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Full Length Research Paper

The uncertain probabilistic weighted average and its application in the theory of expertons

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A new model for dealing with decision making under risk by considering subjective and objective information in the same formulation is here presented. The uncertain probabilistic weighted average (UPWA) is also presented. Its main advantage is that it unifies the probability and the weighted average in the same formulation and considering the degree of importance that each case has in the analysis. Moreover, it is able to deal with uncertain environments represented in the form of interval numbers. We study some of its main properties and particular cases. The applicability of the UPWA is studied and it is seen that it is very broad because all the previous studies that use the probability or the weighted average can be revised with this new approach. Focus is placed on a multi-person decision making problem regarding the selection of strategies by using the theory of expertons.

Key words: Interval numbers, probability, weighted average, theory of expertons, decision making.

INTRODUCTION

In the literature, we find a wide range of aggregation operators for decision making. A very common one is the weighted average (Beliakov et al., 2007). It aggregates the information weighting the importance of each argument. Another very common aggregation is the probabilistic aggregation. It aggregates the information by using probabilities in the analysis. Moreover, we can analyze a lot of other aggregation operators such as those that use the OWA operator (Emrouznejad and Amin, 2010; Kacprzyk and Zadrozny, 2009; Yager, 1988, 1993; Yager and Kacprzyk, 1997), the Choquet integral (Tan and Chen, 2010), distance measures (Merigó and Casanovas, 2011b), norms (Yager, 2010), logarithm aggregations (Zhou and Chen, 2010), heavy aggregations (Merigó and Casanovas, 2010c) and induced aggregation operators (Merigó and Gil-Lafuente, 2009).

Usually, when dealing with decision making problems, we assume that the information is clearly known and can be assessed with exact numbers. However, in real world situations this is not so common because our world is very complex and the information is not so clearly known. Thus, we need to use other techniques for assessing the information such as the use of interval numbers. Therefore, we can assess the information considering the lowest and the highest result that may occur and sometimes also the result with the highest possibility of occurrence.

When using interval numbers in an aggregation process, we form the uncertain aggregation operators such as the uncertain weighted average (UWA) and the uncertain probabilistic aggregation (UPA). Thus, we can aggregate the information considering the use of interval numbers in the analysis. In the literature, we find a lot of studies dealing with interval numbers (Jin and Liu, 2010; Liu, 2009; 2010; Merigó and Casanovas, 2011c; Wei, 2009). Note that there are a lot of other aggregation operators that deals with other sources of information such as fuzzy numbers (Merigó and Casanovas, 2010a, 2010b; Srekumar and Mahapatra, 2009; Wang et al., 2009; Wei, 2010; Wei et al., 2010; Yang et al., 2010), linguistic variables (Merigó and Casanovas, 2010c) and with grey information (Liu and Liu, 2010).

Recently, Merigó (2009a) has suggested the probabilistic weighted average (PWA). It is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance of each concept in the analysis. Thus, we can assess a problem considering the objective and the subjective information of the environment.

The aim of this paper is to introduce a new decision making approach based on the uncertain probabilistic weighted average (UPWA). It is an aggregation operator that unifies the uncertain probabilistic aggregation and the UWA in the same formulation and taking into account the degree of importance of each concept in the analysis. Moreover, it also uses interval numbers in the aggregation process. Therefore, it can assess complex environments where the information is very imprecise and cannot be assessed with exact numbers but it is possible to use interval numbers. We study some of its main properties and particular cases including the uncertain average (UA), the UPA, the UWA, the uncertain arithmetic weighted average (UA-WA) and the uncertain arithmetic probabilistic aggregation (UA-PA).

We study the applicability of the UPWA operator and we see that it is very broad because all the previous studies that use the probability or the weighted average can be revised and extended with this new approach. We focus on an uncertain multi-person decision making problem regarding the selection of strategies by using the theory of expertons (Kaufmann, 1988). The use of the UPWA operator, permits to form a new decision making process by using objective and subjective probabilities in the analysis. That is, uncertain decision making under objective risk and under subjective risk. We see that this new decision making framework is more complete because it can assess objective and subjective information in the problem. By using the theory of expertons we can assess the information in a more complete way because we can consider the opinion of several persons in the analysis.

LITERATURE REVIEW

We briefly revised the interval numbers, the uncertain probabilistic aggregation operators, the uncertain weighted aggregation operators and the probabilistic weighted average.

The interval numbers

Interval numbers (Moore, 1966) provide a very useful and simple technique for representing uncertainty because they can consider the minimum and the maximum results that may occur. They have been used in an astonishingly wide range of applications and can be defined as follows.

Definition 1: Let $a = [a_1, a_2] = \{x \mid a_1 \leq x \leq a_2\}$, then, a is called an interval number. Note that a is a real number if $a_1 = a_2$.

