New Macroscopic Forces mediated by Axion Interactions

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Abstract: Very light and weakly-coupled new-physics particles, like axion, may manifest themselves as long-range forces between ordinary matter, such as nucleons and electrons. Such new forces can be searched for by means of a variety of experiments.

In this thesis, we revisit the computation of non-relativistic potentials generated by the axion, a new light pseudo-scalar particle interacting with the Standard Model. We will use a general lagrangian to parametrize the new interactions between axions and ordinary matter. Moreover, we will improve previous results going beyond the leading order Born approximation. We amend previous results and also identify new contact terms that appear in the non-relativistic expansion.

I. INTRODUCTION

After having built the celebrated unifying theory known as Standard Model (SM), physicists are facing several problems which cannot be addressed within the scope of this theoretical construction. For instance, there are some observations such as dark matter (DM), neutrino masses and the cosmological matter-antimatter asymmetry that remain unaccounted. Therefore, alternative theories have been suggested with the intention of solving some of these issues. The example that is going to be discussed in this project is the axion.

The axion, a, is a very light pseudo-scalar particle that couples to the CP violating topological gluon density $(a/f_a + \theta)G\tilde{G}$, where f_a is the scale, θ is the CP violating term, $G = G_{\mu\nu}$ is the gluon field strength tensor and $\tilde{G}_{\mu\nu}$ its dual. Such extension is able to solve the strong CP problem and, consequently, contribute to the DM [1].

The exchange of new light particles by ordinary matter can generate an additional and potentially observable "fifth" force between them, inversely proportional to the mass of the new mediator. Searches for long-range forces have a long history. In this work, we want to revise the theoretical calculation of the long-range force mediated by the axion.

In sec. II we will review the calculations of the tree-level mediated potentials within a general Lagrangian, encoding scalar and pseudo-scalar interactions. The first results for the elastic scattering at leading order in the non-relativistic expansion will be also presented stressing discrepancies with former works. In sec. III experimental constraints for long-range forces will be discussed. In sec. IV the axion-mediated potential will be extended beyond the leading order and the elastic limit. Finally, in sec. V we will summarize the thesis with the key points. An appendix is included at the end with formulae that will be used to obtain the results throughout the work.

II. LONG-RANGE AXION-MEDIATED FORCES

Let us consider the most general Lagrangian parametrizing the interactions of axions (i.e. pseudo-scalar boson) with fermions

$$\begin{aligned} L = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{1}{2}m_a^2 a^2 + \sum_i \bar{\psi}_i (i\partial\!\!\!/ - m_i)\psi_i \\ - \sum_i a\left(g_i^S \bar{\psi}_i \psi_i + g_i^P \bar{\psi}_i i\gamma_5 \psi_i\right) \end{aligned} \quad (1)$$

where i runs on fermions such as p, n and e, the macroscopic matter at large distances or at low energies.

Since the term $\bar{\psi}_i(i\partial\!\!\!/ - m_i)\psi_i$ corresponds to a Dirac's lagrangian, one can consider the fields to be general solutions of the Dirac equation, which read

$$\hat{\psi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_\lambda \left[e^{-ipx}u_\lambda(\vec{p})\hat{a}_\lambda(\vec{p}) + e^{ipx}v_\lambda(\vec{p})\hat{b}^\dagger_\lambda(\vec{p})\right]$$

and taking into account that $\hat{\bar{\psi}} = \hat{\psi}^\dagger\gamma_0$

$$\hat{\bar{\psi}}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E}}\sum_\lambda \left[e^{ipx}\bar{u}_\lambda(\vec{p})\hat{a}^\dagger_\lambda(\vec{p}) + e^{-ipx}\bar{v}_\lambda(\vec{p})\hat{b}_\lambda(\vec{p})\right]$$

where λ labels the different spin polarizations; and $\hat{a}_\lambda(\vec{p})(\hat{b}^\dagger_\lambda(\vec{p}))$ are the (anti)particle creation (annihilation) operators which fulfill anti-commutation relationships

$$\left\{\hat{a}_\lambda(\vec{p}), \hat{a}^\dagger_{\lambda'}(\vec{p'})\right\} = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{p'})\delta_{\lambda\lambda'}$$
$$\left\{\hat{b}_\lambda(\vec{p}), \hat{b}^\dagger_{\lambda'}(\vec{p'})\right\} = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{p'})\delta_{\lambda\lambda'}$$

and the remaining anti-commutators are zero.

