Π_n-ESR(ν, λ) holds for a proper class of λ. But then by Lemma 6.3.5, we have Π_n-ESR(ξ, λ) holds for a proper class of λ. But since $\xi < \kappa$, this contradicts the minimality of κ. Thus κ must be weakly parametrically n-exact for a proper class of cardinals λ. $\square$

Next we analyze the reflective properties of κ and λ when some forms of ESR(κ, λ) holds.

Proposition 6.3.13. For every natural number $n \geq 1$ and every cardinal κ,

1. For every cardinal $\lambda > \kappa$, if $\Pi_1(V_\kappa)$-ESR(κ, λ) holds, then $V_\kappa \prec V_\lambda$.

2. Suppose $\Pi_1(V_\kappa)$-ESR(κ, λ) holds for a proper class of cardinals λ. Then for every cardinal λ, if $\Pi_1(V_\kappa)$-ESR(κ, λ) holds, then $\lambda \in C^{(1)}$. Moreover, $\kappa \in C^{(3)}$.

3. Suppose $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds for a proper class of cardinals λ. Then for every cardinal λ, if $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds, then $\lambda \in C^{(n+1)}$. Moreover, $\kappa \in C^{(n+3)}$.

Proof. (1): If $\Pi_1(V_\kappa)$-ESR(κ, λ) holds, then for any $b_0, \ldots, b_{k-1} \in V_\kappa$, the class of structures of the form $(V_\alpha, \in, b_0, \ldots, b_{k-1})$ is Π_1-definable with parameters $b_0, \ldots, b_{k-1}$, so there must be some elementary embedding $j : V_\kappa \to V_\lambda$ with $b_0, \ldots, b_{k-1}$ fixed by j, so $V_\kappa \models$ "$\phi(b_0, \ldots, b_{k-1})$" if and only if $V_\lambda \models$ "$\phi(b_0, \ldots, b_{k-1})$" for any formula $\phi(y_0, \ldots, y_{k-1})$. It follows that $V_\kappa \prec V_\lambda$.

(2): Suppose $\Pi_1(V_\kappa)$-ESR(κ, λ) holds for a proper class of λ, then we first show that $\kappa \in C^{(1)}$. Suppose $b_0, \ldots, b_{k-1} \in V_\kappa$ and $\phi(a, b_0, \ldots, b_{k-1})$ holds in V for some a, where $\phi(x, y_0, \ldots, y_{k-1})$ is Σ_0, then let $\lambda > rk(a)$ be such that $\Pi_1(V_\kappa)$-ESR(κ, λ) holds. Then V_λ satisfies $\exists x \phi(x, b_0, \ldots, b_{k-1})$, and so does V_κ by (1). It follows that $\kappa \in C^{(1)}$.

Now suppose that $\Pi_1(V_\kappa)$-ESR(κ, λ) holds, then by (1) we have $\lambda \in C^{(1)}$ as well.

Now we can show that $\kappa \in C^{(3)}$. Take any Σ_3-formula $\exists x \forall y \phi(x, y, y_0, \ldots, y_{k-1})$, where ϕ is Σ_1. For every $a, b_0, \ldots, b_{k-1} \in V_\kappa$ and $V_\kappa \models$ "$\forall y \phi(a, y, b_0, \ldots, b_{k-1})$", then if $\forall y \phi(a, y, b_0, \ldots, b_{k-1})$ fails in V, then $\forall y \phi(a, y, b_0, \ldots, b_{k-1})$ fails in some λ sufficiently large and $\Pi_1(V_\kappa)$-ESR(κ, λ) holds. But then by (1) $\forall y \phi(a, y, b_0, \ldots, b_{k-1})$ also fails in V_κ, contradiction. Conversely if $b_0, \dots, b_{k-1} \in V_\kappa$ and $\forall y \phi(a, y, b_0, \dots, b_{k-1})$ holds in V for some a, then let $\lambda > rk(a)$ be such that $\Pi_1(V_\kappa)$-ESR(κ, λ) holds, then since $\lambda \in C^{(1)}$, the Π_2-formula $\forall y \phi(a, y, b_0, \dots, b_{k-1})$ reflects in V_λ, so $\exists x \forall y \phi(x, y, b_0, \dots, b_{k-1})$ holds in V_λ, and also in V_κ, as desired.

