Master In Quantum Science and Technology

Master's thesis

Sufficient separability criteria via quantum maps

July 2024

Jofre Abellanet Vidal

Supervisors:

Prof. Dr. Anna Sanpera Trigueos
Guillem Müller-Rigat

Institution:

Universitat Autònoma de Barcelona
Grup d'informació Quàntica

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my two supervisors, Guillem and Anna, for their infinite help, unwavering support, and belief in me throughout this journey. Their guidance has been invaluable. I would also like to extend my heartfelt thanks to Grzegorz for his valuable assistance and insights.

I am grateful to Jordi, Arnau, David, Irene, and Jennifer for sharing the everyday life of work with me. Your companionship and collaboration made this experience truly enjoyable. I extend this gratitude to every member of the GiQ for creating such a stimulating and supportive environment.

A special thank you to all my master's colleagues who made this year wonderful. Your friendship and camaraderie have been a source of joy and motivation. A particular thanks to Aleix for his expertise in optimizing the numerics. Also thank E. Mateos for insightful discussions.

Finally, I want to thank my family as well as Xavi, and most importantly, Mireia, who have always been by my side. Your constant support and encouragement have been my foundation.

Sufficient separability criteria via quantum maps

Jofre Abellanet Vidal

Supervised by: Prof. Dr. Anna Sanpera Trigueros and Guillem Müller Rigat

Grup d'Informació Quàntica - Universitat Autònoma de Barcelona

July 2024

In this master's thesis, we consider quantum linear maps as a tool to derive new sufficient conditions for separability in bipartite and multipartite systems. In specific, we focus on the so-called reduction map to strengthen the existing criteria for absolute separability in bipartite systems, i.e., for states that remain bi-separable under any global unitary transformation. To this aim, using powerful convex geometry techniques, we introduce tighter volumes and characterization of the set of absolutely separable states w.r.t. any bi-partition for arbitrary dimensions. Furthermore, we derive new conditions on the spectrum of bipartite entanglement witnesses. In addition, we address the multipartite scenario by presenting some non-optimal results. Finally, we provide some insights on the conjecture that having a positive partial transpose from spectrum is equivalent to being separable from spectrum for the symmetric subspace of $N-$qudits, as well as a new criterion for positive partial transpose from spectrum for arbitrary system sizes N.

Keywords: Linear maps, Separability from spectrum, PPT from spectrum, entanglement witnesses, multipartite separability, symmetric subspace.

Jofre Abellanet Vidal: jabellvi70@alumnes.ub.edu

List of acronyms

Abbreviation	Full name
APPT	Absolute Positive Partial Transpose
AS	Absolutely Separable
CH	Convex Hull
CPTP	Completely Positive Trace Preserving
CS	Convex Set
DCP	Disciplined Convex Programming
DS	Diagonal Symmetric
EW	Entanglement Witness
GME	Genuine Multipartite Entanglement
MMS	Maximally Mixed State
NPT	Negative Partial Transpose
PPT	Positive Partial Transpose
PPTES	Positive Partial Transpose Entangled State
PSD	Positive Semidefinite
SAPPT	Absolute positive partial transpose in the Symmetric subspace
SAS	Absolutely Separable in the Symmetric subspace
SEP	Separable
SN	Schmidt Number
SR	Schmidt Rank

Contents

1 Introduction

The 20th century quantum revolution led to massive technological breakthroughs, such as the invention of transistors, microchips, or lasers. However, entanglement and non-locality, two of the most surprising phenomena of quantum physics, were seen, for a long time, as an inconsistency of quantum mechanics[EPR35]. Entanglement, non-locality and superpositions are, nowadays, widely acknowledged to be the most significant quantum features that allow to perform tasks in the processing of information that otherwise will not be possible. Typical examples are secure quantum key distribution [LCT14], quantum teleportation [BBC+93] or secure quantum cryptography using Bell correlations [Eke91], just to name a few. Recently, the experimental advances in controlling and manipulating quantum systems at the micro and nanoscale (from single atoms to many body systems, ions, mechanical resonators or photons) has made possible the preparation of entangled states in various systems [LKS+05, HLC23, JKP01], allowing to test and develop novel quantum information protocols [FGBL22]. Moreover, to unleash the true power of quantum computation, quantum correlations are required. For all these reasons, the characterization, verification, and detection of entanglement, keep being one of the most fundamental research areas in the field of quantum information. In particular, the separability problem, that is, determining whether a state is separable or entangled, remains one of the most relevant open problems in the broad field of quantum information, with major implications for the development of quantum technologies.

Nevertheless, detecting and quantifying entanglement is typically extremely challenging, and determining if a given quantum state is entangled or not has been proven to be an NP-Hard problem [Gha10, Gur03]. As a consequence, the separability problem is still an open question whose solution as a set of analytical sufficient and necessary conditions exists only for a few simple cases. A necessary condition for a statement to be true must be satisfied, but satisfying it alone does not guarantee that the statement is true. A sufficient condition, if met, guarantees that the statement is true, but not meeting it does not necessarily mean that the statement is false. For pure states, entanglement and non-locality are equivalent, meaning that all entangled states violate a Bell inequality and vice versa; all states that violate a Bell inequality are entangled. For bipartite pure states there exists, moreover, the so-called Schmidt decomposition, which easily provides a way to check if a pure state is entangled or not. However, for mixed states, necessary and sufficient criteria are only established for low dimensions [Per96, HHH96]. Sufficient conditions for entanglement, such as entanglement witnesses or Bell inequalities, have been exhaustively scrutinized in the literature. Conversely, sufficient conditions for separability remain much less explored. In the present thesis, we focus on the latter. Remarkably, it has been recently shown that quantum maps can provide sufficient criteria (not necessary) for separability both in the case of bipartite systems [LAC+16, HHH01] and also in the multipartite scenario, where different types of entanglement exist [LMRTS22, FMJ17].

This master's thesis focuses on the latest line of research: quantum maps, and connects it with the concept of absolute separability (AS) or separability from spectrum [KZ01, GB02], i.e., states whose separability is certified with only the eigenvalues of the density matrix and therefore are invariant in front of unitary transformations. Specifically, we derive extreme sufficient linear conditions on the eigenvalues of a state for AS that provide a better bound than the existing criteria. By convexity arguments, we combine our new conditions with existing criteria to provide a tighter characterization of the AS set for general dimensions. Our contribution extends also to the spectral properties of entanglement witnesses, since we also derive a lower and higher bounds for their minimal and maximal eigenvalues, respectively. The novel conditions can be cast as a standard convex program, which, in particular, allows one to test compatibility with bipartite AS states of a given set of experimentally-inferred mean values. Finally, we address the multipartite scenario and, in particular, symmetric states. We derive new sufficient criteria to certify a positive partial transpose from spectrum for general systems of $N-$qudits.

The structure of this thesis is as follows: in Section 2 we introduce all preliminary concepts used and needed for the derivation of our results. In Section 3, we focus on the use of quantum maps as sufficient criteria for separability, and we present some of our results, regarding simplexes of AS states and the properties that can be derived from the corresponding entanglement witnesses. In Section 4, we report on the improvement obtained with our results as compared with other criteria already existing in the literature. We also deploy various convex techniques to combine all these criteria, leading to stronger conditions for bipartite AS. We finish this master's thesis, by analyzing the application of our results in multipartite systems and symmetric states in Section 5. Conclusions and open questions resulting from our study are stated in the Section 6.

Some of the proofs of our results are quite involved and, for the easiness of reading, we include them as complementary appendices at the end.

2 Preliminaries

The main objective of this master's thesis is to approach the problem of separability of mixed states using quantum maps. In this first section, we present some necessary concepts used through this work.

2.1 Quantum states

According to the postulates of quantum mechanics, the state of any physical system $\mathcal{S}$, is encoded into a unit trace *positive semidefinite* (PSD) operator, $\rho_S \in \mathcal{B}(\mathcal{H}_S)$, where $\mathcal{B}(\mathcal{H}_S)$ denotes the set of bounded operators acting on the associated Hilbert space $\mathcal{H}_S$. Isolated quantum systems have a simpler description given by a unit vector $|\psi_k\rangle \in \mathcal{H}_S$, and are called *pure*. Equivalently, a state is pure, if and only if, $\mathrm{Tr}(\rho_S^2) = 1$, corresponding to rank one operators $\rho_S = |\psi_k\rangle\langle\psi_k|$. Otherwise, the state is called *mixed*, and can be represented as a convex combination of projectors onto pure states. Let $\mathcal{D}(\mathcal{H}_S) = \{\rho_S \in \mathcal{B}(\mathcal{H}_S) | \rho_S \succeq 0, \mathrm{Tr}(\rho_S) = 1\}$, denote the space of density matrices. Notice that $\mathcal{D}(\mathcal{H}_S)$ is a convex set, with pure states being its extreme points. The *maximally mixed state* (MMS) is the state with the smallest purity: $\mathrm{Tr}(\rho^2) = 1/D$, and thus it is proportional to the identity operator, $\mathbb{1}_D$, where D is the dimension of the corresponding Hilbert space.

The state space of composite systems is given by the tensor product of the individual state spaces. For example, for bipartite composite systems AB, the global Hilbert space can be expressed as $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The subsystems are usually called *parties*. Considering $\{A_i\}$, $\{B_i\}$ to be Hermitian operators of the subsystems A and B, respectively, in the density matrix representation, the state of the composite system can be represented as

$$\rho_{AB} = \sum_i A_i \otimes B_i. \tag{1}$$

The density matrix of the subsystems $\rho_A \in \mathcal{B}(\mathcal{H}_A)$ and $\rho_B \in \mathcal{B}(\mathcal{H}_B)$ can be obtained through the partial trace of the composite state. Specifically, $\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_i \mathrm{Tr}(B_i) A_i$ and similarly for $\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \sum_i \mathrm{Tr}(A_i) B_i$. In addition, one can also define the partial transpose w.r.t. subsystem A as $\rho_{AB}^{T_A} = \sum_i A_i^T \otimes B_i$, where T is the usual transposition (the definition is analogue for B).

2.2 Quantum maps

A *quantum map* or super-operator $\Lambda : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ transforms operators $A_1 \in \mathcal{B}(\mathcal{H}_1)$ onto operators $A_2 = \Lambda(A_1) \in \mathcal{B}(\mathcal{H}_2)$. Thus, maps are used to describe state transformations when using the density matrix formalism. For a map to correspond to a *quantum channel*, i.e., a physical transformation that converts a physical system $\rho_1 \in \mathcal{D}(\mathcal{H}_1)$ into another physical state $\rho_2 = \Lambda(\rho_1) \in \mathcal{D}(\mathcal{H}_2)$, it must comply with the following properties. (i) Linearity $\Lambda(\sum_i p_i \rho_i) = \sum_i p_i \Lambda(\rho_i)$, (ii) hermiticity-preserving $\Lambda(\rho_1^\dagger) = \Lambda(\rho_1)^\dagger$, (iii) positivity $\Lambda(\rho_1) \geq 0, \forall \rho_1 \in \mathcal{D}(\mathcal{H}_1)$, (iv) trace preserving $\mathrm{Tr}(\Lambda(\rho_1)) = \mathrm{Tr}(\rho_1) = 1$ and (v) complete positivity. That is, the extended map $\mathbb{1} \otimes \Lambda$, where $\mathbb{1}$ is the identity map, is also positive. The maps that fulfill the above conditions and are trace preserving are called *completely positive trace preserving* (CPTP). In this thesis, we make use of positive maps which are not necessarily CPTP to derive sufficient conditions for separability. In this context, the map should be understood as a mathematical tool to derive the aforementioned conditions.

2.3 Entanglement

The quantum formalism admits parties to be correlated in ways that cannot be explained by classical statistical mechanics. In order to describe such phenomena, we have to introduce the notion of entanglement. We begin the discussion with the simplest scenario of just two parties.

2.3.1 Bipartite entanglement

Let us consider two subsystems A, B in a global pure state $|\phi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B := \mathcal{H}_{AB}$. Below, we provide the definition of separability for such states.

Definition 2.1. *A pure state of a composite system $|\phi_{AB}\rangle \in \mathcal{H}_{AB}$ is said separable if and only if it can be written as $|\phi_{AB}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$.*

Clearly, not all states in the composite Hilbert space are separable – those that are not are called *entangled*. The previous definition is extended to mixed states as follows:

Definition 2.2. *A given state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ is called to be separable if and only if it can be written as $\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}$, where $\rho_k^{(i)} \in \mathcal{H}_k$ and $\{p_i\}$ forms a probability distribution. The set of separable states is denoted by $SEP(A|B)$. States that do not admit such a decomposition are called entangled.*

Although this previous notion of entanglement can be generalized to an arbitrary number of parties, one needs to be careful when defining multipartite entanglement because its characterization is very subtle [GT09, HHHH09]. For example, it is not clear how to define a maximally entangled state, since different states maximize different entanglement measures.

2.3.2 Multipartite entanglement

The extension of Def. 2.2 to the multipartite case can be cast as follows.