In order to study the axion potential between two fermions we have to consider the following process mediated by the axion

$$\psi_1(p_1)\,\psi_2(p_2) \to \psi_1(p_1')\,\psi_2(p_2')\,. \quad (2)$$

with $p_i^{(\prime)}$ the incoming (outgoing) external momenta. Given the Lagrangian in Eq. (1), this process results in three possible amplitudes characterised by the two terms mediating interactions between the scalar field a and the fermionic fields ψ_i. Therefore, one obtains scalar×scalar ($g_1^S - g_2^S$), scalar×pseudo-scalar ($g_1^S - g_2^P$) and pseudo-scalar×pseudo-scalar ($g_1^P - g_2^P$) interactions whose diagrams are presented in Fig. 1. The resultant amplitudes read

$$
\begin{aligned}
\mathcal{M}_1 &= -g_1^S g_2^S \frac{[\bar{u}_{\lambda_1'}(\vec{p}_1')u_{\lambda_1}(\vec{p}_1)][\bar{u}_{\lambda_2'}(\vec{p}_2')u_{\lambda_2}(\vec{p}_2)]}{q^2 - m_a^2} \\
\mathcal{M}_2 &= -ig_1^P g_2^S \frac{[\bar{u}_{\lambda_1'}(\vec{p}_1')\gamma_5 u_{\lambda_1}(\vec{p}_1)][\bar{u}_{\lambda_2'}(\vec{p}_2')u_{\lambda_2}(\vec{p}_2)]}{q^2 - m_a^2} \\
\mathcal{M}_3 &= g_1^P g_2^P \frac{[\bar{u}_{\lambda_1'}(\vec{p}_1')\gamma_5 u_{\lambda_1}(\vec{p}_1)][\bar{u}_{\lambda_2'}(\vec{p}_2')\gamma_5 u_{\lambda_2}(\vec{p}_2)]}{q^2 - m_a^2}
\end{aligned}
\tag{3}
$$

where $q = p_1' - p_1 = p_2 - p_2'$ is the momentum transfer.

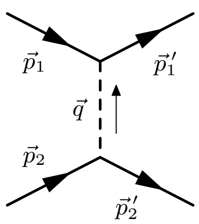

FIG. 1: Feynman diagrams of the interaction in Eq. (2). Depending on the kind of interaction mentioned, the vertices will be $\{g_i^S, i\gamma_5 g_i^P\}$

A. Axion-mediated potentials at leading order in the relativistic limit

The first step of our treatment focuses on the calculation of the non-relativistic potentials for the elastic scattering in the leading order (LO) in the non-relativistic expansion. In our calculation we confirm and amend some results present in literature, as in [2, 3].

Now, in order to calculate the axion potential between two fermions we have considered the Fourier transform of the amplitudes in Eq.(3). In the elastic limit the following kinematic conditions, consistent with the non-relativistic expansion, apply

$$q^0 = 0 \to |\vec{p}_i| = |\vec{p}_i'| \text{ and } q^2 = -|\vec{q}|^2.$$

By using the non-relativistic expansion for the scalar and pseudo-scalar fermionic bilinear in App. A, the non-relativistic potentials in the different sectors at LO in non-relativistic limit are listed below.

• Scalar×Scalar Interaction

$$
\begin{aligned}
\mathcal{M}_1^{el} &= \frac{g_1^S g_2^S}{|\vec{q}|^2 + m_a^2} \to \\
V_{SS}^{el} &= \int \frac{d^3\vec{q}}{(2\pi)^3} \mathcal{M}_1 e^{i\vec{q}\cdot\vec{x}} = -\frac{g_1^S g_1^S}{4\pi r} e^{-m_a r}
\end{aligned}
\tag{4}
$$

