A novel approach to cosmological fermion production

Author: Jordi Ruiz Ponsell
*Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.**

Advisor: Bartomeu Fiol Núñez

Abstract: This work extends a recent approach to gravitational particle production in asymptotically flat spacetimes, originally applied to scalar particles, to explore the cosmological production of Dirac fermions. This method avoids the necessity of solving the equation of motion by examining monodromies around singular points in the equations. We derive a general expression for the amount of spin-1/2 particle production, and elucidate to which scale factors it applies. In contrast to the scalar case, which exhibits three regimes, Dirac fermions present a single regime of production. Simple scale factor examples are examined, recovering easily all known results and producing new ones.

I. INTRODUCTION

Gravitational particle production is one of the earliest outcomes emerging from the application of quantum field theory in curved spacetimes. This was already predicted by Schrödinger [1] and then established by Parker in the late 60s [2, 3]. Nowadays, the interest on this topic is driven by its relevance to applications such as baryogenesis and the understanding of dark matter's origin.

To recall the basic phenomenon consider a 3+1 dimensional universe undergoing a homogeneous and isotropic expansion, as described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 = a(\eta)^2(-d\eta^2 + d\vec{x}^2) \qquad (1)$$

and place a quantum field on this background, taken to be in the vacuum state at very early times. Since the scale factor $a(t)$ is a function of time, the gravitational field is time dependent and the time translation symmetry is broken. Hence, the spacetime no longer possesses Poincaré invariance. Then, the conservation of energy for quantum fields is no longer imperative because they can exchange energy with the gravitational background. Observers at asymptotic late times will describe the field not to be in their notion of vacuum, but in an excited state.

The traditional approach to cosmological particle production [3, 4] requires solving the equation of motion around two different times, which severely limits the amount of fully solved examples. Recently, a new perspective has emerged, capable of providing an expression for the creation of scalar particles without the need to solve the equation of motion [5]. Our goal in this work is to extend this formalism to the study of gravitational production of Dirac fermions.

The structure of this work is as follows. In section II scalar production [5] is reviewed, exposing the possibility of different regimes of particle production. In section III we present our results on cosmological production of Dirac fermions, concluding that there is only one regime of production. Sections IV and V are devoted to the study of analytically solvable profiles of expansion corresponding to equations with three and four regular singular points (RSPs). Finally, in section VI the most important results are summarized.

II. SCALAR FIELD WARM UP

In order to study scalar production, we consider a real massive scalar field coupled to the gravitational field through the Ricci scalar, which leads to the following equation of motion

$$\Box\phi - m^2\phi - \xi R\phi = 0 \qquad (2)$$

Any solution to a second order linear differential equation can be expressed as a sum over a complete set of positive and negative norm solutions,

$$\phi = \sum_{\vec{k}} (a_{\vec{k}} f_{\vec{k}} + a^{\dagger}_{\vec{k}} f^{*}_{\vec{k}}) \qquad (3)$$

In the canonical quantization field theory the coefficients $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$ are annihilation and creation operators, respectively. There are two useful and particularly simple choices for the coupling constant ξ, since these do not introduce derivatives of the scale factor on the equation of motion: the minimally coupled massless field $(\xi, m = 0)$ and the conformal coupling $(\xi = 1/6)$.

If one writes the solution for the conformal coupling case as [6]

$$f_{\vec{k}}(\eta) = \frac{e^{i\vec{k}\vec{x}}\chi_{\vec{k}}(\eta)}{a(\eta)\sqrt{V}} \qquad (4)$$

equation (2) is reduced to the following mode equation

$$\chi''(\eta) + [k^2 + m^2 a^2(\eta)]\chi(\eta) = 0 \qquad (5)$$

A similar procedure can be performed for the minimal coupling case, which would be the appropriate one to study cosmological graviton production.

