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Title: The COS method for pricing European options

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Abstract

The COS method exploits the relation between the characteristic function of a random variable and the series coefficients of the Fourier-cosine expansion of the density function. After the mathematical introduction and the derivation of the Black-Scholes formula, we introduce with all the details the COS method. We compare, in terms of absolute error and in CPU time, its performance when pricing European options with a Monte Carlo scheme and with the Black-Scholes value of the derivative. An error analysis of COS method is also provided. Numerical experiments confirm the fast convergence and the precision of the COS method.
Keywords: Option pricing, European options, Cosine expansion, Fourier inverse transform

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Contents

1 Introduction

When it appeared in 1973 the Black-Scholes model was the cornerstone of option pricing. It assumed constant volatility and a log-normal distribution of asset prices. However, empirical evidence has shown that asset prices exhibit features like volatility clustering, jumps, and fat tails, which the Black-Scholes model cannot capture. Since then, the mathematical models used in finance and in particularly for pricing financial instruments such as options have become progressively more complex. This complexity arises from the need to capture more realistic market behaviours and account for factors such as stochastic volatility (stochastic volatility models such as Merton model address this issue), jumps in asset prices (Merton model and Merton-Kou model are of this type), and other intricate dynamics that simple models cannot adequately represent. The fact that the density of a random variable and its characteristic function form a Fourier pair has had important implications in Finance. The characteristic function, which is the Fourier transform of the probability density function, provides a way to handle complex models analytically. For many advanced models, the density function can not be computed while the characteristic function can be derived explicitly. So instead of applying the direct discounted expectation approach of computing the integral that involves the product of the terminal payoff and the density function of the process, it may be easier to recover the density from its Fourier transform.

Fourier analysis (that includes the Fourier transform) has been widely studied since the 18th century (see for example [Duo00] or [Gra14]) and is a huge important mathematical field with important applications in many different areas such as signal processing, digital image processing, acoustics and others. In finance, the Fourier inversion method was probability first used in [SS91], in a stochastic volatility model that used the transform method in order to find the distribution of the underlying asset. In [Hes93] it is obtained a closed-form solution for the price of a European call option with time varying volatility of the underlying with a technique based on the characteristic functions. In the 2000, [DPS00] provided an analytical treatment of a class of transforms, including various Laplace and Fourier transforms, that can be applied when treating with affine jump diffusion processes. A numerical approach was given by [CM99]. The authors transformed the problem of pricing options into the Fourier domain and then applied the Fast Fourier Transform (FFT) to evaluate the integral numerically. In the Fourier domain it is possible to solve various derivative contracts, as long as the characteristic function is available. The FFT allows for efficient computation by discretizing the integral into a sum and rapidly computing the result, reducing significantly the computational time compared to traditional numerical integration methods. The method proved to be particularly efficient for handling a wide range of asset price models, including those with complex dynamics such as jumps and stochastic volatility. Finally, [FO09] took a different but related approach. The authors used the cosine-series of the density and the relation between the Fourier series and the characteristic function. The method proved to be accurate and fast.

The structure of this thesis is the following: in Chapter 2 we give a brief review about probability theory as well as setting some of the mathematical framework about stochastic processes in order to be able to explain and prove the main results, in Chapter 3, about the Black-Scholes model. In the fourth chapter we set up the basic Monte Carlo scheme for pricing options that will be used later for comparisons. In the fifth chapter we explain with all the details the COS method. In chapter six we provide numerical results about the COS method, comparing the obtained ones with the Monte Carlo scheme and the Black-Scholes formula. In particular, we will study the accuracy and efficiency of the method in terms of absolute error and CPU time, exploring some of the limitations of the method. Finally in Chapter 7 we give a brief summary of the main conclusions drawn from the thesis alongside with some future research that can be made about the method.

2 Mathematical framework

We begin this work reviewing some important mathematical concepts about probability theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 2.1. *Given the two measurable sets* $(\Omega, \mathcal{F}), (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *a random variable is a function*

$$\begin{aligned} X : \Omega &\longrightarrow \mathbb{R} \\ \omega &\longrightarrow X(\omega) \end{aligned}$$

such that $X^{-1}(B) \in \mathcal{F}$ *for all* $B \in \mathcal{B}(\mathbb{R})$*, where* $\mathcal{B}(\mathbb{R})$ *is the Borel* $\sigma-$*algebra. We also say that* X *is* $\mathcal{F}-$*measurable.*

We could substitute $\mathbb{R}$ with any measurable space $\mathcal{E}$ but in $\mathbb{R}$ we can define quantities such as the expectation and variance of a variable or the distribution function. The law of a random variable X is usually expressed in terms of its cumulative distribution function (CDF) $F_X(x)$ defined as

$$F_X(x) := \mathbb{P}(X \leq x)$$

or in terms of its probability density function (PDF):

Definition 2.2. *We say that a random variable X is absolutely continuous with density* $f_X(x)$ *if its cumulative distribution function (CDF)* $F_X(x)$ *can be written as*

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy$$

with $f_X(x) \geq 0$ for all $x \in \mathbb{R}$ and $\int_{\mathbb{R}} f\, d\mu = 1$ where μ is the Lebesgue measure in $\mathbb{R}$.

Definition 2.3. *Let X be a real-valued continuous random variable with PDF* $f_X(x)$*. The expected value of X, denoted by E[X], is defined as*

$$E[X] = \int_{\mathbb{R}} x f_X(x)\, dx$$

if $\int_{\mathbb{R}} |x| f_X(x)\, dx$ *is finite.*
Moreover, if $E[X^2] < \infty$ *we define the variance of X as*

$$Var[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

Definition 2.4. *Given a random variable* X *and* $n \in \mathbb{N}$*, we say that* X *has finite moment of order* n *if* $E[X^n]$ *exists (i.e is finite).*

We will denote by m_n the nth moment of X. In particular, the first moment of X is by definition the expected value of X.

Definition 2.5. *Given an absolutely continuous random variable X with PDF $f_X(x)$ we define the characteristic function X as*

$$\phi_X(u) : \mathbb{R} \longrightarrow \mathbb{C}$$
$$u \longrightarrow E[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} f_X(x)dx$$

Example 2.6. We compute the characteristic function of a random variable Z that follows a standard normal distribution, $Z \sim \mathcal{N}(0, 1)$. We know that in this case $f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2}$. Then

$$\begin{aligned}\phi_Z(u) = E[e^{iuZ}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2} + iux\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2}(x - iu)^2\right) \exp\left(\frac{-u^2}{2}\right) dx \\ &= \exp\left(\frac{-u^2}{2}\right) \end{aligned} \tag{2.1}$$

Then, as $Y = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$ we also have that

$$\phi_Y(u) = E[e^{(iu(\mu+\sigma Z))}] = e^{iu\mu} E[e^{iu\sigma Z}] = e^{iu\mu}\phi_Z(u\sigma) = e^{iu\mu - \frac{\sigma^2 u^2}{2}} \tag{2.2}$$

We first notice that the characteristic function of a random variable X always exist as $|e^{iut}| = 1$ for all $t \in \mathbb{R}$ (note that all possible values of e^{iut} lie on the unit circle, hence are bounded) as long as X admits a density function $f_X(x)$. This is a big difference with the moment generating function which does not always exist (for example with a log-normal distribution). A direct implication from the definition is

$$\phi_X(0) = 1 \tag{2.3}$$

There is also a relation between the characteristic function of a random variable an its moments m_n:

Theorem 2.7. *Let X be a random variable with nth moment finite. Then the characteristic function ϕ_X is n times continuously differentiable and*

$$\phi_X^k(0) = i^k m_k$$

for every $k = 1, \dots, n$

The proof of the theorem can be found in [NS90].

Definition 2.8. *For a random variable X we define its cumulant characteristic function*[2] *as*

$$G(u) = \log(\phi_X(u)) \tag{2.4}$$

[2]Here we have defined the cumulants in terms of the characteristic function. A widely alternative used definition of the cumulants is in terms of the moment generating function. We define the moment generating function of a random variable X as $M_X(t) = \log(E[\exp(tX)]$. The cumulants are defined then as setting $G(t) = \log(E[\exp(tX)]$ and $\xi_n = \left.\frac{d^n G(t)}{dt^n}\right|_{t=0}$. In particular, given the moment generating function M, the moments of X are defined as $m_k = \left.\frac{d^k M(t)}{dt^k}\right|_{t=0}$.

and the nth cumulant of X as

$$\xi_n = \frac{1}{i^n}\frac{d^nG(u)}{du^n}\bigg|_{u=0} \tag{2.5}$$

For example, the first fourth cumulants of are random variable X are:

$$\xi_1 = \frac{1}{i}\frac{dlog(\phi_X(u))}{du}\bigg|_{u=0} = \frac{1}{i}\frac{1}{\phi_X(u)}\frac{d\phi_X(u)}{du}\bigg|_{u=0}$$

$$\xi_2 = \frac{1}{i^2}\frac{d^2log(\phi_X(u))}{d^2u}\bigg|_{u=0} = \frac{1}{i^2}\left[\frac{-1}{(\phi_X(u))^2}\left(\frac{d\phi_X(u)}{du}\right)^2 + \frac{1}{\phi_X(u)}\frac{d^2\phi_X(u)}{d^2u}\right]_{u=0}$$

$$\xi_3 = \frac{1}{i^3}\frac{d^3log(\phi_X(u))}{d^3u}\bigg|_{u=0} = \frac{1}{i^3}\left[\frac{2}{(\phi_X(u))^3}\left(\frac{d\phi_X(u)}{du}\right)^3 + \frac{-3}{(\phi_X(u))^2}\frac{d\phi_X(u)}{du}\frac{d^2\phi_X(u)}{d^2u}\right.$$
$$\left. + \frac{1}{\phi_X(u)}\frac{d^3\phi_X(u)}{d^3u}\right]_{u=0}$$

$$\xi_4 = \frac{1}{i^4}\frac{d^4log(\phi_X(u))}{d^4u}\bigg|_{u=0} = \frac{1}{i^4}\left[\frac{-6}{(\phi_X(u))^4}\left(\frac{d\phi_X(u)}{du}\right)^4 + \frac{12}{(\phi_X(u))^3}\left(\frac{d\phi_X(u)}{du}\right)^2\frac{d^2\phi_X(u)}{d^2u}\right.$$
$$\left.\frac{-3}{(\phi_X(u))^2}\left(\frac{d^2\phi_X(u)}{d^2u}\right)^2 + \frac{-4}{(\phi_X(u))^2}\frac{d\phi_X(u)}{du}\frac{d^3\phi_X(u)}{d^3u} + \frac{1}{\phi_X(u)}\frac{d^4\phi_X(u)}{d^4u}\right]_{u=0}$$

Using Theorem 2.7 and equation (2.3) we can simplify the cumulant expressions to get:

$$\xi_1 = \frac{1}{i}\frac{1}{\phi_X(0)}\phi_X^1(0) = m_1 = E[X] \tag{2.6}$$

$$\xi_2 = \frac{1}{i^2}\left(-(\phi_X^1(0))^2 + \phi_X^2(0)\right) = -(m_1)^2 + m_2 = -E[X]^2 + E[X^2] = Var[X] \tag{2.7}$$

$$\xi_3 = \frac{1}{i^3}\left(2(\phi_X^1(0))^3 - 3\phi_X^1(0)\phi_X^2(0) + \phi_X^3(0)\right) = 2m_1^3 - 3m_1m_2 + m_3 \tag{2.8}$$

$$\begin{aligned}\xi_4 &= \frac{1}{i^4}\left(-6(\phi_X^1(0))^4 + 12(\phi_X^1(0))^2\phi_X^2(0) - 3(\phi_X^2(0))^2 - 4\phi_X^1(0)\phi_X^3(0) + \phi_X^4(0)\right)\\ &= -6m_1^4 + 12m_1^2m_2 - 3m_2^2 - 4m_1m_3 + m_4\end{aligned} \tag{2.9}$$