(3): This can be proved by induction on n. For $n = 1$, suppose $\Sigma_2(V_\kappa)$-ESR(κ, λ) holds for a proper class of λ, then by (2) we already have $\kappa \in C^{(3)}$. Now if $\Sigma_2(V_\kappa)$-ESR(κ, λ) holds, then if $\lambda \notin C^{(2)}$, then since the class $\mathcal{C}$ of structures of the form $(V_\alpha, \in)$ and $\alpha \notin C^{(2)}$ is Σ_2-definable, the fact that $(V_\lambda, \in) \in \mathcal{C}$ and $\Sigma_2(V_\kappa)$-ESR(κ, λ) imply that $(V_\kappa, \in) \in \mathcal{C}$, which is a contradiction. Thus $\lambda \in C^{(2)}$.

Now we can show that $\kappa \in C^{(4)}$. One direction easily follows from the fact that $\kappa \in C^{(3)}$. For the other direction, if $b_0, \dots, b_{k-1} \in V_\kappa$ and some Σ_4-formula is of the form $\exists x \phi(x, y_0, \dots, y_{k-1})$, with $\phi(a, b_0, \dots, b_{k-1})$ holds in V for some a. Then let $\lambda > rk(a)$ be such that $\Sigma_2(V_\kappa)$-ESR(κ, λ) holds. Thus $\lambda \in C^{(2)}$ and the Π_3-formula $\phi(a, b_0, \dots, b_{k-1})$ reflects in V_λ, implying that $\exists x \phi(x, b_0, \dots, b_{k-1})$ holds in V_λ, and also in V_κ by (1).

The general case can be shown similarly, using inductive hypothesis together with the observation that if $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds then $\lambda \in C^{(n+1)}$. □

Now we can answer Question 6.3.1 in the case $n = 1$. As remarked before, a negative answer is equivalent to the existence a 2-supercompact cardinal. In fact more is true:

Theorem 6.3.14. *The following statements are equivalent:*

1. *There exists a* 2*-supercompact cardinal.*
2. *There exists some cardinal κ such that* Π_1-ESR(κ, λ) *holds for a proper class of cardinals λ, but the least such κ does not have the property that* $\Pi_1(V_\kappa)$-ESR(κ, λ) *holds for a proper class of cardinals λ.*
3. *There exists some cardinal κ such that* Π_1-ESR(κ, λ) *holds for a proper class of cardinals λ, but the least such κ is not weakly parametrically* 1*-exact for a proper class of cardinals λ.*

Proof. (1) implies (2): suppose there exists some 2-supercompact cardinal, then let κ_0 be the least one. By Proposition 6.3.3, there are unboundedly many $\kappa < \kappa_0$ such that Π_1-ESR(κ, λ) holds for a proper class of cardinals λ. Let κ_1 be the least such cardinal less than κ_0. Suppose for contradiction that $\Pi_1(V_{\kappa_1})$-ESR(κ_1, λ) holds for a proper class of cardinals λ. Then by Proposition 6.3.13, $\kappa_1 \in C^{(3)}$. However, note that the statement "there exists a 2-supercompact cardinal" is Σ_3 by Theorem 2.1.5, so there must be some $\kappa_2 < \kappa_1 < \kappa_0$, such that κ_2 is really a 2-supercompact cardinal, contradicting the minimality of κ_0.

(2) implies (3): this follows from Lemma 6.3.11.

(3) implies (1): follows from Lemma 6.3.7. □

In contrast, note that it is relatively consistent with ZFC that the answer to Question 6.3.1 is positive, simply because a negative answer implies large cardinals, so if ZFC alone proves that the answer is negative, then ZFC proves its own consistency.

Although we know that in the presence of large cardinals, the least κ such that Π_1-ESR(κ, λ) holds for a proper class of cardinals λ cannot be weakly parametrically 1-exact for a proper class of cardinals λ, we can still ask a weaker version of Question 6.3.1, namely if the least κ such that $\Pi_1(V_\kappa)$-ESR(κ, λ) holds for a proper class of cardinals λ is weakly parametrically 1-exact for a proper class of cardinals λ. The answer is positive.

