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THE MOUNTAIN PASS THEOREM ON SUBSYSTEMS OF SECOND ORDER ARITHMETIC

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Abstract

The main goal of this work is to formalize the Mountain Pass Theorem of Ambrosetti and Rabinowitz within the formal subsystem of second order arithmetic known as ACA_0. We develop some Analysis within this system to have access to the space of continuous functions from $[0, 1]$ into a separable Banach space and from there built formalized proofs of the basic ingredients of the Mountain Pass Theorem: The deformation lemma and the minimax principle that proves the theorem itself.

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Chapter 1

On the program of Reverse Mathematics

Reverse Mathematics is a crucial subject within Proof Theory needed not only for understanding the foundations of modern mathematical research, but also for unearthing the computational content behind key results in a wide range of branches.

History has witnessed that in the beginning of the twentieth century, mathematics was shaken by a foundational crisis after some paradoxes, most famously Russell's paradox, were discovered within Cantor's *naïve* set theory. Part of the mathematical community of that time argued that the origin of the paradoxes was in the *risky* and uncontrolled treatment of the concept of infinite set. Thus, David Hilbert proposed a sort of common-sense solution: eliminate all *infinitary* reasoning from mathematics. This, some authors argue, gave rise to a completely new branch of Mathematical Logic: Proof Theory, an intrinsically finitistic approach to logical and mathematical reasoning that allegedly could account even for infinitary concepts and thus, *safeguards* them. It is often said that Gödel destroyed almost all of Hilbert's illusions with his impressive Incompleteness Theorems, but even if a full account of Hilbert's *finitist* program is impossible, some partial reckoning of it could be pursued. As Stephen Simpson (one of the fathers of Reverse Mathematics) explains in [19], Hilbert's program has today evolved into Reverse Mathematics, a proof-theoretical research program which safeguards results in a wide variety of fields of classical mathematics by showing that they can be proven without strong infinitary assumptions or, conversely, shows that such assumptions are downright unavoidable.

Reverse Mathematics was explicitly initiated by Harvey Friedman in [12] and continued to a huge extent by Simpson and his students. Concretely, its purpose is to study the role of set-existence axioms (of an increasingly non-finitistic flavour) within the formal system of second order arithmetic, with an eye to determining which axioms are needed in order to prove specific mathematical theorems. Friedman's notion of the right axioms identified to prove a given theorem is one on which not only can the theorem be proved from the axioms, but the axioms can be proved from the theorem. In other words, the proper axioms are necessary in order to prove the theorem, and not merely sufficient. Such equivalences are often proved in the weak base theory RCA_0, a formal subsystem of second order arithmetic that may be viewed as a kind of formalized computable mathematics, with full classical logic but restricted comprehension and induction. RCA_0 is used as a weak base theory because, despite it being only able to build computable sets, it is strong enough to prove equivalences between other set-existence axioms and particular mathematical results.

Traditionally, the subsystems studied in the framework of Reverse Mathematics are five: the aforementioned base subsystem RCA_0 and four others, ordered increasingly according to the strength of their set-existence axioms: WKL_0 which states that every infinite binary tree has an infinite branch and deals directly with the finitistic reductionism of Hilbert; ACA_0 which constructs arithmetical sets and thus is a good account of predicative mathematics; ATR_0 which constructs sets by transfinite recursion and supports predicative reductionism; and Π^1_1-CA_0 a properly impredicative very strong system. For more on finitistic reductionism, predicativity and impredicativity, we recommend Dean and Walsh [7], a systematic study on the philosophical ideas behind each of this subsystems. The technical definitions of these systems will be detailed in the third chapter of our work.

Surprisingly, a very rich body of theorems of classical mathematics has been investigated in the context of Reverse Mathematics and have been shown to be either provable in RCA_0 or equivalent to one of those other four systems over RCA_0. Even quite extensive theories in fields such as Number Theory, Analysis and Algebra have been developed and analysed in Reverse Mathematics. They can be found consolidated by Simpson in [19], the mandatory reference in the field.