Definition 2.3. *A quantum state $\rho \in \mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$ is fully separable if it can be written as*

$$\rho = \sum_i p_i \cdot \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)} \in SEP(\mathcal{H}_1| \cdots |\mathcal{H}_N), \tag{2}$$

where $\rho_i^{(k)} \in \mathcal{D}(\mathcal{H}_k)$ are quantum states acting on the $k-th$ Hilbert space $\mathcal{H}_k$ and the coefficients $\{p_i\}$ form a convex combination. Otherwise, it is called entangled.

Notice that one can take an $N-$partite system of qudits of local dimension d and work with it as a bipartite system acting on a Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes k} \otimes (\mathbb{C}^d)^{\otimes N-k}$ of dimension $d^k \cdot d^{N-k} = d^N$. Nevertheless, $(\mathbb{C}^d)^{\otimes k} \neq \mathbb{C}^{d \cdot k}$. Despite being isomorphic, the two spaces are not equivalent. Moreover, multipartite entangled states can be classified in a wide number of classes that increases with the dimension of the system [DC00]. Next, we introduce two classifications of entanglement for multipartite systems.

A quantum state of a multipartite system $\rho \in \mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$ is *separable w.r.t. the bi-partition* of the state on two groups $A|B$ if it can be written following Def. 2.2 for that partition. This definition applies to all the possible partitions $A_i|B_i$, independently of their local dimensions. Then, *bi-separability* is defined as follows.

Definition 2.4. *A state $\rho \in \mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$ is bi-separable if it can be written as*

$$\rho_{bi\text{-}sep} = \sum_i p_i \cdot \rho^{sep}_{A_i|B_i}, \tag{3}$$

where $\rho^{sep}_{A_i|B_i}$ are PSD operators that are separable in each $A_i|B_i$ possible bi-partition.

One can clearly see that a state that is separable (SEP) w.r.t. a certain bi-partition is already bi-separable, but the converse is not true.

2.3.3 Entanglement criteria for bipartite systems

Entanglement detection is a complex subject and, in fact, the separability problem has been proven to be NP-hard [Gur03]. While it is expected that there is no efficient procedure to determine whether a given mixed state is separable [Gha10], there exist some criteria for certain subsets of states or specific cases. The Schmidt decomposition gives rise to a well known sufficient criteria for pure bipartite systems.

Definition 2.5. *Given $|\Psi_{AB}\rangle \in \mathcal{H}_{AB}$, there exist two orthonormal bases $\{|\psi_i\rangle\} \in \mathcal{H}_A$, $\{|\phi_i\rangle\} \in \mathcal{H}_B$ such that*

$$|\Psi_{AB}\rangle = \sum_{i=1}^{r} \lambda_i |\psi_i\rangle |\phi_i\rangle , \tag{4}$$

where the Schmidt coefficients $\lambda_i \in \mathbb{R}^+$ satisfy $\sum_{i=1}^r \lambda_i^2 = 1$ and $r \leq \min(d_A, d_B)$ is the so-called Schmidt rank.

If the Schmidt rank (SR) of a state is $r = 1$, then the state is separable, otherwise is entangled. For bipartite mixed states, the Schmidt rank can be generalized to the *Schmidt number* (SN) m, however, it is generally not possible to determine the SN of a given state ρ_{AB}, preventing us from a complete characterization of entanglement.

Definition 2.6. *A bipartite mixed state ρ_{AB} has Schmidt number m if: i) for any decomposition $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ at least one of the vectors $\{|\psi_k\rangle\}$ has at least Schmidt rank m, and ii) there exists a decomposition of ρ with all the vectors $\{|\psi_k\rangle\}$ of Schmidt rank at most m.*

As it happens for pure state, any bipartite mixed state is separable if and only if it has Schmidt number $m = 1$, otherwise it is entangled. This condition is key to prove quantum maps as a sufficient separability criterion in Ref. [LAC+16].

On the other hand, for mixed states, the positivity of the partial transpose provides a simple way to detect entanglement with the so-called *postive partial transpose* (PPT) or Peres-Horodecki criterion.

Theorem 1. *[Per96, HHH96] If $\rho_{AB} \in \mathcal{B}(\mathcal{H}_{AB})$ is a bipartite separable state, then it is PPT, i.e., $\rho_{AB}^{T_A} \geq 0$.*

Thus, states with *negative partial transpose* (NPT) are certified as entangled. It also paves the way for the existence of *PPT entangled states* (PPTES), since it is only a necessary condition. However, for low dimensions $dim(\mathcal{H}_{AB}) \leq 6$, PPT becomes also a sufficient condition. Sufficient criteria for separability are substantially more intricate to derive than just necessary conditions, and are the main focus of this master's thesis.

Finally, positive but not completely positive maps are also used to detect entanglement, by finding counterexamples to the following theorem.

Theorem 2. *[HHH96] A state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable if and only if, for every positive map $\Lambda : \mathcal{H}_B \to \mathcal{H}_C$, it holds that $(\mathbb{1} \otimes \Lambda)[\rho_{AB}] \geq 0$.*

2.3.4 Separability from spectrum and PPT from spectrum

Some states have the property that are not entangled under the action of any **global** unitary operation U. As unitary matrices do not change the spectrum of the eigenvalues of the given state ρ, it is possible to infer some separability criteria that only depend on the eigenvalues of the given state [VADM01]. So, if ρ fulfills any separability criterion, $U\rho U^\dagger$ also fulfills it for any global unitary. Moreover, the criterion will be valid for the different partitions of the whole system, as the spectrum does not change. It is specially interesting to characterize such separable states because obtaining the eigenvalues of a system is easier than inferring full tomography [EAO+02, TOK+14]. Obtaining the eigenvalues of a quantum state is easier than full tomography because a density matrix of dimension D has D^2 complex entries to reconstruct, whereas there are only D eigenvalues to determine. On the other hand, determining whether a state is separable from spectrum is also of great importance for quantum enhanced applications in which entanglement constitutes a resource. Such states remain separable under any global unitary map. Unitary transformations are by far the most common method to generate entanglement, e.g., through quantum circuits or quenches. Thus, if a state is certified separable from spectrum, it is apparently useless, as it cannot lead to entanglement in these experiments. Nonetheless, some methods to create entanglement from them without global unitary operations have already been proposed [HMS21].

States that remain separable under the action of any global unitary are called *separable from spectrum* states or *absolutely separable states* (AS). We have followed the notation in [FMJ17], naming them $\mathcal{A}(i|j)$, where i, j are the dimensions of each of the separable partitions. Despite the efforts, characterizing AS states is still one of the most important open problems in quantum theory [Kni13]. It is interesting to note that the set of AS states is convex (see Appendix A), and it is compact [GCM14], the border of the set is included in it. Some sufficient criteria have been derived for such states, as we present next.

Theorem 3. *[GB02] Let $\rho \in \mathcal{D}(\mathbb{C}^M \otimes \mathbb{C}^N)$ be a normalized density matrix and $\{\lambda_i\}$ its eigenvalues. Then, it is AS w.r.t. any partition if*

$$\mathrm{Tr}\left(\rho^2\right) = \sum_{i=0}^{M\cdot N-1} \lambda_i^2 \leq \frac{1}{M \cdot N - 1}. \tag{5}$$

Notice that this equation is actually defining a ball in the space of eigenvalues. The previous theorem is general and holds for arbitrary dimensions, but it does not provide a full characterization of the set of AS states. For partitions of the type $\mathbb{C}^2 \otimes \mathbb{C}^M$, the set $\mathcal{A}(2|M)$ has been completely characterized in Ref. [Joh13], where they obtain a single equation for the ordered eigenvalues $\lambda_0 \leq \cdots \leq \lambda_{2M-1}$ as a necessary and sufficient condition for AS.

Theorem 4. *[Joh13] Let $\rho \in \mathcal{D}(\mathbb{C}^2 \otimes \mathbb{C}^M)$ be a normalized density matrix and $\{\lambda_i\}$ its eigenvalues in non-decreasing order. Then, it is AS w.r.t. any partition $\mathbb{C}^2 \otimes \mathbb{C}^M$ if and only if*

$$\lambda_{2M-1} \leq \lambda_1 + 2 \cdot \sqrt{\lambda_2 \cdot \lambda_0}. \tag{6}$$

It is interesting to note that all AS states that have been characterized are close to the MMS and, indeed, there is a maximum bound on the purity of such set,

$$\mathrm{Tr}\left(\rho^2\right) \leq \frac{9}{M \cdot N + 8}, \tag{7}$$

for which AS states can exist [FMJ17].

The concept of AS can be extended to PPT states, introducing the so-called *PPT from spectra* or *absolutely PPT* states (APPT). These remain PPT for any global unitary U. It has been conjectured in Ref. [AJR15], that being APPT is equivalent to being AS. Also, it has been shown that, at least, being APPT is equivalent to fulfilling many of the necessary conditions for separability. Finally, the conjecture has been proven for $\mathbb{C}^2 \otimes \mathbb{C}^M$ [Joh13]. In the general case $\mathbb{C}^M \otimes \mathbb{C}^N$, APPT has been completely characterized by a series of linear matrix inequalities (LMI) [Hil07b]. However, the number and the dimension of the LMI depend on M, N, making their resolution daunting.

2.3.5 Entanglement criteria for multipartite systems

In this section, we extend the separability criteria to detect states as completely separable, i.e., that can be written as in Def. 2.3. Unlike the bi-separable case, the characterization of full separability from spectrum remains largely unexplored.

Let us start with the simplest multiparty system, namely, a collection of three qubits. It is immediate to notice that $\mathcal{A}(2|2|2) \subseteq \mathrm{SEP}(2|2|2) \subseteq \mathrm{SEP}(2|4)$. It is also possible to verify that, the inclusion $\mathcal{A}(2|2|2) \subseteq \mathcal{A}(2|4)$ holds. However, it is crucial to realize that $\mathcal{A}(2|4) \not\subset \mathrm{SEP}(2|2|2)$, i.e., there exist states that are simultaneously absolutely separable w.r.t. any bi-partition $2|4$ but nonetheless entangled (see also Section 5).

The result of AS from Ref. [GB02] can be extended to separability w.r.t. any bi-partition for multipartite systems. The same author extended the result to full separability of multipartite systems of N parties in Hilbert spaces of dimension d.

Theorem 5. *[GB03] Let $\rho \in \mathcal{D}((\mathbb{C}^d)^{\otimes N})$ be a normalized density matrix of N qudits of local dimension d. Then, ρ is fully separable if*

$$\mathrm{Tr}\left(\rho^2\right) \leq \frac{1}{d^N} \cdot \left(1 + \frac{2^{-N+2}}{d^N - 2^{-(N-2)}}\right). \tag{8}$$

Again, the trace of the matrix squared is defining a ball in the space of eigenvalues. This bound has been improved for the $d = 2$ case [Hil05] as follows:

Theorem 6. *[Hil05] Let $\rho \in \mathcal{D}((\mathbb{C}^2)^{\otimes N})$ be a normalized density matrix of N qubits. Then, ρ is fully separable if*

$$\mathrm{Tr}\left(\rho^2\right) \leq \frac{1}{2^N} \left(1 + \frac{54}{17} \cdot 3^{-N}\right). \tag{9}$$

Interestingly, these two latter results present much smaller balls than Theorem 3 and, the latest Ref. [Hil07a] also establishes an upper bound for the radius of the fully separable ball around the MMS for an $N-$qubit system,

$$\mathrm{Tr}\left(\rho^2\right) \leq \frac{1}{2^N} \cdot \left(1 + 4 \cdot 3^{-N}\right). \tag{10}$$

This ball is much smaller than the one for bi-separability leading to the existence of states that are APPT and AS w.r.t. any bi-partition, full rank and yet, entangled.

3 Quantum maps as sufficient criteria for separability

Having introduced the main concepts related to separability and entanglement, we switch now to the subject of the master's thesis, the use of quantum maps. The properties of a linear quantum map can depend on a set of parameters $\mathbf{p}$, from which the properties of the map can be changed. The action of maps $\Lambda_{\mathbf{p}}(\rho) = \sigma$ onto a state ρ to check the entanglement or separability conditions on the output of the channel σ have been studied (see, e.g., [FMJ17, HHH01]). Here, we focus on the use of the inverse of linear maps $\Lambda_{\mathbf{p}}^{-1}(\sigma) = \rho$ as **sufficient** conditions for entanglement or separability on a given state σ. The starting point of my master's thesis is encoded in the following theorem:

Theorem 7. *[LAC+16] Let S, S' be convex and compact subsets of $\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$, and let $\Lambda_{\mathbf{p}} : S \to S'$ be a family of maps, invertible for almost all $\mathbf{p}$. By, $\mathcal{P}_{SS'}$ we denote the subset of the parameters set $\mathbf{p}$. The maps have the property that, for every $\rho \in S$, $\Lambda_{\mathbf{p}}(\rho) \in S'$ provided $\mathbf{p} \in \mathcal{P}_{SS'}$. Then, if $\Lambda_{\mathbf{p}}^{-1}(\sigma) \in S \to \sigma \in S'$*

In other words, the theorem provides sufficient criteria to certify $\sigma \in S'$. In practice, the choice of $\Lambda_{\mathbf{p}}$ is such that (i) we can easily check that $\Lambda_{\mathbf{p}}^{-1}(S') \subset S$ and (ii) we can prove the assumption that $\Lambda_{\mathbf{p}}(S) \subset S'$. The difficulty on the derivation of such criteria is hidden in demonstrating condition (ii) and in technical difficulties of inverting $\Lambda_{\mathbf{p}}$. In practice, we take S the set of states, or, up to normalization, positive-semi definite (PSD) matrices and S' a subset of states with certain separability conditions to be specified.

