MASTER IN PURE AND APPLIED LOGIC

Master's Thesis

On Languages and a Strictly Positive Fragment of Linear Temporal Logic

Author: Lucas Uzías Acevedo

Supervisor: Dr. Joost Johannes Joosten

Barcelona, September 2024

UNIVERSITAT DE BARCELONA

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

Abstract

This thesis explores various characterizations of regular and star-free languages and introduces a novel syntactic fragment of Linear Temporal Logic (LTL), called Strictly Positive Linear Temporal Logic (SPLTL), inspired by the Reflection Calculus. The opening chapter provides a comprehensive survey of regular languages, characterized by regular expressions, regular grammars, finite automata, and Monadic Second-Order logic over words. We conclude the exposition with a detailed proof of Büchi's Theorem, which bridges automata and logic. The discussion then shifts to star-free languages, emphasizing their representation using LTL. An exhaustive proof of the Completeness Theorem for LTL is also provided.

The principal contribution of this thesis is the definition and analysis of SPLTL, which aims to achieve improved complexity compared to LTL. We establish several foundational results for SPLTL and show its soundness concerning the standard semantic framework of LTL. However, proving the completeness of SPLTL presents difficulties, primarily due to the absence of the disjunction operator in the SPLTL formalization.

Despite these challenges, we think that this thesis introduces valuable insights and results that lay the groundwork for future research. It paves the way for a more in-depth investigation into the completeness of SPLTL and its potential applications.

Acknowledgements

I would like to thank my thesis supervisor, Joost Joosten, for his immense patience and dedication throughout this journey. I truly appreciate his effort to teach me how to write and express myself effectively within the academic ecosystem. Joost is an outstanding teacher, so any flaws in this thesis, if not, simply show that I am not yet an outstanding pupil. Working on that!

I am grateful to Vicent Navarro for his time and support during the initial stages of this thesis. His assistance was crucial in developing the necessary rigor and meticulousness. In addition to my thanks, I probably also owe him a coffee.

I would like to express my appreciation to David Fernández-Duque for his insightful suggestions regarding intuitionistic LTL and completeness. His input undoubtedly enriched my understanding of the subject and showed me just how much more there is to learn.

Contents

Introduction

If asked about the most significant evolutionary advancement that distinguishes humans from other species, we would likely point to *language*. While some might suggest *opposable thumbs*, it is clear that, despite we would need to redesign scissors and gloves, we could more easily manage without them than without language. In this sense, language is not merely an adaptive trait that increases our chances of survival. It is a tool that reflects and enhances our cognitive abilities, such as abstract thinking, problem-solving, and future planning, in addition to serving as a vehicle for thoughts and cultural expression.

Driven by our inherent quest for progress, human beings have achieved remarkable sophistication in our use of languages across their many facets. Since exploring every aspect and refinement would be an endless pursuit, this thesis will focus on providing a glimpse into the realm of formal languages. Interestingly, just as natural language facilitates human communication, formal languages enable interactions with machines. In this sense, although our discussion will be restricted to theoretical aspects, it is important to note that the impact of formal languages is extensive and profound within *Computer Science*, particularly in *Theory of Computation* and *Formal Verification*.

This thesis offers a brief survey of the fascinating world of formal languages and logic: we begin by examining *regular languages*, a fundamental concept in *Automata Theory*. This class of languages is characterized by various representations, including *regular expressions*, *regular grammars*, *finite automata*, and *Monadic Second-Order* (MSO) logic over words. Special attention will be given to *Büchi's Theorem*, *Theorem 1.4.1*, which connects the perspectives of finite automata and MSO logic over words.

In the second chapter, we consider *star-free languages*, a subclass of regular languages. After a concise review of various representations for star-free languages, analogous to our treatment of regular languages, we focus on their characterization through *Linear Temporal Logic* (LTL). Given the importance of this logic, we will delve into it in detail, culminating with a comprehensive proof of the Completeness Theorem of LTL, *Theorem 2.2.24*, concerning the axiomatization given in *Definition 2.2.6*.

As an initial contribution made by this thesis, the first two chapters provide an extensive overview of formal languages in logic and theoretical computer science, shedding light on often underexplored areas in the literature. The main concepts presented are summarized in the following table:

Languages	Regular (1.1.1)	Star-free (2.1.1)
Expressions	Regular (1.1.4)	Star-free (2.1.2)
Grammars	Regular (1.1.8)	(see 2.1, Page 35)
Automata	NFA (1.2.1), DFA (1.2.7)	Counter-free (2.1.4)
Predicate logics	MSO (1.3.4, 1.3.6)	MFO (see 2.1, Page 35)
Validities on words	MSO, $\text{MSO}^{<\omega}$, MSO^{ω} (1.3.18)	(see 2.1, Page 35)
Modal logics	ETL, RLTL (see 2.2.4)	LTL (2.2.1, 2.2.2)

Table 1: Main characterizations of regular and star-free languages studied in the thesis.

It is noteworthy that the elements in the right column are subclasses or fragments of their corresponding components in the left column. Moreover, all concepts listed in the left column equivalently characterize regular languages, while those in the right column do the same for star-free languages. The equivalence between the rows for automata and predicate logics in the left column is given by Büchi's Theorem. Conversely, the equivalence between the rows for predicate and modal logics in the right column is established by Kamp's Theorem, *Theorem 2.2.5*.

Finally, the third chapter introduces a syntactic fragment of LTL, which we have termed Strictly Positive Linear Temporal Logic (SPLTL). This new fragment is inspired by the theory of *strictly positive* fragments of modal logics and the *Reflection Calculus*, seeking to potentially achieve a better complexity compared to LTL. We derive several results for SPLTL, focusing on its soundness with respect to the standard semantic relation for LTL. Furthermore, our research initiates an exploration into proving the completeness of SPLTL.

Besides the comprehensive overview already mentioned, the contributions of this thesis primarily focus on the third chapter and the study of the new system SPLTL. However, the first two chapters also present valuable and detailed proofs usually overlooked in the existing literature. Although not being central to this thesis, the appendix also offers original results that contribute perspective on the topic.

Chapter 1

A Matter of Languages

In this chapter, we define the notions of languages and regular languages and explore diverse characterizations of regular languages. These characterizations will lead us to a variety of logical and computational objects, including regular expressions, context-free grammars, finite automata, and Monadic Second-Order logic over word models. We will conclude this chapter with an in-depth proof of Büchi's Theorem, which establishes the correlation between the automata and logic characterizations of regular languages.

We define a *finite alphabet* Σ as a finite set of symbols, which we call *letters*. A *finite word* over Σ will be a finite string of letters $w = a_0 a_1 \ldots a_{n-1}$, where $a_i \in \Sigma$ for every $i < n$, and we say that $|w| := n$ —using the same notation as cardinality— is the *length* of w. The empty word is usually denoted by ϵ. We will also say that an *infinite word* over Σ is an infinite sequence $w = a_0 a_1 \ldots a_n \ldots$, with $a_i \in \Sigma$ for every $i \in \mathbb{N}$. Infinite words have length ω.

We denote by Σ^* —the Kleene star of Σ— the set of all finite words over the alphabet Σ; and Σ^ω will be the set of all infinite words over Σ. A *language* is a set of finite or infinite words, that is, a subset of either Σ^* or Σ^ω. We do not usually mix finite and infinite words in our languages. Languages of infinite words are also called *ω-languages*. When we state $L \subseteq \Sigma^* \cup \Sigma^\omega$, we mean that L may be a language of finite words or an ω-language without distinction.

1.1 Regular Languages and First Characterizations

We will mainly focus on a particular class of languages, the so-called *regular languages*. Essentially, regular languages are the ones closed under the three following basic operations on languages of finite words:

- For L and M two languages, we denote their *union* by

  $$L \cup M := \{w : w \in L \text{ or } w \in M\}.$$

  For instance, if $L = \{1, 2\}$ and $M = \{\epsilon, 1, 3\}$, then $L \cup M = \{\epsilon, 1, 2, 3\}$.

- The *concatenation* of two languages L and M of finite words, notated by $L \cdot M$ or simply LM, is the language

  $$L \cdot M := \{vw : v \in L \text{ and } w \in M\}.$$

The word vw is the result of concatenating the strings given by v and w. For example, if $L = \{1, 2\}$ and $M = \{\epsilon, 1, 33\}$, then $L \cdot M = \{1, 2, 11, 21, 133, 233\}$ (the first two words are the words in L concatenated with ϵ).

- The *Kleene star* of a language of finite words L is represented by L^* and is the infinite union
$$L^* := \bigcup_{i \geqslant 0} L^i;$$
where $L^0 = \{\epsilon\}$ and $L^i = L^{i-1} \cdot L$ for $i \geqslant 1$. That is, L^* is the language whose words can be formed by a concatenation of any number of words of L. For example, the Kleene star of the language $L = \{1, 2\}$ is the set of all possible finite strings of 1s and 2s, including the empty string.

With these operations in mind, we can define the collection of regular languages:

Definition 1.1.1. *Given a finite alphabet Σ, we inductively define* ***regular languages*** *by:*

- *the empty language $\varnothing$ is regular;*
- *the language $\{a\}$ is regular, for every $a \in \Sigma$;*
- *if L and M are regular languages, then the language $L \cup M$, the language $L \cdot M$, and L^* are also regular;*
- *no other language over Σ is regular.*

Observation 1.1.2. Immediately from the definition, the language $\varnothing^* = \{\epsilon\}$ is regular.

Before introducing formalizations to express, generate, recognize and define regular languages, it is important to note that not all languages are regular. A typical example of a non-regular language, often found in textbooks such as [HMU06; Sip13; Sud05], is $\{a^n b^n : n \geqslant 0\}$, over the alphabet $\{a, b\}$. To prove its non-regularity, we might apply the *Pumping Lemma*, which we will state but omit the proof of, as it requires one of the characterizations we will introduce later on.

Lemma 1.1.3 (Pumping Lemma)**.** *If L is a regular language, then there is some constant $c \geqslant 1$ such that every word $w \in L$ with $|w| \geqslant c$ can be divided into three sub-words $w = xyz$ satisfying:*

- $|xy| \leqslant c$*;*
- $|y| \geqslant 1$*;*
- $xy^n z \in L$ *for every* $n \geqslant 0$*.*

▲

Now we can prove that $L = \{a^n b^n : n \geqslant 0\}$ is not regular by *reductio ad absurdum*. If we assume L to be a regular language, then the premise of the Pumping Lemma holds, so we will have some constant $c \geqslant 1$ satisfying the requirements of the lemma. In particular, we can consider the word $a^c b^c \in L$, which has length $2c > c$. We would have $a^c b^c = xyz$ for some sub-words x, y and z verifying $|xy| \leqslant c$, where $|y| \geqslant 1$, and $xy^n z \in L$ for every $n \geqslant 0$. However, since $|xy| \leqslant c$, the sub-word y must be of the form $y = a \dots a$, with $|y| \geqslant 1$. Therefore, for any $n > 1$ we have that $xy^n z$ contains a different number of letters a and b, specifically, $xy^n z = a^{c+(n-1)\cdot|y|} b^c$, which does not belong to L because $(n-1) \cdot |y| > 0$. This contradiction ensures that L is not a regular language, as we claimed.

1.1.1 Regular Expressions

We can identify regular languages using *regular expressions*, which are defined as follows:

Definition 1.1.4. *Given a finite alphabet Σ, we inductively define* ***regular expressions*** *by:*

- *the constant $\varnothing$;*
- *the constant ϵ;*
- *the constant $\boldsymbol{a}$, for each $a \in \Sigma$;*

If R and S are regular expressions, then

- *$(R|S)$, (RS) and (R^*) are regular expressions.*

It is straightforward to find the characterization of regular languages: the constants $\varnothing$, ϵ and $\mathbf{a}$ express the empty language, the language $\{\epsilon\}$, and the language $\{a\}$, respectively. If R and S are regular expressions that represent the regular languages L and M, respectively, then $(R|S)$ denotes the language $L \cup M$, the notation (RS) expresses the language $L \bullet M$, and (R^*) represents the language L^*. Thus, every regular expression corresponds to a regular language, and we also have that every regular language can be represented by at least one regular expression.

To avoid some unnecessary parentheses in our regular expressions, the Kleene star operation (R^*) is considered to have the highest priority followed by the concatenation (RS), and then the union $(R|S)$. We also omit the outermost parentheses. Additionally, it is common to define other symbols, such as $R^+ := RR^*$, and $R? := R|\epsilon$, for R a regular expression.

> **Example 1.1.5.** We give some regular expressions together with the regular languages they represent, for the finite alphabet $\Sigma = \{0, 1\}$:
>
> $$\begin{aligned}&\mathbf{0}|\mathbf{1}^* \text{ represents the language } \{0, \epsilon, 1, 11, 111, 1111, \dots\};\\ &\mathbf{0}^+\mathbf{1}? \text{ denotes } \{0, 01, 00, 001, 000, 0001, \dots\};\\ &(\mathbf{01})^* \text{ represents } \{\epsilon, 01, 0101, 010101, \dots\};\\ &\mathbf{0}^*\mathbf{1}\mathbf{0}^* \text{ characterizes } \{w \in \Sigma^* : w \text{ contains one single } 1\};\\ &\Sigma = \mathbf{0}|\mathbf{1} \text{ will represent } \{0, 1\};\\ &\Sigma\mathbf{1}\Sigma \text{ represents } \{010, 011, 110, 111\};\\ &\Sigma^*\mathbf{1}\Sigma^* \text{ denotes } \{w \in \Sigma^* : w \text{ contains at least one } 1\}.\end{aligned}$$

For more details and some applications of regular expressions in the *Theory of Computation*, we refer to [HMU06; Sip13].

1.1.2 Regular Grammars

Now we will use *context-free grammars* to generate languages, and we will introduce *regular grammars*, which will give us another characterization of regular languages.

Definition 1.1.6. *A* ***context-free grammar*** *is a quadruple $\mathcal{G} = (V, \Sigma, P, S)$ where V is a finite set of variables (which will be denoted by capital letters) assumed to be disjoint* *from the finite alphabet* Σ*; we call* $S \in V$ *the start symbol, and* P *is a finite set of rules. A* ***rule*** *is an element of* $V \times (V \cup \Sigma)^*$*. The rule* (A, w) *is usually denoted by* $A \to w$*. A rule of the form* $A \to \lambda$*, with* $\lambda = \epsilon \in (V \cup \Sigma)^*$*, is called a lambda rule.*

Intuitively, grammars provide rules that, starting from a given symbol, allow us to construct words, and thus languages, by iteratively applying those rules. Specifically, for a grammar $\mathcal{G} = (V, \Sigma, P, S)$, the process of generating words from $\mathcal{G}$ involves transforming the initial variable S by applying some rules in P repeatedly, until we have no more variables in the resulting string, that is, until we get a word over Σ. We apply a rule $A \to w$ to the variable occurrence A in the string uAv to produce the string uwv, and we denote this procedure by $uAv \Rightarrow uwv$. The prefix u and suffix v are called the *context* in which the variable A occurs. The concept of context-free in grammar refers to the fact that context does not restrict the applicability of a rule. Now, we provide a formal definition of how grammars generate languages:

Definition 1.1.7. *In a given context-free grammar* $\mathcal{G} = (V, \Sigma, P, S)$*, a string* $w \in (\Sigma \cup V)^*$ *is* ***generated*** *from* $v \in (\Sigma \cup V)^*$ *if we can transform* v *into* w *with a finite number of applications of rules of* P*, that is, if we have, for* $w_i \in (\Sigma \cup V)^*$ *and* $i \leqslant n < \omega$*:*

$$v \Rightarrow w_1 \Rightarrow w_2 \Rightarrow \cdots \Rightarrow w_n = w.$$

If this is the case, we denote it by $v \overset{*}{\Rightarrow}_{\mathcal{G}} w$*.*

The ***language*** *generated by a context-free grammar* $\mathcal{G} = (V, \Sigma, P, S)$*, denoted by* $L(\mathcal{G})$*, is given by* $L(\mathcal{G}) := \{w \in \Sigma^* : S \overset{*}{\Rightarrow}_{\mathcal{G}} w\}$*.*

As we mentioned, we are interested in a particular subclass of grammars, the regular grammars, which will provide us with a characterization of regular languages:

Definition 1.1.8. *A* ***regular grammar*** *is a context-free grammar* $\mathcal{G} = (V, \Sigma, P, S)$ *such that each rule in* P *is of one of the following forms, for* $A, B \in V$ *and* $a \in \Sigma$*:*

- $A \to a$*;*
- $A \to aB$*;*
- $A \to \lambda$*.*

Example 1.1.9. Let $\mathcal{G} = (V, \{a, b\}, P, S)$ be a context-free grammar where $V = \{S, A\}$ and P is the set of the rules:

$$S \to aA;$$
$$S \to \lambda;$$
$$A \to bS.$$

We clearly have that $\mathcal{G}$ is a regular grammar, and we see that:

- $\epsilon \in L(\mathcal{G})$, since $S \Rightarrow \lambda$;
- $ab \in L(\mathcal{G})$, because $S \Rightarrow aA \Rightarrow abS \Rightarrow ab$;
- $abab \in L(\mathcal{G})$, considering that $S \overset{*}{\Rightarrow}_{\mathcal{G}} abS \overset{*}{\Rightarrow}_{\mathcal{G}} abab$.

Although we will not give a detailed proof, we claim that we can represent the language $L(\mathcal{G})$ with the regular expression $(\mathbf{ab})^*$.

Notice that we could have built the same (regular) language by using a non-regular grammar, for instance, $\mathcal{G}' = (\{S\}, \{a, b\}, P', S)$ with P' being the set of rules:

$$S \to abS;$$
$$S \to \lambda.$$

For proof of the characterization of regular languages by means of regular grammars, and for more examples of grammars, we refer to [Sud05]. In this thesis, however, we will focus on the equivalent representations using *finite automata* and the logic introduced below.

1.2 Finite Automata

If we have stated that regular expressions represent regular languages, and regular grammars generate them, then we would assert that finite automata allow us to computationally accept or recognize these languages, as demonstrated below. As a note, while we have thus far treated only finite words, we will now start to differentiate between finite and infinite words, with minor distinctions in our definitions.

Definition 1.2.1. *A* ***nondeterministic finite automaton*** *(NFA) $\mathcal{A}$ is a tuple $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ where:*

- *Q is a finite set of states,*
- *Σ is a finite alphabet,*
- *$\delta \subseteq Q \times \Sigma \times Q$ is a transition relation,*
- *$Q_0 \subseteq Q$ is a set of initial states, and*
- *$F \subseteq Q$ is a set of final (or accepting) states.*

The core concept of automata lies in their capability to compute what we refer to as *runs* and *successful runs*. These runs enable automata, when given a word as input, to determine whether to *accept* it or not. In extension, automata can *recognize* languages. As a convention, when we restrict inputs to finite words, we simply refer to the automaton in question as an NFA, and we denote the automaton as *Büchi automaton* when it receives infinite words as input. The primary difference between NFAs and Büchi automata will be the acceptance or success condition:

Definition 1.2.2. *Consider a finite (infinite) word $w = a_0a_1a_2 \cdots \in \Sigma^* \cup \Sigma^\omega$ over a finite alphabet Σ, and $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ an NFA (Büchi automaton). We define a* ***path****, or computation, of $\mathcal{A}$ on w as a finite (infinite) sequence $\langle q_0, q_1, q_2, \dots \rangle$ of states of $\mathcal{A}$ such that $(q_i, a_i, q_{i+1}) \in \delta$ for every value of i. If $q_0 \in Q_0$ we call that path a* ***run****. A run is considered* ***successful*** *if it is either finite —and so w, for $\mathcal{A}$ an NFA— and its last state $q_{|w|}$ belongs to F, or the run is infinite —and so w, for $\mathcal{A}$ a Büchi automaton— and infinitely many of its states belong to F.*

Once we have defined our successful runs, automata are able to accept words and, therefore, recognize languages:

Definition 1.2.3. *A finite (infinite) word w is* ***accepted*** *by an NFA (Büchi automaton) $\mathcal{A}$ if there is a successful run of $\mathcal{A}$ on w. A language of finite words is* ***recognizable*** *by an NFA $\mathcal{A}$, and we denote it by $L(\mathcal{A})$, if it is the set of finite words accepted by $\mathcal{A}$. Analogously, we represent by $L^\omega(\mathcal{A})$ the ω-language recognized by a Büchi automaton $\mathcal{A}$.*

The following proposition gives us a characterization of regular languages through finite automata, as we wanted. We will omit the proof, but readers interested in a comprehensive proof can refer to Section 3.2 of [HMU06], as well as the original presentation by Kleene in [Kle56].

Proposition 1.2.4. *Every language of the form $L(\mathcal{A})$, for $\mathcal{A}$ some NFA, is a regular language. Moreover, every regular language is of the form $L(\mathcal{A})$, for some NFA $\mathcal{A}$.*

▲

Remark. Some authors, such as [HMU06; Sip13], introduce finite automata before defining regular languages, using the characterization provided in the previous proposition as the definition of regular languages.

Although we will focus on regular languages and NFAs, it is worth mentioning that *Proposition 1.2.4* has an analogous result, given in [Büc66], referring to Büchi automata and the so-called ω-*regular languages*, which only contain infinite words:

Definition 1.2.5. *We inductively define* ω-***regular languages*** *by:*

- *Base case: for L a regular language such that $\epsilon \notin L$, the ω-power*

$$L^\omega := \{w_1 w_2 w_3 \cdots : w_i \in L, \text{ for every } i \in \mathbb{N}\},$$

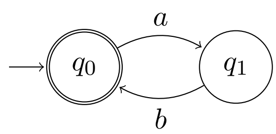
  is an ω-regular language;

- *Inductive cases: if L and M are ω-regular languages and R is a regular language, then $L \cup M$ and $R \cdot L$ are ω-regular languages.*

Example 1.2.6. We consider the finite automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ with:

- $Q = \{q_0, q_1\}$;
- $\Sigma = \{a, b\}$;
- $\delta = \{(q_0, a, q_1), (q_1, b, q_0)\}$;
- $Q_0 = F = \{q_0\}$.

The standard graphical representation for finite automata is to express states as circles and connect them with arrows labeled with a letter of the alphabet. These arrows symbolize transitions, indicating that the triple of the state where the arrow starts, the letter that labels the arrow, and the state where the arrow points, belongs to the transition relation of the automata in question. Also, we will label an arrow with a set of letters to simplify that we would have an arrow —a transition— for each letter of the set. Initial states are represented by an incoming arrow not originating from any state, while final states are marked with a double circle. Then, a graphical representation of the previous automaton $\mathcal{A}$ would be:



From our definitions of δ, Q_0 and F, we easily deduce that the (only) successful runs of $\mathcal{A}$ are

$$\langle q_0 \rangle,\ \langle q_0, q_1, q_0 \rangle,\ \langle q_0, q_1, q_0, q_1, q_0 \rangle,\ \langle q_0, q_1, q_0, q_1, q_0, q_1, q_0 \rangle, \ldots$$

Then, the words accepted by $\mathcal{A}$ are the words of the form

$$\epsilon,\ ab,\ abab,\ ababab, \ldots$$

We observe that the language recognizable by $\mathcal{A}$ could be represented by the regular expression $(\mathbf{ab})^*$, so $L(\mathcal{A})$ is a regular language, as expected.

1.2.1 Some Properties of Finite Automata

As we have seen, languages can be recognized by nondeterministic finite automata. Observe that, in the absence of nondeterminism, constructing a successful run would be merely a matter of computation. This is why, in practice, it will sometimes be preferable to work with so-called *(total) deterministic finite automata* (DFA), defined below, instead of NFA. Fortunately, *Proposition 1.2.13* will allow us to speak almost indistinctly of DFAs and NFAs, by establishing an equivalence between their recognizable languages.

Notation. To improve readability and clarity, we will define, for a transition relation $\delta \subseteq Q \times \Sigma \times Q$, with $Q' \cup \{q\} \subseteq Q$ a set of states, and a letter $a \in \Sigma$:

$$\delta[q, a] := \{p \in Q : (q, a, p) \in \delta\};$$
$$\delta[Q', a] := \bigcup_{q' \in Q'} \delta[q', a].$$

Definition 1.2.7. *Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NFA. This automaton is called a* ***deterministic finite automaton*** *(DFA) if $|Q_0| \leqslant 1$ and $|\delta[q, a]| \leqslant 1$ for every $q \in Q$ and every $a \in \Sigma$. If we have $|Q_0| = 1$ and $|\delta[q, a]| = 1$ for every $q \in Q$ and every $a \in \Sigma$, we say that the DFA is* ***total****.*

Observation 1.2.8. If $|Q_0| = 0$, then the recognizable language of the automaton is empty, so we usually skip this case. Additionally, in the case of total DFAs, note that we could consider the transition relation δ as a function of the form $\delta : Q \times \Sigma \to Q$, rather than just a relation.

By definition, every total DFA is a DFA, so, trivially, for every total DFA there exists a DFA —itself— such that they both recognize the same language. We can prove that the converse is also true:

Lemma 1.2.9. *For every DFA $\mathcal{A}$, there is a total DFA $\mathcal{A}^t$ such that $L(\mathcal{A}) = L(\mathcal{A}^t)$.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \delta, \{q_0\}, F)$ be a DFA. We will pick a fresh state $q^t \notin Q$, and define the following total DFA $\mathcal{A}^t := (Q \cup \{q^t\}, \Sigma, \delta^t, \{q_0\}, F)$, where δ^t is given, for every $q, q' \in Q \cup \{q^t\}$ and every $a \in \Sigma$, by:

$$(q,a,q') \in \delta^t :\Longleftrightarrow \begin{cases} & (q,a,q') \in \delta; \\ (or) & |\delta[q,a]| = 0 \text{ and } q' = q^t; \\ & q = q' = q^t. \end{cases}$$

That is, we extend δ by adding transitions in such a way that for every $q \in Q$ and every $a \in \Sigma$ we achieve $|\delta^t[q,a]| = 1$, where these new transitions will go to the new state q^t. We also add transitions from q^t to q^t to ensure that $|\delta^t[q^t,a]| = 1$ for every $a \in \Sigma$. Then, we observe that $\mathcal{A}^t$ is a total DFA, and we can easily prove that we have $L(\mathcal{A}^t) = L(\mathcal{A})$:

Suppose that there is a successful run of $\mathcal{A}$ on a finite word w, meaning $w \in L(\mathcal{A})$. Since $\delta \subseteq \delta^t$, the same run will also be successful for $\mathcal{A}^t$ on w. Therefore, we conclude $L(\mathcal{A}) \subseteq L(\mathcal{A}^t)$.

On the other hand, if $\langle q_0, \ldots, q_n \rangle$ is a successful run of $\mathcal{A}^t$ on $w = a_0 \ldots a_{n-1}$, we can check that, again, the same run will be a successful run of $\mathcal{A}$ on w. Suppose that the sequence is not a successful run of $\mathcal{A}$ on w: since $\mathcal{A}$ and $\mathcal{A}^t$ have the same initial and accepting states, our supposition implies that there is some $i < n$ such that $(q_i, a_i, q_{i+1}) \in \delta^t$ but $(q_i, a_i, q_{i+1}) \notin \delta$. Note that, by the definition of δ^t, we necessarily have $q_{i+1} = q^t$. If $i+1 = n$ we get a contradiction because $q^t \notin F$ and we assumed $\langle q_0, \ldots, q_n \rangle$ to be a successful run of $\mathcal{A}^t$ on w. And if $i+1 < n$ we know we will have $(q^t, a_{i+1}, q_{i+2}) \in \delta^t$, but this, again, is only possible if $q_{i+2} = q^t$. Then, inductively, we deduce $q_n = q^t$, which gives us the same contradiction as before. This proves, by *reductio ad absurdum*, that $\langle q_0, \ldots, q_n \rangle$ is a successful run of $\mathcal{A}$ on w, and so we get the other inclusion needed to conclude $L(\mathcal{A}^t) = L(\mathcal{A})$. ▲

Observation 1.2.10. The idea behind the construction of $\mathcal{A}^t$, in the previous proof, is that since q^t is not a final state and there are no outgoing transitions from it —all of its transitions loop back to itself—, once we transition to q^t we inevitably enter a loop with no option to accept the word in question. Then, our additions to $\mathcal{A}$ to transform it into the total DFA $\mathcal{A}^t$ do not cause it to accept any word not previously recognized by $\mathcal{A}$.

Analogously, we know that every (total) DFA is an NFA, so for every (total) DFA there is an NFA such that both automata recognize the same language. The converse will also be true, as we prove in *Proposition 1.2.13*. However, before that it will be useful to characterize recognizable languages by extending the transition relations from just letters to finite words, as is done similarly in 2.2 and 2.3 of [HMU06]:

Definition 1.2.11. *Given an NFA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$, we inductively define the relation $\delta^* \subseteq Q \times \Sigma^* \times Q$ by:*

- $(q, \epsilon, q) \in \delta^*$, *for each* $q \in Q$;
- *for* $q, q' \in Q$ *and* $a_j \in \Sigma$ *for every* $j < \omega$,

$$(q, a_0 a_1 \ldots a_i a_{i+1}, q') \in \delta^* :\Longleftrightarrow \text{ there is } q'' \in Q \text{ such that } (q, a_0 a_1 \ldots a_i, q'') \in \delta^* \text{ and } (q'', a_{i+1}, q') \in \delta.$$

We also use the notations, for $Q' \cup \{q\} \subseteq Q$ and $w \in \Sigma^*$:

$$\delta^*[q,w] := \{p \in Q : (q,w,p) \in \delta^*\};$$
$$\delta^*[Q',w] := \bigcup_{q' \in Q'} \delta^*[q',w].$$

Lemma 1.2.12. *Given an NFA* $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$, *we have:*

$$L(\mathcal{A}) = \{w \in \Sigma^* : (q_0, w, q_f) \in \delta^* \text{ for some } q_0 \in Q_0 \text{ and some } q_f \in F\} = \\ = \{w \in \Sigma^* : \delta^*[Q_0, w] \cap F \neq \varnothing\}.$$

Proof. Immediate from breaking down the definition of δ^* since, for $q_0 \in Q_0$ and $q_n \in F$, with $w = a_0 \dots a_n \in \Sigma^*$, the following expressions are equivalent:

- $(q_0, w, q_n) \in \delta^*$;
- $(q_0, a_0 \dots a_{n-1}, q_{n-1}) \in \delta^*$ and $(q_{n-1}, a_n, q_n) \in \delta$, for some $q_{n-1} \in Q$;
- $(q_0, a_0 \dots a_{n-2}, q_{n-2}) \in \delta^*$ and $(q_{n-2}, a_{n-1}, q_{n-1}), (q_{n-1}, a_n, q_n) \in \delta$, for some $q_{n-2} \in Q$;
- $\dots$

Then, inductively, we see that we will have $(q_0, w, q_n) \in \delta^*$ if and only if there are some $q_0, q_1, \dots q_{n-1}, q_n \in Q$ such that $(q_i, a_i, q_{i+1}) \in \delta$, for $i < n$. And this, considering that $q_0 \in Q_0$ and $q_n \in F$, is clearly equivalent to the existence of a successful run of $\mathcal{A}$ on w.

For the case of the empty word, we have $(q_0, \epsilon, q_f) \in \delta^*$ exclusively if $q_0 = q_f$. Therefore, if we require $q_0 \in Q_0$ and $q_f \in F$, we need the existence of some state $(q_0 = q_f =) q_\epsilon \in Q_0 \cap F$. On the other hand, there is a successful run of $\mathcal{A}$ on ϵ if and only if we can start a run already in a final state, that is, if there is some state $q_\epsilon \in Q$ such that $q_\epsilon \in Q_0 \cap F$. ▲

Now we are able to prove the following proposition, which ensures that NFA and DFA are equally expressive. This will mean that we can work with one formalization or the other indistinctly.

Proposition 1.2.13. *For every NFA* $\mathcal{A}_N$, *there is a DFA* $\mathcal{A}_D$ *such that* $L(\mathcal{A}_N) = L(\mathcal{A}_D)$.

Proof. We use the so-called *subset construction* or *powerset construction*. Let $\mathcal{A}_N = (Q, \Sigma, \delta, Q_0, F)$ be an NFA. We will consider the following DFA:

$$\mathcal{A}_D = (\mathcal{P}(Q), \Sigma, \delta_D, \{Q_0\}, F_D),$$

where, for $Q', Q'' \in \mathcal{P}(Q)$ and $a \in \Sigma$:

$$(Q', a, Q'') \in \delta_D :\Longleftrightarrow Q'' = \delta[Q', a],$$

and we take

$$F_D := \{Q' \subseteq Q : Q' \cap F \neq \varnothing\}.$$

It is clear that $\mathcal{A}_D$ is (an NFA and) a DFA, in fact, it is a total DFA, since $|\{Q_0\}| = 1$ and we have, for every $Q' \in \mathcal{P}(Q)$ and every $a \in \Sigma$:

$$|\delta_D[Q', a]| = |\{Q'' \in \mathcal{P}(Q) : (Q', a, Q'') \in \delta_D\}| = |\{\delta[Q', a]\}| = 1.$$

We want to prove $L(\mathcal{A}_N) = L(\mathcal{A}_D)$. Thanks to *Lemma 1.2.12*, for $w \in \Sigma^*$ any finite word we have:

$$w \in L(\mathcal{A}_N) \iff \delta^*[Q_0, w] \cap F \neq \varnothing;$$
$$w \in L(\mathcal{A}_D) \iff \delta_D^*[Q_0, w] \cap F_D \neq \varnothing.$$

Then, it will be enough to prove the double implication:

$$\delta^*[Q_0, w] \cap F \neq \varnothing \Longleftrightarrow \delta_D^*[Q_0, w] \cap F_D \neq \varnothing.$$

We will show something slightly stronger:

$$\{\delta^*[Q_0, w]\} = \delta_D^*[Q_0, w]. \tag{$*$}$$

First, we observe that both sides of ($*$) are a singleton whose element is a subset of Q: on the left side we have the singleton of $\delta^*[Q_0, w] \subseteq Q$, while on the right side, we have set of states or elements of $\mathcal{P}(Q)$. Since $\mathcal{A}_D$ is a total DFA, we deduce that $\delta_D^*[Q_0, w]$ is indeed a singleton. Now we prove the equality ($*$) by induction on $n = |w|$:

For the base case $w = \epsilon$, we have $\delta_D^*[Q_0, \epsilon] = \{Q_0\} = \{\delta^*[Q_0, \epsilon]\}$, since for every $q_0 \in Q_0$ we have $\delta^*[q_0, \epsilon] = \{q_0\}$, so $\delta^*[Q_0, \epsilon] = Q_0$.

For the inductive case, we consider the finite words $w = a_0a_1 \ldots a_{n-1}a_n$, and $w' = a_0a_1 \ldots a_{n-1}$; by our Induction Hypothesis, we can assume we have, for some $r_i \in Q$ with $i \leqslant k$

$$\delta_D^*[Q_0, w'] = \{\delta^*[Q_0, w']\} = \{\{r_0, \ldots, r_k\}\},$$

and we need to show

$$\delta_D^*[Q_0, w] = \{\delta^*[Q_0, w]\}.$$

From the definition of δ^* and our assumption, we have that $(q_0, w', r_i) \in \delta^*$ for $i \leqslant k$, and that no other state $r \neq r_i$, for all $i \leqslant k$, verifies $(q_0, w', r) \in \delta^*$. Then, we can conclude the equality

$$\delta^*[Q_0, w] = \bigcup_{q_0 \in Q_0} \delta^*[q_0, w] = \bigcup_{i \leqslant k} \delta[r_i, a_n]$$

And, similarly, we also have:

$$\delta_D^*[Q_0, w] = \{\delta_D[\{r_0, \ldots, r_k\}, a_n]\}.$$

From the definition of δ_D, we know

$$\delta_D[\{r_0, \ldots, r_k\}, a_n] = \bigcup_{i \leqslant k} \delta[r_i, a_n].$$

Therefore, we have:

$$\delta_D^*[Q_0, w] = \{\delta_D[\{r_0, \ldots, r_k\}, a_n]\} = \left\{\bigcup_{i \leqslant k} \delta[r_i, a_n]\right\} = \{\delta^*[Q_0, w]\}.$$

This shows, by induction on $n = |w|$, that we have $\delta_D^*[Q_0, w] = \{\delta^*[Q_0, w]\}$.

To conclude our proof, we only need to point out that, by the definition of F_D, we have:

$$\delta^*[Q_0, w] \cap F \neq \varnothing \Longleftrightarrow \{\delta^*[Q_0, w]\} \cap F_D = \delta_D^*[Q_0, w] \cap F_D \neq \varnothing.$$

▲

Remark. If an NFA has n states, the resulting DFA from the subset construction, which recognizes the same language, has, in principle, 2^n states. However, in practice, not all of these states are necessarily reachable via the transition relation, so we can drop these non-reachable states from the automaton without affecting its functionality. Even so, NFAs are generally more efficient regarding the number of states and are well-suited for theoretical analysis, whereas DFAs are easier to implement. For an example illustrating this, refer to *Example 2.13* in [HMU06], where an NFA with $n+1$ states is presented, for which every DFA recognizing the same language necessarily has at least 2^n states.

Now we will present some interesting closure properties of recognizable languages, which are closely related to the logical approach we will introduce later. The proofs for these closure properties essentially involve constructing a finite automaton that meets particular requirements.

We observe, however, an alternative method to prove some of these results, by simply applying *Proposition 1.2.4*, which established a correlation between NFAs and regular languages. This method would take advantage of the fact that certain closure properties of regular languages follow directly from their definition. However, since we will apply these lemmas in the context of automata when proving Büchi's Theorem, *Theorem 1.4.1*, we found it preferable to prove them by presenting the corresponding automata.

Lemma 1.2.14 (Closure Under Complement)**.** *Given a total DFA $\mathcal{A}$ over the finite alphabet Σ, there is some total DFA $\overline{\mathcal{A}}$ such that $L(\overline{\mathcal{A}}) = \overline{L(\mathcal{A})} = \Sigma^* \backslash L(\mathcal{A})$.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \delta, \{q_0\}, F)$ be a total DFA. We can show that the also total DFA

$$\overline{\mathcal{A}} = (Q, \Sigma, \delta, \{q_0\}, Q \backslash F),$$

will satisfy $L(\overline{\mathcal{A}}) = \overline{L(\mathcal{A})}$:

The key point of our argument is that, since $\mathcal{A}$ and $\overline{\mathcal{A}}$ are total, that is, we have $|\delta[q, a]| = 1$ for every pair $(q, a) \in Q \times \Sigma$, then for every finite word $w = a_0 a_1 \dots a_{n-1}$, we always can associate to it a unique run $\langle q_0, \dots, q_n \rangle$, which we will denote by $\langle q_0, \dots, q_n \rangle_w$. Therefore, we see that:

$$w \in L(\overline{\mathcal{A}}) \iff \langle q_0, \dots, q_n \rangle_w \text{ verifies } q_n \in Q \backslash F \iff$$
$$\iff \langle q_0, \dots, q_n \rangle_w \text{ verifies } q_n \notin F \iff w \notin L(\mathcal{A}).$$

So, $\overline{\mathcal{A}}$ will accept exclusively the non-accepted words of $\mathcal{A}$. That is, we get $L(\overline{\mathcal{A}}) = \overline{L(\mathcal{A})}$, as intended. ▲

Observation 1.2.15. To conclude the closure under complement, we needed the automata to be total DFAs, as this ensures the existence of a unique run for each word, regardless of whether that word is accepted or not. In general, the automaton presented in our proof does not work for non-total DFAs. However, thanks to *Proposition 1.2.13* and *Lemma 1.2.9*, we know that every NFA has a corresponding total DFA that recognizes the same language. Therefore, we can extend the previous lemma to the class of NFAs, although the appropriate automaton accepting the complement language will be more complex than the one defined above.

In the following definition, we present the *product automaton*, which will be useful in the next proofs:

Definition 1.2.16. *Let $\mathcal{A}_i = (Q_i, \Sigma, \delta_i, Q_{0i}, F_i)$ be NFAs, for $i \in \{1, 2\}$. The* ***product automaton*** *of $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined as:*

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := (Q_1 \times Q_2, \Sigma, \delta_\otimes, Q_{01} \times Q_{02}, F_1 \times F_2),$$

where $\delta_\otimes \subseteq (Q_1 \times Q_2) \times \Sigma \times (Q_1 \times Q_2)$ *is given for every* $q_i, q'_i \in Q_i$ *and every* $a \in \Sigma$*, by:*

$$((q_1, q_2), a, (q'_1, q'_2)) \in \delta_\otimes \;:\Longleftrightarrow\; (q_1, a, q'_1) \in \delta_1 \textit{ and } (q_2, a, q'_2) \in \delta_2. \qquad (\otimes)$$

Lemma 1.2.17 (Closure Under Intersection)**.** *If* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are NFAs over the same finite alphabet, then there is some NFA* $\mathcal{A}$ *such that* $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$*.*

Proof. Let us consider $\mathcal{A}_i = (Q_i, \Sigma, \delta_i, Q_{0i}, F_i)$ be NFAs, for $i \in \{1, 2\}$. It is straightforward to check that the product automaton $\mathcal{A}_1 \otimes \mathcal{A}_2$ is an NFA such that $L(\mathcal{A}_1 \otimes \mathcal{A}_2) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$. For every word $w = a_0 a_1 \ldots a_{n-1}$ we have that the following statements are equivalent:

- $w \in L(\mathcal{A}_1 \otimes \mathcal{A}_2)$;
- $\langle (q_1^0, q_2^0), \ldots, (q_1^n, q_2^n) \rangle$ is a successful run of $\mathcal{A}_1 \otimes \mathcal{A}_2$ on w, for some $q_i^j \in Q_i$ with $j \leqslant n$ and $i \in \{1, 2\}$;
- $q_i^0 \in Q_{0i}$ and $q_i^n \in F_i$, and $((q_1^k, q_2^k), a_k, (q_1^{k+1}, q_2^{k+1})) \in \delta_\otimes$ for all $k < n$, for some $q_i^j \in Q_i$ with $j \leqslant n$ and $i \in \{1, 2\}$;
- $q_i^0 \in Q_{0i}$ and $q_i^n \in F_i$, and $(q_i^k, a_k, q_i^{k+1}) \in \delta_i$ for all $k < n$, for some $q_i^j \in Q_i$ with $j \leqslant n$ and $i \in \{1, 2\}$;
- $\langle q_i^0, \ldots, q_i^n \rangle$ is a successful run of $\mathcal{A}_i$ on w, for both $i = 1$ and $i = 2$;
- $w \in L(\mathcal{A}_1)$ and $w \in L(\mathcal{A}_2)$.

Then, as required, we conclude for every word w:

$$w \in L(\mathcal{A}_1 \otimes \mathcal{A}_2) \Leftrightarrow w \in L(\mathcal{A}_1) \cap L(\mathcal{A}_2).$$

▲

To complete these closure properties over the Boolean operations, we also present the closure of recognizable languages under the union:

Lemma 1.2.18 (Closure Under Union)**.** *If* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are NFAs over the same finite alphabet, then there is some NFA* $\mathcal{A}$ *such that* $L(\mathcal{A}) = L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$*.*

Proof. Consider the NFAs $\mathcal{A}_i = (Q_i, \Sigma, \delta_i, Q_{0i}, F_i)$, for $i \in \{1, 2\}$. We will determine not only one but two automata recognizing the language $L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$. The first one is based on the application of the two previous closure properties, *Lemma 1.2.14* and *Lemma 1.2.17*, and the instance of a De Morgan law:

$$L(\overline{\overline{\mathcal{A}'_1} \otimes \overline{\mathcal{A}'_2}}) = \overline{\overline{L(\mathcal{A}'_1)} \cap \overline{L(\mathcal{A}'_2)}} = L(\mathcal{A}'_1) \cup L(\mathcal{A}'_2);$$

where $\mathcal{A}'_i$ are total DFAs such that $L(\mathcal{A}'_i) = L(\mathcal{A}_i)$ for $i \in \{1, 2\}$. We know that these $\mathcal{A}'_i$ total DFAs exist by *Proposition 1.2.13* and *Lemma 1.2.9*, as we already mentioned in *Observation 1.2.15*. Then, we have proved our lemma, since the automaton $\overline{\overline{\mathcal{A}'_1} \otimes \overline{\mathcal{A}'_2}}$ satisfies:

$$L(\overline{\overline{\mathcal{A}'_1} \otimes \overline{\mathcal{A}'_2}}) = L(\mathcal{A}_1) \cup L(\mathcal{A}_2).$$

Notice that if the given $\mathcal{A}_i$ were already total DFAs, then the automaton we have built would be:

$$\mathcal{A} = (Q_1 \times Q_2, \Sigma, \delta_\otimes, Q_{01} \times Q_{02}, F),$$

with $\delta_\otimes$ defined as in ($\otimes$), and $F := (F_1 \times Q_2) \cup (Q_1 \times F_2)$, that is, for $q_i \in Q_i$, we have

$$(q_1, q_2) \in F \iff q_1 \in F_1 \text{ or } q_2 \in F_2.$$

The second automaton recognizing $L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$ that we will present is simpler: we just need Q_1 and Q_2 to be disjoint, which we can assume without loss of generality, and we build the automaton

$$\mathcal{A} := (Q_1 \cup Q_2, \Sigma, \delta_1 \cup \delta_2, Q_{01} \cup Q_{02}, F_1 \cup F_2).$$

It is immediate to check that if some word is accepted by $\mathcal{A}_1$ or by $\mathcal{A}_2$, then it is also accepted by $\mathcal{A}$. Conversely, suppose that $\langle q_0, \ldots, q_n \rangle$ is a successful run of $\mathcal{A}$ on some finite word $w = a_0 \ldots a_{n-1}$. Our assumption that Q_1 and Q_2 are disjoint together with the condition $(q_i, a_i, q_{i+1}) \in \delta_1 \cup \delta_2$ for $i < n$, imply that we have for every $i < n$:

$$\text{either } (q_i, a_i, q_{i+1}) \in \delta_1 \text{ or } (q_i, a_i, q_{i+1}) \in \delta_2.$$

Then, we conclude that q_i are all in Q_1 or in Q_2, respectively. This gives us that the same sequence $\langle q_0, \ldots, q_n \rangle$ will constitute a successful run of either $\mathcal{A}_1$ or $\mathcal{A}_2$ on w, as desired. ▲

Now we will prove two closure properties of recognizable languages over Cartesian product alphabets. Even though their proofs are almost immediate, it is worth studying the following two lemmas in this level of detail, as they will be useful in the proof of Büchi's Theorem.

