Reiterman‘s theorem for pseudovarieties

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CHAPTER 1

INTRODUCTION

In universal algebra, varieties are classes of similar algebras that are closed under subalgebras, homomorphic images and direct products. Concrete examples of these structures arise and are used in many different contexts, for instance, they provide appropriate semantics for many relevant logics. In this context, the semantics of classical propositional logic is given by the variety of Boolean algebras. Similiarly, the semantics of intuitionistic logic is given by the variety of Heyting algebras. Many other different and more exotic examples of varieties can be found in the literature (see [8], [14],[15], [5] for instance), nevertheless, variety theory can be very useful when studying classes of algebras of any kind (not necessarily related to logic) such as groups, rings and fields among others (see [19] and [16] for example).

One of the central results regarding the characterization and description of varieties is due to Birkhoff, who showed in 1935 in his paper *On the Structure of Abstract Algebras* ([4]) that this classes coincide with the equational classes, which are the ones defined by a set of identities. Some years later, in 1946, Tarski proved in [26] that a variety $\mathcal{V}$ indeed coincides with the class $\mathbb{HSP}(\mathcal{V})$, meaning that every algebra in $\mathcal{V}$ is a homomorphic image of a subalgebra of a product of members in $\mathcal{V}$. In fact, we can consider more generally a class $\mathcal{K}$ of algebras which is not a variety, and it turns out that the least variety $\mathbb{V}(\mathcal{K})$ containing $\mathcal{K}$ is the same as $\mathbb{HSP}(\mathcal{K})$.

In this thesis we will restrict our attention only to classes of finite algebras. Besides the inherent motivation in studying finite algebras, they appear and are used in many other fields. For example, finite semigroups and monoids are very useful in the theory of automata and rational languages (see [22] and [9]). In particular, the classes considered in this context have some special properties, namely, they are closed under homomorphic images, under subsemigoups and under finite products. This motivates the general definition of a *pseudovariety*, which is a class of finite algebras closed under homomorphic images, under subalgebras and under finite products.

We may ask if it is possible for a pseudovariety to be a variety. It turns out that this question has a negative answer: no pseudovariety is a variety. This is a direct consequence of the fact that every nontrivial variety contains infinite algebras [I]. This means that no pseudovariety is equational (in the sense that it is not axiomatizable by a set of equations)[II] and that Birkhoff's theorem is not valid for classes of finite algebras. Due to this, one may also think that pseudovarieties can be obtained considering only the finite members of a variety, and although this is sometimes the case (what we call them *equational pseudova-*

[I]Every nontrivial variety contains at least one algebra. Then, due to the closure of arbitrary products, it suffices to take the infinite power of this algebra to construct an infinite one.

[II]Otherwise, if there is some set of identities axiomatizing a pseudovariety, this becomes immediately into a variety due to Birkhoff' theorem and therefore it must contain an infinite algebra, which is not possible.

rieties[III]), it is not true in general. For instance, the class of finite groups in the language of semigroups is not one of these equational pseudovarieties. Many other examples (and more sophisticated), can be found in [23], [3] and in [1], as the classes of finite nilpotent semigroups, finite cancellation monoids, finite abelian p-groups and semigroups where the idempotent elements are left-zeros among others. We refer the reader especially to [1] since the examples are more developed. We want to emphasize that, although some of these examples can be somehow unfamiliar, this phenomenom affects to classes as natural and useful as finite groups, which are widely studied. Therefore, the lack of a characterization for this classes is not a trivial issue.

Some different approaches have been adopted to resolve this problem (see [10],[2] and [3]); nevertheless, we are particularly interested in Reiterman's work of his paper [23]. Here, he introduce the concept of an *implicit operation*, which can be viewed as a generalization of the usual notion of term-definable function. In a few words, given a pseudovariety $\mathcal{V}$ of some specific language, an n-ary implicit operation (relative to $\mathcal{V}$) is a family of n-ary mappings that commute with homomorphism between members in $\mathcal{V}$. A *pseudoidentity for* $\mathcal{V}$ is defined as a pair of two implicit operations (both of the same arity), and it can be viewed as a generalization of the usual identities (as implicit operations for term-definable functions). With this two ingredients (and the help of topological machinery), Reiterman stated in his celebrated theorem that pseudovarieties (contained in a larger pseudovariety) coincide with the classes of finite algebras axiomatized by a set of pseudoidentities, i.e., a pseudovariety $\mathcal{V}$ can be defined by a set of pseudoidentities relative to some pseudovariety containing $\mathcal{V}$, and every class defined by a set of pseudoidentities is a pseudovariety. Roughly speaking, Reiterman's theorem is a version of Birkhoff theorem for classes of finite algebras.

One of the typical examples of implicit operations (relative to the pseudovariety $\mathcal{S}$ of all finite semigroups in this case) is the unary mapping denoted by x^ω, which helps to understand what an implicit operation is. For each finite semigroup $\mathbf{S}$, this function sends every element s in $\mathbf{S}$ to some power of s that is idempotent (see [1] and [22]). This implicit operation appears in many axiomatizations of pseudovarieties. As it happens, in the axiomatization of the examples of pseudovarieties mentioned above the implicit operation s^ω plays a significant role.

The main aim of the following text is to state and prove Reiterman's theorem for pseudovarieties, but there are many things to develop before achieving that objective, namely, all the theoretical framework and machinery and tools for proving it. To do that, and trying to keep a transparent exposition, we will divide the text in mainly four blocks or chapters. It is worthy of mention that great part of the material exposed here (if not all) follows

[III]The class of all finite semigroups, for example

Almeida's book [1]. We have tried to make a more detailed exposition of what appears in [1] making all necessary steps explicit.

In chapter 2 we will introduce some preliminaries. Although it may be quite compressed, the chapter will include all the needed background (algebraic and topological) for, at least, understand what follows. On the one hand, basic notions in universal algebra will be defined. Even if some of them are not strictly related to what we are going to develop, they are a kind of their "predecessors" and are fundamental for understanding the motivations for Reiterman's theorem. Some of these concepts involve the formal definitions of an algebra and a variety, and, of course, the Birkhoff's theorem. To conclude the preliminaries regarding algebra, we will give some examples of particular algebras as well as a their few properties. On the other hand, all the necessary topologial notions will be introduced: topological space, continuous functions, compactness and Stone spaces among others. In addition, every relevant result will be clearly stated.

In chapter 3 we will provide a desired framework where Reiterman's theorem and other results (as auxiliary lemmas and propositions for proving Reiterman's main theorem) are stated. Pseudovarieties will be introduced in the first place along with the analogous result for the $\mathbb{HSP}$ theorem. Once we have defined pseudovarieties, we will be in a good shape to define properly implicit operations. After this, the chapter will be splitted into two sections: algebraic and topological aspects. In the first part, we will see how an algebraic structure can be defined over the set of all n-ary implicit operations relative to some pseudovariety $\mathcal{V}$ ($\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$) and we will prove some properties of such algebra, as its relation with the respective algebra of term-definable implicit operations (called explicit operations). Regarding the topological part, we first endow $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ with a topology and then some properties of this space are studied. For instance, we will prove that it is a Stone space (as a topological space) and that it also forms a topological algebra. Moreover, we will see that also a metric can be defined and that the topology that it defines coincides with the first one considered.

In chapter 4 we finally tackle the main theorem, however, some technical lemmas need to be stated and proved before undertaking the proof. Roughly speaking, we study the behaviour of continuous homomorphisms regarding the algebras of a pseudovariety $\mathcal{V}$ and the algebra of n-ary implicit operations. After this work, we conclude with Reiterman's theorem and its proof.

Finally, chapter 5 will be dedicated to give some examples of pseudovarieties. We introduce the notion of equational pseudovariety and provide a couple of simple examples of them. To finish, we show that the pseudovariety of finite groups is not equational and therefore it must be defined by what we call pseudoidentities. In particular, the unary implicit operation denoted by s^{ω} will play a significant role as we have mentioned along this introduction.

CHAPTER 2

PRELIMINARIES

2.1 Some algebraic notions

In this section, some basic definitions and properties of universal algebra will be stated. Moreover, we will also introduce a particular type of algebras, namely (finite) semigroups. As mentioned in the introduction, these algebras, very related to automata theory, form interesting examples of pseudovarieties and therefore it is worthwhile showing their definitions and properties. We refer the reader to [7], [1] and [22] for more information and details.

The very first notion in the area of universal algebra is the one of an *algebraic language* or *type*.

Definition 2.1. *A* language *or* type *of algebras is a set τ of function symbols such that for each $\sigma \in \tau$ a nonnegative integer n is assigned to σ. We call this integer the* arity *of σ, and σ is said to be an n-ary function symbol.*

We will not consider neither infinite nor infinitary languages along this text, so whenever appears a type it would be assumed to be finite and finitary. In addition, for a language τ, we will adopt the notation n_σ for the arity of each $\sigma \in \tau$.

Definition 2.2. *Given a set X of variables and a language τ, we define the* terms *of τ with variables in X by recursion as follows:*

i) Each member of X is a term.

ii) For every $\sigma \in \tau$, given terms $t_1(\bar{x}), \ldots, t_{n_\sigma}(\bar{x})$ the expression $\sigma(t_1(\bar{x}), \ldots, t_{n_\sigma}(\bar{x}))$ is also a term.

The set of all terms of τ with variables in X is denoted by $T(X)$.

Definition 2.3. *Let τ be a language. An* algebra $\mathbf{A}$ *of type τ is an ordered pair $\langle A, \tau \rangle$ where A is a nonempty set and such that for every $\sigma \in \tau$ the* interpretation *of σ in $\mathbf{A}$ is a function $\sigma^{\mathbf{A}} : \mathbf{A}^{n_\sigma} \to \mathbf{A}$. The set A is called the* universe *of $\mathbf{A}$.*

Given a langauge τ and a set X, we can define an algebraic structure $\mathbf{T}(X) := \langle T(X), \tau \rangle$ where for each $\sigma \in \tau$ its interpretation is defined as

$$\sigma^{\mathbf{T}(\mathbf{X})}(t_1, \ldots, t_{n_\sigma}) = \sigma(t_1, \ldots, t_{n_\sigma})$$

for every $t_1, \ldots, t_{n_\sigma} \in T(X)$. In case $X = \{x_1, \ldots, x_n\}$ we will use the notation $\mathbf{T}(x_1, \ldots, x_n)$ instead of $\mathbf{T}(\{x_1, \ldots, x_n\})$.

Definition 2.4. *Let $\mathbf{A}$ be an algebra of type τ and let X be a set. An* assigment *h of X on $\mathbf{A}$ is a mapping $h : X \to A$.*

Definition 2.5. *Given two algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *of the same type* τ*, we say that* $\mathbf{B} := \langle B, \tau \rangle$ *is a* subalgebra *of* $\mathbf{A}$ *and denote by* $\mathbf{B} \leqslant \mathbf{A}$ *if* $B \subseteq A$ *and if for every* $\sigma \in \tau$ *we have* $\sigma^{\mathbf{B}} := \sigma^{\mathbf{A}} \restriction B$*.*

Definition 2.6. *Let* $\mathbf{A}$ *be an algebra of type* τ *and let* $B \subseteq A$ *be nonempty.* B *is said to be a* subuniverse *of* $\mathbf{A}$ *if for every* $\sigma \in \tau$ *and every* $b_1, \ldots, b_{n_\sigma} \in B$ *we have* $\sigma^{\mathbf{A}}(b_1, \ldots, b_{n_\sigma}) \in B$*. We say that* B *is* closed under the operations in τ *in that case.*

Definition 2.7. *Let* $\mathbf{A}$ *be an algebra of type* τ *and let* $B \subseteq A$ *be nonempty. The* subalgebra of $\mathbf{A}$ generated by B *is defined as*

$$[\mathbf{B}]_{\mathbf{A}} := \langle [B]_{\mathbf{A}}, \tau \rangle$$

where $[B]_{\mathbf{A}}$ *is the least subuniverse of* $\mathbf{A}$ *containing* B*. When* B *is finite, we say that it is finitely generated.*

Lemma 2.8. *Let* $\mathbf{A}$ *be an algebra of type* τ *and let* $B \subseteq A$ *be nonempty. Then, taking* X *to be a set such that* $|X| = |B|$ *and taking an enumeration* $\langle b_i : i < \alpha \rangle$ *of* B *for some ordinal* α*, we have*

$$[B]_{\mathbf{A}} = \{ p^{\mathbf{A}}(\langle b_i : i < \alpha \rangle) : p \in \mathbf{T}(X) \}$$

Proof sketch. The inclusion $\supseteq$ is proved by induction on the construction of terms. The members of B constitute the base cases and the inductive cases follows from Definition 2.7. The inclusion $\subseteq$ follows from the easy observation that the set $\{ p^{\mathbf{A}}(\langle b_i : i < \alpha \rangle) : p \in \mathbf{T}(X) \}$ is closed under the operations in τ. Then, we just apply Definition 2.7.

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Definition 2.9. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *two algebras of the same type and a mapping* $h : \mathbf{A} \to \mathbf{B}$*. We say that* h *is a* homomorphism *if for every* $\sigma \in \tau$ *and every* $a_1, \ldots, a_{n_\sigma} \in A$ *the following equality:*

$$h(\sigma^{\mathbf{A}}(a_1, \ldots, a_n)) = \sigma^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) \ .$$

We will write $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$ when h is a surjective homomorphism between the algebras $\mathbf{A}$ and $\mathbf{B}$. Similarly, we write $h : \mathbf{A} \hookrightarrow \mathbf{B}$ when h is injective.

Definition 2.10. *Let* h *be a homomorphism* $h : \mathbf{A} \to \mathbf{B}$ *for some algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *of the same type. We say that* h *is an* isomorphism *if it is injective and surjective. In this case, we write* $\mathbf{A} \cong \mathbf{B}$*.*

Lemma 2.11 ([7] Theorem 6.5)**.** *Let* $h_1 : \mathbf{A} \to \mathbf{B}$ *and* $h_2 : \mathbf{B} \to \mathbf{C}$ *be two homomorphisms for some algebras* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *of the same type* τ*. Then, the mapping* $h_2 \circ h_1$ *is a homomorphism.*

Definition 2.12. *Let* $\mathbf{A}$ *be an algebra of type* τ *and let* $\theta \subseteq A^2$*. We say that* θ *is a* congruence *on* $\mathbf{A}$ *if the following holds:*

i) $\langle a, a \rangle \in \theta$ *for every* $a \in A$*.*

ii) For every $a, b \in A$ *if* $\langle a, b \rangle \in \theta$ *then* $\langle b, a \rangle \in \theta$ *and viceversa.*

iii) For every $a, b, c \in A$ if $\langle a, b\rangle \in \theta$ and $\langle b, c\rangle \in \theta$ then we have $\langle a, c\rangle \in \theta$.

iv) For each $\sigma \in \tau$ and for every $a_1, \dots, a_{n_\sigma}, b_1, \dots, b_{n_\sigma} \in A$ if $\langle a_i, b_i\rangle \in \theta$ for every $1 \leqslant i \leqslant n_\sigma$ then $\langle \sigma^{\mathbf{A}}(a_1, \dots, a_{n_\sigma}), \sigma^{\mathbf{A}}(b_1, \dots, b_{n_\sigma})\rangle \in \theta$.

The set of all congruences on $\mathbf{A}$ *is denoted by* $Con(\mathbf{A})$.

Definition 2.13. *Let h be a homomorphism between two algebras $\mathbf{A}$ and $\mathbf{B}$ of the same type. Then, we define the* kernel *of h as the set* $\ker h := \{\langle a, b\rangle \in A^2 : h(a) = h(b)\}$.

Lemma 2.14 ([7] pg. 49 Theorem 6.8)**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of the same type and h a homomorphism between them. Then* $\ker(h)$ *is a congruence on* $\mathbf{A}$.

Definition 2.15. *Let $\mathbf{A}$ be an algebra of type τ and θ a congruence on $\mathbf{A}$. The* quotient algebra of $\mathbf{A}$ by θ, *denoted by $\mathbf{A}/\theta$, is the algebra whose universe is A/θ and where the interpretation of the function symbols is defined as*

$$\sigma^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) := \sigma^{\mathbf{A}}(a_1, \dots, a_n)/\theta$$

for each n-ary function symbol σ and for every $a_1, \dots, a_n \in A$.

For an algebra $\mathbf{A}$ and a congruence θ on $\mathbf{A}$, we can define the natural (projection) mapping $\rho_\theta : \mathbf{A} \to \mathbf{A}/\theta$, which is in fact a sujective homomorphism (see [7]).

Theorem 2.16 (Homomorphism theorem)**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of the same type and h be a surjective homomorphism from $\mathbf{A}$ onto $\mathbf{B}$. Then, the mapping $h \circ \rho_{\ker h}$ is an isomorphism between $\mathbf{A}/(\ker h)$ and $\mathbf{B}$.*

This theorem is quite standard an it can be found in many texts. Nevertheless, see [7] pg. 50 Theorem 6.12.

VARIETIES

Before giving the definition of a variety, it is required to define the operators $\mathbb{I}, \mathbb{H}, \mathbb{P}$ and $\mathbb{S}$ over a given class $\mathcal{K}$ of algebras of the same type.

Definition 2.17. *Given a class of algebras $\mathcal{K}$ of the same type, we define the previous operators as follows:*

$$\mathbb{I}(\mathcal{K}) := \{\mathbf{B} : \mathbf{A} \cong \mathbf{B} \text{ for some } \mathbf{A} \in \mathcal{K}\}$$

$$\mathbb{H}(\mathcal{K}) := \{\mathbf{B} : \mathbf{A} \twoheadrightarrow \mathbf{B} \text{ for some } \mathbf{A} \in \mathcal{K}\}$$

$$\mathbb{S}(\mathcal{K}) := \{\mathbf{B} : \mathbf{B} \leqslant \mathbf{A} \text{ for some } \mathbf{A} \in \mathcal{K}\}$$

$$\mathbb{P}(\mathcal{K}) := \{\mathbf{B} : \mathbf{B} = \prod_{i \in I} \mathbf{A}_i \text{ for some } \varnothing \neq \{\mathbf{A}_i : i \in I\} \subseteq \mathcal{K}\}.$$

If we restrict $\mathbb{P}$ to finite products, we get a new operator denoted by $\mathbb{P}_{fin}$. Notice that, if we have two classes $\mathcal{K}_1$ and $\mathcal{K}_2$ of similar algebras such that $\mathcal{K}_1 \subseteq \mathcal{K}_2$, then $\mathbb{O}(\mathcal{K}_1) \subseteq \mathbb{O}(\mathcal{K}_2)$ if $\mathbb{O}$ is one of the operators previously named.

Definition 2.18. *Let $\mathcal{K}$ be a class of algebras of the same type. We say that $\mathcal{K}$ is a* variety *if it is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. That is,*

$$\mathbb{S}(\mathcal{K}) \subseteq \mathcal{K}, \quad \mathbb{H}(\mathcal{K}) \subseteq \mathcal{K}, \quad \mathbb{P}(\mathcal{K}) \subseteq \mathcal{K}.$$

Let $\mathcal{K}$ be some class of algebras of the same type. The least variety containing $\mathcal{K}$ is denoted by $\mathbb{V}(\mathcal{K})$ and we call it the *variety generated by* $\mathcal{K}$.

Proposition 2.19 (Tarski)**.** *For every class $\mathcal{K}$ of algebras of the same type it holds that*

$$\mathbb{V}(\mathcal{K}) = \mathbb{HSP}(\mathcal{K}).$$

Proposition 2.19 is essential in the theory of varieties and it originally appeared in [26]. Nevertheless, the reader can also resort to [7] (pg. 67 Theorem 9.5) and [17] (pg. 8) for a more transparent and updated formulation.