3.1 The reduction map

In this thesis, we focus on the family of reduction maps, as a function of a single parameter α, which can be related to the robustness as introduced in [VT99]. The expression

$$\Lambda_\alpha(\rho) = \mathrm{Tr}(\rho) \cdot \mathbb{1} + \alpha \cdot \rho \tag{11}$$

defines positive but not completely positive maps. Let us also fix $S' \subset S$ as the set of separable states w.r.t. a certain bi-partition. States $\rho \in \mathcal{D}(\mathcal{H})$ fulfill $\mathrm{Tr}(\rho) = 1$, but one might consider the action of the map on a partition of a state or even on a non-normalized state, so that, in general, the $\mathrm{Tr}(\rho)$ term is not necessarily equal to one.

Inverting a map is, in general, an involved task. However, for the reduction map Eq. (11), the inverse can be obtained in a few steps (see Appendix B). It reads

$$\Lambda_\alpha^{-1}(\sigma) = \frac{1}{\alpha}\left(\sigma - \frac{\mathrm{Tr}(\sigma)\cdot\mathbb{1}}{D+\alpha}\right). \tag{12}$$

After the presentation of the inverse, we outline its associated separability theorem:

Theorem 8. *[LAC+16] Given a state $\rho \in \mathcal{D}(\mathcal{H}_M \otimes \mathcal{H}_N)$, let $\Lambda_\alpha(\rho) = \mathrm{Tr}(\rho)\mathbb{1} + \alpha\rho$ be the family of maps, and $-1 \le \alpha \le m+1$. Then, $\rho \ge 0 \to \Lambda_\alpha(\rho) =: \sigma$ has at most Schmidt number m.*

Similarly, if $\sigma \in \mathcal{D}(\mathbb{C}^M \otimes \mathbb{C}^N) \ge 0$, and $\rho = \Lambda_\alpha^{-1}(\sigma) \ge 0$, then σ has SN at most m.

It is important to mention here that the theorem is only giving us an upper bound for the SN, so a separable state will also be detected for any SN. Then, it is more natural to focus on the smallest of such sets, namely the separable one with $m = 1$.

Corollary 8.1. *[LAC+16] Given a state $\sigma \in \mathcal{D}(\mathbb{C}^M \otimes \mathbb{C}^N)$ and $\alpha \in [-1, 2]$. If $\Lambda_\alpha^{-1}(\sigma) = \frac{1}{\alpha}(\sigma - \frac{\mathrm{Tr}(\sigma)\mathbb{1}}{M\cdot N+\alpha}) \ge 0$, then σ is separable.*

In order to verify the bi-separability condition $\Lambda_\alpha^{-1}(\sigma) \geq 0$, it is sufficient to check that the minimal eigenvalue of $\Lambda_\alpha^{-1}(\sigma)$ is non-negative. This observation, leads to our first result:

Result 1. *Given a state $\sigma \in \mathcal{D}(\mathbb{C}^D)$, the inverse map $\Lambda_\alpha^{-1}(\sigma) = \frac{1}{\alpha}\left(\sigma - \frac{\mathrm{Tr}(\sigma)\cdot\mathbb{1}}{D+\alpha}\right)$ provides the two following conditions for the state σ to be separable w.r.t. any bi-partition:*

$$\lambda_{\min}(\sigma) \geq \frac{1}{D+2}, \qquad \lambda_{\max}(\sigma) \leq \frac{1}{D-1}, \tag{13}$$

where $\lambda_{\min}, \lambda_{\max}$ indicates the smallest and largest eigenvalues of σ respectively.

Proof. The result follows Corollary 8.1 applied to a system of total dimension D. For $2 \geq \alpha > 0$, one has $\sigma - \mathbb{1}/(D+\alpha) \geq 0$ (i.e. $\lambda_{\min}(\sigma) - 1/(D+\alpha) \geq 0$) is sufficient for separability w.r.t. any bi-partition. On the other hand, for $0 > \alpha \geq -1$, the condition reads $\mathbb{1}/(D+\alpha) - \sigma \geq 0$, or equivalently, $1/(D+\alpha) - \lambda_{\max}(\sigma) \geq 0$. The extreme values $\alpha = 2$ and $\alpha = -1$ yield the announced result. Note that it can be extended to sufficient conditions for Schmidt number $SN \leq m$ by considering $\alpha = 1 + m$. □

The previous separability criterion is basis-independent and is only based on the extreme eigenvalues of the density matrix. Thus, in particular, it constitutes a condition for absolute separability.

3.2 Spectral properties of entanglement witnesses

From the previous results, it is possible to derive new conditions on the spectra of entanglement witnesses (EW) that might detect entangled states close to the MMS. We recall that an entanglement witness is defined as an operator W such that $\mathrm{Tr}(W\rho) \geq 0$ for all SEP state ρ and $\mathrm{Tr}(W\sigma) < 0$ for some entangled state σ [Ter98, LKCH00]. Witnesses arise as a consequence of the Hann-Banach theorem and are related to convex sets. The so-called decomposable EW can be expressed as $W = P + Q^{T_A}$. It has been proven that decomposable EW cannot detect PPTES, whereas non-decomposable EW can [HHH99, Ter02].

Recent studies have focused on the spectral properties of entanglement witnesses [CK09, CJMP22, JP18], just as the spectral properties of separable states discussed earlier. An interesting property that has been derived [JP18, Ran13] (see Appendix D) is that all decomposable bipartite EW fulfill the relation

$$\lambda_{min}(W) \geq -\frac{\mathrm{Tr}(W)}{2}. \tag{14}$$

So far we have focused on the separability problem on states σ, working mostly on their spectrum. This characterization can also be tackled by studying the spectral properties of the EW that provide sufficient conditions for a state to be entangled. Specifically, it is possible to compute the relations for any EW W to detect as entangled states σ that fulfill $\Lambda_{\alpha\pm}^{-1}(\sigma) \geq 0$. The EW is restricted, as derived in Appendix E, by the following inequalities

$$\alpha_+ > -\frac{\mathrm{Tr}(W)}{\lambda_{min}(W)}, \qquad \alpha_- < -\frac{\mathrm{Tr}(W)}{\lambda_{max}(W)}. \tag{15}$$

From these relations, we derive the next result on the spectra of any bipartite witness.

Result 2. *Given a bipartite entanglement witness W, either decomposable or non-decomposable, its spectrum fulfills the following relations on the minimal and maximal eigenvalues:*

$$\lambda_{min}(W) \geq -\frac{\mathrm{Tr}(W)}{2}, \qquad \lambda_{max}(W) \leq \mathrm{Tr}(W).$$

Proof. By contradiction, no bipartite EW can certify as entangled the separable bipartite states detected with $\alpha_+ \in (0,2]$ or $\alpha_- \in [-1,0)$. Thus, the relation in Eq. (15), combined with the results from Theorem 8, restrict the conditions of the spectrum of any bipartite EW, whether decomposable or non-decomposable. □

An example of decomposable EW saturating the previous lower bound is $W = X\otimes X + Y\otimes Y + Z\otimes Z + \mathbb{1}$, where $\{X, Y, Z\}$ are the Pauli matrices. Such a witness is maximally violated by the spin singlet $|\psi\rangle = (|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle)/\sqrt{2}$.

3.3 Characterizing the set of AS states: simplexes and a geometrical approach

In this section, we proceed to characterize the set of states detected as AS from the criteria derived from our Result 1. We aim at characterizing geometrically the set of states detected by the AS criteria. It is also relevant to evaluate its volume, as it has been done with similar sets [BZ06, KZ01]. According to Ref. [ZHSL98], the set of states $\mathcal{S}$ can be interpreted as the Cartesian product of two sets:

$$\mathcal{S} = \mathcal{P} \times \Delta, \tag{16}$$

where $\mathcal{P}$ is the set of complete families of orthogonal projectors (eigenvectors) and Δ the set of eigenvalues. Since our conditions are basis-independent, we will focus on the latter. The set of normalized eigenvalues $\vec{\lambda} := \{\lambda_i \geq 0\}_{i=0}^{D-1}$, $\sum_i \lambda_i = 1$ can be seen as a real vector. In such a geometrical picture, the set of possible eigenvalues forms a $(D-1)-$dimensional regular *simplex*. A simplex can be understood as a generalization of a triangle in higher dimensions and is described either by its extreme points (vertices) or facets. In our case, the vertices represent pure states (or rank-1 projectors), with eigenvalues of the structure $\vec{\lambda} = (0, ..., 0, 1, 0, ...0)$. The volume of such an object is given by:

$$V_{SN} = \frac{\sqrt{D}}{(D-1)!}. \tag{17}$$

As it turns out, the subsets of eigenvalues detected by our conditions based on linear maps (Result 1) forms also simplexes. A schematic representation of these sets for $D = 3$ is depicted in Figure 1. In such a Figure, we offer two views, in the original coordinates (eigenvalues) and in barycentric coordinates $\{\tilde{\lambda}_i\}$, where the MMS is at the origin.

The inverse of the reduction map, leads to the inequalities of Eq. (13), which in turn lead to the D linear conditions of $\lambda_i \geq 1/(D+\alpha_+)$ or $\lambda_i \leq 1/(D-|\alpha_-|)$, depicted as dashed red and dotted orange edges in Figure 1. However, if we restrict to the space of non-decreasing ordered eigenvalues (specifically for $D = 3$ $\lambda_0 \leq \lambda_1 \leq \lambda_2$), just one condition is enough, as presented in the text. As already announced, the two sets of inequalities enclose two $(D-1)-$dimensional simplexes in the space of normalized eigenvalues, with each of the previous inequalities being one of the facets. The two simplexes are dual of each other (see Appendix C), one of them being inverted w.r.t. the other one. The vertices of the simplexes, $\mathbf{v}$, correspond to all permutations of the vectors of eigenvalues

$$\mathbf{v}_+(\alpha_+) = \left(\frac{1}{D+\alpha_+}, \cdots, \frac{1}{D+\alpha_+}, 1-\frac{D-1}{D+\alpha_+}\right), \quad (18)$$

$$\mathbf{v}_-(\alpha_-) = \left(1-\frac{D-1}{D-|\alpha_-|}, \frac{1}{D-|\alpha_-|}, \cdots, \frac{1}{D-|\alpha_-|}\right), \quad (19)$$

considering them ordered $\lambda_0 \leq \cdots \leq \lambda_{D-1}$ in the example given. Notice that in the limit $\alpha_+ \to \infty$ we recover the whole simplex of normalized states. The volume of the set of states detected by the map condition decays fast with the dimension of the system, according to $V_\alpha/V_{SN} = [\alpha_\pm/(D+\alpha_\pm)]^{(D-1)} \sim \mathcal{O}\left(D^{-D}\right)$. This decay is not surprising, since it is known that the volume of separable states tends exponentially to 0 as the dimension of the Hilbert space increases [ZHSL98].

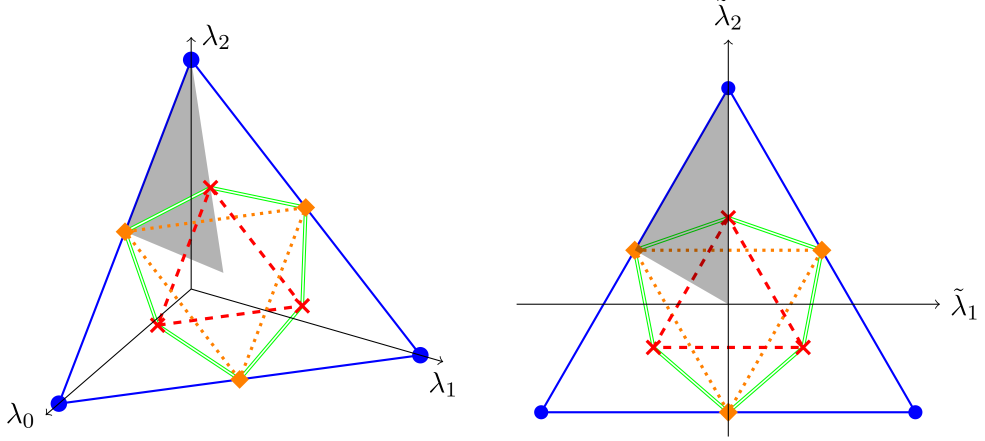

Figure 1: Schematic representation of the set of states in the D and $(D-1)-$dimensional representation for $D=3$ (left and right panels respectively). The normalized set of states is represented in blue. There are also plotted the simplexes of detected states with the condition $\Lambda^{-1}_{\alpha=2} \geq 0$ (red dashed line) and the set with $\Lambda^{-1}_{\alpha=-1} \geq 0$ (orange dotted line). The double green line displays the convex hull of both simplexes. Finally, the shaded zone fulfills the ordering $\lambda_0 \leq \lambda_1 \leq \lambda_2$.