Example 2.9. Let us consider a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. Then it is clear that $\xi_1 = \mu$ and $\xi_2 = \sigma^2$. To compute ξ_3, ξ_4 we use the characteristic function computed in Example 2.6:

$$\phi_X^1(u) = ie^{iu\mu + \frac{-\sigma^2u^2}{2}}(\mu + iu^2t)$$

$$\phi_X^2(u) = -e^{iu\mu + \frac{-\sigma^2u^2}{2}}(\mu^2 + \sigma^2 + \sigma^4(-t^2) + 2i\mu\sigma^2t)$$

$$\phi_X^3(u) = e^{iu\mu + \frac{-\sigma^2u^2}{2}}(-i\mu^3 + \sigma^6(-t^3) + 3i\mu\sigma^2(\sigma^2t^2 - 1) + 3\mu^2\sigma^2t + 3\sigma^4t)$$

$$\begin{aligned}\phi_X^4(u) = e^{iu\mu + \frac{-\sigma^2u^2}{2}}&(\mu^4 + \mu^2(6\sigma^2 - 6\sigma^4t^2) - 4i\mu\sigma^4t(\sigma^2t^2 - 3) + \sigma^4(\sigma^4t^4 - 6\sigma^2t^2 + 3)\\ &+ 4i\mu^3\sigma^2t)\end{aligned}$$

So

$$\begin{aligned}
m_2 &= \mu^2 + \sigma^2 \\
m_3 &= \frac{1}{i^3} \frac{d^3\phi(u)}{d^3u}|_{u=0} = \frac{1}{i^3}(-i\mu^3 - 3i\mu\sigma^2) = \mu^3 + 3\mu\sigma^2 \\
m_4 &= \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4
\end{aligned}$$

and clearly

$$\begin{aligned}
&\xi_3 = 2\mu^3 - 3\mu(\mu^2 + \sigma^2) + \mu^3 + 3\mu\sigma^2 = 0 \\
&\xi_4 = -6\mu^4 + 12\mu^2(\mu^2 + \sigma^2) - 3(\mu^2 + \sigma^2)^2 - 4\mu(\mu^3 + 3\mu\sigma^2) + \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 = 0
\end{aligned}$$

From the Definition 2.5 if a random variable X admits a density $f_X(x)$ then its characteristic function $\phi_X(u)$ is its Fourier transform with the sign reversal in the complex exponential. In fact, the density $f_X(x)$ and the characteristic function $\phi(u)$ form the following Fourier transform pair:

$$\phi(u) = \int_{\mathbb{R}} e^{ixu} f_X(x)dx \tag{2.10}$$

$$f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \phi(u)du \tag{2.11}$$

2.1 Stochastic processes

In order to be able to model natural phenomenons or the financial markets we need to consider a sequence of random variables over the time. We can define then a stochastic process as a sequence of random variables indexed by some set $\mathbb{T}$ (called the index set, that usually has the meaning of time). Formally,

Definition 2.10. *A stochastic process X is a function*

$$\begin{aligned}
X : \Omega \times \mathbb{T} &\longrightarrow \mathbb{R} \\
(\omega, t) &\longrightarrow X_t(\omega)
\end{aligned}$$

measurable, that is such that

$$X^{-1}(B) \in \mathcal{F} \otimes \mathcal{B}(\mathbb{T}) \tag{2.12}$$

for all $B \in \mathcal{B}(\mathbb{R})$ where $\mathbb{T} := \{0, 1, ...T\}$ when we are modelling a discrete process or $\mathbb{T} = [0, T]$ when we model a continuous process over the time.

Usually we use the notation $\{X(t) : t \in \mathbb{T}\}$, $\{X(t, \omega) : t \in \mathbb{T}\}$ or $\{X_t : t \in \mathbb{T}\}$ to denote an stochastic process. In that sense, the second option is a better choice for a notation, as it also emphasises that fact that a stochastic process is a function of two variables $t \in \mathbb{T}$ and $\omega \in \Omega$. That is, for every t we have a different random variable while for a fixed state $\omega \in \Omega$ we have a map $X(\cdot, \omega) : \mathbb{T} \longrightarrow \mathbb{R}$ that represents a path for that state. For sake of simplicity, in this work we will use the notation X_t

or $X(t)$.
There is a vast bibliography about stochastic processes: we refer the reader to [KT75], [Law06], [Ros19], for more information about general theory and some examples of stochastic processes. Among the most famous stochastic processes there is the Wiener process also known as Brownian motion:

Definition 2.11. *A standard Wiener process is a stochastic process $\{W_t; t \geq 0\}$ such that:*
(i) $W_0 = 0$ almost surely.
(ii) The function $t \longrightarrow W_t := W(t, \omega)$ is, with probability 1, a continuous function.
(iii) The process has independent increments, that is, for $0 = t_0 < t_1 < \ldots < t_m$, the random variables

$$W_{t_1} = W_{t_1} - W_{t_0}, \cdots, W_{t_m} - W_{t_{m-1}}$$

are independent.
(iv) For all $0 \leq s \leq t, W_t - W_s \sim \mathcal{N}(0, t-s)$. In particular, $W_t \sim \mathcal{N}(0, t)$.

Definition 2.12. *Let $X = \{X_t; t \geq 0\}$ be a stochastic process. We say that X is an Itô process if it can be written as*

$$X_t = X_0 + \int_0^t u_s ds + \int_0^t v_s dW_s, \quad t \in [0, T] \tag{2.13}$$

where X_0 is an F_0- measurable random variable (usually a constant), u is an $L^1(\Omega \times [0, T])$ process, v is an $L^2(\Omega \times [0, T])$ and $W = \{W_t; t \geq 0\}$ is a standard Brownian motion. In particular, we have that

- $E \int_0^T |u_s| ds < \infty$
- $E \int_0^T |v_s|^2 ds < \infty$

The notation

$$dX_t = u_t dt + v_t dW_t \tag{2.14}$$

is shorthand for equation (2.13) with X_0 as an initial condition

Remark 1. *It can be proven (see [Shr04]) that the quadratic variation of Brownian motion up to time t is given by t implying that, with probability 1, the paths of the Brownian motion have infinite variation on any finite interval. Moreover, Brownian Motion is nowhere differentiable for any t (see [CKT12]). As a consequence, if $v = \{v_t; t \geq 0\}$ is a process with continuous paths, the Riemann-Stieltjes integral*

$$\int_0^T v_t(\omega) dW_t(\omega) \tag{2.15}$$

does not exist with probability 1.
The integral $\int_0^t v_s dW_s$ is an stochastic Itô integral with respect to the Brownian motion. We will not develop here the theory that is behind stochastic integration (see for example [LL08],[Shr04]). Here we will just state that, for every t, $\int_0^t v_s dW_s$ is a random variable with expectation 0, meaning that $E \int_0^t v_s dW_s = 0$, variance $E\Big[\Big(\int_0^t v_s dW_s\Big)^2\Big] = E\Big[\Big(\int_0^t |v_s|^2 dt\Big)^2\Big]$ and continuous trajectories. Hence every Itô process has also continuous trajectories.

The following theorem, known as Itô's lemma is the equivalent of the chain rule for stochastic calculus:

Theorem 2.13. *Let X be an Itô process and let $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a $C^{1,2}$-function. That is, F is a function for which $F_t(t,x) = \partial_t F(t,x), F_x = \partial_x F(t,x), F_{xx} = \partial_{xx} F(t,x)$ are defined and continuous. Then for every $t \geq 0$*

$$F(t,X_t) = F(0,X_0) + \int_0^t F_s(s,X_s)ds + \int_0^t F_x(s,X_s)dX_s + \frac{1}{2}\int_0^t F_{xx}(s,X_s)v_s^2 ds \tag{2.16}$$

$$= F(0,X_0) + \int_0^t F_s(s,X_s)ds + \int_0^t F_x(s,X_s)u_s ds + \int_0^t F_x(s,X_s)v_s dW_s + \frac{1}{2}\int_0^t F_{xx}(s,X_s)v_s^2 ds \tag{2.17}$$

where the second equality comes from the fact that X is an Itô process.

Proof of the theorem can be found in [LL08] and [Shr04].
In particular, if $X_0 = 0, u \equiv 0, v \equiv 1$ we have that $X_t = W_t$ and we can rewrite Itô's lemma as

$$F(t,W_t) = F(0,0) + \int_0^t F_s(s,W_s)ds + \int_0^t F_x(s,W_s)dW_s + \frac{1}{2}\int_0^t F_{xx}(s,W_s)ds \tag{2.18}$$

Moreover, if $F(t,x) = F(x)$ then

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)u_s ds + \int_0^t F_x(s,X_s)v_s dW_s + \frac{1}{2}\int_0^t F''(s,X_s)v_s^2 ds \tag{2.19}$$

In differential notation Itô's lemma can be written as

$$dF(t,X_t) = (\partial_t F)(t,X_t)dt + (\partial_x F)(t,X_t)dX_t + \frac{1}{2}(\partial_{xx} F)(t,X_t)v_t^2 dt \tag{2.20}$$

with $F(0,X_0)$ initial condition.

3 The Black-Scholes model

The first application of the Brownian motion to finance was given by Louis Bachelier in 1900. His doctoral thesis *Théorie de la spéculation* (see [Bac00]). He proposed that stock prices follow a continuous-time stochastic process with normally distributed returns. His work laid the foundations for stochastic calculus, a mathematical framework essential for modelling and analysing random processes. He also anticipated the efficient market hypothesis by suggesting that market prices reflect all available information and follow a random walk, making it impossible to consistently predict future price movements. Bachelier's work initially met with limited recognition until the mid-20th century, when it was rediscovered and gained recognition. In 1959 M. F. M. Osborne presented a work ([Osb59]) in which he proposed that stock prices follow a random walk and assuming price changes are independent and identically distributed. Osborne demonstrated that while absolute stock prices do not follow a normal distribution, the logarithms of stock prices do. Some years later, in 1973, [Sam73] presented the argument that geometric Brownian motion is a good model for stock prices. The most significant breakthrough in option pricing came in 1973, when Fischer Black, Myron Scholes (alongside with Robert Merton) developed the Black-Scholes model in a famous publication in the *Journal of Political Economy* called *The Pricing of Options and Corporate Liabilities* (see [BS73]). Their work provided arbitrage techniques for pricing and hedging options and a mathematical formula to price European call and put options, revolutionizing financial theory and practice

3.1 Derivatives

There are many types of derivatives such as forward contracts, swaps or futures, but in this work we will focus on **options**. There are two types of options: a call option gives the holder the right to buy an underlying asset whereas a put option gives the holder the right to sell the underlying asset both by a certain date for a certain price. The price of the option, also known as premium, compensates the fact that the owner of the call or the put option is not obliged to buy or sell the underlying asset: this is the fundamental characteristic that distinguishes options from other derivatives such as forwards and futures, where the holder is obligated to buy or sell the underlying asset at fixed price and date. The price in the contract (price at which the underlying will be bought/sold) is known as the exercise price or strike price, and the date on which the sale or purchase of the underlying asset will be executed is known as the expiration date or maturity.

Options can be either American or European. American options can be exercised at any time up to the expiration date, whereas European options can be exercised only on the expiration date itself. Most of the options that are traded on exchanges are American, and usually American options are more traded with stocks as underlying assets while European options are usually traded with stock market index. The main focus on this work will be European options, as under the Black-Scholes model we can price them with closed formulas.

The profit (also known as payoffs) of European call and put options is given respectively by

$$\text{Payoff}_{Call} = (S_T - K)^+ = max(S_T - K, 0) \tag{3.1}$$

$$\text{Payoff}_{Put} = (K - S_T)^+ = max(K - S_T, 0) \tag{3.2}$$

An important relation between the prices C and P of a European call and a European put both with identical strike prices and expiration date is the **Call-Put parity**:

Theorem 3.1. *Consider a put and a call with the same maturity time T and exercise price K on the same underlying asset which is worth S_t at time t. Furthermore, we assume that it is possible to borrow or invest money at a constant rate r. Let us denote by C_t and P_t respectively the prices of the call and the put at time t. Then for all $t < T$ is true that*

$$C_t - P_t = S_t - Ke^{-r(T-t)} \tag{3.3}$$

The proof of it can be found in [LL08] and it based on the absence of an arbitrage opportunity.
Consider a European call option on a stock whose price at time t is denoted by S_t. Let us call T the expiration date and K the strike value. At time $t = T$ the owner of the option has the right to buy the stock at price K. If $S_T \leq K$ the owner can go to the market and buy the stock at price S_T so it will have no interest in exercising the option. However, if at time $t = T$ we have that $S_T > K$ the owner of the option can exercise its right, buy the stock at price K and sell it immediately back on the market at a price S_T making a profit of $S_T - K$. Two questions arises:

1. How much should the buyer pay for the option? In other words, how should we price at time $t = 0$ an asset worth $(S_T - K)^+$ at time T? That is the problem of pricing the option and is a central challenge in financial markets.