Definition 2.20. *Let τ be a language and X a set of variables. An* identity *over τ is a pair (ε, δ) (or $\varepsilon \approx \delta$) such that $\varepsilon(x_1, \dots, x_m), \delta(x_1, \dots, x_m) \in \mathbf{T}(X)$. Given an algebra $\mathbf{A}$ of type τ we say that $\mathbf{A}$ satisfies the identity $\varepsilon \approx \delta$ and denote by $\mathbf{A} \models \varepsilon \approx \delta$ if for every assigment $h : X \to A$ we have*

$$\varepsilon^{\mathbf{A}}(h(x_1), \dots, h(x_m)) = \delta^{\mathbf{A}}(h(x_1), \dots, h(x_m)).$$

If we have a variety $\mathcal{V}$, we say that $\mathcal{V} \models \varepsilon \approx \delta$ if for every $\mathbf{A} \in \mathcal{V}$ we have $\mathbf{A} \models \varepsilon \approx \delta$. We define the set of identities of $\mathcal{V}$ as

$$Id(\mathcal{V}) := \{(\varepsilon, \delta) : \mathcal{V} \models \varepsilon \approx \delta\}.$$

Definition 2.21. *Let $\mathcal{K}$ be a class of algebras of the same type τ and let $\mathbf{U}(X)$ be an algbera of the type τ generated by some set X. We say that $\mathbf{U}(X)$ has* universal mapping property for $\mathcal{K}$ over *X if for every $\mathbf{A} \in \mathcal{K}$ and for every map $h : X \to A$ there is a homomorphism $\bar{h} : \mathbf{U}(X) \to \mathbf{A}$ extending h, i.e., $h(x) = \bar{h}(x)$ for every $x \in X$. In this case, we call X a set of* free generators *of $\mathbf{U}(X)$ and we say that $\mathbf{U}(X)$ is* freely generated *by X.*

Remark 2.22. In fact, given an algebra $\mathbf{U}(X)$ with the universal mapping property and an algebra $\mathbf{A}$ of the same type, if we have a mapping $h : X \to A$ then the homomorphism $\bar{h}$ from the above definition is unique (see [7] Lemma 10.6).

Remark 2.23. Notice that every assigment $h : X \to A$ can be uniquely extended to a homomorphism $\bar{h} : \mathbf{T}(X) \to \mathbf{A}$ where $\bar{h}$ is defined by recursion as follows:

- For each member $x \in X$ define $\bar{h}(x) := h(x)$.

- If we have a term $t \in \mathbf{T}(X)$ such that $t = \sigma(t_1, \ldots, t_n)$ for some n-ary function symbol $\sigma \in \tau$ and some $t_1, \ldots, t_n \in \mathbf{T}(X)$ where $\bar{h}(t_i)$ is defined for every $i \leqslant n$, then $\bar{h}(t) := \sigma^{\mathbf{A}}(\bar{h}(t_1), \ldots, \bar{h}(t_n))$.

Definition 2.24. *Let $\mathcal{K}$ be a class of algebras of the same type τ and let X be a set. We define the congruence $\theta_{\mathcal{K}}$ on $\mathbf{T}(X)$ by*

$$\theta_{\mathcal{K}}(X) := \bigcap\{\phi \in Con(\mathbf{T}(X)) : \mathbf{T}(X)/\phi \in \mathbb{IS}(\mathcal{K})\}.$$

*Then, define $\mathbf{F}_{\mathcal{K}}(\bar{X})$ the $\mathcal{K}$-*free algebra over *$\bar{X}$ as*

$$\mathbf{F}_{\mathcal{K}}(\bar{X}) := \mathbf{T}(X)/\theta_{\mathcal{K}}(X)$$

and where $\bar{X} := X/\theta_{\mathcal{K}}(X)$.

For the sake of clarity, we will write $\theta_{\mathcal{V}}$ if the set of variables is known from the context.

Theorem 2.25 (([7] pg. 72 Theorem 10.8))**.** *For every $\mathcal{K}$ classes of algebras of a fixed type τ and and every a nonempty set X, the algebra $\mathbf{T}(X)$ has the universal mapping property for $\mathcal{K}$ over X*[I]

Given a class $\mathcal{K}$ of algebras of the type τ and a set X, this algebra exists whenever the term algebra $\mathbf{T}(X)$ exists, i.e., if $X \neq \varnothing$ or if the type has 0-ary function symbols (constants). In case $\mathbf{F}_{\mathcal{K}}(\bar{X})$ exists, we know due to Birkhoff the following property.

Theorem 2.26 (Birhoff ([7] pg. 73 Theorem 10.10))**.** *Let $\mathcal{K}$ be a class of algebras of type τ and X a nonempty set. Then, the algebra $\mathbf{F}_{\mathcal{K}}(\bar{X})$ has the universal mapping property for $\mathcal{K}$ over $\bar{X}$.*

Theorem 2.27 (Birkhoff ([7] pg. 74 Theorem 10.12))**.** *Let $\mathcal{K}$ be a nontrivial class of algebras of the same type τ and let X be a set. If $\mathbf{T}(X)$ exists, then $\mathbf{F}_{\mathcal{K}}(\bar{X}) \in \mathbb{ISP}(\mathcal{K})$.*

This means that for an arbitrary class $\mathcal{K}$ of similar algebras the $\mathcal{K}$-free algebra may not be a member of $\mathcal{K}$. In particular, this is does not happen in the case of varieties since these free algebras are always in them.

Since $\mathbf{F}_{\mathcal{K}}(\bar{X})$ is unique up to isomorphism ([7] Theorem 10.7), we will denote the $\mathcal{K}$-free algebra over n free generators by by $\mathbf{F}_n\mathcal{K}$.

Theorem 2.28 (Birkhoff)**.** *Let $\mathcal{K}$ be a class of algebras of the same type. $\mathcal{K}$ is a variety if an only if $\mathcal{K} = \{\mathbf{A} : \mathbf{A} \models \Sigma\}$ for some set Σ of identities.*

In the next, we give a brief introduction to semigroups.

[I]For each mapping $h : X \to A$ for some $\mathbf{A} \in \mathcal{K}$, the homomorphism $\bar{h} : \mathbf{T}(X) \to \mathbf{A}$ extending h is constructed by recursion from the set X, i.e., for each $p(\vec{x}) \in \mathbf{T}(X)$ we have $\bar{h}(p(\vec{x})) = p^{\mathbf{A}}(h(\vec{x}))$. Again, see [7] pg. 72 theorem 10.8 for more details.

SEMIGROUPS

Definition 2.29. *A* semigroup *is an algebra* $\langle S, \cdot \rangle$ *where the operation* $\cdot$ *is binary and associative.*

To denote products, we will often use the natural notation $x \cdot y$, however, there will be some cases in where we will need to use the notation $\cdot(x, y)$, which is in fact the formal one.

Definition 2.30. *For a given semigroup* **S***, an element* $v \in S$ *is called a* unit *if for every* $s \in S$ *we have*

$$s \cdot v = v \cdot s = s \,.$$

Definition 2.31. *For a given semigroup* **S***, an element* $v \in S$ *is a* zero *if for every* $s \in S$ *we have*

$$s \cdot v = v \cdot s = v \,.$$

Definition 2.32. *For a given monoid* **S** *and a member* s *in* **S***, an element* $v \in S$ *is said to be the* inverse of s *(denoted by* s^{-1}*) if*

$$s \cdot v = 1 \,.$$

Remark 2.33. Proving that a semigroup can have at most one unit and zero elements is a simple exercise, so, in case they exist, we will denote them 1 and 0 respectively. For more details, see [22].

Definition 2.34. *A* monoid *is a semigroup with unit element. In a monoid* **S***, for every* $s, v \in S$ *we say that* s *is the* inverse *of* v *if*

$$s \cdot v = v \cdot s = 1 \,.$$

When every element in a monoid S *has an inverse element, we say that* **S** *is a* group.

Proposition 2.35 ([22] Proposition 3.12)**.** *Let* **G** *be a finite group. Then, for every* $g \in G$ *we have* $g^{|G|} = 1$.

Proposition 2.36 ([22] Proposition 3.13)**.** *A nonempty subsemigroup of a finite group is a subgroup.*

Definition 2.37. *Let* **S** *be a semigroup. We say that an element* $e \in S$ *is* idempotent *if* $e \cdot e = e$.

Lemma 2.38. *Let* **G** *be a finite group. The unit is the unique idempotent element in* **G***.*

Proof. Suppose that there is some $g \in \mathbf{G}$ that is idempotent. Then,

$$\begin{aligned} g \cdot g &= g \\ g \cdot g \cdot g^{-1} &= g \cdot g^{-1} \\ g &= 1 \end{aligned}$$

⊠

Proposition 2.39 ([22] Proposition 6.33)**.** *Let* **S** *be a finite semigroup. Then, there is some* $n \in \omega$ *such that for every* $s \in S$ *the element* s^n *is idempotent. We will denote this element by* s^ω*.*

Remark 2.40. In fact, if **S** is a finite semigroup, each member s has an idempotent power (see [22] Proposition 6.31). Thus, the number n (in the previous proposition) can be defined as

$$n := lcm(m_s : s \in S)$$

where for each $s \in S$ the power s^{m_s} is idempotent (see the proof of Proposition 2.39 in [22] for more details).

Example 2.41.

i) The structure $(\mathbb{Z}^+, +)$ *with the usual interpretation for the sum forms a semigroup (but not a monoid).*

ii) The structures $(\mathbb{N}, +)$ *and* $(\mathbb{N}, \cdot)$ *with the usual sum and product are monoids (but not groups). Moreover, in the case of* $(\mathbb{N}, \cdot)$ *the* 0 *is a zero element.*

iii) The structure $(\mathbb{Z}, +)$ *with the usual sum forms a group.*

2.2 Some topological aspects

We now refresh some concepts and results of topology. For general definitions and properties see [20], [13] and [18]. Regarding the content about metric spaces, although they also appear in the previous references, we suggest the reader to see [24] since it is dedicated to this kind of spaces in particular. Finally, for Stone spaces we resort the reader to [12].

TOPOLOGICAL SPACES

Definition 2.42. *Given a set* X*, a* topology T *on* X *is a set of subsets of* X *satisfying the following properties:*

i) $X \in T$ *and* $\varnothing \in T$*.*

ii) For every $U \subseteq T$ *we have* $\cup U \in T$*.*

iii) For every finite $U \subseteq T$ *we have* $\cap U \in T$*.*

The sets in T are called *open sets*.

Definition 2.43. *Given a set* X *and a topology on* X*, the pair* (X, T) *is a* topological space.

Sometimes, one wants to compare topologies defined on a same set X, so we introduce the following definition.

Definition 2.44. *Let X be a nonempty set and T_1 and T_2 topologies on X. We say the topologies T_1 and T_2* coincide *if $T_1 = T_2$, i.e, for every $Y \subseteq X$ we have $Y \in T_1$ if and only if $Y \in T_2$.*

There are many different possible definitions for a closed set, however, we consider the more usual one.

Definition 2.45. *Let (X, T) be a topological space. Then, a set $A \subseteq X$ is* closed *if and only if $X \setminus A \in T$. Moreover, if a set $A \subseteq X$ is open and closed at the same time, we say that A is* clopen.

Lemma 2.46 ([18] Proposition 17.1)**.** *Let (X, T) be a topological space. Then, the following hold:*

i) X and $\varnothing$ are closed.

ii) Finite unions of closed sets are closed.

iii) Arbitrary intersections of closed sets are closed.

Definition 2.47. *Let (X, T) be a topological space and $A \subseteq X$. If $I_A := \{C \subseteq X : C \text{ is closed and } A \subseteq C\}$, we define the* closure of *A as $cl(A) := \bigcap_{C \in I_A} C$.*

Remark 2.48. It follows immediately from Definition 2.47 that for every $A \subseteq X$ it holds $A \subseteq cl(A)$.

Definition 2.49. *Let (X, T) be a topological space. A* basis *for the topology on X is a collection $\mathcal{B}$ of subsets in T such that every $U \in T$ is the union of elements of $\mathcal{B}$.*

In general, whenever we have a set X whithout a topology and a collection $\mathcal{B}$ of subsets of X satifying the properties

i) For each $x \in X$ there is some $B \in \mathcal{B}$ such that $x \in B$,

ii) For each $x \in X$, if $x \in B_1 \cap B_2$ for some $B \in \mathcal{B}$, then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$,

we can define a topology T on X by means of $\mathcal{B}$ as follows: for every $U \subseteq X$ we say that $U \in T$ if for each $x \in U$ there is some $B \in \mathcal{B}$ such that $x \in B$. That is, $\mathcal{B}$ is a basis of the called *topology generated by* $\mathcal{B}$. In fact, a basis is sometimes defined as a collection of open subsets of a topological space (X, T) satisfying the properties i) and ii). For more details see [18] (pg. 76-77).

Definition 2.50. *Let (X, T) be a topological space. A* subbasis *for the topology on X is a collection $\mathcal{B}$ of subsets in T such that the collection $\mathcal{B}'$ of finite intersections of members in $\mathcal{B}$ is a base.*

This definition means that if we have a subbasis $\mathcal{B}$ of a topological space (X, T), then every open set is equal to an arbitrary union of finite intersections of members in $\mathcal{B}$. Moreover, if we have a set X and a collection $\mathcal{B}$ of subsets of X such that $X = \bigcup_{B \in \mathcal{B}} B$, then we can endow X with a topology T such that $\mathcal{B}$ is a subbasis (see [18] pg. 82).

Definition 2.51. *Given a topological space (X, T) and some $Y \subseteq X$, we say that Y is a* dense *subset if and only if for every $U \in T$ we have that $Y \cap U \neq \varnothing$.*

Lemma 2.52. *Let (X, T) be a topological space and a dense subset $A \subseteq X$. Then, $\bar{A} = X$.*

Proof. The inclusion $\bar{A} \subseteq X$ is trivial, so we need to focus on the other one. Assume towards a contradiction that $X \setminus \bar{A} \neq \varnothing$. Observe that, by definition, $\bar{A}$ is closed, so $X \setminus \bar{A}$ is open. Then, by Remark 2.48 we deduce that $A \cap X \setminus \bar{A}$. However, this is not possible since A is dense in X; hence, we conclude that $\bar{A} = X$. ⊠

Definition 2.53. *Given a topological space (X, T), a family $\mathcal{U}$ of open sets is called* open covering *of X if $X = \cup_{U \in \mathcal{U}} U$.*

Definition 2.54. *We say that a topological space (X, T) is* compact *if for every open covering $\mathcal{U}$ of X there is a finite family $\mathcal{U}' \subseteq \mathcal{U}$ such that $X = \cup_{U \in \mathcal{U}'} U$*

Notice that an immediate consequence of the definition is that every finite topological space is compact.

Lemma 2.55 ([18] Theorem 26.2)**.** *Let (X, T_X) be a compact topological space. If $C \subseteq X$ is closed, then it is also compact.*

Definition 2.56. *We say that a topological space (X, T) is* Hausdorff *if for every pair of distinct elements $x, y \in X$ there are some open sets U_x and U_y containing x and y respectively, and such that $U_x \cap U_y = \varnothing$.*

Lemma 2.57 ([18] Theorem 26.3)**.** *Let (X, T_X) be a Hausdorff space. If $C \subseteq X$ is compact, then it is closed.*

Usually Hausdorff spaces are also called T_2 spaces. In fact, this notation is adopted to name many kind of topological spaces coming from different separation axioms. For instance, *Kolmogorov spaces* are also called T_0 and are formally defined as follows.

Definition 2.58. *A topological space (X, T) is T_0 if for every $x, y \in X$ if $x \neq y$ then there is an open set $U \in T$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.*

Notice that every T_2 space is automatically a T_0 space.

Definition 2.59. *A topological space (X, T) is* zero-dimensional *if it has a basis of clopen sets.*

There are many examples and types of topological spaces, but a few of them will be considered along this text. Nevertheless, they are basic examples

Example 2.60. ***Discrete topology:*** *Given a set X, the set of all subsets of X compounds a topology T_D on X and it is called the* discrete topology. *Notice that the set $\{\{x\} : x \in X\}$ is a basis for this topology. The discrete topology on a set X is an example of a zero-dimensional Hausdorff space. It is clear that it is T_2* [II]. *To see that it is zero-dimensional, recall that every subset U of X is open and therefore $X \setminus U$ is open. Since $U = X \setminus (X \setminus U)$ we obtain that U is also closed. Hence, in particular, the singletons are clopen sets and (X, T_D) is zero-dimensional.*

Example 2.61. ***Product topology:*** *Let I be some set of indexes and let (X_i, T_i) be a topological space for each $i \in I$. Considering the cartesian product $\prod_{i\in I} X_i$, for each $i \in I$ we denote with p_i the* natural projection functions *$p_i : \prod_{i\in I} X_i \to X_i$. The* product topology *on $\prod_{i\in I} X_i$, denoted by T_P, is the topology generated by the basis $\{\prod_{i\in I} U_i : \forall i \in I \;\; U_i \in T_i\}$. In fact, it can be proven that the collection*

$$\bigcup_{i\in I}\{p_i^{-1}[U] : U \in T_i\}$$

is a subbasis for the space $(\prod_{i\in I} X_i, T_P)$ (see [18] pg. 88 Theorem 15.2).
Moreover, if we have a basis $\mathcal{B}_i$ for each (X_i, T_i) the collection

$$\bigcup_{\substack{J\subseteq I\\ |J|<\omega}} \{\prod_{j\in J} B_j \times \prod_{i\in I\setminus J} X_i : \forall j \in J \;\; B_j \in \mathcal{B}_j\} \tag{2.1}$$

forms a basis of $(\prod_{i\in I} X_i, T_P)$ (see [18] pg. 116 Theorem 19.2).

Example 2.62. ***Subspace topology:*** *Given a topological space (X, T) and a subset $Y \subseteq X$, the* subspace topology *(Y, T_Y) is defined by*

$$T_Y := \{Y \cap U : U \in T\}.$$

Moreover, if we have a basis $\mathcal{B}$ of (X, T) then the collection $\{B \cap Y : B \in \mathcal{B}\}$ is a basis of (Y, T_Y) ([18] pg. 87 Lemma 16.2).

We state now some properties of these spaces.

Proposition 2.63 ([18] Theorem 16.3)**.** *Let (X, T_X) and (Y, T_Y) be topological spaces and $A \subseteq X$ and $B \subseteq Y$. Then, in $A \times B$ the subspace topology inherited from the product $X \times Y$ (with the product topology) coincides with the product topology of the subspaces topologies in A and B respectively.*

[II] For every $x, y \in X$ such that $x \neq y$ just take $U_x := \{x\}$ and $U_y := \{y\}$.

Proposition 2.64 ([18] Theorem 17.11)**.**

- *Let (X, T) be a Hausdorff space and $Y \subseteq X$. The space (Y, T_Y) (with the subspace topology) is Hausdorff.*
- *Let (X_i, T_i) be Hausdorff spaces for some set I and every $i \in I$. Then, the space $(\prod_{i \in I} X_i, T_P)$ is Hausdorff.*

Theorem 2.65 (Tychonoff)**.** *Let (X_i, T_i) be compact spaces for some set I and every $i \in I$. Then, the space $(\prod_{i \in I} X_i, T_P)$ is compact.*

Proposition 2.66. *([18] pg. 165 Theorem 26.2) Let (X, T) be a compact space and $Y \subseteq X$ a closed set. Then, the space (Y, T_Y) (with the subspace topology) is compact.*

CONTINUOUS FUNCTIONS

Definition 2.67. *Let (X, T_X) and (Y, T_Y) be topological spaces. A function $f : (X, T_X) \to (Y, T_Y)$ is continuous if for every $U \in T_Y$ we have $f^{-1}[U] \in T_X$.*

Proposition 2.68. *Let (X, T_X) and (Y, T_Y) be topological spaces and $f : (X, T_X) \to (Y, T_Y)$ a function. If $\mathcal{B}_Y$ is a basis of Y, then the following are equivalent:*

i) f is continuous.

ii) For every $B \in \mathcal{B}_Y$ we have $f^{-1}[B] \in T_X$.

iii) For every closed subset A of Y, $f^{-1}[A]$ is closed in X.