• Pseudo-scalar×Scalar Interaction

$$
\begin{aligned}
\mathcal{M}_2^{el} &= -i \frac{g_1^P g_2^S}{|\vec{q}|^2 + m_a^2} \frac{\vec{q}\cdot\vec{\sigma}_{\lambda_1'\lambda_1}}{2m_1} \to \\
V_{PS}^{el} &= -\frac{g_1^P g_2^S}{8\pi m_1} \hat{r}\cdot\vec{\sigma}_{\lambda_1'\lambda_1} \left(\frac{m_a}{r} + \frac{1}{r^2}\right) e^{-m_a r}
\end{aligned}
\tag{5}
$$

where we have introduced the three-component vector $\vec{\sigma}_{\lambda_i'\lambda_i} = \chi_{\lambda_i'}^\dagger \vec{\sigma} \chi_{\lambda_i}$ for each particle types ($i = 1, 2$) and for each spinor polarizations, $(\lambda_i,\, \lambda_i') = (1, 2)$. Moreover $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ is the vector containing the 2×2 Pauli matrices.

• Pseudo-scalar×Pseudo-scalar Interaction

$$
\begin{aligned}
\mathcal{M}_3^{el} &= -\frac{g_1^P g_2^P}{|\vec{q}|^2 + m_a^2} \frac{(-\vec{q}\cdot\vec{\sigma}_{\lambda_1'\lambda_1})(q\cdot\vec{\sigma}_{\lambda_2'\lambda_2})}{4m_1 m_2} \to \\
V_{PP}^{el} &= \frac{g_1^P g_2^P}{4m_1 m_2} \vec{\sigma}^i_{\lambda_1'\lambda_1} \vec{\sigma}^j_{\lambda_2'\lambda_2} \partial_i \partial_j \frac{e^{-m_a r}}{4\pi r}
\end{aligned}
$$

which, using the results in App. B

$$
\begin{aligned}
V_{PP}^{el} = -\frac{g_1^P g_2^P}{16\pi m_1 m_2} \Bigg[&(\vec{\sigma}_{\lambda_1'\lambda_1}\cdot\vec{\sigma}_{\lambda_2'\lambda_2}) \left(\frac{m_a}{r^2} + \frac{1}{r^3} + \frac{4\pi}{3}\delta^3(r)\right) \\
&-(\vec{\sigma}_{\lambda_1'\lambda_1}\cdot\hat{r})(\vec{\sigma}_{\lambda_2'\lambda_2}\cdot\hat{r}) \left(\frac{m_a^2}{r} + 3\frac{m_a}{r^2} + \frac{3}{r^3}\right)\Bigg] e^{-m_a r}
\end{aligned}
\tag{6}
$$

Here we highlight a list of discrepancies with respect to the previous literature about the calculation of the non-relativistic potentials. The calculations were also checked with little programs in Mathematica to avoid possible errors given the length of the formulae.

• Scalar×Pseudo-scalar Potential (Eq. (5))
We find an overall minus sign with respect to the result of Refs. [2, 3] arising from an inconsistency of the conventions. As reported in App. C, the Feynman diagrams used in the thesis are compatible with the positive exponent of the Fourier transforms. Eq. (5) is derived from a t-channel diagram in which the momentum q is transferred from the scalar to the pseudo-scalar current.

• Pseudo-scalar×Pseudo-scalar Potential (Eq. (6))
We find an overall minus sign with respect to the result of Refs. [2, 3]. This probably originates from a missing sign in the derivation of the non-relativistic limit of the pseudo-scalar bilinears. Recalling the expression $q = p_1' - p_1 = p_2 - p_2'$ (Fig. 1), the explicit expressions of the fermion bilinears (App. A) differ by

$$
\begin{aligned}
\bar{u}_{\lambda_1'}(\vec{p}_1')\gamma_5 u_{\lambda_1}(\vec{p}_1) &= -\frac{q\cdot\vec{\sigma}_{\lambda_1'\lambda_1}}{2m_1} \\
\bar{u}_{\lambda_2'}(\vec{p}_2')\gamma_5 u_{\lambda_2}(\vec{p}_2) &= +\frac{q\cdot\vec{\sigma}_{\lambda_2'\lambda_2}}{2m_1}
\end{aligned}
$$

Moreover, note the contact interaction represented by $\delta^3(r)$ in Eq. (6). This contact arises from the singular part of the second derivatives of $1/r$ and was missed in previous studies. It belongs to the microscopic world and cannot be probed at classical scales. However, it is a new effect that may cause shifts on the energies of quantum systems.