2. The seller of the option must be able to deliver a stock at price K meaning that must generate an amount $(S_T - K)^+$ at maturity. That is the problem of hedging the option. Hedging aims to mitigate potential losses due to adverse price movements in the underlying asset. However, effective hedging is complex due to the dynamic and often unpredictable nature of financial markets.

3.2 Asset dynamics

The main purpose of this section is to develop the Black-Scholes model. We start by making some assumptions about the market:

1. The market is efficient: we assume that financial markets are efficient. This means that asset prices fully reflect all available information at any given time.

2. Absence of arbitrage opportunities. In other words, the model assumes that there are no opportunities for riskless profit in the market. If arbitrage opportunities existed, traders would exploit them until they disappeared, ensuring

that the model's pricing remains accurate. Consequently, it is impossible to consistently achieve returns that outperform the market through arbitrage or other strategies.

3. No dividends. The original Black-Scholes model assumes that the underlying asset does not pay dividends during the option's life. Extensions of the model incorporate dividend payments, adjusting the option pricing accordingly.

4. Continuous trading. We assume that trading of the underlying asset and the option occurs continuously without any interruptions. This implies that the markets are open at all times, allowing for the instant execution of trades.

5. Frictionless markets. The Black-Scholes model assumes that markets are frictionless, meaning there are no transaction costs, taxes, or other trading impediments. This assumption ensures that hedging strategies, such as delta hedging, can be implemented perfectly and without cost.

The Black-Scholes model assumes a market with two underlying securities.

- A risk-free asset, that represents a money-market account, described by a deterministic function
$$dA_t = rA_t dt \tag{3.4}$$
with A(0)=1 for convenience, and $r > 0$ a constant risk-free rate. Note that this is an ordinary differential equation $A'(t) = rA(t)$ with a unique solution
$$A(t) = e^{rt}$$
but for consistency with stochastic calculus and Itô processes notation we use differential notation.

- A risky asset (thought of as a stock) represented by an stochastic differential equation known as geometric Brownian motion (GBM) expressed as
$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.5}$$
with S_0 given, $\mu \in \mathbb{R}$ is called the drift rate, and $\sigma > 0$ is the volatility of the stock price S. In particular we have that
$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s dW_s \tag{3.6}$$
so we are assuming that $S = \{S_t; t \geq 0\}$ is an Itô process with $u_s = \mu S_s$ and $v_s = \sigma S_s$.

The equation (3.5) is widely used in financial mathematics. An informal interpretation of it is the following: the change dS_t that the stock prices experiment in an increment of time dt is given by a first term completely deterministic (and thus predictable) plus a random fluctuation given by a Brownian motion W_t.
We start studying the dynamics of the log-asset price:

Example 3.2. Let $X_t = \log(S_t)$. If $F(x) = \log(x)$ then $\partial_t F(x) = 0, \partial_x F(x) = \frac{1}{x}, \partial_{xx} F(x) = \frac{-1}{x^2}$. Applying Itô's lemma we have that

$$\log(S_t) = \log(S_0) + \int_0^t \frac{1}{S_s} dS_s + \frac{1}{2} \int_0^t \frac{-1}{S_s^2} \sigma^2 S_s^2 ds \tag{3.7}$$

$$= \log(S_0) + \int_0^t \mu ds + \int_0^t \frac{1}{S_s} \sigma S_s dW_s + \frac{-1}{2} \int_0^t \sigma^2 ds \tag{3.8}$$

$$= \log(S_0) + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW_s \tag{3.9}$$

So

$$\log(S_t) = \log(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \tag{3.10}$$

From (3.10) is clear that under the Black-Scholes model (assuming (3.5)) the log-asset price follows normal distribution with mean $log(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$. Taking this into account it seems that

$$S_t = S_0 \exp\left\{ (\mu - \frac{\sigma^2}{2})t + \sigma W_t \right\} \tag{3.11}$$

is a solution of (3.5). In fact, we can check it applying again Itô's lemma. We have $S_t = F(t, W_t)$ with

$$F(t, x) = S_0 \exp\left\{ (\mu - \frac{\sigma^2}{2})t + \sigma x \right\}$$

so

$$\begin{aligned} S_t &= F(t, W_t) \\ &= F(0, W_0) + \int_0^t (\partial_s F)(s, W_s) ds + \int_0^t (\partial_x F)(s, W_s) dW_s + \frac{1}{2} \int_0^t (\partial_{xx} F)(s, W_s) ds \\ &= S_0 + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) S_s ds + \int_0^t \sigma S_s dW_s + \frac{1}{2} \int_0^t \sigma^2 S_s ds \end{aligned}$$

Regrouping terms we get that

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma S_s ds$$

All in all, we conclude that under the Black-Scholes model, the stock price of an asset is a log-normally distributed random variable with mean $S_0 e^{\mu t}$ and variance $S_0^2 e^{2\mu t}(e^{\sigma^2 - 1})$. Furthermore, as the log-asset price follows a normal distribution we have that, if $X_t = \log(S_t)$ then the characteristic function of the log asset price is given by

$$\phi_X(u) = e^{iu(\log(S_0) + (\mu - \frac{\sigma^2}{2})t) - \frac{\sigma^2 t u^2}{2}} \tag{3.12}$$

3.3 Black-Scholes equation

Assume that the value of a derivative (for example a call or a put option) with expiration time T can be written as $V(T, S_T) = h(S_T)$. In particular, assume that at any time $0 \leq t \leq T$ we can write the value of the derivative as $V(t, S_t)$ for some deterministic function $V(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$. Assuming that the risky asset follows (3.5) if we apply Itô's lemma to $V(t, S_t)$ we obtain that

$$\begin{aligned} dV(t, S_t) &= \partial_t V(t, S_t)dt + \partial_x V(t, S_t)dS_t + \frac{1}{2}\partial_{xx} V(t, S_t)\sigma^2 S_t^2 dt && (3.13) \\ &= [\partial_t V(t, S_t) + \mu S_t \partial_x V(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx} V(t, S_t)]dt + [\sigma S_t \partial_x V(t, S_t)]dW_t && (3.14) \end{aligned}$$

Now we consider a self-financing trading strategy where at each time t we hold x_t units of the risk-free asset and y_t units of the stock. We denote by P_t, the value at time t of this strategy. In particular, P_t satisfies that

$$P_t = x_t A_t + y_t S_t \tag{3.15}$$

By self-financing we mean that any gains or losses on the portfolio are due entirely to gains or losses in the underlying securities (the risk-free asset and the stock) and not due to changes in the holdings x_t and y_t. The self-financing assumption implies that at an infinitesimal time t x_t and y_t do not change and that implies that

$$\begin{aligned} dP_t &= x_t r A_t dt + y_t dS_t \\ &= x_t r A_t dt + y_t(\mu S_t dt + \sigma S_t dW_t) \\ &= (r x_t A_t + y_t \mu S_t)dt + y_t \sigma S_t dW_t && (3.16) \end{aligned}$$

The idea is to choose our strategy weights x_t and y_t in such a way that the strategy replicates the value of the option, so we equate equations (3.14) and (3.16) so we set

$$\begin{aligned} y_t &= \partial_x V(t, S_t) && (3.17) \\ r x_t A_t &= \partial_t V(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx} V(t, S_t) && (3.18) \end{aligned}$$

We notice that when choosing $y_t = \partial_x V(t, S_t)$ we are in fact eliminating the possible effects that the randomness of the underlying asset could have in our strategy: we are making delta neutral our portfolio. If we set $V(0, S_0) = P_0$, the initial value of our strategy, then $V(t, S_t) = P_t$ for all t since $V(t, S_t)$ and P_t have the same dynamics as by construction we have equated terms in (3.14) with terms in (3.16). Replacing (3.17) and (3.18) in (3.14) and setting $V(t, S_t) = P_t$ we get that

$$\partial_t V(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx} V(t, S_t) + r S_t \partial_x V(t, S_t) - rV(t, S_t) = 0 \quad t \geq 0, x \geq 0 \tag{3.19}$$

with the final condition $V(T, S_T) = h(S_T)$ and some other particular conditions depending on the derivative. In particular, the price of a European call option $V(t, S_t) = C(t, S_t)$ we have that

$$\partial_t C(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{xx} C(t, S_t) + r S_t \partial_x C(t, S_t) - rC(t, S_t) = 0 \quad t \geq 0, x \geq 0 \tag{3.20}$$

with the boundary conditions $C(T, S_T) = (S_T - K)^+, C(t, 0) = 0$ for all t and $C(t, S_t) \to S_t$ as $S_t \to \infty$. Equation (3.20) is the famous Black-Scholes partial differential equation for a European call option.
Equations (3.19) and (3.20) are partial differential equations that do not involve probability and that both holds regardless of which path the stock price follows: if the initial price is positive, then the stock price is always positive, and it can take a positive value. If the initial stock price is zero, then the subsequent stock prices are all zero. We are considering both cases when we set $x \geq 0$.
In terms of differential equations the equation (3.20) is a backward parabolic equation with final data given by $t = T$ and it can be reconverted into the heat equation which has a closed form solution. The derivation can be found in [WHD95] Chapter 5 section 4. After all, we get the **Black-Scholes formula for a call option**:

Theorem 3.3. *The price C, at t=0, of a European call option with strike price K and maturity date T on a stock with initial price S_0, volatility σ and with a given risk-free rate r is*

$$C(S_0, \sigma, T, r, K) = S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2) \tag{3.21}$$

where

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad d_2 = d_1 - \sigma\sqrt{T}$$

The price of the corresponding put option can be obtained using the call-put parity (3.3):

$$P(S_0, \sigma, T, r, K) = Ke^{-rT} - S_0 + C(S_0, \sigma, T, r, K) = e^{-rT} K \Phi(-d_2) - S_0 \Phi(-d_1) \tag{3.22}$$

4 Monte Carlo framework

The principle of risk-neutral valuation (see [Shr04] or [CK13]) can be summarized by

$$V(0, S_0) = \mathbb{E}_Q[e^{-rT} V(T, S_T)] = e^{-rT} \mathbb{E}_Q[V(T, S_T)] \tag{4.1}$$

where $V(0, S_0)$ is the price of the derivative, $\mathbb{E}_Q$ represents the expectation under a risk-neutral measure and r is the risk-free interest rate. So the price of the derivative today can be computed as an expectation of the possible terminal values of the underlying asset. In particular, for a European call option we have that $V(T, S_T) = (S_T - K)^+$.

As by definition we have that $W_t \sim \mathcal{N}(0, t)$ we may rewrite equation (3.11) as

$$S_t = S_0 \exp\left\{(\mu - \frac{\sigma^2}{2})t + \sigma\sqrt{t}Z\right\} \tag{4.2}$$

where Z is a standard normal random variable. Therefore, the problem of generating a random sample for the terminal price of the underlying asset is reduced to generate a random sample from a standard normal distribution which can be done using for example the inverse transform sampling method. Given such a sample $Z_1, Z_2, \ldots$, and taking into account the Law of large numbers we can estimate $\mathbb{E}_Q[(S_T - K)^+]$ using the following algorithm:

Algorithm 1 European call option pricing under the Black-Scholes dynamics using a Monte Carlo framework

1: **for** `i=1,...,n` **do**
2:     `generate` $Z_i \sim \mathcal{N}(0, 1)$
3:     $S_i(T) \leftarrow S(0)exp\left\{(r - \sigma^2/2)T + \sigma\sqrt{T}Z_i\right\}$
4:     $C_i \leftarrow e^{-rT}[S_i(T) - K]^+$
5: **end for**
6: $\hat{C}_n = (C_1 + ... + C_n)/n$

Note that in step 3 we have substituted μ for r, the risk-free rate. Without given much detail we will just state that when doing so we are implicitly describing the risk-neutral dynamics of the stock price (see [CK13]).

