Estimation and statistics control of earthquakes in mixed systems

Author: Adrià Argemí Muriel
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.
Advisor: Jordi Baró Urbea

Abstract: In this paper we will study the statistical properties and computational modelling of earthquakes using a Olami-Feder-Christensen based model to simulate the tectonic faults. Our focus will be in the analysis of the Gutenberg-Richter law, which relates earthquakes magnitude and frequency, and see if it complies with our model and the changes we apply to the system.

I. INTRODUCTION

Earthquakes can be described as the tremors and shakings of Earth's terrestrial surface. They are generally caused by the sudden release of energy due to the friction and movement of the tectonic plates and can vary greatly in intensity and frequency, from weak earthquakes that go unfelt to large ones that cause severe damage.

In physics the term avalanche is often used to refer to collective events that happen in a sudden way from an initial system at rest, caused by a small change in the system. In this case earthquakes too can be described as avalanches, where the force that is being accumulated is the stress, until it reaches a threshold point, causing a slide to quickly propagate, in an event that can range widely in size and duration. The study of avalanches is of special interest in statistical physics as it allows better understanding of the internal dynamics of some systems and is also used to predict and prepare for events such as natural disasters. Avalanches are generally governed by power laws due to their scale-invariance behaviour, which means that the properties of avalanches remain the same for different scales. This a consequence of the criticality of the system, since it is located at an instability point that leads to events of all sizes. In the case of earthquakes, a power law of interest is the Gutenberg-Richter law, first proposed by Charles Francis Richter and Beno Gutenberg in 1944[1], that expresses the relationship between the magnitude M and the total number of earthquakes N for any region and time:

$$\mathrm{N} \propto M^{-b} \tag{1}$$

Due to the limitations of both to recreate the tectonic motions that cause earthquakes in a controlled environment for experimentation and to observe the structure of real-life faults, the study of earthquake phenomenology is often made with simple computational models that simulate the faults of tectonic plates through blocks and springs. In this work, we will make use of a lattice model largely based on the one introduced by Olami, Feder and Christensen, making use of the programming language Python. By changing parameters such as the dissipation and size of the system, we will analyse how this affects the Gutenberg-Richter law and if there is a point where the system no longer obeys the power law and try to determine the reasons why.

II. COMPUTATIONAL MODELS FOR EARTHQUAKE PHENOMENOLOGY

The first model to make use of blocks and springs was the one introduced by Burridge and Knopoff (the BK model)[2]. It consists of a series of blocks resting on a surface, which are connected to each other with springs of elastic constant K_C and to a "loading plate" that moves with constant velocity V with springs of elastic constant K_L. Thus, the surface where the blocks are located simulate the fault, with the moving plate representing the tectonic plate that moves relative to another, and which causes the stress on the fault due to friction.

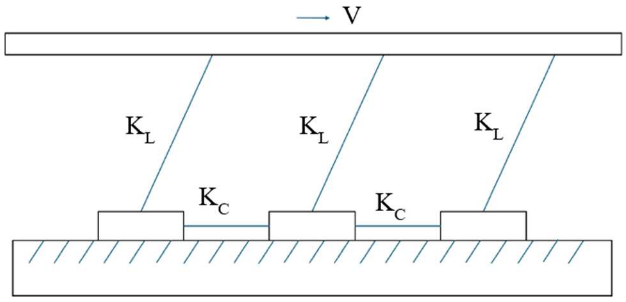

FIG. 1: Representation of a fault system using springs and blocks such as in the BK and RJB model

The Olami, Feder and Christensen model[3], is largely similar but is instead a square lattice model where each point of the lattice represents a block, which are again connected to their first-time neighbours and to the "loading plate" with the same springs. The total force, that is, the stress, on each block will be, according to Hooke's Law:

$$\sigma_{i,j} = K_C\left[4 \cdot \Delta x_{i,j} - \Delta x_{i-1,j} - \Delta x_{i+1,j} - \Delta x_{i,j-1} - \Delta x_{i,j+1}\right] + K_L \Delta x_{i,j-1} \tag{2}$$

When the tectonic plate, that is the "loading plate", moves it will cause an increase in the stress of the blocks until there is a block where its local stress will reach a threshold value $\sigma_{i,j} \geq \sigma_F$, beginning a process of relaxation, $\sigma_{i,j} \rightarrow 0$, a slip, and the energy will be transferred to its neighbours. Equalling (2) to 0, we will get:

$$\Delta x_{i,j} = \frac{(\Delta x_{i-1,j} + \Delta x_{i+1,j} + \Delta x_{i,j-1} + \Delta x_{i,j+1})}{K_L + 4 \cdot K_C} \tag{3}$$