*Electronic address: jruizpon12@alumnes.ub.edu

In generic spacetimes, as opposed to Minkowski spacetimes, the impossibility to define a family of observers perceiving identical quantum states is notable. Specifically, there is no common vacuum state from which the concept of particle emerges, as the explicit identification of a particle arises from the application of the creation operator to the vacuum state. Consequently, the emphasis of this study will be on asymptotically flat spacetimes, which approaches Minkowski geometry both in the remote past and future.

For a spacetime of that nature, sets of plane wave solutions can be identified in the asymptotic regions:

$$\chi_{\mathrm{in},\vec{k}} \sim \frac{e^{-iw_{\mathrm{in}}\eta}}{\sqrt{2w_{\mathrm{in}}}} \qquad \chi^*_{\mathrm{in},\vec{k}} \sim \frac{e^{iw_{\mathrm{in}}\eta}}{\sqrt{2w_{\mathrm{in}}}} \tag{6a}$$

$$\chi_{\mathrm{out},\vec{k}} \sim \frac{e^{-iw_{\mathrm{out}}\eta}}{\sqrt{2w_{\mathrm{out}}}} \qquad \chi^*_{\mathrm{out},\vec{k}} \sim \frac{e^{iw_{\mathrm{out}}\eta}}{\sqrt{2w_{\mathrm{out}}}} \tag{6b}$$

where w_{in} and w_{out} are the asymptotic past and future values for the frequency term of the equation, respectively.

Since the order of the differential equation defines the dimension of the solution vector space, any pair of solutions of equation (5) can be expressed as a linear combination of any other pair of linearly independent solutions. Solutions between Minkowski regions are connected by Bogoliubov coefficients:

$$\chi_{in,\vec{k}} = \alpha_k \chi_{out,\vec{k}} + \beta_k \chi^*_{out,\vec{k}} \tag{7a}$$

$$\chi^*_{in,\vec{k}} = \beta^*_k \chi_{out,\vec{k}} + \alpha^*_k \chi^*_{out,\vec{k}} \tag{7b}$$

These are the same coefficients that relate the creation and annihilation operators in both regions. Besides, the importance of the computation of the β_k coefficient (over solving the equation) is further accentuated as it enables the calculation of the number of particles created with mode $\vec{k}$ [2]

$$\langle N_{\vec{k}} \rangle = {}_{\mathrm{in}}\langle 0 | b^\dagger_k b_k | 0 \rangle_{\mathrm{in}} = \left|\beta_{\vec{k}}\right|^2 \tag{8}$$

where $b^\dagger_k, b_k$ are the creation and anihilation operators at asymptotic late times.

As expected for an ODE with real coefficients, solutions are complex conjugated of each other. Furthermore, if we work with the plane wave normalized solutions, the connection matrix belongs to $\mathrm{SL}(2,\mathbb{C})$. Thus the determinant is simply

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{9}$$

It can be proved that if equation (5) presents only RSPs, then $a(\eta)$ tends to constant values in the infinite past and future, and thus the spacetime is asymptotically Minkowski. We will focus our interest on equations of this type, denoted as Fuchsian. The most general second order Fuchsian ODE is

$$\frac{d^2\Phi(z)}{dz^2} + \left(\sum_{k=1}^{n} \frac{A_k}{z-z_k}\right)\frac{d\Phi(z)}{dz} + \sum_{k=1}^{n}\left(\frac{B_k}{(z-z_k)^2} + \frac{C_k}{z-z_k}\right)\Phi(z) = 0 \tag{10}$$

This can be transformed into a time-dependent harmonic oscillator-type equation, like the mode equation, by implementing the following transformations: a particular Möbius transformation ($z \to z + z_1$), an index transformation ($A_1 = 1, A_2 = ... = A_n = 0$) and a change of variables $z = e^{\eta/s}$. The resulting equation can be written as

$$\frac{d^2\chi^2(\eta)}{d\eta^2} + w^2(\eta)\chi(\eta) = 0 \tag{11}$$

with a general scale factor for the scalar case of the form of

$$w^2(\eta) = \frac{w_f e^{2(n-1)\eta/s} + ... + (c_1 c_2 ... c_{n-1})^2 w_i^2}{(e^{\eta/s}+c_1)^2 ... (e^{\eta/s}+c_{n-1})^2} \tag{12}$$