III. EXPERIMENTAL PROSPECT

A possible way to detect light particles coupled with ordinary matter, is to look for new macroscopic forces mediated by the exchange of such light new particles. The range of the force is related to Compton wavelength $\lambda = 1/m_a$ of new particle. It can, therefore, be tested in laboratory searches for m_a below eV, as reported in Fig. 2, where the best experimental constraints on their parameter space are highlighted. Fig. 2 has been generated making use of the python code in [4].

As mentioned in the previous sections, the lagrangian in Eq.(1) enables three possible interactions: monopole-monopole ($g^S_{N,e} - g^S_{N,e}$), monopole-dipole ($g^S_{N,e} - g^P_{N,e}$) and dipole-dipole ($g^P_{N,e} - g^P_{N,e}$). Monopole-monopole forces in Eq. (4) at LO represent long-range spin-independent potentials. Their detection can be carried out via precision measurements of Newton's law, searched for violations of the equivalence principle with torsion balance techniques. Bounds from the potential in Eq.(4) are shown on the left plot of Fig. 2. The axion develops also spin-dependent thanks to scalar and pseudo-scalar couplings to nucleons and electron, namely monopole-dipole forces in Eq. (5). The eperimental bounds are shown on the left plot of Fig. 2. Finally, dipole-dipole forces mediated from axion in Eq. (6) can also be searched for in laboratory experiments, but their results are much less restrictive than the corresponding astrophysical limits.

IV. AXION-MEDIATED POTENTIALS BEYOND THE LEADING ORDER

Until now we have been concerned with reproducing the calculation of the potentials, limiting ourselves to LO contributions in the non-relativistic expansion and elastic scattering. In this section we consider, instead, higher-order non-relativistic corrections of the elastic process and in the next we comment about relaxing as well the hypothesis of elastic scattering. Using the non-relativistic expansion at next LO in App. A we have for each sector

- Scalar×Scalar corrections

$$\begin{aligned}\mathcal{M}_1^{el} &= \frac{g_1^S g_2^S}{|\vec{q}|^2 + m_a^2}\left(1 + \frac{|\vec{q}|^2}{8m_1^2} + i\frac{(\vec{P}_1 \times \vec{q}) \cdot \vec{\sigma}_{\lambda_1\lambda_1}}{4m_1^2}\right) \\ &\quad \left(1 + \frac{|\vec{q}|^2}{8m_2^2} - i\frac{(\vec{P}_2 \times \vec{q}) \cdot \vec{\sigma}_{\lambda_2\lambda_2}}{4m_2^2}\right) \\ &= \frac{g_1^S g_2^S}{|\vec{q}|^2 + m_a^2}\left(1 + \frac{|\vec{q}|^2}{8\mu^2} + \frac{i}{4}\vec{K} \cdot \vec{q}\right)\end{aligned} \tag{7}$$

where

$$\frac{1}{\mu^2} = \frac{m_1^2 + m_2^2}{m_1^2 m_2^2},\ \vec{K} = \frac{\vec{\sigma}_{\lambda_1\lambda_1} \times \vec{P}_1}{m_1^2} - \frac{\vec{\sigma}_{\lambda_2\lambda_2} \times \vec{P}_2}{m_2^2}$$

and the corresponding Fourier transform is

$$\begin{aligned}V_{SS}^{el} = -\frac{g_1^S g_2^S}{16\pi r}\Bigg[1 - \frac{1}{8\mu^2}\left(m_a^2 - 4\pi r\delta^3(x)\right) \\ - \frac{1}{4}\left(m_a + \frac{1}{r}\right)\vec{K} \cdot \hat{r}\Bigg] e^{-m_a r}\end{aligned} \tag{8}$$

- Pseudo-scalar×Scalar corrections

$$\begin{aligned}\mathcal{M}_2^{el} = -i\frac{g_1^P g_2^S}{|\vec{q}|^2 + m_a^2}\frac{\vec{q} \cdot \vec{\sigma}_{\lambda_1'\lambda_1}}{2m_1}\Bigg(1 + \frac{|\vec{q}|^2}{8m_2^2} \\ -i\frac{(\vec{P}_2 \times \vec{q}) \cdot \vec{\sigma}_{\lambda_2\lambda_2}}{4m_2^2}\Bigg)\end{aligned} \tag{9}$$