Proposition 6.3.15. *If κ is the least cardinal such that $\Pi_1(V_\kappa)$-ESR(κ, λ) holds for a proper class of cardinals λ, then for any cardinal λ, if $\Pi_1(V_\kappa)$-ESR(κ, λ) holds, then κ is weakly parametrically* 1*-exact for* λ.

Proof. Suppose κ satisfies the assumption and $\Pi_1(V_\kappa)$-ESR(κ, λ) holds, but κ is not weakly parametrically 1-exact for λ. Then as in Lemma 6.3.7, we may choose some $A \in V_{\lambda+1}$ such that there is no transitive, $\Pi_n(V_{\kappa+1})$-upwards correct M with $V_\kappa \cup \{\kappa\} \subseteq M$, with some cardinal λ' greater than $\beth_\lambda$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $A \in ran(j)$, and $j(crit(j)) = \kappa$. Take $\lambda' \in C^{(1)}$ greater than λ. We have by Lemma 6.3.6 some transitive, $\Pi_n(V_{\kappa+1})$-upwards correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A, \kappa \in ran(j)$. Thus $j(crit(j)) < \kappa$. Let $crit(j) = \mu$, and let $j(\eta) = \kappa$.

Claim 6.3.16. There is some Σ_2-2-supercompact cardinal κ' less than κ.

Proof. First note that by (2) of Proposition 6.3.13 we have $\kappa \in C^{(3)}$ and $\lambda \in C^{(1)}$, thus $V_\lambda \models$ "$\kappa \in lim(C^{(2)})$", so by elementarity $V_\kappa \models$ "$\eta \in lim(C^{(2)})$". Thus it is true in V that $\eta \in lim(C^{(2)})$.

Now we claim that $\mu \in C^{(2)}$. Otherwise let $\xi < \mu$ such that there are no $C^{(2)}$ cardinals between ξ and μ, and let $\nu < \eta$ be the least element of $C^{(2)}$ above μ. Then

$$V_\eta \models \text{"}\nu \text{ is the least element of } C^{(2)} \text{ above } \xi\text{"}$$

and since $j(\xi) = \xi$, $V_\kappa \models$ "$j(\nu)$ is the least element of $C^{(2)}$ above ξ". But then $j(\nu) = \nu$, as both η and κ are in $C^{(2)}$. Thus $j\restriction V_{\nu+2}$ contradicts the Kunen Inconsistency.

Thus it holds in V_η that $\mu \in C^{(2)}$, so by elementarity it holds in V_κ, and in V, that $j(\mu) \in C^{(2)}$.

Now for any $\xi < min\{\eta, j(\mu)\}$, we have $j\restriction V_{j(\mu+\xi)} : V_{j(\mu+\xi)} \to V_{j^2(\mu+\xi)}$ witnesses that μ is ξ-2-fold extendible, and since the statement "μ is ξ-2-fold extendible" is Σ_2 with parameters μ and ξ, it also holds in both V_η and $V_{j(\mu)}$ that μ is ξ-2-fold extendible. Now if $\eta \leq j(\mu)$, we have $V_\eta \models$ "μ is a 2-fold extendible cardinal". Otherwise, if $\eta > j(\mu)$, then it holds in $V_{j(\mu)}$ that μ is 2-fold extendible. Let $i \geq 2$ be the least natural number such that $j^i(\mu) \geq \eta$. We have by elementarity that it holds in $V_{j^i(\mu)}$ that $j^{(i-1)}(\mu)$ is 2-fold extendible. It follows that it is also true in V_η that $j^{(i-1)}(\mu)$ is 2-fold extendible. So either way we have $V_\eta \models$ "There is a 2-fold extendible cardinal". By elementarity, it holds in V_κ that there is some 2-fold extendible cardinal κ', which is true in V given that $\kappa \in C^{(3)}$, and "κ' is 2-fold extendible" is Π_3-definable. Now by Theorem 5.1.3, κ' is Σ_2-2-supercompact. ☐

