Extremality in Reissner-Nordström black holes

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Abstract: The Reissner-Nordström (RN) solution describes the most general static black hole of the Einstein-Maxwell system. The solution is characterized by its mass and electric charge and, depending on their relative values, it can possess two, one, or no horizons at all. After surveying the geometric and thermodynamic properties of RN black holes, we focus on the "extremal" case, corresponding to solutions with a single horizon and zero temperature. Near-extremal black holes present peculiar features which make them seemingly incompatible with classical gravity and thermodynamics. We comment on these features and explain how they can be resolved by making use of an effective two-dimensional theory of gravity which provides a quantum description of the black holes at ultra-low temperatures. We briefly comment on how supersymmetry dramatically alters the description, saving the day for the string-theoretical counting of extremal black hole microstates.

I. INTRODUCTION

Black holes are regions of spacetime characterized by the present of an event horizon, namely, a spacetime frontier which causally disconnects from the exterior anything that falls inside. Although the Schwarzschild solution to the Einstein equations is known since 1916 [1], black holes were not taken seriously as possible physical objects till the late 1960s [2, 3]. The experimental evidence for their existence, which started appearing in the 1970s, is by now overwhelming —see [4, 5] for recent outstanding experimental breakthroughs. Despite remarkable progress in the theoretical understanding of black holes —particularly as thermodynamic objects [6]— many fundamental questions remain unsolved.

Schwarzschild black holes are the simplest ones. They are spherically symmetric, static, neutral, non-rotating and fully characterised by their mass M. Reissner-Nordström (RN) black holes [7, 8] generalise them by introducing electric charge Q as a new parameter. This gives rise to two horizons: an outer event horizon and an inner "Cauchy horizon". RN black holes become "extremal" when their charge is —in natural units— equal to their mass. In that situation, the two horizons merge into one and the black hole has zero temperature. In this project we will review and explain the resolution to various puzzles associated to the properties of extremal RN black holes, which involve incompatibilities with classical physics and violations of the 3rd law of thermodynamics.

In section II we introduce the Einstein-Hilbert and Maxwell actions from which the solutions that describe black holes arise. We also present the "No-hair theorem" along with the description of Reissner-Nordström black holes and the analysis of their properties depending on the relative values of their mass and charge, as well as their corresponding Penrose diagrams. In section III we present the laws of black hole thermodynamics along with their classical interpretation and some thermodynamic properties of RN black holes. We focus on the case in which black holes become near-extremal in section IV, analyzing two puzzles associated to their special properties. In section V we comment on how the two-dimensional "Jackiw-Teitelboim gravity" provides a resolution to these puzzles and in VI we explain how supersymmetric black holes have a very different fate. We conclude that classical extremal black holes only exist in the presence of supersymmetry.

II. THE REISSNER-NORDSTRÖM FAMILY

The Einstein-Maxwell system is defined by minimally coupling the Einstein-Hilbert and Maxwell actions as

$$\mathcal{S} = \int \sqrt{|g|} \left(\frac{R}{2k} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \mathrm{d}^4 x \, . \qquad (1)$$

In this expression, $k \equiv 8\pi G$ is the gravitational constant, R the Ricci scalar associated to the spacetime metric and $F_{\mu\nu} \equiv 2\partial_{[\mu} A_{\nu]}$ is the Faraday tensor.

The equations of motion of this theory admit many types of solutions. A particularly relevant one corresponds to the "Kerr–Newman family". This describes the spacetime and electromagnetic field of a rotating charged object with mass M, charge Q and angular momentum J [9, 10]. While black holes can be formed from the collapse of all kinds of matter, the "No-hair theorem" establishes that all stationary black hole solutions of the Einstein–Maxwell system are fully characterised by their mass M, charge Q and angular momentum J [11, 12]. Hence, all information characterizing the matter which collapsed to form the black hole or which falls inside is completely lost for any outside observer.