The interval numbers can be expressed in different forms. For example, assume a 4-tuple $[a_1, a_2, a_3, a_4]$, that is, a quadruplet, and let a_1 and a_4 represent the minimum and the maximum of the interval number, respectively, and a_2 and a_3 represent the interval with the highest probability or possibility, depending on how we plan to use the interval numbers. Note that $a_1 \leq a_2 \leq a_3 \leq a_4$. If $a_1 = a_2 = a_3 = a_4$, then the interval number is an exact number. If $a_2 = a_3$, it is a 3-tuple known as triplet, and if $a_1 = a_2$ and $a_3 = a_4$, it is a simple 2-tuple interval number.

We review some basic interval number operations. Let A and B be two triplets, where $A = [a_1, a_2, a_3]$ and $B = [b_1, b_2, b_3]$. Then:

1. $A + B = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$.
2. $A - B = [a_1 - b_3, a_2 - b_2, a_3 - b_1]$.
3. $A \times k = [k \times a_1, k \times a_2, k \times a_3]$, for $k > 0$.
4. $A \times B = [a_1 \times b_1, a_2 \times b_2, a_3 \times b_3]$, for R^+.
5. $A \times B = [\min (a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3), a_2 \times b_2, \max (a_1 \times b_1, a_1 \times b_3, a_3 \times b_1, a_3 \times b_3)]$, for R.
6. $A \div B = [a_1 \div b_3, a_2 \div b_2, a_3 \div b_1]$, for R^+.
7. $A \div B = [\min (a_1 \div b_1, a_1 \div b_3, a_3 \div b_1, a_3 \div b_3), a_2 \div b_2, \max (a_1 \div b_1, a_1 \div b_3, a_3 \div b_1, a_3 \div b_3)]$, for R.

Note that in some cases, it is not clear which interval number is higher, so we must establish an additional criterion for ranking the interval numbers. For simplicity, we recommend the following criteria:

1. For 2-tuples, calculate the arithmetic mean of the interval, with $(a_1 + a_2) / 2$.
2. For 3-tuples and above, calculate a weighted average that yields more importance to the central values. That is, for 3-tuples, $(a_1 + 3a_2 + a_3) / 5$.
3. For 4-tuples, we calculate: $(a_1 + 3a_2 + 3a_3 + a_4) / 8$.
4. And so on.

In the case of a tie between the intervals, we select the interval with the lowest difference, that is, $(a_2 - a_1)$. For 3-tuples and above odd-tuples, we select the interval with the highest central value. Note that for 4-tuples and above even-tuples, we must calculate the average of the central values following the initial criteria. To understand the usefulness of this method, let us look into an example.

Example 1: Assume we want to rank the following interval numbers: A = (13, 26, 47), B = (22, 28, 35) and C = (17, 27, 39). Initially, it is not clear which is higher. As explained before, we assume in this paper a ranking based on $(a_1 + 3a_2 + a_3) / 5$. Thus:

$A = (13 + 3 \times 26 + 47) / 5 = 27.6$.
$B = (22 + 3 \times 28 + 35) / 5 = 28.2$.
$C = (17 + 3 \times 27 + 39) / 5 = 27.4$.

With these results, we can reorder the interval numbers such that $B > A > C$. Note that other operations and ranking methods could be studied (Moore, 1966) but in this paper we focus on those discussed above.

Uncertain probabilistic aggregation operators

Probabilistic aggregation operators are those functions that use probabilistic information in the aggregation process. Some examples are the aggregation with simple probabilities, the aggregation with belief structures (Merigó and Casanovas, 2009; Merigó et al., 2010) and the concept of immediate probabilities (Engemann et al., 1996; Merigó 2010; Yager, 1995). For example, the classical uncertain probabilistic aggregation (UPA) can be defined as follows. Note that the UPA can also be seen as the uncertain expected value.

Definition 1: Let Ω be the set of interval numbers. An UPA operator of dimension n is a mapping UPA: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector P, with $\tilde{p}_i \in [0, 1]$ and $\sum_{i=1}^{n} \tilde{p}_i = 1$, such that:

$$UPA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{i=1}^{n} \tilde{p}_i \tilde{a}_i \tag{1}$$

Where $\tilde{a}_i$ is an interval number representing the ith argument variable.

Another example of uncertain probabilistic aggregation is the uncertain immediate probability (IP-UOWA) that uses OWAs and probabilities in the same formulation. It can be defined as follows.

Definition 2: Let Ω be the set of interval numbers. An IP-UOWA operator of dimension n is a mapping IP-UOWA: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector W of dimension n such that $\tilde{w}_j \in [0, 1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$, according to the following formula:

$$IP\text{-}UOWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} \hat{p}_j b_j \tag{2}$$

Where b_j is the jth largest of the $\tilde{a}_i$, each argument $\tilde{a}_i$ is an interval number and has a probability $\tilde{p}_i$ with $\sum_{i=1}^{n} \tilde{p}_i = 1$ and $\tilde{p}_i \in [0, 1]$, $\hat{p}_j = (\tilde{w}_j \tilde{p}_j / \sum_{j=1}^{n} \tilde{w}_j \tilde{p}_j)$ and $\tilde{p}_j$ is the probability $\tilde{p}_i$ ordered according to b_j, that is, according to the jth largest of the $\tilde{a}_i$.