The main objective of the present work is to formalize within some of the mentioned formal subsystems (it will turn out to be ACA_0) a particular result of Non Linear Analysis known as the Mountain Pass Theorem (in short MPT), first presented in 1973 by Ambrosetti and Rabinowitz [2]. Broadly speaking, the MPT provides necessary conditions to ensure the existence of critical points of differentiable functionals with domain defined in a Banach space and image in the real numbers. We will focus on it in the following chapter.

The main application of the Mountain Pass Theorem is as a tool to guarantee the existence of weak solutions of semilinear elliptic partial differential equations (PDEs). In fact, it was with the motivation of finding solutions of such semilinear problems that Ambrosetti and Rabinowitz studied the theorem in the first place. A wide range of PDEs can be solved using the MPT indeed; in the referenced paper [2], nonlinear problems of eigenvalues and eigenvectors are solved using this technique; even more, along with small modifications of the MPT and the inclusion of various concepts and different hypotheses, it is possible to speak of a multiplicity of solutions to those problems. They also provide applications of their theorem to integral equations. A very practical but excellent summary of those possibilities can be found in the second part of Rabinowitz's monograph [18].

The MPT that we will analyze here has undergone modifications of various types and has been widely generalized. The methods that arise from it or have some relationship with its structure are called methods of minimax, and among other things, they are useful for instance to find solutions of Hamiltonian systems. The applications are numerous and frequent in contemporary research, so it will be impossible to detail them here; the interested reader is addressed to Ambrosetti and Rabinowitz [2], Badiale and Serra [5], Evans [9], Jabri [15] or Rabinowitz [18].

From a practical point of view, one can not stress enough the importance that PDEs and related problems have as a subject inside Mathematical Analysis. Since its origins as a device to address certain problems in Physics, the mathematical study of techniques that can guarantee the existence of solutions of such equations has always been a fruitful enterprise. Thus, we consider clearly justified the study within the field of Reverse Mathematics of one of the principal theorems sustaining the theory of PDEs. However, in this work we will only formalize the theorem within ACA_0 and not pursue its reversal. The technical difficulties of the proof of the theorem itself make the formalization a program interesting enough to be self contained here. With no more delay, we begin our research by getting to know the classical Mountain Pass Theorem in what follows.

Chapter 2

The Classical Mountain Pass Theorem

As already stated in the introduction, the objective of this work is to formalize the classical proof of the Mountain Pass Theorem by Ambrosetti and Rabinowitz [2] within a subsystem of Second Order Arithmetic towards a future analysis in the context of Reverse Mathematics.

In this chapter we introduce the theorem in a non-formalized context to get acquaintance with all the nuances that the mathematical metalanguage can sustain. We will only discuss theoretical aspects of the MPT but it is worth mentioning again that the theorem is a very popular result in the field of Nonlinear Analysis due to several interesting applications regarding the solutions of nonlinear PDEs.

For starters, we enunciate the statement of the theorem in its most known form.

Theorem 2.1 (Mountain Pass Theorem). *Let E be a Banach space and let $I \in C^1(E, \mathbb{R})$ be a continuously differentiable functional that satisfies the compactness property of Palais-Smale (PS)[1]. Suppose that $I(0) = 0$ and that there exist $\rho, \alpha > 0$ such that:*

1. *If $\|u\| = \rho$ then $I(u) \geq \alpha$,*
2. *There is $v \in E$ such that $\|v\| > \rho$ and $I(v) \leq 0$.*

Then I has a critical value $c \geq \alpha$. Moreover, c can be characterized as:

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, \, g(1) = v\}.$$

[1]See Definition 2.10 in the next section.