Throughout this project we will be interested in various features of extremal black holes which are already manifest for the subclass of Kerr-Newmann solutions which have a vanishing angular momentum, $J = 0$. The $J \neq 0$

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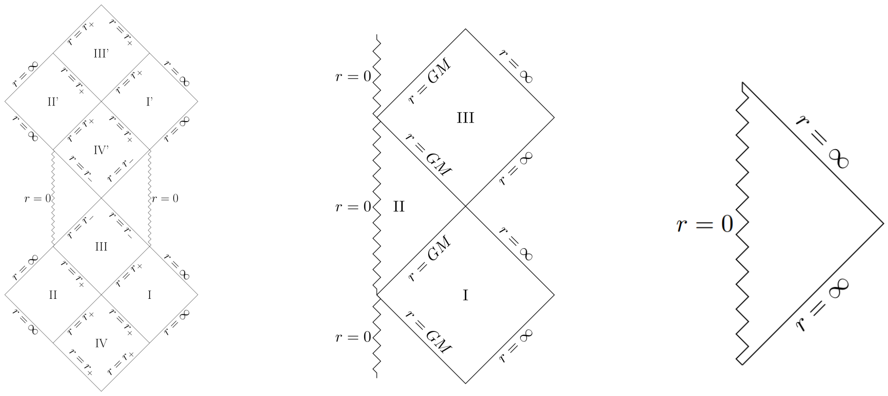

Figure 1: Penrose diagrams of Reissner-Nordström solutions corresponding respectively: to a non-extremal black hole, $M > M^{\rm ext}$, (left), an extremal black hole, $M = M^{\rm ext}$, (middle) and a naked singularity, $M < M^{\rm ext}$, (right). In all cases, time runs upwards, light rays propagate following straight lines at 45 degrees everywhere and zig-zag lines denote curvature singularities.

case will not modify our conclusions qualitatively, so we will stick to the non-rotating case for the sake of simplicity. Setting $J = 0$ in the Kerr-Newmann solution gives rise to the Reissner-Nordström one, which reads [7, 8]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \,, \quad A_\mu = \frac{Q}{r}\delta^t_\mu \,, \tag{2}$$

where

$$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \,. \tag{3}$$

The metric has a curvature singularity at $r = 0$, as well as coordinate singularities whenever $f(r) = 0$. These signal the presence of horizons and, in the general case, they correspond to the following two values of the radial coordinate:

$$r_\pm = G\left[M \pm \sqrt{M^2 - \frac{Q^2}{G}}\right] \,. \tag{4}$$

Observe that $f(r)$ can now be written as $f(r) = (r - r_+)(r - r_-)/r^2$. The region $r = r_+$ is an "event horizon" which causally disconnects the exterior from the interior of the black hole. On the other hand, $r = r_-$ is a "Cauchy horizon", beyond which the predictability of Einstein equations seems to break down.

From (4) it follows that in order for r to be real, the black hole is required to have a mass greater than certain "extremall mass", $M^{\rm ext}$. There exist different situations —see Fig. 1 for Penrose diagrams of the various cases— which can be summarized as follows,

$$M^{\rm ext} \equiv \frac{|Q|}{\sqrt{G}} \,, \qquad \begin{cases} M > M^{\rm ext} \leftrightarrow \text{Non-extremal} \\ M \gtrsim M^{\rm ext} \leftrightarrow \text{Near-extremal} \\ M = M^{\rm ext} \leftrightarrow \text{Extremal} \\ M < M^{\rm ext} \leftrightarrow \text{Naked singularity} \end{cases}$$

Black holes with $M > M^{\rm ext}$ are called "non-extremal". In particular, for $Q = 0$ we recover the Schwarzschild solution [1]. Solutions with mass M slightly bigger than $M^{\rm ext}$ are called "near-extremal" and the ones satisfying $M = M^{\rm ext}$ are "extremal". Extremal black holes satisfy $r_+ = r_-$. In that case, the event and Cauchy horizons are merged into a single one.

In the situation with $M < M^{\rm ext}$ there is no event horizon and the solution describes a "naked singularity". A naked singularity involves a lack of predictability of the theory, since it is not possible to provide sensible initial conditions to any field on a spacetime singularity. This leads to the "weak cosmic censorship conjecture", which says that physical singularities must always be hidden behind event horizons [13]. Looking at the Penrose diagrams shown in Fig. 1, we observe that crossing the Cauchy horizon leads to a region of spacetime whose causal past contains the singularity, so we face an analogous loss-of-predictability issue. This leads to the "strong cosmic censorship conjecture", which roughly asserts that evolution across a Cauchy horizon should not be possible. In other words, the conjecture would imply that Cauchy horizons are in fact unstable and all singularities are either spacelike or null. Violations of both conjectures in particular situations have been reported [14, 15].