Uncertain weighted aggregation operators

Uncertain weighted aggregation operators are those functions that weight the aggregation process by using the weighted average and the information is represented with interval numbers. The uncertain weighted average (UWA) can be defined as follows.

Definition 3: Let Ω be the set of interval numbers. A WA operator of dimension n is a mapping WA: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector W, with $\tilde{w}_i \in [0, 1]$ and $\sum_{i=1}^{n} \tilde{w}_i = 1$, such that

$$UWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{i=1}^{n} \tilde{w}_i \tilde{a}_i \tag{3}$$

Where $\tilde{a}_i$ is an interval number representing the ith argument variable.

Other extensions of the UWA are those that use it with the UOWA operator (Xu and Da, 2003) such as the WOWA operator (Torra, 1997; Torra and Narukawa, 2007) and the hybrid averaging (HA) operator (Xu and Da, 2003; Zhao et al., 2009, 2010). Note that in this case we get the uncertain WOWA (UWOWA) and the uncertain HA (UHA) operator. Recently, Merigó (2009b) suggested another approach called the OWA weighted average (OWAWA) operator. Its main advantage is that it unifies the OWA and the WA considering the degree of importance that each concept has in the aggregation. For the case with interval numbers it is called the uncertain OWAWA (UOWAWA) operator and it can be defined as follows.

Definition 4: Let Ω be the set of interval numbers. An UOWAWA operator of dimension n is a mapping $UOWAWA$: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector W of dimension n such that $\tilde{w}_j \in [0, 1]$ and $\sum_{j=1}^{n} \tilde{w}_j = 1$, according to the following formula:

$$UOWAWA\ (\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{4}$$

Where b_j is the jth largest of the $\tilde{a}_i$, each argument $\tilde{a}_i$ is an interval number and has an associated weight (WA) $\tilde{v}_i$ with $\sum_{i=1}^{n} \tilde{v}_i = 1$ and $\tilde{v}_i \in [0, 1]$, $\hat{v}_j = \tilde{\beta} \tilde{w}_j + (1 - \tilde{\beta}) \tilde{v}_j$ with $\tilde{\beta} \in [0, 1]$ and $\tilde{v}_j$ is the weight (WA) $\tilde{v}_i$ ordered according to b_j, that is, according to the jth largest of the $\tilde{a}_i$.

The probabilistic weighted average

The probabilistic weighted averaging (PWA) operator (Merigó, 2009a) is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation. It is defined as follows.

Definition 5: A PWA operator of dimension n is a mapping PWA: $R^n \rightarrow R$ such that:

$$PWA\ (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_i a_i \tag{5}$$

Where the a_i are the argument variables, each argument a_i has an associated weight (WA) $\tilde{v}_i$ with $\sum_{i=1}^{n} \tilde{v}_i = 1$ and $\tilde{v}_i \in [0, 1]$, and a probabilistic weight $\tilde{p}_i$ with $\sum_{i=1}^{n} \tilde{p}_i = 1$ and $\tilde{p}_i \in [0, 1]$, $\hat{v}_i = \tilde{\beta}\tilde{p}_i + (1-\tilde{\beta})\tilde{v}_i$ with $\tilde{\beta} \in [0, 1]$ and $\hat{v}_i$ is the weight that unifies probabilities and WAs in the same formulation.

Note that it is also possible to formulate the PWA operator separating the part that strictly affects the probabilistic information and the part that affects the WAs. The PWA is monotonic, bounded and idempotent.

THE UNCERTAIN PROBABILISTIC WEIGHTED AVERAGE

The uncertain probabilistic weighted averaging (UPWA) operator is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation process. Moreover, it is also able to deal with uncertain environments that can be assessed with different types of interval numbers. Thus, we could also call the UPWA as the interval probabilistic weighted average (IPWA). However, we will follow the usual notation used in the literature (Xu and Da, 2002; Merigó and Casanovas, 2011). It is defined as follows.

Definition 6: Let Ω be the set of interval numbers. An UPWA operator of dimension n is a mapping $UPWA$: $\Omega^n \rightarrow \Omega$ such that:

$$UPWA\,(\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} \hat{v}_j \tilde{a}_j \tag{6}$$

Where the $\tilde{a}_i$ are the argument variables represented in the form of interval numbers, each argument $\tilde{a}_i$ has an associated weight (WA) $\tilde{v}_i$ with $\sum_{i=1}^{n} \tilde{v}_i = 1$ and $\tilde{v}_i \in [0, 1]$, and a probabilistic weight $\tilde{p}_i$ with $\sum_{i=1}^{n} \tilde{p}_i = 1$ and $\tilde{p}_i \in [0, 1]$, $\hat{v}_i = \tilde{\beta}\tilde{p}_i + (1-\tilde{\beta})\tilde{v}_i$ with $\tilde{\beta} \in [0, 1]$ and it is also an interval number and $\hat{v}_i$ is the weight that unifies probabilities and WAs in the same formulation.