Lemma 1.2.19 (Closure Under Projection)**.** *Given an NFA $\mathcal{A}'$ over the finite alphabet $\Sigma \times \Sigma'$, there is some NFA $\mathcal{A}$ over Σ such that*

$$L(\mathcal{A}) = \{a_0 \ldots a_{n-1} \in \Sigma^* : (a_0, b_0) \ldots (a_{n-1}, b_{n-1}) \in L(\mathcal{A}'), \text{ for some } b_0, \ldots, b_{n-1} \in \Sigma'\}.$$

Proof. Let $\mathcal{A}' = (Q, \Sigma \times \Sigma', \delta', Q_0, F)$ be an NFA over $\Sigma \times \Sigma'$. We will consider the NFA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ with, for $q, p \in Q$ and $a \in \Sigma$:

$$(q, a, p) \in \delta \;:\Longleftrightarrow\; (q, (a, b), p) \in \delta', \text{ for some } b \in \Sigma'.$$

From this definition, the following two statements are equivalent:

- $\langle q_0, \ldots, q_n \rangle$ is a successful run of $\mathcal{A}$ on $a_0 \ldots a_{n-1} \in \Sigma^*$;
- $\langle q_0, \ldots, q_n \rangle$ is a successful run of $\mathcal{A}'$ on $(a_0, b_0) \ldots (a_{n-1}, b_{n-1}) \in (\Sigma \times \Sigma')^*$, for $b_i \in \Sigma'$ such that $(q_i, (a_i, b_i), q_{i+1}) \in \delta'$ for all $i < n$.

This ensures what we were looking for, that

$$a_0 \ldots a_{n-1} \in L(\mathcal{A}) \iff (a_0, b_0) \ldots (a_{n-1}, b_{n-1}) \in L(\mathcal{A}'), \text{ for some } b_0, \ldots, b_{n-1} \in \Sigma'.$$

▲

Lemma 1.2.20 (Closure Under Padding)**.** *Given an NFA $\mathcal{A}$ over the finite alphabet Σ, and given a finite alphabet Σ', there is some NFA $\mathcal{A}'$ over $\Sigma \times \Sigma'$ such that*

$$L(\mathcal{A}') = \{(a_0, b_0) \dots (a_{n-1}, b_{n-1}) \in (\Sigma \times \Sigma')^* : a_0 \dots a_{n-1} \in L(\mathcal{A})\}.$$

Proof. Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NFA over Σ, and let Σ' be some finite alphabet. We consider the NFA $\mathcal{A}' = (Q, \Sigma \times \Sigma', \delta', Q_0, F)$, where we define δ', for $q, p \in Q$ and $(a, b) \in \Sigma \times \Sigma'$, by:

$$(q, (a, b), p) \in \delta' :\Longleftrightarrow (q, a, p) \in \delta.$$

This definition directly gives us that the following statements are equivalent, for every $a_0, \dots, a_{n-1} \in \Sigma$ and every $b_0, \dots, b_{n-1} \in \Sigma'$:

- $(a_0, b_0) \dots (a_{n-1}, b_{n-1}) \in L(\mathcal{A}')$;
- there is a successful run $\langle q_0, \dots, q_n \rangle$ of $\mathcal{A}'$ on $(a_0, b_0) \dots (a_{n-1}, b_{n-1}) \in (\Sigma \times \Sigma')^*$;
- there is a successful run (the same as before) $\langle q_0, \dots, q_n \rangle$ of $\mathcal{A}$ for $a_0 \dots a_{n-1} \in \Sigma^*$;
- $a_0 \dots a_{n-1} \in L(\mathcal{A})$.

Then, we obtain the characterization of $L(\mathcal{A}')$ we were looking for. ▲

In summary, we have proved various closure properties of the class of recognizable languages of finite words. We will not enter into details, but we also have closure properties for the recognizable languages of infinite words, that is, for the ω-regular languages. The proofs of the closures under union, intersection, projection, and padding are similar to their finite analogs. The closure under complement needs more work, we refer to [Büc66; McN66] for two different proofs.

All these closure properties, particularly those concerning Boolean operations, guide us towards the logical approach we will introduce in the next section.

1.3 Words and Logic

Regular languages are not only applicable from an automata and computational perspective but can also be defined in logical terms, employing logical formulas. To do that, we will first formalize, roughly speaking, what it means to be a word, and then build formulas over that formalization, in the framework of *Monadic Second-Order Logic.*

Note that over a finite alphabet, if we want to determine whether two words are equal, we could first check if they have the same number of letters. If they do not, then the words are different. If they have the same length, or if they are infinite, we would need to verify whether the letters at each position in the words are identical. In this sense, the length and the identification of letters at each position seem to be the key parameters for formalizing words.

Taking this into account, for a given finite alphabet Σ, we could formalize a finite or infinite word w over Σ by some model $(\mathrm{dom}(w), \{Q_a^w\}_{a \in \Sigma})$. The domain $\mathrm{dom}(w)$ would be the set of (letter) positions of w, so $|\mathrm{dom}(w)| = |w|$; and each Q_a^w would be a unary predicate capturing which positions of w host the letter $a \in \Sigma$. Note that we are also interested in incorporating a model for the empty word, therefore, we will allow models with empty domains.

Although the unary predicates Q_a^w will be useful by themselves, we will need to add some binary ones, besides the always-assumed equality, to be able to state properties about words and, in some sense, *do logic* with them. In summary, we will formalize words by the *word models* defined below:

Definition 1.3.1. *Let Σ be a finite alphabet, and let $w = a_0a_1a_2\ldots$ be a word over Σ. We define a **word model** for w as $\underline{w} = (\mathrm{dom}(w), S^w, <^w, (Q_a^w)_{a\in\Sigma})$, where:*

- $\mathrm{dom}(w) := \begin{cases} \{0,\ldots,|w|-1\} & \textit{if } w \textit{ is finite,} \\ \mathbb{N} & \textit{if } |w| = \omega; \end{cases}$
- *S^w is the successor relation on $\mathrm{dom}(w)$, that is, $S^w = \{(i, i+1) : 0 \leqslant i < |w|-1\}$;*
- *$<^w$ is the natural less-than order on $\mathrm{dom}(w)$, in symbols, $<^w := <_{\upharpoonright \mathrm{dom}(w)}$; and*
- *Q_a^w, for each $a \in \Sigma$, is the unary predicate $Q_a^w := \{i \in \mathrm{dom}(w) : a_i = a\}$.*

If $w = \epsilon$, we have $|w| = 0$, and so we consider $\mathrm{dom}(\epsilon) = \varnothing$.

With these word models in mind, we will now build Monadic Second-Order (MSO) formulas. We will also show how those formulas, interpreted over our words, can define languages.

Firstly, we define a supply set of first-order variables $Var^0 := \{x_i : i \in \mathbb{N}\}$, and a set of second-order variables $Var^1 := \{X_i : i \in \mathbb{N}\}$. Semantically, we will understand first-order variables to represent individual positions within a given word, and second-order variables will denote sets of positions.

Definition 1.3.2. *Given a finite alphabet Σ, the set of (MSO-)**atomic formulas**, which we will notate by $\mathtt{At}_{\mathrm{S}(\Sigma)}$, is the set whose elements are of the form, for $x, y \in Var^0$:*

$x =^0 y$;	$S(x,y)$;	$x <^0 y$;
$Q_a(x)$ *for* $a \in \Sigma$;	$x \in X_i$ *for* $X_i \in Var^1$.	

Observation 1.3.3. Since we do not have any function nor constant in our signature, our set of terms is equal to Var^0. That is why we can directly use the set of first-order variables to define the atomic formulas.

The intended semantic meaning of $S(x, y)$ would be that the position y immediately succeeds the position x. The Q_a operators formalize the positions that carry the letter $a \in \Sigma$, so $Q_a(x)$ represents the predicate "position x contains the letter a". And $x \in X_i$ would translate as to say that the position x belongs to the subset of positions X_i. From the set of atomic formulas, we define our MSO-formulas:

Definition 1.3.4. *Given a finite alphabet Σ, we inductively define the set of MSO-**formulas**, and we denote it by $\mathtt{Fm}_{\mathrm{S}(\Sigma)}$, as:*

- $\mathtt{At}_{\mathrm{S}(\Sigma)} \subseteq \mathtt{Fm}_{\mathrm{S}(\Sigma)}$;
- *if $\varphi, \psi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$ then, for $x \in Var^0$ and $X \in Var^1$:*

$$\neg\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}; \quad (\varphi \vee \psi) \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}; \quad \exists^0 x\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}; \quad \exists^1 X\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}.$$

As usual, from $\neg, \vee, \exists^0$, and $\exists^1$ we can define $\wedge, \rightarrow, \leftrightarrow, \forall^0$, and $\forall^1$ by, for $\varphi, \psi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$ with $x \in \mathit{Var}^0$ and $X \in \mathit{Var}^1$:

$$\begin{aligned}(\varphi \wedge \psi) &:= \neg(\neg\varphi \vee \neg\psi);\\ (\varphi \rightarrow \psi) &:= (\neg\varphi \vee \psi);\\ (\varphi \leftrightarrow \psi) &:= ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi));\\ \forall^0 x\varphi &:= \neg\exists^0 x\neg\varphi;\\ \forall^1 X\varphi &:= \neg\exists^1 X\neg\varphi.\end{aligned}$$

Notation. From now on, we will remove the superscripts of $=^0$, $<^0$, $\exists^0$, $\forall^0$, $\exists^1$ and $\forall^1$. This can be done without ambiguity by systematically using lowercase letters for first-order variables and uppercase letters for second-order variables. Also, to avoid unnecessary parentheses, we follow the standard convention of omitting the outermost parentheses and using the precedence of connectives: $\leftrightarrow$ over $\rightarrow$ over $\wedge$ and $\vee$. Furthermore, strictly speaking, we should remark that we are working with MSO logic including S, $=$, and $<$. However, since this will not cause ambiguity in our context, we will simplify our notation and refer to the logic as MSO (over words).

The formulas within the MSO logic we have constructed allow us to represent properties of words and languages. We aim to interpret these formulas over the word models defined earlier, to be able to study which words satisfy the represented properties. We will need to determine the notions of assignment and satisfiability, in the context of word models:

Definition 1.3.5. *Given a finite alphabet Σ and a word $w \in \Sigma^* \cup \Sigma^\omega$, we say that a function of the form*

$$\alpha : \mathit{Var}^0 \cup \mathit{Var}^1 \rightarrow \mathrm{dom}(w) \cup \mathcal{P}(\mathrm{dom}(w))$$

*is an (MSO-)**assignment**, or interpretation, of the set of variables on the word model $\underline{w}$ if*

$$\alpha_{\restriction \mathit{Var}^0} : \mathit{Var}^0 \rightarrow \mathrm{dom}(w) \quad \text{and} \quad \alpha_{\restriction \mathit{Var}^1} : \mathit{Var}^1 \rightarrow \mathcal{P}(\mathrm{dom}(w)).$$

We denote the set of assignments on $\underline{w}$ by $\mathrm{As}(w)$.

Given an assignment $\alpha \in \mathrm{As}(w)$, given $x \in \mathit{Var}^0$, and $i \in \mathrm{dom}(w)$, we denote by $\alpha[x \leftarrow i]$ the assignment, for every $\mathcal{X} \in \mathit{Var}^0 \cup \mathit{Var}^1$:

$$\alpha[x \leftarrow i](\mathcal{X}) := \begin{cases} \alpha(\mathcal{X}) & \text{if } \mathcal{X} \in (\mathit{Var}^0 \backslash \{x\}) \cup \mathit{Var}^1; \\ i & \text{if } \mathcal{X} = x. \end{cases}$$

Similarly, we can define the assignment $\alpha[X \leftarrow I]$ with $X \in \mathit{Var}^1$ and $I \in \mathcal{P}(\mathrm{dom}(w))$ as, for $\mathcal{X} \in \mathit{Var}^0 \cup \mathit{Var}^1$:

$$\alpha[X \leftarrow I](\mathcal{X}) := \begin{cases} \alpha(\mathcal{X}) & \text{if } \mathcal{X} \in \mathit{Var}^0 \cup (\mathit{Var}^1 \backslash X); \\ I & \text{if } \mathcal{X} = X. \end{cases}$$

Definition 1.3.6. *Consider a finite alphabet Σ, a word $w \in \Sigma^* \cup \Sigma^\omega$, a formula $\phi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$, and an assignment $\alpha \in \mathrm{As}(w)$. We denote by $\underline{w} \models^\alpha \phi$ the **satisfaction** of ϕ in $\underline{w}$ under α. We inductively define it by, for $x, y \in \mathit{Var}^0$ with $X \in \mathit{Var}^1$ and $\varphi, \psi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$:*

- $\underline{w} \models^\alpha x = y :\Longleftrightarrow \alpha(x) = \alpha(y)$;

- $\underline{w} \vDash^{\alpha} S(x,y) :\Longleftrightarrow (\alpha(x), \alpha(y)) \in S^w \Longleftrightarrow \alpha(x) + 1 = \alpha(y)$;
- $\underline{w} \vDash^{\alpha} x < y :\Longleftrightarrow \alpha(x) <^w \alpha(y) \Longleftrightarrow \alpha(x) < \alpha(y)$;
- *for* $a \in \Sigma$, *we have* $\underline{w} \vDash^{\alpha} Q_a(x) :\Longleftrightarrow \alpha(x) \in Q_a^w$;
- $\underline{w} \vDash^{\alpha} x \in X :\Longleftrightarrow \alpha(x) \in \alpha(X)$;
- $\underline{w} \vDash^{\alpha} \neg\varphi :\Longleftrightarrow \underline{w} \nvDash^{\alpha} \varphi$;
- $\underline{w} \vDash^{\alpha} \varphi \vee \psi :\Longleftrightarrow \underline{w} \vDash^{\alpha} \varphi$ *or* $\underline{w} \vDash^{\alpha} \psi$;
- $\underline{w} \vDash^{\alpha} \exists x\varphi :\Longleftrightarrow$ *there is some* $i \in \mathrm{dom}(w)$ *such that* $\underline{w} \vDash^{\alpha[x\leftarrow i]} \varphi$;
- $\underline{w} \vDash^{\alpha} \exists X\varphi :\Longleftrightarrow$ *there is some* $I \subseteq \mathrm{dom}(w)$ *such that* $\underline{w} \vDash^{\alpha[X\leftarrow I]} \varphi$.

Observation 1.3.7. Over word models, equality is definable in terms of $<$: for $x, y \in Var^0$, the equality $x = y$ could be replaced by $\neg(x < y) \wedge \neg(y < x)$. Similarly, $<$ is definable in terms of S: for $x, y \in Var^0$, we see that $x < y$ is equivalent over word models (they are satisfied in the same words, under every given assignment), to

$$(x < y \vee y < x) \wedge \forall X \left(x \in X \wedge \forall z \forall z' \left(z \in X \wedge S(z, z') \rightarrow z' \in X\right) \rightarrow y \in X\right).$$

Let $\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$ be a formula, we denote by $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$ that the variables that occur free —not under the scope of quantifiers— in φ are at most the first-order and second-order variables $x_1, \ldots, x_n, X_1, \ldots, X_m$. The following lemma is immediate yet relevant:

Lemma 1.3.8 (Coincidence Lemma)**.** *Consider a finite alphabet* Σ*, a word* $w \in \Sigma^* \cup \Sigma^\omega$*, and we pick a formula* $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m) \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$*. For every* $\alpha, \alpha' \in \mathrm{As}(w)$ *such that* α *and* α' *coincide in the assignments of the variables* $x_1, \ldots, x_n, X_1, \ldots, X_m$*, we have:*

$$\underline{w} \vDash^{\alpha} \varphi(x_1, \ldots, x_n, X_1, \ldots, X_m) \iff \underline{w} \vDash^{\alpha'} \varphi(x_1, \ldots, x_n, X_1, \ldots, X_m).$$

Proof. By induction on the construction of the formula φ. It becomes straightforward when considering *Definition 1.3.6*. ▲

We say that a formula is a *sentence* if it has no (first-order nor second-order) free variables, that is, all the variables that occur in the formula are under the scope of some quantifier. Notice that given a word w, an assignment $\alpha \in \mathrm{As}(w)$, and a sentence φ, we have that, as a direct consequence of *Lemma 1.3.8*, the satisfiability of $\underline{w} \vDash^{\alpha} \varphi$ is entirely independent of the assignment α. When a formula ψ holds in a word model $\underline{w}$ under every assignment, we will simplify our notation to $\underline{w} \vDash \psi$.

Finally, we are able to define languages using MSO-formulas —sentences— and their interpretation on word models, as we were looking for:

Definition 1.3.9. *Given a finite alphabet* Σ *and a sentence* $\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$*, the* ***language defined by*** φ *is*

$$L(\varphi) := \{w \in \Sigma^* : \underline{w} \vDash \varphi\}.$$

We say that a language $L \subseteq \Sigma^*$ *is MSO*(Σ)*-**definable** if there is some sentence* $\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$ *such that* $L = L(\varphi)$*. Analogously, the* ω*-**language defined** by* φ *is*

$$L^\omega(\varphi) := \{w \in \Sigma^\omega : \underline{w} \vDash \varphi\}.$$

Example 1.3.10. We consider the alphabet $\Sigma = \{a, b\}$. We will check which language is defined by the sentence:

$$\varphi := \exists X\,(\text{Prefix}(X) \wedge \forall x(x \in X \leftrightarrow Q_a(x)))\,;$$

where

$$\text{Prefix}(X) := \exists y(\forall x(x \in X \leftrightarrow \neg(x < y))) \vee \forall x(\neg(x \in X)).$$

For simplicity, we will only consider the finite words case. For w a finite word over Σ, we observe that $\underline{w} \vDash^{\alpha} \text{Prefix}(X)$ holds for all assignments α such that $\alpha(X) = \{0, 1, \ldots, k\}$, for some $k < |w|$, or $\alpha(X) = \varnothing$. Also, note that the subformula $\forall x(x \in X \leftrightarrow Q_a(x))$ from φ holds whenever, semantically speaking, the positions represented by X carry the letter a, and any other position has the letter b. That is, we see that:

$$\underline{w} \vDash \varphi \iff w = \overbrace{a \ldots a}^{n}\overbrace{b \ldots b}^{m}\,, \text{ where } n, m \geqslant 0 \text{ and } n + m = |w|.$$

Therefore, we conclude that $L(\varphi) = \{a^n b^m \in \Sigma^* : n, m \geqslant 0\}$. This language can be represented by the regular expression $\mathbf{a}^*\mathbf{b}^*$.

1.3.1 MSO and MSO$_0$

As we have seen, MSO-formulas allow for two sorts or types of variables: first-order and second-order. Generally, this is not an issue, but it may be preferable to have only one type of variable in certain frameworks. In fact, this will be the case when we want to prove Büchi's Theorem, *Theorem 1.4.1*.

This section presents a translation from the MSO-formulas over word models into the so-called MSO$_0$-formulas. This new formalization will not contain —quantification over— first-order variables, while, as we prove below, MSO$_0$-formulas will be as expressive as MSO-formulas.

We define MSO$_0$-formulas and their satisfiability over word models similarly to how we defined them in the MSO case:

Definition 1.3.11. *For Σ a finite alphabet, the set of MSO$_0$-**atomic formulas**, denoted by $\mathtt{At}_{\mathrm{S0}(\Sigma)}$, is composed exclusively by elements of the form, for $X, Y \in Var^1$ and for every $a \in \Sigma$:*

$$X \subseteq Y; \quad Sing(X); \quad Succ(X, Y); \quad Q_a[X].$$

The intended semantic meaning of $X \subseteq Y$ will be that the subset of positions represented by X is included in Y's. The unary predicate $Sing(X)$ says that "X is a singleton". The binary relation $Succ(X, Y)$ would express $X = \{x\}$ and $Y = \{y\}$, and $S(x, y)$. The expression $Q_a[X]$ will refer to the predicate "the set of positions X is a subset of Q_a", where Q_a, as before, would represent the set of positions that carry the letter $a \in \Sigma$. From these atomic formulas, we define the set of MSO$_0$-formulas:

Definition 1.3.12. *For Σ a finite alphabet, the set of MSO$_0$-**formulas**, $\mathtt{Fm}_{\mathrm{S0}(\Sigma)}$, is the smallest set containing $\mathtt{At}_{\mathrm{S0}(\Sigma)}$ such that verifies $\neg\varphi, (\varphi \vee \psi), \exists X\varphi \in \mathtt{Fm}_{\mathrm{S0}(\Sigma)}$, for every $\varphi, \psi \in \mathtt{Fm}_{\mathrm{S0}(\Sigma)}$ and every $X \in Var^1$.*

As we did with the MSO-formulas, from the connectives $\neg$, $\vee$ and the quantifier $\exists^{(1)}$, we can induce the connectives $\wedge, \rightarrow, \leftrightarrow$, and the quantifier $\forall^{(1)}$. As before, we follow the standard convention of omitting unnecessary parentheses.

We now define the notion of MSO_0-*assignments*, which will be simpler than their MSO counterpart, since now we only have one type of variable:

Definition 1.3.13. *For a given finite alphabet Σ and a word w over Σ, an MSO_0-**assignment** on the word model $\underline{w}$ is a function of the form $\alpha : Var^1 \to \mathcal{P}(\mathrm{dom}(w))$.*

Observation 1.3.14. By *Definition 1.3.5*, for every MSO-assignment $\alpha \in \mathrm{As}(w)$, we find that $\alpha_{\upharpoonright Var^1}$ is an MSO_0-assignment. Also, every MSO_0-assignment is of the form $\alpha_{\upharpoonright Var^1}$, for some $\alpha \in \mathrm{As}(w)$.

Now we can define the satisfiability of MSO_0-formulas over words:

Definition 1.3.15. *Consider a finite alphabet Σ, a word w over Σ, an MSO_0-formula φ_0, and an MSO_0-assignment α. We denote the **satisfaction** of φ_0 in $\underline{w}$ under α by $\underline{w} \vDash_0^\alpha \varphi_0$. We inductively define it, for $X, Y \in Var^1$ and $\varphi_0, \psi_0 \in \mathtt{Fm}_{\mathrm{S0}(\Sigma)}$, by:*

- $\underline{w} \vDash_0^\alpha X \subseteq Y :\Longleftrightarrow \alpha(X) \subseteq \alpha(Y)$;
- $\underline{w} \vDash_0^\alpha Sing(X) :\Longleftrightarrow \alpha(X) = \{i\}$, *for some* $i \in \mathrm{dom}(w)$;
- $\underline{w} \vDash_0^\alpha Succ(X,Y) :\Longleftrightarrow \alpha(X) = \{i\}$, $\alpha(Y) = \{i+1\}$, *for some* $i < |w| - 1$;
- *for* $a \in \Sigma$, *we have* $\underline{w} \vDash_0^\alpha Q_a[X] :\Longleftrightarrow \alpha(X) \subseteq Q_a^w$;
- $\underline{w} \vDash_0^\alpha \neg\varphi_0 :\Longleftrightarrow \underline{w} \nvDash_0^\alpha \varphi_0$;
- $\underline{w} \vDash_0^\alpha \varphi_0 \vee \psi_0 :\Longleftrightarrow \underline{w} \vDash_0^\alpha \varphi_0$ *or* $\underline{w} \vDash_0^\alpha \psi_0$;
- $\underline{w} \vDash_0^\alpha \exists X \varphi_0 :\Longleftrightarrow$ *there is some* $I \subseteq \mathrm{dom}(w)$ *such that* $\underline{w} \vDash_0^{\alpha[X \leftarrow I]} \varphi_0$.

We will check that MSO and MSO_0 are equally expressive, in the sense that their sentences satisfy or define the same languages. To do that, we will find two *translations*, from MSO_0 to MSO-formulas —*Lemma 1.3.16*— and from MSO to MSO_0-formulas —*Lemma 1.3.17*—. These translations will preserve the validity of sentences over word models, so the corresponding languages will also be preserved.

Lemma 1.3.16. *Given a finite alphabet Σ, for every MSO_0-formula $\varphi_0 \in \mathtt{Fm}_{\mathrm{S0}(\Sigma)}$, we can find some MSO-formula $\varphi_0^+ \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$ such that for every $w \in \Sigma^* \cup \Sigma^\omega$ and every MSO-assignment $\alpha \in \mathrm{As}(w)$, we have:*

$$\underline{w} \vDash_0^{\alpha_{\upharpoonright Var^1}} \varphi_0 \iff \underline{w} \vDash^\alpha \varphi_0^+. \qquad (+)$$

Proof. By induction on the construction of the MSO_0-formula φ_0. Such induction will become immediate by the translation from MSO_0 to MSO-formulas we are going to define:

By unfolding the definitions of $\vDash^\alpha$ and $\vDash_0^{\alpha_{\upharpoonright Var^1}}$, we have the equivalences, for $X, Y \in Var^1$ and $\alpha \in \mathrm{As}(w)$:

$$\begin{aligned}
&\underline{w} \vDash_0^{\alpha_{\upharpoonright Var^1}} X \subseteq Y \iff \underline{w} \vDash^\alpha \forall x(x \in X \to x \in Y);\\
&\underline{w} \vDash_0^{\alpha_{\upharpoonright Var^1}} Sing(X) \iff \underline{w} \vDash^\alpha \exists x(x \in X \wedge \forall y(y \in X \to x = y));\\
&\underline{w} \vDash_0^{\alpha_{\upharpoonright Var^1}} Succ(X,Y) \iff \underline{w} \vDash^\alpha \exists x \exists y(Sin(X,x) \wedge Sin(Y,y) \wedge S(x,y));\\
\text{for } a \in \Sigma, \quad &\underline{w} \vDash_0^{\alpha_{\upharpoonright Var^1}} Q_a[X] \iff \underline{w} \vDash^\alpha \forall x(x \in X \to Q_a(x));
\end{aligned}$$

where $Sin(Z,z) = z \in Z \wedge \forall z'(z' \in Z \rightarrow z = z')$, for every $Z \in Var^1$ and every $z \in Var^0$. Taking into account these correlations, we inductively define the following translation, for $X, Y \in Var^1$ and $\psi_0, \phi_0 \in \mathtt{Fm}_{S0(\Sigma)}$:

- $(X \subseteq Y)^+ := \forall x(x \in X \rightarrow x \in Y)$;
- $(Sing(X))^+ := \exists x(x \in X \wedge \forall y(y \in X \rightarrow x = y))$;
- $(Succ(X,Y))^+ := \exists x \exists y(Sin(X,x) \wedge Sin(Y,y) \wedge S(x,y))$;
- for $a \in \Sigma$, we have $(Q_a[X])^+ := \forall x(x \in X \rightarrow Q_a(x))$;
- $(\neg\psi_0)^+ := \neg\psi_0^+$;
- $(\psi_0 \vee \phi_0)^+ := \psi_0^+ \vee \phi_0^+$;
- $(\exists X \psi_0)^+ := \exists X \psi_0^+$.

Now, by induction on the construction of φ_0, it is immediate that we have $(+)$, with φ_0^+ the MSO-formula built by the presented translation. ▲

In particular, it is easy to see that if φ_0 is a sentence, then φ_0^+ will also be a sentence, and we will have $L(\varphi_0) = L(\varphi_0^+)$, as we wanted. Also note that, in this context, the interpretation of first-order variables is meaningless, since no translated formula has free first-order variables. That is why on the left side of $(+)$ we can consider MSO-assignments restricted over Var^1.

On the other hand, we also have the following:

Lemma 1.3.17. *Given a finite alphabet Σ and $\varphi \in \mathtt{Fm}_{S(\Sigma)}$, we can build some $\varphi_0 \in \mathtt{Fm}_{S0(\Sigma)}$ such that for every MSO-assignment $\alpha \in \mathrm{As}(w)$ and every $w \in \Sigma^* \cup \Sigma^\omega$, we have, for every MSO$_0$-assignment $\tilde{\alpha}$:*

$$\underline{w} \vDash^\alpha \varphi \iff \underline{w} \vDash_0^{\tilde{\alpha}} \varphi_0. \qquad (\sim)$$

Proof. As in the previous lemma, we will immediately show $(\sim)$ by induction on the construction of the MSO-formula φ, thanks to the translation from MSO to MSO$_0$-formulas we will provide. This translation will be based on the key idea of the MSO$_0$ formalization: we essentially replace MSO first-order variables x by singleton predicates or, specifically, by second-order variables X such that $Sing(X)$ holds. In this sense, we want the interpretation of X to be the singleton of the interpretation of x.

We know that variables can be subscripted based on our supply sets $Var^0 = \{x_i : i \in \mathbb{N}\}$ and $Var^1 = \{X_i : i \in \mathbb{N}\}$. Moreover, we can consider second-order variables of the form Y_i, which will be used to avoid ambiguity in the following suggested translation. For $x_i, x_j \in Var^0$, for $X_i, X_j, Y_k \in Var^1$ with $i, j, k \in \mathbb{N}$, and for $\psi, \phi \in \mathtt{Fm}_{S(\Sigma)}$, we define:

- $(S(x_i, x_j))_0 := Succ(X_i, X_j)$;
- $(x_i \in Y_k)_0 := X_i \subseteq Y_k \wedge Sing(X_i)$;
- for $a \in \Sigma$, we have $(Q_a(x_i))_0 := Q_a[X_i] \wedge Sing(X_i)$;
- $(\neg\psi)_0 := \neg\psi_0$;
- $(\psi \vee \phi)_0 := \psi_0 \vee \phi_0$;

- $(\exists x_i \psi)_0 := \exists X_i(Sing(X_i) \wedge \psi_0)$;
- $(\exists Y_k \psi)_0 := \exists Y_k \psi_0$.

Note that for the equality and the less-than cases, we can take advantage of *Observation 1.3.7*, and establish their translation as the translation of their equivalent formulas.

An easy induction on the construction of φ will give us that ($\sim$) holds, by unraveling the definitions of $\vDash^\alpha$ and $\vDash_0^{\tilde{\alpha}}$, using the translation we have stated. ▲

Again, it is easy to show that if $\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$ is a sentence, then $\varphi_0 \in \mathtt{Fm}_{\mathrm{S0}(\Sigma)}$ is also a sentence. Thus, we have $L(\varphi) = L(\varphi_0)$, as intended.

Moreover, note that for both translations we have presented, the growth of the translated formulas (whether in terms of the number of symbols, quantifiers, the nesting of connectives, or virtually any other measure we could consider) is linear with respect to the original formulas. This observation is easy to prove since, in all cases, we add only a constant to the original measure.

In conclusion, the last two lemmas show that for every MSO-sentence there is an MSO_0 one that defines the same language, and vice versa. So, we have that MSO_0 and MSO are equally expressive, their sentences define the same languages.

1.3.2 MSO Axiomatization Over Words

Before stating and proving Büchi's Theorem in the next section, we will pause to consider MSO-validities on words. Specifically, given a finite alphabet Σ, we will explore which formulas $\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$ satisfy $\underline{w} \vDash^\alpha \varphi$ for every word w over Σ and every assignment α. To achieve this, we will introduce, based on [Rib12], three axiomatizations: $\mathrm{MSO}, \mathrm{MSO}^{<\omega}$, and MSO^ω. These formalizations will be complete concerning our satisfiability definition over word models.

Definition 1.3.18. *Given a finite alphabet Σ, we denote the* ***axiomatization*** *of the MSO logic over words by* MSO. *We define* MSO *as the set of MSO-formulas built from the following axioms and inference rules. For every $\varphi, \psi, \phi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$ and $\mathcal{X}, \mathcal{X}_1, \ldots, \mathcal{X}_k, \ldots \mathcal{X}_n \in Var^0 \cup Var^1$ variables with $\mathcal{X}$ and $\mathcal{X}_k$ being of the same type (that is, $\mathcal{X}, \mathcal{X}_k \in Var^0$ or $\mathcal{X}, \mathcal{X}_k \in Var^1$), we have:*
Axioms and Axioms Schemes:

$$\forall x[\neg(x < x)]; \tag{LO1}$$

$$\forall x \forall y \forall z(x < y \wedge y < z \to x < z); \tag{LO2}$$

$$\forall x \forall y(x < y \vee x = y \vee y < x); \tag{LO3}$$

$$\forall X[\forall x(\forall y(y < x \to y \in X) \to x \in X) \to \forall x(x \in X)]; \tag{IndP}$$

$$\forall x(\exists y(y < x) \to \exists y(S(y, x))); \tag{IP}$$

$$\forall x \left(\bigvee_{a \in \Sigma} Q_a(x) \right); \tag{Q1}$$

$$\forall x(Q_a(x) \to \neg Q_b(x)), \text{ for every } a, b \in \Sigma \text{ with } a \neq b; \tag{Q2}$$

$$\varphi \vee \neg\varphi; \tag{LEM}$$

$$\exists X \forall x(x \in X \leftrightarrow \varphi), \text{ for } X \text{ not a free variable in } \varphi. \tag{CAS}$$

Inference Rules:

$$\frac{\varphi}{\neg\neg\varphi} \qquad \frac{\varphi}{\varphi \vee \psi} \qquad \frac{\varphi}{\psi \vee \varphi} \qquad \frac{\varphi(\mathcal{X}_1, \ldots, \mathcal{X}_k, \ldots, \mathcal{X}_n)}{\exists \mathcal{X} \varphi(\mathcal{X}_1, \ldots, \mathcal{X}, \ldots, \mathcal{X}_n)} \text{ for } \mathcal{X}_k \text{ a free variable in } \varphi$$

We define the axiomatization of the MSO logic over finite words, $\mathrm{MSO}^{<\omega}$*, by adding to* MSO *the axiom:*

$$\exists x(x = x) \to \exists y \forall z(\neg(y < z)).$$

And the axiomatization of the MSO logic over infinite words, MSO^{ω}*, by adding to* MSO *the axiom:*

$$\forall x \exists y(x < y).$$

Notice that the first three axioms express that our words are linearly ordered by $<$. Axiom (IndP) gives us an induction principle. Axiom (IP) would tell us that every position having some predecessor has an immediate predecessor. The next two axioms, (Q1) and (Q2), would express that each position carries some and at most one letter, respectively. And then we have two axiom schemes, where (LEM) would be the *Law of Excluded Middle*, and the last one can be understood as a *Comprehension Axiom Scheme*. The inference rules are clear, as they would characterize the connectives $\neg$ and $\vee$, as well as the existential quantification over first and second-order variables. Regarding $\mathrm{MSO}^{<\omega}$ and MSO^{ω}, we see that the axioms we add would express that our words are either empty or finite, or that they have an infinite number of positions, respectively.

For simplicity, we will present *Theorem 1.3.19* with reference only to the axiomatization MSO. However, it can be easily adapted to $\mathrm{MSO}^{<\omega}$ by restricting the domain of words to finite words, and to MSO^{ω} by restricting it to infinite words.

Theorem 1.3.19. *Given a finite alphabet* Σ*, for every* $\varphi \in \mathtt{Fm}_{\mathrm{S}(\Sigma)}$*, the following holds:*

$$\varphi \in \mathrm{MSO} \iff \text{for every } w \in \Sigma^* \cup \Sigma^\omega, \text{ and every } \alpha \in As(w), \text{ we have } \underline{w} \vDash^\alpha \varphi.$$

▲

We will not enter into the details of the proof, however, note that the left-to-right implication, the soundness implication, is almost immediate, since all axioms of *Definition 1.3.18* hold in all word models, and all the rules preserve the satisfiability. We refer to [Rib12] for proof of the other implication and a more exhaustive exposition of the axiomatization of the MSO logic over words.

1.4 Büchi's Theorem

Büchi's Theorem, as mentioned before, establishes a connection between the two main approaches we have presented regarding the characterization of regular languages: the logical perspective based on word models, and the automata and computational one. The significance of this theorem could be summarized in two key points: 1) it bridges a purely computational object, automata, with logic, as well as indirectly links regular languages with logic; and 2) this relationship ensures that finite automata have considerable expressive power, as we have:

Theorem 1.4.1 (Büchi [Büc60], Elgot [Elg61])**.** *Given a finite alphabet* Σ*, an (*ω*-)language* $L \subseteq \Sigma^* \cup \Sigma^\omega$ *is recognizable by a (Büchi automaton) nondeterministic finite automaton if and only if* L *is MSO*(Σ)*-definable.*

Proof. We will focus on languages of finite words, and we will follow the structure outlined in [Tho96], delving into all the details. Essentially, we have to prove the effectiveness of both implications. First, we demonstrate the **direction from left to right** (only if):

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be an NFA. We can assume $Q = \{0, \ldots, k\}$ without loss of generality. Given a finite word $w = a_0 \ldots a_{n-1}$, we want to find an MSO-sentence expressing, once interpreted in the word model $\underline{w}$, if $\mathcal{A}$ accepts w. Therefore, we require a sentence stating the existence of a successful run $\langle q_0, \ldots, q_n \rangle$ of $\mathcal{A}$, with $(q_i, a_i, q_{i+1}) \in \delta$ for $i < n$, and $q_0 \in Q_0$, and $q_n \in F$. We prove the first one of the two implications we need:

Claim 1. If the finite automaton $\mathcal{A} = (\{0, \ldots, k\}, \Sigma, \delta, Q_0, F)$ accepts the non-empty word w, then we have:

$$\underline{w} \vDash \exists X_0 \ldots \exists X_k (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4);$$

where:

$$\varphi_1 = \bigwedge_{\substack{i,j \leqslant k \\ i \neq j}} \neg \exists x \, (x \in X_i \wedge x \in X_j) \, ;$$

$$\varphi_2 = \forall x \left(\neg \exists y \, (S(y, x)) \rightarrow \bigvee_{i \in Q_0} x \in X_i \right) ;$$

$$\varphi_3 = \forall x \forall y \left(S(x, y) \rightarrow \bigvee_{(i,a,j) \in \delta} (x \in X_i \wedge Q_a(x) \wedge y \in X_j) \right) ;$$

$$\varphi_4 = \forall x \left(\neg \exists y \, (S(x, y)) \rightarrow \bigvee_{\substack{(i,a,j) \in \delta \\ j \in F}} (x \in X_i \wedge Q_a(x)) \right) .$$

Proof of Claim 1. Let us assume that $\langle q_0, \ldots, q_n \rangle$ is a successful run of $\mathcal{A}$ on w, and we need to check that we have $\underline{w} \vDash \exists X_0 \ldots \exists X_k (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$. That is, we need to find some $P_i \subseteq \{0, \ldots, n-1\}$, for $i \leqslant k$, such that $\underline{w} \vDash^{\alpha} \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4$ for α any assignment verifying $\alpha(X_i) = P_i$ for every $i \leqslant k$.

We will show we can define our P_i to be the following sets of positions of w:

$$P_i \coloneqq \{j \leqslant n-1 : \text{ within the successful run } \langle q_0, \ldots, q_n \rangle, \text{ we have } q_j = i\}.$$

From this definition, it is clear that the P_i sets are pairwise disjoint. Then, if $\alpha(X_i) = P_i$ for every $i \leqslant k$, we have that $\underline{w} \vDash^{\alpha} \varphi_1$ holds.

Also, as $\langle q_0, \ldots, q_n \rangle$ is a successful run, we have $q_0 \in Q_0$. Then the first position of w, position 0, will be included in P_i for some $i \in Q_0$. If $\alpha(X_i) = P_i$ for every $i \leqslant k$, we deduce $\underline{w} \vDash^{\alpha} \varphi_2$: position 0 is the only one that is not a successor of any other position, so $\neg \exists y (S(y, x))$ holds if and only if we interpret x as 0; and $\bigvee_{i \in Q_0} x \in X_i$ will also be satisfied since, as we mentioned, $0 \in P_i$ for some $i \in Q_0$.

Another consequence of $\langle q_0, \ldots, q_n \rangle$ being a successful run is having $(q_i, a_i, q_{i+1}) \in \delta$ for every $i < n$. Now, directly from their definitions, for every position $i < n$ of w we have $i \in P_{q_i}$ and $(i+1) \in P_{q_{i+1}}$, and $i \in Q^w_{a_i}$. Then, we see that φ_3 will hold, interpreted over the word model $\underline{w}$, under any assignment α such that $\alpha(X_i) = P_i$ for every $i \leqslant k$.

Finally, we also know $q_n \in F$ and $(q_{n-1}, a_{n-1}, q_n) \in \delta$. Similarly as before, we have $(n-1) \in P_{q_{n-1}}$ and $(n-1) \in Q^w_{a_{n-1}}$, where the position $n-1$ is the last one of w. This leads us to deduce that we also have $\underline{w} \vDash^\alpha \varphi_4$ for every α verifying $\alpha(X_i) = P_i$.

In conclusion, by the satisfiability of the conjunction function, we have, for every assignment α such that $\alpha(X_i) = P_i$:

$$\underline{w} \vDash^\alpha \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4.$$

Then, as we wanted, we get

$$\underline{w} \vDash \exists X_0 \dots \exists X_k(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4).$$

△

Now we are going to prove the other implication, namely:

Claim 2. Following the definitions of Claim 1, if we have $\underline{w} \vDash \exists X_0 \dots \exists X_k(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$, for some non-empty finite word w, then $\mathcal{A}$ accepts w.

Proof of Claim 2. We assume we have $\underline{w} \vDash \exists X_0 \dots \exists X_k(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$, for some finite word w with $|w| = n > 0$. We need to find a successful run $\langle q_0, \dots, q_{n-1}, q_n \rangle$ of $\mathcal{A}$ on w. From our assumption, we have that there are some sets $P_i \subseteq \{0, \dots, n-1\} = \mathrm{dom}(w)$ for $i \leqslant k$ such that $\underline{w} \vDash^\alpha \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4$, for α any assignment sending each X_i to P_i.

We will determine the states $q_0, \dots, q_{n-1}$ by, for $m < n$ and $i \leqslant k$:

$$q_m = i :\Longleftrightarrow m \in P_i.$$

We first need to show that these q_m are well-defined, specifically, that there is one and only one possible value i such that $m \in P_i$, for each $m < n$. We prove it by induction on m.

Regarding the base case $m = 0$, we require $0 \in P_j$ for exactly one possible value of $j \in \{0, \dots, k\}$. By the satisfiability of φ_2, we have $0 \in P_{i_0}$ for some $i_0 \in Q_0$. At the same time, by φ_1, we deduce that our P_i are pairwise disjoint, and so 0 does not belong to any other P_i than P_{i_0}. Then, we conclude that there is exactly one $i_0 = j \in \{0, \dots, k\}$ —with $j \in Q_0$— such that $0 \in P_j$, as we wanted.

Now let us assume q_m to be well-defined, for $m < n-1$, we want to show that q_{m+1} is also well-defined. By definition, we have $(m, m+1) \in S^w$ so, by the satisfiability of φ_3 over $\underline{w}$, we get that there is some $j \leqslant k$ and some $a' \in \Sigma$ such that:

$$(q_m, a', j) \in \delta, \qquad m \in Q^w_{a'}, \text{ and} \qquad m+1 \in P_j.$$

This leads us to conclude that there is at least one value $j \leqslant k$ such that $m+1 \in P_j$. To check that this j is unique, we refer again to the satisfaction of φ_1.

Then, by induction on m, we have shown that our q_m are well-defined. Not only that, we have also seen that $q_0 \in Q_0$, by applying φ_2, and that we have $(q_i, a'_i, q_{i+1}) \in \delta$ and $i \in Q^w_{a'_i}$ for $i < n-1$, from the satisfiability of φ_3. Those last expressions $i \in Q^w_{a'_i}$ ensure that we can state $a'_i = a_i$ for $i < n-1$, with a_i the letters of $w = a_0 \dots a_{n-2} a_{n-1}$.

To find the successful run we want, it only remains to define $q_n \in F$ in such a way that $(q_{n-1}, a_{n-1}, q_n) \in \delta$. We already know that the position $n-1$ is the last position of w, therefore, it is the only position that has no successor. Then, by the satisfiability of φ_4 we conclude that there is some $j \in F$ such that $(i, a, j) \in \delta$, with $n-1 \in P_i$ and $n-1 \in Q_a^w$, that is, with $i = p_{n-1}$ and $a = a_{n-1}$. In this way, we have found that there is at least one $j \leqslant k$ such that $(q_{n-1}, a_{n-1}, j) \in \delta$. We can define the state q_n as the minimum such j.

In conclusion, we have built a sequence $\langle q_0, \ldots, q_{n-1}, q_n \rangle$ of states of $\mathcal{A}$ such that $q_0 \in Q_0$ and $q_n \in F$, and $(q_i, a_i, q_{i+1}) \in \delta$ for $i < n$. That is, $\langle q_0, \ldots, q_{n-1}, q_n \rangle$ is a successful run of $\mathcal{A}$ on w, so $\mathcal{A}$ accepts the non-empty word w. △

We observe that Claims 1 and 2 together imply that the finite automaton $\mathcal{A}$ accepts the non-empty word w if and only if we have $\underline{w} \models \exists X_0 \ldots \exists X_k (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$. We only need to consider the empty words case. We immediately see that

$$\underline{\epsilon} \models \exists X_0 \ldots \exists X_k (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4),$$

by taking the interpretations of X_i to be $P_i = \varnothing$ for every $i \leqslant k$. If $\mathcal{A}$ does not accept the empty word, we only need to add —via conjunction— the expression $\exists x (x = x)$ to the suggested sentence. This addition ensures that the empty word model will not satisfy the sentence anymore, without affecting the satisfiability of other word models, as every non-empty model satisfies $\exists x (x = x)$.

In summary, we have seen that if the language $L \subseteq \Sigma^*$ is recognizable by $\mathcal{A}$, then L is also MSO(Σ)-definable, since we have

$$L = \begin{cases} L(\exists X_0 \ldots \exists X_k (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)), & \text{if } \epsilon \in L; \\ (or) \quad L(\exists X_0 \ldots \exists X_k (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \exists x (x = x))), & \text{if } \epsilon \notin L. \end{cases}$$

This proves the left-to-right implication (only if) of our theorem.

Let us show now the **direction from right to left** (if):

Given an MSO-sentence φ defining a language $L = L(\varphi) \subseteq \Sigma^*$, we want to prove that there is some finite automaton $\mathcal{A}_\varphi$ such that $L(\mathcal{A}_\varphi) = L$. The natural step would be to prove it by induction on the construction of φ. However, consider the inductive case $\varphi = \exists X \psi$: we observe that as ψ is not in general a sentence, we cannot apply our Induction Hypothesis over ψ, and we end up with our hands tied unable to prove what we need.

This is why we will use a different formalization seeking to, in some sense, get track not only of the sentences, but also of the formulas with free variables, and their interpretations. Our first move in this direction will be to translate our MSO-formulas into MSO$_0$-formulas. In Subsection 1.3.1, we already demonstrated that this translation can be performed without losing expressiveness, and it has the additional advantage of simplifying the set of free variables by only allowing second-order variables.