Near singular points, solutions of ODEs usually cannot be expressed as power series. Nevertheless, the Frobenius method provides a basis of solutions near each RSP of the form

$$\phi_\pm(u) = u^{r_\pm} \sum_k c_k u^k \tag{13}$$

where u is a coordinate that vanishes at the RSP and $r_\pm$ are the solutions of the indicial equation associated to the RSP. It is straightforward to check that the indicial equation for the $z = 0$ and $z = \infty$ RSPs are

$$r^2 + s^2 w_i^2 = 0 \qquad r^2 + s^2 w_f^2 = 0 \tag{14}$$

using $u = z$ at $z = 0$ and $u = \frac{1}{z}$ at $z = \infty$. By imposing $z = e^{\eta/s}$, the asymptotic solutions are recovered, except for the normalization. Since the bases of solutions of the asymptotic regions have been identified, the problem is to obtain $|\beta_{\vec{k}}|^2$ in the change of basis (7). The key idea [5] is to consider elementary and composite monodromies for these two RSPs. The monodromy matrices corresponding to elementary monodromies around these RSPs in their Frobenius bases are

$$\begin{pmatrix} e^{2\pi i r_+} & 0 \\ 0 & e^{2\pi i r_-} \end{pmatrix} \tag{15}$$

The composite monodromy matrix corresponding to the path encircling the two asymptotic RSPs, which is SL(2,$\mathbb{C}$), can be brought to upper triangular form with the diagonal components being $\{-e^{2\pi i\sigma}, -e^{-2\pi i\sigma}\}$. Using the invariance of the trace ($T = -2\cos 2\pi\sigma$), as well as equation (9), an expression for $|\beta_k|^2$ can be found [5].

For the conformal coupled case:

$$|\beta_k|^2 = \frac{a_f^2}{a_i^2} \frac{\cos^2 \pi\sigma + \sinh^2 s\pi(w_f - w_i)}{\sinh 2\pi s w_i \sinh 2\pi s w_f} \tag{16}$$

For the massless minimally coupled, the prefactor a_f^2/a_i^2 does not appear.

All the characteristics of the expansion profile are encoded in σ. It is a complex function of the parameters that appear in the scale factor $a(\eta)$ and it is mathematically related to the positions and residues of the remaining RSPs. Also, σ must be such that $|\beta_k|^2$ is real and positive. Furthermore, considering that the monodromy matrix belongs to $\mathrm{SL}(2,\mathbb{C})$ and the classes of these matrices are determined by their trace squared (T^2), the possible classes given a real trace are: elliptic ($T^2 < 4$), parabolic ($T^2 = 4$) and hyperbolic ($T^2 > 4$). These possibilities result in three alternative scenarios for σ. Notice that since $|\beta_k|^2$ depends on T, the conditions for the regimes are not the same as for the classes of the matrix.

For real σ (elliptic regime or regime I) the production is bounded,

$$|\beta_k|_I^2 = \frac{a_f^2}{a_i^2}\frac{\cos^2\pi\sigma + \sinh^2 s\pi(w_f - w_i)}{\sinh 2\pi s w_i \sinh 2\pi s w_f} \tag{17}$$

Alternatively, $\sigma = i\Delta$ (hyperbolic regime or regime II) leads to an enhanced particle production, given the exponential resulting dependence,

$$|\beta_k|_{II}^2 = \frac{a_f^2}{a_i^2}\frac{\cosh^2\pi\Delta + \sinh^2 s\pi(w_f - w_i)}{\sinh 2\pi s w_i \sinh 2\pi s w_f} \tag{18}$$

A complex σ would also be possible, but it would be restricted to the form of $\sigma = 1/2 + i\Delta$ (another hyperbolic regime or regime III),

$$|\beta_k|_{III}^2 = \frac{a_f^2}{a_i^2}\frac{\sinh^2 s\pi(w_f - w_i) - \sinh^2\pi\Delta}{\sinh 2\pi s w_i \sinh 2\pi s w_f} \tag{19}$$

The condition $s(w_f - w_i) > \Delta$ must also be true. No examples within this regime have been found.