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$$\Delta\sigma = K_C \Delta x_{i,j} = \frac{K_C}{K_L + 4 \cdot K_C} \cdot \sigma_i = \alpha\sigma_i \qquad (4)$$

Where $\Delta\sigma$ will be the stress transferred to its neighbours and α will be a parameter constant, that will have a value between 0 and 0.25 depending on the dissipation of the system. If $\alpha = 0.25$ then the system will be conservative, and no energy will be lost by dissipation. If $\alpha < 0.25$, then the energy lost by dissipation will be:

$$\Delta\sigma_{diss} = (1 - 4 \cdot \alpha) \cdot \sigma_i \qquad (5)$$

To simulate this mechanical process, in our code each point will have a threshold stress σ_F, common to all sites, and a randomly assigned initial stress $\sigma_i < \sigma_F$. Instead of the "loading plate" we will locate the site with the maximum stress $\sigma_{i,max}$ and to all sites of the lattice we will add the difference between the threshold stress and the maximum stress $\Delta\sigma = \sigma_F - \sigma_{i,max}$. In this way, only one point in the lattice will reach the failure point, making it the origin of the earthquake. The energy released from the failure of this site will be transferred to its neighbours as per eq. (4). The number of neighbours each site will have will really depend on which boundary conditions we will use. In open boundaries, we assume that the sites in the borders have four neighbours, and they will transfer part of its stress to the void, while in closed boundaries we will have three or two neighbours in the borders. For periodic boundary conditions, the borders are connected, and we will always have four neighbours. In both open and closed boundaries, we will find finite size problems that constricts the size and range of the earthquake, while in periodic boundary conditions we can find problems with earthquakes that prolong almost indefinitely, increasing the time of the simulation. In our simulations we will use open boundaries. Since the same site can fail more than once in one simulation, we will make a distinction between the size of the earthquake, where we will count the number of failures even if the same site has failed more than once, and the area of the earthquake, where we will count the number of sites that have failed at least once. Making use of the same reloading process of finding the maximum, we will repeat the simulation, where the initial state will be the final state of the prior one.

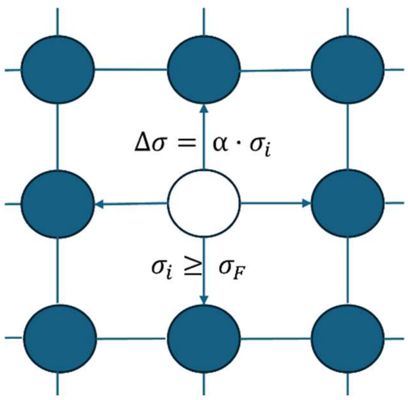


FIG. 2: Representation of the OFC model. If $\sigma_i \geq \sigma_F$ then the site suffers a relaxation and transfers $\Delta\sigma$ to its neighbours