Given a finite alphabet Σ, we will call the alphabet $\Sigma' = \Sigma \times \{0, 1\}^n$, for some $n < \omega$, an *expanded alphabet*. So, $w' \in (\Sigma')^*$ will be an *expanded word*, with *expanded letters* of the form $(a, b_1, \ldots, b_n)$, where $a \in \Sigma$ is carried in the 0-*th component* of the expanded word, and $b_i \in \{0, 1\}$ in its i-th component for $i \leqslant n$. For an expanded word w', we will also speak about the *expanded word model* $\underline{w'} = (\underline{w}, P_1, \ldots, P_n)$, with $w \in \Sigma^*$ and $P_1, \ldots, P_n \subseteq \text{dom}(w)$. In such a case, for every expanded letter $(a, b_1, \ldots, b_n)$ of w' in position p, we have that the position p of w carries the letter $a \in \Sigma$; and for every $i \leqslant n$:

$$p \in P_i \iff b_i = 1.$$

Consider an MSO$_0$-formula $\varphi(X_1, \ldots, X_n)$, where X_i for $i \leqslant n$ are its free second-order variables, and let $w \in \Sigma^*$ be a finite word. We will denote by

$$(\underline{w}, P_1, \ldots, P_n) \models_0 \varphi(X_1, \ldots, X_n),$$

if φ is satisfied in $\underline{w}$ by taking $P_1, \ldots, P_n$ as interpretations of the variables $X_1, \ldots, X_n$, respectively. Notice that, ultimately, this notation is simply summarizing

$$\underline{w} \vDash_0^\alpha \varphi\left(X_1, \ldots, X_n\right),$$

for α any assignment satisfying $\alpha(X_i) = P_i$ for $i \leqslant n$. Note that this implies

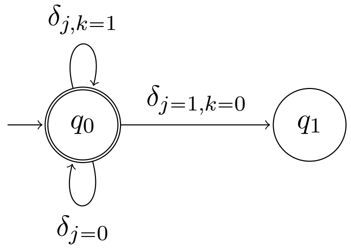
$$\underline{w} \vDash_0 \exists X_1 \ldots \exists X_n \varphi.$$

We will see how using expanded alphabets we avoid the problems mentioned before regarding existential sentences. We show it in the following claim:

Claim 3. Consider a finite alphabet Σ. For every formula $\psi(X_1, \ldots, X_n)$ only containing the variables $X_1, \ldots, X_n$ —free or bound—, there is some NFA $\mathcal{A}_\psi$ accepting precisely those expanded words $w' \in \Sigma \times \{0,1\}^n$ that verify $\underline{w'} \vDash_0 \psi(X_1, \ldots, X_n)$.

Proof of Claim 3. By induction on the construction of the formula ψ. We can propose simple finite automata for the MSO_0-atomic formulas cases. All these automata will be defined over the expanded alphabet $\Sigma' = \Sigma \times \{0,1\}^n$. For $j, k \leqslant n$:

- A finite automaton checking whether $X_j \subseteq X_k$ holds in $w' \in (\Sigma')^*$ has to verify that if 1 occurs in the j-th component of some expanded letter of w', it also occurs in the k-th component. For instance, we could consider the automaton:



Where

$$\delta_{j=0} := \{(a, b_1, \ldots, b_n) : a \in \Sigma \text{ and } b_i \in \{0,1\} \text{ for } i \leqslant n, \text{ with } b_j = 0\};$$

$$\delta_{j,k=1} := \{(a, b_1, \ldots, b_n) : a \in \Sigma \text{ and } b_i \in \{0,1\} \text{ for } i \leqslant n, \text{ with } b_j = b_k = 1\};$$

$$\delta_{j=1,k=0} := \{(a, b_1, \ldots, b_n) : a \in \Sigma \text{ and } b_i \in \{0,1\} \text{ for } i \leqslant n, \text{ with } b_j = 1 \text{ and } b_k = 0\}.$$

We see that any successful run of this automaton cannot include the state q_1, as there are no transitions originating from of it, nor is it a final state. Then, this automaton does not accept any expanded word with some expanded letter in $\delta_{j=1,k=0}$, which would have a 1 in its j-th component but a 0 in its k-th. This is precisely the only requirement for stating that $X_j \subseteq X_k$ does not hold, since we would have, once we interpret our variables, that there is some element of P_j that does not belong to P_k, so $P_j \nsubseteq P_k$. Then, we conclude that the presented automaton,

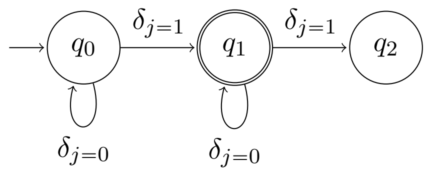
$$\langle \{q_0, q_1\}, \Sigma', \delta, \{q_0\}, \{q_0\} \rangle,$$

where

$$\delta := \{(q_0, a', q_0) : a' \in \delta_{j=0} \cup \delta_{j,k=1}\} \cup \{(q_0, a', q_1) : a' \in \delta_{j=1,k=0}\},$$

accepts only the expanded words where $X_j \subseteq X_k$ is satisfied, as we wanted.

- $Sing(X_j)$: the desired NFA has to check that there is precisely one expanded letter such that its j-th component is 1. We could consider the automaton:

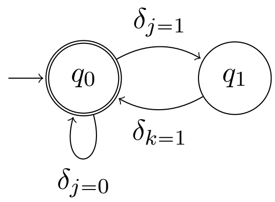


  The set $\delta_{j=0}$ is the same as before, and:

$$\delta_{j=1} := \{(a, b_1, \ldots, b_n) : a \in \Sigma \text{ and } b_i \in \{0,1\} \text{ for } i \leqslant n, \text{ with } b_j = 1\}.$$

  Note that any successful run of the suggested automaton on an expanded word w' has to start with the state q_0 and finish with q_1. So, we need at least one expanded letter of w' to be in $\delta_{j=1}$. However, once we have that letter in $\delta_{j=1}$, our w' cannot have any other expanded letter in $\delta_{j=1}$. Otherwise, we would transition to state q_2, but that is not possible for a successful run because there are no transitions originating from q_2, and it is not a final state either. So, the automaton accepts those expanded words such that the j-th components of their expanded letters are all 0 except for exactly one of them. This means that the interpretation of X_j, the set P_j, has only one element. In this way, the suggested automaton only accepts the words satisfying $Sing(X_j)$.

- $Succ(X_j, X_k)$: we consider an automaton verifying that if the j-th component of an expanded letter in the position p is 1, then the expanded letter in position $p+1$ has a 1 in its k-th component. We will denote this automaton by $\mathcal{A}_S$, and we see it could be represented by:



  The sets $\delta_{j=1}$ and $\delta_{j=0}$ are defined as in the previous cases, and:

$$\delta_{k=1} := \{(a, b_1, \ldots, b_n) : a \in \Sigma \text{ and } b_i \in \{0,1\} \text{ for } i \leqslant n, \text{ with } b_k = 1\}.$$

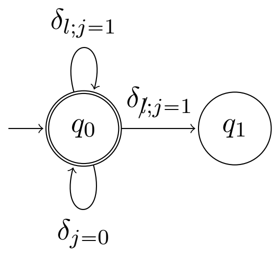
  Let us assume that $\mathcal{A}_S$ accepts an expanded word w', and that w' has some expanded letter having its j-th component equal to 1. Therefore, the next expanded letter of w', in the consecutive position, necessarily carries a 1 in its k-th component. Otherwise, a successful run would be impossible because we would become "stuck" in the state q_1, which is not a final state.

  Now, consider the automata $\mathcal{A}_{Sing(X_j)}$ and $\mathcal{A}_{Sing(X_k)}$ to accept exclusively those words satisfying $Sing(X_j)$ and $Sing(X_k)$, respectively. We know there are automata like that thanks to the previous case in the induction. By *Lemma 1.2.17*, we determine that the product automaton $\mathcal{A}_{Sing(X_j)} \otimes \mathcal{A}_{Sing(X_k)}$ exclusively accepts expanded words such that their expanded letters only have one j-th and one k-th component equal to 1, and the remaining j-th and k-th components are 0.

Consequently, the product automaton $(\mathcal{A}_{Sing(X_j)} \otimes \mathcal{A}_{Sing(X_k)}) \otimes \mathcal{A}_S$ will exclusively accept expanded words such that their expanded letters have only one 1 in their j-th components, say in the letter in position p, and only the expanded letter in position $p+1$ will have 1 in its k-th component. Thus, the resulting automaton precisely accepts those expanded words for which the assignments of X_j and X_k are $P_j = \{p\}$ and $P_k = \{p+1\}$. This means that the suggested automaton only accepts the expanded words satisfying $Succ(X_j, X_k)$, as we wanted.

Considering the automata of the form $\mathcal{A}_{Sing(X)}$ as proposed in the previous case, each with three states, we find that our automaton checking whether $Succ(X_j, X_k)$ holds requires $(3 \times 3) \times 2 = 18$ states.

- $Q_l[X_j]$, for $l \in \Sigma$: the required NFA has to verify that whenever 1 occurs in the j-th component of some expanded letter of w', we also have $l \in \Sigma$ in its 0th component. An automaton with this behavior could be:



The set $\delta_{j=0}$ has the same definition as in the previous cases, and:

$$\delta_{l;j=1} := \{(l, b_1, \ldots, b_n) : b_i \in \{0,1\} \text{ for } i \leqslant n, \text{ with } b_j = 1\};$$
$$\delta_{\not{l};j=1} := \{(a, b_1, \ldots, b_n) : l \neq a \in \Sigma \text{ and } b_i \in \{0,1\} \text{ for } i \leqslant n, \text{ with } b_j = 1\}.$$

Similarly to the case $X_j \subseteq X_k$, we deduce that the suggested automaton does not accept any expanded word with some expanded letter in $\delta_{\not{l};j=1}$, a letter that would have 1 in its j-th component and carry some letter $a \in \Sigma$ different from l in its 0th component. We see that expanded letters of such form would give us that there is some element of P_j, the assignment of X_j, that does not belong to Q_l^w, the set of positions that their 0th component is l. In symbols, $P_j \nsubseteq Q_l^w$. Then, we conclude that the presented automaton accepts only the expanded words where we have $P_j \subseteq Q_l^w$, that is, where $Q_l[X_j]$ is satisfied.

For the inductive step, we have to consider the cases of the connectives $\neg$, $\vee$, and the existential quantification over (monadic) second-order variables. These three cases follow from the closure properties of the class of recognizable languages we proved in Subsection 1.2.1:

- Negation case: assuming that the language defined by the formula $\phi(X_1, \ldots, X_n)$ over the expanded alphabet $\Sigma \times \{0,1\}^n$ is recognized by the automaton $\mathcal{A}_\phi$, we want to find an automaton corresponding to the formula $\neg\phi(X_1, \ldots, X_n)$. It is enough to take the complementary automaton $\overline{\mathcal{A}_\phi}$ (if needed, we know we can translate our NFA $\mathcal{A}_\phi$ into an equivalent total DFA), built as is done in *Lemma 1.2.14*. By definition, for every expanded word w', we have

$$\underline{w}' \vDash_0 \neg\phi\,(X_1,\dots,X_n) \iff \underline{w}' \nvDash_0 \phi\,(X_1,\dots,X_n) \overset{IH}{\iff} w' \notin L(\mathcal{A}_\phi) \iff w' \in L(\overline{\mathcal{A}_\phi}).$$

- Disjunction case: suppose that the language defined by the formula $\phi\,(X_1,\dots,X_n)$ over the expanded alphabet $\Sigma \times \{0,1\}^n$ is recognized by the automaton $\mathcal{A}_\phi$, and that the language defined by the formula $\phi'\,(X_1,\dots,X_n)$, over the same expanded alphabet, is recognized by the automaton $\mathcal{A}_{\phi'}$. We want to find an automaton for the formula $(\phi \vee \phi')\,(X_1,\dots,X_n)$. We apply *Lemma 1.2.18* to $\mathcal{A}_\phi$ and $\mathcal{A}_{\phi'}$ to get an automaton $\mathcal{A}_{\phi\vee\phi'}$ such that $L(\mathcal{A}_{\phi\vee\phi'}) = L(\mathcal{A}_\phi) \cup L(\mathcal{A}_{\phi'})$. This $\mathcal{A}_{\phi\vee\phi'}$ will accept just those words which satisfy $\phi \vee \phi'$ since, for every expanded word w', we have:

$$\underline{w}' \vDash_0 (\phi \vee \phi')(X_1,\dots,X_n) \iff \underline{w}' \vDash_0 \phi(X_1,\dots,X_n) \text{ or } \underline{w}' \vDash_0 \phi'(X_1,\dots,X_n) \overset{IH}{\iff}$$
$$\overset{IH}{\iff} w' \in L(\mathcal{A}_\phi) \text{ or } w' \in L(\mathcal{A}_{\phi'}) \iff w' \in L(\mathcal{A}_\phi) \cup L(\mathcal{A}_{\phi'}) = L(\mathcal{A}_{\phi\vee\phi'}).$$

- Existential case: we assume that the language defined by the formula $\phi\,(X_1,\dots,X_n)$ over the expanded alphabet $\Sigma \times \{0,1\}^n$ is recognized by the automaton $\mathcal{A}_\phi$. We have to find an automaton corresponding to the formula $\exists X_i \phi\,(X_1,\dots,X_n)$ for some $i \leqslant n$. We can suppose, without loss of generality, that we have $i = n$, and so we consider $\phi'(X_1,\dots,X_{n-1}) = \exists X_n \phi\,(X_1,\dots,X_n)$.

  Since the variable X_n is not free in $\phi'(X_1,\dots,X_{n-1})$, and so the interpretations of X_n are not relevant anymore, it could be tempting to consider our desired automaton to be defined over the expanded alphabet $\Sigma \times \{0,1\}^{n-1}$. Then, we could say we only need to apply *Lemma 1.2.19*, the closure under projections, over $\mathcal{A}_\phi$. However, changing our alphabet this way would be problematic, especially for our previous $\vee$ case where we need two automata over the same language to compute their union.

  Then, our required automaton, which we will call $\mathcal{A}_{\phi'}$, must be defined over the alphabet $\Sigma \times \{0,1\}^n$, even though the last bit becomes almost vacuous. Particularly, if $\mathcal{A}_\phi$ accepts the expanded word $w' = (a_0, b_0^1, \dots, b_0^n) \dots (a_m, b_m^1, \dots, b_m^n) \in (\Sigma \times \{0,1\}^n)^*$, our $\mathcal{A}_{\phi'}$ will accept the expanded words

$$(a_0, b_0^1, \dots, b_0^{n-1}, b_0) \dots (a_m, b_m^1, \dots, b_m^{n-1}, b_m) \in (\Sigma \times \{0,1\}^n)^*, \quad \text{for every } b_0, \dots, b_m \in \{0,1\}.$$

  If w' is accepted by $\mathcal{A}_\phi$, it is clear that w' will also be accepted by $\mathcal{A}_{\phi'}$. Conversely, if some expanded word w' is accepted by $\mathcal{A}_{\phi'}$, then there is some sequence of 0s and 1s —the bits $b_0^n, \dots, b_m^n$— such that $\mathcal{A}_\phi$ accepts the expanded word resulting of replacing the last coordinate of the expanded letters of w' by the elements of the sequence.

  We have assumed $\mathcal{A}_\phi$ to exist and being defined over the alphabet $\Sigma \times \{0,1\}^n$. Then, by *Lemma 1.2.19*, from $\mathcal{A}_\phi$ we can get a new automaton $\mathcal{A}'_\phi$ over the alphabet $\Sigma \times \{0,1\}^{n-1}$. If now we apply *Lemma 1.2.20*, the closure under padding, to $\mathcal{A}'_\phi$, we obtain another automaton, $\mathcal{A}''_\phi$, over the alphabet $\Sigma \times \{0,1\}^n$. Notice that, in this way, $\mathcal{A}''_\phi$ will be defined as $\mathcal{A}_\phi$ but, if δ_ϕ is the transition relation of $\mathcal{A}_\phi$, then the transition relation of $\mathcal{A}''_\phi$ is:

$$\delta''_\phi = \{(p, (a, b_1, \dots, b_{n-1}, b_n), q) : (p, (a, b_1, \dots, b_{n-1}, 0), q) \in \delta_\phi \text{ or } (p, (a, b_1, \dots, b_{n-1}, 1), q) \in \delta_\phi\}.$$

Now we show that we can consider $\mathcal{A}_{\phi'} = \mathcal{A}''_{\phi}$, that is, $\mathcal{A}''_{\phi}$ only accepts the words satisfying the formula $\phi' = \exists X_n \phi$:

To simplify notation, we will denote an expanded word w' over the alphabet $\Sigma \times \{0,1\}^n$ as $w' = (w, P_1, \ldots, P_n)$, whenever $\underline{w'} = (\underline{w}, P_1, \ldots, P_n)$. We see that the following expressions are equivalent:

i) $(\underline{w}, P_1, \ldots, P_n) \vDash_0 \exists X_n \phi$;

ii) $\underline{w} \vDash_0^{\alpha} \exists X_n \phi$ for every MSO$_0$-assignment α verifying $\alpha(X_i) = P_i$, for $i \leqslant n$;

iii) there is some $P'_n \subseteq \mathrm{dom}(w)$ such that $\underline{w} \vDash_0^{\alpha[X_n \leftarrow P'_n]} \phi$, for every α verifying $\alpha(X_i) = P_i$ for $i \leqslant n$;

iv) $(\underline{w}, P_1, \ldots, P_{n-1}, P'_n) \vDash_0 \phi$ for some $P'_n \subseteq \mathrm{dom}(w)$;

v) $(w, P_1, \ldots, P_{n-1}, P'_n) \in L(\phi)$ for some $P'_n \subseteq \mathrm{dom}(w)$;

vi) $(w, P_1, \ldots, P_{n-1}, P'_n) \in L(\mathcal{A}_{\phi})$ for some $P'_n \subseteq \mathrm{dom}(w)$;

vii) $(w, P_1, \ldots, P_{n-1}) \in L(\mathcal{A}'_{\phi})$;

viii) $(w, P_1, \ldots, P_{n-1}, P''_n) \in L(\mathcal{A}''_{\phi})$ for all $P''_n \subseteq \mathrm{dom}(w)$.

The equivalences among the first five items are directly derived from previous definitions and notations. By Induction Hypothesis, we have the equality $L(\phi) = L(\mathcal{A}_{\phi})$, ensuring us $v) \Leftrightarrow vi)$. The equivalence between $vi)$ and $vii)$ is given by the construction of $\mathcal{A}'_{\varphi}$, based in *Lemma 1.2.19*. Similarly, we have $vii) \Leftrightarrow viii)$ by the construction of $\mathcal{A}''_{\varphi}$, thanks to *Lemma 1.2.20*.

We observe that, in particular, by considering $P''_n = P_n$, the equivalence between $i)$ and $viii)$ shows us what we wanted to check, that $\mathcal{A}''_{\phi}$ accepts precisely the expanded words such that satisfy $\exists X_n \phi$.

Then, we have proved, by induction on the construction of ψ, that there is some NFA $\mathcal{A}_{\psi}$ such that

$$L(\mathcal{A}_{\psi}) = \{w' \in \Sigma \times \{0,1\}^n : \underline{w'} \vDash_0 \psi(X_1, \ldots, X_n)\}.$$

This finishes our proof of Claim 3. △

In particular, Claim 3 ensures that there is some NFA $\mathcal{A}_{\varphi}$ such that $L(\mathcal{A}_{\varphi}) = L(\varphi)$, for φ our initial sentence. This automaton $\mathcal{A}_{\varphi}$ is defined over the expanded alphabet $\Sigma \times \{0,1\}^n$, with n being the number of bound variables of φ. By *Lemma 1.2.19*, we know that there is some NFA $\mathcal{A}'_{\varphi}$ over the alphabet Σ such that:

$$L(\mathcal{A}'_{\varphi}) = \{a_0 \ldots a_m \in \Sigma^* : (a_0, b_0^1, \ldots b_0^n) \ldots (a_m, b_m^1, \ldots, b_m^n) \in L(\mathcal{A}_{\varphi}), \text{ for some } b_i^j \in \{0,1\} \text{ with } 0 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n\}.$$

Then, we see that the language defined by φ is recognizable by $\mathcal{A}'_{\varphi}$, which concludes the proof of the direction from right to left of the double implication of the theorem.

Since we have demonstrated both implications, effectively building both conversions from automata to sentences and vice versa, we conclude that a language over the finite alphabet Σ is recognizable by an NFA if and only if it is MSO(Σ)-definable, as we were looking for.

In the context of infinite words and Büchi automata, the proof follows an analogous structure to the finite words case. The main difference is that, to establish the direction from right to left, we apply the closure properties of ω-regular languages instead of those of regular languages. ▲

Chapter 2

A Matter of Time

So far, we have defined languages, the class of regular languages, and we have presented substantially different yet equivalent characterizations for regular languages: using regular expressions, regular grammars, finite automata, and employing monadic second-order logic over words.

In this second chapter, we will focus on *star-free languages*, which can be viewed as a fragment of regular languages. Star-free languages are significant due to their equivalence or characterization by *Linear Temporal Logic* (LTL), a propositional modal logic introduced in [Pnu77], widely used in *Model Checking* in the context of *Formal Verification* (see, for example, [BK08; HV18]). Where in the first chapter we specifically looked at Büchi's Theorem, we will now delve into LTL and its completeness result. However, we cannot fail to mention the following results:

2.1 Star-free Languages and their Characterizations

Definition 2.1.1. *Given a finite alphabet Σ, we inductively define the* ***star-free languages*** *over Σ by:*

- *the empty language $\varnothing$ is star-free;*
- *the language $\{a\}$ is star-free for every $a \in \Sigma$;*
- *if L is a star-free language, then its complement $\overline{L} = \Sigma^* \backslash L$ is also a star-free language;*
- *if L and M are star-free languages, then the union language $L \cup M$ and the concatenation $L \cdot M$ are also star-free;*
- *no other language over Σ is star-free.*

Remember that *Lemma 1.2.14* has shown us that the languages recognized by deterministic finite automata are closed under complement. By the characterizations we have given, this result also tells us that regular languages are closed under complement. Then, by the previous definition, we deduce that every star-free language is a regular language.

Similar to what we had with regular and ω-regular languages, we can define star-free languages of infinite words by allowing the ω-power operation. We refer to [Lad77] for the first formalization of the star-free languages of infinite words.

We can now define the notion of *star-free expressions*. It is immediate to see that the class of star-free expressions is a fragment of the class of regular expressions:

Definition 2.1.2. *Given a finite alphabet* Σ*, we inductively define* ***star-free expressions*** *by:*

- *the constant* $\varnothing$*;*
- *the constant* ϵ*;*
- *the constant* $\mathbf{a}$*, for each* $a \in \Sigma$*;*

if R *and* S *are star-free expressions, then*

- $\overline{R}$ *is a star-free expression;*
- $(R|S)$ *is a star-free expression;*
- (RS) *is a star-free expression.*

Now we see that the constants $\varnothing$, ϵ and $\mathbf{a}$, for every $a \in \Sigma$, represent the empty language, $\{\epsilon\}$ and the language $\{a\}$, respectively. Also, if R and S are star-free expressions representing the (star-free) languages L and M, respectively, then $\overline{R}$ denotes the language $\overline{L}$, the expression $(R|S)$ represents the language $L \cup M$, and (RS) the language $L \cdot M$. This gives us a characterization of star-free languages by star-free expressions.

Example 2.1.3. Let $\Sigma = \{a, b\}$ be a finite alphabet. We will briefly show that the languages represented by the expressions $\mathbf{a}^*$ and $(\mathbf{ab})^*$ are star-free. Essentially, we need to see that they can be represented without using the Kleene star but complement. Specifically, we have:

$$\mathbf{a}^* = \overline{\Sigma^*\mathbf{b}\Sigma^*} = \overline{\overline{\varnothing}\mathbf{b}\overline{\varnothing}};$$
$$(\mathbf{ab})^* = \overline{\mathbf{b}\overline{\varnothing}|\overline{\varnothing}\mathbf{a}|\overline{\varnothing}\mathbf{aa}\overline{\varnothing}|\overline{\varnothing}\mathbf{bb}\overline{\varnothing}}.$$

We prove the first equality, and the second is done analogously. As a notation, when a word w belongs to the language expressed by some expression R, we denote it by $w \in R$. Suppose that $w \in \mathbf{a}^*$. We need to show that we also have $w \in \overline{\overline{\varnothing}\mathbf{b}\overline{\varnothing}}$:

Since the language expressed by $\mathbf{a}^*$ is $\{\epsilon, a, aa, aaa, \dots\}$, we know that w must be of the form $w = a \dots a$ with $|w| = n < \omega$. We consider $w = \epsilon$ if $n = 0$. For any value of n, it is clear that $w \notin \{b\}$, which is the language expressed by $\mathbf{b}$, nor does any sub-word of w, since w contains no letter b in any of its positions. Therefore, although trivially $w \in \overline{\varnothing} = \Sigma^*$, we conclude that $w \notin \overline{\varnothing}\mathbf{b}\overline{\varnothing}$. This gives us that $w \in \overline{\overline{\varnothing}\mathbf{b}\overline{\varnothing}}$, as we intended.

On the other hand, if $w \in \overline{\overline{\varnothing}\mathbf{b}\overline{\varnothing}}$, then $w \notin \overline{\varnothing}\mathbf{b}\overline{\varnothing}$. This means that w cannot contain any letter b. Otherwise, if w is of the form $w = w_1 b w_2$, for some sub-words w_1 and w_2, we would have $w \in \overline{\varnothing}\mathbf{b}\overline{\varnothing}$, as $w_1, w_2 \in \overline{\varnothing} = \Sigma^*$ and clearly $b \in \{b\}$. This is a contradiction, since we had $w \notin \overline{\varnothing}\mathbf{b}\overline{\varnothing}$. Since w does not contain any b, our word is of the form $w = a \dots a$, and so $w \in \mathbf{a}^*$, as we wanted.

Notice that intuitively, we are changing the perspective of the languages, saying that the language expressed by $\mathbf{a}^*$ is also the language of all words that do not have in any position any other letter different from a. In this sense, the language represented by $(\mathbf{ab})^*$ is equivalent to the languages whose words do not start with b, do not finish with a, and do not contain any sub-word of the form aa nor bb.

An example of a regular language that is not a star-free language is the one expressed by $(\mathbf{aa})^*$, for every finite alphabet containing the letter a. Note that the characterizations used in our previous example would not work, as they do not reflect the condition of having an even number of a's. We will not go into details, but to prove that $(\mathbf{aa})^*$ does not express a star-free language, it is useful to apply the characterization of star-free languages given by *aperiodic finite monoids*, introduced in [Sch65].

From the results of this paper, it can also be shown that star-free languages are exactly the regular languages L for which there exists a constant c_L such that for all words x, y, z and all integers $n \geqslant c_L$, we have

$$xy^n z \in L \Longleftrightarrow xy^{c_L} z \in L.$$

Thus, the star-free condition is equivalent to this *counter-free* or aperiodic condition, which also lightens a bit of the characterization of star-free languages by automata, as stated below. Furthermore, this aperiodicity contrasts with the Pumping Lemma, *Lemma 1.1.3*, which is weaker since it imposes a condition on the length of the words. This difference in the aperiodicity and repetitiveness between regular and star-free languages makes sense, considering that star-free languages are not closed under the Kleene star operation.

Given an automaton $\mathcal{A}$, we will define $L_{p,q}$, for p and q states of $\mathcal{A}$, as the (maybe empty) set of words w such that there is some (finite) path of $\mathcal{A}$ on w starting with the state p and ending with the state q. This notation will simplify the definition of the so-called *counter-free automata*:

Definition 2.1.4. *An NFA (Büchi automaton) $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ is called **counter-free**, or aperiodic, if, for every finite word —or sub-word— $w \in \Sigma^*$, we have that $w^n \in L_{p,p}$ implies $w \in L_{p,p}$ for all states $p \in Q$ and all $n \geqslant 1$.*

We claim that counter-free automata form the class of automata that accepts exactly the star-free languages. For proof and a more extensive presentation on counter-free automata, we refer to [MP71; Tho81].

Since regular grammars generate the class of regular languages, it would seem reasonable to have some subclass of regular grammars generating the star-free languages. Unfortunately, these specific grammars have not been widely discussed in the literature, as their other characterizations have been more useful and tractable. This is not central to our discussion, so we will not delve into the details either, but we refer interested readers to [MPR23; CGM78] for further information on the topic.

Analogously to regular languages, we can also characterize star-free languages using logic. Instead of MSO logic over words, now we will restrict ourselves to the *Monadic First-Order* (MFO) logic over words, which is defined equally to MSO but without allowing the use of second-order variables, so we do not have $\exists X \varphi$ nor $x \in X$ in our syntax. Proofs demonstrating how this logic captures the class of star-free languages can be found in [Lad77; MP71; Tho79; Tho90]. Similarly to the grammar case, there is little literature on the validities or on the axiomatization of MFO logic over words. However, we could address this gap considering the axiomatization of LTL as presented in *Definition 2.2.1*, using the translation from LTL-formulas to MFO-formulas provided in the proof of *Theorem 2.2.5*.

As mentioned, we will focus on the completeness of LTL, in the next sections. For a more comprehensive and exhaustive presentation on star-free languages and their equivalent characterizations, we recommend some short surveys on the topic such as [DG07; Pin95; Pin20], in addition to the references already suggested.

2.2 Linear Temporal Logic

As we will state in *Theorem 2.2.5*, Kamp's Theorem, Monadic First-Order logic over words —with a restriction on its variables— is as expressive as Linear Temporal Logic. Consequently, we obtain another characterization for star-free languages, using LTL. Besides its application in Model Checking, the crucial aspect for our purposes is that LTL is a propositional modal logic whose modal operators provide a linear and discrete notion of time. This allows us to describe the different states of a system over time and, in our context, the runs and successful runs produced by an automaton.

Let us first define the syntax and semantics of LTL. In general, we build LTL-formulas over a set of propositional variables. In practice, we will use the same finite alphabet notion as before.

Definition 2.2.1. *Let Σ be a set of propositional variables. The set of LTL-**formulas** over Σ, denoted by $\mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, is inductively generated by:*

- $\Sigma \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$;
- *if* $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, *then* $\neg\varphi$, $(\varphi \vee \psi)$, $\mathcal{X}\varphi$, $(\varphi\,\mathcal{U}\,\psi) \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$.

Notation. Similarly to how we simplified the notation to MSO, we denote the logic for the previous set of formulas as LTL. Strictly speaking, however, we should specify that we are referring to LTL with the operators $\mathcal{X}$ and $\mathcal{U}$.

We will give the operators $\mathcal{X}$ and $\mathcal{U}$ meaning with the following definition for the semantics of LTL. Under the LTL framework, we will work with infinite words over the alphabet $\mathcal{P}(\Sigma)$, so each letter of our words will be a subset of Σ. Also, instead of considering assignments as with the MSO and MSO_0 logics, we will only pick $i \in \mathbb{N}$. Each index i refers to a point in time, and the letter at position i represents the system state at that specific moment.

Definition 2.2.2. *Consider a set of propositional variables Σ, a formula $\phi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, and an infinite word $\sigma = a_0a_1a_2\cdots \in \mathcal{P}(\Sigma)^\omega$. We denote by $\sigma, i \vDash_{\mathsf{LTL}} \phi$ or simply $\sigma, i \vDash \phi$ the **satisfaction** of ϕ in σ under $i \in \mathbb{N}$. We inductively define it by, for $p \in \Sigma$ and $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:*

- $\sigma, i \vDash p :\Longleftrightarrow p \in a_i$;
- $\sigma, i \vDash \neg\varphi :\Longleftrightarrow \sigma, i \nvDash \varphi$;
- $\sigma, i \vDash (\varphi \vee \psi) :\Longleftrightarrow \sigma, i \vDash \varphi$ *or* $\sigma, i \vDash \psi$;
- $\sigma, i \vDash \mathcal{X}\varphi :\Longleftrightarrow \sigma, i+1 \vDash \varphi$;

- $\sigma, i \models (\varphi \mathcal{U} \psi) :\iff$ *there is* $j \geqslant i$ *such that* $\sigma, j \models \psi$ *and* $\sigma, k \models \varphi$ *for all* $i \leqslant k < j$.

If we have $\sigma, i \models_{\mathsf{LTL}} \varphi$ *for every* $\sigma \in \mathcal{P}(\Sigma)^\omega$ *and every* $i \in \mathbb{N}$*, then we say that* φ *is an (LTL-)****tautology****.*

Notation. If $\sigma, i \models_{\mathsf{LTL}} \varphi$ for all $i \in \mathbb{N}$, we abbreviate it by $\sigma \models_{\mathsf{LTL}} \varphi$. Also, if we have $\sigma \models_{\mathsf{LTL}} \varphi$ for all $\sigma \in \mathcal{P}(\Sigma)^\omega$, that is, φ is a tautology, we notate it by $\models_{\mathsf{LTL}} \varphi$. Furthermore, for $\Gamma \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, by $\sigma, i \models_{\mathsf{LTL}} \Gamma$ we mean that $\sigma, i \models_{\mathsf{LTL}} \gamma$ for all $\gamma \in \Gamma$.

In this way, we see that the intended meaning of $\mathcal{X}\varphi$ is to consider the satisfaction of φ in the *next* position of our word. And $(\varphi \mathcal{U} \psi)$ would express that φ holds *until* ψ holds.

From the connectives $\neg$ and $\vee$ we can syntactically define $\wedge$, $\rightarrow$ and $\leftrightarrow$ as usual. Again, to omit some parentheses we follow the standard preference hierarchy, with $\mathcal{U}$ having the highest priority among the binary connectives.

We can also define $\bot := p \wedge \neg p$ and $\top := \neg\bot$, for $p \in \Sigma$. We immediately see that $\bot$ is never satisfied, whereas $\top$ always holds. Moreover, we will consider two additional modal operators, for every $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:

$$\Diamond\varphi := \top \mathcal{U} \varphi; \qquad\qquad \Box\varphi := \neg\Diamond\neg\varphi.$$

By unraveling their definitions, we will have that for $\sigma \in \mathcal{P}(\Sigma)^\omega$ and $i \in \mathbb{N}$:

$$\begin{aligned} \sigma, i \models \Diamond\varphi &\iff \text{there is some } j \geqslant i \text{ such that } \sigma, j \models \varphi; \\ \sigma, i \models \Box\varphi &\iff \text{for every } j \geqslant i \text{ we have } \sigma, j \models \varphi. \end{aligned}$$

This is why we will call $\Diamond$ the *eventually* or *future* modality, and $\Box$ the *always* or *henceforth* modality.

Now, from a given LTL-formula, we will define a language over the alphabet $\mathcal{P}(\Sigma)$, similarly as we did for the MSO-formulas case:

Definition 2.2.3. *Given an LTL-formula* φ *over the set of propositional variables* Σ*, the* ***language defined by*** φ *is*

$$L_{LTL}(\varphi) := \{\sigma \in \mathcal{P}(\Sigma)^\omega : \sigma, 0 \models \varphi\}.$$

We say that a language $L \subseteq \mathcal{P}(\Sigma)^\omega$ *is LTL-****definable*** *if there is some LTL-formula* φ *such that* $L = L_{LTL}(\varphi)$.

Since we will use it in the completeness proof, it is worth mentioning that we can also understand the infinite words $\sigma \in \mathcal{P}(\Sigma)^\omega$ as the word of a *path* of a *Kripke structure*. We remind that a relation $R \subseteq A \times B$ over the sets A and B is called *left-total* if, for all $a \in A$, exists $b \in B$ such that $(a, b) \in R$.

Definition 2.2.4. *A* ***(serial) Kripke structure*** *over the set of propositional variables* Σ *is a tuple of the form* $K = (S, R, V)$*. The set* S *is a set of states or worlds,* $R \subseteq S \times S$ *is a left-total relation, and* $V : S \to \mathcal{P}(\Sigma)$ *is a valuation or interpretation. A* ***path*** *of* K *is a sequence* $\rho = \langle s_0, s_1, s_2, \dots \rangle$ *such that* $s_i \in S$ *and* $(s_i, s_{i+1}) \in R$ *for every* $i < \omega$*. We define the* ***word*** *of the path* ρ *as* $\sigma_\rho := V(s_0)V(s_1)V(s_2)\cdots \in \mathcal{P}(\Sigma)^\omega$.

Remark. Since our Kripke structures are serial, that is, the accessibility relations R are left-total, it is always possible to construct an (infinite) path of a Kripke structure, and so an infinite word of that path. Also, some authors, as can be seen in [HV18], include in the formalization of Kripke structures a subset $I \subseteq S$ of initial states, and if a path starts with an initial state, they refer to that path as a *run*.

We can easily see that every word $\sigma \in \mathcal{P}(\Sigma)^\omega$ can be given by a Kripke structure, while every Kripke structure provides us with at least one path and one correspondent word in $\mathcal{P}(\Sigma)^\omega$ for each state of the structure. For a path $\rho = \langle s_0, s_1, s_2, \dots \rangle$ of a Kripke structure K, if we define the suffix paths $\rho^i := \langle s_i, s_{i+1}, \dots \rangle$ for $i < \omega$ —notice that we have $\rho^0 = \rho$—, we can characterize the satisfiability notion of *Definition 2.2.2* also with the formalization

$$K, \rho^i \Vdash \varphi :\Longleftrightarrow \sigma_\rho, i \vDash \varphi;$$

for φ any LTL-formula. Unraveling the definitions, it is easy to check that we get the following equivalences for $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:

- $K, \rho^i \Vdash a \Longleftrightarrow a \in V(s_i)$, for every $a \in \Sigma$;
- $K, \rho^i \Vdash \neg\varphi \Longleftrightarrow K, \rho^i \nVdash \varphi$;
- $K, \rho^i \Vdash \varphi \vee \psi \Longleftrightarrow K, \rho^i \Vdash \varphi$ or $K, \rho^i \Vdash \psi$;
- $K, \rho^i \Vdash \mathcal{X}\varphi \Longleftrightarrow K, \rho^{i+1} \Vdash \varphi$;
- $K, \rho^i \Vdash \varphi\, \mathcal{U}\, \psi \Longleftrightarrow$ there is $j \geqslant i$ so that $K, \rho^j \Vdash \psi$ and $K, \rho^k \Vdash \varphi$ for all $i \leqslant k < j$;
- $K, \rho^i \Vdash \Diamond\varphi \Longleftrightarrow$ there is some $j \geqslant i$ such that $K, \rho^j \Vdash \varphi$;
- $K, \rho^i \Vdash \Box\varphi \Longleftrightarrow$ for every $j \geqslant i$ we have $K, \rho^j \Vdash \varphi$.

The following theorem, usually called Kamp's Theorem, is an improvement of Kamp's dissertation results in [Kam68] (see [Rab14] for a simpler and shorter treatment), and connects LTL with the MFO logic over words. Before stating it, we need to make some minor clarifications:

Although we originally defined it only for sentences, in the proof of Büchi's Theorem we have already seen that languages can be defined using formulas —not sentences— by extending the alphabet. Also, as we mentioned, the formalization of MFO is similar to the MSO one, without the second-order character. However, since we are now working over words $\sigma = p_0 p_1 p_2 \cdots \in \mathcal{P}(\Sigma)^\omega$ instead of words in Σ^ω, the predicates Q_a^σ, for $a \in \Sigma$, of our word models will slightly change to $Q_a^\sigma := \{i \in dom(\sigma) = \mathbb{N} : a \in p_i\}$.

Theorem 2.2.5 (Gabbay et al. [Gab+80], Kamp [Kam68])**.**

i) For every LTL-formula φ, there is some MFO-formula φ^{MFO} with a single free variable such that $L_{LTL}(\varphi) = L^\omega(\varphi^{MFO})$.

ii) For every MFO-formula ψ with a single free variable, there is some LTL-formula ψ^{LTL} such that $L^\omega(\psi) = L_{LTL}(\psi^{LTL})$.

Proof. We will only state a translation from LTL-formulas to MFO-formulas with one free variable, which ensures the first part of the theorem. That translation is given inductively for $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, with Σ a set of propositional variables, as follows:

- $a^{MFO} := Q_a(x)$, for every $a \in \Sigma$;
- $(\neg\varphi)^{MFO} := \neg\varphi^{MFO}$;
- $(\varphi \vee \psi)^{MFO} := \varphi^{MFO} \vee \psi^{MFO}$;
- $(\mathcal{X}\varphi)^{MFO} := \exists y(x < y \wedge \forall z(z \leqslant x \vee y \leqslant z) \wedge \varphi^{MFO}(y))$;
- $(\varphi\,\mathcal{U}\,\psi)^{MFO} := \exists y \left(x \leqslant y \wedge \psi^{MFO}(y) \wedge \forall z(x \leqslant z \wedge z < y \wedge \to \varphi^{MFO}(z))\right)$.

We claim that the translation verifies, for every word $\sigma \in \mathcal{P}(\Sigma)^\omega$ and all $i \in \mathbb{N}$:

$$\sigma, i \vDash_{\mathsf{LTL}} \varphi \iff \underline{\sigma} \vDash^{\alpha}_{MFO} \varphi^{MFO}(x) \text{ for every (MFO-)assignment } \alpha \text{ with } \alpha(x) = i.$$

This would immediately give us *i)*.

Note that in the presented translation, we have only one free variable and at most two additional bound variables. Therefore, we deduce that MFO logic, without restrictions on its variables, is more expressive than LTL, which can be translated or reduced into a fragment of MFO logic using only three variables, with only one being free. This suggests, as indeed it does, that the converse translation —from MFO-formulas with one free variable to LTL-formulas—, needed to prove *ii)*, is not as straightforward as the previous one.

For complete proofs of the theorem, we refer to [Gab+80], which also simplifies Kamp's original proof, and to [Hod95]. This last article follows [Gab+80] but uses game theory tactics. ▲

2.2.1 Axiomatization of LTL

The following definition is based on [Gab+80], where the first explicit axiomatization of LTL is presented, and [Bur82; KM08]:

Definition 2.2.6. *Let Σ be a set of propositional variables. We define the* ***axiomatization*** *of LTL as the set of LTL-formulas given, for every $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, by:*
Axiom Schemes:

$$\textit{Any classical propositional tautology;} \tag{prop}$$
$$\mathcal{X}(\varphi \to \psi) \to (\mathcal{X}\varphi \to \mathcal{X}\psi); \tag{$K_{\mathcal{X}}$}$$
$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi); \tag{$K_{\Box}$}$$
$$\neg\mathcal{X}\varphi \leftrightarrow \mathcal{X}\neg\varphi; \tag{Lin}$$
$$\Box(\varphi \to \mathcal{X}\varphi) \to (\varphi \to \Box\varphi); \tag{Ind}$$
$$\varphi\,\mathcal{U}\,\psi \to \Diamond\psi; \tag{U1}$$
$$\varphi\,\mathcal{U}\,\psi \leftrightarrow (\psi \vee (\varphi \wedge \mathcal{X}(\varphi\,\mathcal{U}\,\psi))). \tag{U2}$$

Inference Rules:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\,MP \qquad \frac{\varphi}{\mathcal{X}\varphi}\,N_{\mathcal{X}} \qquad \frac{\varphi}{\Box\varphi}\,N_{\Box}$$

The formulas that can be obtained by repeatedly applying the presented axioms and rules are called ***theorems*** *of LTL. If φ is a theorem of LTL, we denote it by $\vdash_{\mathsf{LTL}} \varphi$.*

Observe that ($K_{\mathcal{X}}$) and ($K_{\Box}$) can be understood as the distributive property of $\mathcal{X}$ and $\Box$ over the connective $\rightarrow$, respectively. Axiom (Lin) would give us a kind of linearity for $\mathcal{X}$. Axiom (Ind) provides us with an induction principle, saying "If the satisfaction of φ implies that it always holds in the next state (position), then the satisfaction of φ implies its satisfaction in all future states". Finally, axioms (U1) and (U2) express the behavior of the binary operator $\mathcal{U}$. Regarding the inference rules, we have the well-known *Modus Ponens* (MP), and the *Necessitation* rules for both $\mathcal{X}$ and $\Box$.

The presented axiomatization of LTL is likely the easiest to work with, but it can be improved a bit. In *Appendix A* we discuss a variation that, through a syntactical relation we define, reduces the number of required axioms.

Notation. In the following proofs and examples, when we need to call to a particular instance of a propositional tautology to immediately apply MP over it and some previous formula, we will usually only state the conclusion of that MP, and abbreviate its justification with the notation (prop)+i.(+...). Index i is the number labeling the previous formula we need to use in the MP application. By adding more labels, we refer to the concatenation of multiple MP applications over the corresponding formulas.

Similarly to the standard Classical Propositional Logic framework, we will have the *Substitution Lemma*, or replacement lemma, for LTL-formulas:

Definition 2.2.7. *Let Σ be a set of propositional variables. Consider $\phi_1, \phi_2 \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. We say that ϕ_1 and ϕ_2 are (LTL)-**equivalent** if $\vdash_{\mathsf{LTL}} \phi_1 \leftrightarrow \phi_2$.*

Lemma 2.2.8 (Substitution Lemma for LTL)**.** *Let Σ be a set of propositional variables. Consider $\phi_1, \phi_2, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. If ϕ_1 and ϕ_2 are equivalent, then:*

$$\vdash_{\mathsf{LTL}} \psi[a \leftarrow \phi_1] \leftrightarrow \psi[a \leftarrow \phi_2];$$

where $a \in \Sigma$, and the notation $\psi[a \leftarrow \phi]$ refers to the formula obtained by substituting all occurrences of a in ψ by ϕ.

Proof. By induction on the construction of the formula ψ. If ψ is a propositional formula, then it is clear since we have:

1.	$\phi_1 \leftrightarrow \phi_2$;	by hypothesis
2.	$\phi_1 \leftrightarrow \phi_2 \rightarrow \psi[a \leftarrow \phi_1] \leftrightarrow \psi[a \leftarrow \phi_2]$;	instance of a prop. tautology
3.	$\psi[a \leftarrow \phi_1] \leftrightarrow \psi[a \leftarrow \phi_2]$.	MP over 1. and 2.

That is, we take advantage of the classical propositional Substitution Lemma.

If ψ is of the form $\psi = \mathcal{X}\psi'$, then, by Induction Hypothesis, we know:

$$\vdash_{\mathsf{LTL}} \psi'[a \leftarrow \phi_1] \leftrightarrow \psi'[a \leftarrow \phi_2].$$

To simplify notation, we consider $\phi_1' := \psi'[a \leftarrow \phi_1]$ and $\phi_2' := \psi'[a \leftarrow \phi_2]$. We have:

1.	$\phi_1' \leftrightarrow \phi_2'$;	by Induction Hypothesis
2.	$\phi_1' \rightarrow \phi_2'$;	(prop)+1.
3.	$\phi_2' \rightarrow \phi_1'$;	(prop)+1.
4.	$\mathcal{X}(\phi_1' \rightarrow \phi_2')$;	$N_{\mathcal{X}}$ over 2.

5. $\mathcal{X}(\phi_2' \to \phi_1')$; $N_{\mathcal{X}}$ over 3.
6. $\mathcal{X}(\phi_1' \to \phi_2') \to (\mathcal{X}\phi_1' \to \mathcal{X}\phi_2')$; instance of ($K_{\mathcal{X}}$)
7. $\mathcal{X}(\phi_2' \to \phi_1') \to (\mathcal{X}\phi_2' \to \mathcal{X}\phi_1')$; instance of ($K_{\mathcal{X}}$)
8. $\mathcal{X}\phi_1' \to \mathcal{X}\phi_2'$; MP over 4. and 6.
9. $\mathcal{X}\phi_2' \to \mathcal{X}\phi_1'$; MP over 5. and 7.
10. $\mathcal{X}\phi_1' \leftrightarrow \mathcal{X}\psi_2'$. (prop)+8.+9.