Lemma 2.69 ([25] Proposition 8.4)**.** *Let (X, T_X), (Y, T_Y) and (Z, T_Z) be topological spaces and let $f : (X, T_X) \to (Y, T_Y)$ and $g : (Y, T_Y) \to (Z, T_Z)$ be continuous functions. Then, the composition $g \circ f : (X, T_X) \to (Z, T_Z)$ is continuous.*

Definition 2.70. *Let (X, T_X) and (Y, T_Y) be topological spaces and $f : (X, T_X) \to (Y, T_Y)$ a continuous function. We say that f is* closed *if for every closed subset of X then $f[X]$ is closed.*

Lemma 2.71 ([18] Theorem 26.5)**.** *Let (X, T_X) and (Y, T_Y) be topological spaces and $f : (X, T_X) \to (Y, T_Y)$ a continuous function. If $C \subseteq X$ is compact then $f[C]$ is compact.*

Proposition 2.72. *Let (X, T_X) be a compact space, (Y, T_Y) a Hausdorff space and $f : X \to Y$ a continuous function. Then, f is a closed function.*

Proof. Let $C \subseteq X$ be a closed subset. Since (X, T_X) is compact, by Lemma 2.57 the set C is compact. Then, by Lemma 2.71 we have that $f[C]$ is compact and therefore, by Lemma 2.55 $f[C]$ is closed. ⊠

Lemma 2.73. *Let (X_1, T_1) and (X_2, T_2) be topological spaces. Then, the projection function $p_i : X_1 \times X_2 \to X_i$ is a continuous surjective function for each $i \in \{1, 2\}$.*

Proof. We will proof that p_1 is continuous and surjective. The case for p_2 is completely identical. Let $U \in T_1$. Then, $p_1^{-1}[U] = U \times X_2$ and therefore it is open because both U and X_2 are open in the topologies T_1 and T_2 respectively. Hence, p_1 is continuous. Lastly, notice that for every $x \in X_1$ we have $p_1(x, y) = x$ where y is an arbitrary element in X_2. Therefore, p_1 is surjective. ⊠

Remark 2.74. Notice that Lemma 2.73 can be easily generalized to arbitrary products (the proof is analogous to Lemma 2.73).

Lemma 2.75 ([25] Proposition 10.11)**.** *Let $(X_1, T_1), (X_2, T_2)$ and (Y, T_Y) be topological spaces. A mapping $f : Y \to X_1 \times X_2$ is continuous if and only if the mappings $p_1 \circ f : Y \to X_1$ and $p_2 \circ f : Y \to X_2$ are continuous.*

Proposition 2.76 ([6] pg. 76 Corollary 1)**.** *Let (X, T_X) and (Y, T_Y) be topological Hausdorff spaces, $f, g : (X, T_X) \to (Y, T_Y)$ be continuous function and $Y \subseteq X$ a dense subset. If for every $y \in Y$ we have $f(y) = g(y)$ then $f = g$.*

Lemma 2.77. *Let (X, T_X) and (Y, T_Y) be discrete topological spaces. Then, the product $(X \times Y, T_P)$ is a discrete topological space.*

Proof. It suffices to check that the singletons of members in $X \times Y$ are open [III]. Fix some arbitrary $(x, y) \in X \times Y$. Certainly, $\{(x, y)\} = p_1^{-1}[\{x\}] \cap p_2^{-1}[\{y\}]$ where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projection functions. Since these functions p_1 and p_2 are continuous (Lemma 2.73), and since by assumption $\{x\}$ and $\{y\}$ are open sets in the topologies T_X and T_Y respectively, we have that $\{(x, y)\}$ is open since it is a finite intersection of open sets. ⊠

From Lemma 2.77 we deduce the following corollary.

Corollary 2.78. *Any finite product of discrete topologies is also discrete.*

Lemma 2.79. *Let (X_i, T_i) be zero-dimensional spaces for some set I and every $i \in I$. Then, the space $(\prod_{i \in I} X_i, T_P)$ is zero-dimensional.*

Proof. We give a proof sketch. We have a basis $\mathcal{B}_i$ of clopen sets of (X_i, T_i) for each $i \in I$ by assumption, so we just need to check that the members of the basis 2.1 are clopen. To this end, use Proposition 2.68 with respect to each projection map p_i and then apply Lemma 2.46. ⊠

[III] If this is the case, then every subset of $X \times Y$ is open since it is equal to the union of all the singletons of its members.

Sometimes, algebra and topology meet and we can endow the universe of an algebra $\mathbf{A}$ of type τ with a topology. Under some assumptions, we call these structures *topological algebras*.

Definition 2.80. *Let $\mathbf{A}$ be an algebra of type τ and (A, T) a topological space. We say that $\mathbf{A}$ is a* topological algebra *if for each $\sigma \in \tau$ the mapping $\sigma^{\mathbf{A}} : A^{n_\sigma} \to A$ is continuous (with the prodcut topology on A^{n_σ}).*

Proposition 2.81. *Let $\mathbf{A}$ be a topological algebra (with the topology T) and $\mathbf{B} \leqslant \mathbf{A}$. Then, $\mathbf{B}$ is also a topological algebra with respect to the subspace topology T_S.*

Proof. We want to check that for every $\sigma \in \tau$ the mapping $\sigma^{\mathbf{B}}$ is continuous, so pick some $\sigma \in \tau$ and assume without loss of generality that it is a m-ary symbol. Let $U \subseteq B$ be an open set so that U is of the form $B \cap V$ for some $V \subseteq A$ and $A \in T$. Notice that

$$
\begin{aligned}
(\sigma^{\mathbf{B}})^{-1}[U] &= \{(b_1, \dots, b_m) \in B^m : \sigma^{\mathbf{B}}(b_1, \dots, b_m) \in U\} \\
&= \{(b_1, \dots, b_m) \in B^m : \sigma^{\mathbf{B}}(b_1, \dots, b_m) \in B \cap V\} \\
&= \{(b_1, \dots, b_m) \in B^m : \sigma^{\mathbf{A}}(b_1, \dots, b_m) \in B \cap V\} \\
&= B^m \cap \{(b_1, \dots, b_m) \in A^m : \sigma^{\mathbf{A}}(b_1, \dots, b_m) \in V\} \\
&= B^m \cap (\sigma^{\mathbf{A}})^{-1}[V]
\end{aligned}
$$

By hypothesis we know that the set $(\sigma^{\mathbf{A}})^{-1}[V]$ is open in $\mathbf{A}^m$ and therefore, by Proposition 2.63 the set $B^m \cap (\sigma^{\mathbf{A}})^{-1}[V]$ is open in $\mathbf{B}^m$. Hence, $(\sigma^{\mathbf{B}})^{-1}[U]$ is open and we are done. ⊠

METRIC SPACES

Definition 2.82. *A* metric *on a set X is a function $d : X^2 \to \mathbb{R}$ satisfying the following properties:*

i) For every $x \in X$ it holds $d(x, x) = 0$.

ii) For every $x, y \in X$, if $x \neq y$ then $d(x, y) > 0$.

iii) For every $x, y \in X$ it holds $d(x, y) = d(y, x)$.

iv) (Triangle inequality) For every $x, y, z \in X$ it holds $d(x, z) \leqslant d(x, y) + d(y, z)$.

Definition 2.83. *If d is a metric on a set X, we say that (X, d) is a* metric space.

Remark 2.84. Given a set X and a metric d over it, if we define the *balls* $B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}$, then the collection $T := \{B_\varepsilon(x) : x \in X \text{ and } \varepsilon > 0\}$ defines a topology over X (see [24]) denoted by T_X^d.

We say that a metric space is *compact* if the topological space that it defines (as above) is compact.

Definition 2.85. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : (X, d_X) \to (Y, d_Y)$ is* uniformly continuous *if for every $\varepsilon > 0$ there is some $\delta > 0$ such that for every $x, y \in X$ if $d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.*

Proposition 2.86. *[Heine-Cantor] ([18] pg. 176 theorem 27.6) Let (X, d_X) and (Y, d_y) be a metric spaces and (X, d_x) compact. If a function $f : (X, d_X) \to (Y, d_Y)$ is continuous then it is uniformly continuous.*

Definition 2.87. *An* ultrametric *on a set X is a function $d : X^2 \to \mathbb{R}$ satisfying the following properties:*

i) For every $x \in X$ it holds $d(x, x) = 0$.

ii) For every $x, y \in X$, if $x \neq y$ then $d(x, y) > 0$.

iii) For every $x, y \in X$ it holds $d(x, y) = d(y, x)$.

iv) (Strong triangle inequality) For every $x, y, z \in X$ it holds $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$.

See [21] for more details about ultrametrics and ultrametric spaces.

Definition 2.88. *If d is an ultrametric on a set X, we say that (X, d) is an* ultrametric space*.*

Observe that every ultrametric space is in fact a metric space since the strong triangle inequality implies the triangle inequality.

STONE SPACES

Stone spaces conform another example of topological spaces in which we will be concerned. This spaces, although they are not as simple as the examples considered before, they are quite relevant since they link topology, algebra and logic thanks to the *Stone Representation Theorem* (see section 4 of chapter IV of [7]).

Definition 2.89. *A topological space (X, T) is called a* Stone space *if it is compact, T_0 and zero-dimensional.*

Proposition 2.90. *Let (X, T) be a Stone space and $Y \subseteq X$. If Y is closed in (X, T) then topological space (Y, T_Y) is a Stone space.*

Proof. By Proposition 2.64 and Proposition 2.66 we obtain that (Y, T_Y) is Hausdorff and compact, so it suffices to check that it is zero-dimensional. To this end, let $Y \cap U$ be an arbitrary member of the basis of (Y, T_Y). We need to prove that $Y \cap U$ is closed, i.e., that $Y \setminus (Y \cap U)$ is open. Notice that $Y \setminus (Y \cap U) = Y \cap (X \setminus U)$. Now, by assumption we know that U is closed in (X, T) and therefore $X \setminus U$ is open in (X, T). Thus, $Y \cap (X \setminus U)$ is open in (Y, T_Y) and we are done. ⊠

Example 2.91. Given a set I of indexes, possibly infinite, and for each $i \in I$ a finite space (X_i, T_i) with the discrete topology, the product space $\prod_{i \in I} X_i$ with the product topology forms a compact, Hausdorff and zero-dimensional space. This follows from the facts that each X_i is compact, Hausdorff and zero-dimensional space and using the Tychonoff theorem (2.65), Proposition 2.64 and Lemma 2.79. Consequently, $(\prod_{i \in I} X_i, T_p)$ is a Stone space.

CHAPTER 3

PSEUDOVARIETIES AND THEIR IMPLICIT OPERATIONS

This chapter can be divided in two different part. On the one hand, we will introduce pseudovarieties and their respective implicit operations. Some algebraic properties will be also stated and proved regarding the nature of these objects. More precisely, we will endow the set of all the implicit operations of a given pseudovariety $\mathcal{V}$ with the structure of an algebra and study some of its basic properties. On the other hand, topological features will be considered, namely, a topology over the algebra mentioned above. This will be crucial on the study and the proof of Reiterman's Theorem later on.

3.1 Pseudovarieties and the algebra of implicit operations

Definition 3.1. *We call a class of similar finite algebras $\mathcal{K}$ a* pseudovariety *if it is closed under homomorphic images, finite products, and subalgebras.*

Definition 3.2. *Let $\mathcal{V}$ be a pseudovariety of type τ and $\mathcal{W} \subseteq \mathcal{V}$. We say that $\mathcal{W}$ is a* subpseudovariety of *$\mathcal{V}$ if $\mathcal{W}$ is itself a pseudovariety.*

We define the pseudovariety generated by some class of finite similar algebras $\mathcal{K}$ as the least pseudovariety containing it and we denote it by $p\mathbb{V}(\mathcal{K})$. If $\mathcal{K}$ is finite, we say that it is *finitely generated*, and if $\mathcal{K} = \{\mathbf{A}_1, \ldots, \mathbf{A}_m\}$, then we write for the sake of simplicity $p\mathbb{V}(\mathbf{A}_1, \ldots, \mathbf{A}_m)$ instead of $p\mathbb{V}(\{\mathbf{A}_1, \ldots, \mathbf{A}_m\})$. Moreover, such as for varieties, the existence of such pseudovariety is justified by the following proposition, which is analogous to the Tarki's HPS theorem ([26]). However, in order to prove it, we will make use of the construction involving homomorphisms from below.

If we are considering some homomorphism $h : \mathbf{A} \to \mathbf{B}$ for some algebras $\mathbf{A}$ and $\mathbf{B}$, we can define the homomorphism $h^n : \mathbf{A}^n \to \mathbf{B}^n$ by the rule $h^n(a_1, ..., a_n) = (h(a_1), ..., h(a_n))$. The fact that it is a homomorphism is a easy verification. In fact, we can consider more genarally the algebras $\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}_1, \ldots, \mathbf{B}_n$ and for each $i \leqslant n$ a homomorphism $h_i : \mathbf{A}_i \to \mathbf{B}_i$, and define the homomorphism $\langle h_1, \ldots, h_n \rangle : \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \to \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$ by the rule $\langle h_1, \ldots, h_n \rangle(a_1, \ldots, a_n) = (h_1(a_1), \ldots, h_n(a_n))$. Again, to check that it is a homomorphism is an easy verification.

Having this in mind, we are in a good shape to prove the next proposition.

Proposition 3.3. *Let $\mathcal{K}$ be a class of finite similar algebras. Then,*

$$p\mathbb{V}(\mathcal{K}) = \mathbb{HSP}_{fin}(\mathcal{K})$$

Proof. We first check that $\mathbb{HSP}_{fin}(\mathcal{K})$ is a pseudovariety. Recall that, if we have an algebra $\mathbf{A}$ then $\mathbf{A} \in \mathbb{HSP}_{fin}(\mathcal{K})$ if and only if there are some $\mathbf{B}_1, \ldots, \mathbf{B}_k \in \mathcal{K}$ (for some $k < \omega$) such that $\mathbf{A}$ is a homomorphic image of some $\mathbf{D} \leqslant \mathbf{B}_1 \times \cdots \times \mathbf{B}_k$.

The inclusion $\mathbb{H}\mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K}) \subseteq \mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K})$ is almost trivial since we only need to take the composition of both homomorphism. The inclusion $\mathbb{S}\mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K}) \subseteq \mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K})$ follows from the fact that $\mathbb{S}\mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K}) \subseteq \mathbb{H}\mathbb{S}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K}) \subseteq \mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K})$(see [7] Lemma 9.2).

To prove that it is closed under finite products, pick some $\mathbf{A}_1, \ldots, \mathbf{A}_k \in \mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K})$. Then, there are for each $i \leqslant k$ an algebra $\mathbf{D}_i \leqslant \mathbf{B}^i_1 \times \cdots \times \mathbf{B}^i_{r_i}$ and a surjective homomorphism $h_i : \mathbf{D}_i \to \mathbf{A}_i$. Thus, we have $\mathbf{D}_i \times \cdots \times \mathbf{D}_k \leqslant \prod_{i \leqslant k} \mathbf{B}^i_1 \times \cdots \times \mathbf{B}^i_{r_i}$ and a homomorphism $\langle h_1, \ldots, h_k \rangle : \mathbf{D}_1 \times \cdots \times \mathbf{D}_k \to \mathbf{A}_1 \times \cdots \times \mathbf{A}_k$ described before. The surjectivity of h' follows from the surjectivity of each h_i. Therefore, we conclude that $\mathbf{A}_1 \times \cdots \times \mathbf{A}_k \in \mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K})$.

We claim now that $\mathbb{H}\mathbb{S}\mathbb{P}_{fin}(\mathcal{K})$ is the least pseudovariety containing $\mathcal{K}$. Of course, notice that if $\mathcal{W}$ is an arbitrary pseudovariety containing $\mathcal{K}$, then, ⊠

From now on, we will fix an arbitrary pseudovariety $\mathcal{V}$ of a type τ and some $n \in \omega$. Notice that a pseudovariety is also closed under the operator $\mathbb{I}$ since $\mathbb{I}(\mathcal{V}) \subseteq \mathbb{H}(\mathcal{V})$. Therefore, and since the cardinality of $\mathbb{I}(\mathcal{V})$ can not be bounded by any cardinal, any pseudovariety is a proper class [I]. This is something we want to avoid in the definition that follows, so we restrict to the set of representative of the class $\mathbb{I}(\mathcal{V})$, which we denote by $\mathcal{V}_0$. Observe that since we are working with finite algebras, we have the equality $\mathcal{V}_0 = \cup_{n \in \omega} \mathcal{V}^n_0$ where $\mathcal{V}^n_0 = \{\mathbf{A} \in \mathcal{V}_0 : |A| = n\}$. Then, since there are only finitely non-isomorphic finite-many algebras of size n (because τ is assumed to be finite), we have $|\mathcal{V}^n_0| < \aleph_0$ for each $n \in \omega$ and therefore $|\mathcal{V}_0| \leqslant \aleph_0$.

Definition 3.4. *An n-ary* implicit operation *over $\mathcal{V}$ is a tuple $f = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathcal{V}_0 \rangle$ of functions $f^{\mathbf{A}} : \mathbf{A}^n \to \mathbf{A}$ such that for every homomorphism $h : \mathbf{B} \to \mathbf{C}$ between members in $\mathcal{V}_0$ we have $h \circ f^{\mathbf{B}} = f^{\mathbf{C}} \circ h^n$.*

The spirit of this definition is to weaken the idea of term-definable functions, that is, an implicit operation may be a mapping commuting with homomorphisms that can not be expressed with the operations of the language. On the other side, every term defines an implicit operation (as we will see later).

Definition 3.5. *We define the operator $\bar{\Omega}_n$ which assigns to a given pseudovariety $\mathcal{V}$ the set of n-ary implicit operations over $\mathcal{V}$. We will denote this set with $\bar{\Omega}_n\mathcal{V}$.*

[I]This follows from the question of whether the collection of sets of a certain cardinality is a set or a proper class. The answer is that it is a proper class. For instance, the collection of set of cardinality 1 is already a proper class, just consider the bijection from the class of all sets to the collection of sets of cardinality 1 that sends a set x to the singleton $\{x\}$.

Definition 3.6. *Let $\mathcal{V}$ be a pseudovariety. We define the algebra $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} = \langle \bar{\Omega}_n \mathcal{V}, \tau \rangle$ where for each m-ary operation $\sigma \in \tau$ and every $f_1, ..., f_m \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$, the m-ary implicit operation $\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}} f_1, ..., f_m$ is defined for each algebra $\mathbf{A} \in \mathcal{V}_0$ by the rule*

$$(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}} f_1, ..., f_m)^{\mathbf{A}}(a_1, ..., a_n) = \sigma^{\mathbf{A}}(f_1^{\mathbf{A}}(a_1, ..., a_n), ..., f_m^{\mathbf{A}}(a_1, ..., a_n)).$$

It is not difficult to check that $\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, ..., f_m)$ defined as above is indeed an implicit operation. It is a straightforward verification of the commutative property using that each f_i satisfies it.