For any $n \geq 1$ the estimator $\hat{C}_n$ is unbiased, in the sense that its expectation is the target quantity:

$$E[\hat{C}_n] = C \equiv E[e^{-rT}(S_T - K)^+].$$

as $\hat{C}_n$ is defined as the mean of n independent and identically distributed C_i.

Moreover, if we denote $Var(\hat{C}_n) = \sigma_C^2 < \infty$, the central limit theorem guarantees us that as the number of replications n increases we have the following convergence in distribution:

$$\frac{\hat{C}_n - C}{\sigma_C/\sqrt{n}} \Rightarrow \mathcal{N}(0, 1) \tag{4.3}$$

which can be expressed, as we are interested in the distribution of the error in our simulations, as

$$\hat{C}_n - C \approx \mathcal{N}(0, \sigma_C^2/n)$$

The convergence in distribution can be expressed in terms of limits of distributions functions. In that sense, (4.3) implies that for $x \in \mathbb{R}$

$$P\left(\frac{\hat{C}_n - C}{\sigma_C/\sqrt{n}} \leq x\right) \longrightarrow \Phi(x)$$

where Φ is the standard cumulative normal distribution We can replace σ_C with the sample variance s_C defined as

$$s_C = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(C_i - \hat{C}_n)^2}$$

This is useful because σ_C is rarely known in practice but s_C can be easily computed for every sample. The fact that we can replace σ_C with s_C without changing the limit in distribution follows from the following fact: as s_C^2 is a consistent estimator of σ^2, that is $\lim_{n\to+\infty} s_C^2 = \sigma^2$, by the continuous map theorem and as the convergence in distribution is implied by convergence in probability we have that $s_C \Rightarrow \sigma$ and $\sigma/s_C \Rightarrow 1$. Finally, the Slutsky's Theorem guarantees us that

$$\frac{\hat{C}_n - C}{s_C/\sqrt{n}} \Rightarrow \mathcal{N}(0,1) \tag{4.4}$$

or equivalently,

$$\sqrt{n}(\hat{C}_n - C) \approx \mathcal{N}(0, s_C^2) \tag{4.5}$$

A $1-\delta$ confidence interval for C is then

$$\hat{C}_n \pm z_{\delta/2}\frac{s_C}{\sqrt{n}} \tag{4.6}$$

with $\Phi(z_\delta) = 1-\delta$ (for a 95% confidence interval $\delta = 0.05$ and $z_\delta =\approx 1.96$). From (4.4) it is clear that the standard deviation shrinks at a rate of $\frac{1}{\sqrt{n}}$ which brings that the error of the method is of convergence $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$. This implies that reducing the error of the method in half requires increasing the number of points by a factor of four which increases the complexity of the calculations.

5 COS method

As we have pointed out in the introduction, plenty of methods have been developed to price options using the relation between the characteristic function and the density function of a random variable (Fourier pair (2.10),(2.11)). In this chapter we develop another method described in [FO09]. This method, call the COS method, is another powerful Fourier transform-based technique known for its efficiency and accuracy. While the Carr-Madan method focuses on transforming the payoff function and then applying the inverse Fourier transform (with the FFT) to retrieve the option price, the COS method directly works with the density function and its cosine series expansion in order to approximate the probability density function. Although the COS method can be implemented for more complex models such as Levy processes we will focus our work in how it can be implemented to price European options.

5.1 Density approximation via Fourier transform

The Fourier series is an approximation of a periodic function using a sum of sines and cosines. In general, the Fourier series is an approximation of the original function, but there are cases where the Fourier series is exactly equal to the original function across its entire range. This occurs when the original function is periodic and satisfies certain conditions, such as being piece wise smooth or having a finite number of discontinuities within one period. In such cases, the Fourier series converges to the original function and is equal to it across its entire range.
Remember that the Fourier series involving both sines and cosines, given that these two functions form a complete orthogonal system, over $[-\pi, \pi]$ of a function $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} A_k \cos(kx) + \sum_{k=1}^{\infty} B_k \sin(kx) \tag{5.1}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{5.2}$$

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \tag{5.3}$$

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \tag{5.4}$$

In particular we are interested in the Fourier cosine series of the function f when f is even, that is when $b_n = 0$ and the Fourier series collapses to

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} A_k \cos(kx) \tag{5.5}$$

with

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx \tag{5.6}$$

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(kx)dx \tag{5.7}$$

where the second equality comes from the equality

$$f(x)\cos(nx) = f(-x)\cos(-nx)$$

as we are working with an even function $f(x)$ and $\cos(x)$ is an even function too. To summarize: for a function supported on $[-\pi, \pi]$, the cosine expansion reads

$$f(\theta) = {\sum_{k=0}^{\infty}}' A_k \cos(k\theta) \text{ with } A_k = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\cos(k\theta)d\theta \tag{5.8}$$

where $\sum'$ indicates that the first term in the summation is weighted by one-half.

Remark 2. *Given a not even function $g : [0, \pi] \to \mathbb{R}$ we can extend it to $[-\pi, \pi]$ to become an even function as*

$$\bar{g}(x) = \begin{cases} g(x) & if \quad x \geq 0 \\ g(-x) & if \quad x < 0 \end{cases}$$

We can rewrite equation (5.8) for functions supported on any other finite interval $[a, b] \in \mathbb{R}$ via the following change of variables:

$$\theta = \frac{x-a}{b-a}\pi; \quad x = \frac{b-a}{\pi}\theta + a$$

Then:

$$f(x) = {\sum_{k=0}^{\infty}}' A_k \cos\left(k\pi\frac{x-a}{b-a}\right) \tag{5.9}$$

with

$$A_k = \frac{2}{b-a}\int_{a}^{b} f(x)\cos\left(k\pi\frac{x-a}{b-a}\right)dx \tag{5.10}$$

Suppose that with the chosen interval $[a, b]$ the truncated integral approximates the infinite counterpart (the Fourier transform) very well, i.e.

$$\hat{\phi}(\omega) := \int_{a}^{b} e^{i\omega y} f_X(y)dy \approx \int_{\mathbb{R}} e^{i\omega y} f_X(y)dy = \phi(\omega) \tag{5.11}$$

We have then the following equalities:

$$\hat{\phi}\left(\frac{k\pi}{b-a}\right)e^{-i\frac{k\pi a}{b-a}} = e^{-i\frac{k\pi a}{b-a}}\int_{a}^{b} e^{i\frac{k\pi}{b-a}y} f_X(y)dy = \tag{5.12}$$

$$= \int_{a}^{b} e^{ik\pi\frac{y-a}{b-a}} f_X(y)dy = \tag{5.13}$$

$$= \int_{a}^{b} \left(\cos\left(k\pi\frac{y-a}{b-a}\right) + isin\left(k\pi\frac{y-a}{b-a}\right)\right) f_X(y)dy \tag{5.14}$$

where we have used the Euler formula $e^{ix} = \cos(x) + i\sin(x)$. If we compare this last expression with equation (5.10) we can conclude that

$$A_k \equiv \frac{2}{b-a} Re\left\{ \hat{\phi}\left(\frac{k\pi}{b-a}\right) \exp\left(-i\frac{k\pi a}{b-a}\right)\right\} \tag{5.15}$$

where $Re\{\cdot\}$ denotes the real part of the argument. It then follows from (5.11) that $A_k \approx F_k$ with

$$F_k := \frac{2}{b-a} Re\left\{ \phi\left(\frac{k\pi}{b-a}\right) \exp\left(-i\frac{k\pi a}{b-a}\right)\right\} \tag{5.16}$$

where we recall ϕ is the characteristic function of X. We can now replace A_k by F_k in the Fourier cosine series of $f_X(y)$ on $[a, b]$:

$$\hat{f}_X(y) \approx {\sum_{k=0}^{\infty}}' F_k \cos\left(k\pi\frac{y-a}{b-a}\right) \tag{5.17}$$

Finally we can also truncate this summation such that

$$\tilde{f}_X(y) \approx {\sum_{k=0}^{N-1}}' F_k \cos\left(k\pi\frac{y-a}{b-a}\right) \tag{5.18}$$

5.2 Option pricing with the COS method

This section is mostly based on [FO09] with some additional notes on my own. The starting point is the valuation of a derivative under a risk neutral measure:

$$v(x, t_0) = e^{-r\tau}\mathbb{E}_Q[v(y, T)|x] = e^{-r\tau}\int_{\mathbb{R}} v(y, T) f(y|x)dy \tag{5.19}$$

where $v(y, T)$ denotes the option value at the expiration date T, $\tau = T - t_0$ where t_0 is the valuation moment, $\mathbb{E}_Q$ is the expectation under the risk neutral probability Q. $f(y|x)$ is the probability density of y given x and r is the risk-neutral interest rate. x and y are state variables at time t_0 and T respectively: usually x is the value of the log-asset price at time t and y is the value of the log-asset price at the expiration date. That is, $x = X(t)$ and $y = X(T)$ where $X(t) := \log(S_t)$. As $f(y|x)$ in (5.19) decays to zero fast as $y \to \pm\infty$ we truncate the infinite integration to range to $[a, b] \in \mathbb{R}$ without losing significant accuracy, getting a first approximation to (5.19) :

$$v(x, t_0) \approx v_1(x, t_0) = e^{-r\tau}\int_a^b v(y, T) f(y|x)dy \tag{5.20}$$

In most of the cases we will not know the explicit form of $f(y|x)$, although the characteristic function can be obtained explicitly. We will start by replacing the density $f(y|x)$ with its cosine-series expansion in y. If we denote

$$f(y|x) = {\sum_{k=0}^{\infty}}' A_k \cos\left(k\pi\frac{y-a}{b-a}\right) \tag{5.21}$$

with

$$A_k = \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dx \tag{5.22}$$

we have that

$$v_1(x,t_0) = e^{-r\tau}\int_a^b v(y,T)\sum_{k=0}^{\infty}{}' A_k\cos\left(k\pi\frac{y-a}{b-a}\right)dy \tag{5.23}$$

We interchange the summation and the integration and insert the definition

$$V_k = \frac{2}{b-a}\int_a^b v(y,T)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \tag{5.24}$$

so that

$$v_1(x,t_0) = \frac{b-a}{2}e^{-r\tau}\sum_{k=0}^{\infty}{}' A_kV_k \tag{5.25}$$

Note that the V_k are the cosine-series coefficients of the payoff function $v(y,T)$. So we have transformed the product of two functions $v(y,T)$, $f(y|x)$ (the second one usually unknown) into the product of the coefficients of their respective cosine-series expansion. We can now apply what we have seen in the last section: we can truncate the summation into a finite sum of N terms to get another approximation because of the rapid decay rate of the coefficients:

$$v_2(x,t_0) = \frac{b-a}{2}e^{-r\tau}\sum_{k=0}^{N-1}{}' A_kV_k \tag{5.26}$$

Finally, we can approximate A_k, the cosine-series coefficients of the density $f(y|x)$ with F_k as in (5.16) to get

$$v(x,t_0) \approx v_3(x,t_0) = \frac{b-a}{2}e^{-r\tau}\sum_{k=0}^{N-1}{}' F_kV_k \tag{5.27}$$

$$= e^{-r\tau}\sum_{k=0}^{N-1}{}' Re\left\{\phi_X\left(\frac{k\pi}{b-a}\right)exp\left(-i\frac{k\pi a}{b-a}\right)\right\}V_k \tag{5.28}$$

Equation (5.28) **is the COS formula for general underlying processes** with $\tau = T - t_0$, and x a function of $S(t_0)$.