Thus, $\vdash_{\mathsf{LTL}} \mathcal{X}\phi_1' \leftrightarrow \mathcal{X}\psi_2'$, which means that we have $\vdash_{\mathsf{LTL}} \mathcal{X}\psi'[a \leftarrow \phi_1] \leftrightarrow \mathcal{X}\psi'[a \leftarrow \phi_2]$. That is, we have shown $\vdash_{\mathsf{LTL}} \psi[a \leftarrow \phi_1] \leftrightarrow \psi[a \leftarrow \phi_2]$, what we wanted.

The last case is $\psi = \psi' \,\mathcal{U}\, \psi''$. We need to derive, considering ϕ_1 and ϕ_2 to be equivalent, the theorem

$$\vdash_{\mathsf{LTL}} (\psi' \,\mathcal{U}\, \psi'')[a \leftarrow \phi_1] \leftrightarrow (\psi' \,\mathcal{U}\, \psi'')[a \leftarrow \phi_2].$$

Maintaining the notations $\phi_i' = \psi'[a \leftarrow \phi_i]$ and $\phi_i'' = \psi''[a \leftarrow \phi_i]$ for $i \in \{1, 2\}$, we also consider $\mu \coloneqq \phi_1' \,\mathcal{U}\, \phi_1'' \wedge \neg(\phi_2' \,\mathcal{U}\, \phi_2'')$. Due to space limitations, we first state the derived theorems, and then we list their justifications:

1. $\phi_1' \leftrightarrow \phi_2'$;
2. $\phi_1'' \leftrightarrow \phi_2''$;
3. $\phi_1' \,\mathcal{U}\, \phi_1'' \to (\phi_1'' \vee (\phi_1' \wedge \mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1'')))$;
4. $(\phi_2'' \vee (\phi_2' \wedge \mathcal{X}(\phi_2' \,\mathcal{U}\, \phi_2''))) \to \phi_2' \,\mathcal{U}\, \phi_2''$;
5. $\mu \to \big[(\phi_1'' \vee (\phi_1' \wedge \mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1''))) \wedge \neg(\phi_2'' \vee (\phi_2' \wedge \mathcal{X}(\phi_2' \,\mathcal{U}\, \phi_2'')))\big]$;
6. $\mu \to \big[(\phi_1'' \vee (\phi_1' \wedge \mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1''))) \wedge \neg\phi_2'' \wedge (\neg\phi_2' \vee \neg\mathcal{X}(\phi_2' \,\mathcal{U}\, \phi_2''))\big]$;
7. $\neg\phi_2'' \to \neg\phi_1''$;
8. $\mu \to \big[(\phi_1'' \vee (\phi_1' \wedge \mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1''))) \wedge \neg\phi_1'' \wedge (\neg\phi_2' \vee \neg\mathcal{X}(\phi_2' \,\mathcal{U}\, \phi_2''))\big]$;
9. $\mu \to \big[\phi_1' \wedge \mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1'') \wedge \neg\phi_1'' \wedge (\neg\phi_2' \vee \neg\mathcal{X}(\phi_2' \,\mathcal{U}\, \phi_2''))\big]$;
10. $\mu \to \big[\phi_1' \wedge \mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1'') \wedge \neg\phi_1'' \wedge \neg\mathcal{X}(\phi_2' \,\mathcal{U}\, \phi_2'')\big]$;
11. $\mu \to \big(\mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1'') \wedge \neg\mathcal{X}(\phi_2' \,\mathcal{U}\, \phi_2'')\big)$;
12. $\neg\mathcal{X}(\phi_2' \,\mathcal{U}\, \phi_2'') \leftrightarrow \mathcal{X}\neg(\phi_2' \,\mathcal{U}\, \phi_2'')$;
13. $\mu \to \big(\mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1'') \wedge \mathcal{X}\neg(\phi_2' \,\mathcal{U}\, \phi_2'')\big)$;
14. $\big(\mathcal{X}(\phi_1' \,\mathcal{U}\, \phi_1'') \wedge \mathcal{X}\neg(\phi_2' \,\mathcal{U}\, \phi_2'')\big) \to \mathcal{X}\mu$;
15. $\mu \to \mathcal{X}\mu$;
16. $\Box(\mu \to \mathcal{X}\mu)$;
17. $\mu \to \Box\mu$;
18. $\Box\mu \to (\Box(\phi_1' \,\mathcal{U}\, \phi_1'') \wedge \Box\neg(\phi_2' \,\mathcal{U}\, \phi_2''))$;
19. $\mu \to \big(\Box(\phi_1' \,\mathcal{U}\, \phi_1'') \wedge \Box\neg(\phi_2' \,\mathcal{U}\, \phi_2'')\big)$;
20. $\mu \to \Box\neg(\phi_2' \,\mathcal{U}\, \phi_2'')$;
21. $\phi_2'' \to \phi_2' \,\mathcal{U}\, \phi_2''$;
22. $\neg(\phi_2' \,\mathcal{U}\, \phi_2'') \to \neg\phi_2''$;
23. $\Box(\neg(\phi_2' \,\mathcal{U}\, \phi_2'') \to \neg\phi_2'')$;
24. $\Box\neg(\phi_2' \,\mathcal{U}\, \phi_2'') \to \Box\neg\phi_2''$;
25. $\mu \to \Box\neg\phi_2''$;
26. $\phi_1' \,\mathcal{U}\, \phi_1'' \to \Diamond\phi_1''$;

27. $\mu \to (\Box\neg\phi''_2 \wedge \Diamond\phi''_1)$;
28. $\Box(\phi''_1 \to \phi''_2)$;
29. $\Box(\phi''_1 \to \phi''_2) \to (\Diamond\phi''_1 \to \Diamond\phi''_2)$;
30. $\Diamond\phi''_1 \to \Diamond\phi''_2$;
31. $\mu \to (\Box\neg\phi''_2 \wedge \Diamond\phi''_2)$;
32. $(\Box\neg\phi''_2 \wedge \Diamond\phi''_2) \to \bot$;
33. $\mu \to \bot$;
34. $\neg\mu$;
35. $\phi'_1\,\mathcal{U}\,\phi''_1 \to \phi'_2\,\mathcal{U}\,\phi''_2$.

Each derivation is given by:

1. Induction Hypothesis
2. Induction Hypothesis
3. Instance of axiom (U2)
4. Instance of axiom (U2)
5. (prop)+3.+4.
6. (prop)+5.
7. (prop)+2.
8. (prop)+6.+7.
9. (prop)+8.
10. (prop)+9.
11. (prop)+10.
12. Instance of (Lin)
13. (prop)+11.+12.
14. See *Example 2.2.10*
15. (prop)+13.+14.
16. Application of rule $N_\Box$ over 15.
17. MP over 16. and an instance of axiom (Ind)
18. Theorem of $\mathbf{K}_\Box$, and so of LTL, as explained below
19. (prop)+17.+18.
20. (prop)+19.
21. (prop)+4.
22. (prop)+21.
23. Application of rule $N_\Box$ over 22.
24. MP over 23. and an instance of axiom ($K_\Box$)
25. (prop)+20.+24.
26. Instance of axiom (U1)
27. (prop)+25.+26.
28. Application of rule $N_\Box$ over 2.
29. Theorem of $\mathbf{K}_\Box$, and so of LTL, as explained below
30. MP over 28. and 29.

31. (prop)+27.+30.
32. Instance of a propositional tautology, considering the definition of $\diamond$
33. (prop)+31.+32.
34. (prop)+33.
35. (prop)+34.

Then, we have concluded the theorem of LTL

$$\vdash_{\mathsf{LTL}} \phi_1' \,\mathcal{U}\, \phi_1'' \to \phi_2' \,\mathcal{U}\, \phi_2''.$$

Interchanging the roles of ϕ_1' and ϕ_2', and of ϕ_1'' and ϕ_2'', we deduce the other implication, so we have the equivalence

$$\vdash_{\mathsf{LTL}} \phi_1' \,\mathcal{U}\, \phi_1'' \leftrightarrow \phi_2' \,\mathcal{U}\, \phi_2''.$$

For every formula ϕ, we know that

$$(\psi' \,\mathcal{U}\, \psi'')[a \leftarrow \phi] = \psi'[a \leftarrow \phi] \,\mathcal{U}\, \psi''[a \leftarrow \phi],$$

so the previous equivalence is indeed the theorem

$$\vdash_{\mathsf{LTL}} (\psi' \,\mathcal{U}\, \psi'')[a \leftarrow \phi_1] \leftrightarrow (\psi' \,\mathcal{U}\, \psi'')[a \leftarrow \phi_2],$$

which provides us with the property we were looking for in the case $\psi = \psi' \,\mathcal{U}\, \psi''$.

In conclusion, we have proved, by induction on the construction of ψ, that if ϕ_1 and ϕ_2 are equivalent, then:

$$\vdash_{\mathsf{LTL}} \psi[a \leftarrow \phi_1] \leftrightarrow \psi[a \leftarrow \phi_2].$$

▲

Therefore, if we know that $\psi(\phi_1)$ and $\phi_1 \leftrightarrow \phi_2$ are theorems of LTL for $\psi, \phi_1, \phi_2 \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, then we have:

1.	$\psi(\phi_1)$;	assumed
2.	$\phi_1 \leftrightarrow \phi_2$;	assumed
3.	$\psi(\phi_1) \leftrightarrow \psi(\phi_2)$;	Substitution Lemma using ψ and 2.
4.	$\psi(\phi_1) \to \psi(\phi_2)$;	(prop)+3.
5.	$\psi(\phi_2)$.	MP over 3. and 4.

In our derivations, we will skip these steps by stating only the conclusion, the formula of 5. in this case. We will justify the step by "Substitution Lemma on i. by j.", where i. would be the label for the formula of the form $\psi(\phi_1)$, and j. the label for the equivalence of the form $\phi_1 \leftrightarrow \phi_2$. Note that if ψ is propositional, we could simply state "(prop)+i.+j.".

Example 2.2.9. We can show that, for φ an LTL-formula, we have

$$\vdash_{\mathsf{LTL}} \square\varphi \to (\varphi \wedge \mathcal{X}\square\varphi). \qquad (\square\mathrm{Unr})$$

To verify this, we must first remember that $\square\varphi$ is an abbreviation of $\neg(\top \,\mathcal{U}\, \neg\varphi)$. We consider the following sequence of theorems of LTL:

1. $\top \,\mathcal{U}\, \neg\varphi \leftrightarrow (\neg\varphi \vee (\top \wedge \mathcal{X}(\top \,\mathcal{U}\, \neg\varphi)))$; instance of axiom (U2)

2. $\top\mathcal{U}\neg\varphi \leftrightarrow (\neg\varphi \vee \mathcal{X}(\top\mathcal{U}\neg\varphi))$;	(prop)+1.
3. $(\neg\varphi \vee \mathcal{X}(\top\mathcal{U}\neg\varphi)) \to \top\mathcal{U}\neg\varphi$;	(prop)+2.
4. $\neg(\top\mathcal{U}\neg\varphi) \to \neg(\neg\varphi \vee \mathcal{X}(\top\mathcal{U}\neg\varphi))$;	(prop)+3.
5. $\neg(\top\mathcal{U}\neg\varphi) \to (\neg\neg\varphi \wedge \neg\mathcal{X}(\top\mathcal{U}\neg\varphi))$;	(prop)+4.
6. $\varphi \leftrightarrow \neg\neg\varphi$;	instance of prop. tautology
7. $\neg(\top\mathcal{U}\neg\varphi) \to (\varphi \wedge \neg\mathcal{X}(\top\mathcal{U}\neg\varphi))$;	Substitution Lemma on 5. by 6.
8. $\neg\mathcal{X}(\top\mathcal{U}\neg\varphi) \leftrightarrow \mathcal{X}\neg(\top\mathcal{U}\neg\varphi)$;	instance of axiom (Lin)
9. $\neg(\top\mathcal{U}\neg\varphi) \to (\varphi \wedge \mathcal{X}\neg(\top\mathcal{U}\neg\varphi))$;	Substitution Lemma on 7. by 8.
10. $\Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi)$.	by 9. and the definition of $\Box$

Then, we conclude that $\vdash_{\mathsf{LTL}} \Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi)$. Semantically, this theorem of LTL could be understood as an unraveling or a definition of the operator $\Box$. It is worth mentioning that this theorem is often presented as an axiom [Bur82; Gab+80; KM08], since different formalizations may adopt varying definitions for $\Box$.

We take the opportunity to also derive the converse implication:

1. $\Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi)$;	instance of ($\Box$Unr)
2. $\mathcal{X}(\Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi))$;	$N_{\mathcal{X}}$ over 1.
3. $\mathcal{X}(\Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi)) \to (\mathcal{X}\Box\varphi \to \mathcal{X}(\varphi \wedge \mathcal{X}\Box\varphi))$;	instance of ($K_{\mathcal{X}}$)
4. $\mathcal{X}\Box\varphi \to \mathcal{X}(\varphi \wedge \mathcal{X}\Box\varphi)$;	MP over 2. and 3.
5. $(\varphi \wedge \mathcal{X}\Box\varphi) \to \mathcal{X}(\varphi \wedge \mathcal{X}\Box\varphi)$;	(prop)+4.
6. $\Box[(\varphi \wedge \mathcal{X}\Box\varphi) \to \mathcal{X}(\varphi \wedge \mathcal{X}\Box\varphi)]$;	$N_{\Box}$ over 5.
7. $\Box[(\varphi \wedge \mathcal{X}\Box\varphi) \to \mathcal{X}(\varphi \wedge \mathcal{X}\Box\varphi)] \to ((\varphi \wedge \mathcal{X}\Box\varphi) \to \Box\varphi)$;	instance of (Ind)
8. $(\varphi \wedge \mathcal{X}\Box\varphi) \to \Box\varphi$.	MP over 6. and 7.

Recall that a modal logic $\mathbf{L}_{\overline{\Box}}$ with the unary operator $\overline{\Box}$ is called *normal* if:

- $\mathbf{L}_{\overline{\Box}}$ contains all propositional tautologies;
- $\mathbf{L}_{\overline{\Box}}$ contains all formulas of the form $\overline{\Box}(\varphi \to \psi) \to (\overline{\Box}\varphi \to \overline{\Box}\psi)$, for modal formulas φ and ψ;
- $\mathbf{L}_{\overline{\Box}}$ is closed under substitutions of variables;
- $\mathbf{L}_{\overline{\Box}}$ is closed under the rule MP; and
- $\mathbf{L}_{\overline{\Box}}$ is closed under the necessitation rule of $\overline{\Box}$.

For a normal modal logic $\mathbf{L}_{\overline{\Box}}$, we also define the operator $\overline{\Diamond}$ by $\overline{\Diamond}\varphi := \neg\overline{\Box}\neg\varphi$, for φ a modal formula.

Now observe that LTL is, in fact, a *bi-modal logic*, as we have defined two not inter-definable modal operators, $\mathcal{X}$ and $\Box$, besides the $\mathcal{U}$ and the $\Diamond$ operators related to $\Box$. Moreover, the fragments of LTL with the operators $\overline{\Box} = \mathcal{X}$ and $\overline{\Box} = \Box$ will be normal. This is clear considering we have the axioms (prop), ($K_{\mathcal{X}}$) and ($K_{\Box}$), and the inference rules MP, $N_{\mathcal{X}}$ and $N_{\Box}$. Note that, by axiom (Lin), we know that $\neg\mathcal{X}\neg\varphi$ is equivalent to $\mathcal{X}\neg\neg\varphi$, which is also equivalent to $\mathcal{X}\varphi$. Then, for $\overline{\Box} = \mathcal{X}$, the corresponding operator $\overline{\Diamond}$ defined by $\overline{\Diamond}\varphi = \neg\mathcal{X}\neg\varphi$ is $\mathcal{X}$ itself. No need to mention that, for $\overline{\Box} = \Box$, we have $\overline{\Diamond} = \Diamond$.

Given this, we can embed or derive within LTL all the theorems of the modal logic $\mathbf{K}_{\overline{\square}}$, for both cases $\overline{\square} = \mathcal{X}$ and $\overline{\square} = \square$. In fact, for the case of the $\square$ operator, we can derive all the theorems of the logic **S4**, as it is not difficult to prove that $\vdash_{\mathsf{LTL}} \square\varphi \to \varphi$ and $\vdash_{\mathsf{LTL}} \square\varphi \to \square\square\varphi$:

1.	$\square\varphi \to (\varphi \wedge \mathcal{X}\square\varphi)$;	instance of ($\square$Unr)
2.	$\square\varphi \to \varphi$.	(prop)+1.

So $\vdash_{\mathsf{LTL}} \square\varphi \to \varphi$, and:

1.	$\square\varphi \to (\varphi \wedge \mathcal{X}\square\varphi)$;	instance of ($\square$Unr)
2.	$\square\varphi \to \mathcal{X}\square\varphi$;	(prop)+1.
3.	$\square(\square\varphi \to \mathcal{X}\square\varphi)$;	$N_\square$ over 2.
4.	$\square(\square\varphi \to \mathcal{X}\square\varphi) \to (\square\varphi \to \square\square\varphi)$;	instance of axiom (Ind)
5.	$\square\varphi \to \square\square\varphi$.	MP over 3. and 4.

Leading us to conclude that $\vdash_{\mathsf{LTL}} \square\varphi \to \square\square\varphi$.

To illustrate the usefulness of our findings, note that the theorem derived in the following example can now be readily accepted, given that it is derivable within logic $\mathbf{K}_{\mathcal{X}}$.

Example 2.2.10. We will explicitly prove that, for every $\varphi, \psi \in \mathtt{Fm}_{LT(\Sigma)}$, we have

$$\vdash_{\mathsf{LTL}} \neg(\mathcal{X}\varphi \to \neg\mathcal{X}\psi) \to \mathcal{X}\neg(\varphi \to \neg\psi).$$

We consider the following sequence:

1.	$\mathcal{X}(\varphi \to \neg\psi) \to (\mathcal{X}\varphi \to \mathcal{X}\neg\psi)$;	instance of axiom ($K_{\mathcal{X}}$)
2.	$\neg\mathcal{X}\psi \leftrightarrow \mathcal{X}\neg\psi$;	instance of axiom (Lin)
3.	$\mathcal{X}(\varphi \to \neg\psi) \to (\mathcal{X}\varphi \to \neg\mathcal{X}\psi)$;	Substitution Lemma on 1. by 2.
4.	$\neg(\mathcal{X}\varphi \to \neg\mathcal{X}\psi) \to \neg\mathcal{X}(\varphi \to \neg\psi)$;	(prop)+3.
5.	$\neg\mathcal{X}(\varphi \to \neg\psi) \leftrightarrow \mathcal{X}\neg(\varphi \to \neg\psi)$;	instace of (Lin)
6.	$\neg(\mathcal{X}\varphi \to \neg\mathcal{X}\psi) \to \mathcal{X}\neg(\varphi \to \neg\psi)$.	Substitution Lemma on 4. by 5.

That is, we have shown $\vdash_{\mathsf{LTL}} \neg(\mathcal{X}\varphi \to \neg\mathcal{X}\psi) \to \mathcal{X}\neg(\varphi \to \neg\psi)$, as we wanted. Notice that, by the definition of the connective $\wedge$, the given theorem is equivalent to

$$\vdash_{\mathsf{LTL}} \mathcal{X}\varphi \wedge \mathcal{X}\psi \to \mathcal{X}(\varphi \wedge \psi).$$

2.2.2 Completeness of LTL

We seek a completeness result for the syntactical axiomatization introduced in *Definition 2.2.6* with respect to the semantic satisfaction given in *Definition 2.2.2*. This will be formalized in *Theorem 2.2.24*. To demonstrate this result, we will adapt the standard method to prove completeness: the soundness implication is shown directly by induction, and the completeness implication by contraposition using a *canonical model* built from *maximal consistent sets*. However, we first need to introduce some definitions and lemmas:

Definition 2.2.11. *A set of LTL-formulas* Γ *is (LTL-)**inconsistent** if* $\vdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma_0} \neg\gamma$ *for some finite* $\Gamma_0 \subseteq \Gamma$*. Otherwise,* Γ *is **consistent**. An LTL-formula* φ *is considered inconsistent (consistent) if* $\{\varphi\}$ *is so.*

We can establish some characterizations for inconsistency and consistency, which will be useful to work with:

Lemma 2.2.12. *The set of LTL-formulas* Γ *is inconsistent if and only if*

$$\vdash_{\mathsf{LTL}} \bigwedge_{\gamma\in\Gamma_0} \gamma \to \bot \quad \textit{for some finite } \Gamma_0 \subseteq \Gamma.$$

Proof. Immediate from the definition: by instances of propositional tautologies, we know that $\bigvee_{\gamma\in\Gamma_0} \neg\gamma$, for some finite $\Gamma_0 \subseteq \Gamma$, is equivalent to ($\neg \bigwedge_{\gamma\in\Gamma_0} \gamma$ and to) $\bigwedge_{\gamma\in\Gamma_0} \gamma \to \bot$. That is, Γ is inconsistent if and only if $\vdash_{\mathsf{LTL}} \bigwedge_{\gamma\in\Gamma_0} \gamma \to \bot$. ▲

Since it also holds in Classical Propositional Logic, it is easy to prove by instances of propositional tautologies, that we also have the so-called *Principle of Contradiction*, or *Principle of Inconsistency*, in LTL: $\bot$ is equivalent to the conjunction of any LTL-formula and its negation. Then, from the previous lemma and the Principle of Contradiction, we deduce that if Γ is inconsistent, then

$$\vdash_{\mathsf{LTL}} \bigwedge_{\gamma\in\Gamma_0} \gamma \to (\varphi \wedge \neg\varphi),$$

for some finite $\Gamma_0 \subseteq \Gamma$ and every LTL-formula φ. On the other hand, we will also have that if

$$\vdash_{\mathsf{LTL}} \bigwedge_{\gamma\in\Gamma_0} \gamma \to (\varphi \wedge \neg\varphi),$$

for some finite $\Gamma_0 \subseteq \Gamma$ and some LTL-formula φ, then Γ is inconsistent.

Lemma 2.2.13. *If* Γ *is a consistent set of LTL-formulas, then* $\nvdash_{\mathsf{LTL}} \neg\gamma$ *for every* $\gamma \in \Gamma$.

Proof. Immediate by *reductio ad absurdum*: consider Γ to be a consistent set, and let $\gamma' \in \Gamma$ be such that $\vdash_{\mathsf{LTL}} \neg\gamma'$. Since $\{\gamma'\}$ is a finite subset of Γ and we have assumed $\vdash_{\mathsf{LTL}} \neg\gamma'$, we get that Γ has to be inconsistent, contradicting our hypothesis. ▲

Observe that the converse implication of the previous lemma is trivially true for single formulas: if $\nvdash_{\mathsf{LTL}} \neg\gamma$, then γ is consistent. However, it does not hold for sets of more than one formula. For example, consider the set $\Gamma = \{p, \neg p\}$, for p a propositional variable. We have $\nvdash_{\mathsf{LTL}} \neg p$ and $\nvdash_{\mathsf{LTL}} \neg\neg p$, but Γ is clearly not consistent since $\vdash_{\mathsf{LTL}} \neg p \vee \neg\neg p$.

Lemma 2.2.14. *A finite set of LTL-formulas* Γ *is inconsistent if and only if* $\vdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma} \neg\gamma$. *Dually, a finite* Γ *is consistent if and only if* $\nvdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma} \neg\gamma$.

Proof. If Γ is a finite inconsistent set, then there is some $\Gamma_0 \subseteq \Gamma$ such that $\vdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma_0} \neg\gamma$. By instances of propositional tautologies, we easily deduce $\vdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma} \neg\gamma$. On the other hand, if $\vdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma} \neg\gamma$, as we trivially know that $\Gamma \subseteq \Gamma$, by definition we get that Γ is inconsistent.

The dual statement does not need proof, as it is the contraposition of what we have just shown, but we will demonstrate it directly anyway: if Γ is a finite consistent set, then

$$\nvdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma_0} \neg\gamma \quad \text{for every } \Gamma_0 \subseteq \Gamma.$$

In particular, we have $\nvdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma} \neg\gamma$. Conversely, if $\nvdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma} \neg\gamma$, then $\nvdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma_0} \neg\gamma$ for every $\Gamma_0 \subseteq \Gamma$. Otherwise, by instances of propositional tautologies, we would get $\vdash_{\mathsf{LTL}} \bigvee_{\gamma\in\Gamma} \neg\gamma$, a contradiction. Therefore, we conclude that Γ has to be consistent. ▲

The following lemma is almost immediate, but it is worth mentioning as it will simplify some of our proofs, and serves as a generalization of the previous lemma:

Lemma 2.2.15. *If a set of LTL-formulas* Γ *is consistent, then every subset* $\Gamma' \subseteq \Gamma$ *is consistent. Dually, if the set of LTL-formulas* Δ *is inconsistent, then every* $\Delta' \supseteq \Delta$ *is inconsistent.*

Proof. Let Γ be a consistent set and let Γ' be a subset of Γ. By the consistency of Γ, we know $\nvdash_{\mathsf{LTL}} \bigvee_{\gamma \in \Gamma_0} \neg\gamma$ for every finite $\Gamma_0 \subseteq \Gamma$. In particular, we also have $\nvdash_{\mathsf{LTL}} \bigvee_{\gamma' \in \Gamma'_0} \neg\gamma'$ for every finite $\Gamma'_0 \subseteq \Gamma'$. Thus, we deduce that Γ_0 is also consistent.

We also prove the dual statement: if Δ is inconsistent, then $\vdash_{\mathsf{LTL}} \bigvee_{\delta \in \Delta_0} \neg\delta$ for some finite $\Delta_0 \subseteq \Delta$. For every $\Delta' \supseteq \Delta$, we clearly have $\Delta_0 \subseteq \Delta'$, so we conclude that Δ' is also inconsistent. ▲

The following lemma is required for proving *Lemma 2.2.20*, which will be crucial for the completeness proof:

Lemma 2.2.16. *Let* Γ *be a consistent set of LTL-formulas, and let* φ *be an LTL-formula. Then* $\Gamma \cup \{\varphi\}$ *is consistent or* $\Gamma \cup \{\neg\varphi\}$ *is consistent, or both.*

Proof. Let us assume that Γ is consistent but that both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are inconsistent, and we will see that we get a contradiction. From our assumptions, we have:

$$\vdash_{\mathsf{LTL}} \bigvee_{\gamma \in \Gamma_0} \neg\gamma \qquad \text{and} \qquad \vdash_{\mathsf{LTL}} \bigvee_{\gamma' \in \Gamma'_0} \neg\gamma',$$

for some finite $\Gamma_0 \subseteq \Gamma \cup \{\varphi\}$ and some finite $\Gamma'_0 \subseteq \Gamma \cup \{\neg\varphi\}$. By these theorems and instances of propositional tautologies, we can derive, for $\Gamma'' = (\Gamma_0 \cup \Gamma'_0) \backslash \{\varphi, \neg\varphi\}$:

$$\vdash_{\mathsf{LTL}} \bigvee_{\gamma \in \Gamma''} \neg\gamma \vee \neg\varphi \qquad \text{and} \qquad \vdash_{\mathsf{LTL}} \bigvee_{\gamma \in \Gamma''} \neg\gamma \vee \neg\neg\varphi.$$

From an instance of a propositional tautology and MP, we also deduce

$$\vdash_{\mathsf{LTL}} \left(\bigvee_{\gamma \in \Gamma''} \neg\gamma \vee \neg\varphi \right) \wedge \left(\bigvee_{\gamma \in \Gamma''} \neg\gamma \vee \neg\neg\varphi \right).$$

Now, using an instance of the form

$$\left[\left(\bigvee_{\phi \in \Phi} \phi \vee \xi \right) \wedge \left(\bigvee_{\phi \in \Phi} \phi \vee \neg\xi \right) \right] \to \left(\bigvee_{\phi \in \Phi} \phi \right),$$

for $\Phi \cup \{\xi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$ (notice that it would express the *Resolution rule*), and MP again, we can conclude $\vdash_{\mathsf{LTL}} \bigvee_{\gamma \in \Gamma''} \neg\gamma$. Since Γ'' is a finite subset of Γ, we find that Γ has to be inconsistent, contradicting our first assumption. This proves, by *reductio ad absurdum*, that if Γ is consistent, then $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$, or both, are also consistent. ▲

Now we define the notion of maximal consistent set (MCS) of LTL-formulas. We will also state some lemmas characterizing MCSs, which will be relevant in the proof of completeness:

Definition 2.2.17. *We say that a consistent set of LTL-formulas* Γ *is* ***maximal*** *(MCS) if it is not a proper subset of any other consistent set.*

Lemma 2.2.18. *For every MCS* Γ *and every LTL-formula* $\varphi \notin \Gamma$*, the set* $\Gamma \cup \{\varphi\}$ *is inconsistent.*

Proof. By *reductio ad absurdum*: suppose Γ to be an MCS such that $\Gamma \cup \{\varphi\}$ is consistent. We immediately get a contradiction, since we assumed Γ not to be a proper subset of any other consistent set, but trivially $\Gamma \subset \Gamma \cup \{\varphi\}$. ▲

Lemma 2.2.19. *If* Γ *is an MCS, then, for every LTL-formula* φ*, either* $\varphi \in \Gamma$ *or* $\neg\varphi \in \Gamma$*, but not both.*

Proof. Let Γ be an MCS. We cannot have $\varphi, \neg\varphi \in \Gamma$ at the same time for any $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. Otherwise, since $\vdash_{\mathsf{LTL}} \neg\varphi \vee \neg\neg\varphi$, we would get a contradiction with the assumption that Γ is consistent.

We cannot have both $\varphi, \neg\varphi \notin \Gamma$ either: since Γ is an MCS, we would deduce, by *Lemma 2.2.18*, that $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent. Given that Γ is consistent, this leads to a contradiction, as we established in the proof of *Lemma 2.2.16*. Then we conclude that, for every LTL-formula φ, exclusively $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. ▲

The following lemma ensures that every consistent set of formulas can be extended or completed to be an MCS. This lemma closely parallels the well-known *Lindenbaum's Lemma*.

Lemma 2.2.20. *If* Γ *is a consistent set, then there is some MCS* Δ *such that* $\Gamma \subseteq \Delta$*.*

Proof. Using a numerable set of propositional variables Σ, we know that we can enumerate all LTL-formulas, so let us consider $\mathtt{Fm}_{\mathrm{LT}(\Sigma)} = \{\varphi_i : i < \omega\}$. We build a sequence of sets of formulas $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$ with the following rule:

$$\Gamma_{i+1} := \begin{cases} \Gamma_i \cup \{\varphi_i\}, & \text{if } \Gamma_i \cup \{\varphi_i\} \text{ is consistent;} \\ \Gamma_i \cup \{\neg\varphi_i\}, & \text{otherwise.} \end{cases}$$

Note that for every $i < \omega$, there exists some $j < \omega$ such that $\neg\varphi_i = \varphi_j$, and there are infinitely many equivalent formulas for each. Thus, in some sense, this construction might be considered quite inefficient.

In any case, we define $\Delta := \bigcup_{i<\omega} \Gamma_i$, and we will show that Δ is an MCS, which clearly verifies $\Gamma = \Gamma_0 \subseteq \Delta$. By *Lemma 2.2.16*, as $\Gamma = \Gamma_0$ is consistent, we see that Γ_1 is also consistent because if $\Gamma_0 \cup \{\varphi_0\}$ is not, then $\Gamma_0 \cup \{\neg\varphi_0\}$ has to be consistent. Inductively, we find that Γ_i is consistent for every $i < \omega$. Therefore, Δ will also be consistent: for every finite $\Delta_0 \subseteq \Delta$ there is some $k < \omega$, specifically, we could consider $k = 1 + max\{i < \omega : \varphi_i \in \Delta_0\}$, such that $\Delta_0 \subseteq \Gamma_k$. Then, by *Lemma 2.2.15*, we deduce that Δ_0 is consistent and, by *Lemma 2.2.14*, we have $\nvdash_{\mathsf{LTL}} \bigvee_{\delta\in\Delta_0} \neg\delta$. Since this applies to all finite subsets of Δ, we find that Δ is consistent, as we have $\nvdash_{\mathsf{LTL}} \bigvee_{\delta\in\Delta_0} \neg\delta$ for every finite $\Delta_0 \subseteq \Delta$.

It only remains to show that Δ is maximal: consider Δ' to be a consistent set such that $\Delta \subseteq \Delta'$. By *Lemma 2.2.15*, we can assume, without loss of generality, that $\Delta' = \Delta \cup \{\varphi_k\}$, for some LTL-formula φ_k with $k < \omega$. We need to see that necessarily $\Delta' = \Delta$, equivalently, that $\varphi_k \in \Delta$. We know that $\Gamma_{k+1} \subseteq \Delta$ contains either φ_k or $\neg\varphi_k$, but not both. Since every subset of Δ' is consistent, in particular $\Gamma_k \cup \{\varphi_k\} \subseteq \Delta \cup \{\varphi_k\} = \Delta'$ is consistent. Therefore, by our construction rule, we have $\Gamma_{k+1} := \Gamma_k \cup \{\varphi_k\}$, and then $\varphi_k \in \Gamma_{k+1} \subseteq \Delta$. In conclusion, we get $\varphi_k \in \Delta$, so $\Delta' = \Delta$. This shows that Δ is maximal. ▲

As we have mentioned, and as is usual in completeness proofs, we will build a canonical model using MCSs. Additionally, we will use a variation of the so-called *Fischer-Ladner closure*, originally introduced in [FL79]:

Definition 2.2.21. *Given a set of propositional variables Σ, consider $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. We define the **Fischer-Ladner closure** of φ as the least set $FL(\varphi)$ of LTL-formulas such that, for any $\psi, \phi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:*

- *$\varphi \in FL(\varphi)$;*
- *$FL(\varphi)$ is closed under subformulas;*
- *if $\psi \in FL(\varphi)$ and ψ is not of the form $\psi = \neg\phi$, then $\neg\psi \in FL(\varphi)$;*
- *$\neg\mathcal{X}\psi \in FL(\varphi)$ if and only if $\mathcal{X}\neg\psi \in FL(\varphi)$;*
- *if $\psi\,\mathcal{U}\,\phi \in FL(\varphi)$, then $\top\,\mathcal{U}\,\phi,\ \mathcal{X}(\psi\,\mathcal{U}\,\phi) \in FL(\varphi)$.*

We declare without proof, as it is almost immediate, that $FL(\varphi)$ is a finite set for every LTL-formula φ. In this sense, note that the condition in the third item, that ψ is not a negation, prevents us from obtaining infinitely equivalent formulas in $FL(\varphi)$, ensuring that we cannot concatenate negations. An alternative way to state the third item is to say that $FL(\varphi)$ is closed under negations modulo equivalences.

It is also trivial to check that if ψ is a subformula of φ, then $FL(\psi) \subseteq FL(\varphi)$, as we have $\psi \in FL(\varphi)$.

We observe that, roughly speaking, $FL(\varphi)$ consists of the formulas that are "relevant" to determine the satisfiability of φ. In practice, $FL(\varphi)$ can be seen as the set of subformulas of φ, its negations, and the formulas derived by applying MP over those subformulas and negations and some instance of the LTL axioms (Lin), (U1) or (U2).

With this intuition, we could expect that, for instance, if $\Box\varphi \in FL(\psi)$, for any LTL-formulas φ and ψ, then $\varphi, \mathcal{X}\Box\varphi \in FL(\psi)$, following our derived theorem ($\Box$Unr). Although it is not explicitly given in the previous definition, we can see that it is the case. We must recall, however, that the $\Box$ and $\Diamond$ modal operators are not primitive. We should consider their definitions based on primitive connectives to build the Fischer-Ladner closure properly. We see that $FL(\Box\varphi)$, for some LTL-formula φ, is given by:

$$\begin{aligned}FL(\Box\varphi) &= FL(\neg\Diamond\neg\varphi) = FL(\neg(\top\,\mathcal{U}\,\neg\varphi)) = \\ &= \{\neg(\top\,\mathcal{U}\,\neg\varphi), \top\,\mathcal{U}\,\neg\varphi, \top, \neg\varphi, \mathcal{X}(\top\,\mathcal{U}\,\neg\varphi), \neg\top, \varphi, \neg\mathcal{X}(\top\,\mathcal{U}\,\neg\varphi), \mathcal{X}\neg(\top\,\mathcal{U}\,\neg\varphi)\} = \\ &= \{\Box\varphi, \Diamond\neg\varphi, \top, \neg\varphi, \mathcal{X}\Diamond\neg\varphi, \bot, \varphi, \neg\mathcal{X}\Diamond\neg\varphi, \mathcal{X}\Box\varphi\}.\end{aligned}$$

Therefore, we have both $\varphi, \mathcal{X}\Box\varphi \in FL(\Box\varphi)$. This gives us that, following our expectations, if $\Box\varphi \in FL(\psi)$, for any LTL-formulas φ and ψ, then $\varphi, \mathcal{X}\Box\varphi \in FL(\psi)$.

Before stating and proving the completeness result we seek, we need to consider the following construction and the proposition distilled from it:

Definition 2.2.22. *For Σ a set of propositional variables, and for $\Gamma \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we define the following set of formulas:*

$$\mathcal{X}^{-1}\Gamma := \{\gamma : \mathcal{X}\gamma \in \Gamma\}.$$

Proposition 2.2.23. *Let Σ be a set of propositional variables, and consider $\Gamma \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. If Γ is consistent, then $\mathcal{X}^{-1}\Gamma$ is also consistent.*

Proof. By contraposition: we assume $\mathcal{X}^{-1}\Gamma$ to be inconsistent, and aim to demonstrate that Γ must also be inconsistent. We first prove an auxiliary result, for every $\varphi_1, \varphi_2 \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:

1.	$\mathcal{X}(\neg\varphi_1 \to \varphi_2) \to (\mathcal{X}\neg\varphi_1 \to \mathcal{X}\varphi_2)$;	instance of ($K_{\mathcal{X}}$)
2.	$\mathcal{X}\neg\varphi_1 \leftrightarrow \neg\mathcal{X}\varphi_1$;	instance of (Lin)
3.	$\mathcal{X}(\neg\varphi_1 \to \varphi_2) \to (\neg\mathcal{X}\varphi_1 \to \mathcal{X}\varphi_2)$;	Substitution Lemma on 1. by 2.
4.	$\mathcal{X}(\neg\neg\varphi_1 \vee \varphi_2) \to (\neg\neg\mathcal{X}\varphi_1 \vee \mathcal{X}\varphi_2)$;	definition of $\to$
5.	$\neg\neg\varphi_1 \leftrightarrow \varphi_1$;	instance of prop. tautology
6.	$\neg\neg\mathcal{X}\varphi_1 \leftrightarrow \mathcal{X}\varphi_1$;	instance of prop. tautology
7.	$\mathcal{X}(\varphi_1 \vee \varphi_2) \to (\neg\neg\mathcal{X}\varphi_1 \vee \mathcal{X}\varphi_2)$;	Substitution Lemma on 4. by 5.
8.	$\mathcal{X}(\varphi_1 \vee \varphi_2) \to (\mathcal{X}\varphi_1 \vee \mathcal{X}\varphi_2)$.	Substitution Lemma on 7. by 6.

That is, we see that $\vdash_{\mathsf{LTL}} \mathcal{X}(\varphi_1 \vee \varphi_2) \to (\mathcal{X}\varphi_1 \vee \mathcal{X}\varphi_2)$. We could have also derived this directly, as it is a theorem of the logic $\mathbf{K}_{\mathcal{X}}$.

Now, inductively, it is easy to check that our result can be extended to

$$\vdash_{\mathsf{LTL}} \mathcal{X}\left(\bigvee_{\gamma\in\Gamma}\gamma\right) \to \left(\bigvee_{\gamma\in\Gamma}\mathcal{X}\gamma\right),$$

for every finite $\Gamma \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$.

The assumption of $X^{-1}\Gamma$ being inconsistent gives us $\vdash_{\mathsf{LTL}} \bigvee_{\delta\in\Delta_0} \neg\delta$ for some finite $\Delta_0 \subseteq \mathcal{X}^{-1}\Gamma$. Now we can derive:

1.	$\bigvee_{\delta\in\Delta_0} \neg\delta$;	derivable by hypothesis
2.	$\mathcal{X}\left(\bigvee_{\delta\in\Delta_0} \neg\delta\right)$;	rule $N_{\mathcal{X}}$ over 1.
3.	$\mathcal{X}\left(\bigvee_{\delta\in\Delta_0} \neg\delta\right) \to \left(\bigvee_{\delta\in\Delta_0} \mathcal{X}\neg\delta\right)$;	from our previous result
4.	$\bigvee_{\delta\in\Delta_0} \mathcal{X}\neg\delta$;	MP over 2. and 3.
5_δ.	$\neg\mathcal{X}\delta \leftrightarrow \mathcal{X}\neg\delta$;	for every $\delta \in \Delta_0$, instances of (Lin)
6.	$\bigvee_{\delta\in\Delta_0} \neg\mathcal{X}\delta$;	Substitution Lemma on 4. by all the 5_δ.

Note that $\delta \in \Delta_0 \subseteq \mathcal{X}^{-1}\Gamma$ implies, by *Definition* 2.2.22, that we have $\mathcal{X}\delta \in \Gamma$. Then, it is clear that the set of formulas defined as $\Delta_{\mathcal{X}} := \{\mathcal{X}\delta : \delta \in \Delta_0\}$ will be a finite subset of Γ and

$$\vdash_{\mathsf{LTL}} \bigvee_{\delta' \in \Delta_{\mathcal{X}}} \neg\delta',$$

thanks to our previous derivation $\vdash_{\mathsf{LTL}} \bigvee_{\delta \in \Delta_0} \neg\mathcal{X}\delta$. The theorem of LTL we have found gives us that Γ is inconsistent, as we needed to see. This proves our lemma by contraposition. ▲

With all the results presented, we now have the tools needed to prove the completeness theorem:

Theorem 2.2.24 (Completeness of LTL)**.** *Given a set of propositional variables* Σ*, for every* $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$ *we have:*

$$\vdash_{\mathsf{LTL}} \varphi \iff \vDash_{\mathsf{LTL}} \varphi.$$

Proof. Firstly, we consider the implication from left to right, the *soundness* implication. It follows by induction on the construction of the theorems of LTL. We only need to check that every axiom scheme of the axiomatization of LTL is a tautology, and that the inference rules preserve validity. For instance, in the MP case we should prove that if $\vDash_{\mathsf{LTL}} \{\varphi_1, \varphi_1 \to \varphi_2\}$, then $\vDash_{\mathsf{LTL}} \varphi_2$, which follows easily from the definition of satisfiability.

We prove the other implication, the completeness implication, by contraposition. Hence, we need to show that $\nvdash_{\mathsf{LTL}} \varphi$ implies $\nvDash_{\mathsf{LTL}} \varphi$, where that last expression means that there is some $\sigma \in \mathcal{P}(\Sigma)^\omega$ and some $i \in \mathbb{N}$ such that $\sigma, i \nvDash \varphi$.

Since $\neg\neg\psi$ is equivalent to ψ, we can assume without loss of generality $\varphi = \neg\psi$. Note that the statement $\nvdash_{\mathsf{LTL}} \neg\psi$ tells us that ψ is a consistent formula. Moreover, $\nvDash_{\mathsf{LTL}} \neg\psi$ holds if there is some $\sigma \in \mathcal{P}(\Sigma)^\omega$ and some $i \in \mathbb{N}$ such that $\sigma, i \vDash \psi$, that is, if ψ is satisfiable. So, in essence, what we need to prove is that every consistent formula is satisfiable.

Consider the formula ψ to be consistent; we must prove its satisfiability. From what we have seen in the previous section, we know it is enough to provide a Kripke structure K over Σ, and some path ρ of K such that $K, \rho \Vdash \psi$ holds.

We will build a canonical model, in this case, a Kripke structure, by employing MCSs and the Fischer-Ladner closure: we consider the Kripke structure $K = (S, R, V)$, where $S := \{\Gamma \cap FL(\psi) : \Gamma \text{ is an MCS}\}$, the relation $R \subseteq S \times S$ is given by:

$$(\Delta, \Delta') \in R \;:\Longleftrightarrow\; \mathcal{X}^{-1}\Delta \subseteq \Delta';$$

and we define $V(\Delta) := \Delta \cap \Sigma$, that is, $V(\Delta)$ is the set of propositional variables appearing in Δ as (atomic) formulas. Note that we have defined one K for each formula ψ.

Suppose $\Delta \in S$; by definition and by *Lemma 2.2.15*, the state Δ is a consistent set included in $FL(\psi)$. The set Δ is not necessarily maximal, but it has a maximal aspect or remnant within the formulas of $FL(\psi)$. By *Proposition 2.2.23*, since Δ is consistent, we have that $\mathcal{X}^{-1}\Delta$ is also consistent. Moreover, $FL(\psi)$ is closed under subformulas, in particular under the $\mathcal{X}$ operator, so $\mathcal{X}^{-1}\Delta \subseteq FL(\psi)$, too. By *Lemma 2.2.20*, we know that there is some MCS containing $\mathcal{X}^{-1}\Delta$. Therefore, there is always some Δ' such that $(\Delta, \Delta') \in R$, that is, the relation R is left-total. This gives us that K is certainly a Kripke structure. Not only that but, since $FL(\psi)$ is a finite set, we know that K has a finite number of states, specifically, a maximum of $2^{FL(\psi)}$ states.

Now we aim to find a path ρ of the Kripke structure K such that $K, \rho \Vdash \psi$. Firstly, we enumerate the formulas of the form $\phi \mathcal{U} \phi'$ within our Kripke structure, for some $n < \omega$:

$$\{\phi \mathcal{U} \phi' : \phi \mathcal{U} \phi' \in FL(\psi)\} = \{\phi_0 \mathcal{U} \phi'_0, \ldots, \phi_{n-1} \mathcal{U} \phi'_{n-1}\}.$$

We inductively define the path $\rho_\psi = \langle \Delta_0, \Delta_1, \ldots \rangle$ via the following procedure:

- Step 0: Δ_0 is such that $\psi \in \Delta_0$;
- Step $i+1$: assume that we have already defined the first k states $\Delta_0, \Delta_1, \ldots, \Delta_{k-1}$ of our path ρ_ψ, and let $j \equiv i \pmod n$:
  - if $\phi_j \mathcal{U} \phi'_j \in \Delta_{k-1}$ and $\phi'_j \notin \Delta_{k-1}$, then we consider some sequence of states $\Delta'_k, \ldots, \Delta'_l$ such that $(\Delta_{k-1}, \Delta'_k), (\Delta'_m, \Delta'_{m+1}) \in R$ for every $k \leqslant m < l$, and $\phi'_j \in \Delta'_l$. We choose $\Delta_k := \Delta'_k, \ldots, \Delta_l := \Delta'_l$.
  - otherwise, we pick Δ_k to be any arbitrary state such that $(\Delta_{k-1}, \Delta_k) \in R$.

We will see that ρ_ψ satisfies $K, \rho_\psi \Vdash \psi$, as we wanted. This will follow from the *Truth Lemma* given in Claim 4 (in Page 56). However, before proving this, we must show that ρ_ψ is well-defined.