Notice that for each $i \leqslant n$, we can define an n-ary implicit operation $\hat{x}_i = \langle \hat{x}_i^{\mathbf{A}} : \mathbf{A} \in \mathcal{V}_0 \rangle$ where $\hat{x}_i^{\mathbf{A}}$ is the i-th projection function, that is, for every $a_1, \ldots, a_n \in A$ the map $\hat{x}_i^{\mathbf{A}}$ is defined as $\hat{x}_i^{\mathbf{A}}(a_1, \ldots, a_n) = a_i$. Due to obvious reasons, we will call these implicit operations *projections*. But, of course, it remains to check that for every homomorphism $h : \mathbf{B} \to \mathbf{C}$ between member in $\mathcal{V}_0$ it holds $h \circ \hat{x}_i^{\mathbf{B}} = \hat{x}_i^{\mathbf{C}} \circ h^n$.

Lemma 3.7. *For each $i \leqslant n$, the map $\hat{x}_i$ is an implicit operation over $\mathcal{V}$.*

Proof. Consider a homomorphism h between some $\mathbf{A}, \mathbf{B} \in \mathcal{V}_0$. Then, for every $a_1, \ldots, a_n \in A$ we have, on the one hand,

$$\begin{aligned} h \circ \hat{x}_i^{\mathbf{A}}(a_1, ..., a_n) &= h(\hat{x}_i^{\mathbf{A}}(a_1, \ldots, a_n)) \\ &= h(a_i) \end{aligned}$$

and on the other hand,

$$\begin{aligned} \hat{x}_i^{\mathbf{B}} \circ h^n(a_1, \ldots, a_n) &= \hat{x}_i^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) \\ &= h(a_i) \end{aligned}$$

as desired. ⊠

Moreover, we claim that every term $t(x_1, \ldots, x_n)$ induces a n-ary implicit operation $\langle t^{\mathbf{A}} : \mathbf{A} \in \mathbf{V}_0 \rangle$ if we thought each variable x_i as the projection $\hat{x}_i$ described above. In this case, Lemma 3.7 becomes a particular case of the following lemma.

Lemma 3.8. *Every term $t(x_1, \ldots, x_n)$ in the language τ induces an n-ary implicit operation over $\mathcal{V}$.*

Proof. The base case is just Lemma 3.7. For the inductive case, we have a term of the form

$$t(x_1, \ldots, x_n) = \sigma(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n))$$

for some m-ary operation f and such that for each $i \leqslant m$ the term t_i induces an implicit operation. To show that $\langle t^{\mathbf{A}} : \mathbf{A} \in \mathcal{V}_0 \rangle$ is an implicit operation, consider some homomorphism $h : \mathbf{A} \to \mathbf{B}$ and fix some arbitrary $a_1, \ldots, a_n \in A$. On the one hand,

$$\begin{aligned} h \circ t^{\mathbf{A}}(a_1, \ldots, a_n) &= h(\sigma^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots t_m^{\mathbf{A}}(a_1, \ldots, a_n))) \\ &= \sigma^{\mathbf{B}}(h(t_1^{\mathbf{A}}(a_1, \ldots, a_n)), \ldots, h(t_m^{\mathbf{A}}(a_1, \ldots, a_n))) \end{aligned}$$

since h is a homomorphism. Moreover, by induction hypothesis we also have

$$\sigma^{\mathbf{B}}(h(t_1^{\mathbf{A}}(a_1, \ldots, a_n)), \ldots, h(t_m^{\mathbf{A}}(a_1, \ldots, a_n))) = \sigma^{\mathbf{B}}(t_1^{\mathbf{B}}(h(a_1), \ldots, h(a_n)), \ldots, t_m^{\mathbf{B}}(h(a_1), \ldots, h(a_n))).$$

On the other hand,

$$t^{\mathbf{B}} \circ h^n(a_1, \ldots, a_n) = t^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \sigma^{\mathbf{B}}(t_1^{\mathbf{B}}(h(a_1), \ldots, h(a_n)), \ldots, t_m^{\mathbf{B}}(h(a_1), \ldots, h(a_n))).$$

Therefore, we have $h \circ t^{\mathbf{A}} = t^{\mathbf{B}} \circ h^n$.

⊠

For the sake of clarity, given a term $t(x_1, \ldots, x_n)$,, the n-ary implicit operation induced by t described above will be denoted by $\hat{t}$. We illustrate now the previous lemma using the next simple example.

Example 3.9. *Let $\mathcal{V}$ be an arbitrary pseudovariety of semigroups and let $t(x, y) := x \cdot y$ be a term. Then, $\langle t^{\mathbf{A}} : \mathbf{A} \in \mathcal{V}_0 \rangle$ is a binary implicit operation over $\mathcal{V}$. To see how it is computed, fix an arbitrary semigroup $\mathbf{A} \in \mathcal{V}_0$ and let a and b be some elements in A. Then,*

$$\begin{aligned} t^{\mathbf{A}}(a, b) &= \cdot^{\mathbf{A}}(x^{\mathbf{A}}(a, b), y^{\mathbf{A}}(a, b)) \\ &= \cdot^{\mathbf{A}}(a, b) \end{aligned}$$

Definition 3.10. *We define ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ as the subalgebra of $\bar{\Omega}_n^{\mathcal{V}}$ generated by $\{\hat{x}_1, \ldots, \hat{x}_n\} \subseteq \bar{\Omega}_n\mathcal{V}$. The members in ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ are called* n-ary explicit operations *and the universe of $\Omega_n^{\mathcal{V}}$ is denoted by $\Omega_n\mathcal{V}$.*

Remark 3.11. Notice that by Lemma 2.8 we have that

$$\Omega_n\mathcal{V} = \{t^{\bar{\Omega}_n\mathcal{V}}(\hat{x}_1, \ldots, \hat{x}_n) : t \in T(x_1, \ldots, x_n)\}.$$

Lemma 3.12. *Let h be a mapping from $\{x_1, \ldots, x_n\}$ to ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ asigning x_i to $\hat{x}_i$ for each $i \leqslant n$. Then there is a unique homomorphism $\bar{h} : \mathbf{T}(x_1, \ldots, x_n) \to {\Omega_{\mathbf{n}}}^{\mathcal{V}}$ extending h.*

Proof. From Theorem 2.25 we know that such homomorphism exists, and by Remark 2.22 it is unique. The surjectivity follows from Remark 3.11 since for every $f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}}$ there is some term $p \in \mathbf{T}(x_1, \ldots, x_n)$ such that $f = p^{\bar{\Omega}_n\mathcal{V}}(\hat{x}_1, \ldots, \hat{x}_n)$ and therefore $\bar{h}(p) = f$ (see the footnote of Theorem 2.25). ⊠

Definition 3.13. *Let $f, g \in \bar{\Omega}_n^{\mathcal{V}}$. We say that the pair (f, g) is a* pseudoidentity *for $\mathcal{V}$ and we denoted by $f \approx g$. We say that an algebra $\mathbf{A} \in \mathcal{V}_0$ satisfies the pseudoidentity (f, g) if the functions $f^{\mathbf{A}}$ and $g^{\mathbf{A}}$ are equal, and we denote it by $\mathbf{A} \Vdash f \approx g$. Given $\mathcal{K}$ a subclass of $\mathcal{V}_0$ we write $\mathcal{K} \Vdash f \approx g$ if $\mathbf{A} \Vdash f \approx g$ for every $\mathbf{A} \in \mathcal{K}$.*

In general terms, if we do not focus on the parameter n, pseudoidentities for a pseudovariety $\mathcal{V}$ are members in the set $\cup_{n\in\omega}(\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}})^2$. In fact, observe that for each $f \approx g$ we have $\mathcal{V}_0 \Vdash f \approx g$ if and only if $\mathcal{V} \Vdash f \approx g$[II], thus we define the set

$$pId(\mathcal{V}) := \{f \approx g \in \cup_{n\in\omega}(\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}})^2 : \mathcal{V}_0 \Vdash f \approx g\}.$$

In addition, the notation $\Vdash$ is adopted here in order to avoid any ambiguity with the notation $\models$ defined in the preliminaries. Nevertheless, both validity concepts are closely related (and coincide under some assumptions) as the following lemma shows.

Lemma 3.14. *Let $p, q \in \mathbf{T}(x_1, \ldots, x_n)$, then $p \approx q \in Id(\mathcal{V})$ if and only if $\hat{p} \approx \hat{q} \in pId(\mathcal{V})$.*

Proof. Recall that $\hat{p} = \langle p^{\mathbf{A}} : \mathbf{A} \in \mathcal{V}_0\rangle$ and the same for $\hat{q}$. If $p \approx q \in Id(\mathcal{V})$, then for each algebra $\mathbf{A} \in \mathcal{V}$ and every $\vec{a} \in A^n$ we have $p^{\mathbf{A}}(\vec{a}) = q^{\mathbf{A}}(\vec{a})$. Therefore, $\mathcal{V}_0 \Vdash \hat{p} \approx \hat{q}$ and hence $\hat{p} \approx \hat{q} \in pId(\mathcal{V})$. The other implication is symetric. ⊠

Proposition 3.15. *$\Omega_{\mathbf{n}}{}^{\mathcal{V}}$ is isomorphic to $F_n\mathcal{V}$.*

Proof. To construct the isomorphism, let h be a mapping $h : \mathbf{T}(x_1, \ldots, x_n) \to \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ which for every $t \in \mathbf{T}(x_1, \ldots, x_n)$ the image $h(t)$ is the n-ary implicit operation $\hat{t}$ induced by the term t. Clearly this function is well defined. To see that it is an homomorphism, pick some m-ary function symbol σ and terms $t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n)$. Notice that

$$h(\sigma(t_1, \ldots, t_m)) = \langle \sigma^{\mathbf{A}}(t_1^{\mathbf{A}}, \ldots, t_m^{\mathbf{A}}) : \mathbf{A} \in \mathcal{V}_0\rangle .$$

So, for every $\mathbf{A} \in \mathcal{V}_0$ and every $a_1, \ldots, a_n \in A$ we have

$$(h(\sigma(t_1, \ldots, t_m)))^{\mathbf{A}}(a_1, \ldots, a_n) = \sigma^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, t_m^{\mathbf{A}}(a_1, \ldots, a_n))$$

Now, recall that from Definition 3.6 we have for every $\mathbf{A} \in \mathcal{V}_0$ and every $a_1, \ldots, a_n \in A$ the following:

$$\begin{aligned}(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(h(t_1), \ldots, h(t_m)))^{\mathbf{A}}(a_1, \ldots, a_n) &= \sigma^{\mathbf{A}}(h(t_1)(a_1, \ldots, a_n), \ldots, h(t_m)(a_1, \ldots, a_n)) \\ &= \sigma^{\mathbf{A}}(t^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, t^{\mathbf{A}}(a_1, \ldots, a_n))\end{aligned}$$

Therefore, for every $\mathbf{A} \in \mathcal{V}_0$ the equality

$$(h(\sigma(t_1, \ldots, t_m)))^{\mathbf{A}} = (\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(h(t_1), \ldots, h(t_m)))^{\mathbf{A}}$$

[II]This holds since $\mathcal{V}_0$ is the class of isomoprhic members of $\mathcal{V}$.

and hence
$$h(\sigma(t_1,\dots,t_m)) = (\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(h(t_1),\dots,h(t_m))).$$
That is, h is a homomorphism. Notice that, due to Remark 3.11, the image of h is precisely ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$. Moreover, $\ker h = \theta_{\mathcal{V}}$ where $\theta_{\mathcal{V}}$ is the congruence from Definition 2.24. To see this, observe that

$$\begin{aligned}
\ker h &= \{(p,q) \in \mathbf{T}(x_1,\dots,x_n)^2 : h(p) = h(q)\}\\
&= \{(p,q) \in \mathbf{T}(x_1,\dots,x_n)^2 : \hat{p} = \hat{q}\}\\
&= \{(p,q) \in \mathbf{T}(x_1,\dots,x_n)^2 : p^{\mathbf{A}} = q^{\mathbf{A}} \textit{ for all } \mathbf{A} \in \mathcal{V}_0\}\\
&= \{(p,q) \in \mathbf{T}(x_1,\dots,x_n)^2 : \mathcal{V}_0 \Vdash \hat{p} \approx \hat{q}\}\\
&= \{(p,q) \in \mathbf{T}(x_1,\dots,x_n)^2 : \mathcal{V} \Vdash \hat{p} \approx \hat{q}\}\\
&= \{(p,q) \in \mathbf{T}(x_1,\dots,x_n)^2 : \mathcal{V} \models p \approx q\}\\
&= \theta_{\mathcal{V}}.
\end{aligned}$$

The last equality holds due to the fact that $(p,q) \in \theta_{\mathcal{V}}(x_1,\dots,x_n)$ if and only if $\mathcal{V} \models p \approx q$ for every $p,q \in \mathbf{T}(x_1,\dots,x_n)$ (see [1] Proposition 1.3.6). Hence, by the homomorphism theorem (Theorem 2.16), we have $\mathbf{T}(x_1,\dots,x_n)/\theta_{\mathcal{V}} \cong {\Omega_{\mathbf{n}}}^{\mathcal{V}}$ as desired[III]. ⊠

Corollary 3.16. *The algebra ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ has the universal mapping property for $\mathcal{V}$ over $\{\hat{x}_1,\dots,\hat{x}_n\}$.*

Proof. For every $\mathbf{A} \in \mathcal{V}$ and every mapping $h_1 : \{\hat{x}_1,\dots,\hat{x}_n\} \to A$, define the function $h_2 : \{x_1/\theta_{\mathcal{V}},\dots,x_n/\theta_{\mathcal{V}}\} \to A$ sending each $x_i/\theta_{\mathcal{V}}$ to $h_1(\hat{x}_i)$. Since $\mathbf{F_n}\mathcal{V}$ has the universal mapping property for $\mathcal{V}$ over $\{x_1,\dots,x_n\}/\theta_{\mathcal{V}}$, there is a unique homomorphism $h_2' : \mathbf{F_n}\mathcal{V} \to \mathbf{A}$ extending h_2. Let $h : {\Omega_{\mathbf{n}}}^{\mathcal{V}} \to \mathbf{F_n}\mathcal{V}$ be an isomorphism. Then, we just define the mapping $h_1' : {\Omega_{\mathbf{n}}}^{\mathcal{V}} \to \mathbf{A}$ by the rule
$$h_1'(f) = h_2'(h(f)).$$
That is, $h_1' = h_2' \circ h$ and by Lemma 2.11 h_1' is a homomorphism. ⊠

The first lemmas we will prove describe the character of both algebras (${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ and $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$) in relation to the nature and characteristics of the pseudovariety we are considering. In particular, the following Lemma says basically that in case our pseudovariety $\mathcal{V}$ is, in some sense, simple, then we obtain that the algebra ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ falls into $\mathcal{V}$.

Lemma 3.17. *If $\mathcal{V}$ is generated by a single algebra, then for every $n \in \omega$ we have ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \in \mathcal{V}$.*

Proof. Let $\mathbf{A}$ be a finite algebra such that $\mathcal{V} = p\mathbb{V}(\mathbf{A})$ and fix some $n \in \omega$. We can assume without loss of generality that $\mathbf{A} \in \mathcal{V}_0$. We first observe that every implicit operation

III Recall that $F_n\mathcal{V} = \mathbf{T}(x_1,\dots,x_n)/\theta_{\mathcal{V}}$.

$f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ is completely determined by $f^{\mathbf{A}}$. To see the reason behind this, we will compute $f^{\mathbf{B}}$ where $\mathbf{B} \in \mathbb{P}_{fin}(\mathbf{A})$, that is, $\mathbf{B}$ is of the form $\mathbf{A}^m$ for some $m > 0$. The other cases are completely analogous and are left to the reader. For every $j \leqslant m$ let $p_j : \mathbf{A}^m \to \mathbf{A}$ be the j-th natural projection. Then, we have the following diagram

$$\begin{array}{ccc} \mathbf{A}^m \times \cdots \times \mathbf{A}^m & \rightarrow_{f^{\mathbf{B}}} & \mathbf{A}^m \\ \downarrow_{(p_j)^n} & & \downarrow_{p_j} \\ \mathbf{A} \times \cdots \times \mathbf{A} & \rightarrow_{f^{\mathbf{A}}} & \mathbf{A} \end{array}$$

where $(p_j)^n$ is the homomorphism obtained from the construction described at the beginning of this chapter. Due to Definition 3.4, the diagram commutes and therefore $p_j \circ f^{\mathbf{B}} = f^{\mathbf{A}} \circ (p_j)^n$. So, for every $\vec{a}_1, \ldots, \vec{a}_n \in \mathbf{A}^m$ the j-th component of $f^{\mathbf{B}}(\vec{a}_1, \ldots, \vec{a}_n)$ is obtained as follows:

$$\begin{aligned} (f^{\mathbf{B}}(\vec{a}_1, \ldots, \vec{a}_n))_j &= (f^{\mathbf{A}} \circ (p_j)^n)(\vec{a}_1, \ldots, \vec{a}_n) \\ &= f^{\mathbf{A}}((\vec{a_1})_j, \ldots, (\vec{a_n})_j) \end{aligned}$$

Thus, in general we have that

$$f^{\mathbf{B}}(\vec{a}_1, \ldots, \vec{a}_n) = (f^{\mathbf{A}}((\vec{a_1})_1, \ldots, (\vec{a_n})_1), \ldots, f^{\mathbf{A}}((\vec{a_1})_m, \ldots, (\vec{a_n})_m)).$$

Keeping this in mind, we tackle the main proof. The strategy is to define a homomorphism h from ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ to $\mathbf{A}^{\mathbf{A}^{\mathbf{n}}}$ in a natural way by the rule $h(f) = f^{\mathbf{A}}$ for every $f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}}$ [IV]. Notice that, by obvious reasons, we have that $\mathbf{A}^{\mathbf{A}^n} \in \mathcal{V}$; hence, if we prove that h is injective then we obtain by the Theorem 2.16 that ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \cong h[{\Omega_{\mathbf{n}}}^{\mathcal{V}}] \leqslant \mathbf{A}^{\mathbf{A}^n}$ concluding that ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \in \mathcal{V}$ as desired (because, in that case, ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \in \mathbb{HS}(\mathbf{A}^{\mathbf{A}^n}) \subseteq \mathcal{V}$).

It is immediate from the definition of h that it is a well-defined homomorphism [V], so it only remains to show that is is in fact injective. Observe that if $f \neq g$ then necessarily $f^{\mathbf{A}} \neq g^{\mathbf{A}}$; otherwise, since for every $\mathbf{B} \in \mathcal{V}_0$ the mappings $f^{\mathbf{B}}$ and $g^{\mathbf{B}}$ are completely determined by $f^{\mathbf{A}}$ and $g^{\mathbf{A}}$, we have that $f^{\mathbf{B}} = g^{\mathbf{B}}$ and this contradicts the assumption $f = g$. Thus if we have n-ary implicit operations over $\mathcal{V}$ such that $f \neq g$ then

$$h(f) = f^{\mathbf{A}} \neq g^{\mathbf{A}} = h(g)$$

and so, h is injective. We conclude then, as explained in the previous paragraph, that ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \in \mathcal{V}$ ⊠

Moreover, thanks to the following lemma, we can extend the previous result to arbitrary finitely generated pseudovarieties.

[IV] Strictly speaking, $h(f)$ shoud be defined as $f^{\mathbf{A}^{\mathbf{A}^{\mathbf{n}}}}$, but due to the observation done at the beginning of the proof, the mapping $f^{\mathbf{A}}$ is enough to contstruct $f^{\mathbf{A}^{\mathbf{A}^{\mathbf{n}}}}$.

[V] If we have $f, g \in {\Omega_{\mathbf{n}}}^{\mathcal{V}}$ such that $f = g$, then necessarily $f^{\mathbf{A}} = g^{\mathbf{A}}$.