But by Proposition 6.3.9, κ' is weakly parametrically 2-exact for a proper class of cardinals γ, which implies by Lemma 6.3.11 that $\Pi_1(V_{\kappa'})$-ESR(κ', γ) holds for a proper class of cardinals γ, contradicting the minimality of κ. Thus κ is weakly paramtrically 1-exact for λ. ☐

The following theorem follows from the proposition above:

Theorem 6.3.17. *The following are equivalent for any cardinal κ:*

1. *κ is the least cardinal such that $\Pi_1(V_\kappa)$-ESR(κ, λ) holds for a proper class of cardinals λ.*

2. *κ is the least cardinal that is weakly parametrically* 1*-exact for a proper class of cardinals λ.*

Here are two lemmas analogous to Lemma 6.3.6 and Lemma 6.3.11, respectively:

Lemma 6.3.18 ([6], Proposition 6.4*). *The following are equivalent for all cardinals $\kappa < \lambda$ and natural number $n \geq 1$:*

1. Σ_{n+1}-ESR(κ, λ) *holds.*

2. *For any $A_0, \ldots, A_{k-1} \in V_{\lambda+1}$ there is some cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\kappa$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, some $X \prec H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ such that $j(\kappa) = \lambda$ and $A_0, \ldots, A_{k-1} \in ran(j)$.*

Lemma 6.3.19 ([6], Proposition 6.3). *For any natural number $n \geq 1$, if κ is parametrically n-exact for λ, then $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds.*

We can also prove a lemma analogous to Lemma 6.3.7:

Lemma 6.3.20. *For every natural number $n \geq 1$, if κ is the least cardinal such that Σ_{n+1}-ESR(κ, λ) holds for a proper class of cardinals λ, but κ is not parametrically n-exact for a proper class of cardinals λ, then there exists some Σ_n-2-supercompact cardinal.*

Proof. Suppose the assumptions are satisfied. Then let $\delta \in C^{(n+1)}$ be greater than κ, and so there is no $\alpha < \kappa$ such that there is some $\beta \geq \delta$ where Σ_{n+1}-ESR(α, β) holds. Let $\lambda > \delta$ be such that Σ_{n+1}-ESR(κ, λ) holds, but κ is not parametrically n-exact for λ, with some $B \in V_{\lambda+1}$ as a counterexample. It follows from Lemma 6.3.18 that there is some cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\kappa$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, some $X \prec H_{\kappa'} = V_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $A, \kappa, \delta \in ran(j)$. Let $crit(j) = \mu$. We have $j(\mu) < \kappa$, since $A \in ran(j)$ implies $j(crit(j)) \neq \kappa$. Let $j(\eta) = \kappa$ and $j(\xi) = \delta$. Let $i \geq 2$ be the least natural number such that $j^i(\mu) > \kappa$. It follows by similar argument as in Lemma 6.3.7 that $j^i(\mu) < \delta$.

Since it holds in $V_{\lambda'}$ that $\delta \in C^{(n)}$, by elementarity it holds in X that $\xi \in C^{(n)}$, thus $\xi \in C^{(n)}$, given that $X \prec H_{\kappa'} = V_{\kappa'} \prec_{\Sigma_n} V$. Suppose for contradiction that there is no Σ_n-2-supercompact cardinal. Then in particular there is no Σ_n-i-supercompact cardinal, and this fact reflects in V_δ, given that $\delta \in C^{(n+1)}$. So by elementarity $V_\xi \models$ "$\exists \nu \in C^{(n)}\ j^{(i-1)}(\mu)$ is not ν-Σ_n-i-supercompact", witnessed by some element $y \in V_\nu$. Since $\xi \in C^{(n)}$, $\nu \in C^{(n)}$, so by elementarity, there should not be some elementary embedding $k : V_\nu \to V_{j(\nu)}$ with $j(y) \in ran(k)$ and $k^i(crit(k)) = j^i(\mu)$, contradicting the existence of $j \restriction V_\nu$. □

Now we can show that negative answers to Question 6.3.2 for every $n \geq 1$ are also equivalent to the existence of large cardinals.