Since we assumed ψ to be consistent and trivially $\psi \in FL(\psi)$, by *Lemma 2.2.20* we know that there is some element $\Delta_0 \in S$ such that $\psi \in \Delta_0$. Also, considering that R is left-total, for every $\Delta_{k-1} \in S$ there is some (arbitrary) $\Delta_k \in S$ that ensures $(\Delta_{k-1}, \Delta_k) \in R$. Then, to see that ρ_ψ is well-defined, we have to prove that the sequence of states $\Delta'_k, \ldots, \Delta'_l$ of our definition certainly exists:

Consider an element $\Delta^0 \in S$ with $\phi \mathcal{U} \phi' \in \Delta^0$ and $\phi' \notin \Delta^0$. Then, we need to demonstrate the existence of a sequence $\Delta^1, \ldots, \Delta^l \in S$ such that $(\Delta^m, \Delta^{m+1}) \in R$ for every $0 \leqslant m < l$, and $\phi' \in \Delta^l$. We will prove it by *reductio ad absurdum.*

To find the required contradiction, we will deduce some auxiliary theorems of LTL related to the states of S, presented in the following three claims. For clarity, we will use some abbreviations. The first one, for Δ a finite set of formulas:

$$\widehat{\Delta} := \left(\bigwedge_{\delta \in \Delta} \delta \right).$$

Claim 1. For every (finite) $\Delta' \subseteq \Delta \in S$, we have:

$$\vdash_{\mathsf{LTL}} \widehat{\Delta'} \to \left(\bigvee_{\substack{\Delta'' \in S \\ \Delta' \subseteq \Delta''}} \widehat{\Delta''} \right). \tag{C1}$$

Proof of Claim 1. For every $\phi \in FL(\psi)$, we have, by instances of propositional tautologies:

$$\vdash_{\mathsf{LTL}} \widehat{\Delta'} \leftrightarrow \left((\widehat{\Delta'} \wedge \phi) \vee (\widehat{\Delta'} \wedge \neg\phi) \right). \tag{C1P}$$

Since $\Delta \in S$ is consistent, its subset Δ' is also consistent, by *Lemma 2.2.15*. Then, thanks to *Lemma 2.2.20*, we know that we can find some MCS containing Δ'. Also, by our construction, every state $\Delta'' \in S$ such that $\Delta' \subseteq \Delta''$ will be of the form $\Delta'' =$

$\Delta' \cup \{\phi_1, \phi_2, \ldots, \phi_m\}$ for $\phi_i \in FL(\psi)\backslash\Delta'$, with $i \leqslant m < \omega$. By induction on m, using (C1P), we can derive (C1) without difficulty. △

From now on, we will also use the abbreviation, for $\Delta \in S$:

$$R(\Delta) := \left(\bigvee_{(\Delta,\Delta')\in R} \widehat{\Delta'} \right).$$

Claim 2. For every $\Delta \in S$, we have:

$$\vdash_{\mathsf{LTL}} \widehat{\Delta} \to \mathcal{X}R(\Delta).$$

Proof of Claim 2. Remember that $(\Delta, \Delta') \in R$ means that we have $\mathcal{X}^{-1}\Delta \subseteq \Delta'$. Then, by the previous result (C1), we deduce:

$$\vdash_{\mathsf{LTL}} \widehat{\mathcal{X}^{-1}\Delta} \to R(\Delta).$$

Also, observe that if we have $\Delta_{\mathcal{X}} := \{\mathcal{X}\varphi : \varphi \in \mathcal{X}^{-1}\Delta\} \subseteq \Delta$, then by instances of propositional tautologies we derive

$$\vdash_{\mathsf{LTL}} \widehat{\Delta} \to \widehat{\Delta_{\mathcal{X}}}.$$

By generalizing the theorem given in *Example 2.2.10*, which ensures the distributivity of $\mathcal{X}$ over the connective $\wedge$, we can also deduce

$$\vdash_{\mathsf{LTL}} \widehat{\Delta_{\mathcal{X}}} \to \mathcal{X}\left(\widehat{\mathcal{X}^{-1}\Delta}\right).$$

From the last two theorems, by an instance of a proportional tautology and MP, we have:

$$\vdash_{\mathsf{LTL}} \widehat{\Delta} \to \mathcal{X}\left(\widehat{\mathcal{X}^{-1}\Delta}\right).$$

Now we can see:

1.	$\widehat{\mathcal{X}^{-1}\Delta} \to R(\Delta)$;	derived from Claim 1
2.	$\mathcal{X}\left(\widehat{\mathcal{X}^{-1}\Delta} \to R(\Delta)\right)$;	$N_{\mathcal{X}}$ over 1.
3.	$\mathcal{X}\left(\widehat{\mathcal{X}^{-1}\Delta}\right) \to \mathcal{X}R(\Delta)$;	MP over instance of ($K_{\mathcal{X}}$) and 2.
4.	$\widehat{\Delta} \to \mathcal{X}\left(\widehat{\mathcal{X}^{-1}\Delta}\right)$;	derived above
5.	$\widehat{\Delta} \to \mathcal{X}R(\Delta)$.	(prop)+3.+4.

That is, we have derived the theorem $\vdash_{\mathsf{LTL}} \widehat{\Delta} \to \mathcal{X}R(\Delta)$, as we wanted. △

The previous two claims will be used to show the following one. We will consider the relation $R^* \subseteq S \times S$ to be the *reflexive transitive closure* of R, that is, we inductively define R^* as:

- $R \subseteq R^*$;
- R^* is reflexive: $(\Delta, \Delta) \in R^*$ for every $\Delta \in S$;
- R^* is transitive: for every $\Delta_i \in S$ with $i \leqslant 2$, if $(\Delta_0, \Delta_1), (\Delta_1, \Delta_2) \in R^*$, then $(\Delta_0, \Delta_2) \in R^*$.

Similarly as before, we will use the abbreviation

$$R^*(\Delta) := \left(\bigvee_{(\Delta,\Delta')\in R^*} \widehat{\Delta'} \right).$$

Claim 3. For every $\Delta \in S$, we have:

$$\vdash_{\mathsf{LTL}} \widehat{\Delta} \to \Box R^*(\Delta). \tag{C3}$$

Proof of Claim 3. Given $\Delta \in S$, we can apply Claim 2 on every $\Delta' \in S$ such that $(\Delta, \Delta') \in R^*$, to deduce:

$$\vdash_{\mathsf{LTL}} \widehat{\Delta'} \to \mathcal{X}R(\Delta'); \quad \text{for every } \Delta' \text{ verifying } (\Delta, \Delta') \in R^*.$$

From instances of propositional tautologies of the form

$$(\gamma \to \gamma') \to ((\delta \to \delta') \to ((\gamma \vee \delta) \to (\gamma' \vee \delta'))),$$

we can derive, by applying MP over these instances and the previous theorems:

$$\vdash_{\mathsf{LTL}} R^*(\Delta) \to \left(\bigvee_{(\Delta,\Delta')\in R^*} \mathcal{X}R(\Delta') \right).$$

Now, as it holds in logic $\mathbf{K}_{\mathcal{X}}$ —distributivity of $\mathcal{X}$ over $\vee$—, we can state the theorem

$$\vdash_{\mathsf{LTL}} \left(\bigvee_{(\Delta,\Delta')\in R^*} \mathcal{X}R(\Delta') \right) \to \mathcal{X} \left(\bigvee_{(\Delta,\Delta')\in R^*} R(\Delta') \right).$$

From the previous two theorems, we can deduce, by an instance of a propositional tautology and MP, that we have

$$\vdash_{\mathsf{LTL}} R^*(\Delta) \to \mathcal{X} \left(\bigvee_{(\Delta,\Delta')\in R^*} R(\Delta') \right). \tag{C3P}$$

On the other hand, as we clearly have $Id_S \subseteq R^*$ and $R^* \circ R \subseteq R^*$, with $\circ$ denoting the composition of the relations, we deduce:

$$\vdash_{\mathsf{LTL}} \widehat{\Delta} \to R^*(\Delta);$$

$$\vdash_{\mathsf{LTL}} \left(\bigvee_{(\Delta,\Delta')\in R^*} R(\Delta') \right) \to R^*(\Delta).$$

And so,

1. $\left(\bigvee_{(\Delta,\Delta')\in R^*} R(\Delta') \right) \to R^*(\Delta);$ newly derived

2. $\mathcal{X} \left[\left(\bigvee_{(\Delta,\Delta')\in R^*} R(\Delta') \right) \to R^*(\Delta) \right];$ $N_{\mathcal{X}}$ rule over 1.

3. $\mathcal{X}\left(\bigvee_{(\Delta,\Delta')\in R^*} R(\Delta')\right) \to \mathcal{X}R^*(\Delta);$ MP over an instance of ($K_{\mathcal{X}}$) and 2.
4. $R^*(\Delta) \to \mathcal{X}R^*(\Delta);$ (prop)+3.+(C3P)
5. $\Box\left(R^*(\Delta) \to \mathcal{X}R^*(\Delta)\right);$ $N_\Box$ over 4.
6. $R^*(\Delta) \to \Box R^*(\Delta);$ MP over an instance of (Ind) and 5.
7. $\widehat{\Delta} \to R^*(\Delta);$ derived above
8. $\widehat{\Delta} \to \Box R^*(\Delta).$ (prop)+6.+7.

This proves Claim 3. △

With this third claim, we are finally able to find the contradiction we require for the *reductio ad absurdum* proof. For our canonical Kripke structure $K = (S, R, V)$, we consider some state $\Delta^0 \in S$ such that $\phi\,\mathcal{U}\,\phi' \in \Delta^0$ and $\phi' \notin \Delta^0$. We want to prove that there is a sequence $\Delta^1, \ldots, \Delta^l \in S$ such that $(\Delta^m, \Delta^{m+1}) \in R$ for every $0 \leqslant m < l$, and $\phi' \in \Delta^l$. Since R is left-total, the existence of a sequence verifying the first condition of being R-related is always true. Then, for the sake of contradiction, we suppose $\phi' \notin \Delta^l$ for every $l < \omega$ and for every possible sequence $\Delta^1, \ldots, \Delta^l$.

Observe that $\phi' \notin \Delta^l \in S$ implies $\neg\phi' \in \Delta^l$ —or some equivalent formula—, by the maximal aspect of the states of S. Then, since this applies for every l, we deduce that $R^*(\Delta^0) = \bigvee_{(\Delta',\Delta^0)\in R^*} \widehat{\Delta'}$ is indeed of the form:

$$R^*(\Delta^0) = \bigvee_{\substack{(\Delta',\Delta^0)\in R^*\\ \Delta''=\Delta'\setminus\{\neg\phi'\}}} \left(\widehat{\Delta''} \wedge \neg\phi'\right).$$

So, it is a routine check to derive the theorem of LTL:

$$\vdash_{\mathsf{LTL}} R^*(\Delta^0) \leftrightarrow (\neg\phi' \wedge \tilde{\Delta}); \qquad (\tilde{*})$$

where we consider

$$\tilde{\Delta} := \bigvee_{\substack{(\Delta^0,\Delta')\in R^*\\ \Delta''=\Delta'\setminus\{\neg\phi'\}}} \widehat{\Delta''}.$$

Now we can state:

1. $\widehat{\Delta^0} \to \Box R^*(\Delta^0);$ by Claim 3
2. $\widehat{\Delta^0} \to \Box\left(\neg\phi' \wedge \tilde{\Delta}\right);$ Substitution Lemma on 1. by ($\tilde{*}$)
3. $\Box\left(\neg\phi' \wedge \tilde{\Delta}\right) \to \left(\Box\neg\phi' \wedge \Box\tilde{\Delta}\right);$ from logic $\mathbf{K}_\Box$
4. $\widehat{\Delta^0} \to \left(\Box\neg\phi' \wedge \Box\tilde{\Delta}\right);$ (prop)+2.+3.
5. $\widehat{\Delta^0} \to \Box\neg\phi'.$ (prop)+4.

So, we have derived the LTL theorem $\vdash_{\mathsf{LTL}} \widehat{\Delta^0} \to \Box\neg\phi'$.

Now, recalling that we assumed $\phi\,\mathcal{U}\,\phi' \in \Delta^0$, we can build the following sequence of theorems of LTL:

1. $\phi\,\mathcal{U}\,\phi' \to \Diamond\phi';$ instance of axiom (U1)

2. $\widehat{\Delta^0} \to \Diamond\phi'$; (prop)+1.

3. $\widehat{\Delta^0} \to \Box\neg\phi'$; derived above

4. $\widehat{\Delta^0} \to (\Diamond\phi' \wedge \Box\neg\phi')$; (prop)+2.+3.

5. $\widehat{\Delta^0} \to (\Diamond\phi' \wedge \neg\Diamond\neg\neg\phi')$; definition of operator $\Box$

6. $\neg\neg\phi' \leftrightarrow \phi'$; instance of prop. tautology

7. $\widehat{\Delta^0} \to (\Diamond\phi' \wedge \neg\Diamond\phi')$; Substitution Lemma on 5. by 6.

8. $(\Diamond\phi' \wedge \neg\Diamond\phi') \to \bot$; instance of prop. tautology

9. $\widehat{\Delta^0} \to \bot$. (prop)+7.+8

That is, we have $\vdash_{\mathsf{LTL}} \widehat{\Delta^0} \to \bot$. By *Lemma 2.2.12*, it follows that Δ^0 is inconsistent. This leads to a contradiction, given that we assumed $\Delta^0 \in S$, which implies that Δ^0 has to be consistent.

In conclusion, we have demonstrated, by *reductio ad absurdum*, that there must necessarily exist a sequence of states in which $\phi' \in \Delta^l$ for some $l \geqslant 0$, with Δ^l being the last element in this sequence. Note that $l > 0$, as we initially assumed $\phi' \notin \Delta^0$.

So far, we have shown that the path ρ_ψ is well-defined. Let us now prove that it satisfies the following result:

Claim 4 (Truth Lemma)**.** Given the canonical Kripke structure K and the path $\rho_\psi = \langle \Delta_0, \Delta_1, \Delta_2, \ldots \rangle$ of K as defined above, we have for every $\phi \in FL(\psi)$ and every $i \in \mathbb{N}$:

$$K, \rho_\psi^i \Vdash \phi \iff \sigma_{\rho_\psi}, i \vDash_{\mathsf{LTL}} \phi \iff \phi \in \Delta_i.$$

Proof of Claim 4. The first double implication is clear by definition. We need to prove the second double implication. We do it by induction on the construction of the LTL-formula ϕ. Note that the proof of case $i = 0$ also induces satisfaction for every $i < \omega$, since we can work with suffix paths (remember that ρ^i is equal to the path ρ but starting from its i-th element).

- We first assume $\phi = p \in \Sigma$ to be a propositional variable. Since $FL(\psi)$ is closed under subformulas and negations modulo equivalences, we know that both $p, \neg p \in FL(\psi)$. Also, by *Lemma 2.2.19*, every MCS will contain p or $\neg p$, but not both. Then, we deduce that for every $\Delta \in S$ we will have $p \in V(\Delta) \subseteq \Delta$ or $\neg p \in \Delta$. This gives us that every path of K starting with some state Δ_0 such that $p \in \Delta_0$, in particular also the path ρ_ψ, will ensure $K, \rho \Vdash p$, since $p \in V(\Delta_0)$. Moreover, if the path starts with some state not containing p —so it contains $\neg p$—, we will have $K, \rho \nVdash p$, because $p \notin V(\Delta_0)$. This gives us the implications

$$K, \rho_\psi^i \Vdash p \iff p \in \Delta_i.$$

- We suppose ϕ_1 to verify $K, \rho_1 \Vdash \phi_1$ if and only if the path ρ_1 starts with some state containing ϕ_1. If we have $\phi = \neg\phi_1$, we can see that, for every path ρ starting with Δ_0 such that $\phi \in \Delta_0$, we will have $K, \rho \Vdash \phi$. Since $\neg\phi_1 \in \Delta_0$ and Δ_0 has to be consistent, we know that $\phi_1 \notin \Delta_0$. Then, by our Induction Hypothesis, we have $K, \rho \nVdash \phi_1$, which means that $K, \rho \Vdash \neg\phi_1$, as we wanted.

  The other direction also holds: if $\phi = \neg\phi_1 \notin \Delta_0$, then ϕ_1, which is equivalent to the negation of ϕ, will belong to Δ_0. By Induction Hypothesis, we have $K, \rho \Vdash \phi_1$, and so $K, \rho \nVdash \phi$.

As a particular case of what we have shown, we see:

$$K, \rho_\psi^i \Vdash \neg\phi_1 \iff \neg\phi_1 \in \Delta_i.$$

- Now we assume

$$K, \rho_k \vDash \phi_k \iff \phi_k \in \Delta_{0k},$$

for Δ_{0k} the first state of the paths ρ_k, and $k \in \{1, 2\}$. Consider $\phi = \phi_1 \vee \phi_2$. we can see that for every path ρ starting with Δ_0 such that $\phi \in \Delta_0$, we will have $K, \rho \vDash \phi$: by our definition of the Fischer-Ladner closure, we know $\phi_1, \phi_2, \neg\phi_1, \neg\phi_2 \in FL(\psi)$. Since the elements of S are conjunctions of MCSs and $FL(\psi)$, we have, by *Lemma 2.2.19*, that

$$\text{either } \phi_1 \in \Delta_0 \text{ or } \neg\phi_1 \in \Delta_0, \text{ and either } \phi_2 \in \Delta_0 \text{ or } \neg\phi_2 \in \Delta_0.$$

If we assume $\phi_1 \vee \phi_2 \in \Delta_0$, then we deduce that we cannot have $\neg\phi_1, \neg\phi_2 \in \Delta_0$ at the same time as

$$\neg(\phi_1 \vee \phi_2) \vee \neg\neg\phi_1 \vee \neg\neg\phi_2,$$

is equivalent to

$$\neg(\phi_1 \vee \phi_2) \vee (\phi_1 \vee \phi_2),$$

which is an instance of a propositional tautology, and so a theorem of LTL. This would contradict the assumption that Δ_0 is consistent.

Then, we have $\phi_1 \in \Delta_0$ or $\phi_2 \in \Delta_0$, or both. Without loss of generality, we can assume $\phi_1 \in \Delta_0$. By our Induction Hypothesis, we have $K, \rho \vDash \phi_1$, which immediately also gives us $K, \rho \vDash \phi_1 \vee \phi_2$.

If we now assume that the path ρ starts with some Δ_0 not containing ϕ, then we know that $\neg\phi \in \Delta_0$, recalling that $\neg\phi \in FL(\psi)$ and the maximal aspect of the state Δ_0. Also, see that $\neg\phi = \neg(\phi_1 \vee \phi_2)$ is equivalent to $\neg\phi_1 \wedge \neg\phi_2$, which might not be included in $\Delta_0 \subseteq FL(\psi)$, but it shows us that necessarily $\phi_1, \phi_2 \notin \Delta_0$, by the consistency of Δ_0. By Induction Hypothesis, we have $K, \rho \nVdash \phi_1$ and $K, \rho \nVdash \phi_2$, and so $K, \rho \nVdash \phi_1 \vee \phi_2$, as intended.

- Let us consider the case $\phi = \mathcal{X}\phi_1$. We pick a path ρ starting with some state Δ_0 verifying $\mathcal{X}\phi_1 \in \Delta_0$. By the definition of the relation R, we have that ϕ_1 will belong to any other $\Delta \in S$ such that $(\Delta_0, \Delta) \in R$. Then, whatever is the second element of our path ρ, call it Δ_1, we have $\phi_1 \in \Delta_1$. By Induction Hypothesis, we know that $K, \rho^1 \vDash \phi_1$ holds. This ensures $K, \rho \vDash \mathcal{X}\phi_1$, as we were looking for.

  On the other hand, if $\mathcal{X}\phi_1 \notin \Delta_0$, then $\neg\mathcal{X}\phi_1 \in \Delta_0$. By axiom (Lin), we know $\neg\mathcal{X}\phi_1$ is equivalent to $\mathcal{X}\neg\phi_1$. By the definition of the Fischer-Ladner closure and the consistency and maximality of Δ_0, we deduce that if $\neg\mathcal{X}\phi_1 \in \Delta_0$ then we also have $\mathcal{X}\neg\phi_1 \in \Delta_0$. Therefore, as we have just shown, we will have $\neg\phi_1 \in \Delta_1$ and so $\phi_1 \notin \Delta_1$. By our Induction Hypothesis, this means $K, \rho^1 \nVdash \phi_1$. Thus, we immediately get $K, \rho^0 \nVdash \mathcal{X}\phi_1$, that is, $K, \rho \nVdash \phi$.

- The case $\phi = \phi_1 \mathcal{U} \phi_2$ is the difficult one, and the most delicate to handle. Denoting by Δ_i the state of the position i of the path ρ_ψ, by Induction Hypothesis it suffices to prove that $\phi_1 \mathcal{U} \phi_2 \in \Delta_0$ if and only if $\phi_2 \in \Delta_j$ for some $j \geqslant 0$ and $\phi_1 \in \Delta_k$ for all $0 \leqslant k < j$.

  First, we prove that for every path starting with $\phi_1 \mathcal{U} \phi_2 \in \Delta_0$, the second condition —to have $\phi_1 \in \Delta_k$ for all $0 \leqslant k < j$— always holds: since $FL(\psi)$ is closed under subformulas and negations modulo equivalences, we know $\phi_1, \phi_2, \neg\phi_1, \neg\phi_2 \in FL(\psi)$. Also, by *Lemma 2.2.19* we have that

$$\text{either } \phi_1 \in \Delta_0 \text{ or } \neg\phi_1 \in \Delta_0, \text{ and either } \phi_2 \in \Delta_0 \text{ or } \neg\phi_2 \in \Delta_0.$$

If $\phi_2 \in \Delta_0$, the proof is complete, as we can pick $j = 0$ so $\phi_2 \in \Delta_j$, and the requisite $0 \leqslant k < j$ will be void, making the second condition trivially hold. We assume, then, $\neg\phi_2 \in \Delta_0$. By the consistency of Δ_0, we can show that necessarily $\phi_1 \in \Delta_0$. Otherwise, we would have $\neg\phi_1 \in \Delta_0$, which would give us a contradiction as shown below. By iteratively using the Substitution Lemma and equivalences given by instances of propositional tautologies, we see that the following formulas are equivalent:

- $\neg\neg\phi_1 \vee \neg\neg\phi_2 \vee \neg(\phi_1 \,\mathcal{U}\, \phi_2)$;
- $\phi_1 \vee \phi_2 \vee \neg(\phi_1 \,\mathcal{U}\, \phi_2)$;
- $\phi_1 \vee \phi_2 \vee \neg(\phi_2 \vee (\phi_1 \wedge \mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2)))$, by the equivalence given in axiom (U2);
- $\phi_1 \vee \phi_2 \vee (\neg\phi_2 \wedge \neg(\phi_1 \wedge \mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2)))$;
- $\phi_1 \vee \phi_2 \vee (\neg\phi_2 \wedge (\neg\phi_1 \vee \neg\mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2)))$;
- $\phi_1 \vee \phi_2 \vee (\neg\phi_2 \wedge \neg\phi_1) \vee (\neg\phi_2 \wedge \neg\mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2))$;
- $(\phi_1 \vee \phi_2) \vee \neg(\phi_2 \vee \phi_1) \vee (\neg\phi_2 \wedge \neg\mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2))$.

Since we know $\vdash_{\mathsf{LTL}} (\phi_1 \vee \phi_2) \vee \neg(\phi_2 \vee \phi_1)$, we would deduce that Δ_0 is inconsistent, which is a contradiction. Then, we necessarily have $\neg\phi_1 \notin \Delta_0$, that is, $\phi_1 \in \Delta_0$.

A similar argument would show us that if $\phi_1 \,\mathcal{U}\, \phi_2, \neg\phi_2, \phi_1 \in \Delta_0$, then we necessarily also have $\mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2) \in \Delta_0$. Therefore, we can deduce that $\phi_1 \,\mathcal{U}\, \phi_2 \in \Delta_1$. As before, we will have either $\phi_2 \in \Delta_1$, so we could choose $j = 1$, or $\neg\phi_2, \phi_1, \mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2) \in \Delta_1$. Inductively, we conclude that if there is some $j \geqslant 0$ such that $\phi_2 \in \Delta_j$, then $\phi_1 \in \Delta_k$ for all $0 \leqslant k < j$; and also $\phi_1 \in \Delta_k$ for all $0 \leqslant k < \omega$ if there is no j verifying $\phi_2 \in \Delta_j$.

Now we need to prove that, in our specific path ρ_ψ, there is such j with $\phi_2 \in \Delta_j$. This will follow from our curated definition of ρ_ψ. In our case, we can assume that we have already defined Δ_0, and that some subsequent states of the path are given by Step 1 of the path definition procedure. However, for clarity and generality, we will consider the last defined state to be Δ_{k-1}, with subsequent states along the path provided by Step i.

By assumption $\phi_1 \,\mathcal{U}\, \phi_2 \in FL(\psi)$, so we know that this formula is of the form $\phi_1 \,\mathcal{U}\, \phi_2 = \phi_m \,\mathcal{U}\, \phi'_m$ for some $m < n$, by the enumeration presented when defining ρ_ψ. Now we need to distinguish between two possible cases: $m \equiv i \pmod n$ and $m \not\equiv i \pmod n$.

If $m \equiv i \pmod n$, then we know that Step i has defined the subsequent states of the path through a sequence of states, ensuring that $\phi_2 = \phi'_m$ belongs to the last state in the sequence. Thus, our objective is achieved.

If $m \not\equiv i \pmod n$, then Step i has provided the path with some sequence of states, the last of them including $\phi'_{j'}$ for some $j' \equiv i \pmod n$, or it has added one arbitrary state R-related to Δ_{k-1}, our last considered state of ρ_ψ. Then, we know that Step i will add at least one state, let us call it Δ_k, and potentially some more states, $\Delta_{k+1}, \ldots, \Delta_{l'}$ for some $k \leqslant l' < \omega$. In any case, the procedure does not explicitly guarantee that some of these states will contain the formula $\phi_2 = \phi'_m$. However, we can see that the states $\Delta_k, \dots, \Delta_{l'}$ that have been added still preserve the satisfaction of $\phi_1 \mathcal{U} \phi_2$, that is, they contain ϕ_1 until some of them include ϕ_2, or they all contain ϕ_1 and $\phi_1 \mathcal{U} \phi_2$. We prove it by induction on l':

We recall a previous argument: $\phi_m \mathcal{U} \phi'_m \in \Delta \in S$ implies that,

$$\phi'_m \in \Delta \text{ or } \neg\phi'_m, \phi_m, \mathcal{X}(\phi_m \mathcal{U} \phi'_m) \in \Delta.$$

Observe that $\mathcal{X}(\phi_m \mathcal{U} \phi'_m) \in \Delta$ gives us that $\phi_m \mathcal{U} \phi'_m \in \Delta'$ for every possible Δ' such that $(\Delta, \Delta') \in R$.

Then, as we assumed $\phi_1 \mathcal{U} \phi_2 \in \Delta_{k-1}$, we deduce that we may have $\phi_2 \in \Delta_{k-1}$, so our proof is finished, or $\phi_1 \in \Delta_{k-1}$ and $\phi_1 \mathcal{U} \phi_2 \in \Delta_k$. Again, we have that either $\phi_2 \in \Delta_k$, and so we achieve our goal, or $\phi_1 \in \Delta_k$ and $\phi_1 \mathcal{U} \phi_2 \in \Delta_{k+1}$, which also shows the base case of our induction.

Now we assume that the satisfiability of $\phi_1 \mathcal{U} \phi_2$ is preserved until the state $\Delta_{k'}$. We can suppose $\phi_1 \mathcal{U} \phi_2 \in \Delta_{k'}$ and that no previous state contains ϕ_2, otherwise the argument is settled. Then we know that either $\phi_2 \in \Delta_{k'}$, and so $\phi_1 \mathcal{U} \phi_2$ is satisfied in this state, or we have $\phi_1 \in \Delta_{k'}$ and $\phi_1 \mathcal{U} \phi_2 \in \Delta_{k'+1}$. This last option ensures that the satisfiability of $\phi_1 \mathcal{U} \phi_2$ is also extended to $\Delta_{k'+1}$.

In conclusion, we have shown, by induction on l', that the added states $\Delta_k, \dots, \Delta_{l'}$ preserve the satisfaction of $\phi_1 \mathcal{U} \phi_2$, in the sense that they contain ϕ_1 until some of them includes ϕ_2, or all states contain ϕ_1 and $\phi_1 \mathcal{U} \phi_2$.

In this way, we see that although Step i adds states that do not necessarily include the formula $\phi_2 = \phi'_m$, if these states do not contain ϕ_2 then, roughly speaking, we propagate the formula $\phi_m \mathcal{U} \phi'_m$ to them.

In this last case, we have $\phi_m \mathcal{U} \phi'_m \in \Delta_{l'}$. The subsequent states of ρ_ψ will be defined by Step $i+1$ of our path definition procedure. The same argument as above applies, so we will continue to preserve the satisfiability of the formula $\phi_m \mathcal{U} \phi'_m$, either by propagating it or by ensuring that ϕ_2 is contained in some state. In the propagation case, this process will repeat until we reach the state or states defined by Step m', where $m \equiv m' \pmod{n}$. With the application of Step m', we will finally obtain a state in ρ_ψ that contains $\phi'_m = \phi_2$, with the preceding states including $\phi_m = \phi_1$.

In conclusion, we have proved that for every $i \in \mathbb{N}$ we have:

$$\phi_1 \mathcal{U} \phi_2 \in \Delta_i \implies K, \rho^i_\psi \Vdash \phi_1 \mathcal{U} \phi_2.$$

As we have done so far, we can demonstrate the other implication by contraposition. We consider a path ρ such that $\phi = \phi_1 \mathcal{U} \phi_2 \notin \Delta_0$, and we need to prove that $K, \rho \nVdash \phi_1 \mathcal{U} \phi_2$. That is, for any path $\rho = \langle \Delta_0, \Delta_1, \dots \rangle$ of K with $\phi_1 \mathcal{U} \phi_2 \notin \Delta_0$, we must show that if $\phi_2 \in \Delta_j$ for some $j \geqslant 0$, then there is some Δ_k, for $0 \leqslant k < j$, such that $\neg\phi_1 \in \Delta_k$. We prove this by *reductio ad absurdum*: suppose we have $\phi_1 \mathcal{U} \phi_2 \notin \Delta_0$ with $\phi_2 \in \Delta_j$ for some $j \leqslant 0$, and that $\neg\phi_1 \notin \Delta_k$ for every $k < j$; we will get a contradiction.

First, we recall that $\phi_1 \mathcal{U} \phi_2 \notin \Delta_0$ and $\neg\phi_1 \notin \Delta_k$ imply that we have $\neg(\phi_1 \mathcal{U} \phi_2) \in \Delta_0$ and $\phi_1 \in \Delta_k$, respectively. Now, observe that the following formulas are equivalent in LTL:

- $\neg(\phi_1 \mathcal{U} \phi_2)$;

- $\neg(\phi_1 \vee (\phi_1 \wedge \mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2)))$;
- $\neg\phi_2 \wedge \neg(\phi_1 \wedge \mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2))$;
- $\neg\phi_2 \wedge (\neg\phi_1 \vee \neg\mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2))$;
- $\neg\phi_2 \wedge (\phi_1 \to \neg\mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2))$.

From that last formula and our assumption $\phi_1 \in \Delta_0$, we can deduce $\neg\mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2) \in \Delta_0$, which also means $\mathcal{X}\neg(\phi_1 \,\mathcal{U}\, \phi_2) \in \Delta_0$. This is true by the definition of the Fischer-Ladner closure and by the maximal aspect of Δ_0. Then, by our definition of the relation R, we will have $\neg(\phi_1 \,\mathcal{U}\, \phi_2) \in \Delta_1$.

Iteratively repeating the previous argument, we can conclude $\neg(\phi_1 \,\mathcal{U}\, \phi_2) \in \Delta_k$ for all $k \leqslant j$. As we supposed $\phi_2 \in \Delta_j$, we get a contradiction, because Δ_j must be consistent, but $\{\phi_2, \neg(\phi_1 \,\mathcal{U}\, \phi_2)\} \subseteq \Delta_j$ is not: $\neg\phi_2 \vee \neg\neg(\phi_1 \,\mathcal{U}\, \phi_2)$ is equivalent to $\neg\phi_2 \vee \phi_2 \vee (\phi_1 \wedge \mathcal{X}(\phi_1 \,\mathcal{U}\, \phi_2))$, which is clearly derivable since $\neg\phi_2 \vee \phi_2$ is so. That is, by *Lemma 2.2.15* we find that Δ_j is inconsistent, contradicting our hypothesis.

Then, by *reductio ad absurdum*, we have shown that if the first state of a path ρ does not contain $\phi_1 \,\mathcal{U}\, \phi_2$, then $K, \rho \not\Vdash \phi_1 \,\mathcal{U}\, \phi_2$. And this gives us, by contrapositive, the implication:

$$K, \rho_\psi^i \Vdash \phi_1 \,\mathcal{U}\, \phi_2 \implies \phi_1 \,\mathcal{U}\, \phi_2 \in \Delta_i.$$

This ends the inductive case proof for $\phi = \phi_1 \,\mathcal{U}\, \phi_2$.

In conclusion, we have demonstrated, by induction on the construction of $\phi \in FL(\psi)$, that we have the Truth Lemma, for every $i \in \mathbb{N}$:

$$K, \rho_\psi^i \Vdash \phi \iff \sigma_{\rho_\psi}, i \vDash_{\mathsf{LTL}} \phi \iff \phi \in \Delta_i.$$

△

In particular, we have $K, \rho_\psi \Vdash \psi$ and $\sigma_{\rho_\psi}, 0 \vDash_{\mathsf{LTL}} \psi$, because $\psi \in \Delta_0$ by definition. That is, we have found some model where ψ holds, where ψ is satisfiable, as required. This concludes the proof of the completeness theorem for LTL. ▲

Observation 2.2.25. Consider the so-called *weak until* operator $\mathcal{W}$, defined for every $\sigma \in \mathcal{P}(\Sigma)^\omega$, every $i \in \mathbb{N}$, and $\phi_1, \phi_2 \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, as follows:

$$\sigma, i \vDash_{\mathsf{LTL}} \phi_1 \mathcal{W} \phi_2 \;:\iff\; \sigma, i \vDash_{\mathsf{LTL}} (\phi_1 \,\mathcal{U}\, \phi_2) \vee \Box\phi_1.$$

That is, $\mathcal{W}$ is similar to $\mathcal{U}$, but $\phi_1 \mathcal{W} \phi_2$ does not require the existence of a future state where ϕ_2 holds. Clearly, we have that $\phi_1 \,\mathcal{U}\, \phi_2$ implies $\phi_1 \mathcal{W} \phi_2$. If we were working with $\mathcal{W}$ instead of the usual $\mathcal{U}$, the definition of the path ρ_ψ in the previous proof of completeness could have been simplified to consider an initial state containing ψ and then concatenate arbitrary R-related states. This is because, in the induction used to prove Claim 4, the Truth Lemma, the only point where we needed to reference the definition of the path ρ_ψ was to show that if $\phi_1 \,\mathcal{U}\, \phi_2$ belongs to some state of the path, then there is a later state containing ϕ_2. As can be seen from our proof, all other conditions we needed to satisfy were guaranteed for every path of K with only the initial formula constraint.

Remark. In the previous proof, the use of the Fischer-Ladner closure was unnecessary. We could have defined a single canonical model with an infinite number of states, specifically, with $S = \{\Gamma : \Gamma \text{ is an MCS}\}$. Such an infinite Kripke structure would work just as well for any ψ, similar to the finite structures we have employed, only requiring a slight adjustment in the path definition. However, these finite structures pave the way for further results, such as the decidability of LTL. Nevertheless, these topics are beyond the scope of this thesis.

2.2.3 On the Strong Completeness of LTL

We will extend the weak completeness result we found in *Theorem 2.2.24* to a finitary strong completeness. Essentially, we only need to define the usual consequence relations distilled from $\vdash_{\mathsf{LTL}}$ and $\vDash_{\mathsf{LTL}}$, and show that they present the *Deduction Theorem*. Then the finitary strong completeness will easily follow from the weak completeness.

Regarding the semantic relation, we will define it as follows:

Definition 2.2.26. *Consider a set of propositional variables Σ, and a set of LTL-formulas $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. We define $\vDash'_{\mathsf{LTL}} \subseteq \mathcal{P}(\mathtt{Fm}_{\mathrm{LT}(\Sigma)}) \times \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$ by:*

$$\Gamma \vDash'_{\mathsf{LTL}} \varphi \;:\Longleftrightarrow\; \text{for } \sigma \in \mathcal{P}(\Sigma)^{\omega} \text{ and } i \in \mathbb{N}, \text{ if } \sigma, i \vDash_{\mathsf{LTL}} \Gamma \text{ then } \sigma, i \vDash_{\mathsf{LTL}} \varphi.$$

Observation 2.2.27. We clearly have $\varnothing \vDash'_{\mathsf{LTL}} \varphi \iff \vDash_{\mathsf{LTL}} \varphi$.

We can find a semantic Deduction Theorem for $\vDash'_{\mathsf{LTL}}$, without major difficulties:

Theorem 2.2.28 (Semantic Deduction Theorem)**.** *Consider Σ a set of propositional variables. For $\Gamma \cup \{\varphi, \psi\} \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we have:*

$$\Gamma \cup \{\varphi\} \vDash'_{\mathsf{LTL}} \psi \iff \Gamma \vDash'_{\mathsf{LTL}} \varphi \to \psi.$$

Proof. By definition, $\Gamma \vDash'_{\mathsf{LTL}} \varphi \to \psi$ gives us that, for every $\sigma \in \mathcal{P}(\Sigma)$ and every $i \in \mathbb{N}$, if $\sigma, i \vDash_{\mathsf{LTL}} \Gamma$, then $\sigma, i \vDash_{\mathsf{LTL}} \varphi \to \psi$. By the definition of satisfiability:

$$\sigma, i \vDash_{\mathsf{LTL}} \varphi \to \psi \iff \text{if } \sigma, i \vDash_{\mathsf{LTL}} \varphi \text{ then } \sigma, i \vDash_{\mathsf{LTL}} \psi.$$

It is not difficult to see that the following statements are equivalent:

- if $\sigma, i \vDash_{\mathsf{LTL}} \Gamma$ then $\sigma, i \vDash_{\mathsf{LTL}} \varphi \to \psi$;
- if $\sigma, i \vDash_{\mathsf{LTL}} \Gamma$ and $\sigma, i \vDash_{\mathsf{LTL}} \varphi$ then $\sigma, i \vDash_{\mathsf{LTL}} \psi$;
- if $\sigma, i \vDash_{\mathsf{LTL}} \Gamma \cup \{\varphi\}$ then $\sigma, i \vDash_{\mathsf{LTL}} \psi$.

The last expression can be translated to $\Gamma \cup \{\varphi\} \vDash'_{\mathsf{LTL}} \psi$. This proves the Semantic Deduction Theorem. ▲

Now we define a syntactical consequence relation from the axiomatization of LTL, and we will also see that the Deduction Theorem holds for it:

Definition 2.2.29. *Consider Σ a set of propositional variables. For $\Gamma \cup \{\varphi\} \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we define the relation $\vdash'_{\mathsf{LTL}} \subseteq \mathcal{P}(\mathtt{Fm}_{\mathrm{LT}(\Sigma)}) \times \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$ by:*

$$\Gamma \vdash'_{\mathsf{LTL}} \varphi \;:\Longleftrightarrow\; \vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \varphi \;\; \text{for some finite } \Gamma_0 \subseteq \Gamma.$$

Observation 2.2.30. It is trivial to see that $\varnothing \vdash'_{\mathsf{LTL}} \varphi \iff \vdash_{\mathsf{LTL}} \varphi$. Also, due to instances of propositional tautologies, for a finite Γ it suffices to consider only the case where $\Gamma_0 = \Gamma$. Moreover, by definition, it is clear that $\vdash'_{\mathsf{LTL}}$ is compact: if $\Gamma \vdash_{\mathsf{LTL}} \varphi$, then there is some finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathsf{LTL}} \varphi$.

Theorem 2.2.31 (Syntactic Deduction Theorem)**.** *Consider* Σ *a set of propositional variables. For* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$*, we have:*

$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{LTL}} \psi \iff \Gamma \vdash'_{\mathsf{LTL}} \varphi \to \psi.$$

Proof. The proof is similar to the given for *Theorem 2.2.28*, its semantic counterpart. We see that:

$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{LTL}} \psi \iff \vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma \wedge \varphi\right) \to \psi \text{ for some finite } \Gamma_0 \subseteq \Gamma.$$

To establish this, we have considered that, by an instance of a propositional tautology and using MP, we have:

$$\text{if} \vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \psi, \text{ then} \vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma \wedge \varphi\right) \to \psi.$$

Now, we can deduce, again by an instance of a propositional tautology and an application of MP:

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma \wedge \varphi\right) \to \psi \iff \vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to (\varphi \to \psi).$$

And, by the definition of $\vdash'_{\mathsf{LTL}}$ and recalling that $\Gamma_0 \subseteq \Gamma$, we see:

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to (\varphi \to \psi) \iff \Gamma \vdash'_{\mathsf{LTL}} \varphi \to \psi.$$

Combining all the correlations we have presented, we conclude,

$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{LTL}} \psi \iff \Gamma \vdash'_{\mathsf{LTL}} \varphi \to \psi.$$

▲

Now we can show the finitary strong completeness theorem we were looking for:

Theorem 2.2.32 (Finitary Strong Completeness of LTL)**.** *Given a set of propositional variables* Σ*, for every finite* $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$ *we have:*

$$\Gamma \vdash'_{\mathsf{LTL}} \varphi \iff \Gamma \vDash'_{\mathsf{LTL}} \varphi.$$

Proof. We observe that, by our Deduction Theorems, *Theorem 2.2.28* and *Theorem 2.2.31*, and considering that $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is finite, we have

$$\Gamma \vDash'_{\mathsf{LTL}} \varphi \iff \varnothing \vDash'_{\mathsf{LTL}} \psi \iff \vDash_{\mathsf{LTL}} \psi;$$

$$\Gamma \vdash'_{\mathsf{LTL}} \varphi \iff \varnothing \vdash'_{\mathsf{LTL}} \psi \iff \vdash_{\mathsf{LTL}} \psi;$$

where $\psi := \gamma_n \to (\gamma_{n-1} \to (\ldots(\gamma_1 \to \varphi))\ldots)$. So, we have reduced our task to proving the statement:

$$\vDash_{\mathsf{LTL}} \psi \iff \vdash_{\mathsf{LTL}} \psi.$$

And we know this holds by the weak completeness of LTL, *Theorem 2.2.24*. Thus, we conclude the proof of the Finitary Strong Completeness of LTL. ▲

A more compelling result than the one presented above would be achieving strong completeness for LTL, rather than just finitary strong completeness. However, this will not be the case as $\vDash'_{\mathsf{LTL}}$ is not compact. We provide a counterexample to demonstrate that we do not have strong completeness when the set of assumptions is infinite:

Consider the infinite set $\Gamma = \{\varphi \to \psi, \varphi \to \mathcal{X}\psi, \varphi \to \mathcal{X}\mathcal{X}\psi, \dots\}$, for φ and ψ LTL-formulas. Unraveling the satisfiability definitions, we can see that $\Gamma \vDash'_{\mathsf{LTL}} \varphi \to \Box\psi$ holds. However, it is not possible to deduce $\Gamma \vdash'_{\mathsf{LTL}} \varphi \to \Box\psi$, considering that no finite subset $\Gamma_0 \subseteq \Gamma$ will satisfy

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma \right) \to (\varphi \to \Box\psi).$$

2.2.4 Beyond Classical LTL

At this point, it is worth noting that this thesis has focused solely on the classical LTL framework. As we have seen, this classical approach does not yield a strong completeness theorem. However, strong completeness results can be achieved with *intuitionistic* LTL formalizations, as demonstrated in works such as [Dav96; Ewa86; Fer18; Mai04]. We also refer to papers such as [BDF19; CM21; KW10; KI11; Hir82] for more details on calculi and axiomatizations related to intuitionistic LTL that are strongly complete regarding their semantic counterparts.

In this quest for completeness, note that we have presented characterizations of star-free languages in this second chapter that are analogous to those provided in the first chapter for regular languages. We have also identified a characterization using LTL, a propositional modal logic that captures a notion of time. Then, a discerning reader might wonder whether there exists a characterization of regular languages using some modal temporal logic that, when suitably restricted, would lead us to LTL, similar to how regular languages lead to free-star languages through a restriction on the application of the Kleene star.

We will not delve into the details but highlight two logics that extend LTL and are as expressive as MSO logic over words, thus characterizing regular languages. The first is *Extended Temporal Logic*, introduced by Wolper in [Wol83], which takes inspiration from the theory of context-free and regular grammars. The second is *Regular Linear Temporal Logic*, presented by Leucker and Sánchez in [LS07], which is built upon a variant of regular expressions.

Chapter 3

A Matter of Being Positive

The LTL *Satisfiability Problem* refers to, given an LTL-formula $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, answering if φ is satisfiable in LTL or not. That is, if there is a word $\sigma \in \mathcal{P}(\Sigma)^\omega$ and some $i \in \mathbb{N}$ such that $\sigma, i \models \varphi$. Observe that the satisfiability problem is equivalent to the complement of the LTL *Validity Problem*, which can be stated as: given $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, answering if $\models_{\mathsf{LTL}} \varphi$.

It is well known that both the LTL Satisfiability Problem and the Validity Problem are PSPACE-complete, as is proved in [SC85], and in textbooks as [BK08; HV18]. Several authors, see [Art+13; Bau+09; CL93; DS02; DFK07; FG16; Hem05; Mar04; ON80; SR99], have searched for better complexity results by studying syntactic fragments of LTL. Some of these fragments include operators related to the past, in contrast to our work, which has focused solely on operators related to the future. As an example of an improvement in the complexity, in [ON80] it is demonstrated that for the LTL fragment with only the operators $\Box$ and $\Diamond$, the Satisfiability Problem becomes NP-complete, instead of PSPACE-complete.

In this final chapter, we consider another syntactical fragment of LTL. Our fragment will be based or inspired by the so-called *strictly positive* fragments of modal logics, specifically, the *Reflection Calculus* introduced in [Bek12; Das12]. Articles such as [AJ23a; AJ23b] suggest that the strictly positive character of the fragment will ensure a more favorable complexity than that of the standard LTL.

Next, we will present the syntax for what we will call the *Strictly Positive Linear Temporal Logic* (SPLTL). We will seek a syntactic calculus that defines the consequence relation in which we are interested, and we will prove the soundness between the given syntactical and semantic relations of SPLTL.