Lemma 3.18. *If $\mathcal{V}$ is finitely generated, then it is also generated by a single algebra.*

Proof. Let $\mathcal{V} = p\mathbb{V}(\mathbf{A}_1, \ldots, \mathbf{A}_m)$. We claim that $\mathcal{V} = p\mathbb{V}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_m)$. The inclusion $\mathcal{V} \supseteq p\mathbb{V}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_m)$ is trivial since $\mathbf{A}_1 \times \cdots \times \mathbf{A}_m \in \mathcal{V}$, so all we need to do is to prove that $\mathcal{V} \subseteq p\mathbb{V}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_m)$. Moreover, to see that it suffices to check that $\mathbf{A}_i \in p\mathbb{V}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_m)$ for each $i \leqslant m$. But this is quite clear once we notice that the projection mappings $p^i : \mathbf{A}_1 \times \cdots \times \mathbf{A}_m \to \mathbf{A}_i$ are surjective homomorphisms. Therefore, for every $i \leqslant m$ we have $\mathcal{A}_i \in p\mathbb{V}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_m)$ and hence $\mathcal{V} \subseteq p\mathbb{V}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_m)$. ⊠

Then, from Lemma 3.17 and Lemma 3.18 we deduce the following corollary.

Corollary 3.19. *If $\mathcal{V}$ is finitely generated, then for any $n \in \omega$ we have ${\Omega_\mathbf{n}}^{\mathcal{V}} \in \mathcal{V}$.*

Continuing with finitely generated pseudovarieties, we have another helpful characterization for $\bar{\Omega}_\mathbf{n}^{\mathcal{V}}$. But first, we need the following lemma to prove it.

Lemma 3.20. *If $\mathcal{V}$ is finitely generated, then $\bar{\Omega}_\mathbf{n}^{\mathcal{V}} = {\Omega_\mathbf{n}}^{\mathcal{V}}$.*

Proof. Due to Lemma 3.18, we can assume without loss of generality that $\mathcal{V}$ is generated by a single algebra $\mathbf{A}$ and therefore, by Lemma 3.17 we know that ${\Omega_\mathbf{n}}^{\mathcal{V}} \in \mathcal{V}$. Since the inclusion ${\Omega_\mathbf{n}}^{\mathcal{V}} \subseteq \bar{\Omega}_\mathbf{n}^{\mathcal{V}}$ is trivial, it suffices to check the other inclusion.

Pick an arbitrary $f \in \bar{\Omega}_\mathbf{n}^{\mathcal{V}}$ and recall from Lemma 3.12 there is a surjective homomorphism $\bar{h}$ from $\mathbf{T}(x_1, \ldots, x_n)$ to ${\Omega_\mathbf{n}}^{\mathcal{V}}$ extending the mapping $h : \{x_1, \ldots, x_n\} \to A$ defined by the rule $h(x_i) = \hat{x}_i$ for every $i \leqslant n$. Recall that ${\Omega_\mathbf{n}}^{\mathcal{V}} \in \mathcal{V}$; so, there is an isomorphic copy of ${\Omega_\mathbf{n}}^{\mathcal{V}}$ in $\mathcal{V}_0$, which we will denote also by ${\Omega_\mathbf{n}}^{\mathcal{V}}$ for simplicity. Then, it is clear that $f^{{\Omega_\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \ldots, \hat{x}_n) \in {\Omega_\mathbf{n}}^{\mathcal{V}}$. Thus, since $\bar{h}$ is surjective, there is some term $t \in \mathbf{T}(x_1, \ldots, x_n)$ such that

$$\bar{h}(t) = f^{{\Omega_\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \ldots, \hat{x}_n).$$

Recall from Lemma 3.12 that $\bar{h}(t) = t^{{\Omega_\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \ldots, \hat{x}_n)$, so for every $\mathbf{A} \in \mathcal{V}_0$ and every $a_1, \ldots, a_n \in A$ we have

$$(f^{{\Omega_\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \ldots, \hat{x}_n))^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n).$$

Now, recall from Definition 3.6 that

$$\begin{aligned}(f^{{\Omega_\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \ldots, \hat{x}_n))^{\mathbf{A}}(a_1, \ldots, a_n) &= f^{\mathbf{A}}(\hat{x}_1(a_1, \ldots, a_n), \ldots, \hat{x}_n(a_1, \ldots, a_n)) \\ &= f^{\mathbf{A}}(a_1, \ldots, a_n).\end{aligned}$$

Therefore, for all $\mathbf{A} \in \mathcal{V}_0$ we have that $f^{\mathbf{A}} = t^{\mathbf{A}}$ and therefore $f = \hat{t}$, so we conclude that $f \in {\Omega_\mathbf{n}}^{\mathcal{V}}$. ⊠

In general, there is no precise characterization for the algebra of n-ary implicit operations over a pseudovariety $\mathcal{V}$. However, as we have seen in Lemmas 3.19 and 3.20, when considering finitely generated pseudovarieties (one thing that we will repeatedly do along proofs) we obtain a good description of $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ with well-known properties.

3.2 The space of implicit operations

Until now we have studied only the algebraic aspects of the algebra $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$, but it is also possible to define a topology on its universe $\bar{\Omega}_n\mathcal{V}$ and get the structure of a topological algebra in $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ (as we will see later). To do that, observe first that each $f \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ can be seen as an element of $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ and therefore $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \subseteq \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$. Now, we consider the discrete topology on A^{A^n} for each algebra $\mathbf{A}$ in $\mathcal{V}_0$. Define for each $\mathbf{B} \in \mathcal{V}_0$ the projection mapping $\pi_{\mathbf{B}} : \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n} \to B^{B^n}$ defined by the rule $\pi_{\mathbf{B}}(\langle f^{\mathbf{A}} : \mathbf{A} \in \mathcal{V}_0\rangle) := f^{\mathbf{B}}$. Then, we have on $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ the product topology generated by the subbase

$$\{\pi_{\mathbf{A}}^{-1}[X] : \mathbf{A} \in \mathcal{V}_0 \textit{ and } X \subseteq A^{A^n}\}^{\mathrm{VI}}.$$

Observe that, due to Remark 2.74, the mapping $\pi_{\mathbf{A}}$ is continuous and surjective for each $\mathbf{A} \in \mathbf{V}_0$. Thus, we define on $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ the subspace topology T_S, which is generated by the subbase (see Example 2.62)

$$\{\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \cap \pi_{\mathbf{A}}^{-1}[X] : \mathbf{A} \in \mathcal{V}_0 \;\; \textit{and } X \subseteq A^{A^n}\}. \tag{3.1}$$

Therefore, every open set U in $\bar{T}_S$ is of the form

$$\bigcup_{i\in I}\Big(\bigcap_{\mathbf{A}\in\mathcal{K}_i}(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \cap \pi_{\mathbf{A}}^{-1}[X_{\mathbf{A}}])\Big)$$

where I is an arbitrary set of indexes, and for each $i \in I$ the class $\mathcal{K}_i$ is a finite subclass of $\mathcal{V}_0$ and $X_{\mathbf{A}} \subseteq A^{A^n}$ for every $\mathbf{A} \in \mathcal{K}_i$. Moreover, due to the commutativity of the intersection and the distributivity of the union over the intersection, we have that

$$U = \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \bigcap\Big(\bigcup_{i\in I}(\cap_{\mathbf{A}\in\mathcal{K}_i}\pi_{\mathbf{A}}^{-1}[X_{\mathbf{A}}])\Big). \tag{3.2}$$

Of course, notice that since it holds that ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \subseteq \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$, we can also consider on ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ a topology, namely, the subspace topology inherited by the topology on $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$. More precisely, we have the topological space $({\Omega_{\mathbf{n}}}^{\mathcal{V}}, T'_S)$ for which the family

$$\{{\Omega_{\mathbf{n}}}^{\mathcal{V}} \cap \pi_{\mathbf{A}}^{-1}[X] : \mathbf{A} \in \mathcal{V}_0 \;\; \textit{and } X \subseteq A^{A^n}\}$$

[VI]see Example 2.61.

forms a subbase.

We begin proving that our space $(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}, \bar{T}_s)$ is a Stone space and therefore a compact, T_0 and zero-dimensional space. Moreover, since $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ with the product topology is Hausdorff, we obtain also that $(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}, T_s)$ is T_2 and not only T_0 (see Example 2.62 and Proposition 2.64).

Theorem 3.21. *$\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ is a Stone space.*

Proof. Recall that for each $\mathbf{A} \in \mathcal{V}_0$ we are considering the discrete topology on A^{A^n},so, due to Example 2.91 we know that $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ is a Stone space. Therefore, by Proposition 2.90 it suffices to check that it is closed in $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$.

Pick some $f \in \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n} \setminus \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ and recall that $f = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathcal{V}_0 \rangle$. We want to find a clopen set containing f but not elements in $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$. Since $f \notin \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ there are some algebras $\mathbf{B}, \mathbf{C} \in \mathcal{V}_0$, a homomorphism $h : \mathbf{B} \to \mathbf{C}$ and some $\vec{b} \in B^n$ such that

$$f^{\mathbf{C}} h^n(\vec{b}) \neq h(f^{\mathbf{B}}\vec{b}) \tag{3.3}$$

We define now $X := (\pi_{\mathbf{B}})^{-1}[\{f^{\mathbf{B}}\}] \cap (\pi_{\mathbf{C}})^{-1}[\{f^{\mathbf{C}}\}]$. Notice that, since the spaces A^{A^n} and B^{B^n} are discrete, $\{f^{\mathbf{B}}\}$ and $\{f^{\mathbf{B}}\}$ are clopen sets in B^{B^n} and C^{C^n} respectively (see Example 2.60), and since both $\pi_{\mathbf{B}}$ and $\pi_{\mathbf{C}}$ are continuous (see Lemma 2.73), we have that $(\pi_{\mathbf{B}})^{-1}[\{f^{\mathbf{B}}\}]$ and $(\pi_{\mathbf{C}})^{-1}[\{f^{\mathbf{C}}\}]$ are open sets. Therefore, X is an open set and it clearly contains f. Recall that $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ is zero-dimensional, so there is a clopen U such that $f \in U \subseteq X$. To finish the proof, we need to see that $U \cap \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} = \varnothing$.

It is obvious that if $X \cap \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} = \varnothing$ then $U \cap \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} = \varnothing$, so it suffices to check that $X \cap \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} = \varnothing$. Let some $g \in X$. Due to how X is defined we have that $\pi_{\mathbf{B}}(g) = f^{\mathbf{B}}$ and $\pi_{\mathbf{C}}(g) = f^{\mathbf{C}}$. Observe that the condition 3.3 forces that $g \notin \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$, thus we get $X \cap \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} = \varnothing$ concluding the proof. ⊠

In fact, the algebra $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ equipped with the topology described above forms also a topological algebra as we show here below. Recall that, for each m-ary function symbol $\sigma \in \tau$, its interpretation in $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ is defined pointwise as

$$(\sigma^{\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}}(f_1, \ldots, f_m))^{\mathbf{B}}(b_1, \ldots, b_n) := \sigma^{\mathbf{B}}(f_1^{\mathbf{B}}(b_1, \ldots, b_n), \ldots, f_m^{\mathbf{B}}(b_1, \ldots, b_n)) \tag{3.4}$$

for every $f_1, \ldots, f_m \in \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$, for every $\mathbf{B} \in \mathcal{V}_0$ and every $b_1, \ldots, b_n \in B$. For each $\mathbf{A} \in \mathcal{V}_0$ we also consider the algebra $\langle A^{A^n}, \tau \rangle$ where for every $\sigma \in \tau$ and every $f_1^{\mathbf{A}}, \ldots, f_{m_\sigma}^{\mathbf{A}} \in A^{A^n}$ we have

$$(\sigma^{\mathbf{A}^{\mathbf{A}^{\mathbf{n}}}}(f_1^{\mathbf{A}}, \ldots, f_m))(a_1, \ldots, a_n) := \sigma^{\mathbf{A}}(f_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, f_m^{\mathbf{A}}(a_1, \ldots, a_n)). \tag{3.5}$$

Lemma 3.22. *The space $(\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}, T)$ is a topological algebra.*

Proof. Pick an arbitrary mm-ary function symbol $\sigma \in tau$ and fix some $\mathbf{B} \in \mathcal{V}_0$. We define for each $i \leqslant m$ the mapping $h_i : \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n} \to B^{B^n}$ as the composition $\pi_{\mathbf{B}} \circ p_i$. Clearly, both $\pi_{\mathbf{B}}$ and p_i are continuous functions; so, each h_i is continuous by Lemma 2.69. Then, the mapping $h : \prod_{\mathbf{A}\in\mathcal{V}_0} \times \ldots, \times \prod_{\mathbf{A}\in\mathcal{V}_0} \to B^{B^n} \times \cdots \times B^{B^n}$ defined by the rule $h(f_1, \ldots, f_m) = (h_1(f_1, \ldots, f_m), \ldots, h_m(f_1, \ldots, f_m))$ is a continuous function (see [18] Theorem 19.6). It is easy to see that

$$\pi_{\mathbf{B}} \circ \sigma^{\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}} = \sigma^{\mathbf{B}^{\mathbf{B}^{\mathbf{n}}}} \circ h, \tag{3.6}$$

we just need to apply the definitions 3.4 and 3.5 of the interpretations of σ. Recall that we are considering on B^{B^n} the discrete topology, therefore $B^{B^n} \times \cdots \times B^{B^n}$ is also a discrete space (see Corollay 2.78). Hence, the mapping $\sigma^{\mathbf{B}^{\mathbf{B}^{\mathbf{n}}}} : B^{B^n} \times \cdots \times B^{B^n} \to B^{B^n}$ is continuous. Again, the composition of continuous functions is continuous, so $\sigma^{\mathbf{B}^{\mathbf{B}^{\mathbf{n}}}} \circ h$ is continuous. Due to equality 3.6 we have that $pi_{\mathbf{B}} \circ \sigma^{\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}}$ is continuous, and since we already know that $\pi_{\mathbf{B}}$ is continuous, the mapping $\sigma^{\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}}$ is necessarily continuous. ⊠

Theorem 3.23. *$(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}, T_s)$ is a topological algebra.*

Proof. From Lemma 3.22 we know that $(\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}, T)$, so if we prove that $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \leqslant \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ then by Proposition 2.81 we obtain automatically what we want. Recall that the universe of $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ is a subset of $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$, therefore, by Definition 3.6 we have that for every m-ary operation $\sigma \in \tau$ and every $f_1, \ldots, f_m \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ we have

$$\sigma^{\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}}(f_1, \ldots, f_m) = \sigma^{\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}}(f_1, \ldots, f_m)\,.$$

Hence, $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ is indeed a subalgebra of $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$.

⊠

Concerning subpseudovarieties, consider a subpseudovariety $\mathcal{W}$ of $\mathcal{V}$. Let $\mathcal{W}_0$ be the set of representatives of the isomorphic class of $\mathcal{W}$. Then, we can construct the corresponding space $\bar{\Omega}^{\mathcal{W}}_{\mathbf{n}}$ as we have done with $\mathcal{V}_0$. Moreover, notice that $\mathcal{W}_0$ can be assumed to be without loss of generality a subset $\mathcal{W}_0 \subseteq \mathcal{V}_0$, so, this will be the case from now on. Therefore, it makes sense to define a projection mapping

$$\begin{array}{cccc} p_{\mathcal{W}_0} : & \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} & \to & \bar{\Omega}^{\mathcal{W}}_{\mathbf{n}} \\ & \langle f^{\mathbf{A}} : \mathbf{A} \in \mathcal{V}_0 \rangle & \to & \langle f^{\mathbf{A}} : \mathbf{A} \in \mathcal{W}_0 \rangle. \end{array}$$

Whenever the subpseudovariety is clear from the context, we will use simply the notation p instead of $p_{\mathcal{W}_0}$. Notice also that, until now, we have used the notation p_i for the i-th projections considering arbitrary direct products. In addition, it is important to emphasize that when speaking about the mappings $\pi_{\mathbf{A}}$ may not be clear if it corresponds to the one with domain $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ or with domain $\prod_{\mathbf{A}\in\mathcal{W}_0} A^{A^n}$. Therefore, in case it is not clear from the context, we will make that explicit using $\pi_{\mathbf{A}}^{\mathcal{V}_0}$, for instance. However, this does not create confusion with the convention adopted in the previous phrase since the notation for these projections will always have the subscript of the corresponding coordinate.

Lemma 3.24. *The mapping* $\pi^* : \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n} \to \prod_{\mathbf{A}\in\mathcal{W}_0} A^{A^n}$ *defined by the rule*

$$\pi^*(f) = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathcal{W}_0\rangle$$

is a continuous[VII] *and surjective mapping.*

Proof. Notice that, for every $\mathbf{A} \in \mathcal{W}_0$ and for every $f \in \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ it holds that $\pi_{\mathbf{A}}^{\mathcal{V}_0}(\pi^*(f)) = \pi_{\mathbf{A}}^{\mathcal{V}_0}(f)$, i.e, $\pi_{\mathbf{A}}^{\mathcal{V}_0} \circ \pi^* = \pi_{\mathbf{A}}^{\mathcal{V}_0}$. Since for each $\mathbf{A} \in \mathcal{W}_0$ the mapping $\pi_{\mathbf{A}}^{\mathcal{V}_0}$ is continuous, we have by Lemma 2.75 that π^* is continuous. Surjectivity follows from the fact that for each $f \in \prod_{\mathbf{A}\in\mathcal{W}_0} A^{A^n}$ there is some $f' \in \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ such that $\pi^*(f') = f$ and where f' is defined as then $(f')^{\mathbf{A}} = f^{\mathbf{A}}$ for every $\mathbf{A} \in \mathcal{W}_0$, and $(f')^{\mathbf{A}}(a_1,\ldots,a_n) := a_1$ for every $\mathbf{A} \in \mathcal{V}_0 \setminus \mathcal{W}_0$ and for every $a_1,\ldots,a_n \in A$[VIII]. ⊠

Lemma 3.25. *Given a subpseudovariety* $\mathcal{W}$ *of* $\mathcal{V}$*, the natural projection* $p : \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \to \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$ *defined by the rule* $p(f) = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathcal{W}_0\rangle$ *is a continuous homomorphism.*

Proof. Recall from previous lemma that the mapping $\pi^* : \prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n} \to \prod_{\mathbf{A}\in\mathcal{W}_0} A^{A^n}$ defined by the rule $\pi^* f = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathcal{W}_0\rangle$ is a continuous and surjective mapping. Moreover, we also have that $\pi^* \restriction \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} = p$.

To check that it is continuous, take (due to Propositon 2.68) an open set $U \subseteq \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$ of the form $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \cap (\bigcap_{\mathbf{A}\in\mathcal{K}} \pi_{\mathbf{A}}^{-1}[X_{\mathbf{A}}])$ for some finite $\mathcal{K} \subseteq \mathcal{W}_0$ and $X_{\mathbf{A}} \subseteq A^{A^n}$ for each $\mathbf{A} \in \mathcal{W}_0$. Then,

$$\begin{aligned} p^{-1}[U] &= \{f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} : (p(f))^{\mathbf{A}} \in X_{\mathbf{A}} \textit{ for all } \mathbf{A} \in \mathcal{K}\} \\ &= \bigcap_{\mathbf{A}\in\mathcal{K}} \{f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} : (p(f))^{\mathbf{A}} \in X_{\mathbf{A}}\} \end{aligned}$$

If $U_{\mathbf{A}} := \{f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} : (p(f))^{\mathbf{A}} \in X_{\mathbf{A}}\}$, we claim that for each $\mathbf{A} \in \mathcal{K}$ it holds $U_{\mathbf{A}} = \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \cap (\pi^*)^{-1}[(\pi_{\mathbf{A}}^{\mathcal{W}_0})^{-1}[X_{\mathbf{A}}]]$:

$$\begin{aligned} \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \cap (\pi^*)^{-1}[(\pi_{\mathbf{A}}^{\mathcal{W}_0})^{-1}[X_{\mathbf{A}}]] &= \{f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} : \pi^*(f) \in (\pi_{\mathbf{A}}^{\mathcal{W}_0})^{-1}[X_{\mathbf{A}}]\} \\ &= \{f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} : p(f) \in (\pi_{\mathbf{A}}^{\mathcal{W}_0})^{-1}[X_{\mathbf{A}}]\} \\ &= \{f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} : (p(f))^{\mathbf{A}} \in X_{\mathbf{A}}\} \\ &= U_{\mathbf{A}}. \end{aligned}$$

[VII] Considering the corresponding product topology in each space.