Remark 2.2. *To intuitively understand the reason why the Mountain Pass Theorem has such a name, consider the case when $E = \mathbb{R}^2$; there we can think of the graph of the functional I as the surface of a very smooth mountain having the points $(0,0)$ and $(v, I(v))$ as the bottoms of two valleys (the latter being at most as deep as the former). If we take all the continuous paths between those points and consider the highest point of each of those paths, the theorem says that in the path where this point has the lowest height, we would find a spot (the critical point) where we can pass through from one valley to another, i.e., a mountain pass. Figure 2.1 illustrates this situation.*

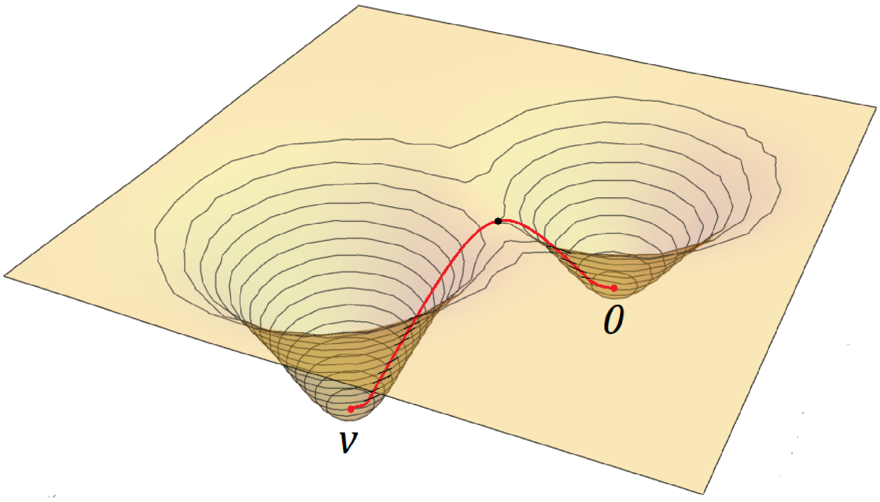

Figure 2.1: Graph of a function that satisfies the MPT hypotheses. In the highest point of the red path between the points $I(0) = 0$ and $I(v) < 0$ we can find a mountain pass.

To fully understand the statement of the theorem and to perform its non-formalized proof, we first consider some prerequisites regarding the analytic and topological tools needed to that aim.

2.1 Analytical and Topological Prerequisites

We assume some basic undergraduate knowledge of Mathematical Analysis to present some important definitions in what follows.

2.1.1 Continuity and differentiability review

In the first place, we recall some necessary concepts regarding metric spaces. Let (X, d) denote a metric space, where as usual X is a set and $d : X \times X \to \mathbb{R}$ is a metric or distance. We also consider the usual notations for the open ball of center $x \in X$ and radius $r > 0$:

$$B_r(x) = \{y \in X : d(y,x) < r\},$$

and its closure:

$$\overline{B}_r(x) = \overline{B_r(x)} = \{y \in X : d(y,x) \leq r\}.$$

The stepping stone of Analysis in metric spaces is the definition of continuous function, so we recall its most known definition.

Definition 2.3 (continuous function). *Let (X,d) and $(Y,\tilde{d})$ be metric spaces. A function $F : X \to Y$ is said to be* continuous *if for every $x_0 \in X$ we have that for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$d(x,x_0) < \delta \Rightarrow \tilde{d}(F(x),F(x_0)) < \varepsilon.$$

We profit from this definition to present a characterization of continuity that will inspire the formalization of this concept within subsystems of second order arithmetic in the next chapter.

Theorem 2.4. *Let (X,d) and $(Y,\tilde{d})$ be metric spaces. A function $F : X \to Y$ is continuous iff for every value $F(x)$ and each open ball $B \subseteq Y$ containing $F(x)$ there is an open ball $\tilde{B} \subseteq X$ including x such that $F(\tilde{B}) \subseteq B$.*