3.1 Strictly Positive LTL

We start by defining the syntax, the set of formulas, of SPLTL:

Definition 3.1.1. *Given a set of propositional variables Σ, we inductively define the set of SPLTL-formulas, $\mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, as follows:*

- $\top \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$*;*
- $p \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$ *for every* $p \in \Sigma$*;*

- *If* $\varphi, \psi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$*, then* $(\varphi \wedge \psi)$*,* $\mathcal{X}\varphi$*,* $(\varphi \,\mathcal{U}\, \psi) \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$*.*

3.1.1 Calculus of SPLTL

We suggest a syntactic calculus that defines a consequence relation

$$\vdash^+ \subseteq \mathcal{P}(\mathtt{Fm}_{\mathrm{SP}(\Sigma)}) \times \mathtt{Fm}_{\mathrm{SP}(\Sigma)}.$$

For $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, we use the notation $\varphi \vdash^+ \psi$ instead of $\{\varphi\} \vdash^+ \psi$. We denote $\Gamma \vdash^+ \Delta$ to indicate that we have $\Gamma \vdash^+ \delta$ for every $\delta \in \Delta$. And by $\Gamma \dashv\vdash^+ \varphi$ we mean that we have $\Gamma \vdash^+ \varphi$ and $\varphi \vdash^+ \Gamma$:

Definition 3.1.2. *Given a set of propositional variables* Σ*, we define the* ***calculus*** *of SPLTL by the following axioms and rules, for every* $\Gamma \cup \{\varphi, \psi, \phi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$*:*
Axiom Schemes:

$$\varphi \vdash^+ \varphi; \tag{Id}$$
$$\varnothing \vdash^+ \top; \tag{Top}$$
$$\varphi \wedge \psi \vdash^+ \varphi; \tag{$\wedge$L}$$
$$\varphi \wedge \psi \vdash^+ \psi; \tag{$\wedge$R}$$
$$\mathcal{X}(\varphi \wedge \psi) \dashv\vdash^+ \mathcal{X}\varphi \wedge \mathcal{X}\psi; \tag{DistX}$$
$$\varphi \,\mathcal{U}\, \psi \vdash^+ \top \,\mathcal{U}\, \psi; \tag{U1+}$$
$$\varphi \,\mathcal{U}\, \varphi \vdash^+ \varphi; \tag{UId}$$
$$\psi \vdash^+ \varphi \,\mathcal{U}\, \psi; \tag{U21}$$
$$\varphi \wedge \mathcal{X}(\varphi \,\mathcal{U}\, \psi) \vdash^+ \varphi \,\mathcal{U}\, \psi; \tag{U22}$$
$$\varphi \,\mathcal{U}\, (\varphi \,\mathcal{U}\, \psi) \vdash^+ \varphi \,\mathcal{U}\, \psi; \tag{UR}$$
$$(\varphi \,\mathcal{U}\, \psi) \,\mathcal{U}\, \psi \dashv\vdash^+ \varphi \,\mathcal{U}\, \psi; \tag{UL}$$
$$\varphi \,\mathcal{U}\, (\varphi \wedge \psi) \dashv\vdash^+ \varphi \wedge \mathcal{X}\varphi \,\mathcal{U}\, \psi; \tag{M}$$
$$\varphi \,\mathcal{U}\, (\psi \wedge \phi) \vdash^+ \varphi \,\mathcal{U}\, \psi \wedge \varphi \,\mathcal{U}\, \phi; \tag{DistUR}$$
$$(\varphi \wedge \psi) \,\mathcal{U}\, \phi \dashv\vdash^+ \varphi \,\mathcal{U}\, \phi \wedge \psi \,\mathcal{U}\, \phi; \tag{DistUL}$$
$$\mathcal{X}\varphi \,\mathcal{U}\, \mathcal{X}\psi \dashv\vdash^+ \mathcal{X}(\varphi \,\mathcal{U}\, \psi). \tag{DistUX}$$

Inference Rules:

$$\frac{\Gamma \vdash^+ \psi}{\Gamma' \vdash^+ \psi}\ \textit{Weakening} \qquad \frac{\Gamma \vdash^+ \psi \qquad \Gamma \vdash^+ \phi}{\Gamma \vdash^+ \psi \wedge \phi}\ I\wedge \qquad \frac{\Gamma \vdash^+ \psi \qquad \psi \vdash^+ \phi}{\Gamma \vdash^+ \phi}\ \textit{Cut}$$

$$\frac{\Gamma \vdash^+ \psi}{\mathcal{X}\Gamma \vdash^+ \mathcal{X}\psi}\ N^{\mathcal{X}} \qquad \frac{\varphi \vdash^+ \psi}{\phi_1 \,\mathcal{U}\, \varphi \vdash^+ \phi_1 \,\mathcal{U}\, \psi}\ N_1^{\mathcal{U}} \qquad \frac{\varphi \vdash^+ \psi}{\varphi \,\mathcal{U}\, \phi_1 \vdash^+ \psi \,\mathcal{U}\, \phi_1}\ N_2^{\mathcal{U}}$$

Where we have considered $\Gamma' \supseteq \Gamma$ *and* $\mathcal{X}\Gamma := \{\mathcal{X}\gamma : \gamma \in \Gamma\}$*.*

Observation 3.1.3. The axiom (M) is called this way because the formula $\varphi \,\mathcal{U}\, (\varphi \wedge \psi)$ is conventionally also stated as $\varphi \mathcal{M} \psi$, where $\mathcal{M}$ is the so-called *strong release* operator. Semantically, the formula $\varphi \mathcal{M} \psi$ indicates that φ is true until and including the first state where ψ holds, which must occur in the present state or some future state.

Notation. We will refer to the expressions of the form $\Gamma \vdash^+ \varphi$ that can be proved using the previous calculus as *theorems of SPLTL*.

The axioms and rules of the SPLTL calculus have been chosen to verify the Soundness implication shown in *Theorem 3.1.16*. Before proving that result, we will present some secondary, yet still interesting, theorems and derivable rules for SPLTL, which will help us to understand the given system.

3.1.2 Some theorems and results for SPLTL

Now we introduce some expressions and properties that hold in the calculus of SPLTL. We will prove the most relevant results but only sketch some straightforward, routine proofs.

Lemma 3.1.4. *For every SPLTL-formulas φ and ψ, and for every $n < \omega$, we have:*

$$\varnothing \vdash^{+} \mathcal{X}^{n}\top;$$

Where we take $\mathcal{X}^{0}\varphi := \varphi$ and $\mathcal{X}^{i+1}\varphi := \mathcal{X}^{i}\mathcal{X}\varphi$ for every $i < \omega$.

Proof. By induction on n: for $n = 0$ is clear, as it is the axiom (Top). If we assume $\varnothing \vdash^{+} \mathcal{X}^{n}\top$, then we can apply the $N^{\mathcal{X}}$ rule to get $\mathcal{X}\varnothing \vdash^{+} \mathcal{X}^{n+1}\top$. It is immediate to see that $\mathcal{X}\varnothing = \varnothing$. ▲

Lemma 3.1.5. *For every SPLTL-formulas φ and ψ, and for every $j, k < \omega$ such that $j \leqslant k + 1$, we have:*

i) $\mathcal{X}^{j}\varphi \vdash^{+} \top\,\mathcal{U}\,\varphi$;

ii) $\mathcal{X}^{k}\varphi \wedge \mathcal{X}^{j}\psi \vdash^{+} (\top\,\mathcal{U}\,\varphi)\,\mathcal{U}\,\psi$;

iii) $\varphi \wedge \mathcal{X}\varphi \wedge \cdots \wedge \mathcal{X}^{k}\varphi \wedge \mathcal{X}^{k+1}(\varphi\,\mathcal{U}\,\psi) \vdash^{+} \mathcal{X}^{j}(\varphi\,\mathcal{U}\,\psi)$;

iv) $\varphi \wedge \mathcal{X}\varphi \wedge \cdots \wedge \mathcal{X}^{k}\varphi \wedge \mathcal{X}^{k+1}\psi \vdash^{+} \mathcal{X}^{j}(\varphi\,\mathcal{U}\,\psi)$;

v) $\varphi\,\mathcal{U}\,(\varphi \wedge \mathcal{X}\varphi \wedge \cdots \wedge \mathcal{X}^{k}\varphi \wedge \mathcal{X}^{k+1}(\varphi\,\mathcal{U}\,\psi)) \vdash^{+} \mathcal{X}^{j}(\varphi\,\mathcal{U}\,\psi)$;

vi) $\varphi\,\mathcal{U}\,(\varphi \wedge \mathcal{X}\varphi \wedge \cdots \wedge \mathcal{X}^{k}\varphi \wedge \mathcal{X}^{k+1}\psi) \vdash^{+} \mathcal{X}^{j}(\varphi\,\mathcal{U}\,\psi)$.

Proof. We prove *i)* by induction on j. For $j = 0$ is clear, since we have the instance of the axiom (U1+):

$$\varphi \vdash^{+} \top\,\mathcal{U}\,\varphi.$$

For the inductive case, we can deduce:

1.	$\mathcal{X}^{j}\varphi \vdash^{+} \top\,\mathcal{U}\,\varphi$;	Induction Hypothesis
2.	$\mathcal{X}^{j+1}\varphi \vdash^{+} \mathcal{X}(\top\,\mathcal{U}\,\varphi)$;	$N^{\mathcal{X}}$ over 1.
3.	$\mathcal{X}^{j+1}\varphi \vdash^{+} \top$;	Weakening over instance of (Top)
4.	$\mathcal{X}^{j+1}\varphi \vdash^{+} \top \wedge \mathcal{X}(\top\,\mathcal{U}\,\varphi)$;	I$\wedge$ over 2. and 3.
5.	$\top \wedge \mathcal{X}(\top\,\mathcal{U}\,\varphi) \vdash^{+} \top\,\mathcal{U}\,\varphi$;	instance of (U22)
6.	$\mathcal{X}^{j+1}\varphi \vdash^{+} \top\,\mathcal{U}\,\varphi$.	Cut over 4. and 5.

This shows, by induction on j, that $\mathcal{X}^{j}\varphi \vdash^{+} \top\,\mathcal{U}\,\varphi$ for every $j < \omega$.

We also prove *ii)* by induction on j. Consider $j = 0$, for every $k < \omega$, we see:

1.	$\psi \vdash^+ (\top \mathcal{U} \varphi) \mathcal{U} \psi$;	instance of the axiom (U21)
2.	$\mathcal{X}^k \varphi \wedge \psi \vdash^+ \psi$;	instance of the axiom ($\wedge$R)
3.	$\mathcal{X}^k \varphi \wedge \psi \vdash^+ (\top \mathcal{U} \varphi) \mathcal{U} \psi$.	Cut over 1. and 2.

Now we assume that $\mathcal{X}^k \varphi \wedge \mathcal{X}^j \psi \vdash^+ (\top \mathcal{U} \varphi) \mathcal{U} \psi$ holds, for $k \geqslant j - 1 \geqslant 0$. We need to show

$$\mathcal{X}^{k'} \varphi \wedge \mathcal{X}^{j+1} \psi \vdash^+ (\top \mathcal{U} \varphi) \mathcal{U} \psi,$$

for every $k' \geqslant j > 0$. Taking into account that $k' - 1 \geqslant j - 1 \geqslant 0$, we can derive:

1.	$\mathcal{X}^{k'-1} \varphi \wedge \mathcal{X}^j \psi \vdash^+ (\top \mathcal{U} \varphi) \mathcal{U} \psi$;	Induction Hypothesis
2.	$\mathcal{X}(\mathcal{X}^{k'-1} \varphi \wedge \mathcal{X}^j \psi) \vdash^+ \mathcal{X}((\top \mathcal{U} \varphi) \mathcal{U} \psi)$;	$N^{\mathcal{X}}$ over 1.
3.	$\mathcal{X}^{k'} \wedge \mathcal{X}^{j+1} \psi \vdash^+ \mathcal{X}(\mathcal{X}^{k'-1} \varphi \wedge \mathcal{X}^j \psi)$;	instance of (DistX)
4.	$\mathcal{X}^{k'} \varphi \wedge \mathcal{X}^{j+1} \psi \vdash^+ \mathcal{X}((\top \mathcal{U} \varphi) \mathcal{U} \psi)$;	Cut over 2. and 3.
5.	$\mathcal{X}^{k'} \varphi \vdash^+ \top \mathcal{U} \varphi$;	derived above
6.	$\mathcal{X}^{k'} \varphi \wedge \mathcal{X}^{j+1} \psi \vdash^+ \top \mathcal{U} \varphi$;	Cut over 5. and an instance of ($\wedge$L)
7.	$\mathcal{X}^{k'} \varphi \wedge \mathcal{X}^{j+1} \psi \vdash^+ \top \mathcal{U} \varphi \wedge \mathcal{X}((\top \mathcal{U} \varphi) \mathcal{U} \psi)$;	I$\wedge$ over 4. and 6.
8.	$\top \mathcal{U} \varphi \wedge \mathcal{X}((\top \mathcal{U} \varphi) \mathcal{U} \psi) \vdash^+ (\top \mathcal{U} \varphi) \mathcal{U} \psi$;	instance of axiom (U22)
9.	$\mathcal{X}^{k'} \varphi \wedge \mathcal{X}^{j+1} \psi \vdash^+ (\top \mathcal{U} \varphi) \mathcal{U} \psi$.	Cut over 7. and 8.

So, we achieved the expression we wanted. This concludes the proof, by induction on j, of the second statement of the lemma.

We will not provide explicit proofs for the rest of the items. The statement *iii)* can be shown by induction on k, similarly to *ii)*. Using an instance of the axiom (U21), we can derive *iv)* from *iii)*. By instances of axioms (UR) and (DistUX), we can prove *v)* by induction on k. Finally, thanks again to (U21) we can derive *vi)* from *v)*. ▲

We also find interesting results by generalizing conjunctions to handle any finite number of arguments:

Lemma 3.1.6. *Consider a finite set of SPLTL-formulas $\Gamma \cup \{\varphi\}$. For every $\Gamma_i \subseteq \Gamma$ and every $i \leqslant n < \omega$, we define the formulas $\widehat{\gamma}_i := \left(\bigwedge_{\gamma \in \Gamma_i} \gamma\right)$. The following expressions hold:*

i) $\bigwedge_{\gamma \in \Gamma} \gamma \vdash^+ \gamma'$, *for every* $\gamma' \in \Gamma$;

ii) $\bigwedge_{\gamma \in \Gamma} \mathcal{X}\gamma \dashv\vdash^+ \mathcal{X}\left(\bigwedge_{\gamma \in \Gamma} \gamma\right)$;

iii) $\varphi \mathcal{U} \left(\bigwedge_{\gamma \in \Gamma} \gamma\right) \vdash^+ \varphi \mathcal{U} \gamma'$, *for every* $\gamma' \in \Gamma$;

iv) $\left(\bigwedge_{\gamma \in \Gamma} \gamma\right) \mathcal{U} \varphi \vdash^+ \gamma' \mathcal{U} \varphi$, *for every* $\gamma' \in \Gamma$;

v) $\bigwedge_{\gamma \in \Gamma} \gamma \vdash^+ \widehat{\gamma}_1 \mathcal{U} (\widehat{\gamma}_2 \mathcal{U} (\widehat{\gamma}_3 \mathcal{U} (\dots \widehat{\gamma}_{n-1} \mathcal{U} \widehat{\gamma}_n) \dots)$;

vi) $\left(\bigwedge_{\gamma \in \Gamma} \gamma\right) \mathcal{U} \varphi \vdash^+ \widehat{\gamma}_1 \mathcal{U} (\widehat{\gamma}_2 \mathcal{U} (\widehat{\gamma}_3 \mathcal{U} (\dots \widehat{\gamma}_n \mathcal{U} \varphi) \dots)$;

vii) $\varphi\,\mathcal{U}\left(\bigwedge_{\gamma\in\Gamma}\gamma\right) \vdash^{+} \varphi\,\mathcal{U}\,(\widehat{\gamma}_1\,\mathcal{U}\,(\widehat{\gamma}_2\,\mathcal{U}\,(\widehat{\gamma}_3\,\mathcal{U}\,(\ldots\widehat{\gamma}_{n-1}\,\mathcal{U}\,\widehat{\gamma}_n)\ldots)).$

Proof. The first two items are shown by induction on the cardinality of Γ, using the axioms ($\wedge$R) and (DistX), respectively. The relations *iii)* and *iv)* are derivable from applying the rules $N_1^{\mathcal{U}}$ and $N_2^{\mathcal{U}}$ over *i)*, respectively. The expression *v)* can be proved by induction on the cardinality of Γ with a secondary induction on n. The expression *vi)* can be easily shown by an instance of (DistUL) and then concatenate instances of axiom (U21). The last statement derives from *v)* by employing the rule $N_1^{\mathcal{U}}$. ▲

It is important to note that the strictly positive fragment of any modal logic is typically defined using the diamond operator, see [Bek12]. However, in our definition of SPLTL, we use the $\mathcal{U}$ operator, which offers greater expressivity. Nevertheless, as with LTL, we can define $\Diamond\varphi := \top\,\mathcal{U}\,\varphi$ for $\varphi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$. As the following lemma suggests, the operator $\Diamond$ exhibits some interesting properties. Certain expressions in this lemma can be generalized as theorems where $\mathcal{U}$ substitutes $\Diamond$, replacing the occurrences of $\top$ with an arbitrary formula.

Lemma 3.1.7. *For every SPLTL-formulas φ and ψ, and every $k, n, m < \omega$ with $m > 0$, we have:*

i) $\varphi \vdash^{+} \Diamond\varphi$;

ii) $\Diamond\varphi \vdash^{+} \Diamond\Diamond\varphi$;

iii) $\Diamond\Diamond\varphi \vdash^{+} \Diamond\varphi$;

iv) $\Diamond^{n}\varphi \vdash^{+} \Diamond^{m}\varphi$;

v) $\varphi\,\mathcal{U}\,(\Diamond^{n}\psi) \vdash^{+} \varphi\,\mathcal{U}\,(\Diamond^{m}\psi)$;

vi) $(\Diamond^{n}\varphi)\,\mathcal{U}\,\psi \vdash^{+} (\Diamond^{m}\varphi)\,\mathcal{U}\,\psi$;

vii) $\Diamond\varphi\,\mathcal{U}\,\varphi \vdash^{+} \Diamond\varphi$;

viii) $\mathcal{X}^{k}\varphi \vdash^{+} \Diamond^{m}\varphi$;

ix) $\Diamond^{n}\mathcal{X}^{k}\varphi \vdash^{+} \Diamond^{m}\varphi$;

x) $\mathcal{X}^{k}\Diamond^{n}\varphi \vdash^{+} \Diamond^{m}\varphi$;

xi) $\varphi\,\mathcal{U}\,\psi \vdash^{+} \Diamond^{m}\psi$.

Proof. The first two expressions come from an instance of the axiom (U21). The third is an instance of (UR). The expression *iv)* is given by induction on $0 \leqslant n < \omega$ with a secondary induction on $0 < m < \omega$, using the previous three relations. To deduce *v)* and *vi)* we only need to apply the rules $N_1^{\mathcal{U}}$ and $N_2^{\mathcal{U}}$ over *iv)*, respectively. The relation *vii)* is given by

1.	$\varphi \vdash^{+} \Diamond\varphi$;	*i)*
2.	$\Diamond\varphi\,\mathcal{U}\,\varphi \vdash^{+} \Diamond\varphi\,\mathcal{U}\,\Diamond\varphi$;	rule $N_1^{\mathcal{U}}$ over 1.
3.	$\Diamond\varphi\,\mathcal{U}\,\Diamond\varphi \vdash^{+} \Diamond\varphi$;	instance of (UId)
4.	$\Diamond\varphi\,\mathcal{U}\,\varphi \vdash^{+} \Diamond\varphi$.	Cut over 2. and 3.

Theorem *viii)* can be shown by induction on k: for $k = 0$, we know $\varphi \vdash^+ \varphi$ from axiom (Id), and $\varphi \vdash^+ \diamond\varphi$ from (U21). Applying the Cut rule over the last expression and *iv)*, we deduce $\varphi \vdash^+ \diamond^m\varphi$. On the other hand, if we assume $\mathcal{X}^k\varphi \vdash^+ \diamond^m\varphi$, then we can derive:

1.	$\mathcal{X}^k\varphi \vdash^+ \diamond^m\varphi$;	Induction Hypothesis
2.	$\mathcal{X}^{k+1}\varphi \vdash^+ \mathcal{X}\diamond^m\varphi$;	$N^{\mathcal{X}}$ over 1.
3.	$\varnothing \vdash^+ \top$;	instance of (Top)
4.	$\mathcal{X}^{k+1}\varphi \vdash^+ \top$;	Weakening over 3.
5.	$\mathcal{X}^{k+1}\varphi \vdash^+ \top \wedge \mathcal{X}\diamond^m\varphi$;	I$\wedge$ over 2. and 4.
6.	$\top \wedge \mathcal{X}\diamond^m\varphi \vdash^+ \diamond^m\varphi$;	instance of (U22)
7.	$\mathcal{X}^{k+1}\varphi \vdash^+ \diamond^m\varphi$.	Cut over 5. and 6.

The elements *ix)* and *x)* are proved similarly to the previous one, and *xi)* is given by an application of Cut over an instance of (U1+) and *iv)*. ▲

Observation 3.1.8. Expression *iv)* indicates that it is redundant to concatenate the $\diamond$ operator, as any concatenation of $\diamond$ is equivalent to a single application.

Although we could prove additional theorems of SPLTL, let us now focus on some meta-theorems. We can first consider compactness:

Lemma 3.1.9. *Given a set of propositional variables* Σ*, we have, for* $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$*:*

$$\Gamma \vdash^+ \varphi \iff \Gamma_0 \vdash^+ \varphi \quad \textit{for some finite } \Gamma_0 \subseteq \Gamma.$$

Proof. Immediate, as we defined $\vdash^+$ using a syntactic calculus. The right-to-left implications is immediate by Weakening. We can demonstrate the other implication by induction on the derivation $\Gamma \vdash^+ \varphi$. If $\Gamma \vdash^+ \varphi$ an instance of an axiom, then there clearly exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash^+ \varphi$, since Γ itself is finite. Inductive cases, where the application of a rule leads to $\Gamma \vdash^+ \varphi$, easily follow by considering the corresponding induction hypothesis. The only case that requires some work is the rule I$\wedge$:

Suppose that $\Gamma \vdash^+ \varphi$ is of the form $\Gamma \vdash^+ \varphi_1 \wedge \varphi_2$, and we know that $\Gamma \vdash^+ \varphi_1$ and $\Gamma \vdash^+ \varphi_2$. By Induction Hypothesis, there are some finite $\Gamma_1, \Gamma_2 \subseteq \Gamma$ such that:

$$\Gamma_1 \vdash^+ \varphi_1 \text{ and } \Gamma_2 \vdash^+ \varphi_2.$$

By Weakening, we easily derive that

$$\Gamma_1 \cup \Gamma_2 \vdash^+ \varphi_1 \text{ and } \Gamma_1 \cup \Gamma_2 \vdash^+ \varphi_2.$$

An application of I$\wedge$ over these two last expressions ensures that $\Gamma_1 \cup \Gamma_2 \vdash^+ \varphi_1 \wedge \varphi_2$. Since Γ_1 and Γ_2 are finite subsets of Γ, we know that $\Gamma_1 \cup \Gamma_2$ also is. ▲

As a direct consequence of compactness, we can derive the following:

Lemma 3.1.10. *Given a set of propositional variables* Σ*, we have, for* $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$*:*

$$\Gamma \vdash^+ \varphi \iff \bigwedge_{\gamma \in \Gamma_0} \gamma \vdash^+ \varphi \quad \textit{for some finite } \Gamma_0 \subseteq \Gamma.$$

Proof. By compactness, it suffices to show that, for every finite $\Gamma_0 \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, we have:

$$\Gamma_0 \vdash^+ \varphi \iff \bigwedge_{\gamma \in \Gamma_0} \gamma \vdash^+ \varphi.$$

To prove the right-to-left implication, we use axiom (Id) and Weakening to deduce that $\Gamma_0 \vdash^+ \gamma$ for every $\gamma \in \Gamma_0$. Then iteratively applying the rule $I\wedge$, we have $\Gamma_0 \vdash^+ \bigwedge_{\gamma \in \Gamma_0} \gamma$. The Cut rule leads us to conclude that $\bigwedge_{\gamma \in \Gamma_0} \gamma \vdash^+ \varphi$ implies $\Gamma_0 \vdash^+ \varphi$, as desired.

The left-to-right implication can be shown similarly: applying the axioms ($\wedge$L) and ($\wedge$R) we derive $\bigwedge_{\gamma \in \Gamma_0} \gamma \vdash^+ \Gamma_0$. Then, if we assume $\Gamma_0 \vdash^+ \varphi$, by Cut we get the expression $\bigwedge_{\gamma \in \Gamma_0} \gamma \vdash^+ \varphi$. ▲

Perhaps the most relevant meta-theorems are the *Substitution Lemma for SPLTL* and the *Positive Replacement Lemma.* Recall that $\varphi[p \leftarrow \phi]$ refers to the formula obtained by substituting the variable p with the formula ϕ in φ.

Lemma 3.1.11 (Substitution Lemma for SPLTL)**.** *Consider Σ a set of propositional variables. For $\varphi, \psi, \phi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$ and $p \in \Sigma$, we have:*

$$\textit{if } \varphi \vdash^+ \psi, \textit{ then } \varphi[p \leftarrow \phi] \vdash^+ \psi[p \leftarrow \phi].$$

Proof. Immediate, as all axioms and rules of the calculus of SPLTL are closed under substitutions. ▲

Lemma 3.1.12 (Positive Replacement Lemma)**.** *Consider Σ a set of propositional variables. For $\varphi, \psi, \phi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$ and $p \in \Sigma$, we have:*

$$\textit{if } \varphi \vdash^+ \psi \textit{ then } \phi[p \leftarrow \varphi] \vdash^+ \phi[p \leftarrow \psi].$$

Proof. By induction on the construction of ϕ. If $\phi = \top$ or $\phi = p \in \Sigma$ is clear. Consider $\varphi \vdash^+ \psi$ and $\phi = \phi_1 \wedge \phi_2$ with, this is our Induction Hypothesis,

$$\phi_i[p \leftarrow \varphi] \vdash^+ \phi_i[p \leftarrow \psi] \quad \text{for } i \in \{1, 2\}.$$

To simplify the notation, we will use the abbreviations $\varphi_i := \phi_i[p \leftarrow \varphi]$ and $\psi_i := \phi_i[p \leftarrow \psi]$ for $i \in \{1, 2\}$. Then we have:

1.	$\varphi_1 \wedge \varphi_2 \vdash^+ \varphi_1$;	by axiom ($\wedge$L)
2.	$\varphi_1 \wedge \varphi_2 \vdash^+ \varphi_2$;	by axiom ($\wedge$R)
3.	$\varphi_1 \vdash^+ \psi_1$;	Induction Hypothesis
4.	$\varphi_2 \vdash^+ \psi_2$;	Induction Hypothesis
5.	$\varphi_1 \wedge \varphi_2 \vdash^+ \psi_1$;	Cut on 1. and 3.
6.	$\varphi_1 \wedge \varphi_2 \vdash^+ \psi_2$;	Cut on 2. and 4.
7.	$\varphi_1 \wedge \varphi_2 \vdash^+ \psi_1 \wedge \psi_2$.	I$\wedge$ on 5. and 6.

As $\phi[p \leftarrow \varphi] = \varphi_1 \wedge \varphi_2$ and $\phi[p \leftarrow \psi] = \psi_1 \wedge \psi_2$, we deduce:

$$\phi[p \leftarrow \varphi] \vdash^+ \phi[p \leftarrow \psi].$$

The case $\phi = \mathcal{X}\phi_1$ follows easily from the rule $N^{\mathcal{X}}$, while the case $\phi = \phi_1 \,\mathcal{U}\, \phi_2$ is given by the rules $N_1^{\mathcal{U}}$ and $N_2^{\mathcal{U}}$. The proofs for both cases are straightforward and similar; therefore, we will show only the case where $\phi = \phi_1 \,\mathcal{U}\, \phi_2$:

1.	$\varphi_1 \vdash^+ \psi_1$;	Induction Hypothesis
2.	$\varphi_2 \vdash^+ \psi_2$;	Induction Hypothesis
3.	$\varphi_1 \,\mathcal{U}\, \varphi_2 \vdash^+ \psi_1 \,\mathcal{U}\, \varphi_2$;	$N_2^{\mathcal{U}}$ on 1.
4.	$\psi_1 \,\mathcal{U}\, \varphi_2 \vdash^+ \psi_1 \,\mathcal{U}\, \psi_2$;	$N_1^{\mathcal{U}}$ on 2.
5.	$\varphi_1 \,\mathcal{U}\, \varphi_2 \vdash^+ \psi_1 \,\mathcal{U}\, \psi_2$.	Cut on 3. and 4.

In conclusion, by induction on the construction of ϕ, we have $\phi[p \leftarrow \varphi] \vdash^+ \phi[p \leftarrow \psi]$, as intended. ▲

Now we can provide more results, whose proofs would come directly from the last two lemmas. In some sense, the following rules are just examples of the power or expressiveness given by the Substitution Lemma and the Positive Replacement Lemma.

Lemma 3.1.13. *Given a set of propositional variables Σ, for $\varphi, \psi, \phi_1, \phi_2 \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$ and $p \in \Sigma$, we have:*

i) if $\varphi \dashv\vdash^+ \psi$ and $\phi_1[p \leftarrow \varphi] \vdash^+ \phi_2[p \leftarrow \varphi]$, then $\phi_1[p \leftarrow \psi] \vdash^+ \phi_2[p \leftarrow \psi]$;

ii) if $\varphi \vdash^+ \psi$, then $\varphi \wedge \phi_1 \vdash^+ \psi \wedge \phi_1$ and $\phi_1 \wedge \varphi \vdash^+ \phi_1 \wedge \psi$;

iii) if $\varphi \vdash^+ \psi$ and $\phi_1 \vdash^+ \phi_2$, then $\varphi \wedge \phi_1 \vdash^+ \psi \wedge \phi_2$;

iv) if $\varphi \vdash^+ \psi$ and $\phi_1 \vdash^+ \phi_2$, then $\varphi \,\mathcal{U}\, \phi_1 \vdash^+ \psi \,\mathcal{U}\, \phi_2$. ▲

3.1.3 Soundness of SPLTL

We will present a soundness result between the consequence relation $\vdash^+$, defined through the calculus given in *Definition 3.1.2*, and the semantic relation $\vDash^+$ given below.

Definition 3.1.14. *Consider a set of propositional variables Σ, and a set of SPLTL-formulas $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$. We define $\vDash^+ \subseteq \mathcal{P}(\mathtt{Fm}_{\mathrm{SP}(\Sigma)}) \times \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$ by:*

$$\Gamma \vDash^+ \varphi \;:\Longleftrightarrow\; \text{for } \sigma \in \mathcal{P}(\Sigma)^\omega \text{ and for } i \in \mathbb{N}, \text{ if } \sigma, i \vDash \Gamma \text{ then } \sigma, i \vDash \varphi.$$

The notation $\sigma, i \vDash \Gamma$ is shorthand for stating that $\sigma, i \vDash \gamma$ for every $\gamma \in \Gamma$.

Observation 3.1.15. This relation $\vDash^+$ can be seen as a restriction of the relation $\vDash'_{\mathsf{LTL}}$, from *Definition 2.2.26*, over the set of SPLTL-formulas. This is because we are using the same satisfaction relation $\vDash$ that was introduced for LTL in *Definition 2.2.2*.

Notation. As we have done so far, we will denote $\{\psi\} \vDash^+ \varphi$ by $\psi \vDash^+ \varphi$, for $\varphi, \psi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$. Also, $\Gamma \vDash^+ \Delta$ refers to $\Gamma \vDash^+ \delta$ for every $\delta \in \Delta$, for $\Gamma, \Delta \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$.

As in the LTL formalization, the characterization of $\vDash^+$ using Kripke structures will also be useful. We determine it by, for $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$:

$$\Gamma \vDash^+ \varphi \Longleftrightarrow \text{for } K \text{ a Kripke structure and } \rho \text{ a path of } K, \text{ if } K, \rho \Vdash \Gamma \text{ then } K, \rho \Vdash \varphi;$$

where we consider $K, \rho \Vdash \Gamma$ to mean that we have $K, \rho \Vdash \gamma$ for every $\gamma \in \Gamma$.

Observe that for every finite $\Gamma_0 \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, it is immediate that $\Gamma_0 \vDash^+ \varphi$ is equivalent to $\bigwedge_{\gamma\in\Gamma_0} \gamma \vDash^+ \varphi$. So, working with single formulas is equivalent to doing it with finite sets.

With the definitions introduced, we can now prove soundness:

Theorem 3.1.16 (Soundness of SPLTL)**.** *Consider a set of propositional variables* Σ*. For every* $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$*, we have:*

$$\Gamma \vdash^+ \varphi \quad \Longrightarrow \quad \Gamma \vDash^+ \varphi.$$

Proof. By induction on the assumed derivation of $\Gamma \vdash^+ \varphi$. It suffices to show that all axioms of *Definition 3.1.2* are valid and that the rules preserve validity. To prove this, we will use the following chain of implications, where Γ_0 is some finite subset of Γ:

$$\Gamma \vdash^+ \varphi \overset{(1)}{\Longleftrightarrow} \Gamma_0 \vdash^+ \varphi \overset{(2)}{\Longrightarrow} \vdash_{\mathsf{LTL}} \bigwedge_{\gamma\in\Gamma_0} \gamma \to \varphi \overset{(3)}{\Longleftrightarrow} \vDash_{\mathsf{LTL}} \bigwedge_{\gamma\in\Gamma_0} \gamma \to \varphi \overset{(4)}{\Longleftrightarrow}$$

$$\overset{(4)}{\Longleftrightarrow} \bigwedge_{\gamma\in\Gamma_0} \gamma \vDash'_{\mathsf{LTL}} \varphi \overset{(5)}{\Longleftrightarrow} \Gamma_0 \vDash'_{\mathsf{LTL}} \varphi \overset{(6)}{\Longrightarrow} \Gamma \vDash'_{\mathsf{LTL}} \varphi \overset{(7)}{\Longleftrightarrow} \Gamma \vDash^+ \varphi.$$

The first double implication is due to the compactness from *Lemma 3.1.10*. The implication (2) is the one we need to prove. (3) is satisfied thanks to the completeness theorem for LTL, *Theorem 2.2.24*. The correlation (4) is an application of the Semantic Deduction Theorem, *Theorem 2.2.28*. The expressions (5) and (6) are immediately proved by the definition of the relation $\vDash'_{\mathsf{LTL}}$. And (7) is straightforward by the definition of $\vDash^+$, as also indicates *Observation 3.1.15*.

We can prove (2) by induction on the derivation $\Gamma_0 \vdash^+ \varphi$. As we will see below, referring to the completeness of LTL and unraveling the semantic relation simplifies the proof. However, we want to work with our SPLTL calculus, therefore, we will limit the semantic translation to selected cases, and prove the remaining expressions syntactically within the calculus.

If $\Gamma_0 \vdash^+ \varphi$ is an instance of some of our first four axioms, from *Definition 3.1.2*, then the proof is settled since for every $\psi, \phi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, we know from propositional tautologies that $\vdash_{\mathsf{LTL}} \psi \to \psi$, that $\vdash_{\mathsf{LTL}} \top$, and $\vdash_{\mathsf{LTL}} (\psi \wedge \phi) \to \psi$ and $\vdash_{\mathsf{LTL}} (\psi \wedge \phi) \to \phi$. If $\Gamma_0 \vdash^+ \varphi$ is an instance of axiom (DistX), then we recall that

$$\vdash_{\mathsf{LTL}} \mathcal{X}(\psi \wedge \phi) \to (\mathcal{X}\psi \wedge \mathcal{X}\phi);$$

$$\vdash_{\mathsf{LTL}} (\mathcal{X}\psi \wedge \mathcal{X}\phi) \to \mathcal{X}(\psi \wedge \phi);$$

as they are theorems of $\mathbf{K}_{\mathcal{X}}$ (see also *Example 2.2.10*).

If $\Gamma_0 \vdash^+ \varphi$ is an instance of axiom (U1+), we only need to consider an instance of the LTL axiom (U1):

$$\vdash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi \to \top \,\mathcal{U}\, \phi.$$

If $\Gamma_0 \vdash^+ \varphi$ is given by the axiom (UId), then we need to show that we have

$$\vdash_{\mathsf{LTL}} \varphi \,\mathcal{U}\, \varphi \to \varphi.$$

As we mentioned, by the completeness of LTL, or equivalently by the correlation (3), we know it is enough to prove its semantic counterpart. For every $\sigma \in \mathcal{P}(\Sigma)^\omega$ and every $i \in \mathbb{N}$, we see:

$$\sigma, i \vDash_{\mathsf{LTL}} \varphi \,\mathcal{U}\, \varphi \iff \text{there is } j \geqslant i \text{ that for every } i \leqslant k \leqslant j, \text{ we have } \sigma, k \vDash_{\mathsf{LTL}} \varphi.$$

This gives us that, if $\sigma, i \vDash_{\mathsf{LTL}} \varphi \,\mathcal{U}\, \varphi$, then, in particular, $\sigma, i \vDash_{\mathsf{LTL}} \varphi$. That is, we see that we have $\sigma, i \vDash_{\mathsf{LTL}} \varphi \,\mathcal{U}\, \varphi \to \varphi$. Since this applies for every $\sigma \in \mathcal{P}(\Sigma)^\omega$ and every $i \in \mathbb{N}$, we deduce $\vDash_{\mathsf{LTL}} \varphi \,\mathcal{U}\, \varphi \to \varphi$, as needed.

Now consider $\Gamma_0 \vdash^+ \varphi$ to be given by an instance of the axiom (U21). This means that we have $\Gamma = \{\phi\}$ and $\varphi = \psi \,\mathcal{U}\, \phi$, for some $\psi, \phi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$. We derive:

1.	$\vdash_{\mathsf{LTL}} \phi \vee (\psi \wedge \mathcal{X}(\psi \,\mathcal{U}\, \phi)) \to \psi \,\mathcal{U}\, \phi$;	instance of (U2)
2.	$\vdash_{\mathsf{LTL}} \phi \to \phi \vee (\psi \wedge \mathcal{X}(\psi \,\mathcal{U}\, \phi))$;	instance of propositional tautology
3.	$\vdash_{\mathsf{LTL}} \phi \to \psi \,\mathcal{U}\, \phi$.	(prop)+1.+2.

So we deduce $\vdash_{\mathsf{LTL}} \bigwedge_{\gamma \in \Gamma_0} \gamma \to \varphi$, as required. If $\Gamma_0 \vdash^+ \varphi$ follows from the axiom (U22), then a similar argument to the previous one demonstrates that $\vdash_{\mathsf{LTL}} (\psi \wedge \mathcal{X}(\psi \,\mathcal{U}\, \phi)) \to \psi \,\mathcal{U}\, \phi$, as needed.

Now consider $\Gamma_0 \vdash^+ \varphi$ to be an instance of the axiom (UR). As previously established, it is enough to deduce

$$\vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\psi \,\mathcal{U}\, \phi) \to \psi \,\mathcal{U}\, \phi.$$

We do prove it by unraveling the semantic satisfaction definition. For every $\sigma \in \mathcal{P}(\Sigma)^\omega$ and every $i \in \mathbb{N}$, we see that

$$\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\psi \,\mathcal{U}\, \phi) \iff \text{there is some } j \geqslant i \text{ such that } \sigma, j \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi, \quad \text{and } \sigma, k \vDash_{\mathsf{LTL}} \psi \text{ for every } i \leqslant k < j;$$

$$\sigma, j \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi \iff \text{there is some } j' \geqslant j \text{ such that } \sigma, j' \vDash_{\mathsf{LTL}} \phi, \quad \text{with } \sigma, k' \vDash_{\mathsf{LTL}} \psi \text{ for every } j \leqslant k' < j'.$$

Then, if $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\psi \,\mathcal{U}\, \phi)$ we see that there is some $j' \geqslant j \geqslant i$ such that $\sigma, j' \vDash_{\mathsf{LTL}} \phi$ and $\sigma, k'' \vDash_{\mathsf{LTL}} \psi$ for every $i \leqslant k'' < j'$. That is, we have $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi$. This gives us that the relation $\vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\psi \,\mathcal{U}\, \phi) \to \psi \,\mathcal{U}\, \phi$ holds.

If $\Gamma_0 \vdash^+ \varphi$ is an instance of axiom (UL), then we proceed similarly as before:

$$\sigma, i \vDash_{\mathsf{LTL}} (\psi \,\mathcal{U}\, \phi) \,\mathcal{U}\, \phi \iff \text{there is some } j \geqslant i \text{ such that } \sigma, j \vDash_{\mathsf{LTL}} \phi, \quad \text{and } \sigma, k \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi \text{ for every } i \leqslant k < j.$$

If such j is $j = i$, we have $\sigma, i \vDash_{\mathsf{LTL}} \phi$, and so $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi$. If $j > i$, then we also have $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi$ since trivially $i \leqslant i < j$. Therefore, we conclude

$$\vDash_{\mathsf{LTL}} (\psi \,\mathcal{U}\, \phi) \,\mathcal{U}\, \phi \to \psi \,\mathcal{U}\, \phi.$$

On the other hand, suppose that we have $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi$, that is, there is some $j \geqslant i$ such that $\sigma, j \vDash_{\mathsf{LTL}} \phi$, and $\sigma, k \vDash_{\mathsf{LTL}} \psi$ for every $i \leqslant k < j$. Then, we easily observe that $\sigma, k \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi$ also holds. This gives us that if $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi$, then there is some

$j \geqslant i$ such that $\sigma, j \vDash_{\mathsf{LTL}} \phi$, and $\sigma, k \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi$ for every $i \leqslant k < j$. Therefore, we have $\sigma, i \vDash_{\mathsf{LTL}} (\psi \,\mathcal{U}\, \phi) \,\mathcal{U}\, \phi$. This shows the satisfiability of

$$\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi \to (\psi \,\mathcal{U}\, \phi) \,\mathcal{U}\, \phi.$$

Since this applies for every $\sigma \in \mathcal{P}(\Sigma)^\omega$ and every $i \in \mathbb{N}$, we deduce $\vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi \to (\psi \,\mathcal{U}\, \phi) \,\mathcal{U}\, \phi$, and then $\vdash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi \to (\psi \,\mathcal{U}\, \phi) \,\mathcal{U}\, \phi$, as we wanted.

Now suppose $\Gamma_0 \vdash^+ \varphi$ to be an instance of the axiom (M). We have:

$$\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\psi \wedge \phi) \iff \text{there is some } j \geqslant i \text{ such that } \sigma, j \vDash_{\mathsf{LTL}} \phi, \text{ and } \sigma, k \vDash_{\mathsf{LTL}} \psi \text{ for every } i \leqslant k \leqslant j.$$

If $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\psi \wedge \phi)$ holds, in particular we have $\sigma, i \vDash_{\mathsf{LTL}} \psi$, and

$$\sigma, i \vDash_{\mathsf{LTL}} \mathcal{X}\psi \,\mathcal{U}\, \phi \iff \text{there is some } j \geqslant i \text{ such that } \sigma, j \vDash_{\mathsf{LTL}} \phi, \text{ and } \sigma, k \vDash_{\mathsf{LTL}} \psi \text{ for every } i < k \leqslant j;$$

will also hold. Then, we can conclude that

$$\vdash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\psi \wedge \phi) \to (\psi \wedge \mathcal{X}\psi \,\mathcal{U}\, \phi).$$

A similar unraveling of the semantic satisfiability definition also ensures the theorem

$$\vdash_{\mathsf{LTL}} (\psi \wedge \mathcal{X}\psi \,\mathcal{U}\, \phi) \to \psi \,\mathcal{U}\, (\psi \wedge \phi).$$

For the case where $\Gamma_0 \vdash^+ \varphi$ is an instance of (DistUR), suppose for every $\sigma \in \mathcal{P}(\Sigma)^\omega$, every $i \in \mathbb{N}$, and for $\psi, \phi, \xi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, that we have $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\phi \wedge \xi)$. Then, there is some $j \geqslant i$ such that $\sigma, j \vDash_{\mathsf{LTL}} \phi \wedge \xi$, with $\sigma, k \vDash_{\mathsf{LTL}} \psi$ for every $i \leqslant k < j$. In particular, $\sigma, j \vDash_{\mathsf{LTL}} \phi$ and $\sigma, j \vDash_{\mathsf{LTL}} \xi$, and so we conclude that $\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \phi \wedge \psi \,\mathcal{U}\, \xi$ also holds. That is,

$$\sigma, i \vDash_{\mathsf{LTL}} \psi \,\mathcal{U}\, (\phi \wedge \xi) \to (\psi \,\mathcal{U}\, \phi \wedge \psi \,\mathcal{U}\, \xi).$$

We now consider $\Gamma_0 \vdash^+ \varphi$ to be of the form $(\psi \wedge \phi) \,\mathcal{U}\, \xi \vdash^+ \psi \,\mathcal{U}\, \xi \wedge \phi \,\mathcal{U}\, \xi$, an instance of one of the sides of (DistUL), for $\psi, \phi, \xi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$. By definition, for $\sigma \in \mathcal{P}(\Sigma)^\omega$ and $i \in \mathbb{N}$, if $\sigma, i \vDash_{\mathsf{LTL}} (\psi \wedge \phi) \,\mathcal{U}\, \xi$, then there is some $j \geqslant i$ such that $\sigma, j \vDash_{\mathsf{LTL}} \xi$, and $\sigma, k \vDash_{\mathsf{LTL}} \psi \wedge \phi$ for every $i \leqslant k < j$. Since $\sigma, k \vDash_{\mathsf{LTL}} \psi \wedge \phi$ gives us that $\sigma, k \vDash_{\mathsf{LTL}} \psi$ and $\sigma, k \vDash_{\mathsf{LTL}} \phi$, we deduce, as desired, that we have

$$\vDash_{\mathsf{LTL}} (\psi \wedge \phi) \,\mathcal{U}\, \xi \to (\psi \,\mathcal{U}\, \xi \wedge \phi \,\mathcal{U}\, \xi).$$

If $\Gamma_0 \vdash^+ \varphi$ is an instance of (DistUL) of the form $\psi \,\mathcal{U}\, \xi \wedge \phi \,\mathcal{U}\, \xi \vdash^+ (\psi \wedge \phi) \,\mathcal{U}\, \xi$, a similar argument as before ensures that we have:

$$\vDash_{\mathsf{LTL}} (\psi \,\mathcal{U}\, \xi \wedge \phi \,\mathcal{U}\, \xi) \to (\psi \wedge \phi) \,\mathcal{U}\, \xi.$$

Therefore, we conclude that $\vdash_{\mathsf{LTL}} (\psi \,\mathcal{U}\, \xi \wedge \phi \,\mathcal{U}\, \xi) \to (\psi \wedge \phi) \,\mathcal{U}\, \xi$.