III. BLACK HOLE THERMODYNAMICS

Black holes are thermodynamic objects. In particular, they posses an entropy proportional to the surface area of their event horizons, a temperature proportional to their surface gravity and an internal energy equal to their mass. In strict analogy with the laws of thermodynamics, black holes satisfy four laws which encapsulate these aspects [6]. Let us review them.

Zeroth Law: The surface gravity κ is always constant over the event horizon of a stationary black hole [6]. For an observer at infinity, κ can be understood as the acceleration experienced on a body of mass m towards the black hole at the event horizon. If the observer at infinity has a rope tied to the body, the tension force holding it would be $F = m\kappa$. In the case of a static black hole like (2), the surface gravity is given by $\kappa = f'(r_+)/2$. Comparing it to the zeroth law of thermodynamics which states that the temperature T is uniform everywhere in thermal equilibrium, the connection between κ and T is clear. More precisely, the temperature of a black hole is given by $T = \kappa/(2\pi)$ [16].

First Law: The energy conservation principle of a black hole is expressed in terms of small variations of its mass, dM, its horizon area, dA, its angular momentum dJ, and its charge dQ through [6]

$$\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A + \Omega\mathrm{d}J + \Phi\mathrm{d}Q\,. \tag{5}$$

In this remarkable relation, κ is the surface gravity, Ω the angular velocity and Φ the electrostatic potential.

In classical thermodynamics, the first law reflects the conservation of energy of an isolated system: $\delta U = \mathrm{d}Q + \delta W$, where δU is the change in internal energy, dQ is the heat added to the system and δW is the work done by the system. Combining it with the second law we can obtain an expression very similar to equation (5), $\mathrm{d}U = T\mathrm{d}S + \ldots$ where the dots stand for additional possible terms [17]. Identifying the internal energy with the black hole mass, T with the surface gravity and S with the surface area, the analogy becomes complete.

Second Law: The second law of black hole mechanics establishes that the area A of an event horizon does not decrease with time under any physical process [18],

$$\delta A \geq 0\,. \tag{6}$$

This is again remarkably similar to the second law of thermodynamics, which states that the entropy of an isolated physical system either increases or remains constant: $\Delta S \geq 0$. Both concepts are indeed connected, as the entropy of a black hole is given by the Bekenstein-Hawking formula [19, 20]

$$S = \frac{c^3 A}{4G\hbar}\,, \tag{7}$$

where we momentarily reinstated c and $\hbar$. This is often considered as the first formula of quantum gravity, as it combines the three fundamental constants of nature.

Black holes will evaporate over time via Hawking radiation, losing entropy in the process [20]. It may seem that Hawking radiation violates the second law but here is where the "generalized second law" steps in [21]. According to this, the total entropy of a system is the ordinary entropy plus the total entropy of all the black holes of a given system. The total entropy cannot decrease, $\Delta S_{\text{total}} \geq 0$, where $S_{\text{total}} = S_{\text{BH}} + S_{\text{stuff}}$.

Third Law: In thermodynamics, the third law states that as the temperature of a system approaches absolute zero, the entropy of the system is minimized and it is not possible to reach the absolute zero with a number of finite steps [17]. On the other hand, black holes with the minimum possible mass that is compatible with its charge and angular momentum are called "extremal" and they have $\kappa = 0$ while often maintaining a finite area (and therefore entropy). However, the analogy remains in that it is not possible for a black hole to reach $\kappa \to 0$ ($T \to 0$) by a finite number of physical processes. Extremal black holes do not emit Hawking radiation as they have $T = 0$.