Proof. ($\Rightarrow$) Let $F(x) \in ran(F)$ and let $B = B_r(y)$ be an open ball such that $F(x) \in B$. Thus, $\tilde{d}(F(x),y) < r$. Now, let $\varepsilon = r - \tilde{d}(F(x),y) > 0$. Since F is continuous we know that there is $\delta > 0$ such that

$$d(x',x) < \delta \Rightarrow \tilde{d}(F(x'),F(x)) < \varepsilon.$$

Take $\tilde{B} = B_\delta(x)$. We claim that $F(\tilde{B}) \subseteq B$. Let $x_0 \in F(\tilde{B})$. Then, there is $x' \in \tilde{B}$ such that $F(x') = x_0$. Since $x' \in \tilde{B}$, we have that $d(x',x) < \delta$ and therefore $\tilde{d}(F(x'),F(x)) < \varepsilon$. Now, to show that $x_0 \in B$ we have to prove that $\tilde{d}(x_0,y) < r$. We have this because:

$$\begin{aligned}\tilde{d}(x_0,y) &= \tilde{d}(F(x'),y) \\ &\leq \tilde{d}(F(x'),F(x)) + \tilde{d}(F(x),y) \\ &< \varepsilon + \tilde{d}(F(x),y) = r.\end{aligned}$$

($\Leftarrow$) Let $x_0 \in X$ and let $\varepsilon > 0$. Since $F(x_0) \in B_\varepsilon(F(x_0))$, then there is an open ball $\tilde{B} = B_r(y)$ such that $F(\tilde{B}) \subseteq B_\varepsilon(F(x_0))$ and $x_0 \in \tilde{B}$ (so $d(x_0,y) < r$). Take

$\delta = r - d(x_0, y) > 0$. Then, given $d(x, x_0) < \delta$ we can prove that $\tilde{d}(F(x), F(x_0)) < \varepsilon$ as follows. First, notice that

$$\begin{aligned} d(x,y) &\leq d(x,x_0) + d(x_0,y) \\ &< \delta + d(x_0,y) = r. \end{aligned}$$

Therefore, whenever $d(x, x_0) < \delta$, we have that $x \in \tilde{B}$, whence $F(x) \in F(\tilde{B}) \subseteq B_\varepsilon(F(x_0))$ and thus $\tilde{d}(F(x), F(x_0)) < \varepsilon$. □

Now we bring our attention to complete normed spaces. We start recalling some basic definitions.

Definition 2.5 (Banach space). *A normed space $(E, \|\cdot\|)$ is said to be* Banach *if it is complete with respect to the metric defined by the norm, i.e.,*

$$d(x,y) = \|x - y\|\,, \quad x, y \in E.$$

Recall that we say that a metric space (E, d) is complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of elements of E has a limit $x \in E$, i.e.,

$$\forall \varepsilon > 0\, \exists N > 0 \text{ such that } n \geq N \to d(x_n, x) < \varepsilon.$$

Definition 2.6 (dual space). *Let $(E, \|\cdot\|)$ be a real Banach space. We define the (topological)* dual *of E, denoted by E', as the class of all the continuous linear maps with domain E and range in $\mathbb{R}$, i.e.:*

$$E' = \{f : E \to \mathbb{R} : f \text{ is a continuous linear map}\}\,.$$

E' is a Banach space (even if E is not one) if we consider the norm:

$$\|f\|_{E'} = \sup\left\{\frac{|f(x)|}{\|x\|} : x \in E,\, x \neq 0\right\}.$$

For $U \subseteq E$, a *functional* is any map $I : U \to \mathbb{R}$, not necessarily linear or continuous.

Definition 2.7 (Fréchet differential). *Let E be a Banach space, U be an open subset of E, and $I : U \to \mathbb{R}$ be a functional. We say that I is* Fréchet differentiable *in $x \in U$ if there is $f \in E'$ such that*

$$\lim_{\|h\|\to 0} \frac{I(x+h) - I(x) - f(h)}{\|h\|} = 0, \tag{2.1}$$

which using the "little-O" notation of Landau can be written as follows:

$$I(x+h) = I(x) + f(h) + o(\|h\|)$$

whenever $\|h\| \to 0$.

It can be shown that if this f *exists, then it is unique. This way, for each* $I : U \to \mathbb{R}$ *differentiable in* $x \in U$*, the unique* $f \in E'$ *that satisfies (2.1) is called the Fréchet differential (or simply differential) of* I *in* x*, and is denoted by* $I'(x)$*. Therefore,*

$$I(x+h) = I(x) + I'(x)(h) + o(\|h\|) \tag{2.2}$$

whenever $\|h\| \to 0$.