From Figure (1) we conclude that some states detected by $\Lambda^{-1}_{\alpha=2} \geq 0$ are not detected by $\Lambda^{-1}_{\alpha=-1} \geq 0$ and viceversa. However, crucially, one can consider the convex hull of the two simplexes (delimited by the green solid line) to build stronger separability criteria than both conditions alone. The new criterion is represented as a *polytope* in the space of eigenvalues, i.e. a convex set with finite number of extreme points. We investigate this approach in Section 4.

4 Improved sufficient criteria for separability under convexity arguments

We have seen the set of states detected by $\Lambda^{-1}_\alpha \geq 0$ for different values of α are not necessarily equivalent. As the set $\mathcal{A}(N|M)$ is convex and compact, merging the different criteria with extreme values of $\alpha_\pm$ in a single convex hull can yield to better separability criteria.

4.1 Convex hull of the simplexes obtained with positive and negative α

Let us consider the convex hull of $\Lambda^{-1}_{\alpha=2}(\sigma) \geq 0$ and $\Lambda^{-1}_{\alpha=-1}(\sigma) \geq 0$. We have derived a linear condition on the eigenvalues from the polytope, resulting from the convex hull of both simplexes. The derivation is long, and we move it to Appendix C, while here we state directly the result. Considering the eigenvalues in non-decreasing order as $\lambda_0 \leq \cdots \leq \lambda_{D-1}$, it is possible to write the inequality as the following Theorem.

Result 3. *Let ρ be a bipartite state acting in a global Hilbert space of dimension D and $\{\lambda_i\}_{i=0}^{D-1}$ its corresponding eigenvalues in non-decreasing order $\{\lambda_i \leq \lambda_{i+1}\}_{i=0,1,..,D-1}$. If*

$$\left(D + 2 - 3 \cdot \left\lfloor \frac{D+1}{3} \right\rfloor\right) \cdot \lambda_{\lfloor \frac{D+1}{3} \rfloor} + 3 \cdot \sum_{j=0}^{\lfloor \frac{D+1}{3} \rfloor - 1} \lambda_j \geq 1, \tag{20}$$

then, ρ is separable.

Proof. We detail the proof in Appendix C. □

This expression requires our state to be sufficiently mixed, i.e., that the combination of the smaller eigenvalues is big enough. Note that, since $\lambda_{i+1} \geq \lambda_i$, the expression can be relaxed, at the expense of detecting fewer states, depending on how many eigenvalues are actually known. Explicitly, it is possible to substitute λ_{i+1} by λ_i as follows, until recovering the weaker condition Eq. (13):

$$\begin{aligned} c_{i+1} \cdot \lambda_{i+1} + \cdots + c_0 \cdot \lambda_0 \geq 1, \\ (c_{i+1} + c_i) \cdot \lambda_i + \cdots + c_0 \cdot \lambda_0 \geq 1, \\ \vdots \\ (D+2) \cdot \lambda_0 \geq 1, \end{aligned} \tag{21}$$

where $\sum_j c_j = D + 2$, $c_j \geq 0$ are the coefficients of Eq. (20). Considering the two smallest eigenvalues, one can state that $(D-1)\lambda_1 + 3\lambda_0 \geq 1$, which already includes the $2D$ vertices of both simplexes despite not being the best criterion. The whole polytope is enclosed by all the possible permutations on the eigenvalues of the inequality (20), but just as with the case of the simplexes $\Lambda^{-1}_{\alpha\pm}(\sigma) \geq 0$, one is enough for ordered eigenvalues (see Appendix C).

Both simplexes are, indeed, extreme. $\alpha = -1$ is the minimal value α can take, otherwise the states presented in Eq. (18) will have one negative eigenvalue. In fact, $\mathbf{v}_-(-1)$ is located on the boundary of quantum states, as seen in Figure 1. On the other hand, considering $\mathbf{v}_{\alpha=2+0.01}$ a diagonal matrix given by Eq. (18), one can find a global unitary U such that $U(\mathbf{v}_{\alpha=2+0.01})U^\dagger$ is NPT. It is possible to see it since there will be an EW W for which $\mathrm{Tr}\left(U\mathbf{v}_{\alpha=2+\epsilon}U^\dagger W\right) < 0$, $\forall\epsilon \geq 0$ (see Section 3.2).

Thus, these two simplexes are indeed the biggest possible ones that one can fit in the AS set of states. Nevertheless, the presented convex hull (linear) is smaller than the known bound of Theorem 4 (non-linear) when M or N are 2. Thus, it is almost certain that they do not achive the general characterization of the AS set of states.

Despite being AS already implies being APPT, one could wonder if we can find different bounds for α regarding APPT. Since PPT is also a convex set, it is enough to check with the vertex of the two extreme simplexes again. It is possible to see that considering $\alpha = -1$, the states with eigenvalues given by $\mathbf{v}_\pm$ from Eq. (13) also saturate the LMI for APPT given in [Hil07b]. Thus, the biggest possible simplexes in AS are the same as the biggest possible simplexes in APPT in general dimensions M, N. This insight supports the conjecture of AS being the same as APPT [AJR15].

4.2 Convex hull of the simplex with positive alpha and the separable ball

In this subsection we will combine the Gurvits condition (Theorem 3) with our linear condition (Result 1) to construct an even stronger sufficient condition for separability. To begin with, it can be seen that the simplex with $\alpha_- = -1$ is always contained inside the ball from Theorem 3, since for any D,

$$\mathrm{Tr}\left(\mathbf{v}_{\alpha=-1}^2\right) = \frac{D-1}{(D-1)^2} = \frac{1}{D-1} \leq \frac{1}{D-1}\,. \tag{22}$$

For $D \geq 5$ though, the simplex with $\alpha \geq 0$ always has some region outside the ball. Thus, the best CH that can be made is considering the ball and $\Lambda_{\alpha=2}^{-1} \geq 0$. In fact, we verify that the distance of the vertex $\mathbf{v}_+$ from the ball in the limit of infinite D, tends to α_+

$$\lim_{D\to\infty} \frac{||\mathbf{v}_{\alpha_+} - \frac{\mathbb{1}}{D}||_2}{\sqrt{\frac{1}{D-1} - \frac{1}{D}}} = \frac{d}{r'} = \alpha_+, \tag{23}$$

where we also introduced the definitions of d and r' as the numerator and denominator of the expression, respectively. Next, it is convenient to define an $n-$ball as all the points in an Euclidean space that fulfill $B^D = \{\lambda \in \mathbb{R}^D : ||\lambda|| \leq r\}$ and $n-$sphere as the points fulfilling $S^D = \{\lambda \in \mathbb{R}^{D+1} : ||\lambda|| = r\}$.

Here, our goal is to describe the CH of the simplex generated with $\alpha = 2$ and the ball to obtain the biggest possible general criterion for AS for $\mathbb{C}^M \otimes \mathbb{C}^N$ from conditions Eqs. (5), (13). In turn, the $\mathbf{v}_+$ are the vertex of some $(D-1)-$dimensional hypercones whose basis are all the points of the $(D-2)-$dimensional ball whose boundary is the $(D-3)-$dimensional sphere of points on the boundary of the Gurvits ball tangent to $\mathbf{v}_+$ (see the sketch in Appendix F). In Figure 2 we provide a $2D$ sketch of the geometry of such CH and how different regions are detected.

The volume of this CH can be computed, thus, as the sum of the volume of the $(D-1)-$dimensional ball plus the volume of the D different $(D-1)-$dimensional hypercones minus the volume of the D caps of the sphere that are contained both in the sphere and inside the hypercones. The resulting formula is given by the expression

$$\begin{aligned} V_{CH} = {} & \frac{\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2}+1})\cdot r'^{(D-1)} + D\cdot\frac{1}{D-1}\cdot a\cdot\frac{\pi^{(D-2)/2}}{\Gamma(\frac{D-2}{2}+1)}\cdot a''^{(D-2)} - \\ & - D\cdot\int_{r'^2/d}^{r'} dx\cdot\frac{\pi^{(D-2)/2}}{\Gamma(\frac{D-2}{2}+1)}\cdot(r'^2 - x^2)^{(D-2)/2}, \end{aligned} \tag{24}$$

where r' and d have been defined in Eq. (23), $a = d - r'^2/d$ and $a'' = \sqrt{r'^2 - r'^4/d}$. See Appendix F for more details.

The evaluation of Eq. (24) signals that for high dimensionality D, only a marginal improvement in volume of the CH with respect to the volume of the ball can be seen, as the region covered from the CH is localized around the vertices of the map condition, the number of which grows linearly with the dimension D. Such fact results in the volume of the spikes in the whole $D-$dimensional space approaching 0 faster than the ball does. However, for finite systems, we identify a remarkable improvement provided by the CH as quantified in Figure 2 (left panel). In particular, for the specific case of $D = 9$ (i.e., two qutrits), the linear convex hull of the two simplexes provides a a clear enhancement over the linear map or the Gurvits condition on its own.

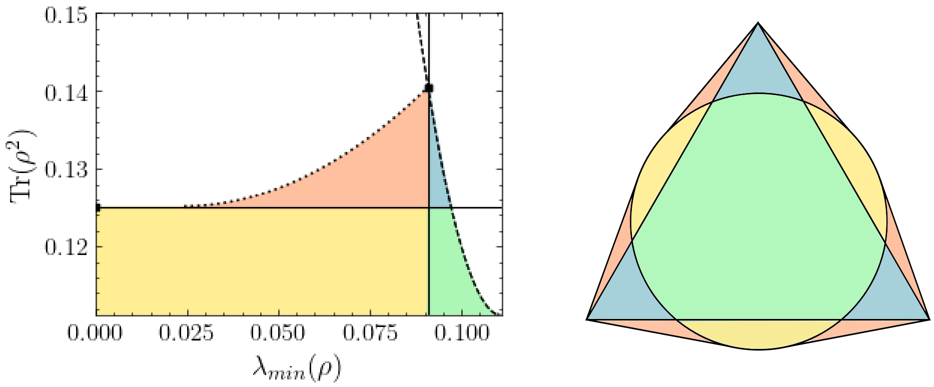

Figure 2: **Left panel:** Plot of the purity $\mathrm{Tr}(\rho^2)$ as a function of the minimal eigenvalue $\lambda_{\min}(\rho)$ for a 2−qutrit system. It is possible to differentiate the states detected as separable from spectrum by the ball (yellow region under the horizontal line), by the inverse of the reduction map (blue area, on the right of the vertical line and above the horizontal one), by both criteria (green region) and with the CH of the simplex and the ball (red area under the dotted curve). The dashed line represents the maximum value of $\mathrm{Tr}(\rho^2)$ given a λ_{min}. Finally, the dotted line represents the presented CH. **Right panel:** Schematic $2D$ representation of the CH of a ball and a simplex in barycentric coordinates (c.f. Figure 1), with the areas shaded as in the right panel.

To recapitulate, we verified through the convexity argument that it is possible to give a better characterization of the set of AS states, and, thus, to provide a much powerful criterion for their detection than the ones that already existed. In the next subsection, we propose a simple numerical method to efficiently test if a state is indeed detectable by our criterion or not.

4.3 Convex program

Since the criterion on the eigenvalues derived from the CH of the separable w.r.t. any bi-partition ball [GB02] and the set of states in the space of eigenvalues that fulfill $\Lambda_2^{-1} \geq 0$ have a long and tedious condition (see Appendix G), we try to derive a numerical method to check if the given state is in fact separable w.r.t. any bi-partition based on convex optimization techniques [SC23, BV04].

Here, we combine the extreme map condition for $\alpha = 2$ (Theorem 8) and the Gurvits criteria based on norms (Theorem 3) to a single *disciplined convex program* (DCP) which will easily test AS in the CH of both criteria. DCPs constitute a class of well-behaved convex optimization problems which can be efficiently solved with available routines [ApS24]. To this end, we decompose the state under scrutiny σ (or just a diagonal density matrix with its eigenvalues) into a convex combination of two (unnormalized) states $\{\sigma_1, \sigma_2\}$ as per $\sigma = \sigma_1 + \sigma_2$ and ask that σ_1 is detected by the map criterion while σ_2 is detected with the norm-based criterion. If one wishes to add more conditions on which to consider the CH, it is just needed to include more addends σ_i in the convex sum. In DCP language, it leads to the following result

Result 4. *Let σ be a bipartite state acting in a global Hilbert space of dimension D. If the* *following DCP is feasible*

$$\begin{array}{lrl} \min_{\sigma_1 \geq 0,\, \sigma_2 \geq 0} & 0 & \\ s.t. & \sigma & = \sigma_1 + \sigma_2 \\ & \sigma_1 - \mathrm{Tr}(\sigma_1)\frac{\mathbb{1}}{D+2} & \geq 0 \\ & ||\sigma_2 - \mathrm{Tr}(\sigma_2)\frac{\mathbb{1}}{D}||_2 & \leq \frac{\mathrm{Tr}(\sigma_2)}{\sqrt{D(D-1)}}, \end{array} \tag{25}$$

then, σ is AS.