5.3 Pricing European options

We notice that, if we are able to compute V_k in (5.24) for the payoff function $v(y,T)$ then (5.28) is straightforward as, under the Black-Scholes model, we know that prices follow a log-normal distribution hence the characteristic function of the log-asset price is known. We represent the payoff as a function of the log-asset price. We denote

$$y(T) = \log\left(\frac{S_T}{K}\right)$$

and we can express a European option payoff as

$$v(y,T) \equiv [\alpha K(e^y - 1)]^+ \text{ with } \alpha = \begin{cases} 1 & \text{for a call} \\ -1 & \text{for a put} \end{cases}$$

Setting $Y_t = \log\left(\frac{S_t}{K}\right)$ it is clear from (3.10) and (3.12) that

$$\phi_X(u) = e^{iu\left(log\left(\frac{S_0}{K}\right)+(\mu-\frac{\sigma^2}{2})t\right)-\frac{\sigma^2 tu^2}{2}} \tag{5.29}$$

The following lemma is the basic tool that helps us to compute V_k:

Lemma 5.1. *For an interval $[c,d] \subset [a,b]$ the cosine series coefficients V_k of $g(y) = e^y$ and $g(y) = 1$ are known analytically. That is, the coefficients*

$$\chi_k(c,d) = \int_c^d e^y \cos\left(k\pi\frac{y-a}{b-a}\right) dy$$

$$\psi_k(c,d) = \int_c^d \cos\left(k\pi\frac{y-a}{b-a}\right) dy$$

have a closed form. In particular, we have that

$$\begin{aligned}\chi_k(c,d) = \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\Bigg[&e^d\cos\left(k\pi\frac{d-a}{b-a}\right) - e^c\cos\left(k\pi\frac{c-a}{b-a}\right) \\ &+ \frac{k\pi}{b-a}e^d\sin\left(k\pi\frac{d-a}{b-a}\right) - \frac{k\pi}{b-a}e^c\sin\left(k\pi\frac{c-a}{b-a}\right)\Bigg]\end{aligned} \tag{5.30}$$

and

$$\psi_k(c,d) = \begin{cases} \left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right]\frac{b-a}{k\pi} & k \neq 0 \\ d-c & k=0 \end{cases} \tag{5.31}$$

The proof of the lemma can be found in Appendix A and can be proved just with basic calculus.
For call and put options we have that

$$V_k^{call} = \frac{2}{b-a}\int_0^b K(e^y-1)\cos\left(k\pi\frac{y-a}{b-a}\right) dy = \frac{2}{b-a}K(\chi_k(0,b) - \psi_k(0,b)) \tag{5.32}$$

$$V_k^{Put} = \frac{2}{b-a}\int_a^0 K(1-e^y)\cos\left(k\pi\frac{y-a}{b-a}\right) dy = \frac{2}{b-a}K(-\chi_k(a,0) + \psi_k(a,0)) \tag{5.33}$$

Remark 3. *Note that, for the call option, while integrating from a to b we are assuming that $a < 0 < b$. If $a < b < 0$ then $V_k^{call} = 0$ while if $0 < a < b$ we need use V_k^{Call} by redefining (5.30) (5.31) with $c \equiv a, d \equiv b$. For the Put options the relations are the opposite: if If $a < b < 0$ then we compute V_k^{Put} with $c \equiv a, d \equiv b$ in (5.30) (5.31) while if $0 < a < b$ then $V_k^{Put} = 0$.*

5.4 Truncation range for the COS method

Fang and Oosterlee proposed in [FO09] the following interval for the range interval $[a, b]$ within the COS method:

$$[a, b] = \left[\xi_1 - L\sqrt{\xi_2 + \sqrt{\xi_4}}, \quad \xi_1 + L\sqrt{\xi_2 + \sqrt{\xi_4}} \right] \tag{5.34}$$

where ξ_n is the nth cumulant of the variable $\log\left(\frac{S_T}{K}\right)$ and the proposed value $L = 10$. In particular as we have seen that $\log\left(\frac{S_T}{K}\right) \sim \mathcal{N}(\log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right) T, \sigma^2 T)$ the cumulants for the integration range are $\xi_1 = \log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right) T$, $\xi_2 = \sigma^2 T$ and $\xi_4 = 0$. The calibration of the parameter L is not obvious. A large interval $[a, b]$ will require larger values to sum in (5.28) to reach a good level of accuracy while a small interval will not work as we will be omitting a part of the density.

5.5 Error analysis

When proceeding as in sections 5.1, 5.2 to arrive to the COS formula (5.28) there were 3 steps where we introduced some kind of approximation error:

1. When truncating the integration range of the density in (5.20). We can express the error introduced as

$$\epsilon_1 = v(x, t_0) - v_1(x, t_0) = \int_{\mathbb{R} \setminus [a,b]} v(y, T) f(y|x) dy \tag{5.35}$$

2. When we substitute the density by its truncated by N cosine series expansion in (5.23), (5.25) and in (5.26). This series truncation error in the interval $[a, b]$ can be expressed as

$$\epsilon_2 = v_1(x, t_0) - v_2(x, t_0) = \frac{b - a}{2} e^{-r\tau} \sum_{k=N}^{\infty} A_k V_k \tag{5.36}$$

   where A_k, V_k are the coefficients defined in (5.22) and (5.24). So basically, ϵ_2 depends on the series coefficients of the density and on the series coefficients of the payoff.

3. The error when approximating A_k with F_k in (5.28) that can be expressed as

$$\epsilon_3 = v_3(x, t_0) - v_2(x, t_0) = e^{-r\tau} {\sum_{k=0}^{N-1}}' Re\left\{ \int_{\mathbb{R} \setminus [a,b]} e^{ik\pi \frac{y-a}{b-a}} f(y|x) dy \right\} V_k \tag{5.37}$$

It can be proved (see [FO09]) that when the truncation range is sufficiently large, the overall error is dominated by ϵ_2 so we will focus our analysis in that error. The density is typically smoother than the payoff functions in finance and the coefficients A_k often decay faster than V_k. Consequently, we can bound ϵ_2 as follows:

$$\left|\sum_{k=N}^{\infty} A_k V_k\right| \leq \sum_{k=N}^{\infty} |A_k| \tag{5.38}$$

Since the PDF is generally unknown in the option pricing problem, the selection of the truncation parameter N is a matter of trial and error. We will follow [Aim+23] to select the appropiate value of N.
We can give a first approximation of error ϵ_2 using as an approximation of A_k the coefficients F_k as in (5.16):

$$\epsilon_2 \approx \bar{\epsilon}_2 := \sum_{k=N}^{\infty} |F_n| \leq \frac{2}{b-a} \sum_{N}^{\infty} \left|\phi\left(\frac{k\pi}{b-a}\right) \exp\left(-i\frac{k\pi a}{b-a}\right)\right| \leq \frac{2}{b-a} \sum_{N}^{\infty} \left|\phi\left(\frac{k\pi}{b-a}\right)\right| \tag{5.39}$$

as $\left|\exp\left(-i\frac{k\pi a}{b-a}\right)\right| = 1$ and we can therefore give an estimation of the error in terms of the modulus of the Fourier transform of f. Under the Black-Scholes model we have already seen that the process $\log\left(\frac{S_T}{K}\right)$ follows a normal distribution with mean $\log\left(\frac{S_0}{K}\right) + (r - \frac{\sigma^2}{2})T$ and variance $\frac{\sigma^2}{2}T$ hence the modulus of the characteristic function of $\log\left(\frac{S_T}{K}\right)$ is

$$\left|\phi_X\left(\frac{k\pi}{b-a}\right)\right| = \exp\left(-\frac{1}{2}\sigma^2 T\left(\frac{k\pi}{b-a}\right)^2\right) \tag{5.40}$$

and we can rewrite (5.39) as

$$\bar{\epsilon}_2 \leq \frac{2}{b-a} \sum_{k=N}^{\infty} e^{\frac{-1}{2}\left(\frac{\sigma\pi}{b-a}\right)^2 T k^2} \leq \frac{2}{b-a} \sum_{k=N}^{\infty} e^{-dn} = \frac{2}{b-a} \frac{e^{-dN}}{1-e^{-d}} \tag{5.41}$$

where $d = \frac{1}{2}\left(\frac{\sigma\pi}{b-a}\right)^2 T$. Expression (5.41) proves the convergence of the infinite series given in (5.39) but it is not enough to obtain an appropriate estimation of the value N. However, since the series (5.39) converges and the terms of the series decrease very rapidly (exponentially) we determine N by means of the first term of the series: given a tolerance error ϵ_N we select the smallest value N satisfying

$$\frac{2}{b-a} e^{\frac{-1}{2}\left(\frac{\sigma\pi}{b-a}\right)^2 T N^2} \leq \epsilon_N \tag{5.42}$$

that is

$$N = \left\lceil \sqrt{\frac{-2}{T}\left(\frac{b-a}{\sigma\pi}\right)^2 \log\left(\frac{b-a}{2}\epsilon_N\right)} \right\rceil \tag{5.43}$$

where $\lceil x \rceil = min\{k \in \mathbb{Z} : k \geq x\}$
In particular, if we choose $[a, b]$ as in (5.34) then $b - a = 2L\sqrt{\xi_2} = 2L\sqrt{\sigma^2 T}$ and then

$$N = \left\lceil \sqrt{\frac{-8L^2}{\pi^2} log\left(\sigma L\sqrt{T}\epsilon_N\right)} \right\rceil \tag{5.44}$$

6 Numerical results

We already know that European type options can be priced via a closed formula, i.e with the Black-Scholes formulas (3.21) and (3.22). So we can compare the values obtained with the COS method and with the Monte Carlo method with the real value of the options. The computer use for numerical experiments has AMD Ryzen 5 3600 6-Core Processor 3.60 GHz. The code has been written in R and can be found in the Appendix B. All CPU times are given in milliseconds.
We start by pricing a European option using a Monte Carlo scheme:

Example 6.1. We price a European call option with the following parameters:

$$S(0) = 11, K = 10, r = 0.03, \sigma = 0.25, T = 0.1$$

where K is the strike price of the option. In order to analyze the convergence of the Monte Carlo method we will use different sample sizes. In particular we will compute the price of the option for $n = 100, 1000, 10000, 100000, 1000000$. Using the Black Scholes formula, the price of a call option with the specified parameters is $1.072382530270\ldots$The following table shows the error of the method as well as a confidence interval for the real price of the option and the CPU time needed:

n	100	1000	10000	100000	1e+06	1e+07
$\hat{C}_n$	1.124517	1.082709	1.070281	1.073236	1.071966	1.072482
CI	(0.975098,1.273936)	(1.032892,1.132527)	(1.054589,1.085973)	(1.068271,1.078202)	(1.070395,1.073536)	(1.071985,1.072979)
error	0.05213448	0.01032688	0.002101151	8.537491e-04	4.169886e-04	9.963915e-05
ThError	0.1	0.03163	0.01	0.003163	0.001	0.0003163
time	0.1358986	0.1568794	0.9379387	16.6940700	76.9190800	857.0421000

Table 1: Results of pricing a European call option using a crude Monte Carlo scheme. $\hat{C}_n$ is the estimated value of the option while CI stands for Confidence Interval of the real value. ThError is the theoretical error for the method given the sample size n.

As we can see, we needed 1e+07 paths in order to get and absolute error < 0.001 which took a CPU time of nearly a second. All in all, the computational effort is very high to get a not very low error considering the sample size. This simple example shows both the strength and the weak points of the method: a Monte Carlo approach can be applied to a wide variety of problems across different fields and the implementation is conceptually simple. However, the method requires many samples for accurate results, which can be computationally expensive and the convergence of the method is slow. There are methods to improve a Monte Carlo scheme and to make it faster but this is above the scope of this work (see for example [Gla10]).
We now start exploring the COS method:

Example 6.2. We start by recovering the density function of a standard normal distribution. Recall that for $X \sim \mathcal{N}(0, 1)$ we have that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2}, \phi(u) = e^{\frac{-1}{2}u^2}$$

We note that $f_X(x)$ is an even function. As we have said the method success relies on, besides other issues, the choice of the interval $[a, b]$. At this moment we will just

choose $[a, b] = [10, 10]$. We also choose the interval $[-5, 5]$ as the domain for studying the error of the approximated density. In particular, we generate a sequence of 1000 points of the interval and calculate the error as $max(|\tilde{f}_X(y) - f_X(y)|$ for every value y in that sequence where $\tilde{f}_X(y)$ is the recovered density with (5.18).
Table 2 shows the error when recovering $f_X(x)$ from $\phi(u)$ using different values for

N	4	8	16	32	64	128
error	0.254	0.108	0.00718	4.04e-01	3.89e-16	3.89e-16

Table 2: COS method maximum error for recovering a normal standard density

the truncation number N. The first thing we can notice is that with 4 terms the approximation is very poor (even given negative values as we can see in Figure 1). But the method convergence improves really fast and with 64 terms the approximation gives an error of order 1e-10 which is extremely accurate considering the numbers of terms we are adding in (5.18).
The next figure plots the approximation results.