A. Thermodynamic properties of Reissner-Nordström black holes

At the event horizon, we can assume $r = r_+$ and $t = 0$. The Reissner-Nordström metric now becomes $\mathrm{d}s^2 = r^2\mathrm{d}\Omega^2$ and the area of the black hole is: $A = \int\sqrt{g}\,\mathrm{d}\theta\mathrm{d}\phi = 4\pi r_+^2$. Thus, using equation (7) the black hole entropy becomes:

$$S = \pi r_+^2 = \pi G\left[M + \sqrt{M^2 - \frac{Q^2}{G}}\right]^2\,, \tag{8}$$

On the other hand, the temperature T and the electrostatic potential Φ read

$$T = \frac{\sqrt{M^2 - \frac{Q^2}{G}}}{2\pi\left[M + \sqrt{M^2 - \frac{Q^2}{G}}\right]^2}\,, \quad \Phi = \frac{Q}{G\left[M + \sqrt{M^2 - \frac{Q^2}{G}}\right]}\,. \tag{9}$$

The First Law shown in (5) can be easily verified.

IV. EXTREMAL BLACK HOLES: TWO PUZZLES

Extremal black holes are interesting due to their peculiar properties. While non-extremal black holes only have a few isometries, near-extremal and extremal black holes develop a new symmetry near the horizon: scale invariance. We have already seen that near-extremal black holes verify $M \gtrsim M^{\text{ext}}$ but they are defined more precisely by the criterion $k_B T \ll 1/\sqrt{A}$.

The geometry around extremal and near-extremal black holes presents a long throat near the horizon. This corresponds to a two-dimensional Anti-de Sitter space along the time and radial directions —see Fig. 2. This can be checked by defining $r' = r_+ + \frac{\lambda}{z}$, where λ is an infinitesimal parameter and z the radial coordinate. In the extremal case, $r_\pm = \sqrt{G}Q$ and expanding $f(r')$ from the equation (2), the AdS_2 metric is obtained. After rescaling the time coordinate as $t = \frac{\sqrt{G}Q}{\lambda}T$, one finds

$$\mathrm{d}s^2 = \frac{\sqrt{G}Q}{z^2}[-\mathrm{d}T^2 + \mathrm{d}z^2] + GQ^2\mathrm{d}\Omega^2\,, \tag{10}$$

which describes a spacetime with geometry $\mathrm{AdS}_2 \times \mathbb{S}^2$.

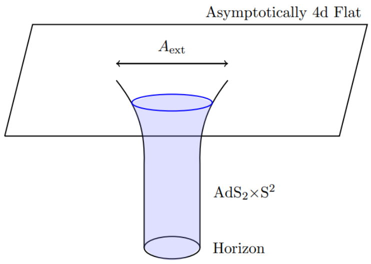

Figure 2: Representation of the Anti-de Sitter geometry in 2 dimensions around a near-extremal black hole (from [22]).

From equations (8) and (9) we can check that when $M = M^{\rm ext}$, the black hole's temperature and entropy become, respectively,

$$T^{\rm ext} = 0\,, \quad S^{\rm ext} = \pi G M^2\,. \tag{11}$$

In order to study black holes as quantum systems, it is necessary to separate the spacetime region of the black hole from its environment. This separation stands out near extremality, in which case the emission of Hawking radiation is suppressed. The $\rm AdS_2$ throat turns out to play a crucial role in such quantum description.

Extremal black holes seem to present two features which are at odds with standard thermodynamics and classical general relativity. These are [22]:

1st puzzle: Extremal black holes have the minimal possible mass M given Q —and J— and have zero temperature. Therefore they do not emit Hawking radiation. If we treat them as quantum systems, they should correspond to ground states. However, as seen in equation (11), their entropy can remain very large even when $T = 0$. This fact is at odds with the third law of classical thermodynamics presented in (III). If real, what does such a great degeneracy of states correspond to?

2nd puzzle: Near-extremal black holes cannot be treated as classical thermodynamic systems for sufficiently small temperatures. The classical description is precise whenever the emission of typical radiation quanta does not change the temperature substantially. This is no longer the case for temperatures lower than [23]

$$T_{\rm breakdown} = \frac{\pi}{G M^{\rm ext} S^{\rm ext}} = \frac{1}{\sqrt{G} Q^3}\,. \tag{12}$$

Additionally, calculations in string theory suggest that an energy gap $E_{\rm gap} \sim T_{\rm breakdown}$ should be present in the spectrum of extremal black holes. However, there is no indication of any such gap in classical general relativity.