III. SIMULATIONS IN A 2-D OFC MODEL

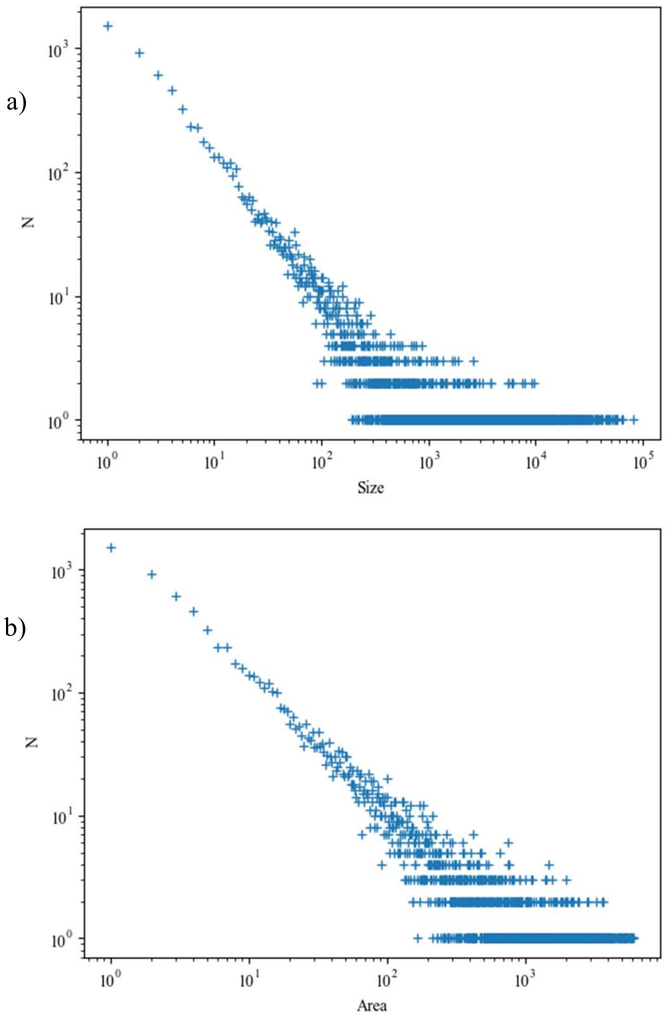


FIG. 3: Number of earthquakes N by its size (a) and area (b) for 10.000 simulations in a lattice 80x80 and $\alpha = 0.25$

We can observe that both variables follow a power law at lower values but in what we will call the "tail" of the function the behaviour diverges slightly, which means that it is possible that at some point there is "cutoff" where the power law isn't fulfilled. To better discern it, we will do a binning of the data observed by creating intervals of the variables size and area and fitting all the frequencies N inside this interval, such that we will have the intervals x and frequency H, with S being the number of simulations:

$$x_0 = 1 \quad x_1 = c \quad x_2 = c^2 \ldots x_M = c^M = x_{max}$$

$$H = \frac{\sum f(x_j \leq x \leq x_{j+1})}{S(x_{j+1} - x_j)} \qquad (6)$$

a)

b)

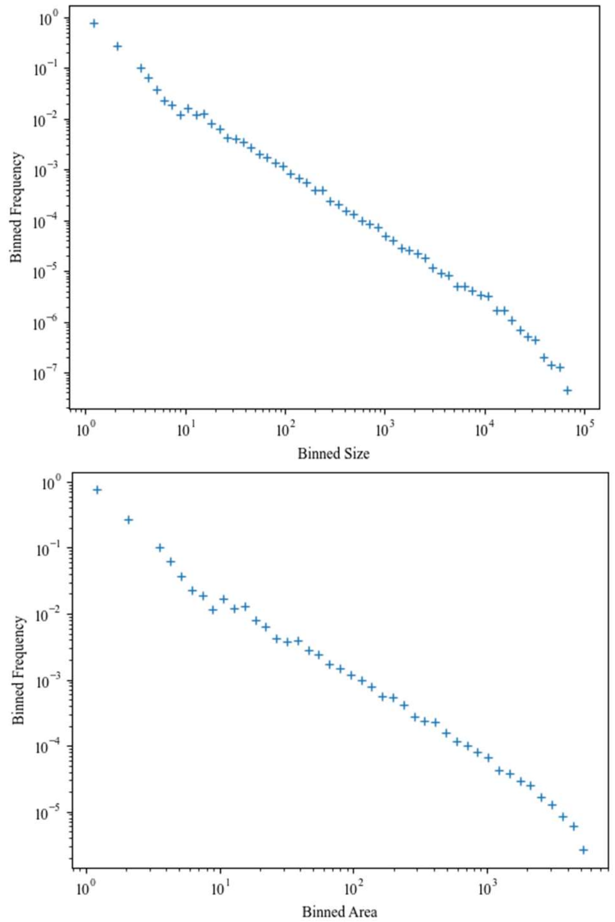
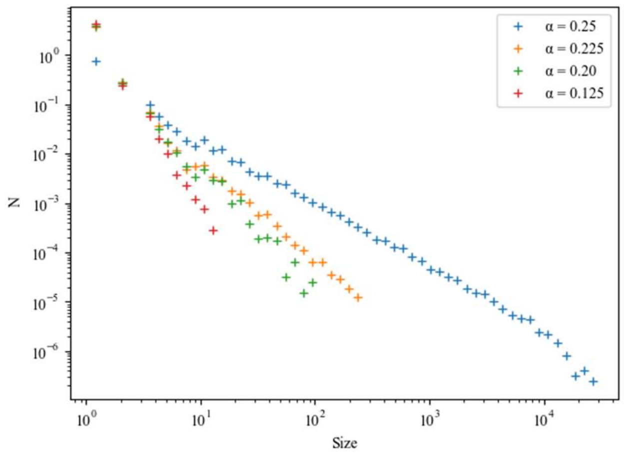


FIG. 4: Binning of the number of earthquakes N by its size (a) and area (b) for the same lattice as the one in figure 3. We can better observe the cutoff at large sizes and areas.