Proof. If the problem is feasible, it implies that the algorithm succeeded in finding $\{\sigma_1, \sigma_2\}$ fulfilling the constraints. Hence, both states $\{\sigma_1, \sigma_2\}$ are certified AS from respective criteria. Finally, by convexity, their sum, i.e. σ, is also AS (see Appendix A). □

The DCP formulation allows us to estimate numerically the volume of states detected by the CH. To do so, we consider an ordinary Monte Carlo method. This approach generates uniformly normalized sets of positive eigenvalues in Eq. (17) and, for each of them, checks if the DCP problem in Eq. (25) has a solution or not. The estimated value of the volume is obtained by dividing the number of points that had a solution over the number of vectors of eigenvalues generated. For modest system sizes, $D = 10$, it provides volumes with a discrepancy w.r.t. Eq. (24) within $\leq 1\%$.

Moreover, we notice that the program defined in Eq. (25) admit also other types of constraints, in particular those that are linear in the state, e.g. $\{\mathrm{Tr}(A_i\sigma) = \langle A_i \rangle\}$, since they do not compromise the convexity of the problem. These constraints could represent the compatibility conditions with experimentally-inferred expectation values against given observables $\{A_i\}$ on an unknown state σ [MRLF22, MRSK+23]. Then, if the DCP is feasible after including these new constraints and leaving σ as a variable, it implies that there exists an AS state compatible with our statistics. This approach could be useful when inferring all eigenvalues of the state is too experimentally costly and one has only access to a restricted set of observables. Finally, it is worth mentioning that the method can be extended to compatibility with states of a given SN m by considering the corresponding criterion (Theorem 8), and minimizing over $\alpha = m + 1$ instead.

5 Multipartite absolute separability

In this section, we are going to discuss the possible applications of the separability criteria based on the inverse of the reduction map on multipartite systems. First, let us consider a system of $N-$qudits and a possible generalization of the Result 1 for multipartite systems. Whenever a given state $\sigma_N \in \mathcal{D}((\mathbb{C}^d)^{\otimes N})$ fulfills the inverse map condition for separability (we restrict now to $\alpha_+ = 2$),

$$\Lambda_2^{-1} = \frac{1}{2} \cdot \left(\sigma - \frac{\mathrm{Tr}(\sigma) \cdot \mathbb{1}}{d^N + 2} \right) \geq 0 \quad \text{and thus} \quad \sigma \geq \frac{\mathbb{1}}{d^N + 2}. \tag{26}$$

Furthermore, any reduced system $\sigma_{N-k} = \mathrm{Tr}_{\mathbf{k}}[\sigma_N] \in \mathcal{D}((\mathbb{C}^d)^{\otimes (N-k)})$ will also be detected as separable w.r.t. any bi-partition by the map in $(\mathbb{C}^d)^{\otimes N-k}$, having

$$\sigma_{N-k} \;\geq\; \frac{d^k \cdot \mathbb{1}}{d^N + 2} \;\geq\; \frac{\mathbb{1}}{d^{N-k} + 2}. \tag{27}$$

However, it is important to note that being separable w.r.t. any bi-partition and being separable w.r.t. any bi-partition in the successive partial traces, is not a sufficient condition for entanglement [DC00]. Even if this result is counter-intuitive, it shows that characterizing full separability is much harder than characterizing bi-separability.

An example of such states is given usually by the *Unextendible Product Basis* (UPB) formalism. For a multipartite quantum system, it is an incomplete orthogonal product basis whose complementary subspace contains no product state [BDM+99]. UPB states that are separable w.r.t. any bi-partition, yet inseparable are non-full rank. However, the states detected with the map need to be full rank to fulfill Eq. (13). Thus, the entanglement that the detected states might have is very subtle, since they are both AS and APPT w.r.t. any bi-partition, and full rank. Most of the common entanglement criteria such as negativity [VW02], the range criterion [Hor97] or decomposable EW [CK09] will not detect them. Finally, applying the map conditions to the reduced states as per Eq. (27) may be useful to evaluate entanglement robustness under particle loss [NMB18].

Neither induction procedures, as in Ref. [GB03] nor by direct inspection as in Ref. [VT99] nor numerical approaches have been useful. The better criteria, namely Theorem 5 for general $N-$qudit systems and Theorem 6 for systems of $N-$qubits, however, can yield to some values of α that represent the biggest simplex that is contained inside the respective balls. For the first case [GB03], it is possible to obtain (see Appendix H)

$$\alpha_{\pm} = \frac{4 \pm 2 \cdot \sqrt{(d^N - 1) \cdot (2^N \cdot d^N - 4)}}{2^N \cdot d^N - 2^N - 4}. \tag{28}$$

Finally, for the specific case of $N-$qubits, it is possible to obtain the expression

$$\alpha_{\pm} = \frac{3 \cdot (\pm 2 \cdot \sqrt{102} \cdot \sqrt{2^{3N} \cdot 3^N - 2^{2N} \cdot 3^N} - 9 \cdot 2^{N+2})}{2 \cdot (17 \cdot 3^{-N} - 17 \cdot 6^N + 54)}. \tag{29}$$

Even though these results do not improve the existing criterion since the simplexes lay inside the already known balls, they yield full separability of a given state using just its smallest or biggest eigenvalue, instead of computing the whole purity, as seen in Eq. (13).

Finally, it is convenient to introduce the concept of *robustness* introduced in Ref. [VT99], which tackles the separability problem from a similar perspective. Regarding the MMS in a space of dimension D, it is defined as

$$\tilde{s} := \min_s \frac{1}{1+s}(\rho + s \cdot \frac{\mathbb{1}}{D}) \qquad \text{such that} \qquad \frac{1}{1+\tilde{s}}(\rho + \tilde{s} \cdot \frac{\mathbb{1}}{D}) \geq 0 \tag{30}$$

is separable. It is possible to see that, in fact, the parameter α can be cast as a function of the robustness $\alpha = D/\tilde{s}$. For the multipartite case, the optimal value of $\tilde{s} = (1 + d^N/2)^{N-1} - 1$ is translated into a much more restrictive value of α than the ones computed in Eqs. (28), (29). Interestingly, from the previous work it is possible to provide a much better value for $\tilde{s}$ than the one given in the original paper.

In the next subsection, we will focus on multipartite states that are totally symmetric under party exchange. We will formalize new relevant criteria for APPT within the symmetric subspace and compare with existing AS conditions.

5.1 Symmetric states

This subsection is devoted to systems of indistinguishable particles, in specific to multi qudit systems. The symmetric subspace of a system of N qudits $\mathcal{S}(\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d)$ is invariant under permutation of any of the composite systems [Har13]. These states are also sometimes referred to as the bosonic states.

Definition 5.1. *A state $\rho \in \mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N)$ is a symmetric state if, for every $\pi \in \mathcal{G}_N$, with $\mathcal{G}_N$ the permutation group of N elements and P_π the unitary operator that performs a given permutation, the following relation holds*

$$\rho = P_\pi \rho P_\pi^\dagger = P_\pi \rho = \rho P_\pi^\dagger. \tag{31}$$

Note that the symmetric subspace is a subset of the permutaionally invariant subspace, where only the first inequality of Eq. (31) is required. The dimension of the symmetric subspace for $N-$qudit systems is $\binom{d+N-1}{N} = dim(\mathcal{S}(\mathbb{C}^d)^{\otimes N}))$ and the basis elements considering $\mathbf{k}$ a partition of N elements, (Dicke States for N qudits [Dic54]) read

$$\left\{ \left| D_{\mathbf{k}}^N \right\rangle = \left(\frac{N!}{k_0! \cdot k_1! \cdots k_{d-1}!} \right)^{-1/2} \cdot \sum_{\pi \in \mathcal{G}_N} \pi \left(|0\rangle^{\otimes k_0} \otimes |1\rangle^{\otimes k_1} \otimes \cdots \otimes |d-1\rangle^{\otimes k_{d-1}} \right) \right\}. \tag{32}$$

AS and APPT have been addressed for $\mathbb{C}^M \otimes \mathbb{C}^N$ systems. However, most of the presented criteria do not apply to the symmetric subspace, which has a lower dimension than the global space. Thus, it is convenient to study further the particular cases of *symmetric absolutely separable* states (SAS) and *symmetric absolutely PPT* states (SAPPT). Notice that not all SAS states are AS. Indeed, SAS states remain separable under any *symmetry-preserving* unitary map. However, the action of generic unitary matrices (not necessarily symmetric) could lead to entanglement.

As done previously, the first step is to study the set S' such that $\Lambda_\alpha(S) \in S'$, being S the set of all the symmetric states of $N-$qudits $\mathcal{D}((\mathbb{C}^d)^{\otimes N})$. For the case of 2 qubits $(d, N = 2)$, the bound on α has already been derived.

Theorem 9. *[LMRTS22] Let $\Lambda_\alpha(\rho_S) = \mathrm{Tr}(\rho_S) \cdot \mathbb{1}_S + \alpha \rho_S$ be the family of maps acting on the states of 2 qubits in the symmetric subspace, and $-3/4 \leq \alpha \leq 1$. Then, $\rho_S \geq 0 \rightarrow \Lambda_\alpha(\rho_S) = \sigma_S$ is separable. Similarly, if $\sigma_S \geq 0$, and $\Lambda_\alpha^{-1}(\sigma_S) = \rho_S \geq 0$, then σ_S is separable.*

This result considers SAS, which is equivalent to SAPPT for this case. Again, by characterizing the planes that enclose the CH (see Appendix I) of the states detected by the map with the highest value of α_+ and the smallest value of negative α_- and considering the ordering $\lambda_0 \leq \lambda_1 \leq \lambda_2$, we derive the following result.

Result 5. *Let $\rho \in \mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ be a symmetric state of a $2-$qubit system and $\lambda_0 \leq \lambda_1 \leq \lambda_2$ its corresponding eigenvalues in non-decreasing order. If*

$$5\lambda_1 + 7\lambda_0 \geq 3, \tag{33}$$

then ρ is SAS.

The latest equation provides a much better criterion than the recently published work based on *EW* [SEDM24]. Curiously, the set of detected states has a very similar shape, although it is bigger for the case of this linear map. Moreover, all the vertex saturate the best known (necessary and sufficient) bound for this kind of states $\sqrt{\lambda_0} + \sqrt{\lambda_1} \geq 1$ [SEM23, CJMP22]. Thus, the given simplex is yet again the biggest possible for SAS states of two qubits.

Theorem 10. *[LMRTS22] Let $\Lambda_\alpha(\rho_S) = \mathrm{Tr}(\rho_S)\mathbb{1}_S + \alpha\rho_S$ be the family of maps acting on the states of $N \geq 3$ qubits in the symmetric space, and $-1/N = -\lambda_{min} \leq \alpha \leq 2\lambda_{min} = 2/N$, where $\lambda_{min} > 0$ is the minimal eigenvalue of $\mathbb{1}_S^{T_A}$. Then $\rho_S \geq 0 \rightarrow \Lambda_\alpha(\rho_S) = \sigma_S$ is PPT in any partition 1:3. Similarly, if $\sigma_S \geq 0$ and $\Lambda_\alpha^{-1}(\sigma_S) = \rho_S \geq 0$, then σ_S is PPT in any partition 1:3.*

For $N = 3$ qubits, it is the last case where PPT is necessary but also sufficient for separability in the symmetric subspace [ESBL02]. However, the following condition for being SAPPT is extendible to an arbitrary number of qubits $N > 3$ for any of the partitions $1 : (N-1)$ qubits. There is an existing criterion [BWGB17] for the purity of SAS symmetric states of $N \geq 3$ qubits, given by Eq. (34) in the framework of absolutely classical spin states, stated as the following theorem.

Theorem 11. *[BWGB17] Let $\rho \in \mathcal{S}((\mathbb{C}^2)^{\otimes N})$ be a symmetric state of an $N-$qubit system. Then, it is SAS if*

$$\mathrm{Tr}\left(\rho^2\right) \leq \frac{1}{N+1} \cdot \left(1 + \frac{1}{2 \cdot (2N+1) \cdot \binom{2N}{N} - (N+2)}\right). \tag{34}$$

For the case of $N = 3$ qubits, the 8 vertices of the SAS simplex given by $\Lambda_\alpha^{-1} \geq 0$ lay outside the ball given by Eq. (34). Thus, in this case, the equation given by the CH of $\Lambda_{\alpha=2/3}^{-1} \geq 0$ and $\Lambda_{\alpha=-1/3}^{-1} \geq 0$ becomes a stronger sufficient criterion to detect SAS states for systems of 3 qubits. The best approach, though, is the DCP combination of the three criteria (see Section 4.3). An upper bound of the radius of SAS sates for 3 qubits is also given in Ref. [SEM23], which lead to a bigger ball than both of the proposed simplexes, so the presented results agree with existing criteria.