In the remaining of the section, we illustrate these general results with the simplest examples. The simplest scale factor corresponds to equation (5) having three RSPs. The resulting frequency term can be written as

$$w^2(\eta) = \frac{w_f^2 e^{\frac{2\eta}{s}} + \epsilon e^{\frac{\eta}{s}} + w_i^2}{(e^{\frac{\eta}{s}} + 1)^2} \tag{20}$$

where w_i and w_f are the initial and final frequencies and ϵ controls the symmetry of the profile. This profile, which stands out as the most general function that enables the Fuchsian equation with three RSPs to transition between Minkowsi spacetimes, was already studied by Epstein [7] in a different physical context.

For a conformally coupled massive scalar field, the well-known example is the one considered by Bernard and Duncan [8], in which the expansion is modelized as a hyperbolic tangent by imposing $\epsilon = w_i^2 + w_f^2$. The same profile was used for a massless minimally coupled scalar field by Parker [6]. For this simple case, the composite monodromy around two RSPs is just the monodromy around the third one. This fact simplifies the determination for an indicial equation from which σ can be determined. In this case

$$r(r-1) + s^2(w_f^2 + w_i^2 - \epsilon) = 0 \tag{21}$$

allows to calculate

$$\sigma = \sqrt{\frac{1}{4} + s^2(\epsilon - w_f^2 - w_i^2)} \tag{22}$$

There are two regimes, depending on the sign inside the square root. Notice that σ is real for the hyperbolic tangent case, so it belongs to regime I.

Adding a fourth RSP to the mode equation results in more intricate expressions. The scale factor is of the form

$$w(\eta)^2 = \frac{w_f^2 e^{\frac{4\eta}{s}} + \epsilon_3 e^{\frac{3\eta}{s}} + \epsilon_2 e^{\frac{2\eta}{s}} + \epsilon_1 e^{\frac{\eta}{s}} + c_1^2 c_2^2 w_i^2}{(e^{\frac{\eta}{s}} + c_1)^2 (e^{\frac{\eta}{s}} + c_2)^2} \tag{23}$$

which yields the Fuchsian equation with four RSPs, the Heun equation [5].

Now that we have reviewed the formalism and results for the scalar field, we present our contribution to the study of gravitational production of spin-1/2 particles.

III. FERMION PRODUCTION IN EXPANDING SPACETIMES

If the produced particles are spin-1/2 fermions, the dynamics of the field is given by the Dirac equation in curved spacetime, where the partial derivative is promoted to a covariant one:

$$(i\not{D} - m)\psi = 0 \tag{24}$$

We have used the slashed notation $\not{D} = \gamma^\mu D_\mu$, where γ^μ are the 4×4 Dirac matrices that satisfy

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \tag{25}$$

Using the latter relation as well as the spin connection

$$[\Gamma_\nu, \gamma^\mu(x)] = \frac{\partial\gamma^\mu(x)}{\partial x^\nu} + \Gamma^\mu_{\nu\rho}\gamma^\rho(x) \tag{26}$$

the Dirac equation in FLRW can be expressed as [3, 4]

$$\psi'' + [k^2 + m^2 a^2(\eta) + m\gamma^0 a'(\eta)]\psi = 0 \tag{27}$$

More conveniently, by writing the equation on the eigenspaces of γ^0, the following two mode equations are found

$$\psi''(\eta) + [k^2 + m^2 a^2(\eta) \pm ima'(\eta)]\psi(\eta) = 0 \tag{28}$$

Each of these two equations has a two-dimensional vector space of solutions. However, unlike in the scalar case, the complex conjugate of a solution is not a solution to the same equation but rather to the other one. The bases for the first eigenspace ($+i$) relate through the Bogoliubov transformation as