From now on, we will use this binning procedure to represent our data, for the aforementioned benefits of reducing the number of overlapping points and better observing the cutoff. Now we will show the results obtained by changing the parameter α in a lattice 60x60. We will use the values $\alpha = 0.25$, where the system conserves all its energy; $\alpha = 0.225$, where 10% of the energy is lost to dissipation; $\alpha = 0.20$, now with 20% of dissipation; and finally, $\alpha = 0.125$, with 50% dissipation.

a)



b)

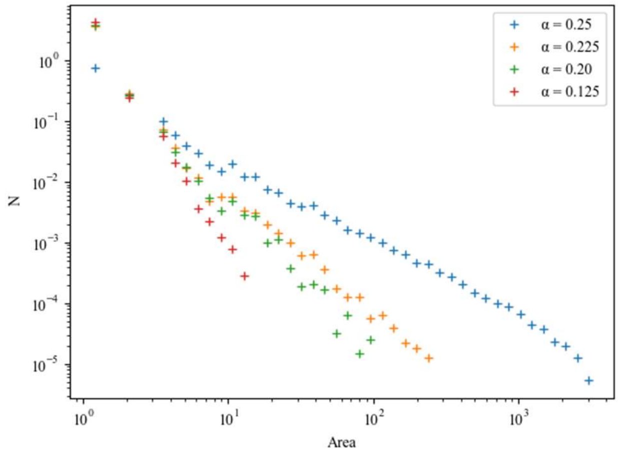


FIG. 5: Binning of the number of earthquakes N by its size (a) and area (b) for lattices 60x60 for different values of the parameter α, and thus different dissipation of the system

In FIG. 5 we can observe that as expected, more dissipation of the energy means that less stress will be transferred to its neighbours, as per eq. (5), reducing the size and area of the earthquake. Only in the conservative case ($\alpha = 0.25$) can we distinguish the power law and cutoff behaviour that we have also seen in FIG. 4, while the non-conservative cases might seem not to follow them at all, though further analysis will have to be made to confirm it.

Now we will show the results obtained by different size lattices (40x40, 80x80 and 120x120) with parameter $\alpha = 0.25$ for all of them.

a)

b)

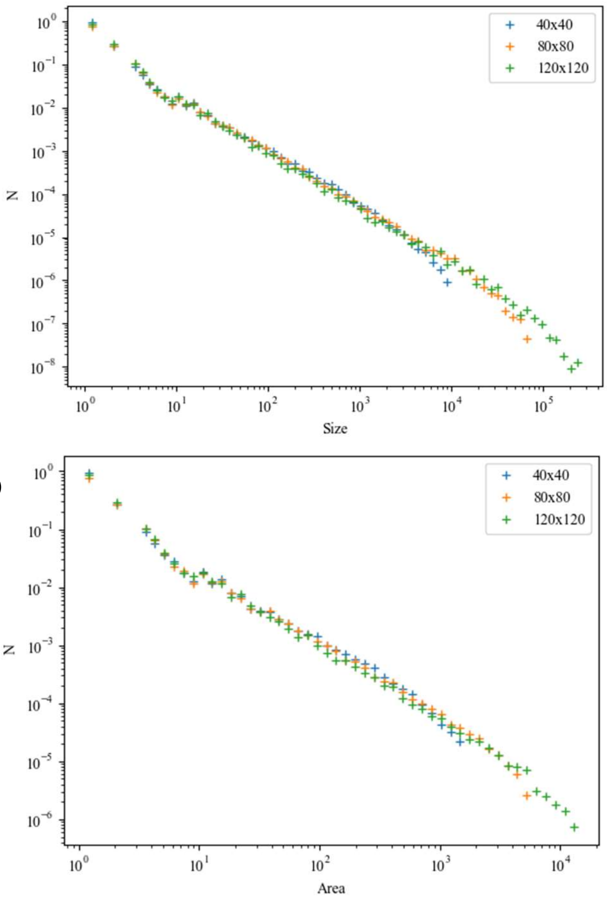


FIG. 6: Binning of the number of earthquakes N by its size (a) and area (b) for lattices of different sizes (40x40, 80x80 and 120x120), all with the same value $\alpha = 0.25$, making the system conservative.