Result 6. *Let $\rho \in \mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ be a symmetric state of a $3-$qubit system and $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$ its corresponding eigenvalues in non-decreasing order. ρ is SAS if*

$$5\lambda_1 + 9\lambda_0 \geq 3, \tag{35}$$

Even though for $N \geq 4$, there exist PPT entangled states [TAH+12], the conjecture of APPT being equivalent to AS has been stated for any kind of systems [AJR15]. Until now, no counterexample has been found, and all the results found in this work seem to back up this conjecture. Again, one can consider the convex hull of the states detected with the map with the biggest α_+ and the smallest α_-, (with a similar derivation than the one in Appendix C) to get the bigger space of states that are SAPPT in any partition $1 : (N-1)$ of the space of $N-$qubits following Theorem 10. However, we have found that, in the example of $N = 4$ qubits, it is possible to construct symmetric states that are SAPPT (APPT under any global unitary matrix that acts on the expression of the state in the symmetric subspace) w.r.t. the partition $1 : 3$ with $\alpha \leq 2/4$ and it is still an NPT state (thus, entangled), since for some global symmetric unitary matrices, the partial transpose on the partition $2 : 2$ has negative eigenvalues [ATSL12]. In other words, it is possible to find some symmetric states that are SAPPT w.r.t. any partition $1 : A.O.$ for $N-$qubit systems, but that are still entangled for some global U. Nevertheless, we do not consider these kinds of states as conjecture-breaking, since in the literature for symmetric systems of $N-$qubits, only states that are PPT w.r.t. any partition are actually considered PPT states. Otherwise, they are directly considered NPT (and thus, entangled) states.

Then, the conjecture is still open to symmetric states that might be SAPPT w.r.t. all partitions. For the specific case of *diagonal symmetric* states of $N-$qubits (states that are diagonal in the Dicke basis), it is known that being PPT w.r.t. the biggest possible bi-partition $\lfloor N/2 \rfloor : N - \lfloor N/2 \rfloor$ implies being PPT w.r.t. all partitions [QRS17] and it implies separability. For this specific case, the conjecture would also remain true.

It is interesting to extend Theorem 10 to SAPPT w.r.t. other partitions and to general dimension in order to expand the separability criteria (SAPPT in this case), so we now present the following new Theorem.

Result 7. *Let $\Lambda_\alpha(\rho_S) = \mathrm{Tr}(\rho_S)\mathbb{1}_S + \alpha\rho_S$ be the family of maps acting on the states of N qudits in the symmetric space, and $-\binom{N}{\lfloor N/2 \rfloor}^{-1} = -\lambda_{min} \leq \alpha \leq 2\lambda_{min} = 2 \cdot \binom{N}{\lfloor N/2 \rfloor}^{-1}$, where $\lambda_{min} > 0$ is the minimal eigenvalue of $\mathbb{1}_S^{T_A}$ and the partition A includes $\lfloor N/2 \rfloor$ qudits. Then $\rho_S \geq 0 \rightarrow \Lambda_\alpha(\rho_S) = \sigma_S$ is SAPPT w.r.t. any bi-partition.*

Similarly, if $\sigma_S \geq 0$ and $\Lambda_\alpha^{-1}(\sigma_S) = \rho_S \geq 0$, then σ_S is SAPPT w.r.t. any bi-partition.

Proof: The proof follows the same structure as the one for Theorem 10. As stated in Theorem 7, we need to prove that $\Lambda_\alpha(S) \in S'$, where S' is the set of PPT symmetric states, which are permutationally invariant. We define k as the number of qudits included in the partition A that is transposed, and assume that $\mathrm{Tr}(\rho) = 1$. Now, we use the linearity of the partial transpose to state that $(\Lambda_\alpha(\rho))^{T_A} = (\mathbb{1}_S)^{T_A} + \alpha\rho^{T_A} \geq 0$. Since the eigenvalues of a transposed PSD density matrix are constrained between $[-1/2, 1]$ [Ran13], the conditions become $\lambda_{min} - \alpha_+/2 \geq 0 \rightarrow \alpha_+ \leq 2 \cdot \lambda_{min}$ and $\lambda_{min} - |\alpha_-| \geq 0 \rightarrow \alpha_- \geq -\lambda_{min}$. Moreover, the span of any transpose matrix of the symmetric subspace of $N-$qudits is contained in $\mathcal{S}((\mathbb{C}^d)^{\otimes k}) \otimes \mathcal{S}((\mathbb{C}^d)^{\otimes N-k})$. From here, it is just needed to consider that the minimal eigenvalue of $\mathbb{1}_S^{T_k}$ is given by $\binom{N}{k}^{-1}$ independently of d and that the biggest partition, given by $k = \lfloor N/2 \rfloor$ includes the rest of them since it provides the smaller value of α. □

We have some numerical evidence that the bounds on α are actually tight, since trying to expand any of the simplexes leads to NPT states for all the d, N. Nevertheless, a formal proof has not been achieved. In the case of SAPPT w.r.t. any bi-partition restricted to qubits ($d = 2$), it is also possible to compute the linear inequality that encloses the CH of both simplexes as explained in the Appendix C and shown in the next result.

Result 8. *Let $\rho \in \mathcal{S}((\mathbb{C}^2)^{\otimes N})$ be a symmetric state of N qubits and $\{\lambda_i\}_{i=1}^{N+1}$ its eigenvalues in non-decreasing order $\{\lambda_i \leq \lambda_{i+1}\}_{i=0,1,..,N}$. If*

$$\left[\binom{N}{\lfloor N/2 \rfloor} \cdot (N + 1 - 3 \cdot \lfloor \frac{N+1}{3} \rfloor) + 2\right] \cdot \lambda_{\lfloor \frac{N+1}{3} \rfloor} + 3 \cdot \binom{N}{\lfloor N/2 \rfloor} \sum_{i=0}^{\lfloor \frac{N+1}{3} \rfloor - 1} \lambda_i \geq \binom{N}{\lfloor N/2 \rfloor}, \tag{36}$$

then ρ is SAPPT.

The previous Eq. (36) does not fully include the ball given by the SAS criterion in Eq. (34) to characterize SAPPT completely. Nevertheless, the DCP approach (see Section 4.3) remains as a valid tool to certify SAPPT w.r.t. any bi-partition on the system of $N-$qubits. Notice that both $\mathbf{v}_\pm$ derived for SAPPT lay outside the SAS ball. In the case that it is possible to prove that these states are actually separable, the characterization will be for the SAS set. On the other hand, if one can prove that these states are entangled, the conjecture SAS$\Longleftrightarrow$SAPPT will be refuted for the symmetric subspace. These questions, however, remain open after the master's thesis. Finally, it is also possible to derive this kind of conditions for the case of general $N-$qudit systems by finding the linear expression of the CH for the different cases. We leave these extensions for the future, as well.

6 Conclusions and outlook

The main objective of this thesis has been the study and understanding of absolute separability (AS) and absolute positive partial transpose (APPT) near the maximally mixed state through the uses of positive linear maps, specifically, the reduction map. In the present work, we succeed in providing stronger sufficient criterion for separability for any dimension of the systems, as opposed to other already existing criteria, which are only necessary or valid in low dimensional systems.

In particular, we employ the inverse of the reduction map to formalize a stronger characterization of the set of absolutely separable states for general dimensions in bipartite systems. To this end, we start by characterizing geometrically the set detected by our criteria and verify that it is extreme. Next, by convexity arguments, we combine different of such conditions to lead to tighter criteria for AS in bipartite systems. The new conditions allow us to certify states as absolutely separable, which cannot be detected by the previous conditions alone. In addition, spectral properties of absolutely separable states can be related to spectral properties of entanglement witnesses. In this work, we make use of this connection to place bounds on the minimal and maximal eigenvalues of any bipartite entanglement witness. In so doing, we generalize results that are only established for decomposable witnesses.

The aforementioned conditions can be cast as standard convex optimization programs, which efficiently test absolute separability by searching over convex combinations of states detected by the different criteria. These methods admit different levels of knowledge of the system. In particular, they are useful when only partial information of the state is available, e.g., in the form of few experimentally-accessible expectation values.

The last part of the thesis is devoted to extend the bipartite conditions to the multipartite setting. In this regard, we present some non-optimal conditions for full separability. Finally, we move to the totally symmetric sector of N qubits. Interestingly, we find examples of symmetric absolutely PPT states (SAPPT) that are not absolutely separable in the symmetric subspace (SAS), even though we do not consider that it properly breaks the $AS \iff APPT$ conjecture as they are NPT in other partitions. Nevertheless, we provide bounds for SAPPT (w.r.t. any partition) for arbitrary system sizes, which would improve notably known SAS criteria provided the validity of the conjecture. This result paves the way to the characterization of APPT in the symmetric sector, which remains unexamined to date.

This work has raised several important questions that require further investigation. It is clear that more tools are needed to detect the states that are AS and APPT w.r.t. any bi-partition, full rank, yet still entangled. Also, for the bipartite case, even though the inverse of the reduction map provides a set of linear conditions, the set of AS states is not linear, and thus the complete characterization is still missing for arbitrary dimensions. It remains an open question to explore if the techniques we have used here with the reduction map can be applied to other maps to obtain a stronger characterization of the problem of AS and its related problem of the equivalence between APPT and AS.

Bibliography

[AJR15] S. Arunachalam, N. Johnston, and V. Russo. Is absolute separability determined by the partial transpose? arXiv preprint, January 2015. arXiv:1405.5853 [quant-ph].

[ApS24] Mosek ApS. The mosek optimization toolbox for python, 2024. Software version 10.2.

[ATSL12] R. Augusiak, J. Tura, J. Samsonowicz, and M. Lewenstein. Entangled symmetric states of N qubits with all positive partial transpositions. Physical Review A, 86(4):042316, October 2012.

[BBC+93] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Physical Review Letters, 70(13):1895–1899, March 1993.

[BDM+99] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal. Unextendible Product Bases and Bound Entanglement. Physical Review Letters, 82(26):5385–5388, June 1999.

[BV04] S. P. Boyd and L. Vandenberghe. Convex optimization. Cambridge University Press, Cambridge, UK ; New York, 2004.

[BWGB17] F. Bohnet-Waldraff, O. Giraud, and D. Braun. Absolutely classical spin states. Physical Review A, 95(1):012318, January 2017.

[BZ06] I. Bengtsson and K. Zyczkowski. Geometry of quantum states: an introduction to quantum entanglement. Cambridge University Press, Cambridge, 2006. OCLC: 70190210.

[Car24] S. Caron. pypoman: Python module for polyhedral geometry, 2024. Software version 1.1.0.

[CJMP22] G. Champagne, N. Johnston, M. MacDonald, and Logan Pipes. Spectral Properties of Symmetric Quantum States and Symmetric Entanglement Witnesses. Linear Algebra and its Applications, 649:273–300, September 2022.

[CK09] D. Chruscinski and A. Kossakowski. Spectral conditions for positive maps. Communications in Mathematical Physics, 290(3):1051–1064, September 2009.

[DC00] W. Dür and J. I. Cirac. Classification of multiqubit mixed states: Separability and distillability properties. Physical Review A, 61(4):042314, March 2000.

[Dic54] R. H. Dicke. Coherence in Spontaneous Radiation Processes. Physical Review, 93(1):99–110, January 1954.

[EAO+02] A.K. Ekert, C.M. Alves, D.K.L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek. Direct Estimations of Linear and Nonlinear Functionals of a Quantum State. Physical Review Letters, 88(21):217901, May 2002.

[Eke91] A.K. Ekert. Quantum cryptography based on Bell's theorem. Physical Review Letters, 67(6):661–663, August 1991.

[EPR35] A. Einstein, B. Podolsky, and N. Rosen. Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? Physical Review, 47(10):777–780, May 1935.

[ESBL02] K. Eckert, J. Schliemann, D. Bruß, and M. Lewenstein. Quantum Correlations in Systems of Indistinguishable Particles. Annals of Physics, 299(1):88–127, July 2002.

[FGBL22] A. K. Fedorov, N. Gisin, S. M. Beloussov, and A. I. Lvovsky. Quantum computing at the quantum advantage threshold: a down-to-business review. arXiv preprint, 2022. arXiv:2203.17181 [quant-ph].

[FMJ17] S.N. Filippov, K.Y. Magadov, and M.A. Jivulescu. Absolutely separating quantum maps and channels. New Journal of Physics, 19(8):083010, August 2017.

[FR94] K. Fukuda and V. Rosta. Combinatorial face enumeration in convex polytopes. Computational Geometry, 4(4):191–198, August 1994.

[GB02] L. Gurvits and H. Barnum. Largest separable balls around the maximally mixed bipartite quantum state. Physical Review A, 66(6):062311, December 2002.