$$\begin{pmatrix}\phi_i^+ \\ \phi_i^{-*}\end{pmatrix} = \begin{pmatrix}\alpha^+ & \beta^+ \\ \beta^{-*} & \alpha^{-*}\end{pmatrix}\begin{pmatrix}\phi_f^+ \\ \phi_f^{-*}\end{pmatrix} \tag{29}$$

and analogously for the second eigenspace $(-i)$ with ϕ_i^-, ϕ_i^{+*} and ϕ_f^-, ϕ_f^{+*}. These Bogoliubov coefficients satisfy the following relations [4]

$$\frac{\alpha^+}{\alpha^-} = \frac{w_i - ma_i}{w_f - ma_f} = \frac{w_f + ma_f}{w_i + ma_i} \tag{30a}$$

$$\frac{\beta^+}{\beta^-} = \frac{w_i - ma_i}{w_f + ma_f} = \frac{w_f - ma_f}{w_i + ma_i} \tag{30b}$$

$$\alpha^- \alpha^{+*} - \beta^- \beta^{+*} = \frac{w_i}{w_f} \tag{30c}$$

where, as for the scalar case

$$w_i(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2 a_i^2} \quad w_f(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2 a_f^2} \tag{31}$$

Similar to the scalar case, for Dirac fermions we will keep the restriction to Fuchsian mode equations, meaning they only present RSPs. Before delving into the discussion of particle production, let us identify which scale factors $a(\eta)$ satisfy this requirement. Comparing equations (28) and (12),

$$m^2 a(\eta)^2 \pm ima'(\eta) = \\ = \frac{\epsilon_{2n} e^{2n\eta/s} + \epsilon_{2n-1} e^{2n-1} + ... + \epsilon_0}{(e^{\eta/s} + c_1)^2 ... (e^{\eta/s} + c_n)^2} \tag{32}$$

Since $a(\eta)$ tends to constant values as $\eta \to \pm\infty$, then in those limits $a'(\eta) \to 0$. This constraints the values of $\epsilon_{2n} = m^2 a_f^2$ and $\epsilon_0 = (c_1 ... c_n)^2 m^2 a_i^2$. In contrast to the scalar case, the values of the remaining ϵ coefficients are complex. With the assumption that $a(\eta)$ is real and considering real values for the coefficients c_i, the general form of the scale factor can be expressed as

$$a(\eta) = a_f - \frac{A_1}{e^{\eta/s} + c_1} - ... - \frac{A_n}{e^{\eta/s} + c_n} \tag{33}$$

with the constraint that $m^2 a(\eta)^2 \pm ima' \to m^2 a_i^2$ as η tends to $-\infty$. These $a(\eta)$ are a subset of the family of scale factors from the scalar case. Notice that for n RSPs, besides the c_i, the scalar scale factors have $2n-5$ free parameters, while the fermionic ones only $n-3$. Additional restrictions between the A_i constants have to be applied if monotonous scale factors are required.

For each of the two mode equations consider a composite monodromy encircling the two RSPs of the asymptotes of the profile. For the present case, the components of the Bogoliubov transformation are not complex conjugate of each other. However, the relations from equation (30) enable the derivation of expressions for $|\beta_k^\pm|^2$. Now, simpler equations result from taking the convention $\{e^{2\pi i\sigma}, e^{-2\pi i\sigma}\}$ for the composite monodromy eigenvalues. Using the notation $w_+ = w_f + w_i$ and $w_- = w_f - w_i$

$$|\beta_k^+|^2 = \frac{w_i}{w_f} \frac{w_f - ma_f}{w_i - ma_i} \frac{\cos 2\pi\sigma - \cosh 2\pi w_-}{\cosh 2\pi w_+ - \cosh 2\pi w_-} \tag{34a}$$

$$|\beta_k^-|^2 = \left(\frac{w_f + ma_f}{w_i - ma_i} \right)^2 |\beta_k^+|^2 \tag{34b}$$

Since $|\beta_k^\pm|^2$ must be real and positive, σ is forced to be imaginary $\sigma = i\Delta$,

$$|\beta_k^+|^2 = \frac{w_i}{w_f} \frac{w_f - ma_f}{w_i - ma_i} \frac{\cosh 2\pi\sigma - \cosh 2\pi w_-}{\cosh 2\pi w_+ - \cosh 2\pi w_-} \tag{35}$$

We have found that, unlike in the scalar case, there is only one predicted regime for fermion creation.