We can observe, as in the FIG. 4 and 5, that conservative systems exhibit a power law behaviour with a cutoff at large sizes and areas. As expected, smaller lattices have smaller avalanches because of their constricted size. Furthermore, for small avalanches, the frequencies of each lattice are generally similar, meaning that the power laws by which they are governed have approximately similar exponents (eq. (1)). However, differences can be found in which point we observe the cutoff, with smaller lattices exhibiting at smaller values of avalanche size and area.

IV. DETERMINATION OF CRITICAL EXPONENTS

The presence of a cutoff in a system that was exhibiting a power law $N \propto M^{-b}$ means that at some point, which we will call x_{min}, the exponent b that was constant will suddenly change its value. The determination of the evolution of this exponent for different avalanches' size and area will give us further information on the system, as well as clearly define the effects of dissipation and size lattice. To find the exponents of all the simulations done and shown in FIG 5&6, we will use the method of the maximum likelihood estimator (MLE)[4].

This method is distinct for continuous and discrete data, us having the latter, but for large avalanches both methods result in approximately the same values, so we will use the continuous method since it is easier to calculate. The Gutenberg-Richter law is a power law, meaning that it has a probability density:

$$p(x)dx = \Pr(x \le X < x + dx) = Cx^{-b}dx \qquad (7)$$

Where X is the observed data and C is the normalization constant, will be:

$$\int_{x_{min}}^{\infty} Cx^{-b}dx = 1 \rightarrow C = (b-1)x_{min}^{(b-1)} \qquad (8)$$

In the MLE method we will make use of the logarithm of the likelihood probability, which represents the probability of the observed data given our model:

$$L = \ln(p(x|b)) = ln\left(\prod_{i=1}^{n} \frac{b-1}{x_{min}}\left(\frac{x}{x_{min}}\right)^{-b}\right) = \sum_{i=1}^{n}\left[ln(b-1) - ln(x_{min}) - b ln\left(\frac{x_i}{x_{min}}\right)\right] = n ln(b-1) - n ln(x_{min}) - b\sum_{i=1}^{n} ln\left(\frac{x_i}{x_{min}}\right) \qquad (9)$$

By equating $\frac{\partial L}{\partial b} = 0$ we will find b and its standard error:

$$\hat{b} = 1 + n\left[\sum_{i=1}^{n} ln\frac{x_i}{x_{min}}\right]^{-1} \qquad (10)$$

$$\sigma = \frac{\hat{b} - 1}{\sqrt{n}} + O(1/n) \qquad (11)$$

Where x_i $(i = 1, \ldots, n)$ will be all values that $x_i \ge x_{min}$. Thus, using this method we will now represent the exponents of the power law to observe at which point, if any, there is the cutoff.

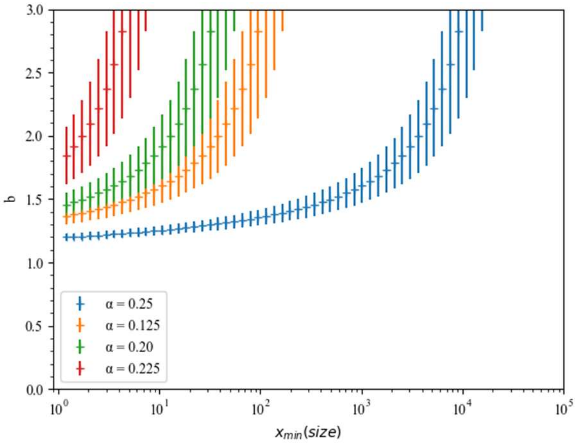


FIG. 7: Critical exponents b in function of the size of the earthquake for the same lattices 60x60 with different parameters α as in FIG 5. The width in the axis y of the points marks its standard error

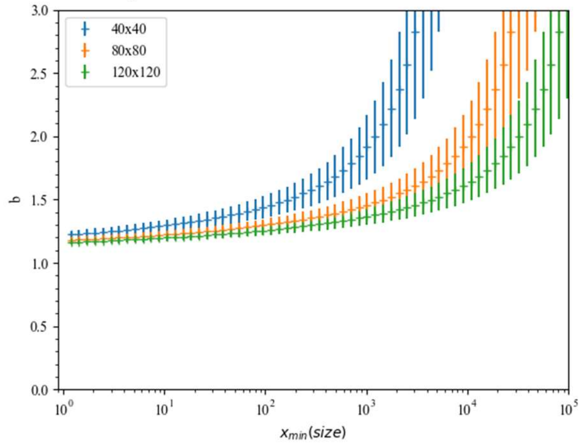


FIG. 8: Critical exponents b in function of the size of the earthquake for the same lattices 40x40, 80x80 and 120x120 in FIG 5, with the same parameter α=0.25. The width in the axis y of the points marks its standard error

We can now observe in greater detail that only in conservative systems can we find power law distributions though at some point these exponents change and the power law is no longer observed. We will try now to fit our power laws distributions with the x_{min} and b values found in FIG 8 to the graph in FIG 6 to see if they coincide as they should. Must be said though, that the exponents grow more quickly than should be expected.