If for every $x \in U$ we have that $I'(x)$ exists, then we say that I is *differentiable on* U or just differentiable. It is important to notice that if I is differentiable, then it is continuous. In this scenario the application

$$\begin{aligned} I' : U &\to E' \\ x &\mapsto I'(x) \end{aligned}$$

is called the *Fréchet derivative* of I, and in general it can be non-linear and discontinuous.
If I' is continuous as a map from U to E', then we say that I is of class C^1 in U and write $I \in C^1(U)$.

2.1.2 Some specific prerequisites for the MPT

We now continue with more definitions to establish the bases of the Mountain Pass Theorem.

Definition 2.8 (sublevels of a functional). *Let* E *be a Banach space and let* $I : E \to \mathbb{R}$ *be a functional. For every* $a \in \mathbb{R}$ *we define the* sublevel *of* I *at* a *as the following set:*

$$I^a = \{x \in E : I(x) \leq a\}.$$

Definition 2.9 (critical value of a differentiable functional). *Let* E *be a Banach space and let* $I : E \to \mathbb{R}$ *be a differentiable functional. We say that* $c \in \mathbb{R}$ *is a* critical value *of* I *if there exists* $u \in E$ *such that*

$$I'(u) = 0 \text{ and } I(u) = c.$$

Since $I'(u)$ *is an element of the dual space of* E*, the expression* $I'(u) = 0$ *means that* $I'(u)(x) = 0$ *for all* $x \in E$*. We say that* u *is a* critical point at level c.

Definition 2.10 (Palais-Smale sequence and (PS) condition). *Let* E *be a Banach space and let* $I : E \to \mathbb{R}$ *be a differentiable functional. We say that a sequence* $(x_k)_{k\in\mathbb{N}}$ *of elements of* E *is* Palais-Smale *if* $(I(x_k))_{k\in\mathbb{N}}$ *is a bounded sequence in* $\mathbb{R}$ *and if the sequence* $(I'(x_k))_{k\in\mathbb{N}}$ *converges to* 0 *(in* E'*). We say that* I *satisfies the* Palais-Smale condition *(in short:* I *satisfies* (PS)*) if every Palais-Smale sequence of elements of* E *has a convergent subsequence.*

Definition 2.11 (lower semi-continuous function). *Let (X, d) be a metric space. We say that $F : X \to \overline{\mathbb{R}}$ is lower semi-continuous if for every $x \in X$ and every $\lambda < f(x)$, there is a $\delta > 0$ such that whenever $d(x, y) < \delta$, it follows that $f(y) > \lambda$.*

With all of these technical definitions, we are ready to understand the statement of the Mountain Pass Theorem. To fully perform its proof we will need some previous results which will be presented in the following section.

2.2 The proof of the MPT

We present the statements of two very important results used to prove the MPT. First, the Deformation Lemma. A detailed proof can be found in [18].

Lemma 2.12 (Deformation Lemma). *Let E be a Banach space and let $I \in C^1(E, \mathbb{R})$ be a continuously differentiable functional that satisfies (PS). Suppose $c \in \mathbb{R}$ and $\bar{\varepsilon} > 0$. If c is not a critical value of I, then there exists $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0, 1] \times E, E)$ such that:*

(a) $\eta(1, I^{c+\varepsilon}) \subseteq I^{c-\varepsilon}$.

(b) $\eta(1, u) = u$ *if* $I(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}.]$

Second, we present a version (in fact a corollary) of Ekelands's ε-variational principle from [3].

Lemma 2.13 (ε-variational principle). *Let X be a complete metric space and $F : X \to \mathbb{R} \cup \{+\infty\}$ a proper lower semi-continuous function bounded below. Then for any $\varepsilon > 0$, there esxists some point y_ε such that:*

(i) $F(y_\varepsilon) \leq \varepsilon + \inf F$.

(ii) $F(x) > F(y_\varepsilon) - \varepsilon d(x, y_\varepsilon)$, *for all* $x \neq y_\varepsilon$.