The UPWA operator can also be formulated separating the part that strictly affects the probabilistic information and the part that affects the UWAs.

Definition 7: Let Ω be the set of interval numbers. An UPWA operator is a mapping $UPWA$: $\Omega^n \rightarrow \Omega$ of dimension n, if it has an associated probabilistic vector P, with $\sum_{i=1}^{n} \tilde{p}_i = 1$ and $\tilde{p}_i \in [0, 1]$ and a weighting vector V that affects the UWA, with $\sum_{i=1}^{n} \tilde{v}_i = 1$ and $\tilde{v}_i \in [0, 1]$, such that:

$$UPWA\,(\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta} \sum_{i=1}^{n} \tilde{p}_i \tilde{a}_i + (1-\tilde{\beta}) \sum_{i=1}^{n} \tilde{v}_i \tilde{a}_i \tag{7}$$

Where the $\tilde{a}_i$ are the argument variables represented in the form of interval numbers and $\tilde{\beta} \in [0, 1]$ and it is also an interval number.

Note that sometimes, it is not clear how to reorder the arguments. Then, it is necessary to establish a criterion for comparing interval numbers.

In order to understand this new approach, let us look to a simple numerical example.

Example 2: Assume the following set of arguments A = ([20, 30], [40, 50], [10, 20], [60, 70]). We assume the following weights for the probability P = (0.2, 0.2, 0.3, 0.3) and for the weighted average V = (0.1, 0.2, 0.3, 0.4); and β = 0.4. By using Eq. (7) we get:

UPWA = 0.4 × (0.2 × [20, 30] + 0.2 × [40, 50] + 0.3 × [10, 20] + 0.3 × [60, 70]) + 0.6 × (0.1 × [20, 30] + 0.2 × [40, 50] + 0.3 × [10, 20] + 0.4 × [60, 70]) = [35.4, 45.4].

Note that if the weighting vector of probabilities or WAs is not normalized, that is, $P = \sum_{i=1}^{n} \tilde{p}_i \neq 1$, or $V = \sum_{i=1}^{n} \tilde{v}_i \neq 1$, then, the UPWA operator can be expressed as:

$$f\,(\tilde{a}_1, \ldots, \tilde{a}_n) = \frac{\tilde{\beta}}{P} \sum_{j=1}^{n} \tilde{p}_i \tilde{a}_i + \frac{(1-\tilde{\beta})}{V} \sum_{i=1}^{n} \tilde{v}_i \tilde{a}_i \tag{8}$$

The UPWA is monotonic, bounded and idempotent. It is monotonic because if $\tilde{a}_i \geq u_i$, for all $\tilde{a}_i$, then, $UPWA\,(\tilde{a}_1, \ldots, \tilde{a}_n) \geq UPWA\,(u_1, u_2 \ldots, u_n)$. It is bounded because the UPWA aggregation is delimitated by the uncertain minimum and the uncertain maximum. That is, $\text{Min}\{\tilde{a}_i\} \leq UPWA\,(\tilde{a}_1, \ldots, \tilde{a}_n) \leq \text{Max}\{\tilde{a}_i\}$. It is idempotent because if $\tilde{a}_i = a$, for all $\tilde{a}_i$, then, $UPWA\,(\tilde{a}_1, \ldots, \tilde{a}_n) = a$.

If B is a vector corresponding to the arguments $\tilde{a}_i$, we shall call this the argument vector and W^T is the transpose of the weighting vector, then, the UPWA operator can be expressed as:

$$UPWA\,(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = W^T B \tag{9}$$

A further interesting result consists in using infinitary aggregation operators (Mesiar and Pap, 2008). Thus, we can represent the aggregation process with an unlimited number of arguments that appear in the aggregation process. Note that $\sum_{i=1}^{\infty} \hat{p}_i = 1$. By using, the UPWA operator we get the infinitary UPWA (∞-UPWA) operator as follows:

$$\infty\text{-}UPWA\,(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{i=1}^{\infty} \hat{p}_i \tilde{a}_i \tag{10}$$

The aggregation process is very complex because we have an unlimited number of arguments (Mesiar and Pap, 2008).

The UPWA operator can also be extended by following (Mesiar and Spirkova, 2006; Torra and Narukawa, 2010). Thus, we can develop a generating function for the arguments of the UPWA operator that represents the formation of this information, such as the use of a multi-person process. We call this formulation the mixture UPWA (MUPWA) operator and it is defined as follows.

Definition 6: Let Ω be the set of interval numbers. A MUPWA operator of dimension n is a mapping *MUPWA*: $\Omega^n \rightarrow \Omega$ that has associated a vector of weighting functions f, s: $\Omega^m \rightarrow \Omega$, such that:

$$MUPWA\,(s_y(\tilde{a}_1),\ldots, s_y(\tilde{a}_n)) = \frac{\sum_{i=1}^{n} f_i(s_y(\tilde{a}_i))s_y(\tilde{a}_i)}{\sum_{i=1}^{n} f_i(s_y(\tilde{a}_i))} \quad (11)$$

Where $\tilde{a}_i$ is the argument variable represented in the form of interval numbers and $s_y(\tilde{a}_i)$ indicates that each argument is formed using a different function.