Finishing with the axiom instances, if $\Gamma_0 \vdash^+ \varphi$ is derived by the axiom (DistUX), it suffices to show that we have:

$$\vDash_{\mathsf{LTL}} \mathcal{X}\psi \,\mathcal{U}\, \mathcal{X}\phi \to \mathcal{X}(\psi \,\mathcal{U}\, \phi) \quad \text{and} \quad \vDash_{\mathsf{LTL}} \mathcal{X}(\psi \,\mathcal{U}\, \phi) \to \mathcal{X}\psi \,\mathcal{U}\, \mathcal{X}\phi.$$

We see, for every $\sigma \in \mathcal{P}(\Sigma)^\omega$ and every $i \in \mathbb{N}$:

$$\sigma, i \vDash_{\mathsf{LTL}} \mathcal{X}\psi\,\mathcal{U}\,\mathcal{X}\phi \iff \text{there is some } j \geqslant i \text{ such that } \sigma, j \vDash_{\mathsf{LTL}} \mathcal{X}\phi, \text{ and } \sigma, k \vDash_{\mathsf{LTL}} \mathcal{X}\psi \text{ for every } i \leqslant k < j.$$

Then, we have:

$$\sigma, i \vDash_{\mathsf{LTL}} \mathcal{X}\psi\,\mathcal{U}\,\mathcal{X}\phi \iff \text{there is some } j \geqslant i+1 \text{ such that } \sigma, j \vDash_{\mathsf{LTL}} \phi, \text{ and } \sigma, k \vDash_{\mathsf{LTL}} \psi \text{ for every } i < k < j.$$

Notice that the last expression is equivalent to state $\sigma, i \vDash_{\mathsf{LTL}} \mathcal{X}(\psi\,\mathcal{U}\,\phi)$. This gives us:

$$\vDash_{\mathsf{LTL}} \mathcal{X}\psi\,\mathcal{U}\,\mathcal{X}\phi \to \mathcal{X}(\psi\,\mathcal{U}\,\phi) \quad \text{and} \quad \vDash_{\mathsf{LTL}} \mathcal{X}(\psi\,\mathcal{U}\,\phi) \to \mathcal{X}\psi\,\mathcal{U}\,\mathcal{X}\phi;$$

as required to conclude

$$\vdash_{\mathsf{LTL}} \mathcal{X}\psi\,\mathcal{U}\,\mathcal{X}\phi \to \mathcal{X}(\psi\,\mathcal{U}\,\phi) \quad \text{and} \quad \vdash_{\mathsf{LTL}} \mathcal{X}(\psi\,\mathcal{U}\,\phi) \to \mathcal{X}\psi\,\mathcal{U}\,\mathcal{X}\phi.$$

Regarding inductive cases: suppose that an application of the Weakening rule derives $\Gamma_0 \vdash^+ \varphi$. Then we know $\Gamma_0' \vdash^+ \varphi$ for some $\Gamma_0' \subseteq \Gamma_0$. By Induction Hypothesis, we have

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0'} \gamma\right) \to \varphi.$$

Thus, by instances of propositional tautologies, we clearly have

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \varphi.$$

If $\Gamma_0 \vdash^+ \varphi$ is given by the rule $\mathrm{I}\wedge$, then φ is of the form $\varphi_1 \wedge \varphi_2$ and, by Induction Hypothesis, we have:

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \varphi_1 \quad \text{and} \quad \vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \varphi_2.$$

By instances of propositional tautologies, it is easy to deduce

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to (\varphi_1 \wedge \varphi_2).$$

A similar argument works for the case where $\Gamma_0 \vdash^+ \varphi$ is derived from the Cut rule.

Suppose $\Gamma_0 \vdash^+ \varphi$ is given by the application of the rule $N^{\mathcal{X}}$. This means that we have some relation $\Gamma_0' \vdash^+ \varphi'$, with $\Gamma_0 = \mathcal{X}\Gamma_0'$ and $\varphi = \mathcal{X}\varphi$. Then, we derive:

1. $\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0'} \gamma\right) \to \varphi'$; Induction Hypothesis
2. $\vdash_{\mathsf{LTL}} \mathcal{X}\left(\left(\bigwedge_{\gamma \in \Gamma_0'} \gamma\right) \to \varphi'\right)$; rule $N_{\mathcal{X}}$ over 1.

3. $\vdash_{\mathsf{LTL}} \mathcal{X}\left(\bigwedge_{\gamma\in\Gamma_0'} \gamma\right) \to \mathcal{X}\varphi'$; *MP* over 2. and an instance of axiom $(K_{\mathcal{X}})$

4. $\vdash_{\mathsf{LTL}} \mathcal{X}\left(\bigwedge_{\gamma\in\Gamma_0'} \gamma\right) \leftrightarrow \left(\bigwedge_{\gamma\in\Gamma_0'} \mathcal{X}\gamma\right)$; from logic $\mathbf{K}_X$

5. $\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma\in\Gamma_0'} \mathcal{X}\gamma\right) \to \mathcal{X}\varphi'$. Substitution Lemma on 3. by 4.

The last expression is equivalent to

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma\in\mathcal{X}\Gamma_0'} \gamma\right) \to \mathcal{X}\varphi'.$$

Therefore, since $\mathcal{X}\Gamma_0' = \Gamma_0$, we get the LTL theorem we wanted.

If $\Gamma_0 \vdash^+ \varphi$ is derived from the rule $N_1^{\mathcal{U}}$, then we know, by Induction Hypothesis, that $\vdash_{\mathsf{LTL}} \gamma \to \varphi'$, with $\Gamma_0 = \{\phi_1 \,\mathcal{U}\, \gamma\}$ and $\varphi = \phi_1 \,\mathcal{U}\, \varphi'$ for some $\phi_1 \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$. We need to show that we have $\vdash_{\mathsf{LTL}} \phi_1 \,\mathcal{U}\, \gamma \to \phi_1 \,\mathcal{U}\, \varphi'$.

We will not explicitly state this verification since it just mirrors the inductive case $\psi = \psi_1 \,\mathcal{U}\, \psi_2$ from the proof of the Substitution Lemma for LTL, *Lemma 2.2.8*. While in the Substitution Lemma we started a sequence of theorems of LTL with the assumptions $\phi_1' \leftrightarrow \phi_2'$ and $\phi_1'' \leftrightarrow \phi_2''$, now we do it with $\gamma \to \varphi'$ and, if not, $\phi_1 \leftrightarrow \phi_1$. Note that we do not require the first assumption to be an equivalence, a double implication, as we only want to prove $\vdash_{\mathsf{LTL}} \phi_1 \,\mathcal{U}\, \gamma \to \phi_1 \,\mathcal{U}\, \varphi'$, not $\vdash_{\mathsf{LTL}} \phi_1 \,\mathcal{U}\, \gamma \leftrightarrow \phi_1 \,\mathcal{U}\, \varphi'$. The case where $\Gamma_0 \vdash^+ \varphi$ is derived using the rule $N_2^{\mathcal{U}}$ can be demonstrated in a similar manner.

In conclusion, we have proved, by induction on the derivation $\Gamma_0 \vdash^+ \varphi$, that we have:

$$\Gamma_0 \vdash^+ \varphi \implies \vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma\in\Gamma_0} \gamma\right) \to \varphi.$$

This proves the implication (2), closing the previously introduced chain of implications. Then, we can finally conclude our soundness theorem: for every $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, we have:

$$\Gamma \vdash^+ \varphi \implies \Gamma \vDash^+ \varphi.$$

▲

3.1.4 On the Completeness of SPLTL

In this section, we will introduce some theories and concepts that may be useful for proving a possible completeness theorem between $\vdash^+$ and $\vDash^+$. Specifically, the objective would be to obtain the reverse implication of the soundness *Theorem 3.1.16*, so that for every $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, we have:

$$\Gamma \vDash^+ \varphi \implies \Gamma \vdash^+ \varphi.$$

In [Das12], by applying results from [CJ97; Dun95], some completeness results are demonstrated for strictly positive fragments of various modal logics. Consequently, it should be relatively straightforward to establish completeness results for the strictly positive fragments of LTL that include only the operators $\Box$, $\Diamond$, or $\mathcal{X}$. To investigate the strictly positive fragment that only considers the modal operators $\mathcal{X}$ and $\Diamond$ could also be particularly interesting, as this logic has already been suggested and utilized within the framework of database queries, see [Jun+24], for instance.

In the conclusions of that same paper [Jun+24], they propose studying the LTL fragment containing the $\mathcal{U}$ operator instead of just $\Diamond$, along with $\mathcal{X}$. This is precisely what we are doing with our introduced SPLTL calculus, although we are exploring SPLTL outside the context of database queries. Unfortunately, because it uses $\mathcal{U}$ rather than $\Diamond$, our logic does not directly benefit from the completeness results mentioned above.

Then, the best way we find to prove a completeness theorem for SPLTL, is by mimicking the argument we gave for the LTL completeness, in *Theorem 2.2.24*, adapting it to the SPLTL framework.

First, we note that the concept of consistency in SPLTL is not expressible, as the formalization of SPLTL does not include the connective $\neg$ or $\bot$. Therefore, instead of maximal consistent sets, we will consider *deductively closed* sets:

Definition 3.1.17. *Given a set of propositional variables Σ, the set $\Gamma \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$ is* ***deductively closed*** *(DCS) if $\Gamma \vdash^+ \varphi$ implies $\varphi \in \Gamma$, for every $\varphi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$.*

We recall that for every numerable set of propositional variables Σ, we can enumerate our SPLTL-formulas. Specifically, we consider $\mathtt{Fm}_{\mathrm{SP}(\Sigma)} = \{\theta_i : i < \omega\}$. The next definition will easily show us that we have a version of Lindenbaum's lemma for DCSs:

Definition 3.1.18. *Let $\Gamma \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)} = \{\theta_i : i < \omega\}$ over the set of propositional variables Σ. We consider the* ***deductive closure*** *of Γ, and we denote it by $DC(\Gamma)$, to be the set*

$$DC(\Gamma) := \bigcup_{i<\omega} \Gamma_i;$$

where we define $\Gamma_0 = \Gamma$ and, for every $i < \omega$:

$$\Gamma_{i+1} := \begin{cases} \Gamma_i \cup \{\theta_i\} & \text{if } \Gamma_i \vdash^+ \theta_i; \\ \Gamma_i & \text{otherwise.} \end{cases}$$

We abbreviate $DC(\{\varphi\})$ by $DC(\varphi)$.

Lemma 3.1.19. *Consider a set of propositional variables Σ. For every set of formulas $\Gamma \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, its deductive closure $DC(\Gamma)$ is a DCS. Moreover, we have $\Gamma \subseteq DC(\Gamma)$.*

Proof. Let $\Gamma \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)} = \{\theta_i : i < \omega\}$ be a set of SPLTL-formulas. By the definition of the deductive closure of Γ, the inclusion $\Gamma \subseteq DC(\Gamma)$ is immediate. To check that $DC(\Gamma)$ is deductively closed, we suppose, for $\varphi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, that we have $DC(\Gamma) \vdash^+ \varphi$, and we need to show that $\varphi \in DC(\Gamma)$.

By compactness, we know that $DC(\Gamma) \vdash^+ \varphi$ implies that there is some finite $\Gamma' \subseteq DC(\Gamma) = \bigcup_{i<\omega} \Gamma_i$ such that $\Gamma' \vdash^+ \varphi$. In addition, there is some $k < \omega$ such that $\Gamma' \subseteq \Gamma_k$. And clearly $\varphi = \theta_j$ for some $j < \omega$. Therefore, if $j \geqslant k$, we have $\varphi \in \Gamma_{j+1} \subseteq DC(\Gamma)$ by definition, since $\Gamma' \vdash^+ \varphi$ implies $\Gamma_k \vdash^+ \varphi$ and so $\Gamma_j \vdash^+ \varphi$, by the Weakening rule.

On the other hand, suppose that $j < k$. By the construction of the sets Γ_i, we know that Γ_k is of the form $\Gamma_k = \Gamma_j \cup \{\theta_{j_1}, \ldots, \theta_{j_n}\}$ for some $n \leqslant k - j$, with $\Gamma_j \vdash^+ \theta_{j_l}$ for every $1 \leqslant l \leqslant n$. In this way, it is easy to derive that we have $\Gamma_j \vdash^+ \Gamma_k$. Since we know that $\Gamma_k \vdash^+ \varphi$, we also have, by the Cut rule, that $\Gamma_j \vdash^+ \varphi$. Then, by definition, $\varphi = \theta_j \in \Gamma_{j+1} \subseteq DC(\Gamma)$. This shows that, in any case, $DC(\Gamma) \vdash^+ \varphi$ implies $\varphi \in DC(\Gamma)$, so $DC(\Gamma)$ is a DCS, as expected. ▲

The following two lemmas could be useful, in addition to providing some intuition on the deductive closure notion:

Lemma 3.1.20. *Consider Σ to be a set of propositional variables, and $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$. We have:*

$$\Gamma \vdash^+ \varphi \iff DC(\Gamma) \vdash^+ \varphi.$$

Proof. If $\Gamma \vdash^+ \varphi$, then it is clear by Weakening that $DC(\Gamma) \vdash^+ \varphi$, as we know $\Gamma \subseteq DC(\Gamma)$ by the previous lemma. We show the other implication by contraposition:

We suppose $\Gamma \nvdash^+ \varphi$. By *Definition 3.1.18*, we know that $DC(\Gamma)$ is of the form $DC(\Gamma) = \bigcup_{i<\omega} \Gamma_i$. Considering that $\vdash^+$ is compact and that, by their definitions, we have $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_k \subseteq \dots$, it suffices to show $\Gamma_k \nvdash^+ \varphi$ for every $k < \omega$. We prove it by induction:

The base case is clear, as we have assumed $\Gamma \nvdash^+ \varphi$. For the inductive case, consider $\Gamma_k \nvdash^+ \varphi$. We have $\Gamma_{k+1} = \Gamma_k$ or $\Gamma_{k+1} = \Gamma_k \cup \{\theta_k\}$, with $\Gamma_k \vdash^+ \theta_k$. If the first option holds, it follows immediately that $\Gamma_{k+1} \nvdash^+ \varphi$.

On the other hand, if $\Gamma_{k+1} = \Gamma_k \cup \{\theta_k\}$ and we suppose $\Gamma_{k+1} \vdash^+ \varphi$, then we get a contradiction by applying the Cut rule: as $\Gamma_k \vdash^+ \gamma$ for every $\gamma \in \Gamma_{k+1}$, and $\Gamma_{k+1} \vdash^+ \varphi$, we deduce $\Gamma_k \vdash^+ \varphi$, which contradicts our Induction Hypothesis. This leads us to deduce $\Gamma_{k+1} \nvdash^+ \varphi$.

This shows, by induction on $k < \omega$, that $\Gamma_k \nvdash^+ \varphi$, and so we have $DC(\Gamma) \nvdash^+ \varphi$. In conclusion, we have proved by contraposition the other implication needed to state

$$\Gamma \vdash^+ \varphi \iff DC(\Gamma) \vdash^+ \varphi.$$

▲

Lemma 3.1.21. *For every $\Gamma \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$ over a set of propositional variables Σ, we have $DC(DC(\Gamma)) = DC(\Gamma)$.*

Proof. The inclusion $DC(\Gamma) \subseteq DC(DC(\Gamma))$ is clear by *Lemma 3.1.19*. We show the other inclusion:

By the previous lemma, we know

$$DC(\Gamma) \vdash^+ \varphi \iff DC(DC(\Gamma)) \vdash^+ \varphi.$$

If $\varphi \in DC(DC(\Gamma))$, then clearly $DC(DC(\Gamma)) \vdash^+ \varphi$, and thus $DC(\Gamma) \vdash^+ \varphi$ by the expression above. Since $DC(\Gamma)$ is a DCS, as we have seen in *Lemma 3.1.19*, we conclude $\varphi \in DC(\Gamma)$, which demonstrates the inclusion $DC(DC(\Gamma)) \subseteq DC(\Gamma)$. ▲

Although we will use DCSs to replace the class of MCSs, it is important to note that DCSs do not exhibit the same maximal behavior as MCSs. For instance, for $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, consider Γ to be a DCS and that we have $\varphi \notin DC(\Gamma)$, so $\Gamma \nvdash^+ \varphi$. Then, thanks to *Lemma 3.1.19*, we have that $DC(\Gamma \cup \{\varphi\})$ will be a DCS containing $\Gamma \cup \{\varphi\}$, so we conclude $\Gamma \subset DC(\Gamma \cup \{\varphi\})$. Therefore, every DCS that does not contain a certain formula is included in a larger DCS. In this sense, the only DCSs with the notion of maximality introduced for MCSs would be the whole $\mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, as it is the only set such that every SPLTL-formula belongs to it.

Regarding the Fischer-Ladner closure that we used in the LTL completeness proof, we point out that, since we do not have the connectives $\vee$ or $\neg$ in the SPLTL context, we can simplify the definition:

Definition 3.1.22. *Consider Σ a set of propositional variables. For $\varphi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, we will denote its **(strictly positive) Fischer-Ladner closure** by the notation $FL^+(\varphi)$. We inductively define it by:*

- $\varphi \in FL^+(\varphi)$;
- *$FL^+(\varphi)$ is closed under subformulas;*
- *if $\varphi_1 \,\mathcal{U}\, \varphi_2 \in FL^+(\varphi)$, then $\mathcal{X}(\varphi_1 \,\mathcal{U}\, \varphi_2) \in FL^+(\varphi)$.*

Clearly, for every $\varphi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, the set $FL^+(\varphi)$ is finite.

Now, following the proof given for LTL, to show the completeness implication:

$$\Gamma \vDash^+ \varphi \implies \Gamma \vdash^+ \varphi;$$

for $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, we proceed by contraposition. We suppose $\Gamma \nvdash^+ \varphi$ and want to demonstrate that $\Gamma \nvDash^+ \varphi$. Then, we need to provide a Kripke structure and a path of it such that Γ holds but φ does not.

We consider picking $K = (S, R, V)$ with

$$S := \{\Delta \cap FL^+(\Gamma \cup \{\varphi\}) : \Delta \text{ is deductively closed}\}.$$

We define the relation R and the valuation V similarly as in the LTL completeness proof, for $\Delta, \Delta' \in S$:

$$(\Delta, \Delta') \in R \;:\Longleftrightarrow\; \mathcal{X}^{-1}\Delta \subseteq \Delta';$$

and $V(\Delta) := \Delta \cap \Sigma$, that is, $V(\Delta)$ is the set of propositional variables appearing in Δ as (atomic) formulas. Note that we have defined a Kripke structure K for every pair (Γ, φ). As in the LTL case, it can be seen that R is left-total, so K is certainly a Kripke structure.

The first difficulty will be to define a path $\rho_{(\Gamma,\varphi)}$ that satisfies a Truth Lemma of the form, for every $\phi \in FL^+(\Gamma \cup \{\varphi\})$ and $\Delta_0 \in S$ the first state of the path:

$$K, \rho_{(\Gamma,\varphi)} \Vdash \phi \iff \phi \in \Delta_0.$$

In fact, we will abandon the search for that Truth Lemma and instead focus on providing a path that specifically ensures that Γ holds while φ does not. We could consider showing a kind of *semi-truth lemma* only satisfying one of the implications of the real Truth Lemma, that is, that for every $\phi \in FL^+(\Gamma \cup \{\varphi\})$ and $\Delta_0 \in S$ being the first state of the path, we have:

$$\phi \in \Delta_0 \implies K, \rho_{(\Gamma,\varphi)} \Vdash \phi.$$

We provide two examples to illustrate why choosing these approaches is preferable to seeking the usual Truth Lemma:

First, consider $\Gamma = \{\psi_1 \mathcal{U} \psi_2\}$. We will see that attempting to preserve the Truth Lemma introduces some complications. We want $K, \rho_{(\Gamma,\varphi)} \Vdash \psi_1 \mathcal{U} \psi_2$, so we include $\psi_1 \mathcal{U} \psi_2 \in \Delta_0$, the first state of the path $\rho_{(\Gamma,\varphi)}$. Now, to make $K, \rho_{(\Gamma,\varphi)} \Vdash \psi_1 \mathcal{U} \psi_2$ to hold, we need $K, \rho^j_{(\Gamma,\varphi)} \Vdash \psi_2$ for some $j \geqslant 0$ and $K, \rho^k_{(\Gamma,\varphi)} \Vdash \psi_1$ for every $0 \leqslant k < j$. Observe that $K, \rho_{(\Gamma,\varphi)} \Vdash \psi_1$ or $K, \rho_{(\Gamma,\varphi)} \Vdash \psi_2$, so either $\psi_1 \in \Delta_0$ or $\psi_2 \in \Delta_0$. We know $\psi_1 \mathcal{U} \psi_2 \nvdash^+ \psi_1$ and $\psi_1 \mathcal{U} \psi_2 \nvdash^+ \psi_2$. Therefore, when constructing the path $\rho_{(\Gamma,\varphi)}$, particularly the first state Δ_0, the role of the formula φ becomes crucial. We must carefully choose the elements of Δ_0 to ensure that $\varphi \notin \Delta_0$; otherwise, φ would be satisfied. In our case, if $\varphi = \psi_1$, then we need $\psi_2 \in \Delta_0$ but $\psi_1 \notin \Delta_0$, and *vice versa* for $\varphi = \psi_2$. But for $\varphi \notin \{\psi_1, \psi_2\}$, we would need more information about ψ_1, ψ_2 and φ in order to properly construct Δ_0. In essence, this shows that defining our path now requires a procedure to determine which formulas should belong to each element of the path and which do not.

In the LTL context, the axioms, along with the consistency and maximality aspect of the path elements, ensured that all possible formulas were included in each state. For instance, thanks to the LTL axiom (U2), if $\psi_1 \mathcal{U} \psi_2 \in \Delta_0$, then it necessarily follows that $\psi_2 \vee (\psi_1 \wedge \mathcal{X}(\psi_1 \mathcal{U} \psi_2)) \in \Delta_0$. Thus, if we knew that $\psi_2 \notin \Delta_0$, consistency required that $\psi_1 \wedge \mathcal{X}(\psi_1 \mathcal{U} \psi_2) \in \Delta_0$, and *vice versa*. In the SPLTL framework, we must actively incorporate or exclude those elements from Δ_0, since we lack an axiom like (U2) and both consistency and maximality of our states. With this in mind, ensuring that a specific formula φ is not satisfied seems more achievable than ensuring that all formulas that should not be satisfiable are indeed not satisfied. This reinforces the strategy of defining a particular path that ensures the satisfiability of Γ while guaranteeing the non-satisfiability of φ, although it might compromise the generality of the Truth Lemma.

The second example is similar but even more direct and clear. Suppose that in the second element of the path $\rho_{(\Gamma,\varphi)}$, call it Δ_1, we have $\psi \in \Delta_1$. Then, we clearly have $K, \rho_{(\Gamma,\varphi)} \Vdash \mathcal{X}\psi$. However, we do not necessarily have $\mathcal{X}\psi \in \Delta_0$. Observe that we have $\mathcal{X}^{-1}\Delta_0 \subseteq \Delta_1$, but not $\mathcal{X}\Delta_1 \subseteq \Delta_0$.

This example, like the first one, does not present an issue in the LTL framework. If $\psi \in \Delta_1$, then we can also derive $\mathcal{X}\psi \in \Delta_0$, in the LTL context. Otherwise, we would have $\neg\mathcal{X}\psi \in \Delta_0$, which is equivalent to $\mathcal{X}\neg\psi$. By the definition of R, this would imply $\neg\psi \in \Delta_1$, leading to a contradiction with the assumption $\psi \in \Delta_1$.

Given these two examples, if we still wish to preserve the Truth Lemma, we essentially have two options: modify the definition of the Kripke structure K or carefully define the path $\rho_{(\Gamma,\varphi)}$ to ensure that the lemma holds. However, pursuing these alternatives would forfeit a valuable tool, outlined below, which we consider indispensable. Nevertheless, we will explicitly discuss some of the alternatives we have considered.

That crucial tool or strategy benefits from the non-maximality of DCSs. Consider $\Delta \in S$ and $\psi \notin \Delta$. According to *Lemma 3.1.19*, we have that

$$\Delta' := DC(\Delta \cup \{\psi\}) \cap FL^+(\Gamma \cup \{\varphi\}) \in S,$$

satisfies $\Delta \cup \{\psi\} \subseteq \Delta'$. Furthermore, if $(\Delta'', \Delta) \in R$ for some $\Delta'' \in S$, then $(\Delta'', \Delta') \in R$ as well, since $\mathcal{X}^{-1}\Delta'' \subseteq \Delta \subseteq \Delta'$. This flexibility allows us to include as many formulas as needed when defining the elements of the path $\rho_{(\Gamma,\varphi)}$. However, note that if $(\Delta, \Delta'') \in R$, we cannot necessarily conclude that $(\Delta', \Delta'') \in R$, because Δ' might include a formula $\mathcal{X}\psi$ not present in Δ, where $\psi \notin \Delta''$.

It is noteworthy that to achieve the semi-truth lemma introduced above, we need to use this method of expanding the elements of the path: as we pointed out before, if we want to satisfy $K, \rho_{(\Gamma,\varphi)} \Vdash \psi_1 \,\mathcal{U}\, \psi_2$, for some $\psi_1, \psi_2 \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, then we need

$$K, \rho_{(\Gamma,\varphi)} \Vdash \psi_1 \wedge \mathcal{X}(\psi_1 \,\mathcal{U}\, \psi_2) \quad \text{or} \quad K, \rho_{(\Gamma,\varphi)} \Vdash \psi_2.$$

That is, we need $\psi_1 \wedge \mathcal{X}(\psi_1 \,\mathcal{U}\, \psi_2) \in \Delta_0$ or $\psi_2 \in \Delta_0$ and, in principle, we cannot derive these memberships from our only premise $\psi_1 \,\mathcal{U}\, \psi_2 \in \Delta_0$ (although in the LTL framework we could, by axiom (U2) and the maximality and consistency of the states). So, we need to expand Δ_0 to include $\psi_1 \wedge \mathcal{X}(\psi_1 \,\mathcal{U}\, \psi_2)$ or ψ_2, while maintaining its deductively closed character.

The core issue is that the alternatives we explore to establish the other implication of the Truth Lemma compromise the expanding method we have developed:

The first easy option would be to adapt the relation R. We could redefine it as, for $\Delta, \Delta' \in S$:

$$(\Delta, \Delta') \in R \;:\Longleftrightarrow\; \mathcal{X}^{-1}\Delta \subseteq \Delta' \text{ and } \mathcal{X}\Delta' \subseteq \Delta.$$

In this way, we easily settle the problem introduced in our second example. However, we would no longer be able to expand our sets, as the new condition would not be preserved during the expansion.

A second option would be to define the path $\rho_{(\Gamma,\varphi)}$ strictly and carefully to ensure the desired implication of the Truth Lemma. We could approach this by defining the elements of $\rho_{(\Gamma,\varphi)}$ to be minimal, taking Δ_0 to be $\Delta_0 := DC(\Gamma)$ and, for every $k \geqslant 1$:

$$\Delta_k := DC(\mathcal{X}^{-1}\Delta_{k-1}) \cap FL^+(\varphi \wedge \psi).$$

These sets would contain the minimum number of formulas necessary to guarantee that they belong to S and are R-related. However, this again prevents us from extending the sets and also does not allow us to ensure the satisfiability of formulas containing the $\mathcal{U}$ operator.

The final alternative we will mention involves accepting that $\rho_{(\Gamma,\varphi)}$ satisfies the semi-truth lemma, and modifying the path definition to also grant the full Truth Lemma. The modification would consist of expanding the already defined states of $\rho_{(\Gamma,\varphi)}$ to include $\mathcal{X}\phi$ for every formula ϕ in the subsequent state of the path, iteratively for each step of the construction. This process should start taking the last two defined states and be repeated for each previous state until Δ_0. Since the Fischer-Ladner closure is finite, there exists a number $n < \omega$ such that no formula is nested by $\mathcal{X}$ more than n times. Therefore, at step $n+1$, the state Δ_0 becomes definitive —no further formulas will be added in subsequent expansions—. Likewise, Δ_1 will be definitive at step $n+2$, and so on.

The issue with this approach is that while we are adding these $\mathcal{X}$ formulas, the deductive closure may introduce additional new formulas. It remains a difficult task to determine whether these additional formulas might affect the satisfiability of the Truth Lemma. For example, problems could arise if a state includes a formula of the form $\varphi_1 \,\mathcal{U}\, \varphi_2$ without also including $\varphi_1 \wedge \mathcal{X}(\varphi_1 \,\mathcal{U}\, \varphi_2)$ or φ_2.

In conclusion, preserving the Truth Lemma proves to be challenging, if not virtually impossible. Therefore, our focus should shift to defining a path that specifically guarantees that Γ holds while ensuring φ does not. However, finding such a path is not easy either.

We try to inductively define the path $\rho_{(\Gamma,\varphi)} = \langle \Delta_0, \Delta_1, \dots \rangle$, as in the LTL case. Step 0, where we define Δ_0, should proceed in this way: we consider $\Delta_0^{(0)} := \Gamma$. Since the Fischer-Ladner closure is finite, for some $m < \omega$ we can establish the enumeration

$$\{\psi_1 \,\mathcal{U}\, \psi'_1, \dots, \psi_m \,\mathcal{U}\, \psi'_m\} = \{\phi \,\mathcal{U}\, \phi' : \phi \,\mathcal{U}\, \phi' \in FL^+(\Gamma \cup \{\varphi\})\} \cap DC(\Gamma).$$

We take, for $i \leqslant m$:

$$\Delta_0^{(i+1)} := \begin{cases} \Delta_0^{(i)} \cup \{\psi_i'\}, & \text{if } \Delta_0^{(i)} \cup \{\psi_i'\} \nvdash^+ \varphi; \\ \Delta_0^{(i)} \cup \{\psi_i, \mathcal{X}(\psi_i \,\mathcal{U}\, \psi_i')\}, & \text{otherwise.} \end{cases}$$

Then, we define $\Delta_0 := DC\left(\Delta_0^{(m)}\right) \cap FL^+(\Gamma \cup \{\varphi\})$. By *Lemma 3.1.19*, we know that $DC\left(\Delta_0^{(m)}\right)$ is a DCS, and so $\Delta_0 \in S$, as needed.

In this way, we ensure that Δ_0 is the smallest set of formulas in S that will allow all formulas in Γ to hold. The purpose of defining the sets $\Delta_0^{(i+1)}$ is to establish the satisfiability of the $\mathcal{U}$ formulas and subformulas of Γ while preventing the satisfaction of φ. However, we encounter a problem: to guarantee the non-satisfiability of φ, we need to prove that it is not possible for both $\Delta_0^{(i)} \cup \{\psi_i'\} \vdash^+ \varphi$ and $\Delta_0^{(i)} \cup \{\psi_i, \mathcal{X}(\psi_i \,\mathcal{U}\, \psi_i')\} \vdash^+ \varphi$ to hold simultaneously. We will refer to this condition as the *Until's disjunction problem.* Additionally, defining Δ_0 as the deductive closure of $\Delta_0^{(m)}$ introduces new formulas, which, as mentioned before, could be problematic. We will not define them, as similar challenges arise when trying to specify the subsequent elements of $\rho_{(\Gamma,\varphi)}$.

We can see that LTL does not have the Until's disjunction problem:

Proposition 3.1.23. *Consider Σ a set of propositional variables. For $\Delta \cup \{\varphi, \psi, \psi'\} \subseteq$ $\mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, the following statements cannot hold at the same time:*

- $\Delta \nvdash'_{\mathsf{LTL}} \varphi$;
- $\Delta \vdash'_{\mathsf{LTL}} \psi \,\mathcal{U}\, \psi'$;
- $\Delta \cup \{\psi'\} \vdash'_{\mathsf{LTL}} \varphi$;
- $\Delta \cup \{\psi, \mathcal{X}(\psi \,\mathcal{U}\, \psi')\} \vdash'_{\mathsf{LTL}} \varphi$.

Proof. We assume the statements to hold, and we will get a contradiction. Specifically, we will derive $\Delta \vdash'_{\mathsf{LTL}} \varphi$ from the last three expressions, which contradicts the first assumption.

For simplicity, we assume Δ to be finite. If Δ is infinite, the argument is the same, due to the compactness stated in *Observation 2.2.30*. We derive:

1.	$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\delta \in \Delta} \delta\right) \to \psi \,\mathcal{U}\, \psi'$;	assumption and *Observation 2.2.30*
2.	$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\delta \in \Delta} \delta \wedge \psi'\right) \to \varphi$;	assumption
3.	$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\delta \in \Delta} \delta \wedge \psi \wedge \mathcal{X}(\psi \,\mathcal{U}\, \psi')\right) \to \varphi$;	assumption
4.	$\vdash_{\mathsf{LTL}} \psi \,\mathcal{U}\, \psi' \leftrightarrow (\psi' \vee (\psi \wedge \mathcal{X}(\psi \,\mathcal{U}\, \psi')))$;	instance of the LTL axiom (U2)
5.	$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\delta \in \Delta} \delta\right) \to (\psi' \vee (\psi \wedge \mathcal{X}(\psi \,\mathcal{U}\, \psi')))$;	Substitution Lemma on 1. by 4.
6.	$\vdash_{\mathsf{LTL}} \left(\bigwedge_{\delta \in \Delta} \delta\right) \to (\psi' \to \varphi)$;	(prop)+2.

7. $\vdash_{\mathsf{LTL}} \left(\bigwedge_{\delta\in\Delta} \delta\right) \to ((\psi \wedge \mathcal{X}(\psi\,\mathcal{U}\,\psi')) \to \varphi);$ (prop)+3.

8. $\vdash_{\mathsf{LTL}} \left(\bigwedge_{\delta\in\Delta} \delta\right) \to \varphi.$ (prop)+5.+6.+7.

The last LTL theorem implies that $\Delta \vdash'_{\mathsf{LTL}} \varphi$. This contradicts our first assumption $\Delta \nvdash'_{\mathsf{LTL}} \varphi$. Then, we deduce that the four statements of the proposition cannot hold simultaneously. ▲

As mentioned, we need the previous proposition to hold within the SPLTL framework. However, in the proof for LTL, we essentially relied on the LTL axiom (U2). In the SPLTL context, we cannot express the disjunction specified by that LTL axiom. We conjecture that SPLTL satisfies the proposition and thus avoids the Until's disjunction problem. However, as part of this conjecture, we currently lack the appropriate tools to provide definitive proof. We are in a similar situation concerning whether deductive closures introduce problematic formulas.

A small glimmer of hope in addressing the disjunction issue is the following identification of a weaker notion of disjunction in SPLTL:

Proposition 3.1.24. *Let Σ be a set of propositional variables. For the Kripke structure K defined in Page 79 and every path $\rho = \langle \Delta_0, \Delta_1, \ldots \rangle$ of K, we have, for $p \in \Sigma$:*

$$K, \rho \Vdash \mathcal{X}p\,\mathcal{U}\,p \iff p \in \Delta_0 \text{ or } p \in \Delta_1.$$

Proof. Almost immediate by the satisfiability definition:

$$K, \rho \Vdash \mathcal{X}p\,\mathcal{U}\,p \iff \text{there is some } j \geqslant 0 \text{ such that } K, \rho^j \Vdash p \text{ and } K, \rho^k \Vdash \mathcal{X}p \text{ for every } 0 \leqslant k < j;$$

$$K, \rho \Vdash \mathcal{X}p\,\mathcal{U}\,p \iff \text{there is some } j \geqslant 0 \text{ such that } p \in V(\Delta_j) \subseteq \Delta_j \text{ and } \mathcal{X}p \in \Delta_k \text{ for every } 0 \leqslant k < j.$$

We see that, if $j = 0$, then $p \in \Delta_0$, and if $j > 0$ then $\mathcal{X}p \in \Delta_0$, and so $p \in \Delta_1$ by the definition of K. This shows the first implication.

On the other hand, if $p \in \Delta_0$, then we have $K, \rho \Vdash p$. This also gives us $K, \rho \Vdash \mathcal{X}p\,\mathcal{U}\,p$ —in this case, j would be 0—. If, instead, we have $p \in \Delta_1$, then we know $K, \rho \Vdash \mathcal{X}p$ and $K, \rho^1 \Vdash p$, so we also deduce $K, \rho \Vdash \mathcal{X}p\,\mathcal{U}\,p$ —now we would have $j = 1$—. ▲

This proposition suggests that the formula $\mathcal{X}p\,\mathcal{U}\,p$ can be interpreted as a disjunction of the form $\mathcal{X}p \vee p$. While this result does not fully capture the disjunction in SPLTL and is specific to the Kripke structure K introduced earlier, this weak notion of disjunction can be generalized and extended to include a wider range of formulas. For instance, if we accept the semi-truth lemma, we can extend the previous proposition to include the formula $\mathcal{X}\varphi\,\mathcal{U}\,\varphi$, for every $\varphi \in \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$. Or we also have, for $p, q \in \Sigma$:

$$K, \rho \Vdash \mathcal{X}p\,\mathcal{U}\,q \iff K, \rho \Vdash q \text{ or } K, \rho \Vdash \mathcal{X}(p\,\mathcal{U}\,(p \wedge q)).$$

Recall that $p\,\mathcal{U}\,(p \wedge q)$ is also known as the strong release notated by $p\mathcal{M}q$.

Besides the weak notion of disjunction, deriving lemmas involving formulas that are syntactically non-derivable could also be beneficial. The following lemma serves as an example of such results:

Lemma 3.1.25. *Let Σ be a set of propositional variables. For every $\Gamma \cup \{\varphi, \psi\} \subseteq \mathtt{Fm}_{\mathrm{SP}(\Sigma)}$, we have:*

i) $\Gamma \nvdash^+ \varphi \Longrightarrow \Gamma \nvdash^+ \varphi \mathcal{U} \varphi$ *and* $\Gamma \nvdash^+ \varphi \wedge \psi$*;*

ii) $\Gamma \nvdash^+ \varphi \mathcal{U} \psi \Longrightarrow \Gamma \nvdash^+ \psi$*;*

iii) $\Gamma \nvdash^+ \varphi \mathcal{U} \psi \Longrightarrow \Gamma \nvdash^+ \varphi \wedge \mathcal{X}(\varphi \mathcal{U} \psi)$*;*

iv) $\Gamma \nvdash^+ \varphi \mathcal{U} \psi \Longrightarrow \Gamma \nvdash^+ \varphi \mathcal{U} (\varphi \wedge \mathcal{X}\psi)$*.*

Proof. All items are shown similarly by contraposition. For *i)*, we assume $\Gamma \vdash^+ \varphi \mathcal{U} \varphi$ or $\Gamma \vdash^+ \varphi \wedge \psi$ and easily conclude, in both cases, that we have $\Gamma \vdash^+ \varphi$. We only need to apply the Cut rule over our assumptions and instances of the SPLTL axioms (UId) and ($\wedge$L), respectively. For *ii)* we repeat the same argument with an instance of the axiom (U21). The statement *iii)* is given by the axiom (U22).

The final expression, *iv)*, requires some additional work. We suppose

$$\Gamma \vdash^+ \varphi \mathcal{U} (\varphi \wedge \mathcal{X}\psi).$$

We can derive:

1.	$\Gamma \vdash^+ \varphi \mathcal{U} (\varphi \wedge \mathcal{X}\psi)$;	assumption
2.	$\varphi \mathcal{U} (\varphi \wedge \mathcal{X}\psi) \vdash^+ \varphi \wedge \mathcal{X}\varphi \mathcal{U} \mathcal{X}\psi$;	instance of (M)
3.	$\Gamma \vdash^+ \varphi \wedge \mathcal{X}\varphi \mathcal{U} \mathcal{X}\psi$;	Cut over 1. and 2.
4.	$\mathcal{X}\varphi \mathcal{U} \mathcal{X}\psi \vdash^+ \mathcal{X}(\varphi \mathcal{U} \psi)$;	instance of (DistUX)
5.	$\varphi \wedge \mathcal{X}\varphi \mathcal{U} \mathcal{X}\psi \vdash^+ \varphi \wedge \mathcal{X}(\varphi \mathcal{U} \psi)$;	*Lemma 3.1.13.ii)* over 4.
6.	$\Gamma \vdash^+ \varphi \wedge \mathcal{X}(\varphi \mathcal{U} \psi)$;	Cut over 3. and 5.
7.	$\varphi \wedge \mathcal{X}(\varphi \mathcal{U} \psi) \vdash^+ \varphi \mathcal{U} \psi$;	instance of (U22)
8.	$\Gamma \vdash^+ \varphi \mathcal{U} \psi$.	Cut over 6. and 7.

This gives us what we wanted, $\Gamma \vdash^+ \varphi \mathcal{U} \psi$, to prove by contraposition the expression *iv)*. ▲

The idea would be to address the Until's disjunction problem in SPLTL using lemmas similar to the one stated above. Instead of applying the last three assumptions from the Until's disjunction problem to derive a contradiction of the first assumption, as we did in *Proposition 3.1.23*, we would use the first expression to obtain a statement that contradicts one of the other assumptions. Although this alternative approach seems promising, we still lack the appropriate theory and tools to fully prove the SPLTL completeness result we are aiming for.

One final consideration is the possibility that the current calculus may not be complete. In principle, this could be addressed by adding new axioms or rules to those specified in *Definition 3.1.2*. For instance, we could add as a rule:

$$\frac{\Gamma \vdash^+ \psi \mathcal{U} \psi' \qquad \Gamma \cup \{\psi'\} \vdash^+ \varphi \qquad \Gamma \cup \{\psi, \mathcal{X}(\psi \mathcal{U} \psi')\} \vdash^+ \varphi}{\Gamma \vdash^+ \varphi}$$

This rule would solve our Until's disjunction problem. However, it does not settle whether this new rule was already admissible, and it also appears somewhat *ad hoc*. Moreover, increasing the number of axioms or rules complicates working with the system, which already includes numerous axioms and rules.

3.2 Conclusions and Open Questions

We began this thesis by examining the class of regular languages and its diverse characterizations through regular expressions, regular grammars, finite automata, and MSO logic over words. We also introduced Büchi's Theorem, which links the automata and logic characterizations, and provided a finely detailed proof.

In the second chapter, we shifted our focus to the subclass of star-free languages, which also has several characterizations, notably including LTL. We centered our attention on the LTL framework, defining its syntax, semantics, and axiomatization, and concluded with an in-depth study of the LTL Completeness Theorem.

In this last chapter, we defined a new syntactic fragment of LTL called SPLTL, inspired by the theory of strictly positive fragments of modal logics. We derived several lemmas on SPLTL and proved soundness with respect to the standard semantic relation. The last section explored the potential completeness of SPLTL. We developed tools and results to adapt the completeness proof of LTL to the SPLTL framework. However, this approach presents challenges, particularly due to the absence of the disjunction connective in SPLTL, which appears crucial for describing the behavior of the $\mathcal{U}$ operator.

In addition to the original content presented in the third chapter, the first two chapters have offered a comprehensive overview of the topic. It is worth noting that we have presented a broad perspective that highlights both foundational concepts and recent developments. Furthermore, we provided detailed proofs for results that are often not thoroughly explored in the literature.

However, our contributions have raised more questions than they have answered. The produced overview stimulates a need for further investigation into the class of context-free grammars that generate star-free languages (see 2.1, Page 35), as well as a more detailed exploration of validities in MFO over words, independent of the translation of LTL tautologies (see 2.1, Page 35). Moreover, a complete explanation of counter-free automata (*Definition 2.1.4*), the characterization of star-free languages by aperiodic finite monoids (see 2.1, Page 35), and MFO logic (see 2.1, Page 35) would also have added significant value to the thesis.

Focusing on the third chapter, several questions remain open regarding the axiomatization and completeness of some strictly positive fragments of LTL. Although we proposed that these fragments might not be particularly difficult to address, given their potential to benefit from existing results, they would still require formal proof. But undoubtedly, the most significant question we would have most liked to resolve is the completeness of our SPLTL calculus. As suggested, such results could be beneficial in the context of databases and related applied areas. We remain *strictly positive* that proving the completeness of SPLTL will be achievable in the future.

References

[AJ23a] A. de Almeida Borges and J. J. Joosten. "An Escape From Vardanyans Theorem". In: *The Journal of Symbolic Logic* 88.4 (2023), pp. 1613–1638. DOI: 10.1017/jsl.2022.38 (cit. on p. 64).

[AJ23b] A. de Almeida Borges and J. J. Joosten. "Strictly Positive Fragments of the Provability Logic of Heyting Arithmetic". In: *arXiv eprint* (2023). DOI: 10.48550/arXiv.2312.14727 (cit. on p. 64).

[Art+13] A. Artale, R. Kontchakov, V. Ryzhikov, and M. Zakharyaschev. "The Complexity of Clausal Fragments of LTL". In: *Logic for Programming, Artificial Intelligence, and Reasoning*. Ed. by K. McMillan, A. Middeldorp, and A. Voronkov. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 35–52. DOI: 10.1007/978-3-642-45221-5_3 (cit. on p. 64).

[Bau+09] M. Bauland, T. Schneider, H. Schnoor, I. Schnoor, and H. Vollmer. "The complexity of generalized satisfiability for linear temporal logic". In: *Logical Methods in Computer Science* Volume 5, Issue 1 (2009). DOI: 10.2168/LMCS-5(1:1)2009 (cit. on p. 64).

[BDF19] J. Boudou, M. Diéguez, and D. Fernández-Duque. "Complete Intuitionistic Temporal Logics in Topological Dynamics". In: *arXiv eprint* (2019). DOI: 10.48550/arXiv.1910.00907 (cit. on p. 63).

[Bek12] L. D. Beklemishev. "Calibrating Provability Logic: From Modal Logic to Reflection Calculus". In: *Advances in Modal Logic 9, Copenhagen, Denmark*. Ed. by T. Bolander, T. Braüner, S. Ghilardi, and L. S. Moss. London, England: College Publications, 2012, pp. 89–94. ISBN: 978-1848900684 (cit. on pp. 64, 68).

[BK08] C. Baier and J.-P. Katoen. *Principles Of Model Checking*. Vol. 950. The MIT Press, 2008. ISBN: 9780262026499 (cit. on pp. 33, 64).

[Büc60] J. R. Büchi. "Weak SecondOrder Arithmetic and Finite Automata". In: *Mathematical Logic Quarterly* 6 (1960). DOI: 10.1002/malq.19600060105 (cit. on p. 24).

[Büc66] J. R. Büchi. "Symposium on Decision Problems: On a Decision Method in Restricted Second Order Arithmetic". In: *Logic, Methodology and Philosophy of Science*. Ed. by E. Nagel, P. Suppes, and A. Tarski. Vol. 44. Studies in Logic and the Foundations of Mathematics. Elsevier, 1966, pp. 1–11. DOI: 10.1016/S0049-237X(09)70564-6 (cit. on pp. 8, 16).

[Bur82] J. P. Burgess. "Axioms for tense logic. I. "Since" and "until"." In: *Notre Dame Journal of Formal Logic* 23.4 (1982), pp. 367–374. DOI: 10.1305/ndjfl/1093870149 (cit. on pp. 39, 44).

[CGM78] S. Crespi-Reghizzi, G. Guida, and D. Mandrioli. “Noncounting Context-Free Languages”. In: *J. ACM* 25.4 (1978), pp. 571–580. DOI: 10.1145/322092.322098 (cit. on p. 35).

[CJ97] S. Celani and R. Jansana. “A New Semantics for Positive Modal Logic”. In: *Notre Dame Journal of Formal Logic* 38.1 (1997), pp. 1–18. DOI: 10.1305/ndjfl/1039700693 (cit. on p. 76).