Now, we reproduce the statement of the Mountain Pass Theorem and give its proof.

Theorem 2.14 (Mountain Pass Theorem). *Let E be a Banach space and let $I \in C^1(E, \mathbb{R})$ be a continuously differentiable functional that satisfies (PS). Suppose that $I(0) = 0$ and that there exist $\rho, \alpha > 0$ such that:*

1. *If $\|u\| = \rho$ then $I(u) \geq \alpha$,*

2. *There is $v \in E$ such that $\|v\| > \rho$ and $I(v) \leq 0$.*

Then I has a critical value $c \geq \alpha$. Moreover, c can be characterized as:

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0,\, g(1) = v\}.$$

Proof. In the first place, we show that $c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u)$ indeed exists, i.e., $c \in \mathbb{R}$. Since for any $g \in \Gamma$ the function $I \circ g : [0,1] \to \mathbb{R}$ is continuous, we know that $\max_{t \in [0,1]} I(g(t))$ does exist. Now, let

$$\begin{aligned} F : \Gamma &\to \mathbb{R}. \\ g &\mapsto \max_{t \in [0,1]} I(g(t)). \end{aligned}$$

Let $g \in \Gamma$. We have that $\|g(0)\| = \|0\| = 0$ and $\|g(1)\| = \|v\| > \rho$. The fact that g and the norm $\|\cdot\|$ are both continuous gives us, due to the intermediate value theorem, the existence of $t_g \in [0,1]$ such that $\|g(t_g)\| = \rho$. Thanks to the Condition 1, we have $I(g(t_g)) \geq \alpha$ and therefore:

$$\max_{t \in [0,1]} I(g(t)) \geq \alpha.$$

Thus, we have that $F(g) \geq \alpha > 0$ for all $g \in \Gamma$. Hence, $\inf_{g \in \Gamma} F(g)$ exists. It is clear that we can write:

$$c = \inf_{g \in \Gamma} F(g),$$

whence we have that $c \in \mathbb{R}$. Moreover, $c \geq \alpha$.

Now, towards proving that c is a critical value of I, we can take two different paths to achieve that aim: via the Deformation Lemma or via the ε-variational principle. We present both in what follows.

Sketch of the Proof Via the Deformation Lemma (See [2]).

By contradiction, suppose that c is not a critical value of I. We apply Lemma 2.12 with $\bar{\varepsilon} = \alpha/2$ and get $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ as in the aforementioned result. By the characterization of the greatest lower bound, we can find $g \in \Gamma$ such that

$$F(g) < c + \varepsilon. \tag{2.3}$$

Consider the notation $\eta_t(x) = \eta(t, x)$. Define

$$\begin{aligned} \tilde{g} : [0,1] &\to E \\ t &\mapsto \eta_1(g(t)). \end{aligned}$$

We want to show that $\tilde{g} \in \Gamma$. Notice that since $\bar{\varepsilon} = \alpha/2$, then $c - \bar{\varepsilon} > 0$. Therefore:

$$I(0) = 0 < c - \bar{\varepsilon} \text{ and } I(v) \leq 0 < c - \bar{\varepsilon}.$$

Thus, by (b) of the Deformation Lemma, we have:

$$\tilde{g}(0) = \eta_1(g(0)) = \eta_1(0) = 0$$

and

$$\tilde{g}(1) = \eta_1(g(1)) = \eta_1(v) = v.$$

Thus, $\tilde{g} \in \Gamma$ and therefore,

$$c \leq F(\tilde{g}). \tag{2.4}$$

By (2.3), $I(g(t)) \leq c + \varepsilon$ for all $t \in [0,1]$. Thus, $g(t) \in I^{c+\varepsilon}$ for all $t \in [0,1]$ and therefore $\eta_1(g(t)) \in \eta_1(I^{c+\varepsilon})$ for all $t \in [0,1]$. By (a) of the Deformation Lemma, we have that $\eta_1(g(t)) \in I^{c-\varepsilon}$ for all $t \in [0,1]$, or in other words $\tilde{g}(t) \in I^{c-\varepsilon}$ for all $t \in [0,1]$. This of course means that $I(\tilde{g}(t)) \leq c - \varepsilon$ for all $t \in [0,1]$; hence:

$$F(\tilde{g}) \leq c - \varepsilon < c. \tag{2.5}$$

This contradicts (2.4), so we are done.