Another interesting issue to analyze are the measures for characterizing the weighting vector $\hat{V}$. The entropy of dispersion (Shannon, 1948; Yager, 1988, 2009) measures the amount of information being used in the aggregation. For the UPWA operator, it is defined as follows.

$$H(\hat{V}) = -\left(\tilde{\beta}\sum_{i=1}^{n}\tilde{v}_i \ln(\tilde{v}_i) + (1-\tilde{\beta})\sum_{i=1}^{n}\tilde{p}_i \ln(\tilde{p}_i)\right) \quad (12)$$

Note that $\tilde{p}_i$ is the *i*th weight of the UPA aggregation and $\tilde{v}_i$ the *i*th weight of the UWA aggregation. As we can see, if $\tilde{\beta} = 1$ or $\tilde{\beta} = 0$, we get the classical Shannon entropy of dispersion (Shannon, 1948) by using interval numbers.

Families of UPWA operators

By using a different manifestation in the weighting vectors, we can obtain a wide range of UPWA operators.

Remark 1: If β = 0, we get the uncertain weighted average (UWA) and if $\tilde{\beta}$ = 1, we get the uncertain probabilistic approach. Note that it is possible to use partial cases such as $\tilde{\beta}$ = (0.9, 1).

Remark 2: If $\tilde{p}_i$ = 1/n and $\tilde{v}_i$ = 1/n, for all i, then, we get the uncertain average (UA). Note that the UA is also found if $\tilde{\beta}$ = 1 and $\tilde{p}_i$ = 1/n, for all i, and if $\tilde{\beta}$ = 0 and $\tilde{v}_i$ = 1/n, for all i.

Remark 3: If $\tilde{v}_i$ = 1/n, for all i, then, we get the uncertain arithmetic probabilistic aggregation (UA-PA).

$$UA\text{-}PA\,(\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\sum_{i=1}^{n}\tilde{p}_i\tilde{a}_i + (1-\tilde{\beta})\frac{1}{n}\sum_{i=1}^{n}\tilde{a}_i \quad (13)$$

Remark 4: If $\tilde{p}_i$ = 1/n, for all i, then, we get the uncertain arithmetic WA (UA-WA).

$$UA\text{-}WA\,(\tilde{a}_1, \ldots, \tilde{a}_n) = \tilde{\beta}\frac{1}{n}\sum_{i=1}^{n}\tilde{a}_i + (1-\tilde{\beta})\sum_{i=1}^{n}\tilde{v}_i\tilde{a}_i \quad (14)$$

Theorem 1: If the interval numbers are reduced to the usual exact numbers, then, the UPWA operator becomes the PWA operator.

Proof: Assume a quadruplet = (a_1, a_2, a_3, a_4). If $a_1 = a_2 = a_3 = a_4$, then (a_1, a_2, a_3, a_4) = a, thus, we get the PWA operator.

Remark 5: In a similar way, we could develop the same proof for all the other types of interval numbers available in the literature.

Remark 6: Note that if the available information is given in different types of interval numbers, then, we have to adapt them to the same structure. Thus, we have to construct an interval that includes all the other ones. For example, if we have one interval with 2-tuples, another one with triplets and the other one with quadruplets, then, we have to convert all of them to quadruplets. The 2-tuple is constructed as follows: [a_1, a_2] = [a_1, a_1, a_2, a_2] and the triplet in the following way: [a_1, a_2, a_3] = [a_1, a_2, a_2, a_3].

Remark 7: In a similar way, we could develop the same analysis with more complex interval numbers such as quintuplets and sextuplets.

Remark 8: Note that similar analysis could be developed for considering situations when the interval numbers are representing linguistic variables, etc.

APPLICATION OF THE UPWA OPERATOR

The UPWA operator can be applied in a lot of fields because all the previous studies that use the weighted average or the probability can be revised and extended with this new approach. Thus, we can apply it:

i. Statistics: The UPWA is a key instrument to revise the majority of the statistical sciences. For example, we can extend it to probability theory and a lot of other related

Table 1. Matrix with states of nature and alternatives.

	S_1	S_i	S_n	UPWA
A_1	$\tilde{a}_{11}$	$\tilde{a}_{1i}$	$\tilde{a}_{1n}$	T_1
A_h	$\tilde{a}_{h1}$	$\tilde{a}_{hi}$	$\tilde{a}_{hn}$	T_h
A_k	$\tilde{a}_{k1}$	$\tilde{a}_{ki}$	$\tilde{a}_{kn}$	T_k
UPWA	Y_1	Y_i	Y_n	

areas such as descriptive statistics, hypothesis testing and inference statistics.
ii. Fuzzy Set Theory and Soft Computing.
iii. Decision Theory and Operational Research.
iv. Business Administration.
v. Economics and Politics.
vi. Biology and Medicine.
vii. Physics and Chemistry.
viii. Other sciences: Many other applications could be developed in a lot of other sciences such as in psychology, sociology, geography and a wide range of disciplines in engineering.