In the upcoming section, particular profiles $a(\eta)$ will be discussed. These represent the simplest cases that lead to Fuchsian equations presenting three and four RSPs.

IV. HYPERGEOMETRIC PROFILES

As for the scalar case, the $a(\eta)$ studied in [4],

$$a(\eta) = \frac{a_i + a_f}{2} + \frac{a_f - a_i}{2} \tanh \frac{\eta}{2s} = a_f - \frac{a_f - a_i}{e^{\eta/s} + 1} \tag{36}$$

is a particular case within this set. Specifically, it is evident that this is the simplest possible one. In order to compute the particle production, it is necessary to determine the composite monodromy exponent σ. This can be done analogously to the scalar case, by solving the indicial equation corresponding to the third RSP,

$$r(r-1) + s^2 m^2 (a_f - a_i)^2 \pm ims(a_f - a_i) = 0 \tag{37}$$

which results in a monodromy exponent as

$$\sigma = \pm ims(a_f - a_i) \tag{38}$$

Note that $\cos 2\pi\sigma = \cosh 2\pi ms(a_f - a_i)$, which matches the result of [4], derived there by a much longer computation, involving the explicit solutions of (28).

V. HEUN PROFILES

Just like for the scalar case, we can identify the scale factors for which the mode equation becomes a Heun equation, *i.e.* one that exhibits four RSPs. For Dirac fermions, this general scale factor takes the form of

$$a(\eta) = a_f - \frac{A_1}{e^{\eta/s} + 1} - \frac{A_2}{e^{\eta/s} + t} \tag{39}$$

with the constraint $a_f - A_1 - A_2/t = a_i$.

If the four RSPs are located at $z = 0, 1, t, \infty$ the Heun equation can be expressed as

$$\frac{d^2\Psi}{dz^2} + \left[\frac{\frac{1}{4} - \theta_0^2}{z^2} + \frac{\frac{1}{4} - \theta_1^2}{(z-1)^2} + \frac{\frac{1}{4} - \theta_t^2}{(z-t)^2} + \right. \\ + \frac{\theta_0^2 + \theta_1^2 + \theta_t^2 - \theta_\infty^2 - \frac{1}{2}}{z(z-1)} + \\ \left. + \frac{(t-1)(-\nu^2 + \theta_t^2 - \theta_1^2 - \frac{1}{4})}{z(z-1)(z-t)} \right] \Psi = 0 \tag{40}$$

where $\frac{1}{2} \pm \theta_0, \frac{1}{2} \pm \theta_1, \frac{1}{2} \pm \theta_t, -\frac{1}{2} \pm \theta_\infty$ are the local exponents related to the singularities and ν is an accessory parameter.

In order to compute the particle production, an expression for the composite monodromy exponent σ in terms of the Heun parameters is needed. This question has surfaced in different physical problems, allowing for solutions through various approaches. However, we are not aware of a closed form expression. One of the strategies is to exploit the relation of the Heun equation and the classical limit of the Virasoro conformal blocks of 2D conformal field theory [9], taking into account that the parameter ν depends on t, which allows the formulation of a perturbative expansion for σ as

$$\sigma(t)^2 = \nu^2 - \sum_{k=1}^{\infty} k W_k t^{-k} \tag{41}$$

where W_k are the known coefficients in the expansion of the classical conformal blocks. It is necessary to rewrite the Heun equation as the mode equation in order to define a map between the Heun parameters and those from the profile. To accomplish that, three transformations must be applied. The first one is an index transformation $\Psi(z) = z^{1/2}(z-1)^{1/2}\chi(z)$ to modify the form of the equation. The second one is the following Möbius transformation in order to use the connection formula from [9], whose RSPs are at $z = 0$ and $z = 1$, between the relevant RSPs for this case: $z = 0$ and $z = \infty$. The last one is a change of variables $z = e^{\eta/s}$. The resulting map between parameters is the following