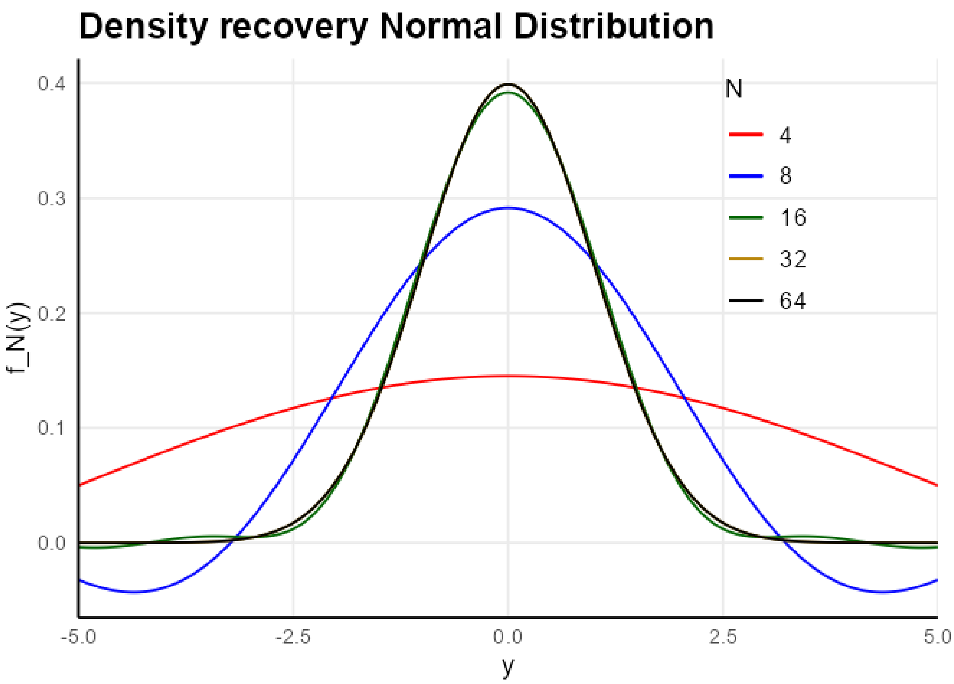

Figure 1: Normal standard density recovered using the characteristic function for different truncation ranges.

The second example is closely related with the first one:

Example 6.3. As we already know, under the Black-Scholes model the stock prices follow a log-normal distribution. Despite the fact that the log-normal distribution has no closed form for the characteristic function we can use the COS method for recover the density the following way: if Y is a random variable that follows a log-normal distribution with parameters μ and σ^2, $Y \sim LN(\mu, \sigma^2)$, we know that

$Y = e^X$ where $X \sim \mathcal{N}(\mu, \sigma^2)$. From general probability theory we know that if g is a one-to-one function then $Y = g(X)$ has density

$$f_Y(y) = f_X(g^{-1}(y))| \ (g^{-1})'(y)|$$

with $g(x) = e^x$ so $g^{-1}(y) = log(y)$. So the density of Y is $f_Y(y) = \frac{1}{y} f_X(log(y))$. All in all, we can used the COS method as in Example 6.2 but evaluating (5.18) at $log(y)$ and then multiplying by $\frac{1}{y}$.
We will use the same truncation interval as before for the characteristic function $[a, b] = [10, 10]$. However, as the log-normal distribution just takes positives values, we also choose the interval $(0, 5]$ as the domain for studying the error of the approximated density. In particular, as before, we generate a sequence a 1000 uniform sample of the interval and calculate the error as $max(|\tilde{f}_X(y) - f_X(y)|$ for every value y in that sequence.
Figure 2 and Table 3 shows the result of the COS method for a log-normal density with parameters $\mu = 0.03$ and $\sigma = 0.1$. As we can see, even with 128 terms the method does not fully recover the log-normal density.

N	4	8	16	32	64	128
error	3.744	3.549	3.163	2.432	1.248	0.211

Table 3: COS method maximum error for recovering a log-normal density of parameters $\mu = 0.03$ and $\sigma = 0.1$

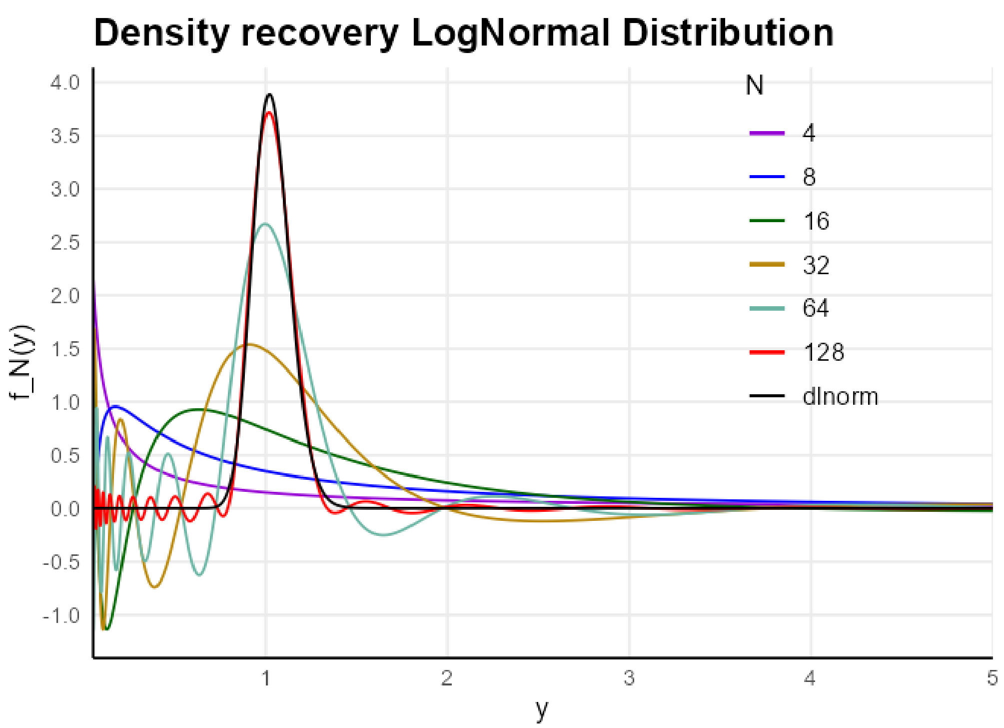

Figure 2: log-normal density with parameters $\mu = 0.03$ and $\sigma = 0.1$ recovered using the COS method with different N truncation ranges.

As we can see in Figure 2 128 terms are not enough for the method to get a low error. We reproduce the method with another parameters for the log-normal density. In particular if we choose $\mu = 0.03$ and $\sigma = 0.4$. This is a density with higher skewness than the log-normal considered before, but with lower kurtosis. Table 4 shows that, in this case, 128 terms are enough to have a maximum error that we can consider acceptable.

N	4	8	16	32	64	128
error	2.136735	0.9508882	0.899878	0.2903176	6.106225e-04	1.587731e-14

Table 4: COS method maximum error for recovering a log-normal density of parameters $\mu = 0.05$ and $\sigma = 0.4$

Figure 3 shows that the red line (128 terms in (5.18)) superpose with the black line (exact values for the density desired).

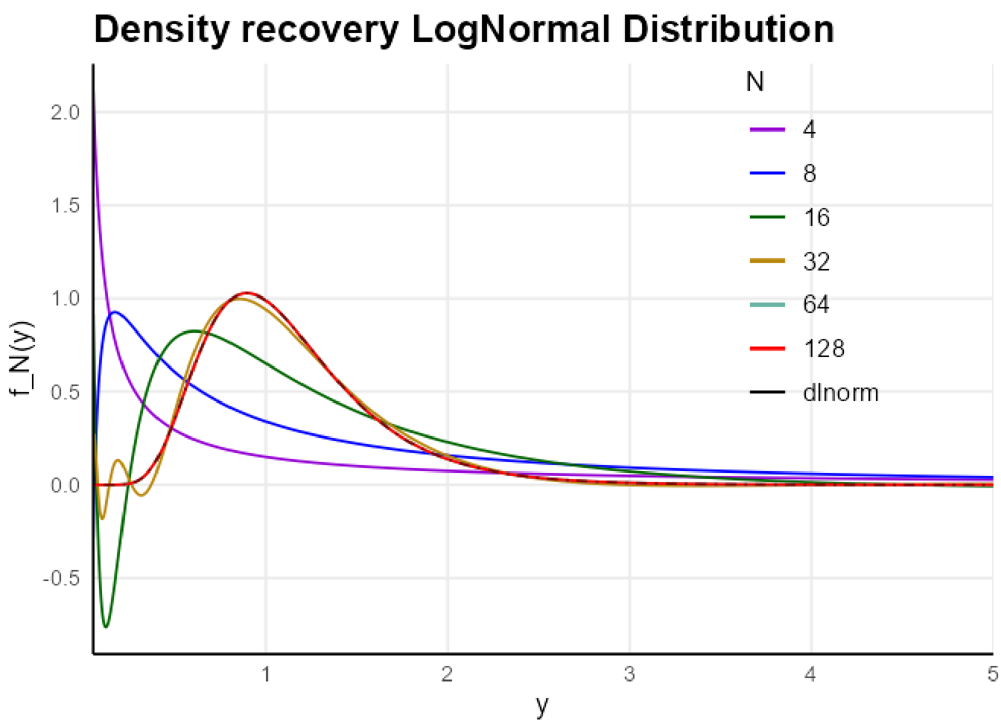

Figure 3: log-normal density with parameters $\mu = 0.03$ and $\sigma = 0.4$ recovered using the COS method with different N truncation ranges.

The core idea of the COS method is to approximate the probability density function (PDF) of the underlying asset's return using a Fourier cosine series expansion and leveraging the characteristic function to approximate the series coefficients. As we have seen in the last two examples, the approximation of a log-normal normal density is pretty accurate and fast so we can expect the method to work well when pricing options:

Example 6.4. We first used the COS method to value a European call option assuming the underlying process follows a GBM. For better comparison we will use the same parameters as in Example 6.1:

$$S(0) = 11, K = 10, r = 0.03, \sigma = 0.25, T = 0.1$$

With this parameters $x = \log(S_0/K) = 0.09531018$ and the resulting integration interval is $[a, b] \approx [-0.6953842, 0.8857546]$. As we can see, this interval includes the 0 so the Remark 3 does not apply. The results are presented in the following table:

n	15	20	25	30	40	45	50
Cos Method	1.07067258	1.07221886	1.07238515	1.07238269	1.07238253	1.07238253	1.07238253
error	1.709949e-03	1.636703e-04	2.622531e-06	1.638291e-07	3.216789e-10	1.005018e-11	4.440892e-16
CPU time	0.0708	0.03915	0.03600	0.03815	0.04101	0.04387	0.04601

Table 5: Error and CPU time for the COS method when pricing a European call option with parameters as in (6.4). The reference value for the Call option with the parameters used and computed with the Black-Scholes formula is 1.072382530270...

As we can observe the COS method significantly outperforms the Monte Carlo method in terms of CPU time and absolute error when it comes to pricing a European call option. With just 30 terms we have achieved an error of magnitude 1e-07: using Monte Carlo (see example 6.1) we couldn't get such a low error even with a sample of size 1e+07. In terms of CPU time, 30 terms took under the COS method $\approx$ 0.04 milliseconds; using a Monte Carlo scheme, generating 1e+07 paths took $\approx$ 857 milliseconds, nearly a second.

The following example illustrates the Remark 3:

Example 6.5. Consider the problem of pricing a European call with the COS method assuming the underlying process follows a GBM and the following parameters:

$$S0 = 100, K = 10, r = 0.03, \sigma = 0.25, T = 0.1 \tag{6.1}$$

Clearly the option is In-The-money. Actually, the option is deep in the money, meaning that the difference between the strike price is significantly below the actual market price of the underlying asset. Furthermore, given the parameters (6.1) we have that $[a, b] \approx [1.511891, 3.09303]$. The Black-Scholes value of this Call option is 90.02995504.... If we use the COS method as in (5.32) we get the following results:

n	15	20	25	30	40	45	50
Cos Method	100.62509119	100.63296140	100.63399724	100.63397721	100.63397619	100.63397620	100.63397620
error	10.59513615	10.60300636	10.60404220	10.60402217	10.60402115	10.60402115	10.60402115

Table 6: Error when using the COS method for a European call option with parameters as in (6.1) and using (5.32)

.