In uncertain decision theory (or interval decision theory), we see that there are a lot of methodologies for doing so including uncertain multiple criteria decision making, uncertain group decision making, uncertain sequential decision making and uncertain game theory. In general terms, we can distinguish between 3 forms of uncertain decision making environments:

1. Uncertain decision making under certainty: We have information regarding what is going to happen in the future. However, the available information is imprecise and represented with interval numbers.
2. Uncertain decision making under risk: We know the possible outcomes by using interval numbers. However, we do not know which of them is going to occur but we can assess the information with uncertain probabilities.
3. Uncertain decision making under uncertainty: We know the possible outcomes by using FNs but we do not know which of them is going to occur in the future and we do not have any probabilistic information.

Furthermore, we can use subjective and objective probabilities in the analysis. Thus, we can formulate two other decision making processes:

1. Uncertain decision making under subjective risk.
2. Uncertain decision making under objective risk.

This general framework has been extended in different ways. Considering the new developments presented in this paper, we can introduce the use of uncertain decision making problems under subjective risk and under objective risk in the same formulation. With the introduction of the UPWA operator, we can assess these two problems in the same formulation and considering the degree of importance that each concept has in the analysis.

Therefore, with the UPWA operator we can formulate a new decision making approach:

1. Uncertain decision making under subjective risk and objective risk.

a. If β = 1, we get uncertain decision making under objective risk.
b. If β = 0, we get uncertain decision making under subjective risk.

A further interesting issue to consider is the meaning of the set of arguments aggregated. In uncertain decision making problems it is very interesting to consider a set of arguments *a* that depend on a set of states of nature *S* and a set of alternatives *A*. This information can be represented in the following matrix shown in Table 1.

As can be seen in Table 1, we can aggregate the information in different ways. In summary, we can summarize the problem in three types of uncertain decision-making methodologies:

1. Uncertain decision making "ex-ante": Select an action and see its potential results (aggregation of a row).
2. Uncertain decision making "ex-post": Assume that a state of nature occurs and see how we can react (aggregation of a column).
3. Uncertain decision making "ex-ante" and "ex-post": Mix both cases in the same uncertain decision making process.

By mixing these concepts, we could consider a wide range of uncertain decision making (UDM) approaches. Thus, we could get the following models shown in Table 2. Note that these approaches could also be studied with a multi-person analysis or more generally with a multi-aggregation process. Thus, we get:

1. Uncertain multi-person decision making under certainty.
2. Uncertain multi-person decision making under risk.

a. Uncertain multi-person decision making under objective risk.
b. Uncertain multi-person decision making under subjective risk.

3. Uncertain multi-person decision making under uncertainty.

Table 2. Uncertain decision making (UDM) approaches.

	UDM – certainty	**UDM under risk**	**UDM – uncertainty**
UDM “ex-ante”	UDM under certainty “ex-ante”	UDM under risk “ex-ante”	UDM – uncertainty “ex-ante”
UDM “ex-post”	UDM under certainty “ex-post”	UDM under risk “ex-post”	UDM – uncertainty “ex-post”
UDM “ex-ante” and “ex-post”	UDM under certainty “ex-ante” and “ex-post”	UDM under risk “ex-ante” and “ex-post”	UDM – uncertainty “ex-ante” and “ex-post”

Table 3. Expert 1.

	S_1	S_2	S_3	S_4
A_1	[0.2, 0.4]	[0.6, 0.7]	[0.3, 0.4]	[0.6, 0.7]
A_2	0.5	[0.3, 0.5]	[0.5, 0.7]	[0.4,0.5]
A_3	[0.1, 0.2]	[0.8, 0.9]	[0.9, 1]	[0.8, 0.9]

Table 4. Expert 2.

	S_1	S_2	S_3	S_4
A_1	[0.3, 0.4]	[0.7, 0.8]	[0.7, 0.9]	[0.4, 0.6]
A_2	[0.7, 0.8]	[0.3, 0.4]	[0.2, 0.3]	[0.6, 0.7]
A_3	[0.6, 0.7]	[0.5, 0.6]	[0.5, 0.6]	[0.3, 0.5]

Finally, note that there are a lot of other methods and techniques for dealing with uncertain decision making that could be considered in the analysis.

APPLICATION OF THE THEORY OF EXPERTONS

In the following, we focus on an uncertain multi-person decision making problem by using the theory of expertons (Kaufmann, 1988; Kaufmann and Gil-Aluja, 1993). Note that an experton is an extension of the concept of probabilistic set (Hirota, 1981) for uncertain environments than cannot be assessed with exact numbers but it is possible to use interval numbers. Thus, we can deal with the opinion of several experts in the analysis in a more efficient way because we can assess the information showing various details on their information and the general tendency of the opinion of the group. Note that in the literature, we can find a lot of other decision making approaches (Demir and Bostanci, 2010; Gil-Lafuente and Merigó, 2010; Jin et al., 2010; Liu, 2011; Liu and Su, 2010; Merigó and Gil-Lafuente, 2010; Wu et al., 2009; Xu and Hu, 2010; Zhang and Liu, 2010).