TABLE I: Size lattices with their respective values of x_{min} (size) and exponent b as approximately found in FIG 8.

Size lattice	x_{min}	b
40x40	30	1.25
80x80	40	1.2
120x120	50	1.15

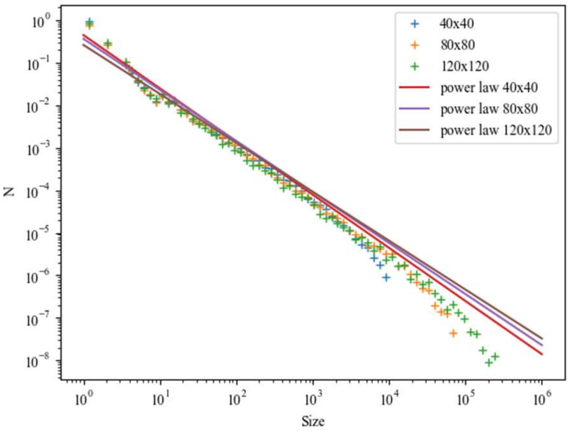


FIG. 9: Same as in FIG 6 a) but with the power laws $N = (b-1)x_{min}^{(b-1)}x^{-b}$ with the values x_{min} and b found in TABLE I

If we do the same procedure for the area instead of the size of the avalanche we will find:

TABLE II: Size lattices with their respective values of x_{min} (area) and exponent b.

Size lattice	x_{min}	**b**
40x40	4	1.3
80x80	7	1.25
120x120	10	1.20

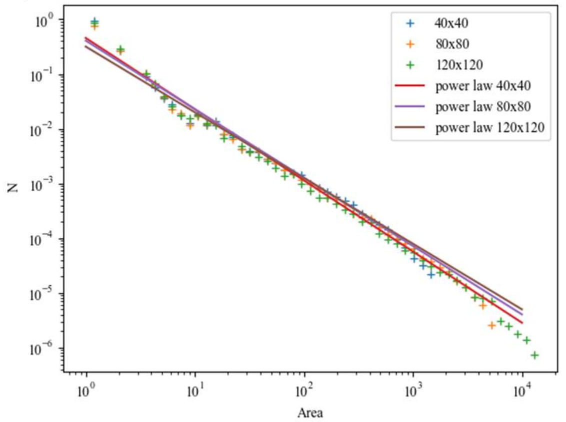


FIG. 10: Same as in FIG 6 b) but with the power laws $N = (b-1)x_{min}^{(b-1)}x^{-b}$ with the values x_{min} and b found in TABLE II

We observe, that though the measured power laws are on the same order as the raw data obtained in the simulations, they don't generally coincide which is normal because as we see in FIG. 8, the exponents are not exactly constants at small sizes and is difficult to assign an exact value to both b and x_{min}.

V. CONCLUSIONS

The conclusions to which we can arrive in this paper, is that the model based on the OFC we have implemented works generally well to simulate real-life tectonic faults and reproduces a power law behaviour similar to the Gutenberg-Richter law, though we couldn't pinpoint an exact value for the critical exponent b, we can approximately say it was in the order of around 1.2. However, this only holds true for conservative systems where $\alpha = 0.25$, and systems where dissipation plays a role, the stress transferred to the blocks is greatly reduced, and the avalanches that are caused are few and small, with the system being non-critical and not obeying the Gutenberg-Richter law, though values sufficiently near 0.25 could be described as almost-critical.

We have also seen that at large avalanches, we observe a cutoff where the system deviates from the Gutenberg-Richter law. This is the consequence of finite size effects, where the size of the system limits the size of the earthquake. Consequently, the value at which the cutoff is seen is higher for larger systems. This is a limitation of the model that has to be considered when trying to analyse real data, especially limiting in the context of prediction and mitigation of real-life large earthquakes. Periodic boundaries might solve this problem but they are computationally exhausting. Another limitation or improvement to these types of models could be the implementation of a 3D models. Must be said also that we designed a simple model in which the threshold stress for a slip was the same for all sites, which isn't generally true in real-life tectonic faults, and there is a great dependence on their geometry.

Acknowledgments

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