[CL93] C.-C. Chen and I.-P. Lin. “The computational complexity of satisfiability of temporal Horn formulas in propositional linear-time temporal logic”. In: *Information Processing Letters* 45.3 (1993), pp. 131–136. DOI: 10.1016/0020-0190(93)90014-Z (cit. on p. 64).

[CM21] S. Chopoghloo and M. Moniri. “A strongly complete axiomatization of intuitionistic temporal logic”. In: *Journal of Logic and Computation* 31.7 (2021), pp. 1640–1659. DOI: 10.1093/logcom/exab041 (cit. on p. 63).

[Das12] E. Dashkov. “On the positive fragment of the polymodal provability logic GLP”. In: *Mathematical Notes* 91.3 (2012), pp. 318–333. DOI: 10.1134/S0001434612030029 (cit. on pp. 64, 76).

[Dav96] R. Davies. “A temporal-logic approach to binding-time analysis”. In: *Logic in Computer Science, Symposium on*. Los Alamitos, CA, USA: IEEE Computer Society, 1996, pp. 184–195. DOI: 10.1109/LICS.1996.561317 (cit. on p. 63).

[DFK07] C. Dixon, M. Fisher, and B. Konev. “Tractable Temporal Reasoning.” In: *20th International Joint Conference on Artificial Intelligence, IJCAI*. Vol. 7. United States: AAAI Press, 2007, pp. 318–323. ISBN: 9781577352983 (cit. on p. 64).

[DG07] V. Diekert and P. Gastin. “First-order definable languages”. In: *Logic and Automata: History and Perspectives*. Ed. by J. Flum, T. Wilke, and E. Grädel. Series Texts in Logic and Games, 2. Amsterdam University Press, 2007, pp. 261–306. ISBN: 9789053565766 (cit. on p. 36).

[DS02] S. Demri and P. Schnoebelen. “The Complexity of Propositional Linear Temporal Logics in Simple Cases”. In: *Information and Computation* 174.1 (2002), pp. 84–103. DOI: 10.1006/inco.2001.3094 (cit. on p. 64).

[Dun95] J. M. Dunn. “Positive Modal Logic”. In: *Studia logical:* 55.2 (1995), pp. 301–317. DOI: 10.1007/BF01061239 (cit. on p. 76).

[Elg61] C. C. Elgot. “Decision problems of finite automata design and related arithmetics”. In: *Transactions of the American Mathematical Society* 98 (1 1961). DOI: 10.1090/s0002-9947-1961-0139530-9 (cit. on p. 24).

[Ewa86] W. B. Ewald. “Intuitionistic tense and modal logic”. In: *Journal of Symbolic Logic* 51.1 (1986), pp. 166–179. DOI: 10.2307/2273953 (cit. on p. 63).

[Fer18] D. Fernández-Duque. “The intuitionistic temporal logic of dynamical systems”. In: *Logical Methods in Computer Science* 14 (3 2018). DOI: 10.23638/LMCS-14(3:3)2018 (cit. on p. 63).

[FG16] V. Fionda and G. Greco. “The Complexity of LTL on Finite Traces: Hard and Easy Fragments”. In: *Proceedings of the AAAI Conference on Artificial Intelligence*. Vol. 30. 1. Palo Alto, California USA: AAAI Press, 2016, pp. 971–977. DOI: 10.1609/aaai.v30i1.10104 (cit. on p. 64).

[FL79] M. J. Fischer and R. E. Ladner. “Propositional dynamic logic of regular programs”. In: *Journal of Computer and System Sciences* 18.2 (1979), pp. 194–211. DOI: 10.1016/0022-0000(79)90046-1 (cit. on p. 49).

[Gab+80] D. Gabbay, A. Pnueli, S. Shelah, and J. Stavi. “On the temporal analysis of fairness”. In: *Proceedings of the 7th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*. New York, NY, USA: Association for Computing Machinery, 1980, pp. 163–173. DOI: 10.1145/567446.567462 (cit. on pp. 38, 39, 44).

[Hem05] E. Hemaspaandra. “The Complexity of Poor Man’s Logic”. In: *arXiv eprints* (2005). DOI: 10.48550/arXiv.cs/9911014 (cit. on p. 64).

[Hir82] N. Hirokazu. “Semantical Analysis of Constructive PDL”. In: *Publications of the Research Institute for Mathematical Sciences* 18.2 (1982), pp. 847–858. DOI: 10.2977/PRIMS/1195183579 (cit. on p. 63).

[HMU06] J. E. Hopcroft, R. Motwani, and J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation (3rd Edition)*. USA: Addison-Wesley Longman Publishing Co., Inc., 2006. ISBN: 0321455363 (cit. on pp. 4, 5, 8, 10, 13).

[Hod95] I. Hodkinson. “Expressive Completeness of Until and Since over Dedekind complete linear time”. In: *Modal logic and process algebra: a bisimulation perspective*. Stanford, CA: Center for the Study of Language and Information, 1995, pp. 171–185. ISBN: 978-1881526964 (cit. on p. 39).

[HV18] T. Henzinger and H. Veith. *Handbook of Model Checking*. Ed. by E. Clarke and R. Bloem. Springer, 2018. DOI: 10.1007/978-3-319-10575-8 (cit. on pp. 33, 38, 64).

[Jun+24] J. C. Jung, V. Ryzhikov, F. Wolter, and M. Zakharyaschev. “Extremal Separation Problems for Temporal Instance Queries”. In: *arXiv eprint* (2024). DOI: 10.48550/arXiv.2405.03511 (cit. on p. 77).

[Kam68] J. A. W. Kamp. “Tense Logic and the Theory of Linear Order”. PhD thesis. University of California, Los Angeles, 1968 (cit. on p. 38).

[KI11] K. Kojima and A. Igarashi. “Constructive linear-time temporal logic: Proof systems and Kripke semantics”. In: *Information and Computation* 209 (2011), pp. 1491–1503. DOI: 10.1016/j.ic.2010.09.008 (cit. on p. 63).

[Kle56] S. C. Kleene. “Representation of Events in Nerve Nets and Finite Automata”. In: *Automata Studies*. Ed. by C. E. Shannon and J. McCarthy. Princeton: Princeton University Press, 1956, pp. 3–42. DOI: 10.1515/9781400882618-002 (cit. on p. 8).

[KM08] F. Kröger and S. Merz. *Temporal Logic and State Systems*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2008. ISBN: 978-3540674016 (cit. on pp. 39, 44, 91).

[KW10] N. Kamide and H. Wansing. “Combining linear-time temporal logic with constructiveness and paraconsistency”. In: *Journal of Applied Logic* 8.1 (2010), pp. 33–61. DOI: 10.1016/j.jal.2009.06.001 (cit. on p. 63).

[Lad77] R. E. Ladner. “Application of model theoretic games to discrete linear orders and finite automata”. In: *Information and Control* 33.4 (1977), pp. 281–303. DOI: 10.1016/S0019-9958(77)90443-0 (cit. on pp. 33, 35).

[LS07] M. Leucker and C. Sánchez. “Regular Linear Temporal Logic”. In: *Theoretical Aspects of Computing – ICTAC 2007*. Ed. by C. B. Jones, Z. Liu, and J. Woodcock. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 291–305. DOI: 10.1007/978-3-540-75292-9_20 (cit. on p. 63).

[Mai04] P. Maier. “Intuitionistic LTL and a New Characterization of Safety and Liveness”. In: *Computer Science Logic*. Vol. 3210. Lecture Notes in Computer Science. Switzerland: Springer, 2004, pp. 295–309. DOI: 10.1007/978-3-540-30124-0_24 (cit. on p. 63).

[Mar04] N. Markey. “Past is for free: On the complexity of verifying linear temporal properties with past”. In: *Acta Informatica* 40.6-7 (2004), pp. 431–458. DOI: 10.1007/s00236-003-0136-5 (cit. on p. 64).

[McN66] R. McNaughton. “Testing and generating infinite sequences by a finite automaton”. In: *Information and Control* 9.5 (1966), pp. 521–530. DOI: 10.1016/S0019-9958(66)80013-X (cit. on p. 16).

[MP71] R. McNaughton and S. Papert. *Counter-free automata*. MIT Press, 1971. ISBN: 0262130769 (cit. on p. 35).

[MPR23] D. Mandrioli, M. Pradella, and S. C. Reghizzi. “Aperiodicity, Star-freeness, and First-order Logic Definability of Operator Precedence Languages”. In: *Logical Methods in Computer Science* Volume 19, Issue 4 (2023). DOI: 10.46298/lmcs-19(4:12)2023 (cit. on p. 35).

[ON80] H. Ono and A. Nakamura. “On the size of refutation Kripke models for some linear modal and tense logics”. In: *Studia Logica* 39 (1980), pp. 325–333. DOI: 10.1007/BF00713542 (cit. on p. 64).

[Pin20] J.-É. Pin. “How to Prove that a Language Is Regular or Star-Free?” In: *Language and Automata Theory and Applications*. Ed. by A. Leporati, C. Martín-Vide, D. Shapira, and C. Zandron. Cham: Springer International Publishing, 2020, pp. 68–88. DOI: 10.1007/978-3-030-40608-0_5 (cit. on p. 36).

[Pin95] J.-É. Pin. “Finite semigroups and recognizable languages: an introduction”. In: *NATO Advanced Study Institute Semigroups, Formal Languages and Groups*. Ed. by J. Fountain. Kluwer academic publishers, 1995, pp. 1–32. ISBN: 978-0792335405 (cit. on p. 36).

[Pnu77] A. Pnueli. “The temporal logic of programs”. In: *18th Annual Symposium on Foundations of Computer Science, sfcs 1977*. 1977, pp. 46–57. DOI: 10.1109/SFCS.1977.32 (cit. on p. 33).

[Rab14] A. Rabinovich. “A Proof of Kamp’s theorem”. In: *Log. Methods Comput. Sci.* 10.1 (2014). DOI: 10.2168/LMCS-10(1:14)2014 (cit. on p. 38).

[Rib12] C. Riba. “A Model Theoretic Proof of Completeness of an Axiomatization of Monadic Second-Order Logic on Infinite Words”. In: *Theoretical Computer Science*. Ed. by J. C. M. Baeten, T. Ball, and F. S. de Boer. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 310–324. DOI: 10.1007/978-3-642-33475-7_22 (cit. on pp. 23, 24).

[SC85] A. P. Sistla and E. M. Clarke. “The complexity of propositional linear temporal logics”. In: *J. ACM* 32.3 (1985), pp. 733–749. DOI: 10.1145/3828.3837 (cit. on p. 64).

[Sch65] M. Schützenberger. "On finite monoids having only trivial subgroups". In: *Information and Control* 8.2 (1965), pp. 190–194. DOI: 10.1016/S0019-9958(65)90108-7 (cit. on p. 35).

[Sip13] M. Sipser. *Introduction to the Theory of Computation*. Third. Boston, MA: Course Technology, 2013. ISBN: 113318779X (cit. on pp. 4, 5, 8).

[SR99] P.-Y. Schobbens and J.-F. Raskin. "The logic of initially and next: Complete axiomatization and complexity". In: *Information Processing Letters* 69.5 (1999), pp. 221–225. DOI: 10.1016/S0020-0190(99)00022-8 (cit. on p. 64).

[Sud05] T. A. Sudkamp. *Languages and Machines: An Introduction to the Theory of Computer Science*. 3rd ed. Pearson international editions. Pearson Addison-Wesley, 2005. ISBN: 978-0321315342 (cit. on pp. 4, 7).

[Tho79] W. Thomas. "Star-free regular sets of ω-sequences". In: *Information and Control* 42.2 (1979), pp. 148–156. DOI: 10.1016/S0019-9958(79)90629-6 (cit. on p. 35).

[Tho81] W. Thomas. "A combinatorial approach to the theory of ω-automata". In: *Information and Control* 48.3 (1981), pp. 261–283. DOI: 10.1016/S0019-9958(81)90663-X (cit. on p. 35).

[Tho90] W. Thomas. "Automata on Infinite Objects". In: *Formal Models and Semantics*. Ed. by J. Van Leeuwen. Handbook of Theoretical Computer Science. Amsterdam: Elsevier, 1990, pp. 133–191. DOI: 10.1016/B978-0-444-88074-1.50009-3 (cit. on p. 35).

[Tho96] W. Thomas. "Languages, Automata, and Logic". In: *Handbook of Formal Languages: Volume 3 Beyond Words*. Ed. by G. Rozenberg and A. Salomaa. Berlin, Heidelberg: Springer Berlin Heidelberg, 1996, pp. 389–455. DOI: 10.1007/978-3-642-59126-6_7 (cit. on p. 25).

[Wol83] P. Wolper. "Temporal logic can be more expressive". In: *Information and Control* 56.1 (1983), pp. 72–99. DOI: 10.1016/S0019-9958(83)80051-5 (cit. on p. 63).

Appendix

A An Alternative Axiomatization of LTL

We will introduce an alternative LTL axiomatization to the standard one given in *Definition 2.2.6*. To show that both axiomatizations are indeed equivalent, we will present a syntactic consequence relation $\vdash'_{\mathsf{ALT}} \subseteq \mathcal{P}(\mathtt{Fm}_{\mathrm{LT}(\Sigma)}) \times \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, and prove a version of the Deduction Theorem, *Theorem A.9*. Both the axiomatization and the proof of the Deduction Theorem are inspired by [KM08].

Actually, the following alternative axiomatization of LTL may not offer more practical benefits than the usual one. However, it is noteworthy as it involves fewer axioms and significantly reduces the occurrences of the operator $\Box$, a derived connective, in the formalization.

Definition A.1. *Let Σ be a set of propositional variables. We define the* ***alternative axiomatization*** *of LTL as the set of LTL-formulas given, for every $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, by:*
Axiom Schemes:

Any classical propositional tautology; (prop)

$$\mathcal{X}(\varphi \to \psi) \to (\mathcal{X}\varphi \to \mathcal{X}\psi); \tag{$K_{\mathcal{X}}$}$$

$$\neg\mathcal{X}\varphi \leftrightarrow \mathcal{X}\neg\varphi; \tag{Lin}$$

$$\varphi\,\mathcal{U}\,\psi \to \Diamond\psi; \tag{U1}$$

$$\varphi\,\mathcal{U}\,\psi \leftrightarrow (\psi \vee (\varphi \wedge \mathcal{X}(\varphi\,\mathcal{U}\,\psi))). \tag{U2}$$

Inference Rules:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\,MP \qquad \frac{\varphi}{\mathcal{X}\varphi}\,N_{\mathcal{X}} \qquad \frac{\varphi \to \psi \qquad \varphi \to \mathcal{X}\varphi}{\varphi \to \Box\psi}\,IndR$$

We will call the elements of the alternative axiomatization of LTL as ***A-theorems*** *of LTL, and if φ is an A-theorem, we denote it by $\vdash_{\mathsf{ALT}} \varphi$.*

Example A.2. We can show that, for φ an LTL-formula, we have

$$\vdash_{\mathsf{ALT}} \Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi). \tag{A$\Box$Unr}$$

To verify this, we must first recall that $\Box\varphi$ is an abbreviation of $\neg(\top\,\mathcal{U}\,\neg\varphi)$. Next, we consider the following sequence, which is essentially the same as the one in *Example $\Box$Unr*:

1. $\top\,\mathcal{U}\,\neg\varphi \leftrightarrow (\neg\varphi \vee (\top \wedge \mathcal{X}(\top\,\mathcal{U}\,\neg\varphi)))$; instance of axiom (U2)

2. $\top \mathcal{U} \neg\varphi \leftrightarrow (\neg\varphi \vee \mathcal{X}(\top \mathcal{U} \neg\varphi))$; (prop)+1. (see *Notation* in Page 40)
3. $(\neg\varphi \vee \mathcal{X}(\top \mathcal{U} \neg\varphi)) \to \top \mathcal{U} \neg\varphi$; (prop)+2.
4. $\neg(\top \mathcal{U} \neg\varphi) \to \neg(\neg\varphi \vee \mathcal{X}(\top \mathcal{U} \neg\varphi))$; (prop)+3.
5. $\neg(\top \mathcal{U} \neg\varphi) \to (\neg\neg\varphi \wedge \neg\mathcal{X}(\top \mathcal{U} \neg\varphi))$; (prop)+4.
6. $\neg\neg\varphi \to \varphi$; instance of prop. tautology
7. $\neg(\top \mathcal{U} \neg\varphi) \to (\varphi \wedge \neg\mathcal{X}(\top \mathcal{U} \neg\varphi))$; (prop)+5.+6.
8. $\neg\mathcal{X}(\top \mathcal{U} \neg\varphi) \leftrightarrow \mathcal{X}\neg(\top \mathcal{U} \neg\varphi)$; instance of axiom (Lin)
9. $\neg(\top \mathcal{U} \neg\varphi) \to (\varphi \wedge \mathcal{X}\neg(\top \mathcal{U} \neg\varphi))$; (prop)+7.+8.
10. $\Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi)$. by 9. and the definition of $\Box$

Then, we conclude $\vdash_{\mathsf{ALT}} \Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi)$.

Our main objective is to prove that the alternative axiomatization of LTL is equivalent to the standard one, so the A-theorems of LTL are exactly the usual theorems of LTL. To demonstrate this, it suffices to verify that the axioms and rules of each axiomatization are derivable from the other. Given that they share most of their axioms and rules, we only need to show the derivability of the rules, for every $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:

$$\frac{\vdash_{\mathsf{LTL}} \varphi \to \psi \qquad \vdash_{\mathsf{LTL}} \varphi \to \mathcal{X}\varphi}{\vdash_{\mathsf{LTL}} \varphi \to \Box\psi} \qquad\qquad \frac{\vdash_{\mathsf{ALT}} \varphi}{\vdash_{\mathsf{ALT}} \Box\varphi}$$

And the A-theorems:

$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi);$$
$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \mathcal{X}\varphi) \to (\varphi \to \Box\varphi).$$

In the case of the rules, we see that they follow easily:

Proposition A.3. *For Σ a set of propositional variables and $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we have that:*

$$\vdash_{\mathsf{LTL}} \varphi \to \psi \textit{ and } \vdash_{\mathsf{LTL}} \varphi \to \mathcal{X}\varphi \textit{ imply } \vdash_{\mathsf{LTL}} \varphi \to \Box\psi.$$

Proof. For $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we derive:

1. $\vdash_{\mathsf{LTL}} \varphi \to \psi$; assumption
2. $\vdash_{\mathsf{LTL}} \Box(\varphi \to \psi)$; $N_\Box$ rule over 1.
3. $\vdash_{\mathsf{LTL}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$; instance of axiom ($K_\Box$)
4. $\vdash_{\mathsf{LTL}} \Box\varphi \to \Box\psi$; *MP* over 2. and 3.
5. $\vdash_{\mathsf{LTL}} \varphi \to \mathcal{X}\varphi$; assumption
6. $\vdash_{\mathsf{LTL}} \Box(\varphi \to \mathcal{X}\varphi)$; $N_\Box$ rule over 5.
7. $\vdash_{\mathsf{LTL}} \Box(\varphi \to \mathcal{X}\varphi) \to (\varphi \to \Box\varphi)$; instance of axiom (Ind)
8. $\vdash_{\mathsf{LTL}} \varphi \to \Box\varphi$; *MP* over 6. and 7.
9. $\vdash_{\mathsf{LTL}} \varphi \to \Box\psi$. (prop)+4.+8.

This leads us to conclude that if we assume $\vdash_{\mathsf{LTL}} \varphi \to \psi$ and $\vdash_{\mathsf{LTL}} \varphi \to \mathcal{X}\varphi$, then we also have $\vdash_{\mathsf{LTL}} \varphi \to \Box\psi$, as we wanted. ▲

Proposition A.4. *For Σ a set of propositional variables and $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we have that:*

$$\vdash_{\mathsf{ALT}} \varphi \textit{ implies } \vdash_{\mathsf{ALT}} \Box\varphi.$$

Proof. We suppose $\vdash_{\mathsf{ALT}} \varphi$ for some $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. We derive:

1.	$\vdash_{\mathsf{ALT}} \varphi$;	assumption
2.	$\vdash_{\mathsf{ALT}} \mathcal{X}\varphi$;	rule $N_{\mathcal{X}}$ over 1.
3.	$\vdash_{\mathsf{ALT}} \varphi \to \mathcal{X}\varphi$;	(prop)+2.
4.	$\vdash_{\mathsf{ALT}} \varphi \to \varphi$;	instance of prop. tautology
5.	$\vdash_{\mathsf{ALT}} \varphi \to \Box\varphi$;	rule $IndR$ over 3. and 4.
6.	$\vdash_{\mathsf{ALT}} \Box\varphi$.	MP over 1. and 5.

This ensures that if we have $\vdash_{\mathsf{ALT}} \varphi$, then we can also derive $\vdash_{\mathsf{ALT}} \Box\varphi$, as required. ▲

Now we only need to show the derivability of the A-theorems:

$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi);$$
$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \mathcal{X}\varphi) \to (\varphi \to \Box\varphi).$$

To prove this, we will define a relation $\vdash'_{\mathsf{ALT}}$, which provides a syntactic deduction system for LTL. The idea is that $\vdash'_{\mathsf{ALT}}$ will satisfy a version of the Deduction Theorem that simplifies the derivation of the two A-theorems we need.

Definition A.5. *For Σ a set of propositional variables and $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we state $\Gamma \vdash'_{\mathsf{ALT}} \varphi$ if there is a* ***proof*** *or derivation of φ by Γ in the alternative axiomatization of LTL. Such proof is a finite sequence $\langle \varphi_0, \dots, \varphi_n \rangle$ where $\varphi_n = \varphi$ and, for every $i \leqslant n$, either:*

- *$\varphi_i \in \Gamma$;*
- *φ_i is an A-theorem of LTL; or*
- *φ_i is obtained from previous formulas in the sequence by applying some rule of the alternative axiomatization of LTL.*

If $\Gamma \vdash'_{\mathsf{ALT}} \varphi$, we say that Γ is the set of ***assumptions****, and φ is a* ***conclusion*** *of Γ.*

This relation $\vdash'_{\mathsf{ALT}}$ is strictly different from the standard consequence relation of LTL, the relation $\vdash'_{\mathsf{LTL}}$ introduced in *Definition 2.2.29*. For instance, consider $\{\varphi\} \vdash'_{\mathsf{ALT}} \Box\varphi$, for $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. The rule derived in *Proposition A.4* makes this derivation clear. However, we do not have $\{\varphi\} \vdash'_{\mathsf{LTL}} \Box\varphi$ since $\nvdash_{\mathsf{LTL}} \varphi \to \Box\varphi$. Yet another example, by the $IndR$ rule, is clear that we have, for $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, the expression $\{\varphi \to \psi, \varphi \to \mathcal{X}\varphi\} \vdash'_{\mathsf{ALT}} \varphi \to \Box\psi$. But it is not true that we have $\{\varphi \to \psi, \varphi \to \mathcal{X}\varphi\} \vdash'_{\mathsf{LTL}} \varphi \to \Box\psi$ in general. Later, thanks to *Proposition A.12*, we will also be able to demonstrate the inclusion $\vdash'_{\mathsf{LTL}} \subset \vdash'_{\mathsf{ALT}}$.

The following three results are immediate yet relevant, and they also help to clarify the behavior of $\vdash'_{\mathsf{ALT}}$ and the A-theorems:

Proposition A.6. *For Σ a set of propositional variables and every $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we have:*

$$\varnothing \vdash'_{\mathsf{ALT}} \varphi \iff \vdash_{\mathsf{ALT}} \varphi.$$

Proof. If φ is a theorem of the alternative axiomatization of LTL, then we easily conclude $\varnothing \vdash'_{\mathsf{ALT}} \varphi$ since the sequence $\langle \varphi \rangle$ would serve as a proof.

On the other hand, if $\varnothing \vdash'_{\mathsf{ALT}} \varphi$, then there exists a sequence, a proof, consisting only of A-theorems and formulas obtained by applying the inference rules to previous formulas in the sequence. Let φ_k be the first element in the sequence obtained in this second manner. Since the previous formulas in the sequence are all A-theorems and we have applied an inference rule from the alternative axiomatization, it follows that φ_k is also an A-theorem. Inductively, we deduce that all elements in the sequence are A-theorems of LTL. In particular, the conclusion φ is also an A-theorem, as desired. ▲

Proposition A.7 (Compactness of $\vdash'_{\mathsf{ALT}}$)**.** *For Σ a set of propositional variables and for all $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, if $\Gamma \vdash'_{\mathsf{ALT}} \varphi$, then there is some finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash'_{\mathsf{ALT}} \varphi$.*

Proof. If $\Gamma \vdash'_{\mathsf{ALT}} \varphi$, then there is some proof $\langle \varphi_0, \dots, \varphi_n \rangle$ of φ by Γ in the alternative axiomatization of LTL. We can simply pick $\Gamma_0 := \{\gamma \in \Gamma : \gamma = \varphi_i \text{ for some } i \leqslant n\}$. Since the proof is a finite sequence of formulas, we have that $\Gamma_0 \subseteq \Gamma$ is finite. Moreover, by the construction of Γ_0, every formula of Γ needed in the proof $\langle \varphi_0, \dots, \varphi_n \rangle$ is also in Γ_0, so the same proof also guarantees $\Gamma_0 \vdash'_{\mathsf{ALT}} \varphi$. ▲

Proposition A.8 (Weakening of $\vdash'_{\mathsf{ALT}}$)**.** *For Σ a set of propositional variables and for $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, if $\Gamma \vdash'_{\mathsf{ALT}} \varphi$, then also $\Gamma' \vdash'_{\mathsf{ALT}} \varphi$ for every $\Gamma' \supseteq \Gamma$.*

Proof. If $\Gamma \vdash'_{\mathsf{ALT}} \varphi$, then there is a sequence, a proof, of φ by Γ. Since every element of Γ is also in every $\Gamma' \supseteq \Gamma$, we clearly have that the same sequence will serve as a proof of φ by Γ'. ▲

Without further delay, let us demonstrate the version of the Deduction Theorem we will need:

Theorem A.9 (Deduction Theorem for $\vdash_{\mathsf{ALT}}$)**.** *Consider Σ to be a set of propositional variables and $\Gamma \cup \{\varphi, \psi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$. We have:*

$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \psi \iff \Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi.$$

Proof. We first show the left-to-right implication: we assume $\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \psi$ and we need to prove that we also have $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi$. We do it by induction on the assumed proof of ψ:

- If ψ is an A-theorem, or if it is equal (or equivalent) to some element of Γ, then we clearly have $\Gamma \vdash'_{\mathsf{ALT}} \psi$, and so we can derive $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi$ by using the instance of a propositional tautology $\psi \to (\Box\varphi \to \psi)$ and *Modus Ponens.*
- Consider $\psi = \varphi$ (or ψ to be equivalent to φ, that is, $\vdash'_{\mathsf{ALT}} \varphi \leftrightarrow \psi$). We need to prove $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \varphi$. This follows essentially from the A-theorem (A□Unr), derived in the previous example. In this way, we have:

1.	$\vdash_{\mathsf{ALT}} \Box\varphi \to (\varphi \wedge \mathcal{X}\Box\varphi)$;	derivable
2.	$\vdash_{\mathsf{ALT}} \Box\varphi \to \varphi$;	(prop)+1.
3.	$\Gamma \vdash_{\mathsf{ALT}} \Box\varphi \to \varphi$.	Weakening of 2.

  So we got what we wanted.

- If the last step of the proof is the application of *Modus Ponens*, then we have $\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \phi$ and $\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \phi \to \psi$, for some LTL-formula ϕ. By Induction Hypothesis, we know $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \phi$ and $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to (\phi \to \psi)$. Thanks to the instance of a propositional tautology

$$(\Box\varphi \to \phi) \to ((\Box\varphi \to (\phi \to \psi)) \to (\Box\varphi \to \psi)),$$

and two consecutive applications of MP, we deduce $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi$, as intended.

- If the final step in the proof of ψ was achieved by applying the rule $N_{\mathcal{X}}$, then ψ is of the form $\psi = \mathcal{X}\phi$ with $\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \phi$, for some LTL-formula ϕ. By our Induction Hypothesis, we also know $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \phi$. We can deduce $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \mathcal{X}\phi$ by:

1.	Γ;	set of assumptions
2.	$\Box\varphi \to \phi$;	derivable from Γ
3.	$\mathcal{X}(\Box\varphi \to \phi)$;	$N_{\mathcal{X}}$ rule over 2.
4.	$\mathcal{X}(\Box\varphi \to \phi) \to (\mathcal{X}\Box\varphi \to \mathcal{X}\phi)$;	instance of axiom ($K_{\mathcal{X}}$)
5.	$\mathcal{X}\Box\varphi \to \mathcal{X}\phi$;	MP over 3. and 4.
6.	$\Box\varphi \to \varphi \wedge \mathcal{X}\Box\varphi$;	A-theorem (A□Unr)
7.	$\Box\varphi \to \mathcal{X}\Box\varphi$;	(prop)+6.
8.	$\Box\varphi \to \mathcal{X}\phi$.	(prop)+5.+7.

- If the last step of the proof of ψ has been given by applying the rule $IndR$, then we know, for some $\phi_1, \phi_2 \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:

$$\psi = \phi_1 \to \Box\phi_2;$$
$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \phi_1 \to \phi_2;$$
$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \phi_1 \to \mathcal{X}\phi_1.$$

By Induction Hypothesis, we get

$$\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to (\phi_1 \to \phi_2);$$
$$\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to (\phi_1 \to \mathcal{X}\phi_1).$$

As in the example with the A-theorem (A□Unr), we can easily adapt the derivation given in *Example 2.2.10* to derive the A-theorem, for $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:

$$\vdash_{\mathsf{ALT}} \mathcal{X}\Box\varphi \wedge \mathcal{X}\psi \to \mathcal{X}(\Box\varphi \wedge \psi). \qquad \text{(ADistX)}$$

Now we can show $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi$ from the following sequence:

1.	Γ;	set of assumptions
2.	$\Box\varphi \to (\phi_1 \to \phi_2)$;	derivable from Γ
3.	$\Box\varphi \to (\phi_1 \to \mathcal{X}\phi_1)$;	derivable from Γ
4.	$\Box\varphi \wedge \phi_1 \to \phi_2$;	(prop)+2.
5.	$\Box\varphi \wedge \phi_1 \to \mathcal{X}\phi_1$;	(prop)+3.
6.	$\Box\varphi \to \varphi \wedge \mathcal{X}\Box\varphi$;	A-theorem (A□Unr)
7.	$\Box\varphi \to \mathcal{X}\Box\varphi$;	(prop)+6.

8.	$\Box\varphi \wedge \phi_1 \to \mathcal{X}\Box\varphi \wedge \mathcal{X}\phi_1$;	(prop)+5.+7.
9.	$\mathcal{X}\Box\varphi \wedge \mathcal{X}\phi_1 \to \mathcal{X}(\Box\varphi \wedge \phi_1)$;	A-theorem, as stated above
10.	$\Box\varphi \wedge \phi_1 \to \mathcal{X}(\Box\varphi \wedge \phi_1)$;	(prop)+8.+9.
11.	$\Box\varphi \wedge \phi_1 \to \Box\phi_2$;	rule *IndR* over 4. and 10.
12.	$\Box\varphi \to (\phi_1 \to \Box\phi_2)$.	(prop)+11.

Thus, we have $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi$, as expected.

In summary, by induction on the proof of ψ by $\Gamma \cup \{\varphi\}$, we have proved that the first implication of the Deduction Theorem holds.

The converse implication of the theorem, namely the direction from right to left, is simpler to demonstrate: on the one hand, if we assume $\Gamma \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi$, then Weakening ensures that we also have

$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi.$$

On the other hand, we clearly have $\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \varphi$, and so, from the $N_\Box$ rule derived in *Proposition A.4*, we can get

$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \Box\varphi.$$

By concatenating the proofs assumed by the A-theorems $\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \Box\varphi \to \psi$ and $\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \Box\varphi$, and applying MP over the formulas $\Box\varphi \to \psi$ and $\Box\varphi$ of the resulting sequence, we deduce

$$\Gamma \cup \{\varphi\} \vdash'_{\mathsf{ALT}} \psi.$$

This shows the lacking implication and finishes the proof of the Deduction Theorem for $\vdash'_{\mathsf{ALT}}$. ▲

This version of the Deduction Theorem will help determine the A-theorems we need to demonstrate the equivalence between our two axiomatizations of LTL:

Proposition A.10. *For Σ a set of propositional variables and $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we have:*

$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi).$$

Proof. By *Proposition A.6*, we know

$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \iff \varnothing \vdash'_{\mathsf{ALT}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi).$$

By the newly derived Deduction Theorem, we see:

$$\varnothing \vdash'_{\mathsf{ALT}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \iff \{\varphi \to \psi\} \vdash'_{\mathsf{ALT}} \Box\varphi \to \Box\psi \iff$$
$$\iff \{\varphi \to \psi, \varphi\} \vdash'_{\mathsf{ALT}} \Box\psi.$$

By MP over $\varphi \to \psi$ and φ, we can easily deduce $\{\varphi \to \psi, \varphi\} \vdash'_{\mathsf{ALT}} \psi$. Finally, from the rule $N_\Box$, derived in *Proposition A.4*, we can conclude $\{\varphi \to \psi, \varphi\} \vdash'_{\mathsf{ALT}} \Box\psi$. This guarantees us the A-theorem:

$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi).$$

▲

Proposition A.11. *For Σ a set of propositional variables and $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we have:*

$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \mathcal{X}\varphi) \to (\varphi \to \Box\varphi).$$

Proof. By *Proposition A.6* and *Theorem A.9*, the Deduction Theorem for $\vdash'_{\mathsf{ALT}}$, we have:

$$\vdash_{\mathsf{ALT}} \Box(\varphi \to \mathcal{X}\varphi) \to (\varphi \to \Box\varphi) \iff \{\varphi \to \mathcal{X}\varphi\} \vdash'_{\mathsf{ALT}} \varphi \to \Box\varphi.$$

Now we can easily build the sequence

$$\langle \varphi \to \mathcal{X}\varphi, \varphi \to \varphi, \varphi \to \Box\varphi \rangle.$$

The first element is our assumption. The second one is an instance of a propositional tautology, so it is an A-theorem. The third item is obtained by applying the *IndR* rule to the previous two elements in the sequence. We see that the defined sequence serves as a proof of $\varphi \to \Box\varphi$ by $\{\varphi \to \mathcal{X}\varphi\}$ in the alternative axiomatization of LTL, that is, we have $\{\varphi \to \mathcal{X}\varphi\} \vdash'_{\mathsf{ALT}} \varphi \to \Box\varphi$. ▲

These last two results, together with *Proposition A.3* and *Proposition A.4*, give us what we were looking for: the alternative axiomatization of LTL given in *Definition A.1* is equivalent to the standard axiomatization stated in *Definition 2.2.6*. This means that the A-theorems of LTL are exactly the theorems of LTL, for every $\varphi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we have:

$$\vdash_{\mathsf{ALT}} \varphi \iff \vdash_{\mathsf{LTL}} \varphi.$$

Although we already know that the consequence relations $\vdash'_{\mathsf{ALT}}$ and $\vdash'_{\mathsf{LTL}}$ are not identical, we observe that our findings reveal that

$$\varnothing \vdash'_{\mathsf{ALT}} \varphi \iff \varnothing \vdash'_{\mathsf{LTL}} \varphi.$$

Also, as advanced before, we have the inclusion $\vdash'_{\mathsf{LTL}} \subset \vdash'_{\mathsf{ALT}}$:

Proposition A.12. *For Σ a set of propositional variables and $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$, we have:*

$$\Gamma \vdash'_{\mathsf{LTL}} \varphi \implies \Gamma \vdash'_{\mathsf{ALT}} \varphi.$$

Proof. If $\Gamma \vdash'_{\mathsf{LTL}} \varphi$, then there is some finite $\Gamma_0 \subseteq \Gamma$ such that $\vdash_{\mathsf{LTL}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \varphi$. Then, we deduce that we also have

$$\vdash_{\mathsf{ALT}} \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \varphi.$$

Now, from the last expression and the (A-)theorem

$$\vdash_{\mathsf{ALT}} \Box\left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right);$$

we easily deduce, by an instance of a propositional tautology and MP, that:

$$\vdash_{\mathsf{ALT}} \Box\left(\bigwedge_{\gamma \in \Gamma_0} \gamma\right) \to \varphi.$$

As we know, by our Deduction Theorem, this will imply the satisfaction of the expression

$$\bigwedge_{\gamma\in\Gamma_0} \gamma \vdash'_{\mathsf{ALT}} \varphi.$$

Since $\Gamma_0 \subseteq \Gamma$, it is not difficult to check that we have $\Gamma \vdash'_{\mathsf{ALT}} \gamma$ for every $\gamma \in \Gamma_0$, and so $\Gamma \vdash'_{\mathsf{ALT}} \bigwedge_{\gamma\in\Gamma_0} \gamma$. Then, the Cut rule allows us to conclude $\Gamma \vdash'_{\mathsf{ALT}} \varphi$. ▲

This proposition ensures $\vdash'_{\mathsf{LTL}} \subseteq \vdash'_{\mathsf{ALT}}$. We know that the relations are not identical, as we have seen that, for instance, $\{\varphi\} \vdash'_{\mathsf{ALT}} \Box\varphi$ but $\{\varphi\} \nvdash'_{\mathsf{LTL}} \Box\varphi$. Therefore, we get:

$$\vdash'_{\mathsf{LTL}} \subset \vdash'_{\mathsf{ALT}} .$$

In *Theorem 2.2.32*, we show a finitary strong completeness result between the syntactic relation $\vdash'_{\mathsf{LTL}}$ and $\vDash'_{\mathsf{LTL}}$, the standard semantic consequence relation of LTL. From our previous result, we can deduce that, for every finite $\Gamma \cup \{\varphi\} \subseteq \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:

$$\Gamma \vDash'_{\mathsf{LTL}} \varphi \implies \Gamma \vdash'_{\mathsf{ALT}} \varphi.$$

But the converse does not hold. This implies that the alternative syntactic relation we have introduced does not achieve finitary strong completeness with respect to the standard semantic relation. While we will not explore this topic further, it is worth noting that this does not invalidate the possibility of defining a different semantic relation that satisfies completeness.

We conclude this appendix with two propositions, to compare or contrast to work with $\vdash'_{\mathsf{LTL}}$ and with our alternative relation $\vdash'_{\mathsf{ALT}}$. In some cases, a relatively straightforward derivation within the $\vdash'_{\mathsf{LTL}}$ framework can become more intricate in the $\vdash_{\mathsf{ALT}}$ context, primarily due to the lack of the standard Deduction Theorem:

Proposition A.13. *Consider Σ a set of propositional variables and $\varphi_i, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$ for $i < n < \omega$. If $\varphi_0, \dots, \varphi_{n-1} \vdash'_{\mathsf{LTL}} \phi$, then $\mathcal{X}\varphi_0, \dots, \mathcal{X}\varphi_{n-1} \vdash'_{\mathsf{LTL}} \mathcal{X}\phi$.*

Proof. By definition of $\vdash'_{\mathsf{LTL}}$, the expression $\varphi_0, \dots, \varphi_{n-1} \vdash'_{\mathsf{LTL}} \phi$ means that we have

$$\vdash_{\mathsf{LTL}} \left(\bigwedge_{i<n} \varphi_i\right) \to \phi.$$

Then, we see:

1. $\left(\bigwedge_{i<n} \varphi_i\right) \to \phi;$ — derivable
2. $\mathcal{X}\left(\left(\bigwedge_{i<n} \varphi_i\right) \to \phi\right);$ — $N_{\mathcal{X}}$ over 1.
3. $\mathcal{X}\left(\bigwedge_{i<n} \varphi_i\right) \to \mathcal{X}\phi;$ — MP over an instance of $(K_{\mathcal{X}})$ and 2.
4. $\left(\bigwedge_{i<n} \mathcal{X}\varphi_i\right) \to \mathcal{X}\left(\bigwedge_{i<n} \varphi_i\right);$ — derivable, see *Example 2.2.10*

5. $\left(\bigwedge_{i<n} \mathcal{X}\varphi_i\right) \rightarrow \mathcal{X}\phi.$ (prop)+3.+4.

Then, by the Deduction Theorem of $\vdash_{\mathsf{LTL}}$, *Theorem 2.2.31*, and the *Observation 2.2.30*, we can deduce

$$\bigwedge_{i<n} \mathcal{X}\varphi_i \vdash'_{\mathsf{LTL}} \mathcal{X}\phi;$$

and so, as we required:

$$\mathcal{X}\varphi_0, \ldots, \mathcal{X}\varphi_{n-1} \vdash'_{\mathsf{LTL}} \mathcal{X}\phi.$$

▲

By *Proposition A.12*, we know that the same statement will also hold within the relation $\vdash'_{\mathsf{ALT}}$, so the following proposition could be shown effortlessly. However, we will effectively prove it, to compare it to the $\vdash'_{\mathsf{LTL}}$ case:

Proposition A.14. *Consider Σ a set of propositional variables and $\varphi_i, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$ for $i < n < \omega$. If $\varphi_0, \ldots, \varphi_{n-1} \vdash'_{\mathsf{ALT}} \phi$, then $\mathcal{X}\varphi_0, \ldots, \mathcal{X}\varphi_{n-1} \vdash'_{\mathsf{ALT}} \mathcal{X}\phi$.*

Proof. The first steps in the argumentation essentially reproduce the proof given in the previous proposition. However, we will need to perform certain modifications to align with the $\vdash'_{\mathsf{ALT}}$ version of the Deduction Theorem.

We suppose $\varphi_0, \ldots, \varphi_{n-1} \vdash'_{\mathsf{ALT}} \phi$. By *Theorem A.9*, the Deduction Theorem for $\vdash'_{\mathsf{ALT}}$, and *Proposition A.6*, we know that we have

$$\vdash_{\mathsf{ALT}} \Box\varphi_0 \rightarrow (\Box\varphi_1 \rightarrow \cdots \rightarrow (\Box\varphi_{n-2} \rightarrow (\Box\varphi_{n-1} \rightarrow \phi) \ldots);$$

$$\vdash_{\mathsf{ALT}} \left(\bigwedge_{i<n} \Box\varphi_i\right) \rightarrow \phi.$$

Then, we see:

1. $\left(\bigwedge_{i<n} \Box\varphi_i\right) \rightarrow \phi;$ derivable
2. $\mathcal{X}\left(\left(\bigwedge_{i<n} \Box\varphi_i\right) \rightarrow \phi\right);$ $N_{\mathcal{X}}$ over 1.
3. $\mathcal{X}\left(\bigwedge_{i<n} \Box\varphi_i\right) \rightarrow \mathcal{X}\phi;$ MP over an instance of ($K_{\mathcal{X}}$) and 2.
4. $\left(\bigwedge_{i<n} \mathcal{X}\Box\varphi_i\right) \rightarrow \mathcal{X}\left(\bigwedge_{i<n} \Box\varphi_i\right);$ derivable, generalization of (ADistX)
5. $\left(\bigwedge_{i<n} \mathcal{X}\Box\varphi_i\right) \rightarrow \mathcal{X}\phi;$ (prop)+3.+4.
6. $\left(\bigwedge_{i<n} \Box\mathcal{X}\varphi_i\right) \rightarrow \left(\bigwedge_{i<n} \mathcal{X}\Box\varphi_i\right);$ derived below
7. $\left(\bigwedge_{i<n} \Box\mathcal{X}\varphi_i\right) \rightarrow \mathcal{X}\phi;$ (prop)+5.+6.

8. $\Box\left(\bigwedge_{i<n}\mathcal{X}\varphi_i\right) \to \left(\bigwedge_{i<n}\Box\mathcal{X}\varphi_i\right)$; derived below

9. $\Box\left(\bigwedge_{i<n}\mathcal{X}\varphi_i\right) \to \mathcal{X}\phi.$ (prop)+7.+8

Then, applying again the Deduction Theorem, we deduce

$$\left(\bigwedge_{i<n}\mathcal{X}\varphi_i\right) \vdash'_{\mathsf{ALT}} \mathcal{X}\phi;$$

from which we can derive, as we were looking for,

$$\mathcal{X}\varphi_0, \ldots, \mathcal{X}\varphi_{n-1} \vdash'_{\mathsf{ALT}} \mathcal{X}\phi.$$

To conclude the proof, we need to derive the expressions:

$$\vdash_{\mathsf{ALT}} \left(\bigwedge_{i<n}\Box\mathcal{X}\varphi_i\right) \to \left(\bigwedge_{i<n}\mathcal{X}\Box\varphi_i\right);$$

$$\vdash_{\mathsf{ALT}} \Box\left(\bigwedge_{i<n}\mathcal{X}\varphi_i\right) \to \left(\bigwedge_{i<n}\Box\mathcal{X}\varphi_i\right).$$

By induction, we can reduce our task to proving the following LTL theorems for $\varphi, \psi \in \mathtt{Fm}_{\mathrm{LT}(\Sigma)}$:

$$\vdash_{\mathsf{ALT}} \Box\mathcal{X}\varphi \to \mathcal{X}\Box\varphi; \tag{$\Box$Lin}$$

$$\vdash_{\mathsf{ALT}} \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi). \tag{$\Box$Dist}$$

Proof of ($\Box$Lin):

1.	$\Box\mathcal{X}\varphi \to (\mathcal{X}\varphi \wedge \mathcal{X}\Box\mathcal{X}\varphi)$;	instance of (A$\Box$Unr)
2.	$(\mathcal{X}\varphi \wedge \mathcal{X}\Box\mathcal{X}\varphi) \to \mathcal{X}(\varphi \wedge \Box\mathcal{X}\varphi)$;	derivable from *Example 2.2.10*
3.	$\Box\mathcal{X}\varphi \to \mathcal{X}(\varphi \wedge \Box\mathcal{X}\varphi)$;	(prop)+1.+2.
4.	$(\varphi \wedge \Box\mathcal{X}\varphi) \to \Box\varphi$;	derivable from *Example $\Box$ Unr*
5.	$\mathcal{X}((\varphi \wedge \Box\mathcal{X}\varphi) \to \Box\varphi)$;	rule $N_{\mathcal{X}}$ over 4.
6.	$\mathcal{X}(\varphi \wedge \Box\mathcal{X}\varphi) \to \mathcal{X}\Box\varphi$;	MP over instance of ($K_{\mathcal{X}}$) and 5.
8.	$\Box\mathcal{X}\varphi \to \mathcal{X}\Box\varphi$.	(prop)+3.+6.

Proof of ($\Box$Dist): by the Deduction Theorem, we only need to show

$$\{\varphi \wedge \psi\} \vdash'_{\mathsf{ALT}} \Box\varphi \wedge \Box\psi.$$

This is clear since $\varphi \wedge \psi$ implies φ and ψ, and, by the derived rule $N_{\Box}$ from *Proposition A.4*, we have both $\Box\varphi$ and $\Box\psi$, so we deduce that there is some proof of $\Box\varphi \wedge \Box\psi$ by $\varphi \wedge \psi$. Alternatively, observe that ($\Box$Dist) holds in $\mathbf{K}_{\Box}$, and therefore also holds in LTL, as discussed in Section 2.2.1, on Page 44. ▲