Theorem 6.3.21. *The following statements are equivalent for every natural number $n \geq 1$:*

1. *There exists a Σ_n-2-supercompact cardinal.*

2. *There exists some cardinal κ such that Σ_{n+1}-ESR(κ, λ) holds for a proper class of cardinals λ, but the least such κ does not have the property that $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds for a proper class of cardinals λ.*

3. *There exists some cardinal κ such that Σ_{n+1}-ESR(κ, λ) holds for a proper class of cardinals λ, but the least such κ is not parametrically n-exact for a proper class of cardinals λ.*

*Similarly as in Lemma 6.3.6, this is a variant of Proposition 6.4 in [6], which can be proven similarly.

Proof. (1) implies (2): suppose that there exists some Σ_n-2-supercompact cardinal, then let κ_0 be the least one. By Proposition 6.3.3, there are unboundedly many $\kappa < \kappa_0$ such that Σ_{n+1}-ESR(κ, λ) holds for a proper class of cardinals λ. Let κ_1 be the least such cardinal less than κ_0. Suppose for contradiction that $\Sigma_{n+1}(V_{\kappa_1})$-ESR$(\kappa_1, \lambda)$ holds for a proper class of cardinals λ. Then by Proposition 6.3.13, $\kappa_1 \in C^{(n+3)}$. However, note that the statement "there exists a Σ_n-2-supercompact cardinal" is Σ_{n+2}, so there must be some $\kappa_2 < \kappa_1 < \kappa_0$, such that κ_2 is really a Σ_n-2-supercompact cardinal, contradicting the minimality of κ_0.

(2) implies (3): this follows from Lemma 6.3.19.

(3) implies (1): this follows from Lemma 6.3.20. □

On the other hand, as in Question 6.3.1, it is relatively consistent with ZFC that the answer to Question 6.3.2 is positive.

As in Theorem 6.3.17, a weaker version of Question 6.3.2 has a positive answer.

Proposition 6.3.22. *For every natural number $n \geq 1$, if κ is the least cardinal such that $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds for a proper class of cardinals λ, then for any cardinal λ, if $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds, then κ is parametrically n-exact for λ.*

Proof. Suppose for contradiction that this fails. One can show similarly as in Lemma 6.3.20 that there is some cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\kappa$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, some $X \prec H_{\kappa'} = V_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $\kappa \in ran(j)$, such that $j(crit(j)) < \kappa$. Let $crit(j) = \mu$. Let $i \geq 2$ be the least natural number such that $j^i(\mu) > \kappa$.

By Lemma 6.3.13, $\kappa \in C^{(n+3)}$ and $\lambda \in C^{(n+1)}$. Now for any $\xi < \kappa$ greater than $j^{(i-1)}(\mu)$ such that $\xi \in C^{(n+2)}$, it holds in V_λ that $\xi \in C^{(n+2)}$, and one can show similarly as in Lemma 6.3.20 that $j^{(i-1)}(\mu)$ is ξ-Σ_{n+2}-i-supercompact. It follows that it holds in V_κ, and thus in V, that $j^{(i-1)}(\mu)$ is Σ_{n+2}-i-supercompact. But then by Lemma 6.3.9, we have $j^{(i-1)}(\mu)$ is weakly parametrically $(n+2)$-exact for a proper class of cardianls ν, which implies that $\Pi_{n+2}(V_{j^{(i-1)}(\mu)})$-ESR$(j^{(i-1)}(\mu), \nu)$ holds, and thus $\Sigma_{n+1}(V_{j^{(i-1)}(\mu)})$-ESR$(j^{(i-1)}(\mu), \nu)$ holds for a proper class of cardinals ν, and so $j^{(i-1)}(\mu) < \kappa$ contradicts the minimality of κ. □

Theorem 6.3.23. *The following are equivalent for every cardinal κ and natural number $n \geq 1$:*

1. *κ is the least cardinal such that $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds for a proper class of cardinals λ.*
2. *κ is the least cardinal that is parametrically n-exact for a proper class of cardinals λ.*