Assume a company that operates in Europe is analyzing its general strategy for the next year and they consider three alternatives:

1. A_1: Expand to the Nigerian market.
2. A_2: Expand to the Kenian market.
3. A_3: Expand to the Algerian market.

In order to evaluate these strategies, the company uses the opinion of five experts that usually assesses the company. They consider that the key factor for the determination of the expected benefits is the economic situation for the next year. They have summarized the possible scenarios as follows:

1. S_1: Bad economic situation.
2. S_2: Regular economic situation.
3. S_3: Good economic situation.
4. S_4: Very good economic situation.

Each expert evaluates the expected benefits of the company according to the economic situation for the next year. They give their opinion in the interval [0, 1] being 0 the lowest expected benefits (or highest loses) and 1 the highest ones. As the available information is very uncertain, the experts provide their information with interval numbers represented in the form of 2-tuples. The results are shown in Tables 3, 4, 5, 6 and 7. With this information, we construct the expertons. The results are shown in Table 8.

Note that in order to calculate the results shown in Table 8, we use the following methodology presented in Table 9 for the experton A_1 with S_1. That is, first we calculate the absolute frequencies (the number of experts that gives each result). Next, we calculate the relative frequencies (we divide the absolute frequencies by the total number of experts) and finally, the accumulated relative frequency of the results (we sum from α = 1 the relative frequencies in an accumulated way until α = 0)

Table 5. Expert 3.

	S_1	S_2	S_3	S_4
A_1	[0, 0.2]	[0.6, 0.8]	[0.3, 0.4]	[0.9, 1]
A_2	[0.3, 0.4]	[0.7, 0.9]	[0.7, 0.8]	[0.7, 0.8]
A_3	[0.6, 0.7]	[0.4, 0.6]	[0.4, 0.6]	[0.5, 0.6]

Table 6. Expert 4.

	S_1	S_2	S_3	S_4
A_1	[0.7, 0.8]	[0.7, 0.8]	[0.8, 0.9]	[0.5, 0.6]
A_2	[0.4, 0.5]	[0.2, 0.4]	[0.4, 0.5]	[0.7, 0.8]
A_3	[0.3, 0.6]	[0.6, 0.7]	[0.3, 0.4]	[0.2, 0.3]

Table 7. Expert 5.

	S_1	S_2	S_3	S_4
A_1	[0.3, 0.4]	[0.6, 0.7]	[0.4, 0.5]	[0.8, 0.9]
A_2	[0.5, 0.6]	[0.5, 0.8]	[0.3, 0.4]	[0.4, 0.5]
A_3	[0.3, 0.5]	[0.2, 0.3]	[0.6, 0.7]	[0.6, 0.7]

Table 8. Expertons for each strategy and state of nature.

	S_1			S_2			S_3			S_4		
	0	**1**		**0**	**1**		**0**	**1**		**0**	**1**	
	0.1	0.8	1	0.1	1	1	0.1	1	1	0.1	1	1
	0.2	0.8	1	0.2	1	1	0.2	1	1	0.2	1	1
	0.3	0.6	0.8	0.3	1	1	0.3	1	1	0.3	1	1
	0.4	0.2	0.8	0.4	1	1	0.4	0.6	1	0.4	1	1
A_1	0.5	0.2	0.2	0.5	1	1	0.5	0.4	0.6	0.5	0.8	1
	0.6	0.2	0.2	0.6	1	1	0.6	0.4	0.4	0.6	0.6	1
	0.7	0.2	0.2	0.7	0.4	1	0.7	0.4	0.4	0.7	0.4	0.6
	0.8	0	0.2	0.8	0	0.6	0.8	0.2	0.4	0.8	0.4	0.4
	0.9	0	0	0.9	0	0	0.9	0	0.4	0.9	0.2	0.4
	1	0	0	1	0	0	1	0	0	1	0	0.2
	0.1	1	1	0.1	1	1	0.1	1	1	0.1	1	1
	0.2	1	1	0.2	1	1	0.2	1	1	0.2	1	1
	0.3	1	1	0.3	0.8	1	0.3	0.8	1	0.3	1	1
	0.4	0.8	1	0.4	0.4	1	0.4	0.6	0.8	0.4	1	1
A_2	0.5	0.6	0.8	0.5	0.4	0.6	0.5	0.4	0.6	0.5	0.6	1
	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.2	0.4	0.6	0.6	0.6
	0.7	0.2	0.2	0.7	0.2	0.4	0.7	0.2	0.4	0.7	0.4	0.6
	0.8	0	0.2	0.8	0	0.4	0.8	0	0.2	0.8	0	0.4
	0.9	0	0	0.9	0	0.2	0.9	0	0	0.9	0	0
	1	0	0	1	0	0	1	0	0	1	0	0
A_3	0.1	1	1	0.1	1	1	0.1	1	1	0.1	1	1
	0.2	0.8	1	0.2	1	1	0.2	1	1	0.2	1	1
	0.3	0.8	0.8	0.3	0.8	1	0.3	1	1	0.3	0.8	1
	0.4	0.4	0.8	0.4	0.8	0.8	0.4	0.8	1	0.4	0.6	0.8

Table 8. Cont'd.