Thus the potential reads as

$$\begin{aligned}V_{PS}^{el} = -\frac{g_1^P g_2^S}{8\pi m_1}\Bigg[\hat{r} \cdot \vec{\sigma}_{\lambda_1'\lambda_1}\left(\frac{m_a}{r} + \frac{1}{r^2}\right) \\ - \frac{1}{8m_2^2}\left(\frac{m_a^3}{r} + \frac{m_a^2}{r^2} - 4\pi m_a\delta^3(x)\right) \\ - \frac{1}{4m_2^2}\left(\frac{m_a}{r^2} + \frac{1}{r^3} + 4\pi\delta^3(x)\right) \\ \vec{\sigma}_{\lambda_1\lambda_1} \cdot \vec{\sigma}_{\lambda_2\lambda_2} \times \vec{P}_2\Bigg] e^{-m_a r}\end{aligned} \tag{10}$$

- Pseudo-scalar×Pseudo-Scalar corrections

Dealing with only pseudo-scalar vertices, the only type of non-relativistic correction arises from the inelastic case. The pseudo-scalar vertices have not got non-relativistic corrections in the elastic limit see App. A.

A. Axion-mediated potentials beyond the elastic limit

Going beyond the elastic scattering is actually required in order to provide a consistent non-relativistic expansion of the potentials, since the elastic condition is automatically enforced at the LO in the non-relativistic expansion. Being the complete calculation of the non-relativistic corrections beyond the elastic list quite cumbersome, here, we only mention the key points to include the inelastic corrections to the potentials. These corrections will imply new terms proportional to $|\vec{p}_i|/m^2$ coming from both the spinor expansion and the axion propagator from amplitudes of Eq. (3).

It is worth to relax our hypothesis in order to study scattering processes where the transferred energy slightly differs from zero. Taking into account that in the inelastic limit

$$(q^0)^2 \simeq \frac{(\vec{P}_1 \cdot \vec{q})^2}{2m_1^2}$$

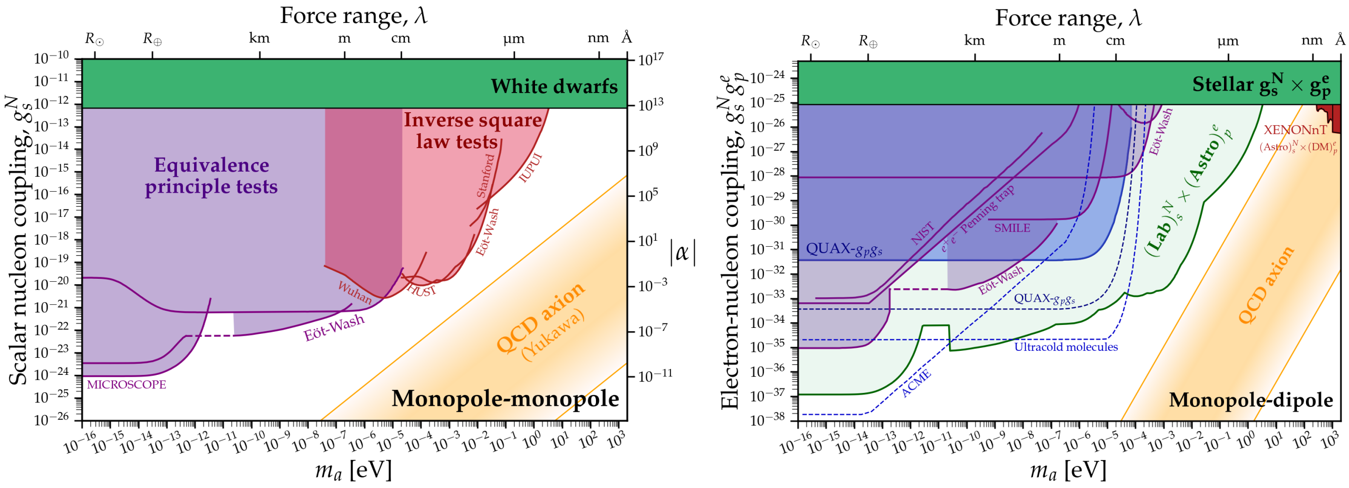

FIG. 2: Using the python code in [4], we show the experimental bounds on the g_N^S nucleon-axion coupling, Eq. (4), and on the $g_N^S g_e^P$ electron- and nucleon-axion couplings, Eq. (5), on the left and the right plots, respectively. The solid lines correspond to existing laboratory bounds, the dotted lines are the future projections while the dashed lines represents the combination of laboratory and astrophysical bounds. The green band corresponds to the astrophysical bound and the theoretical QCD axion band is shown in yellow. Constraints are represented in terms of the axion mass m_a, or its Compton wavelength λ.