6.4 Sequential ESR

In [6], the *sequential* versions of exact structural reflection principles are also introduced and studied. Moreover, questions similar to Question 6.2.4, 6.3.1, 6.3.2 were also open in the sequential cases (for sequences of finite length). Our solutions to Question 6.2.4, 6.3.1, 6.3.2 in the previous sections can be all generalized to the sequential versions of the questions. We record the results in this section. For details of the relevant definitions please consult [6]. In all the cases below, the proofs are obtained by complicating the proofs in the non-sequential situation, but are essentially similar, and make use of our general theory of Σ_n-m-supercompact cardinals, instead of the special case of Σ_n-2-supercompact cardinals.

The following theorem answers the sequential version of Question 6.2.4.

Theorem 6.4.1. *For every natural number $m \geq 1$, if κ is an m-huge cardinal, then for every natural number n, V_κ is a model of ZFC plus the following statement:*

> *"There is a proper class of cardinals μ such that μ is parametrically n-exact for a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i < m \rangle$".*

The following theorem answers the sequential version of Question 6.3.1 in the cases when $n \geq 2$.

Theorem 6.4.2. *For natural numbers $m \geq 1$, $n \geq 2$ and cardinal κ, the following are equivalent:*

1. *κ is the least cardinal that is weakly parametrically n-exact for a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i < m \rangle$.*
2. *κ is the least cardinal such that there is a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that Π_n-ESR$(\vec{\lambda})$ holds and $\lambda_0 = \kappa$.*

The following theorem answers the sequential version of Question 6.3.1 in the case when $n = 1$.

Theorem 6.4.3. *The following statements are equivalent for every natural number $m \geq 1$:*

1. *There exists an $(m+1)$-supercompact cardinal.*
2. *There exists some cardinal κ such that there is a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that Π_1-ESR$(\vec{\lambda})$ holds and $\lambda_0 = \kappa$, but the least such κ does not have the property that there is a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that $\Pi_1(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds and $\lambda_0 = \kappa$.*
3. *There exists some cardinal κ such that there is a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that Π_1-ESR$(\vec{\lambda})$ holds and $\lambda_0 = \kappa$, but the least such κ is not weakly parametrically 1-exact for a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i < m \rangle$.*

A weaker version of the sequential form of Question 6.3.1, in the case $n = 1$, also has a positive answer:

Theorem 6.4.4. *The following are equivalent for every cardinal κ and natural number $m \geq 1$:*

1. *κ is the least cardinal such that there is a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that $\Pi_1(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds and $\lambda_0 = \kappa$.*
2. *κ is the least cardinal that is weakly parametrically 1-exact for a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i < m \rangle$.*

The sequential form of Question 6.3.2 has a negative answer for all $n \geq 1$:

Theorem 6.4.5. *The following statements are equivalent for all natural numbers $n, m \geq 1$:*

1. *There exists a Σ_n-$(m+1)$-supercompact cardinal.*
2. *There exists some cardinal κ such that there is a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that Σ_{n+1}-ESR$(\vec{\lambda})$ holds and $\lambda_0 = \kappa$, but the least such κ does not have the property that there is a proper class of sequences $\vec{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that $\Sigma_{n+1}(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds and $\lambda_0 = \kappa$.*

3. *There exists some cardinal κ such that there is a proper class of sequences $\overrightarrow{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that Σ_{n+1}-ESR$(\overrightarrow{\lambda})$ holds and $\lambda_0 = \kappa$, but the least such κ is not parametrically n-exact for a proper class of sequences $\overrightarrow{\lambda} = \langle \lambda_i : i < m \rangle$.*

Finally, a weaker version of the sequential form of Question 6.3.2 has a positive answer:

Theorem 6.4.6. *The following are equivalent for every cardinal κ and natural numbers $m, n \geq 1$:*

1. *κ is the least cardinal such that there is a proper class of sequences $\overrightarrow{\lambda} = \langle \lambda_i : i \leq m \rangle$ such that $\Sigma_{n+1}(V_{\lambda_0})$-ESR$(\overrightarrow{\lambda})$ holds and $\lambda_0 = \kappa$.*