Sketch of the Proof Via the ε-Variational Principle. (See [15])
We consider Γ as a complete metric space with the metric

$$d(g_1, g_2) = \max_{t \in [0,1]} \|g_1(t) - g_2(t)\|$$

for all $g_1, g_2 \in \Gamma$. It can be shown that $F : \Gamma \to \mathbb{R}$ defined as above is continuous and therefore lower semi-continuous. Also, we already showed that F is bounded bellow. Thus, by Lemma 2.13, we know that for every $\varepsilon > 0$ there is g_ε such that:

$$\begin{cases} F(g_\varepsilon) \leq c + \varepsilon. \\ F(g) \geq F(g_\varepsilon) - \varepsilon d(g - g_\varepsilon), \text{ for all } g \in \Gamma. \end{cases} \tag{2.6}$$

Set $M_\varepsilon = \{t \in [0,1] : I(g_\varepsilon(t)) = \max_{s \in [0,1]} I(g_\varepsilon(s))\}$. Then, using (2.6) and thanks to a quite technical argument from convex analysis (see [3], page 272) we can deduce that there is $t_\varepsilon \in M_\varepsilon$ such that:

$$\left\| I'(g_\varepsilon(t_\varepsilon)) \right\| \leq \varepsilon.$$

Now, we define a sequence $(u_n)_{n \in \mathbb{N}}$ by $u_n = g_{1/n}(t_{1/n})$ for each $n \in \mathbb{N}$. It is easy to prove that this sequence of elements of E is such that $(I(u_n))_n$ is a bounded sequence and such that $I'(u_n) \to 0$. Thus, since I satisfies (PS), we have that there is a subsequence $(u_{n_k})_k$ and $u \in E$ such that $u_{n_k} \to u$. It is straightforward to prove that $I'(u) = 0$ and $I(u) = c$, so c is a critical value. □

The original proof of the MPT (see [2]) follows the first path that we have presented here, i.e., via the Deformation Lemma. Considering this, we will present a formalization of that version of the proof of the MPT. However, since Ekeland's ε-variational principle and its applications have been recently studied in the context of Reverse Mathematics (see [11] and [10]), we encouraged the interested researcher to study the other path.

In the following chapters we will pursue the mentioned formalization, but in the first place we will study in detail the formal theories that we will be working within towards this formalization.

Chapter 3

Second Order Arithmetic

We now turn our eyes to Mathematical Logic. This chapter is divided into two parts. In the first one, we will introduce the formal systems of the tradition of Reverse Mathematics that we will be using to formalize the proof of the MPT. It will be a quite descriptive section. In the second part we will begin the development of ordinary mathematics within those systems to achieve a fair amount of theory that serves as base to state and prove the Deformation Lemma and the MPT itself within this context. We will introduce some basic knowledge from many sources and also give the proofs of some original results.

3.1 Subsystems of Second order Arithmetic

By $\mathbf{Z}_2$ we denote the formal logical system of Second Order Arithmetic which we define in what follows.

Definition 3.1 (language of $\mathbf{Z}_2$). *The* language of $\mathbf{Z}_2$*, denoted L_2, is a two-sorted first-order language. This means that there are two distinct sorts of variables which are intended to range over two distinct kinds of objects. The first sort of variables, known as* number variables *and represented by lower-case letters $i, j, k, m, n, \dots,$ are intended to range over the set $\omega = \{0, 1, 2 \dots\}$ of the natural numbers. The second sort, the* set variables*, are represented by upper-case letters $X, Y, Z, A, B, \dots$ and are intended to range over subsets of ω. We consider as part of the language the binary function symbols $+$ and $\cdot$, the relation symbols $<$ and $\in$ whose intended meanings are the usual ones. The terms and formulas of L_2 are as follows:*