the propagator in the amplitudes of Eq. (3) will be

$$\begin{aligned}\frac{i}{q^2 - m_a^2} &= \frac{i}{(\vec{P}_1\cdot\vec{q})^2/2m_1^2 - |\vec{q}|^2 - m_a^2} \\ &= \frac{-i}{|\vec{q}|^2 + m_a^2}\left(1 + \frac{(\vec{P}_1\cdot\vec{q})^2}{2m_1^2}\frac{1}{|\vec{q}|^2 + M^2}\right)\end{aligned}$$

The factor in bracket in the above equation should enter as a correction into all the scattering amplitudes such as the ones in Eq. (7,8) and the pseudo-scalar×pseudo-scalar one also, which results exactly in the elastic limit.

V. CONCLUSIONS

In this work we critically revisit the calculation of non-relativistic potentials mediated by new light particles, such as axions, coupled to SM fermions. We have reproduced and improved some results present in the literature, going, for example, beyond the LO Born approximation and finding a Dirac delta already missed at LO for pseudo-scalar×pseudo-scalar and at the non-LO for the other amplitudes.
Our next work will be to study the impact of contact terms in the phenomenological analysis.

VI. ACKNOWLEDGMENTS

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Appendix A: Spinor notation and non-relatistic expansion in the elastic limit

First, let us define the set of independent momenta that will be useful for our work for any set of vertex, where only one type of particle play a role.

$$\begin{aligned} q &= p' - p \\ P &= \frac{p'+p}{2} \end{aligned} \quad\rightarrow\quad \begin{aligned} p &= P - \frac{q}{2} \\ p' &= P + \frac{q}{2} \end{aligned}$$

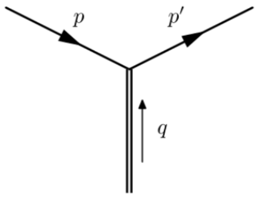

FIG. 3: Interaction vertex and convention for the transfer momentum $q = p' - p$

For simplicity, with respect of sec. II A, we have neglected the label $i = 1, 2$ for each particle type. Now the Dirac spinor is

$$u_\lambda(p) = \frac{1}{\sqrt{2m(E_p + m)}} \begin{pmatrix} (E_p + m)\,\chi_\lambda \\ \vec{p}\cdot\vec{\sigma}\,\chi_\lambda \end{pmatrix}$$
$$v_\lambda(p) = \frac{1}{\sqrt{2m(E_p + m)}} \begin{pmatrix} \vec{p}\cdot\vec{\sigma}\,\chi_\lambda \\ (E_p + m)\chi_\lambda \end{pmatrix}$$

which $\chi_\lambda = (1, 0)$ and $(0, 1)$ for $\lambda = 1$ and 2, respectively. In the spinor normalization the factor $1/2m$ is included to simplify the expression of Feynman amplitudes in sec. II A a factor $1/4m_1m_2$ has to be inserted.

The last assumption that we have made is the elastic limit of the scattering process, meaning that in the center of mass frame

$$q^0 = 0 \to |\vec{p}\,'| = |\vec{p}|,\ \vec{P}\cdot\vec{q} = 0.$$

The bilinear forms which we are interested in sec. II A are in the elastic limit

$$\begin{aligned}\bar{u}(p')_{\lambda'} u(p)_\lambda &= \frac{1}{2m}\Big[\sqrt{E_{p'} + m}\sqrt{E_p + m}\,\delta_{\lambda,\lambda'} \\ &\quad -\chi^T_{\lambda'}\frac{(\vec{p}\,'\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma})}{\sqrt{E_{p'} + m}\sqrt{E_p + m}}\chi_{\lambda'}\Big] \\ &\simeq \left(1 + \frac{|\vec{p}|^2}{4m^2} - \frac{(\vec{p}\cdot\vec{p}\,')}{2m^2}\right)\delta_{\lambda',\lambda} \\ &\quad + i\chi^T_{\lambda'}\frac{(\vec{p}\times\vec{p}\,')\cdot\vec{\sigma}}{4m^2}\chi_\lambda + \mathcal{O}\left(\frac{|\vec{p}|^2}{m^2}\right) \\ &\simeq \left(1 + \frac{|\vec{q}|^2}{8m^2}\right)\delta_{\lambda',\lambda} + i\chi^T_{\lambda'}\frac{(\vec{P}\times\vec{q})\cdot\vec{\sigma}}{4m^2}\chi_\lambda\end{aligned}$$