$$\theta_0^2 = -s^2k^2 - s^2m^2a_i^2, \;\; \theta_\infty^2 = -s^2k^2 - s^2m^2a_f^2 \tag{42a,b}$$

$$\theta_1^2 = \left(imsA_1 \pm \frac{1}{2}\right)^2 \tag{42c}$$

$$\theta_t^2 = \left(ims\frac{A_2}{t} \pm \frac{1}{2}\right)^2 \tag{42d}$$

$$\nu^2 = \left(ismA_1 \pm \frac{1}{2}\right)^2 - \\ - m^2s^2\left(\frac{A_2^2}{t^2} + \frac{2a_iA_2}{t} + 2\frac{A_1A_2}{t-1}\right) \tag{42e}$$

Notice that for either $A_2 \to 0$ or in the limit $t \to \infty$, these relations converge towards the hypergeometric case, with the composite monodromy parameter being $\sigma = \nu - 1/2$, so $\cos 2\pi\sigma = \cosh 2\pi ms(a_f - a_i)$, providing a non trivial check of our results.

VI. CONCLUSIONS

In this work, we have extended a recent approach to gravitational particle production in asymptotically flat spacetimes, originally applied to scalar field particles, to investigate the cosmological production of Dirac fermions. This novel approach sidesteps the need to solve the equation of motion, relying instead on the examination of monodromies around the RSPs of the equations.

We derived a general scale factor that is a subset of the scale factors for the scalar case, as well as an expression to quantify the fermion production, given by the second Bogoliubov coefficient. Unlike the scalar case, characterized by three production regimes, we established that only a single regime exists for Dirac fermions.

Furthermore, we explored simple examples resulting in the mode equation having three and four RSPs. In the case of n=3 RSPs, the scale factor had no free parameters; it has been studied in the past, and it was essentially the only solved example until this work. A perturbative approach based on conformal blocks was applied to study particle production with a scale factor leading to the equation having four RSPs. It was shown that the result reduces to the hypergeometric one under appropriate limits.

Further interesting research could be focused on non asymptotically flat spacetimes, like DeSitter universes, Majorana fermions and higher spin particles.

Acknowledgments

I express my sincere gratitude to Professor Tomeu Fiol for his unwavering dedication and brilliant guidance, to Mickael Komendyak for his exemplary teamwork, and to my family and beloved ones for their constant support.

[1] E. Schrödinger, "The proper vibrations of the expanding universe," Physica 6.7-12 (1939): 899-912.
[2] L. Parker, "Quantized fields and particle creation in expanding universes. 1.," Phys. Rev. 183, 1057-1068 (1969)
[3] L. Parker, "Quantized fields and particle creation in expanding universes. 2.," Phys. Rev. D 3, 346-356 (1971)
[4] A. Duncan, "Explicit Dimensional Renormalization of Quantum Field Theory in Curved Space-Time," Phys. Rev. D 17, 964 (1978)
[5] M. Komendyak, "Cosmological Particle Production and Entanglement", Master Thesis, University of Barcelona (2023).
[6] L. Parker "Thermal radiation produced by the expansion of the Universe," Nature, 261(5555), 20-23 (1976).
[7] P.S. Epstein, "Reflection of waves in an inhomogeneous absorbing medium," Proceedings of the National Academy of Sciences, 16(10), 627-637. (1930)
[8] C. W. Bernard and A. Duncan, "Regularization and Renormalization of Quantum Field Theory in Curved Space-Time," Annals Phys. 107, 201 (1977)
[9] O. Lisovyy and A. Naidiuk, "Perturbative connection formulas for Heun equations," J. Phys. A 55, no.43, 434005 (2022)