0.5	0.4	0.8	0.5	0.6	0.8	0.5	0.6	0.8	0.5	0.6	0.8
0.6	0.4	0.6	0.6	0.4	0.8	0.6	0.4	0.8	0.6	0.4	0.6
0.7	0	0.4	0.7	0.2	0.4	0.7	0.2	0.4	0.7	0.2	0.4
0.8	0	0	0.8	0.2	0.2	0.8	0.2	0.2	0.8	0.2	0.2
0.9	0	0	0.9	0	0.2	0.9	0.2	0.2	0.9	0	0.2
1	0	0	1	0	0	1	0	0.2	1	0	0

Table 9. Experton for A_1 and S_1.

	Absolute frequency				Relative frequency				Experton		
	0	1	0		0	0.2	0		0	1	
	0.1	0	0		0.1	0	0		0.1	0.8	1
	0.2	1	1		0.2	0.2	0.2		0.2	0.8	1
	0.3	2	0		0.3	0.4	0		0.3	0.6	0.8
	0.4	0	3		0.4	0	0.6		0.4	0.2	0.8
A_1	0.5	0	0	⇒	0.5	0	0	⇒	0.5	0.2	0.2
	0.6	0	0		0.6	0	0		0.6	0.2	0.2
	0.7	1	0		0.7	0.2	0		0.7	0.2	0.2
	0.8	0	1		0.8	0	0.2		0.8	0	0.2
	0.9	0	0		0.9	0	0		0.9	0	0
	1	0	0		1	0	0		1	0	0

Table 10. Expected value of the expertons for each strategy and state of nature.

	S_1	S_2	S_3	S_4
A_1	[0.3, 0.44]	[0.64, 0.76]	[0.5, 0.62]	[0.64, 0.76]
A_2	[0.48, 0.56]	[0.4, 0.6]	[0.42, 0.54]	[0.56, 0.66]
A_3	[0.38, 0.54]	[0.5, 0.62]	[0.54, 0.66]	[0.48, 0.6]

Table 11. Aggregated results.

	UA	UPA	UWA	UPWA
A_1	[0.52, 0.645]	[0.564, 0.686]	[0.544, 0.686]	[0.552, 0.686]
A_2	[0.465, 0.59]	[0.478, 0.602]	[0.484, 0.604]	[0.481, 0.603]
A_3	[0.475, 0.605]	[0.492, 0.616]	[0.476, 0.604]	[0.482, 0.608]

(Kaufmann, 1988; Kaufmann and Gil-Aluja, 1993).

Next, we calculate the expected value of the expertons. For doing so, we sum all the levels of membership α excepting the 0 and divide the result by 10. The results are shown in Table 10.

The results obtained in Table 10 can be aggregated in order to obtain a single result that permits us to see the expected benefits by using each alternative according to the experts. Note that in this aggregation process we use several particular cases of the UPWA operator. We consider the UA, the UWA, the UPA and the UPWA operator. We assume that the UWA uses the following weighting vector V = (0.2, 0.2, 0.2, 0.4) with a 60% of importance and the UPA: P = (0.1, 0.2, 0.3, 0.4) with 40% of importance. The results are shown in Table 11.

As we can see, depending on the particular type of UPWA operator used, the results may be different leading to different decisions. In this example, it seems that A_1 is the optimal choice.

Conclusions

We have presented a new approach that unifies uncertain decision making problems under objective risk and subjective risk in the same formulation and considering the degree of importance of each concept in the analysis. For doing so, we have introduced the UPWA operator. It is an aggregation operator that unifies the probability and the weighted average in the same formulation and considering the degree of relevance of each concept in the aggregation. Moreover, it is able to assess uncertain environments by using interval numbers. We have seen some of its main particular cases including the PWA operator, the UA, the UA-PA and the UA-WA operator.

We have studied its applicability and we have seen that it is very broad because all the previous studies that use the probability or the weighted average can be revised and extended with this new approach including statistics, economics and engineering. We have focussed on an application in an uncertain multi-person decision making problem regarding the selection of strategies by using the theory of expertons. Thus, we have been able to assess the opinion of several experts in a more complete way. Moreover, we have seen that with the UPWA operator we can unify decision making problems under objective risk and under subjective risk in the same formulation and considering the degree of importance that each concept has in the analysis.

In future research, we expect to develop further developments by adding more characteristics in the analysis such as the use of generalized aggregation operators and distance measures. We will also consider other sources of information and other applications giving special attention to statistics and decision theory.

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