[GB03] L. Gurvits and H. Barnum. Separable balls around the maximally mixed multipartite quantum states. Physical Review A, 68(4):042312, October 2003.

[GCM14] N. Ganguly, J. Chatterjee, and A. S. Majumdar. Witness of mixed separable states useful for entanglement creation. Physical Review A, 89(5):052304, May 2014.

[Gha10] S. Gharibian. Strong NP-hardness of the quantum separability problem. Quantum Information and Computation, 10(3&4):343–360, March 2010.

[GT09] O. Gühne and G. Tóth. Entanglement detection. Physics Reports, 474(1-6):1–75, April 2009.

[Gur03] L. Gurvits. Classical deterministic complexity of Edmonds' problem and Quantum Entanglement, 2003. STOC 03, page 10–19, New York, NY, USA, 2003.

[Har13] A.W. Harrow. The Church of the Symmetric Subspace. arXiv preprint, August 2013. arXiv:1308.6595 [quant-ph].

[HHH96] M. Horodecki, P. Horodecki, and R. Horodecki. Separability of mixed states: necessary and sufficient conditions. Physics Letters A, 223(1-2):1–8, November 1996.

[HHH99] P. Horodecki, M. Horodecki, and R. Horodecki. Bound Entanglement Can Be Activated. Physical Review Letters, 82(5):1056–1059, February 1999.

[HHH01] M. Horodecki, P. Horodecki, and R. Horodecki. Separability of n-particle mixed states: necessary and sufficient conditions in terms of linear maps. Physics Letters A, 283(1-2):1–7, May 2001.

[HHHH09] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. Reviews of Modern Physics, 81(2):865–942, June 2009.

[Hil05] R. Hildebrand. Cones of ball-ball separable elements. arXiv preprint, 2005. arXiv:quant-ph/0503194.

[Hil07a] R. Hildebrand. Entangled states close to the maximally mixed state. Physical Review A, 75(6):062330, June 2007.

[Hil07b] R. Hildebrand. PPT from spectra. Physical Review A, 76(5):052325, November 2007.

[HLC23] C. M. Holland, Y. Lu, and L. W. Cheuk. On-demand entanglement of molecules in a reconfigurable optical tweezer array. Science, 382(6675):1143–1147, December 2023.

[HMS21] S. Halder, S. Mal, and A. Sen(De). Characterizing the boundary of the set of absolutely separable states and their generation via noisy environments. Physical Review A, 103(5):052431, May 2021.

[Hor97] P. Horodecki. Separability criterion and inseparable mixed states with positive partial transposition. Physics Letters A, 232(5):333–339, August 1997.

[JKP01] B. Julsgaard, A. Kozhekin, and Eugene S. Polzik. Experimental long-lived entanglement of two macroscopic objects. Nature, 413(6854):400–403, September 2001.

[Joh13] N. Johnston. Separability from Spectrum for Qubit-Qudit States. Physical Review A, 88(6):062330, December 2013.

[JP18] N. Johnston and E. Patterson. The Inverse Eigenvalue Problem for Entanglement Witnesses. Linear Algebra and its Applications, 550:1–27, August 2018.

[Kni13] E. Knill. IQOQI - Open Quantum Problems, September 2013.

[KZ01] M. Kus and K. Zyczkowski. Geometry of entangled states. Physical Review A, 63(3):032307, February 2001.

[LAC+16] M. Lewenstein, R. Augusiak, D. Chruściński, S. Rana, and J. Samsonowicz. Sufficient separability criteria and linear maps. Physical Review A, 93(4):042335, April 2016.

[LCT14] H.K. Lo, M. Curty, and K. Tamaki. Secure quantum key distribution. Nature Photonics, 8(8):595–604, August 2014.

[LKCH00] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki. Optimization of entanglement witnesses. Physical Review A, 62(5):052310, October 2000.

[LKS+05] D. Leibfried, E. Knill, S. Seidelin, J. Britton, R. B. Blakestad, J. Chiaverini, D. B. Hume, W. M. Itano, J. D. Jost, C. Langer, R. Ozeri, R. Reichle, and D. J. Wineland. Creation of a six-atom 'Schrödinger cat' state. Nature, 438(7068):639–642, December 2005.

[LMRTS22] M. Lewenstein, G. Müller-Rigat, J. Tura, and A. Sanpera. Linear maps as sufficient criteria for entanglement depth and compatibility in many-body systems. Open Systems & Information Dynamics, 29(03):2250011, September 2022.

[MRLF22] G. Müller-Rigat, M. Lewenstein, and I. Frérot. Probing quantum entanglement from magnetic-sublevels populations: beyond spin squeezing inequalities. Quantum, 6:887, December 2022.

[MRSK+23] G. Müller-Rigat, A.K. Srivastava, S. Kurdziałek, G. Rajchel-Mieldzioć, M. Lewenstein, and I. Frérot. Certifying the quantum Fisher information from a given set of mean values: a semidefinite programming approach. Quantum, 7:1152, October 2023.

[NMB18] A. Neven, J. Martin, and T. Bastin. Entanglement robustness against particle loss in multiqubit systems. Physical Review A, 98(6):062335, December 2018.

[Per96] A. Peres. Separability Criterion for Density Matrices. Physical Review Letters, 77(8):1413–1415, August 1996.

[QRS17] R. Quesada, S. Rana, and A. Sanpera. Entanglement and nonlocality in diagonal symmetric states of N qubits. Physical Review A, 95(4):042128, April 2017.

[Ran13] S. Rana. Negative eigenvalues of partial transposition of arbitrary bipartite states. Physical Review A, 87(5):054301, May 2013.

[SC23] P. Skrzypczyk and D. Cavalcanti. Semidefinite programming in quantum information science. IOP series in quantum technology. IOP Publishing, Bristol, UK, 2023.

[SEDM24] E. Serrano-Ensástiga, J. Denis, and J. Martin. Absolute-separability witnesses for symmetric multiqubit states. Physical Review A, 109(2):022430, February 2024.

[SEM23] E. Serrano-Ensástiga and J. Martin. Maximum entanglement of mixed symmetric states under unitary transformations. SciPost Physics, 15(3):120, September 2023.

[TAH+12] J. Tura, R. Augusiak, P. Hyllus, M. Kuś, J. Samsonowicz, and M. Lewenstein. Four-qubit entangled symmetric states with positive partial transpositions. Physical Review A, 85(6):060302, June 2012.

[Ter98] B.M. Terhal. A Family of Indecomposable Positive Linear Maps based on Entangled Quantum States. Linear Algebra Appl., 323:61–73, 1998.

[Ter02] B. M. Terhal. Detecting quantum entanglement. Theoretical Computer Science, 287(1):313–335, September 2002.

[TOK+14] T. Tanaka, Y. Ota, M. Kanazawa, G. Kimura, H. Nakazato, and F. Nori. Determining eigenvalues of a density matrix with minimal information in a single experimental setting. Physical Review A, 89(1):012117, January 2014.

[VADM01] F. Verstraete, K. Audenaert, and B. De Moor. Maximally entangled mixed states of two qubits. Physical Review A, 64(1):012316, June 2001.

[VT99] G. Vidal and R. Tarrach. Robustness of entanglement. Physical Review A, 59(1):141–155, January 1999.

[VW02] G. Vidal and R. F. Werner. A computable measure of entanglement. Physical Review A, 65(3):032314, February 2002.

[ZHSL98] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein. Volume of the set of separable states. Physical Review A, 58(2):883–892, August 1998.

A The set of AS states is convex

We briefly prove that the set of AS is indeed convex. Let $\rho_1 \in \mathcal{A}(M|N)$ and $\rho_2 \in \mathcal{A}(M|N)$ be AS w.r.t. the same partitions and $\sigma = p \cdot \rho_1 + (1-p) \cdot \rho_2$ a convex combination. Considering an arbitrary operator U, one can see that

$$U\sigma U^\dagger = p \cdot U\rho_1 U^\dagger + (1-p) \cdot U\rho_2 U^\dagger \tag{37}$$

Since ρ_1, ρ_2 are AS, $U\rho_i U^\dagger$ is still separable, which implies that $\sigma \in \mathcal{A}(M|N)$. Hence, AS is convex.

B Inverse of the reduction map

In this Appendix, we explicitly invert the reduction map. First, consider

$$\Lambda_\alpha^{-1}(\Lambda_\alpha(\rho)) = \Lambda_\alpha^{-1}(\mathrm{Tr}(\rho) \cdot \mathbb{1} + \alpha\rho) = \rho, \tag{38}$$

which, by linearity of the inverse, leads to

$$\mathrm{Tr}(\rho) \cdot \Lambda_\alpha^{-1}(\mathbb{1}) + \alpha\Lambda_\alpha^{-1}(\rho) = \rho. \tag{39}$$

Applying both the map and its inverse to $\mathbb{1}$, we obtain

$$\Lambda_\alpha^{-1}(\Lambda_\alpha(\mathbb{1})) = \Lambda_\alpha^{-1}(\mathrm{Tr}(\mathbb{1}) \cdot \mathbb{1} + \alpha\mathbb{1}) = \mathbb{1}. \tag{40}$$

From there, it is possible to isolate $\Lambda_\alpha^{-1}(\mathbb{1}) = \mathbb{1}/(D+\alpha)$. Next, by direct substitution one obtains the linear expression for the inverse

$$\Lambda_\alpha^{-1}(\rho) = \frac{1}{\alpha}\left(\rho - \frac{\mathrm{Tr}(\rho) \cdot \mathbb{1}}{D+\alpha}\right). \tag{41}$$

C Derivation of the inequality Eq.(20)

In this Appendix, we derive the condition Eq. (20) from the main text by evaluating the convex hull of the map conditions for $\alpha = 2, \alpha = -1$ (13). To this end, we use numerical packages that already exist to obtain the equations that enclose the CH and analyze them in order to get the final dependence on D. This procedure is also applied in order to obtain Eqs. (33), (36).

A $(D-1)$-simplex does not enclose a definite volume when embedded in D-dimensional space. For instance, a 2-simplex (a triangle) exists in $2D$ space and encloses an area, but when considered in $3D$ space, it becomes a flat, $2D$ surface that does not enclose any volume. Thus, to characterize the geometrical properties of the CH that actually encloses the set of detected AS, we need to work in $D-1$ dimensions. Following the discussion in the main text, we consider the space of eigenvalues $\vec{\lambda} := \{\lambda_i\}$ of D-dimensional normalized quantum states $\mathrm{Tr}(\rho) = (1, \cdots, 1) \cdot \vec{\lambda} = \sum_i \lambda_i = 1$. The unit trace constraint allows us to eliminate a single variable, $\sum_i \lambda_i := \tilde{\lambda}_0 = 1$. Next, the orthogonal complement to the vector $(1, \cdots, 1)$ is used to complete a basis of the $D-1$ subspace describing the barycentric coordinates (see Figure 1 of the main text).

The first that one can realize of is that, in the $(D-1)-$dimensional space of eigenvalues satisfying $\mathrm{Tr}(\rho) = 1$ of our state, the D vertices of states $\mathbf{v}_\pm$ that are detected as separable w.r.t. any bi-partition by $\Lambda_2^{-1} \geq 0$ form a regular simplex. The states detected by $\Lambda_{-1}^{-1} \geq 0$ form an inverted, smaller simplex. The CH of the two conditions is, then, the polytope generated by the union of the two sets of vertices. The resulting polytope is not regular, but it is symmetric under permutations of the eigenvalues, since the set of vertices is. To analyze it, we introduce the concept of duality. In computational geometry, the $H-$representation (half-space representation) of a polytope involves describing it as an intersection of half-spaces. These half-spaces can be represented by linear inequalities of the form $A\lambda + \mathbf{b} \geq 0$, being A a matrix. As in Eq.(13), all the obtained equations are different permutations of a single expression, so for the ordered eigenvalues, just one formula is actually needed. Another reason to consider is that, indeed, the polyhedron is defined by $D!$ hyperplanes and, in fact, there are $D!$ possible orderings of the eigenvalues [FR94].

We remind here that the vertex for $\Lambda_2^{-1} \geq 0$ are all the possible different permutations of the vector, $(1-\frac{D-1}{D+2}, \frac{1}{D+2}, \cdots, \frac{1}{D+2})$ and the ones for Λ_2^{-1} the same for $(0, \frac{1}{D-1}, \cdots, \frac{1}{D-1})$, following Eq. (18). We also remind that a dual polyhedron is a geometric concept associated to any given polyhedron. This relationship is defined such that the vertices of one polyhedron correspond to the faces of the dual polyhedron, and vice versa. Whereas there is one ordered (in the sense that the eigenvalues follow a non-decreasing order $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{D-1}$) vertex for α_+ and one for α_-, the dual polyhedron of the convex hull only has one ordered vertex.
For low dimensional systems, it is possible to compute the set of linear inequalities analytically by plotting the different vertices as in Figure 1. Given the difficulty of extending the calculations $\forall D$, a numerical approach has been used to compute the equations of the facets of the polyhedron. Moreover, in higher dimensions, the representation on how the facets are constructed (as in $D = 3, 4$) is not possible. To compute the equation of the $(D-2)-$dimensional hyperplanes in the $(D-1)-$dimensional space of $\text{Tr}(\rho) = 1$ using $D-1$ points, we will use the generalized cross product given the set of vertices.