[VIII] f' is just an example of a member in $\prod_{\mathbf{A}\in\mathcal{V}_0} A^{A^n}$ such that $\pi^*(f') = f$, but observe that for each $\mathbf{A} \in \mathcal{V}_0 \setminus \mathcal{W}_0$ we can define $(f')^{\mathbf{A}}$ as we wanted since we do not need to impose any property to f' beyond that $\pi^*(f') = f$ (in contrast with implicit operations).

The second equality holds since $\pi^* \upharpoonright \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} = p$, and the last one is just the definition of $U_{\mathbf{A}}$. Then, by the continuity of π^* and $\pi_{\mathbf{A}}^{\mathcal{W}_0}$ we have that $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \cap (\pi^*)^{-1}[(\pi_{\mathbf{A}}^{\mathcal{W}_0})^{-1}[X_{\mathbf{A}}]]$ is open and so it $U_{\mathbf{A}}$ for each $\mathbf{A} \in \mathcal{K}$. Therefore, since $\mathcal{K}$ is finite and $p^{-1}[U] = \cap_{\mathbf{A}\in\mathcal{K}} U_{\mathbf{A}}$, we conclude that $p^{-1}[U]$ is open and hence p is continuous.

To prove that p is a homomorphism. So, let $\sigma \in \tau$ be a m-ary function symbol and $f_1, \ldots, f_m \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$. We need to check the equality

$$p(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_m)) = \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(f_1), \ldots, p(f_m)). \tag{3.7}$$

From the definition of p, we have that

$$p(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_m)) = \langle (\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_m))^{\mathbf{A}} : \mathbf{A} \in \mathcal{W}_0 \rangle.$$

We want to prove for every $\mathbf{A} \in \mathcal{W}_0$ the equality

$$(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_m))^{\mathbf{A}} = (\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(f_1), \ldots, p(f_m)))^{\mathbf{A}},$$

since in that case, we obtain the equality 3.7. To do that, fix an arbitrary $\mathbf{B} \in \mathcal{W}_0$. Then, from Definition 3.6 we have that for every $a_1, \ldots, a_n \in B$

$$(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1 \ldots f_m))^{\mathbf{B}}(a_1, \ldots, a_n) = \sigma^{\mathbf{B}}(f_1^{\mathbf{B}}(a_1, \ldots, a_n), \ldots, f_m^{\mathbf{B}}(a_1, \ldots, a_n)).$$

Moreover, again due to Definition 3.6, we have that

$$\begin{aligned}(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(f_1) \ldots p(f_m)))^{\mathbf{B}}(a_1, \ldots, a_n) &= \sigma^{\mathbf{B}}((p(f_1))^{\mathbf{B}}(a_1, \ldots, a_n), \ldots, (p(f_m))^{\mathbf{B}})(a_1, \ldots, a_n)) \\ &= \sigma^{\mathbf{B}}(f_1^{\mathbf{B}}(a_1, \ldots, a_n), \ldots, f_m^{\mathbf{B}}(a_1, \ldots, a_n)).\end{aligned}$$

Notice that the last equality from above holds since $f_j^{\mathbf{A}} = (p(f_j))^{\mathbf{B}}$ (recall that $\mathbf{B} \in \mathcal{W}_0$). Thus, for every $\mathbf{A} \in \mathcal{K}$

$$(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_m))^{\mathbf{A}} = (\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(f_1), \ldots, p(f_m)))^{\mathbf{A}},$$

and therefore

$$p(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_m)) = \langle (\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_m))^{\mathbf{A}} : \mathbf{A} \in \mathcal{W}_0 \rangle,$$

showing that p is a homomorphism.

⊠

Proposition 3.26. *$\Omega_{\mathbf{n}}{}^{\mathcal{V}}$ is a dense subset of $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$.*

Proof. To prove that ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ is a dense subset, it suffices to check that for every non-empty open set $U \subseteq \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ it holds that $U \cap {\Omega_{\mathbf{n}}}^{\mathcal{V}} \neq \varnothing$. So, fix an arbitrary nonempty open $U \subseteq \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ and recall that it is of the form $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \bigcap (\bigcup_{i \in I}(\cap_{\mathbf{A} \in \mathcal{K}_i} \pi_{\mathbf{A}}^{-1}[X_{\mathbf{A}}]))$ for some set I, a finite class $\mathcal{K}_i \subseteq \mathcal{V}_0$ for each $i \in I$ and $X_{\mathbf{A}_i} \subseteq A_i^{A_i^n}$ for each $\mathbf{A} \in \mathcal{K}_i$. Observe that

$$\begin{aligned}
{\Omega_{\mathbf{n}}}^{\mathcal{V}} \cap U &= \{f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}} : f \in U\} \\
&= \{f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}} : \exists i \in I \;\; s.t.\; f \in \cap_{\mathbf{A} \in \mathcal{K}_i} \pi_{\mathbf{A}}^{-1}[X_{\mathbf{A}}]\} \\
&= \cup_{i \in I}\{f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}} : f \in \cap_{\mathbf{A} \in \mathcal{K}_i} \pi_{\mathbf{A}}^{-1}[X_{\mathbf{A}}]\} \\
&= \cup_{i \in I}\{f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}} : f \in \pi_{\mathbf{A}}^{-1}[X_{\mathbf{A}}] \; for\ all\ \mathbf{A} \in \mathcal{K}_i\} \\
&= \cup_{i \in I}(\cap_{\mathbf{A} \in \mathcal{K}_i}\{f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}} : f \in \pi_{\mathbf{A}}^{-1}[X_{\mathbf{A}}]\}) \\
&= \cup_{i \in I}(\cap_{\mathbf{A} \in \mathcal{K}_i}\{f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}} : \pi_{\mathbf{A}}(f) \in X_{\mathbf{A}}\}).
\end{aligned}$$

Now, assume towards a contradiction that ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \cap U = \varnothing$. Therefore, for every $i \in I$ there is some $\mathbf{A}_i \in \mathcal{K}_i$ such that $\pi_{\mathbf{A}_i}[{\Omega_{\mathbf{n}}}^{\mathcal{V}}] \subseteq A_i^{A_i^n} \setminus X_{\mathbf{A}_i}$ (since otherwise ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \cap U \neq \varnothing$). Let $\mathcal{K}$ be the class compounded by all such algebras, that is, $\mathcal{K} := \{\mathbf{A}_i : i \in I\}$. Hence, for every $i \in I$ we have $X_{\mathbf{A}_i} \cap \pi_{\mathbf{A}_i}[{\Omega_{\mathbf{n}}}^{\mathcal{V}}] = \varnothing$.

The last step is to prove that indeed for every $i \in I$ we have that $\pi_{\mathbf{A}_i}[{\Omega_{\mathbf{n}}}^{\mathcal{V}}] = \pi_{\mathbf{A}_i}[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]$. With this we would get a contradiction since $X_{\mathbf{A}_i} \cap \pi_{\mathbf{A}_i}[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}] \neq \varnothing$ for every $i \in I$ (observe that for every $i \in I$ we have $\pi_{\mathbf{A}_i}[U] \subseteq X_{\mathbf{A}_i} \cap \pi_{\mathbf{A}_i}[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]$ and $\pi_{\mathbf{A}_i}[U]$ is nonempty because U is nonemtpy).

Fix an arbitrary $i \in I$. To prove that $\pi_{\mathbf{A}_i}[{\Omega_{\mathbf{n}}}^{\mathcal{V}}] = \pi_{\mathbf{A}_i}[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]$, notice that the inclusion $\subseteq$ is trivial because ${\Omega_{\mathbf{n}}}^{\mathcal{V}} \subseteq \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$, so we need to focus on the other one. Pick an arbitrary $a \in \pi_{\mathbf{A}_i}[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]$ for which, from the surjectivity of $\pi_{\mathbf{A}_i}$, there is some $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ such that $\pi_{\mathbf{A}_i}(f) = a$. Let $\mathcal{W} = p\mathbb{V}(\mathbf{A}_i)$ be the pseudovariety generated by $\mathbf{A}_i$. Observe that $p(f) \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$ and that in fact $a = \pi_{\mathbf{A}_i}(p(f))$ because $\mathbf{A}_i \in \mathcal{W}$. Recall that, due to Lemma 3.20, since $\mathcal{W}$ is finitely generated we know that $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} = {\Omega_{\mathbf{n}}}^{\mathcal{W}}$, so there must be some $g \in {\Omega_{\mathbf{n}}}^{\mathcal{W}}$ such that $g = p(f)$. Moreover, since g is an explicit operation, there is some $t \in \mathbf{T}(x_1, \ldots, x_n)$ such that

$$g = \langle t^{\mathbf{A}}(\hat{x}_1^{\mathbf{A}}, \ldots, \hat{x}_n^{\mathbf{A}}) : \mathbf{A} \in \mathcal{W}_0 \rangle.$$

We consider then the mapping

$$g' = \langle t^{\mathbf{A}}(\hat{x}_1^{\mathbf{A}}, \ldots, \hat{x}_n^{\mathbf{A}}) : \mathbf{A} \in \mathcal{V}_0 \rangle,$$

which is, by obvious reasons, an element in ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$. It is immediate then that $p(g') = g$, thus (since $\mathbf{A}_i \in \mathcal{W}_0$)

$$\pi_{\mathbf{A}_i}(g') = \pi_{\mathbf{A}_i}(g) = \pi_{\mathbf{A}_i}(p(f)) = a$$

and we obtain that $a \in \pi_{\mathbf{A}_i}[{\Omega_{\mathbf{n}}}^{\mathcal{V}}]$. In conclusion, for every $i \in I$ it holds the equality $\pi_{\mathbf{A}_i}[{\Omega_{\mathbf{n}}}^{\mathcal{V}}] = \pi_{\mathbf{A}_i}[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]$ and therefore we finish the proof.

⊠

Corollary 3.27. *The equality $cl({\Omega_{\mathbf{n}}}^{\mathcal{V}}) = \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ holds.*

Proof. Just apply Lemma 2.52. ⊠

Proposition 3.28. *Let $\mathcal{W}$ be a subpseudovariety of $\mathcal{V}$. The continuous homomorphism $p : \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \to \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$* [IX] *is surjective.*

Proof. First of all, notice that for every $f \in {\Omega_{\mathbf{n}}}^{\mathcal{W}}$ it is of the form $\langle t^{\mathbf{A}}(\hat{x}_i^{\mathbf{A}}, \ldots, \hat{x}_i^{\mathbf{A}}) : \mathbf{A} \in \mathcal{W}_0\rangle$. Then, the tuple $f' := \langle t^{\mathbf{A}}(\hat{x}_i^{\mathbf{A}}, \ldots, \hat{x}_i^{\mathbf{A}}) : \mathbf{A} \in \mathcal{V}_0\rangle$ is a member in ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ and $p(f') = f$. This means that ${\Omega_{\mathbf{n}}}^{\mathcal{W}} \subseteq p[{\Omega_{\mathbf{n}}}^{\mathcal{V}}] \subseteq p[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]$.

To check that p is surjective it suffices to check that $p[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}] = \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$. The inclusion $p[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}] \subseteq \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$ is clear. For the other inclusion, suppose, towards a contradiction, that $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \setminus p[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}] \neq \varnothing$. Since p is continuous, $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ compact and $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$ Hausdorff, we deduce that p is closed (see Proposition 2.72). Therefore, since $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ is closed, we have that $p[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]$ is a closed subset. We obtain then (by the previous paragraph) that $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \setminus p[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]$ is an open set such that $(\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \setminus p[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}]) \cap {\Omega_{\mathbf{n}}}^{\mathcal{W}} = \varnothing$, which is a contradiction since ${\Omega_{\mathbf{n}}}^{\mathcal{W}}$ is dense in $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$. Thus, we conclude that $p[\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}] = \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$. ⊠

In the following, we will define a metric on $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$. In fact, the distance we will define below defines a stronger notion than a metric, namely, an ultrametric. After, we will define another topology in via this ultrametric, which turns out to be the same as the considered before as we will see later. Due to this fact, no change will be made in a topological sense thanks to this new approach, however, it will be helpful when proving some auxiliary results in the next chapter.

We begin defining the function $r : \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \times \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \to \mathbb{N} \cup \{\infty\}$ by the rule

$$r(f, g) = \min_{\mathbf{A} \in \mathcal{V}_0} \{|A| : f^{\mathbf{A}} \neq g^{\mathbf{A}}\}. \tag{3.8}$$

When $\mathcal{V} \Vdash f \approx g$, that is, when there is no algebra $\mathbf{A} \in \mathcal{V}_0$ such that $f^{\mathbf{A}} \neq g^{\mathbf{A}}$, we define $r(f, g) = \infty$. It is easy to see that r is well-defined. Finally, This function allows us to define the ultrametric d we are looking for, which is defined as

$$d(f, g) = 2^{-r(f,g)} \tag{3.9}$$

[IX] The one defined in Lemma 3.25.

and where for every $f, g \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ we have $r(f,g) = \infty$ if and only if $d(f,g) = 0$. We define the topology generated by the basis

$$\mathcal{B} := \{B_f(\varepsilon) : f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}},\ \varepsilon \in \mathbb{R} \text{ and } \varepsilon > 0\} \tag{3.10}$$

where

$$B_f(\varepsilon) := \{g \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} : d(f,g) < \varepsilon\}.^{\mathrm{X}}$$

These kind of topologies coming from a metric are called *metrizable* topologies (see [18] for example), and, in this cases, both notions of continuity (in the topological and the metric context) coincide ([18] Theorem 21.1).

Proposition 3.29. *The map d satisfies the following properties:*

i) For every $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ we have $d(f,f) = 0$.

ii) For every $f, g \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ if $f \neq g$ then $d(f,g) > 0$.

iii) For every $f, g \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ $d(f,g) = d(g,f)$.

iv) For every $f, g, w \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ $d(f,g) \leqslant \max\{d(f,w), d(w,g)\}$.

v) For every m-ary $\sigma \in \tau$ and every $f_1, \ldots, f_m, g_1, \ldots, g_m \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$

$$d(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1 \ldots f_m), \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(g_1 \ldots g_m)) \leqslant \max_{i \leqslant m}\{d(f_i, g_i)\}.$$

Therefore it is an ultrametric on $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$.

Proof. i) Trivial from the definition of d. Just notice that $r(f,f) = 0$.

ii) If $f \neq g$ then there is some nontrivial $\mathbf{A} \in \mathcal{V}_0$ such that $f^{\mathbf{A}} \neq g^{\mathbf{A}}$. Hence, $r(f,g) < \infty$ and therefore $d(f,g) > 0$.

iii) Trivial since clearly $r(f,g) = r(g,f)$.

iv) Supose that it is not the case and that $d(f,g) > d(f,w), d(w,g)$. This means that

$$2^{-r(f,g)} > 2^{-r(f,w)}, 2^{-r(w,g)}$$

and it follows that $r(f,g) < r(f,w), r(w,g)$. Take $\mathbf{A} \in \mathcal{V}_0$ such that $|A| = r(f,g)$ and $f^{\mathbf{A}} \neq g^{\mathbf{A}}$. Since $|A| < r(f,w), r(w,g)$ we have that $\mathbf{A} \Vdash f \approx w$ and $\mathbf{A} \Vdash w \approx g$. Mixing both results we obtain that $\mathbf{A}\ f \approx g$, which is a contradiction.

[X] See Remark 2.84.

v) The proof of this point is quite similar to the previous one. Assume that

$$d(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1 \dots f_m), \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(g_1 \dots g_m)) > d(f_1, g_1), \dots, d(f_m, g_m).$$

Then, we have that

$$r(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1 \dots f_m), \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(g_1 \dots g_m)) < r(f_1, g_1), \dots, r(f_m, g_m).$$

Let $\mathbf{A} \in \mathcal{V}_0$ such that $|A| = r(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1 \dots f_m), \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(g_1 \dots g_m))$ and $\mathbf{A} \not\Vdash \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1 \dots f_m) \approx \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(g_1 \dots g_m)$. Now, since for every $i \leqslant m$ we have $|A| < r(f_i, g_i)$, it implies that $\mathbf{A} \Vdash f_i \approx g_i$ for each $i \leqslant m$. As above, from these we obtain that $\mathbf{A} \Vdash \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}} f_1 \dots f_m \approx \sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(g_1 \dots g_m)$ which is a contradiction.

⊠

It is convenient to proof that both topologies on $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$, the topology T_S described in the first place and the topology T_d induced by the ultrametric d [XI], coincide. As we have mentioned before, this is in fact the case, and we will prove it in the next theorem. Roughly speaking, this means that both spaces have the same general properties, and therefore there is no difference in considering one topology or the other in what follows. Indeed, the approach from the topology T_d simplify significantly some of the proofs in the next section.

Definition 3.30. *Let $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ and $\varepsilon > 0$. We define*

$$m_\varepsilon := \min\{m \in \omega : 2^{-m} \leqslant \varepsilon\}. \tag{3.11}$$

Lemma 3.31. *Let $f, g \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ and some $\varepsilon > 0$, and consider $\mathcal{K} := \{\mathbf{A} \in \mathcal{V}_0 : |A| < m_\varepsilon\}$. Then, $g \in B_f(\varepsilon)$ if and only if $g \in \cap_{\mathbf{A} \in \mathcal{K}} \pi_{\mathbf{A}}^{-1}[\{f^{\mathbf{A}}\}]$.*

Proof. Suppose that we have $g \in B_f(\varepsilon)$, and assume towards a contradiction that $g \notin \cap_{\mathbf{A} \in \mathcal{K}} \pi_{\mathbf{A}}^{-1}[\{f^{\mathbf{A}}\}]$, i.e., there is some algebra $\mathbf{A} \in \mathcal{K}$ such that $f^{\mathbf{A}} \neq g^{\mathbf{A}}$. Then, by the definition of the function r (see 3.8), we have $r(f, g) \leqslant |A|$ and therefore

$$2^{-|A|} \leqslant 2^{-r(f,g)} < \varepsilon. \tag{3.12}$$

Recall that, by hypothesis, $|A| < m_\varepsilon$, so we also have

$$\varepsilon < 2^{|A|}. \tag{3.13}$$

Putting 3.12 and 3.13 together, we obtain

$$2^{-|A|} \leqslant 2^{-r(f,g)} < \varepsilon < 2^{|A|},$$

which is clearly a contradiction. Therefore, $g \in \cap_{\mathbf{A} \in \mathcal{K}} \pi_{\mathbf{A}}^{-1}[\{f^{\mathbf{A}}\}]$.
Pick now some $g \in \cap_{\mathbf{A} \in \mathcal{K}} \pi_{\mathbf{A}}^{-1}[\{f^{\mathbf{A}}\}]$, then for every $\mathbf{A} \in \mathcal{K}$ we have $f^{\mathbf{A}} = g^{\mathbf{A}}$. This means, due to 3.8, that $m_\varepsilon \leqslant r(f, g)$; hence, $2^{-r(f,g)} \leqslant 2^{-m_\varepsilon} < \varepsilon$. In conclusion, $d(f, g) < \varepsilon$ and $g \in B_f(\varepsilon)$. ⊠

[XI] The topology with basis 3.10.