V. EXTREMAL BLACK HOLES: A RESOLUTION

These puzzles remained unsolved for several decades. Recently, a new approach which makes a careful use of the gravitational path integral to deal with quantum fluctuations on the AdS throat has provided a resolution [24, 25]. This makes use of a two-dimensional theory known as "Jackiw-Teitelboim" (JT) gravity [26, 27]. This is a theory of gravity coupled to a scalar "dilaton" field which captures the dynamics of fluctuations on the $\mathrm{AdS}_2 \times \mathbb{S}^2$ near-horizon metric of extremal black holes. Roughly, JT gravity captures spherically symmetric fluctuations of the $\rm AdS_2$ factor and the dilaton controls fluctuations of the $\mathbb{S}^2$ area. Additional matter can account for fluctuations with non-trivial angular dependence [22].

Quantum fluctuations of extremal black holes can then be accounted for using JT gravity. The effects arise from a mode which becomes light in the extremal regime of black holes, namely, fluctuations of the throat length. As the temperature is lowered, this effect becomes increasingly relevant. The entropy of JT gravity for near-extremal RN black holes reads [24, 25]

$$S(T) \approx S_{\rm Classical} + S_{\rm Quantum}\,, \tag{13}$$

where each term corresponds to

$$S_{\rm Classical} = \frac{A^{\rm ext}}{4G} + \frac{4\pi^2 T}{T_{\rm breakdown}}\,, \tag{14}$$
$$S_{\rm Quantum} = \log\left(\frac{A^{\rm ext}}{4G}\right) c_{\log} + \frac{3}{2}\log\left(\frac{T}{T_{\rm breakdown}}\right)\,.$$

While $S_{\rm Classical}$ comes from classical gravity, $S_{\rm Quantum}$ includes quantum corrections arising from all possible matter fields. In particular for n_S light scalars, n_V vectors and n_F Dirac Fermions, one has [28]: $c_{\log} = (-n_S - 62 n_V - 11 n_F - 964)/180$. Crucially, as opposed to the first term in $S_{\rm Quantum}$, the second —which is the one arising from JT gravity— is universal and depends on T. When the temperature becomes lower than $T_{\rm breakdown}$, this log-T quantum correction dominates over the classical linear-in-T contribution. When this occurs, the description of the system in terms of a classical near-extremal black hole completely breaks down. This addresses the 1st puzzle and is in agreement with the expectation derived from the first part of the 2nd puzzle. Furthermore, at ultra-low temperatures of order $\sim T_{\rm breakdown} \mathrm{e}^{-A^{\rm ext}/(4G)}$ additional saddle points in the path integral can compete with the black hole solution, making the ground state very complicated. Hence, the gravity description is in fact compatible with a ground state with a small or no degeneracy at all. Indeed, as opposed to the naive classical result which suggested a large degeneracy of states at low temperatures, Fig. 3 shows that the density of states actually approaches zero.

VI. SUPERSYMMETRY AND A NEW PUZZLE?

These results pose a new puzzle. This occurs within string theory —the leading candidate for a theory of quantum gravity and all the fundamental interactions.

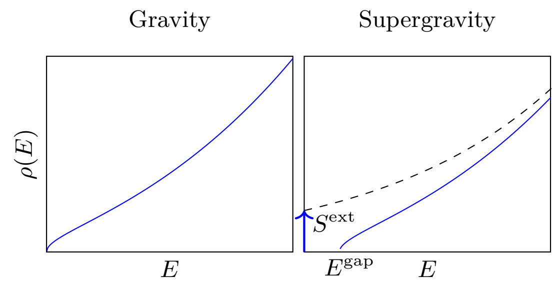

Figure 3: Density of states for a near-extremal black hole in JT gravity and in JT supergravity as a function of the energy.The dashed line represents the classical approximation.

In that context, the semiclassical entropy of certain extremal black holes has been compared with a direct counting of their microstates, finding a remarkable agreement [29]. This is considered to be one of the greatest successes of string theory. But, in view of the above results, how is it possible that the classical entropy formula can be trusted for such black holes? Interestingly, a new key ingredient saves the day: supersymmetry.