Then, it is possible to calculate all the $\binom{2D}{D-1}$ combinations of $D-1$ vertex out of the $2D$ possibilities. Most of them will be far from external, and the majority will be contained inside the others. The condition to determine which were, indeed, extreme, was to find the maximum size of the dual polyhedron that allowed each of its vertex to still saturate an inequality (that is, to verify that the hyperplane contain at least one extreme point in addition to the $D-1$ vertices), thus obtaining the set that actually limited our polyhedron. It is important to notice that each size of the dual polyhedron saturated different sets of inequalities, all of which were permutations of a single one. Later on, an already existing package pypoman [Car24] was used to obtain the same set of linear inequalities, confirming the validity of the numerical method. Moreover, it was more efficient for bigger D than the manual implementation.

By doing so, we can describe our polyhedron as a set of linear inequalities $A\vec{\lambda}^T + b \geq 0$,where A, b depend on the dimension D. By going back to the $D-$dimensional space of eigenvalues, however, it is possible to obtain a set of linear matrix inequalities where b is a constant vector of value s and A is a matrix containing in each row all the different permutations of the vector $(p, \cdots, p, r, -q, \cdots, -q)$. Considering $m(p)$ the *multiplicity* of p in the given vector, the size of the matrix is given by $D \times \binom{D}{\max m(p), m(q)}$. Where p, r, q, s can easily be expressed as a function only of the dimension being $r = 1$, $q = \frac{D+2}{2\cdot(D-1)}$, $p = \frac{(D-2)\cdot D-2}{2\cdot(D-1)}$ and $s = \frac{1}{D-1}$. Considering $\sum_i \lambda_i = 1$, one can eliminate an extra parameter.

Finally, it is only needed to find the expression that corresponds to the ordered eigenvalues, i.e., the one that is saturated by the ordered $\mathbf{v}(\alpha_{\pm})$ and by the ordered vertex of the dual. The equation obtained for general D can be cast as

$$\left[D + 2 - 3 \cdot \lfloor \frac{D+1}{3} \rfloor\right] \cdot \lambda_{\lfloor \frac{D+1}{3} \rfloor} + 3 \cdot \sum_{j=0}^{\lfloor \frac{D+1}{3} \rfloor - 1} \lambda_j \geq 1. \tag{42}$$

It is important to note that this expression has not been derived in a general manner, it comes from an observation that the polyhedron obtained fulfills the same form of inequalities $\forall D \leq 22$, and it has also been checked for any $D = 2^N$ with N up to $N = 7$.

For the case of the symmetric space, the formula is deduced analogously, having different dependencies on d for the parameters p, q, r, s.

D Minimal eigenvalue for decomposable bipartite entanglement witnesses

This property is already commented in [JP18], but since it is not actually proven there, we add the short proof.

First, it is necessary to consider that the eigenvalues of the partial transpose of any given state of $\text{Tr}(\rho) = 1$ are contained in $[-1/2, 1]$ [Ran13]. From here, one can express the decomposable entanglement witness as $W = P + Q^{T_A}$, being Q, P PSD. The optimal EW obtainable is considering $P = 0$ [LKCH00], and restricting to the partial transpose of Q.

Thus, the minimal eigenvalue for W will fulfill $\lambda_{min}(W) \geq -\frac{\text{Tr}(W)}{2}$.

E Translation of the separability problem into entanglement witnesses

Although this derivation is not technical, we have set it in the appendix because it does not provide more insight than the final equations highlighted in the text 14.

We start by considering the effect of the inverse of the map over a specific state.

$$\Lambda_{\alpha}^{-1} = \frac{1}{\alpha}\left(\sigma - \frac{\mathbb{1}}{D+\alpha}\right) \geq 0 \tag{43}$$

Defining $\Delta \geq 0$ as the normalized inverse of the map, one can express a state σ which is detected by the map condition as

$$\sigma = \left(1 - \frac{D}{D+\alpha}\right)\Delta + \frac{\mathbb{1}}{D+\alpha}. \tag{44}$$

Then, being entangled translates into being detected by some entanglement witness W as

$$\text{Tr}(W\sigma) = \frac{\alpha}{D+\alpha}\Delta + \frac{\text{Tr}(W)}{D+\alpha} < 0. \tag{45}$$

Now, it is necessary to make a distinction for positive and negative values of α, because $D/(D+\alpha)$ might be smaller or bigger than one for different values of α.

For $\alpha > 0$, we consider that $\mathrm{Tr}(W\Delta) \geq \lambda_{min}(W)$, where $\lambda_{min}(W) < 0$ in order to detect some entangled states. Then,

$$0 > \frac{\mathrm{Tr}(W)}{D+\alpha} + \frac{\alpha}{D+\alpha} \cdot \mathrm{Tr}(W\Delta) \geq \frac{\alpha}{D+\alpha} \cdot \lambda_{min}(W) + \frac{\mathrm{Tr}(W)}{D+\alpha}. \tag{46}$$

From there, it is possible to obtain the inequality to have an entanglement witness which can detect as entangled some states that are detected by the inverse of our linear map.

$$\alpha > -\frac{\mathrm{Tr}(W)}{\lambda_{min}(W)} \tag{47}$$

For negative values of alpha, since $(1 - \frac{D}{D+\alpha})$ is negative, one can start by $\mathrm{Tr}(W\Delta) \leq \lambda_{max}(W)$ to get

$$0 > \frac{\mathrm{Tr}(W)}{D+\alpha} + \frac{\alpha}{D+\alpha} \cdot \mathrm{Tr}(W\Delta) \geq \frac{\alpha}{D+\alpha} \cdot \lambda_{max}(W) + \frac{\mathrm{Tr}(W)}{D+\alpha}. \tag{48}$$

To finally obtain the desired equation

$$\alpha < -\frac{\mathrm{Tr}(W)}{\lambda_{max}} \tag{49}$$

F Notes on the volume of the convex hull of a simplex and a ball.

Here, we introduce the notion of the formula for the Volume of the CH. In the Figure 3, one can see the basic scheme of the ball and the vertex. We can see, then, that the $n-$ball of Gurvits has a radius $r' = \sqrt{\frac{1}{D-1} - \frac{1}{D}}$. From trigonometric expressions, we can see that the angle of the cone is given by $\gamma = \arcsin \frac{r'}{d}$ and that $a'' = \sqrt{r'^2 - (r' - r'')^2} = \sqrt{r'^2 - \frac{r'^4}{d^2}}$. Finally, the last relevant magnitudes are $r'' = r' - \frac{r'^2}{d}$ and $a = d - r' + r''$.

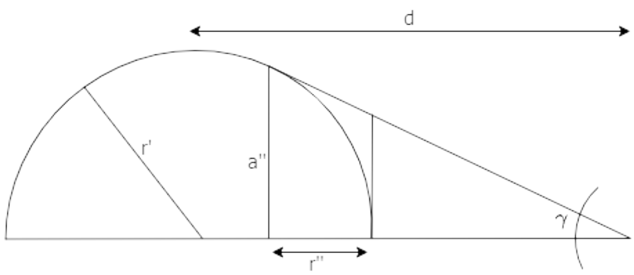

Figure 3: Schematic representation of the trigonometry of the hypercone formed by the CH of GurvitsBall and the simplex $\Lambda^{-1}_{\alpha=2} \geq 0$.

The equation of the volume of an $n-$ball depends only on the radius R as

$$V_{ball}(n, R) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \cdot R^n.$$

The volume of a hypercone in $n-$dimensional space is given by the formula

$$V_{cone} = \frac{1}{n} \cdot V_{basis} \cdot a,$$

where $V_{basis} = V_{ball}(n-1, a'')$ is the volume of the $(n-1)-$dimensional ball of radius a'' that forms its basis, while a is the height of the considered cone.

Finally, the volume of the cap, which is duplicated, can be computed by integrating from $r' - r''$ to r' the volume of successive $(n-1)-$balls of radius $r' - x$, as seen in

$$\int_{r'-r''}^{r'} dx \frac{\pi^{(n-1)/2} \cdot (r'^2 - x^2)^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)}$$

G Notes on the separability criterion of the convex hull of a simplex and a ball

This derivation takes into account the geometrical definitions introduced in Appendix F. Here, it is necessary to divide the detection space according to the value of the smallest eigenvalue. From Figure 3, we can see that the sufficient criterion must be broken in two, the ball and the cone, depending on the region of the space of eigenvalues we are on.

It is possible to see that, in the $(D-1)-$dimensional space in barycentric coordinates, the hypercone that has as vertex the ordered $\mathbf{v}_+$ in decreasing order is given by the normal equation of a hypercone with the vertex in an axis and the basis containing the origin of coordinates

$$x_1^2 + x_2^2 + \cdots + x_{D-2}^2 = \arcsin\frac{r'}{d} \cdot (x_{D-1} - d)^2.$$

It is also possible to see that this equation is the one to apply as long as $x_{D-1} \geq \frac{r'}{d^2}$. Otherwise, it is necessary to check for $\mathrm{Tr}(\sigma^2) \geq \frac{1}{D-1}$.

The expression as a function of the dimension of the changes of variable is not particularly simple, but here we present an example of the change of variable for a system of $D = 4$, which can be written as

$$\begin{aligned}
x_0 &= 0.5 \cdot \lambda_0 + 0.5 \cdot \lambda_1 + 0.5 \cdot \lambda_2 + 0.5 \cdot \lambda_3, \\
x_1 &= -\frac{1}{\sqrt{3}}\lambda_1 + \frac{3+\sqrt{3}}{6}\lambda_2 - \frac{3-\sqrt{3}}{6}\lambda_3, \\
x_2 &= -\frac{1}{\sqrt{3}}\lambda_1 - \frac{3-\sqrt{3}}{6}\lambda_2 + \frac{3+\sqrt{3}}{6}\lambda_3, \\
x_3 &= \frac{\sqrt{3}}{2}\lambda_0 - \frac{1}{2\sqrt{3}}\lambda_1 - \frac{1}{2\sqrt{3}}\lambda_2 - \frac{1}{2\sqrt{3}}\lambda_3.
\end{aligned}$$

H Maximum simplex contained in a ball

To compute the maximum value of α that a simplex given by $\Lambda_\alpha^{-1} \geq 0$ can give, it is necessary to force the vertex from Eq. (18) to have the desired maximum trace given by the ball. Since the expression of the $\mathrm{Tr}(\rho^2)$ is the same for any $\alpha_\pm$ and only depends on the eigenvalues, it is possible to obtain both values with a single equation. For the case of Theorem 5 and considering $D = d^N$, the desired value of α can be obtained by solving the equation

$$\frac{d^N - 1}{(d^N + \alpha)^2} + (1 - \frac{d^N - 1}{d^N + \alpha})^2 = \frac{1}{d^N}\left(1 + \frac{2^{-N+2}}{d^N - 2^{-(N-2)}}\right).$$

For the case of Eq. (9), and considering $D = 2^N$ for the $N-$qubit system, the desired bounds on α can be computed as the solutions of

$$\frac{2^N - 1}{(2^N + \alpha)^2} + (1 - \frac{2^N - 1}{2^N + \alpha})^2 = \frac{1}{2^N} \cdot \left(1 + \frac{54}{17} \cdot 3^{-N}\right).$$

I Derivation of the inequality Eq. (33)

This derivation is an easier and explicit example of the one presented in Appendix C for the case of a 2 qubit symmetric system of $D = 3$. For the bosonic subspaces, the prior derivations do not work, in general, since it has reduced dimension and, in the computational basis, some eigenvalues are just 0. Thus, it is convenient to find new conditions for the specific symmetric subspace, as we present next.

In this case, it is stated that $-3/4 \leq \alpha \leq 1$, so the vertex will be the permutations of the points $(0.5, 0.25, 0.25)$ for α_+ and $(4/9, 4/9, 1/9)$ for α_-. In this case, the hyperplanes are cast from $D - 1 = 2$ of the 6 points, one from each simplex.

Focusing first on the $(4/9, 4/9, 1/9)$ point, in $2D$ space, we compute the equation of the line segment that unites it with all the $\mathbf{v}_{\alpha=2}$ and reverse the change of variables, obtaining

$$\begin{aligned} 5\lambda_1 + 7\lambda_2 &= 3, \\ 5\lambda_0 + 7\lambda_1 &= 3, \\ \lambda_0 - \lambda_1 &= 0. \end{aligned}$$

Clearly, the third equation is discarded since the MMS fulfills it, the hyperplane would pass through the center of the subspace. Doing the same procedure with the rest of the vertices yields similar results with permutations of the eigenvalues. Thus, the 6 final equations are indeed the same, and the one that encloses the ordered space as an inequality that must contain MMS can be cast, considering $\lambda_0 \leq \lambda_1 \leq \lambda_2$ as

$$5\lambda_1 + 7\lambda_0 \geq 3.$$