Theorem 3.32. *The topologies T_S and T_d on $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ coincide, i.e., $T_S = T_d$.*

Proof. $\subseteq$: Pick some $U \subseteq \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ open in T_S. We need to check that U is also open in $(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}, T_d)$. As usual, U has the form $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \cap (\bigcup_{i\in I}(\cap_{\mathbf{A}\in\mathcal{K}_i}\pi^{-1}_{\mathbf{A_i}}[X_i]))$ where I is a set of indexes, $\mathcal{K}_i \subseteq \mathcal{V}_0$ a finite class for each $i \in I$ and $X_{\mathbf{A}_i} \subseteq A_i^{A_i^n}$ for each $\mathbf{A} \in \mathcal{K}_i$. For each $i \in I$ let $m_i = \max\{|\mathbf{A}_i| : i \in I\}$ and define $m := \sup\{m_i : i \in I\}$. We claim that

$$U = \cup_{f\in U} V_f$$

where $V_f = \{g \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} : d(f,g) < 2^{-m}\}$. Notice that $U \subseteq \cup_{f\in U} V_f$ since for every $f \in U$ we have $f \in V_f$. For the other inclusion, pick an arbitrary $g \in \cup_{f\in U} V_f$. Then, there is some $f \in U$ such that $g \in V_f$ and therefore $d(f,g) < 2^{-m}$. This means that, necessarily, for every $i \in I$ and every $\mathbf{A} \in \mathcal{K}_i$ we have $f^{\mathbf{A}} = g^{\mathbf{A}}$ [XII]. Thus, since $f \in U$ we know that $f^{\mathbf{A}} \in X_{\mathbf{A}}$ for some $i \in I$ and every $\mathbf{A} \in \mathcal{K}_i$. Therefore, since $f^{\mathbf{A}} = g^{\mathbf{A}}$ for every $\mathbf{A} \in \mathcal{K}_i$ we obtain $g^{\mathbf{A}} \in X_{\mathbf{A}}$ for each $\mathbf{A} \in \mathcal{K}_i$, i.e., $g \in \cap_{\mathbf{A}\in\mathcal{K}_i}\pi^{-1}_{\mathbf{A}}[X_{\mathbf{A}}]$, so we conclude that $g \in U$. Then, each V_f is open in $(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}, d)$ and thus so is U.

$\supseteq$: Following the same strategy as in the previous inclusion, let $U \subseteq \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ be an open set in the topology induced by the metric d. Then, we know that U has the form $\cup_{i\in I} B_{f_i}(\varepsilon_i)$ for some set I and for some $f_i \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ and some $\varepsilon_i > 0$ for each $i \in I$. We define the class $\mathcal{K}_i := \{\mathbf{A} \in \mathcal{V}_0 : |A| < m_i\}$ for every $i \in I$. Notice that the cardinality of each $\mathcal{K}_i$ is finite since the language of $\mathcal{V}$ is finite. Then, we claim that

$$U = \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \cap (\bigcup_{i\in I}(\cap_{\mathbf{A}\in\mathcal{K}_i}\pi^{-1}_{\mathbf{A}}[\{f_i^{\mathbf{A}}\}])).$$

Pick some $g \in U$. Then, there is some $i \in I$ such that $g \in B_{f_i}(\varepsilon_i)$. By Lemma 3.31 we know that $g \in \cap_{\mathbf{A}\in\mathcal{K}_i}\pi^{-1}_{\mathbf{A}}[\{f_i^{\mathbf{A}}\}]$, hence $g \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \cap (\bigcup_{i\in I}(\cap_{\mathbf{A}\in\mathcal{K}_i}\pi^{-1}_{\mathbf{A}}[\{f_i^{\mathbf{A}}\}]))$ [XIII]. For the other inclusion, pick $g \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \cap (\bigcup_{i\in I}(\cap_{\mathbf{A}\in\mathcal{K}_i}\pi^{-1}_{\mathbf{A}}[\{f_i^{\mathbf{A}}\}]))$, so that there is some $i \in I$ such that $g \in \cap_{\mathbf{A}\in\mathcal{K}_i}\pi^{-1}_{\mathbf{A}}[\{f_i^{\mathbf{A}}\}]$. Again, by Lemma 3.31 we have $g \in B_{f_i}(\varepsilon_i)$. Thus, $g \in U$ and we are done.

$\boxtimes$

[XII] Otherwise, there must be some $i \in I$ and some $\mathbf{A} \in \mathcal{K}_i$ such that $f^{\mathbf{A}} \neq g^{\mathbf{A}}$ and therefore $d(f,g) \geqslant 2^{-|\mathbf{A}|}$, which contradicts the fact that $g \in V_f$.

[XIII] From $g \in b \in U$ we already have $g \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$.

CHAPTER 4

REITERMAN'S THEOREM

Once we have developed all the necessary framework, we can finally undertake our main objective: Reiterman's Theorem for pseudovarieties. Nevertheless, and although we have the sufficient ingredients to state it, some previous work is still needed for the proof of the main Theorem. Therefore, in the following we will set the lemmas (and some Propositions). Recall that on each algebra $\mathbf{A} \in \mathcal{V}_0$ we are considering the discrete topology; so, from now on, everytime some algebra $\mathbf{A} \in \mathcal{V}_0$ appears it would be assumed to be endowed with the discrete topology. In addition, $n \in \omega$ and $\mathcal{V}$ are still fixed.

Lemma 4.1. *Let $\mathbf{A}$ be a finite algebra of the same type of $\mathcal{V}$ and $h : \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \to \mathbf{A}$ a continuous homomorphism. Consider a subpseudovariety $\mathcal{W}$ of $\mathcal{V}$ and assume that for every $f, g \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ whenever we have the equality $p(f) = p(g)$[I] it holds that $h(f) = h(g)$. Then, the mapping $\bar{h} : \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \to \mathbf{A}$ defined by the rule $\bar{h}(p(f)) = h(f)$ is a well-defined continuous homomorphism.*

Proof. First of all, to see that it is well-defined, let $f, g \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$ such that $f = g$. Then, by the surjectivity of p (see Lemma 3.28) there are some $f', g' \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ such that $p(f') = f$ and $p(g') = g$. Since, $f = g$ (that is, $p(f') = p(g')$), we have by assumption that $h(f') = h(g')$. But $\bar{h}(p(f')) = h(f')$ and $\bar{h}(p(g')) = h(g')$, so we conclude that $\bar{h}(f) = \bar{h}(g)$. It remains to check that $\bar{h}$ is a homomorphism and continuous.

We begin seeing that it is a homomorphism. So, let $\sigma \in \tau$ be a k-ary function symbol and pick some $f_1, \ldots, f_k \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$. We need to check the following equality:

$$\bar{h}(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(f_1), \ldots, p(f_k))) = \sigma^{\mathbf{A}}(\bar{h}(p(f_1)), \ldots, \bar{h}(p(f_k))).$$

So,

$$\begin{aligned}
\bar{h}(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(f_1), \ldots, p(f_k))) &= \bar{h}(p(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_k))) \\
&= h(\sigma^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(f_1, \ldots, f_k)) \\
&= \sigma^{\mathbf{A}}(h(f_1), \ldots, h(f_k)) \\
&= \sigma^{\mathbf{A}}(h(p(f_1)), \ldots, h(p(f_k)))
\end{aligned}$$

where the first and the third equalities hold since p and h are homomorphisms. In the other ones we just apply the definition of h.

It remains then to see that h is continuous, so consider an open set $U \subseteq A$. We want to see that $h^{-1}[U]$ is open. The strategy we are going to follow is to first prove the equality

$$h^{-1}[U] = (h^{-1}[U^c])^c \tag{4.1}$$

and then see that $h^{-1}[U^c]$ is indeed a closed set to finish the proof. From now on, thanks to the surjectivity of p, we will refer to elements in $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$ as $p(f)$ for some $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$.

[I]This function p is the same as in Lemma 3.25.

The equality 4.1 is easy to see (actually, it is a general fact):

$$\begin{aligned} p(f) \in h^{-1}[U] &\Leftrightarrow h(p(f)) \in U \Leftrightarrow h(p(f)) \notin U^c \\ &\Leftrightarrow p(f) \notin h^{-1}[U^c] \\ &\Leftrightarrow p(f) \in (h^{-1}[U^c])^c \end{aligned}$$

To see that $h^{-1}[U^c]$ is closed, we claim that $h^{-1}[U^c] = p[h^{-1}[U^c]]$. Observe that since p is a continuous function between compact Hausdorff spaces it is also a closed function (see Lemma 2.72). Therefore, since U^c is closed and h continuous, we have that $h^{-1}[U^c]$ is closed and hence $p[h^{-1}[U^c]]$ is closed. So, we prove the needed equality:

$$\begin{aligned} h^{-1}[U^c] &= \{p(f) : f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}, p(f) \in h^{-1}[U^c]\} \\ &= \{p(f) : f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}, h(p(f)) \in U^c\} \\ &= \{p(f) : f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}, h(f) \in U^c\} \\ &= \{p(f) : f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}, f \in h^{-1}[U^c]\} \\ &= p[h^{-1}[U^c]] \end{aligned}$$

⊠

Proposition 4.2. *Let* $\mathbf{A} \in \mathcal{V}$ *and* $h : \{\hat{x}_1, ..., \hat{x}_n\} \to \mathbf{A}$ *a mapping. Then, there is a unique continuous homomorphism* $\bar{h} : \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \to \mathbf{A}$ *extending* h.

Proof. First, we will show that it is unique. So, consider two extensions $\bar{h}_1$ and $\bar{h}_2$ that are continuous homomorphisms. Since, by assumption, for $j = 1, 2$ and every $i \leqslant n$ we have $\bar{h}_j(\hat{x}_i) = h(\hat{x}_i)$, then certainly $\bar{h}_1 \restriction {\Omega_{\mathbf{n}}}^{\mathcal{V}} = \bar{h}_2 \restriction {\Omega_{\mathbf{n}}}^{\mathcal{V}}$ [II]. Therefore, since ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ is dense in $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ we conclude by Proposition 2.76 that $\bar{h}_1 = \bar{h}_2$.

We turn to prove the existence of such an extension, so let $\mathcal{W} = p\mathbb{V}(\mathbf{A})$ and notice that $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} = {\Omega_{\mathbf{n}}}^{\mathcal{W}}$ (see Lemma 3.20). Let $h_p : \{p(\hat{x}_1), \ldots, p(\hat{x}_n)\} \to \mathbf{A}$ be the mapping defined as $h_p(p(\hat{x}_i)) := h(\hat{x}_i)$ for every $i \leqslant n$. Observe also that, since $\mathbf{A} \in \mathcal{W}$, we have due to Corollary 3.16 that there is a unique homomorphism $h' : \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \to \mathbf{A}$ extending h_p (i.e., $h'(p(\hat{x}_i)) = h_p(\hat{x}_i)$ for every $i \leqslant n$). We define the mapping $\bar{h} = h' \circ p$ and we claim that $\bar{h}$ satisfies the desired conditions. It is clear that it is a homomorphism[III], so it remains to check that it is continuous. To see that $\bar{h}$ is continuous it suffices to check that h' is continuous; so, let $U \subseteq A$ be an arbitrary subset, which is open since we have the discrete topology in $\mathbf{A}$. We define $X := \{\{h(\hat{x}_1)\} \times \cdots \times \{h(\hat{x}_n)\} \times U\}$, which is a open subset of

[II] Recall that ${\Omega_{\mathbf{n}}}^{\mathcal{V}}$ is the subalgebra of $\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ generated by the projection mappings $\hat{x}_1, \ldots, \hat{x}_n$. Then, since every member $f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}}$ is of the form $p^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \ldots, \hat{x}_n)$,

[III] The composition of homomorphism is a homomorphism. See Lemma 2.11.

A^{A^n}[IV]. Then,

$$(h')^{-1}[U] = \{f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} : h'(f) \in U\}$$

We claim that

$$(h')^{-1}[U] = \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \cap \pi_{\mathbf{A}}^{-1}[X]. \tag{4.2}$$

To prove the inclusion $\subseteq$ pick some $f \in (h')^{-1}[U]$. Then, it suffices to check that $f^{\mathbf{A}}(h(\hat{x}_1), \ldots, h(\hat{x}_n)) \in U$. Since $f \in \Omega_{\mathbf{n}}{}^{\mathcal{W}}$, there is some $t \in \mathbf{T}(x_1, \ldots, x_n)$ such that $f = t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \ldots, p(\hat{x}_n))$. So, since h' is a homomorphism we have

$$\begin{aligned} h'(f) &= h(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \ldots, p(\hat{x}_n))) \\ &= t^{\mathbf{A}}(h'(p(\hat{x}_1)), \ldots, p(\hat{x}_n)) \\ &= t^{\mathbf{A}}(h(\hat{x}_1), \ldots, h(\hat{x}_n)). \end{aligned}$$

Finally, notice that

$$\begin{aligned} \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \cap \pi_{\mathbf{A}}^{-1}[X] &= \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \cap \pi_{\mathbf{A}}(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \ldots, p(\hat{x}_n))) \\ &= (t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \ldots, p(\hat{x}_n)))^{\mathbf{A}}, \end{aligned}$$

so

$$(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \ldots, p(\hat{x}_n)))^{\mathbf{A}}(h(\hat{x}_1) \ldots, h(\hat{x}_n)) = t^{\mathbf{A}}(h(\hat{x}_1), \ldots, h(\hat{x}_n)) = h'(f) \in U,$$

which means that $\pi_{\mathbf{A}}(f) \in X$ and therefore $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \cap \pi_{\mathbf{A}}^{-1}[X]$.

For the other inclusion, let $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \cap \pi_{\mathbf{A}}^{-1}[X]$. From we have mentioned above we have

$$(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \ldots, p(\hat{x}_n)))^{\mathbf{A}} \in X,$$

so, by the definition of X, we get that $(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \ldots, p(\hat{x}_n)))^{\mathbf{A}}(h(\hat{x}_1) \ldots, h(\hat{x}_n)) \in U$ and therefore $f \in (h')^{-1}[U]$. Thus, we conclude that $(h')^{-1}[U] \supseteq \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \cap \pi_{\mathbf{A}}^{-1}[X]$.

As $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \cap \pi_{\mathbf{A}}^{-1}[X]$ is an open set in $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}$, the set $(h')^{-1}[U]$ is also open and therefore h' is a continuous homomorphism. ⊠

Lemma 4.3. *Let $h : \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \to \mathbf{A}$ be a surjective continuous homomorphism for some algebra $\mathbf{A} \in \mathcal{V}_0$. Then, for every $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ it holds that*

$$h(f) = f^{\mathbf{A}}(h(\hat{x}_1), \ldots, h(\hat{x}_n)).$$

[IV] Recall that, we are considering on $\mathbf{A}^{\mathbf{A^n}}$ the discrete topology and that $A^{A^n} \subseteq \mathcal{P}(A^{n+1})$ (see [11]).

Proof. Let $\mathcal{W} := p\mathbb{V}(\mathbf{A})$ and define the mapping $h_1 : \{p(\hat{x}_1), \dots, p(\hat{x}_n)\} \to A$ as $h_1(p(\hat{x}_i)) = h(\hat{x}_i)$ for every $i \leqslant n$[V]. Then, by Proposition 4.2 we have a continuous homomorphism $h_1' : \bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} \to \mathbf{A}$ extending h_1. Moreover, since by Lemma 3.20 we have that $\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}} = {\Omega_{\mathbf{n}}}^{\mathcal{W}}$, for every $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ there is some $t \in \mathbf{T}(x_1, \dots, x_n)$ such that $p(f) = t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \dots, p(\hat{x}_n))$. Hence, for every $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$ we have

$$\begin{aligned}(h_1' \circ p)(f) &= h_1'(p(f)) \\ &= h_1'(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \dots, p(\hat{x}_n))) \\ &= t^{\mathbf{A}}(h_1'(p(\hat{x}_1)), \dots, h_1'(p(\hat{x}_n))) \\ &= t^{\mathbf{A}}(h(\hat{x}_1)), \dots, h(\hat{x}_n)) \\ &= f^{\mathbf{A}}(h(\hat{x}_1)), \dots, h(\hat{x}_n)),\end{aligned}$$

where the third equality holds since h_1' is a homomorphism and the fourth one is by the definition of h_1'. The last equality is justified since $f^{\mathbf{A}} = (p(f))^{\mathbf{A}}$ because $\mathbf{A} \in \mathcal{W}_0$. Finally, we want to show that $h_1' \circ p = h$. To do that, pick an arbitrary $f \in {\Omega_{\mathbf{n}}}^{\mathcal{V}}$. Then, we have $f = t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \dots, \hat{x}_n)$ for some $t \in \mathbf{T}(x_1, \dots, x_n)$, and therefore (since p, h_1' and h are homomorphisms)

$$\begin{aligned}(h_1' \circ p)(f) &= (h_1' \circ p)(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \dots, \hat{x}_n)) \\ &= h_1'(p(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}}(\hat{x}_1, \dots, \hat{x}_n))) \\ &= h_1'(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{W}}}(p(\hat{x}_1), \dots, p(\hat{x}_n)))) \\ &= t^{\mathbf{A}}(h_1'(p(\hat{x}_1)), \dots, h_1'(p(\hat{x}_n)))) \\ &= t^{\mathbf{A}}(h(\hat{x}_1), \dots, h(\hat{x}_n)) \\ &= h(t^{\bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}})(\hat{x}_1, \dots, \hat{x}_n) \\ &= h(f).\end{aligned}$$

We have showed that $(h_1' \circ p) \restriction {\Omega_{\mathbf{n}}}^{\mathcal{V}} = h \restriction {\Omega_{\mathbf{n}}}^{\mathcal{V}}$, so by Proposition 2.76 we obtain that $h_1' \circ p = h$[VI]. We conclude then that for every $f \in \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}}$

$$\begin{aligned}h(f) &= (h_1' \circ p)(f) \\ &= f^{\mathbf{A}}(h(\hat{x}_1)), \dots, h(\hat{x}_n)).\end{aligned}$$

$\boxtimes$

Proposition 4.4. *Let* $\mathbf{A}$ *be a finite algebra. Then,* $\mathbf{A} \in \mathcal{V}$ *if and only if there is some* $n \in \omega$ *and a continuous surjective homomorphism* $h : \bar{\Omega}_{\mathbf{n}}^{\mathcal{V}} \to \mathbf{A}$.

[V] Recall that p is the function from Lemma 3.25.

[VI] Recall that the composition of continuous functions is continuous, see Lemma 2.69.

Proof. For the left to right implication, assume that $\mathbf{A} \in \mathcal{V}$. Taking $n = |A|$ we can define a mapping $h : \{\hat{x}_1, \ldots, \hat{x}_n\} \to \mathbf{A}$ so that it becomes surjective. Then, by Proposition 4.2 it extends uniquely to the continuous homormophism $\bar{h} : \bar{\Omega}^{\mathcal{W}}_{\mathbf{n}} \to \mathbf{A}$ that we need.

For the other implication, suppose that we have some surjective and continuous homomorphism $h : \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \to \mathbf{A}$. Since $(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}, d)$ is a compact metric space, we have by Proposition 2.86 that h is also uniform. Hence, there is some $m > 0$ (notice in fact that $m = |\mathbf{A}|$) such that for every $f, g \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$

$$d(f,g) < 2^{-m} \Rightarrow h(f) = h(g) \tag{4.3}$$

Consider the class $\mathcal{K} := \{\mathbf{B} \in \mathcal{V}_0 : |B| \leqslant m\}$ and recall that $|\mathcal{K}| < \omega$. Define $\mathcal{W} := p\mathbb{V}(\mathcal{K})$. Since it is finitely generated, we know by Corollary 3.19 that $\bar{\Omega}^{\mathcal{W}}_{\mathbf{n}} \in \mathcal{W} \subseteq \mathcal{V}$. Hence, if we prove that h there is a mapping $\bar{h} : \bar{\Omega}^{\mathcal{W}}_{\mathbf{n}} \to \mathbf{A}$ such that $\bar{h}(p(f)) = h(f)$ for every $f \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$, we will then have that $\mathbf{A} \in \mathcal{V}$ as desired (because $\mathbf{A} \in \mathbb{H}(\{\bar{\Omega}^{\mathcal{W}}_{\mathbf{n}}\}) \subseteq \mathcal{W}$[VII]).