2. *κ is the least cardinal that is parametrically n-exact for a proper class of sequences $\overrightarrow{\lambda} = \langle \lambda_i : i < m \rangle$.*

Chapter 7

Foundational Reflections

After the endeavors in the previous chapters, let us get back to the foundational considerations in Chapter 0. Recall that we were motivated by Gödel's program of finding well-justified, strong extensions of ZFC, the standard foundation of mathematics. To provide intrinsic justification for large cardinal axioms, the most mathematically important axiom candidates, the central approach is through formulating them in terms of reflection principles. The higher-order approach of Tait is conceptually unwarranted, and falls into the dichotomy theorem of Koellner.

The program of structural reflection initiated by Bagaria promises a viable approach. The basic form of SR, as we have seen, is indeed a plausible principle. Granting this point, it is important to explore the limit of this program, namely the strongest large cardinals that can be formulated in terms of some well-justified form of SR.

We have seen that ESR has several problems in this respect, the most fundamental one being that, as acknowledged by Bagaria and Ternullo, it is not clear that they really are intrinsically justified. Furthermore, their global forms do not satisfy **Conjecture 2**, and they do not exhibit the uniform pattern of correspondence we found at the lower levels of the large cardinal hierarchy.

We thus seek to find natural structural reflection principles and large cardinal notions that are

1. as well-justified as the basic form of SR,
2. stronger than the basic form of SR,
3. satisfies **Conjecture 2**, and
4. exhibits the same pattern as in the equation at the end of section 0.4.

It seems that the principle CSR does fulfill the above expectations. The points (2)-(4) are purely mathematical facts that we have demonstrated in the previous chapters. For (1), recall that SR and CSR use the very same conceptual resources: for any class $\mathcal{C}$, there is some V_α such that, for any $B \in \mathcal{C}$, there is some $A \in \mathcal{C} \cap V_\alpha$ that is structurally very similar to B. We have suggested that the notion of capturing is arguably the correct strengthening of elementary embeddability, and thus CSR is arguably the correct strengthening of SR. This idea is further supported by the fact that m-CSR simply corresponds to higher analogues of supercompact and $C^{(n)}$-extendible cardinals in exactly the same way SR corresponds to supercompact and $C^{(n)}$-extendible cardinals. It seems thus that CSR is as well-justified as SR.

However, one may on the other hand argue that the results actually present a challenge for the current version of the SR program. This is the problem of *extendibility to inconsistency.*

Since it is very plausible to think that δ-CSR correctly extends SR, if SR is intrinsically justified, it would appear that δ-CSR is also justified, and not just for finite δ, but also infinite δ. After all, we have seen that as δ increases, the notion of δ-capturing is simply a stronger formulation of structural similarity. It seems unlikely that there is some principled reason, *based on the concept of the universe of sets*, for us to accept δ-CSR for all finite δ, or even *some particular finite* δ, but reject δ-CSR in general. However, we have seen in Theorem 4.2.3 that δ-CSR is outright inconsistent for infinite δ. This may even be taken as constituting a *Reductio ad absurdum* of the intrinsic justifiability of the basic form of SR.

Moreover, the basic form of SR is already the most conceptually compelling form of SR among all different forms of SR currently available. Thus if the basic form of SR is not intrinsically justified, one may further argue that the whole SR program of intrinsically justifying large cardinals, at least as it is currently formulated, would face a problem.

In conclusion, our evaluation of the program of intrinsically justifying large cardinals by SR is that it can be nicely extended to much higher region in the large cardinal hierarchy, but faces the problem of extendibility to inconsistency if we push too hard. This constitutes a possible objection against the approach, but may not be a fatal issue. Indeed, it is likely that any way of providing theoretical justification for axioms, when inspected carefully enough and pushed to the extremes, will run into various kinds of difficulties, and in the end we simply have to accept our best options. In that case it is also possible that, in the end, we feel that accepting δ-CSR for finite δ but not infinite δ is a good enough and natural enough option to accept.

In any case, given all the investigations, what is clear is the remarkable coherence and power of the concept of *reflection*, which leads us from finitude to infinity, and far, far beyond.

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