As we can see in Table 6 the method does not converge to the true value. When we recalculate with $c \equiv a, d \equiv b$ then we are implicitly integrating from a to b and then the method converges with the same speed and accuracy as before:

n	15	20	25	30	40	45	50
Cos Method	90.02307346	90.02918120	90.02997109	90.02995583	90.02995504	90.02995504	90.02995504
error	6.881589e-03	7.738415e-04	1.604216e-05	7.843294e-07	4.283947e-09	8.333245e-11	5.684342e-14

Table 7: Error when using the COS method for pricing a European call option with parameters as in (6.1) and using (5.32) with $\chi_k(a,b) - \psi_k(a,b)$

The following example shows that the method exhibits some sensitivity regarding the choice of L. This holds specifically for call options with a long time maturity i.e $T = 30, 40$ years. Under this long expiration date the interval of integration $[a, b]$ gets larger. Furthermore, increasing the expiration date T we make the variance of the normal distribution that drives the log-asset price increase and the distribution get fat-tailed (options with long expiration date may be useful when considering insurance products with a long lifetime).

Example 6.6. Consider the problem of pricing a European call with the COS method assuming the underlying process follows a GBM and the following parameters:

$$S0 = 100, K = 100, r = 0.03, \sigma = 0.25, T = 30 \tag{6.2}$$

With these parameters the integration range is $[a, b] \sim [-20.5771, 20.5021]$. The following table shows the absolute error when pricing such an option for different values of N and L:

	16	32	64	128	256
L=5	6.910875e-03	5.732964e-03	5.732964e-03	5.732964e-03	5.732964e-03
L=6	2.441215e-01	6.487593e-05	6.487593e-05	6.487593e-05	6.487593e-05
L=7	1.041984e+01	2.970210e-07	2.841825e-07	2.841825e-07	2.841825e-07
L=8	1.915426e+02	2.317233e-05	5.459668e-10	5.459668e-10	5.459668e-10
L=9	2.160370e+03	6.093369e-03	9.094663e-10	9.094663e-10	9.094663e-10
L=10	1.800639e+04	4.842387e-01	2.356373e-09	2.356373e-09	2.356373e-09
L=11	1.231640e+05	1.762050e+01	3.323336e-09	3.323336e-09	3.323336e-09
L=12	7.360719e+05	3.770185e+02	1.013899e-08	1.013899e-08	1.013899e-08
L=13	3.998071e+06	5.552692e+03	1.191895e-07	4.463409e-07	4.463409e-07
L=14	2.025347e+07	6.240139e+04	8.119612e-05	1.555396e-07	1.555396e-07
L=15	9.738647e+07	5.737040e+05	9.125795e-03	3.533626e-07	3.533626e-07

Table 8: Absolute error of the COS method for different values of L and N with parameters as in (6.2)

As we can see from Table 8 the method seems to be accurate for value $L \in [8, 10]$. Increasing the value L makes the integration interval $[a, b]$ bigger. Larger values of parameter L would require larger N-values to reach the same level of accuracy, but even doubling the sample size does not improve the accuracy. On the contrary, reducing the value of L shrinks the interval $[a, b]$ and gives poor accuracy. All in all, when pricing call options with a long maturity date, the method seems to give good results for Ł $\in [8, 10]$ as we can see also in Figure 4. [FO09] recommends $L \in [7.5, 10]$ when pricing call options or use the call-put parity.

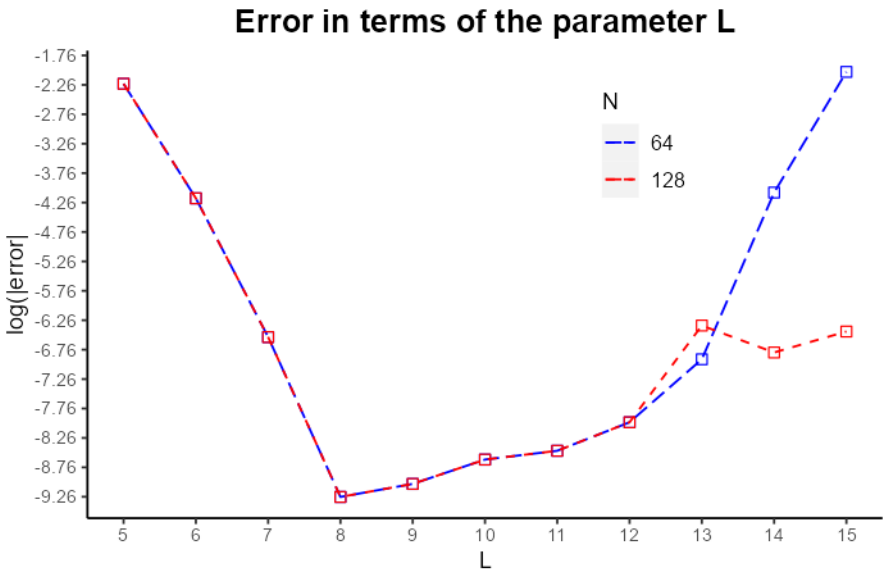

Figure 4: Absolute error of the COS method for different values of L. The logarithm of the error is given is base 10.

The final example illustrates how well the selection of the numbers of terms works using formulas (5.43) or (5.44):

Example 6.7. We first use the parameters in [Aim+23] to price a European call option:

$$S_0 = 100, K = 120, r = 0.05, \sigma = 0.2, T = 0.1 \tag{6.3}$$

For errors $\epsilon_N = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-8}, 10^{-10}$ the formula (5.44) gives $N = 25, 28, 32, 40, 44$. Table 10 shows the absolute error of the COS method when pricing a option with parameters as in (6.3):

N	25	28	32	40	44
ϵ_N	1e-03	1e-04	1e-05	1e-08	1e-10
error	2.268868e-05	4.179924e-07	1.598283e-07	3.865738e-11	4.723669e-13

Table 9: Absolute error for the COS method when pricing a European option with parameters as in (6.3)

As we can see, in this case the absolute error is lower than the theoretical error for the given number of terms. This is not always the case. For example, if we consider an option with the following parameters

$$S_0 = 100, K = 100, r = 0.03, \sigma = 0.25, T = 1 \tag{6.4}$$

and for the same values for the error ϵ_N as before the formula (5.44) gives $N = 23, 26, 30, 38, 43$. For that number of terms, Table 6.4 shows the absolute error of the COS method when pricing a option with parameters as in (6.4).
As we can see, this time the absolute error for $N = 23, 26, 30, 38$ is bigger than the theoretical one. For $N = 44$ the absolute error obtained is lower.

N	25	28	32	40	44
ϵ_N	1e-03	1e-04	1e-05	1e-08	1e-10
error	1.170279e-03	3.864012e-04	1.874864e-05	1.480968e-08	2.150458e-11

Table 10: Absolute error for the COS method when pricing a European option with parameters as in (6.4)

It is important to remark that equation (5.39) it is not a bound but an approximation of the error. In that sense, the formula (5.43) is a good approximation of the needed terms to get a given error. This can be particularly useful when we might need a large number of terms to price options but we do not know, from the earlier, how many, for example when pricing options with long expiration date, as in Example 6.6.

7 Conclusions

In this work, we presented a comprehensive study of option pricing methodologies, focusing on the Black-Scholes model and the COS method, the latter being based on [FO09]. Our primary focus was to evaluate the effectiveness and accuracy of the COS method in comparison to the widely-used Monte Carlo scheme and the Black-Scholes formula. All methods were implemented in R, enabling robust numerical experimentation and analysis.
Firstly, we gave a mathematical approach to Black-Scholes model, which provides an analytical solution for European option pricing. Following [FO09] we provided a detailed explanation of the COS method, highlighting its theoretical basis and practical implementation. The method leverages Fourier cosine expansions to approximate option prices efficiently. We also conducted an error analysis and provided a thorough discussion of the parameters that influence the accuracy and efficiency of the method. Through numerical experiments implemented in R, we compared the performance of the Monte Carlo scheme, the COS method, and the Black-Scholes formula in pricing European options. Our experiments considered various scenarios, including different expiration dates, volatilities and strikes.
While the Monte Carlo scheme is versatile and widely applicable to various financial derivatives, its performance in terms of computational time and convergence rate was outperformed by the COS method for the specific case of European options. The COS method demonstrated superior accuracy and computational efficiency compared to the Monte Carlo scheme. The method's reliance on Fourier series expansions enables rapid convergence and precise results. The findings suggest that the COS method can be a valuable tool for financial practitioners requiring fast and accurate option pricing solutions. Despite the strengths of the COS method, it is important to acknowledge its reliance on the availability of characteristic functions of the underlying asset price distributions. Its implementation is simple when pricing European options and future work could focus on extending the COS method to more complex derivatives or more complex models, such as the Heston model, which incorporates stochastic volatility, and models with jumps in asset prices, such as Merton's jump-diffusion model.

Appendix A

A Derivation of coefficients V_k for European options

The integral

$$\psi_k(c,d) = \int_c^d cos\left(k\pi\frac{y-a}{b-a}\right) dy$$

is direct given as a result (5.31). We focus then on solving

$$\chi_k(c,d) = \int_c^d e^y cos\left(k\pi\frac{y-a}{b-a}\right) dy \tag{.1}$$

We will make us of the two the following trigonometric identities: $cos(\alpha-\beta) = cos(\alpha)cos(\beta) + sin(\alpha)\sin(\beta)$ and $sin(\alpha-\beta) = sin(\alpha)cos(\beta) - cos(\alpha)\sin(\beta)$. First we can rewrite equation (.1) as

$$\chi_k(c,d) = cos\left(\frac{k\pi a}{b-a}\right)\int_c^d e^y cos\left(\frac{k\pi y}{b-a}\right) dy + sin\left(\frac{k\pi a}{b-a}\right)\int_c^d e^y sin\left(\frac{k\pi y}{b-a}\right) dy$$

We denote $A = \int_c^d e^y cos\left(\frac{k\pi y}{b-a}\right) dy$ and $B = \int_c^d e^y sin\left(\frac{k\pi y}{b-a}\right) dy$. Using integration by parts:

$$B = \begin{bmatrix} u = e^y & dv = sin\left(\frac{k\pi y}{b-a}\right) dy \\ du = e^y\, dy & v = -cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi} \end{bmatrix} = \Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d +$$

$$\frac{b-a}{k\pi}\int_c^d cos\left(\frac{k\pi y}{b-a}\right) e^y\, dy = \begin{bmatrix} u = e^y & dv = cos\left(\frac{k\pi y}{b-a}\right) dy \\ du = e^y\, dy & v = sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi} \end{bmatrix} =$$

$$= \Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d + \frac{b-a}{k\pi}\left(\Big[e^y sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d - \frac{b-a}{k\pi}\int_c^d sin\left(\frac{k\pi y}{b-a}\right) e^y\, dy\right) =$$

$$= \Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d + \frac{b-a}{k\pi}\left(\Big[e^y sin\left(\frac{k\pi y}{b-a}\right)\Big]_c^d\right) - \left(\frac{b-a}{k\pi}\right)^2 B$$

So

$$B + \left(\frac{b-a}{k\pi}\right)^2 B = \Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d + \frac{b-a}{k\pi}\left(\Big[e^y sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d\right)$$

and

$$B = \frac{\left(\frac{b-a}{k\pi}\right)^2}{1+\left(\frac{b-a}{k\pi}\right)^2}\Bigg[-e^d cos\left(\frac{k\pi d}{b-a}\right)\frac{k\pi}{b-a} + e^c cos\left(\frac{k\pi c}{b-a}\right)\frac{k\pi}{b-a} + e^d sin\left(\frac{k\pi d}{b-a}\right) -$$

$$e^c sin\left(\frac{k\pi c}{b-a}\right)\Bigg]$$

On the other hand

$$A=\begin{bmatrix} u=e^y & dv=cos\left(\frac{k\pi y}{b-a}\right)dy \\ du=e^y\,dy & v=sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\end{bmatrix}=\Big[e^y sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d-\frac{b-a}{k\pi}\int_c^d sin\left(\frac{k\pi y}{b-a}\right)e^y\,dy=$$

$$=\begin{bmatrix} u=e^y & dv=sin\left(\frac{k\pi y}{b-a}\right)dy \\ du=e^y\,dy & v=-cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\end{bmatrix}=\Big[e^y sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d$$

$$-\frac{b-a}{k\pi}\left(\Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d+\frac{b-a}{k\pi}\int_c^d cos\left(\frac{k\pi y}{b-a}\right)e^y\,dy\right)=$$

$$=\Big[e^y sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d-\frac{b-a}{k\pi}\left(\Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d+\frac{b-a}{k\pi}A\right)=$$

$$=\Big[e^y sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d-\frac{b-a}{k\pi}\left(\Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d\right)-\left(\frac{b-a}{k\pi}\right)^2 A$$