Observe that, by definition of $\mathcal{W}$, the equality $p(f) = p(g)$ implies that $d(f,g) < 2^{-m}$ [VIII]. So, by condition 4.3 we obtain that $p(f) = p(g) \Rightarrow h(f) = h(g)$. Applying Lemma 4.1 we obtain automatically the continuous homomorphism $\bar{h} : \bar{\Omega}^{\mathcal{W}}_{\mathbf{n}} \to \mathbf{A}$ that we want. In addition, in this case, the surjectivity of h implies the surjectivity of $\bar{h}$ too.

$\boxtimes$

Given a subclass $\mathcal{K}$ of $\mathcal{V}$ and a set $\Sigma \subseteq \cup_{n\in\omega}(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}})^2$ of pseudoidentities we write $\mathcal{K} \Vdash \Sigma$ whenever $\mathcal{K} \Vdash f \approx g$ for every $f \approx g \in \Sigma$. Moreover, we define the subclass defined by some Σ as

$$[\Sigma]_{\mathcal{V}} := \{\mathbf{A} \in \mathcal{V} : \mathbf{A} \Vdash \Sigma\}.$$

We can now tackle the main Theorem, that states as follows:

Theorem 4.5 (Reiterman). *Let $\mathcal{W}$ a subclass of a pseudovariety $\mathcal{V}$. $\mathcal{W}$ is a pseudovariety if and only if $\mathcal{W}$ is defined by some set of pseudoidentities on $\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$.*

Proof. We begin with the implication from left to right. So, suppose that $\mathcal{W}$ is a pseudovariety. Define the set of pseudoidentities

$$\Sigma := \{f \approx g \in \cup_{n\in\omega}(\bar{\Omega}^{\mathcal{V}}_{\mathbf{n}})^2 : \mathcal{W} \Vdash f \approx g\}.$$

We claim that $\mathcal{W} = [\Sigma]_{\mathcal{V}}$. The inclusion $\subseteq$ follows from the definition of Σ.
To prove the inclusion $\supseteq$, consider some algebra $\mathbf{A}$ in $[\Sigma]_{\mathcal{V}}$. In view of Proposition 4.4, in order to prove that $\mathbf{A} \in \mathcal{W}$ it suffices to find some continuous surjective homomorphism

[VII] $\mathcal{W}$ is a pseudovariety.

[VIII] Suppose that $d(f,g) \geqslant 2^{-m}$. Then, $r(f,g) \leqslant m$ and therefore there is some algebra $\mathbf{A} \in \mathcal{V}_0$ with $|A| \leqslant m$ such that $f^{\mathbf{A}} \neq g^{\mathbf{A}}$. Since $|A| \leqslant m$ we have $\mathbf{A} \in \mathcal{K}$ and therefore $\mathbf{A} \in \mathcal{W}$. This implies that $p(f) \neq p(g)$, which contradicts the assumption $p(f) = p(g)$.

$\bar{h} : \bar{\Omega}^{\mathcal{W}}_{\mathbf{n}} \to \mathbf{A}$ for some $n \in \omega$. Notice that, also by Proposition 4.4, since $\mathbf{A} \in \mathcal{V}$, there is some $n < \omega$ and some continuous surjective homomorphism $h : \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}} \to \mathbf{A}$.

We want to apply here Lemma 4.1, so we need to show that for every $f, g \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ if $p(f) = p(g)$ then $h(f) = h(g)$. So, let $f, g \in \bar{\Omega}^{\mathcal{V}}_{\mathbf{n}}$ such that $p(f) = p(g)$. Observe that $p(f) = p(g)$ implies that $f \approx g \in \Sigma$. Recall that by Lemma 4.3 we know that $h(f) = f^{\mathbf{A}}(h(\hat{x_1}), \ldots, h(\hat{x_n}))$ and the same for g; hence, since $\mathbf{A} \Vdash f \approx g$

$$h(f) = f^{\mathbf{A}}(h(\hat{x_1}), \ldots, h(\hat{x_n})) = g^{\mathbf{A}}(h(\hat{x_1}), \ldots, h(\hat{x_n})) = h(g)\ .$$

Therefore, due to Lemma 4.1 we obtain a continuous homomorhpism $\bar{h}$, which is surjective because both p and h are surjective[IX]. Finally, since there is a surjective continuous homomorphism $\bar{h} : \bar{\Omega}^{\mathcal{W}}_{\mathbf{n}} \to \mathbf{A}$ we conclude that $\mathbf{A} \in \mathcal{W}$.

For the other implication, if $\mathcal{W}$ is defined by some set of pseudoidentities Σ, that is, $\mathcal{W} = [\Sigma]_{\mathcal{V}}$, what we need to check that $\mathcal{W}$ is closed under the operators $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}_{fin}$.

$\mathbb{S}$: Pick some $\mathbf{A} \in \mathcal{W}$ and let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$. We need to check that $\mathbf{B} \Vdash \Sigma$, so pick some $(f, g) \in \Sigma$ in order to see that $f^{\mathbf{B}} = g^{\mathbf{B}}$. Notice first that we already have that $f^{\mathbf{A}} = g^{\mathbf{A}}$. Moreover, observe that $f^{\mathbf{B}} = f^{\mathbf{A}} \restriction B$ and the same for g, hence we automatically get that $f^{\mathbf{B}} = g^{\mathbf{B}}$.

$\mathbb{H}$: Pick $\mathbf{A} \in \mathcal{W}$ and consider $\mathbf{B} = h[\mathbf{A}]$ for some homomorphism h. By the definition of implicit operations, we have the equalities $h \circ f^{\mathbf{A}} = f^{\mathbf{B}} \circ h^n$ and $h \circ g^{\mathbf{A}} = g^{\mathbf{B}} \circ h^n$. Now, since by assumption we have $f^{\mathbf{A}} = g^{\mathbf{A}}$ we also have the equality $h \circ f^{\mathbf{A}} = h \circ g^{\mathbf{A}}$. Therefore,

$$f^{\mathbf{B}} \circ h^n = h \circ f^{\mathbf{A}} = h \circ g^{\mathbf{A}} = g^{\mathbf{B}} \circ h^n$$

and hence $f^{\mathbf{B}} = g^{\mathbf{B}}$. This means that $\mathbf{B} \Vdash f \approx g$ and that $\mathbf{B} \in \mathcal{W}$.

$\mathbb{P}_{fin}$: Consider $\mathbf{A}_1, ..., \mathbf{A}_m \in \mathcal{W}$ and let $\mathbf{B} = \mathbf{A}_1 \times ... \times \mathbf{A}_m$. Pick some arbitrary $(f, g) \in \Sigma$. From now on, fix an arbitrary $i \leqslant m$ and consider the i-th projection p_i. Of course, p_i is a homomorphism and therefore we get due to the same reason as in the case for homomorphic images that $p_i \circ f^{\mathbf{B}} = f^{\mathbf{A}_i} \circ p_i^n$ and $p_i \circ g^{\mathbf{B}} = g^{\mathbf{A}_i} \circ p_i^n$. Notice that in fact those equalities are valid for every $i \leqslant m$. Now, by assumption, we know that $f^{\mathbf{A}_i} = g^{\mathbf{A}_i}$ for every $i \leqslant m$; hence, we also have that $f^{\mathbf{A}_i} \circ p_i^n = \rho_{\mathbf{A}_i} \circ p_i^n$. Then, for every $i \leqslant m$ we have (similarly as before)

$$p_i \circ f^{\mathbf{B}} = f^{\mathbf{A}_i} \circ p_i^n = g^{\mathbf{A}_i} \circ p_i^n = p_i \circ g^{\mathbf{B}}.$$

That is, $p_i \circ f^{\mathbf{B}} = p_i \circ g^{\mathbf{B}}$ for every $i \leqslant m$, thus we conclude that $f^{\mathbf{B}} = g^{\mathbf{B}}$ and $\mathbf{B} \Vdash f \approx g$, meaning that $\mathbf{B} \in \mathcal{W}$ as desired.

⊠

[IX] For the surjectivity of p see Proposition 3.28.

CHAPTER 5

SOME EXAMPLES OF PSEUDOVARIETIES

5.1 Equational pseudovarieties

An easy way of getting examples of pseudvarieties is to consider what we called equational pseudovarieties. This classes are constituted by the finite algebras of some variety. That is, if we have some variety $\mathcal{V}$, we denote by $\mathcal{V}^F$ the class of all its finite members. It does not require a great effort to notice that $\mathcal{V}^F$ is closed under finite products, subalgebras and homomorphic images due to the fact that $\mathcal{V}$ is a variety. Therefore, it is clear that $\mathcal{V}^F$ is a pseudovariety. We say then that a pseudovariety $\mathcal{W}$ is an equational pseudovariety if $\mathcal{W} = \mathcal{V}^F$ for some variety $\mathcal{V}$.

Consider the language $\{\cdot\}$ of semigroups. The pseudovariety of finite semigroups is the first example one can contemplate. In fact, observe that it is an equational pseudovariety since it is equal to $\mathcal{V}^F$ for the variety $\mathcal{V}$ of semigroups defined by the identity $\Sigma := \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z)\}$. We only need to interpret the terms $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ as explicit operations on $\mathcal{S}$ and see Σ as a set of pseudoidentities.

Sometimes, if we add symbols to our language, psuedovarities that are not equational with respect the original language, when considering the expanded language they become equational. For instance, if we consider the language $\{\cdot, 1\}$ the pseudovariety of finite monoids $\mathcal{V}$ is equational since it is equal to $\mathcal{W}^F$ where $\mathcal{W}$ is the variety of monoids defined by the equations $\{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x \cdot 1 \approx x,\ 1 \cdot x \approx x\}$. Otherwise, it is not equational since the unit is not term-definable. Nevertheless, there are examples where expanding the language does not help as the class of nilpotent semigroups (see [1]).

5.2 Pseudovarieties that are not equational

Let $\mathcal{S}$ be the class of all finite semigroups. The class of finite groups is another example of pseudovariety that can not be defined by a set of identities. Recall that by Proposition 2.39 in every finite semigroup $\mathbf{S}$ for every element there is some power of it (denoted by $s^\omega_{\mathbf{S}}$) that is idempotent. Then, for every finite semigroup S we can define the mapping

$$\begin{array}{rccc} x^\omega_{\mathbf{S}} & : & \mathbf{S} & \to & \mathbf{S} \\ & & s & \mapsto & s^\omega_{\mathbf{S}} \end{array}$$

It turns out that the sequence $x^\omega := \langle x^\omega_{\mathbf{S}} : \mathbf{S} \in \mathcal{S} \rangle$[I] is an implicit operation over $\mathcal{S}$ since it commutes with homomorphisms as we will show now. We take the notation $s^\omega_{\mathbf{S}}$ to denote $x^\omega_S(s)$. Let $\mathbf{S}_1, \mathbf{S}_2 \in \mathcal{S}$ and a homomorphism $h : \mathbf{S}_1 \to \mathbf{S}_2$. To see that x^ω commutes with h, we want to check that for every $s \in \mathbf{S}_1$ it holds $h(s^\omega_{\mathbf{S}_1}) = (h(s))^\omega_{\mathbf{S}_2}$. Recall that we have some $n_{\mathbf{S}_1} > 0$ such that for every $s \in S_1$ we have $s^\omega_{\mathbf{S}_1} = s^{n_{\mathbf{S}_1}}$ and (see Proposition

[I] That is, following the notation used along the text, for every finite semigroup $\mathbf{S} \in \mathcal{S}$ we have $(x^\omega)^{\mathbf{S}} = x^\omega_{\mathbf{S}}$.

2.39). So, in particular, there is some $n_{\mathbf{S}_1} > 0$ such that $s^{\omega}_{\mathbf{S}_1} = s^{n_{\mathbf{S}_1}}$. Observe that, since h is a homomorphism and since $s^{\omega}_{\mathbf{S}_1} \cdot s^{\omega}_{\mathbf{S}_1} = s^{\omega}_{\mathbf{S}_1}$:

$$\begin{aligned} h(s^{\omega}_{\mathbf{S}_1}) &= h(s^{\omega}_{\mathbf{S}_1} \cdot s^{\omega}_{\mathbf{S}_1}) \\ h(s^{n_{\mathbf{S}_1}}) &= h(s^{n_{\mathbf{S}_1}} \cdot s^{n_{\mathbf{S}_1}}) \\ h(s)^{n_{\mathbf{S}_1}} &= h(s)^{n_{\mathbf{S}_1}} \cdot h(s)^{n_{\mathbf{S}_1}}, \end{aligned}$$

so the element $h(s)^{n_{\mathbf{S}_1}} \in \mathbf{S}_2$ is idempotent. Therefore, by Remark 2.40, there is some $k \in \omega$ such that $n_{\mathbf{S}_1} \cdot k = n_{\mathbf{S}_2}$. Hence,

$$h(s)^{n_{\mathbf{S}_2}} = h(s)^{n_{\mathbf{S}_1} \cdot k} = (h(s)^{n_{\mathbf{S}_1}})^k = h(s)^{n_{\mathbf{S}_1}}.$$

We obtain then $h(s^{\omega}_{\mathbf{S}_1}) = (h(s))^{\omega}_{\mathbf{S}_2}$ as desired.

Proposition 5.1. *The class $\mathcal{G}$ of finite groups forms a pseudovariety.*

Proof. We only need to check that $\mathcal{G}$ is closed under subalgebras, homomorphic images and finite products. Clearly, every homomorphic image of a group is again a group. The finite product of finite groups is also a finite group since we only need to define the operation in the product pointwise. Finally, by Proposition 2.36 we know that a subsemigroup of a finite group is a subgroup, so $\mathcal{G}$ is closed under subalgebras too. ⊠

Proposition 5.2. *The class $\mathcal{G}$ of finite groups is not equational.*

Proof. Let $\mathcal{G}$ be the class of finite groups and consider the variety $\mathcal{V} := \mathbb{V}(\mathcal{G})$. We will prove that in $\mathcal{V}$ there is some algebra that is not a group; therefore, $\mathcal{V}$ can not be the variety of all groups.

Consider the 1-generated free semigroup $\mathbf{F}_1\mathcal{V}$ and notice that $\mathbf{F}_1\mathcal{V} \cong (\mathbb{Z}^+, +)$[II]. Pick an arbitrary $n \in \omega$. We can define a sujective homomorphism $h_n : \mathbb{Z}^+ \to \mathbf{Z}_n$ (where $\mathbf{Z}_n$ is the cyclic group of n elements) by the rule $h_n(x) = x(\bmod n)$. The mapping is clearly surjective, and the fact that it is a homomorphism is justified by the property

$$x_1 \equiv x_2(\bmod n),\ y_1 \equiv y_2(\bmod n) \Rightarrow x_1 + y_1 \equiv x_2 + y_2(\bmod n)\ .$$

With all of these homomorphism, we define the homomorphism $h : (\mathbb{Z}^+, +) \to \prod_{n \in \omega} \mathbf{Z}_n$ by the rule

$$h(x) := \langle h_n(x) : n \in \omega \rangle\ .$$

Moreover, it is easy to check that h is in fact injective. Pick some arbitrary $x \neq y$ in $\mathbb{Z}^+$. Then, there is some $n \in \omega$ such that $h_n(x) \neq h_n(y)$ (namely $n = \max\{x, y\}$), and therefore

[II] Although we are considering the language of $\mathcal{V}$ to be $\{\cdot\}$, we use of the additive notation in $(^+, +)$ in order to avoid any possible misunderstanding. We could interpret $\cdot$ in $\mathbb{N}$ as the usual sum, but this could lead to confusion.

$h(x) \neq h(y)$. By the Homomorphism theorem we obtain that $(\mathbb{Z}^+, +)$ is isomrphic to a subalgebra of the product $\prod_{n\in\omega} \mathbf{Z}_n$. Since each $\mathbf{Z}_n \in \mathcal{V}$ and $\mathcal{V}$ is a variety (it is closed under $\mathbb{P}$), we also have that $\prod_{n\in\omega} \mathbf{Z}_n \in \mathcal{V}$. In addition, since $\mathcal{V}$ is closed under $\mathbb{H}$, we obtain that $(\mathbb{Z}^+, +) \in \mathcal{V}$. It is well-known that $\mathbb{Z}^+$ is not a group, so we have found an algebra in $\mathcal{V}$ that is not a group and therefore $\mathcal{V}$ can not be the variety of all groups. ⊠

We have shown that the pseudovariety $\mathcal{G}$ of finite groups is not equational, but due to Reiterman's theorem we know that there must be some set of pseudoidentities defining it. In the following, we will show that $\mathcal{G} = [\Sigma]_{\mathcal{S}}$ where

$$\Sigma := \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x^\omega \cdot y \approx y,\ y \cdot x^\omega \approx y\}.$$

Theorem 5.3. *The pseudovaritey $\mathcal{G}$ of finite groups of type $\{\cdot\}$ is equal to the pseudovariety defined by the set of pseudoidentities $\Sigma := \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x^\omega \cdot y \approx y,\ y \cdot x^\omega \approx y\}$ over $\mathcal{S}$.*

Proof. Let $\mathbf{G} \in \mathcal{G}$ be an arbitrary finite group. Since groups are associative, it is clear that $\mathbf{G} \Vdash (x \cdot y) \cdot z \approx x \cdot (y \cdot z)$. Moreover, there is a unit element $1 \in G$, which is the unique idempotent (see Proposition 2.38). Therefore, for every $s \in G$ the element $s^\omega_{\mathbf{G}}$ is necessarily equal to 1. That is, the mapping $x^\omega_{\mathbf{G}} : \mathbf{G} \to \mathbf{G}$ sends each $g \in \mathbf{G}$ to 1. It is obvious then that $\mathbf{G} \Vdash x^\omega \cdot y \approx y$ and $\mathbf{G} \Vdash y \cdot x^\omega \approx y$. Thus, $\mathbf{G} \in [\Sigma]_{\mathcal{S}}$ and therefore $\mathcal{G} \subseteq [\Sigma]_{\mathcal{S}}$. It remains to check the other inclusion, so pick some $\mathbf{A} \in [\Sigma]_{\mathcal{S}}$. Since $\mathbf{A} \Vdash \Sigma$, it is clear that $\mathbf{A}$ is an asociative semigroup. Moreover, the pseudoidentities $x^\omega \cdot y \approx y$ and $y \cdot x^\omega \approx y$ ensure the existence of a unit element in $\mathbf{A}$ (that we will denote 1 from now on), which by Remark 2.33 is unique. To see that every member in $\mathbf{A}$ has a inverse element, fix some arbitrary $a \in A$. Recall that $a^\omega_{\mathbf{A}} = a^{n_{\mathbf{A}}}$ for some $n_{\mathbf{A}} > 0$. Then,

$$\begin{aligned} a \cdot a^{n_{\mathbf{A}}-1} &= a^{n_{\mathbf{A}}} \\ &= a^\omega_{\mathbf{A}} \\ &= 1. \end{aligned}$$

Therefore, the element $a^{n_{\mathbf{A}}} \cdot a^{n_{\mathbf{A}}-1}$ is an inverse of a and we conclude that $\mathbf{A}$ is a group and $\mathbf{A} \in \mathcal{G}$. Thus, $[\Sigma]_{\mathcal{S}} \subseteq \mathcal{G}$ and we end the proof. ⊠

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