and

$$\bar{u}(p')_{\lambda'}\gamma_5 u(p)_\lambda = \frac{1}{2m}\chi^T_{\lambda'}((\vec{p}\,' - \vec{p})\cdot\vec{\sigma})\chi_\lambda = -\chi^T_{\lambda'}\frac{\vec{q}\cdot\vec{\sigma}}{2m}\chi_\lambda$$

where in our notation, $q = \vec{p}\,' - \vec{p}$.

Appendix B: Derivatives of the Yukawa potential

It is useful to calculate the various derivatives of

$$\frac{e^{-Mr}}{r}$$

emerging from the Fourier transform of all non-relativistic potentials. First let us take

$$\partial_i\partial_j\left(\frac{e^{-Mr}}{r}\right)$$

The second derivative must be considered in two cases where:

- $r \neq 0$:

$$\partial_j\left(\frac{e^{-Mr}}{r}\right) = -x^j\left(\frac{M}{r^2} + \frac{1}{r^3}\right)e^{-Mr},$$
$$\partial_i\partial_j\left(\frac{e^{-Mr}}{r}\right) = -\Big[\delta^{ij}\left(\frac{M}{r^2} + \frac{1}{r^3}\right) - M\frac{x^ix^j}{r}\left(\frac{M}{r^2} + \frac{1}{r^3}\right) - \frac{x^ix^j}{r}\left(\frac{2M}{r^3} + \frac{3}{r^4}\right)\Big]e^{-Mr};$$

- $r = 0$:

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{x}),\quad \partial_i\partial_j\left(\frac{1}{r}\right) = -\frac{4\pi}{3}\delta^{ij}\delta^3(\vec{x})$$

due to the distributional nature of the Coulomb law, we must add this quantity within the term multiplying δ^{ij}. At the end:

$$\vec{\nabla}\left(\frac{e^{-Mr}}{r}\right) = -\left(\frac{M}{r} + \frac{1}{r^2}\right)\hat{r}e^{-Mr},$$
$$\partial_i\partial_j\left(\frac{e^{-Mr}}{r}\right) = -\Big[\delta^{ij}\left(\frac{M}{r^2} + \frac{1}{r^3} + \frac{4\pi}{3}\delta^3(\vec{x})\right) - \hat{r}^i\hat{r}^j\left(\frac{M^2}{r} + \frac{3M}{r^2} + \frac{3}{r^3}\right)\Big]$$
$$\nabla^2\left(\frac{e^{-Mr}}{r}\right) = \left(\frac{M^2}{r} - 4\pi\delta^3(\vec{x})\right)e^{-Mr}$$

Appendix C: Convention on the Fourier transform

Considering the diagram in Fig. (3) our convention on the Fourier transforms for the Feynman propagator is

$$i\Delta(x_1 - x_2) = \int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m^2 + i\varepsilon}e^{-iq\cdot(x_1 - x_2)}$$

The convention cannot be forgotten because it can lead to a mismatch of signs in the literature. Physical observables are not sensitive to convention as long as they are used consistently. The main Fourier transforms appearing in the tree-level work are the following

$$\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{x}}}{|\vec{q}|^2 - M^2} = \frac{1}{4\pi^2 ir}\int dq\frac{qe^{iqr}}{q^2 - M^2} = \frac{e^{-Mr}}{4\pi r},$$
$$\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{x}}}{(|\vec{q}|^2 - M^2)^2} = \frac{1}{4\pi^2 ir}\int dq\frac{qe^{iqr}}{(q^2 - M^2)^2} = \frac{e^{-Mr}}{8\pi M}.$$

where $r \equiv |\vec{x}|$ and $q = |\vec{q}|$.