Supersymmetry is a (not-found-in-nature-so-far) symmetry which transforms bosons into fermions, and viceversa. The aforementioned extremal string theory black holes are supersymmetric. Through a similar supersymmetric version of JT gravity, "JT supergravity", this gives rise to new fermionic light modes which modify the above "bosonic" quantum corrections [30]. As a consequence, a gap is generated —see Fig. 3— the naive ground state degeneracy is in fact present (protected by supersymmetry) and the third law is indeed violated. Hence, extremal black holes only exist when they are supersymmetric.

VII. CONCLUSIONS

We have explored various aspects of RN black holes. After determining their thermodynamic properties, we observed a violation of the third law of thermodynamics associated to a naive large degeneracy of ground states. We explained how this puzzle and a related one associated to the description of ultra-low temperature systems can be addressed by properly accounting for quantum corrections. This is achieved by using JT gravity, which describes fluctuations on the near-horizon geometry of near-extremal RN black holes. Ultimately, the resolution implies that classical extremal black holes do not exist in standard gravity theories and that the naive degeneracy does not really exist. The story changes in the presence of supersymmetry. In that case, the degeneracy does exist and so do extremal black holes.

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[1] Karl Schwarzschild. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1916:189–196, 1916.
[2] David Finkelstein. *Phys. Rev.*, 110:965–967, 1958.
[3] Remo Ruffini and John A. Wheeler. *Phys. Today*, 24(1):30, 1971.
[4] B. P. Abbott et al. *Phys. Rev. Lett.*, 116(6):061102, 2016.
[5] Kazunori Akiyama et al. *Astrophys. J. Lett.*, 875:L1, 2019.
[6] James M Bardeen, Brandon Carter, and Stephen W Hawking. *Communications in mathematical physics*, 31:161–170, 1973.
[7] H. Reissner. *Annalen Phys.*, 355(9):106–120, 1916.
[8] G. Nordström. *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences*, 20:1238–1245, January 1918.
[9] Roy P. Kerr. *Phys. Rev. Lett.*, 11:237–238, 1963.
[10] E T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence. *J. Math. Phys.*, 6:918–919, 1965.
[11] Werner Israel. *Phys. Rev.*, 164:1776–1779, 1967.
[12] B. Carter. *Phys. Rev. Lett.*, 26:331–333, 1971.
[13] Roger Penrose. *Nuovo Cimento Rivista Serie*, 1:252, 1969.
[14] Tomás Andrade, Roberto Emparan, David Licht, and Raimon Luna. *JHEP*, 04:121, 2019.
[15] Mihalis Dafermos and Jonathan Luk. 10 2017.
[16] S. W. Hawking. *Commun. Math. Phys.*, 43:199–220, 1975. [Erratum: Commun.Math.Phys. 46, 206 (1976)].
[17] Francis Weston Sears and Gerhard L Salinger. *Termodinámica teoría cinética y termodinámica estadística*. Reverté, 1978.
[18] Stephen W Hawking. *Communications in Mathematical Physics*, 25:152–166, 1972.
[19] Jacob D. Bekenstein. *Phys. Rev. D*, 7:2333–2346, 1973.
[20] Stephen W Hawking. *Communications in mathematical physics*, 43(3):199–220, 1975.
[21] Jacob D. Bekenstein. *Phys. Rev. D*, 9:3292–3300, 1974.
[22] Gustavo J Turiaci. *arXiv preprint arXiv:2307.10423*, 2023.
[23] John Preskill, Patricia Schwarz, Alfred Shapere, Sandip Trivedi, and Frank Wilczek. *Modern Physics Letters A*, 6(26):2353–2361, 1991.
[24] Henry Maxfield and Gustavo J. Turiaci. *JHEP*, 01:118, 2021.
[25] Luca V. Iliesiu and Gustavo J. Turiaci. *JHEP*, 05:145, 2021.
[26] C. Teitelboim. *Phys. Lett. B*, 126:41–45, 1983.
[27] R. Jackiw. *Nucl. Phys. B*, 252:343–356, 1985.
[28] Ashoke Sen. *Gen. Rel. Grav.*, 44(5):1207–1266, 2012.
[29] Andrew Strominger and Cumrun Vafa. *Phys. Lett. B*, 379:99–104, 1996.
[30] Matthew Heydeman, Luca V. Iliesiu, Gustavo J. Turiaci, and Wenli Zhao. *J. Phys. A*, 55(1):014004, 2022.