So

$$A=\frac{1}{1+\left(\frac{b-a}{k\pi}\right)^2}\left[\Big[e^y sin\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d-\frac{b-a}{k\pi}\Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\frac{b-a}{k\pi}\Big]_c^d\right]$$

$$=\frac{\left(\frac{b-a}{k\pi}\right)^2}{1+\left(\frac{b-a}{k\pi}\right)^2}\left[\Big[\frac{k\pi}{b-a}e^y sin\left(\frac{k\pi y}{b-a}\right)\Big]_c^d-\Big[-e^y cos\left(\frac{k\pi y}{b-a}\right)\Big]_c^d\right]$$

Now

$$\chi_k(c,d)=cos\left(\frac{k\pi a}{b-a}\right)A+sin\left(\frac{k\pi a}{b-a}\right)B=\frac{\left(\frac{b-a}{k\pi}\right)^2}{1+\left(\frac{b-a}{k\pi}\right)^2}\left[\frac{k\pi}{b-a}e^d sin\left(\frac{k\pi d}{b-a}\right)cos\left(\frac{k\pi a}{b-a}\right)\right.$$

$$-\frac{k\pi}{b-a}e^c sin\left(\frac{k\pi c}{b-a}\right)cos\left(\frac{k\pi a}{b-a}\right)+e^d cos\left(\frac{k\pi d}{b-a}\right)cos\left(\frac{k\pi a}{b-a}\right)-e^c cos\left(\frac{k\pi c}{b-a}\right)cos\left(\frac{k\pi a}{b-a}\right)$$

$$-e^d sin\left(\frac{k\pi a}{b-a}\right)cos\left(\frac{k\pi d}{b-a}\right)\frac{k\pi}{b-a}+e^c sin\left(\frac{k\pi a}{b-a}\right)cos\left(\frac{k\pi c}{b-a}\right)\frac{k\pi}{b-a}+e^d sin\left(\frac{k\pi a}{b-a}\right)sin\left(\frac{k\pi d}{b-a}\right)$$

$$\left.-e^c sin\left(\frac{k\pi a}{b-a}\right)sin\left(\frac{k\pi c}{b-a}\right)\right]=\frac{\left(\frac{b-a}{k\pi}\right)^2}{1+\left(\frac{b-a}{k\pi}\right)^2}\left[e^d\Big[\frac{k\pi}{b-a}sin\left(\frac{k\pi d}{b-a}\right)cos\left(\frac{k\pi a}{b-a}\right)+\right.$$

$$cos\left(\frac{k\pi d}{b-a}\right)cos\left(\frac{k\pi a}{b-a}\right)-sin\left(\frac{k\pi a}{b-a}\right)cos\left(\frac{k\pi d}{b-a}\right)\frac{k\pi}{b-a}+sin\left(\frac{k\pi a}{b-a}\right)sin\left(\frac{k\pi d}{b-a}\right)\Big]+$$

$$e^c\Big[-\frac{k\pi}{b-a}sin\left(\frac{k\pi c}{b-a}\right)cos\left(\frac{k\pi a}{b-a}\right)-cos\left(\frac{k\pi c}{b-a}\right)cos\left(\frac{k\pi a}{b-a}\right)+sin\left(\frac{k\pi a}{b-a}\right)cos\left(\frac{k\pi c}{b-a}\right)\frac{k\pi}{b-a}$$

$$\left.-sin\left(\frac{k\pi a}{b-a}\right)sin\left(\frac{k\pi c}{b-a}\right)\Big]\right]$$

Using the two trigonometric relations quoted before we have that

$$\chi_k(c,d) = \frac{\left(\frac{b-a}{k\pi}\right)^2}{1+\left(\frac{b-a}{k\pi}\right)^2}\Bigg[e^d\Big[sin\left(k\pi\frac{d-a}{b-a}\right)\frac{b-a}{k\pi} + cos\left(k\pi\frac{d-a}{b-a}\right)\Big] - e^c\Big[sin\left(k\pi\frac{c-a}{b-a}\right) + cos\left(k\pi\frac{c-a}{b-a}\right)\frac{k\pi}{b-a}\Big]\Bigg]$$

So

$$\chi_k(c,d) = \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\Bigg[cos\left(k\pi\frac{d-a}{b-a}\right)e^d - cos\left(k\pi\frac{c-a}{b-a}\right)e^c + \frac{k\pi}{b-a}sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}sin\left(k\pi\frac{c-a}{b-a}\right)e^c\Bigg]$$

B Code

Recovery density:

```
    install.packages("rbenchmark")
library(ggplot2)
library(dplyr)
library(tidyr)
library(rbenchmark)

#Define the characteristic funtion for the normal distribution
charFunNormalDensity<-function(mu,sigma,u){
  "
  mu= mean of the normal distribution
  sigma= standard deviation of normal distribution
  u=evaluating point
  "
  z<-complex(real=-0.5*(sigma^2) *(u^2) ,imaginary = u*mu)
  return(exp(z))
}
#Cos method
cos_method<-function(x,N,a,b,mu,sigma){
  k<-0:(N-1)
  u<-(k*pi)/(b-a)
  Fk<-(2/(b-a))*Re(charFunNormalDensity(mu,sigma,u)*exp(-1i*u*a))
  Fk[1]<-0.5*Fk[1]
  return (sum(Fk*cos(u*((x-a)))))
}

#parameters for the normal distribution
mu=0
sigma=1
### method parameters
N<-c(4,8,16,32,64,128)
L<-10
a<--1*L #We recover the
b<-L
#y<-seq(from=0.05,to=5,by=0.001)
y<-seq(from=-5,to=5,by=0.001)

#Shortest version
error<-list(0)
time<-list(0)
for (i in 1:length(N)) {
  start <- Sys.time()
  f_y=sapply(y,cos_method,N[i],a,b,mu,sigma)
```

```
  time[i]=Sys.time() - start
  error[i]=max(abs(f_y-dnorm(y)))
  plot(y,f_y,type="l",main=N[i])
}
#recovery lognormal distribution
#y<-seq(from=0.05,to=5,by=0.001)
y<-seq(from=0.05,to=5,by=0.001)
#Shortest version
error<-list(0)
time<-list(0)
###log normal distribution. Short version.
for (i in 1:length(N)) {
  start <- Sys.time()
  f_y=(1/y)*sapply(log(y),cos_method,N[i],a,b,mu,sigma)
  time[i]=Sys.time() - start
  error[i]=max(abs(f_y-dlnorm(y,meanlog=mu,sdlog=sigma)))
  plot(y,f_y,type="l",main=N[i])
}
```

Monte Carlo pricing:

```
    #Valoracion de call europea
K=10 #Strike
S0=11 #Spot
r=0.03 #Interes libre de riesgo
sig=0.25 #Volatilidad
T=0.1 #vencimiento
n=c(10^2,10^3,10^4,10^5,10^6,10^7) #Tamaño muestra Monte Carlo

valoracionMC=function(K,S0,r,sig,T,n)
{
  set.seed(123)
  Z=rnorm(n)
  S=S0*exp((r-0.5*sig^2)*T+sig*sqrt(T)*Z)
  payoff=exp(-r*T)*pmax(S-K,0)
  vcallMC=mean(payoff)
  #IC 95%
  ICl=vcallMC-1.96*sd(payoff)/sqrt(n)
  ICr=vcallMC+1.96*sd(payoff)/sqrt(n)
  return(c(vcallMC,ICl,ICr))
}

valoracionBS=function(K,S0,r,sig,T)
{
  d1=(log(S0/K)+(r+0.5*sig^2)*T)/(sig*sqrt(T))
  d2=d1-sig*sqrt(T)
  vcall=S0*pnorm(d1)-exp(-r*T)*K*pnorm(d2)
```

```
  return(vcall)
}

#Valor call
vBS=valoracionBS(K,S0,r,sig,T)
time<-list(0)
vMC<-list(0)
IC<-list(0)
error<-list(0)
theo_error<-list(0)
for (i  in 1:length(n)){
  start <- Sys.time()
  tmp=valoracionMC(K,S0,r,sig,T,n[i])
  time[i]=Sys.time() - start
  vMC[i]=tmp[[1]]
  IC[[i]]=c(tmp[[2]],tmp[[3]])
  #Error
  #Error MC
  error[i]<-abs(vMC[[i]]-vBS)
  #Error teorico
  theo_error[i]<-1/sqrt(n[i])

}
```

COS method:

```
    #Valoracion de call europea
K=100 #Strike
S0=100 #Spot
r=0.03 #Interes libre de riesgo
sig=0.25 #Volatilidad
T=0.1 #vencimiento
#T=30/365 #Vencimiento
x=log(S0/K)
### method parameters
#N<-c(16,32,64,128)
#N<-c(15,20,25,30,35,40,50)
N<-c(25,28,31,39)
#N<-20
L<-10
#L<-c(5,7.5,10,12,15)
# interval
a<- x + (r-0.5*sig^2)*T -L*sqrt(sig^2 *T)
b<- x + (r-0.5*sig^2)*T +L*sqrt(sig^2 *T)

chi<-function(a,b,c,d,k){
    x<- 1/( 1 + ((k*pi)/(b-a))^2)
```

```
    y<-cos(k*pi*(d-a)/(b-a))*exp(d) - cos(k*pi*(c-a)/(b-a))*exp(c) +
      ((k*pi)/(b-a)) * sin(k*pi*(d-a)/(b-a))*exp(d) - ((k*pi)/(b-a)) *
      sin(k*pi*(c-a)/(b-a))*exp(c)
    return(x*y)

}
psi<-function(a,b,c,d,k){

  ifelse(k==0,(d-c),(sin(k*pi*(d-a)/(b-a)) -
  sin(k*pi*(c-a)/(b-a)))*(b-a)/(k*pi))

}
charFunGBM1<-function(u,x,r,sig,T){
  #u evaluate the fun at point u
  # x=log(S0/K)
  # r= risk-free rate
  # sigma volatility of the stock
  # t=time to maturity
  z<-complex(real=-0.5*(sig^2) * T *(u^2),
  imaginary = u*(x+ (r- 0.5*sig^2)*T))
  return(exp(z))

}
### COS METHOD
### Call pricing
call_cos<-c()
time<-list(0)
for (i in 1:length(N)) {
  start <- Sys.time()
  k<-0:(N[i]-1)
  u<-(k*pi)/(b-a)
  #call pricing v2
  Ak<-Re(charFunGBM1(u,x,r,sig,T)*exp(-1i*k*pi*a/(b-a)))
  # WHEN  0<a<b use instead
  #Vk<-2/(b-a) * K *( chi(a,b,a,b,k) -psi(a,b,a,b,k))
  Vk<-2/(b-a) * K *( chi(a,b,0,b,k) -psi(a,b,0,b,k))


  call_cos[i]<-exp(-r*T)*(sum(Ak*Vk) - 0.5*Ak[1]*Vk[1])
  time[i]=Sys.time() - start

}

valoracionBS=function(K,S0,r,sig,T)
{
  d1=(log(S0/K)+(r+0.5*sig^2)*T)/(sig*sqrt(T))
```

```
  d2=d1-sig*sqrt(T)
  vcall=S0*pnorm(d1)-exp(-r*T)*K*pnorm(d2)
  vput=exp(-r*T)*K*pnorm(-d2)-S0*pnorm(-d1)
  return(c(vcall,vput))
}

#### Error analisis
#### BS
valoracionBS(K,S0,r,sig,T)[1]
sprintf("%.8f",call_cos)
abs(call_cos-valoracionBS(K,S0,r,sig,T)[1])
#benchmarking
time
## sprintf("%.8f",valoracionBS(K,S0,r,sig,T)[1])
```

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