- Numerical terms. *Consisting of the number variables, the constant symbols* 0 *and* 1*, and whenever t_1 and t_2 are numerical terms we recursively define as a numerical terms: $t_1 + t_2$ and $t_1 \cdot t_2$.*

- Atomic formulas. *If t_1 and t_2 are numerical terms and X is a set variable, then the following are atomic formulas: $t_1 = t_2$, $t_1 < t_2$ and $t_1 \in X$.*
- Formulas. *We recursively built up the formulas from atomic formulas by means of propositional connectives $\wedge, \vee, \neg, \rightarrow$, number quantifiers $\forall n, \exists n$ and set quantifiers $\forall X, \exists X$.*

Definition 3.2 (Second Order Arithmetic). *The axioms of* second order arithmetic *consists of the universal closures of the following L_2-formulas:*

(a) basic axioms:

$$n + 1 \neq 0$$
$$m + 1 = n + 1 \rightarrow m = n$$
$$m + 0 = m$$
$$m + (n + 1) = (m + n) + 1$$
$$m \cdot 0 = 0$$
$$m \cdot (n + 1) = (m \cdot n) + m$$
$$\neg m < 0$$
$$m < n + 1 \leftrightarrow (m < n \vee m = n)$$

(b) induction axiom:

$$(0 \in X \wedge \forall n(n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)$$

(c) comprehension axiom scheme:

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

In the comprehension axiom scheme, $\varphi(n)$ may contain free variables in addition to n. These free variables may be referred to as parameters *of this instance of the comprehension axiom scheme.*

By Second Order Arithmetic (Z_2) we mean the formal system in the language of L_2 consisting of the axioms of second order arithmetic, together with all formulas of L_2 which are deducible from those axioms by means of the usual logical axioms and rules including equality axioms and the law of excluded middle. This makes our system a classical, non-intuitionistic system.

By a subsystem of $\mathbf{Z}_2$ we mean any formal system in the language L_2 whose theorems are included in those of $\mathbf{Z}_2$. We recall the most important subsystems used in Reverse Mathematics as well as some known important results.

3.1.1 The system RCA_0

Definition 3.3 (bounded quantifiers). *Let φ be a formula of L_2, let n be a number variable an let t be a numerical term which does not contain n. We abbreviate $\forall n(n < t \rightarrow \varphi)$ as $(\forall n < t)\varphi$ and $\exists n(n < t \wedge \varphi)$ as $(\exists n < t)\varphi$. The quantifiers $\forall n < t$ and $\exists n < t$ are called* bounded number quantifiers.

Definition 3.4 (Σ^0_1 and Π^0_1 formulas). *An L_2-formula is said to be Σ^0_1 (respectively Π^0_1) if is one of the form $\exists n\, \theta$ (respectively $\forall n\, \theta$) where θ is a formula built up from atomic formulas by means of propositional connectives and bounded number quantifiers. Note that although this formulas contain no set quantifiers, they may contain free set variables.*

Definition 3.5 (formal system RCA_0). *The axioms of* RCA_0 *consists on the basic axioms of* $\mathbf{Z}_2$ *plus the following axiom schemes.*

(i) Σ^0_1 *induction axiom:*

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1)) \rightarrow \forall n(\varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula of the language L_2.

(ii) Δ^1_0 *comprehension axiom scheme:*

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and X does not occur freely in $\varphi(n)$.

By RCA_0 *we mean the formal system in the language of L_2 consisting of the previous axioms, together with all formulas of L_2 which are deducible from those axioms by means of the usual logical axioms and rules.*

The scheme of Δ^1_0 comprehension, also known as recursive comprehension axiom (RCA), states the existence of computable sets of natural numbers. This is due to the result by Post [17] that states that a set is computable if and only if it and its complement are computably enumerable, and the fact that computably enumerable sets are Σ^0_1 definable.

Working in RCA_0, we can code a great deal of finite objects like rational numbers, pairs of natural numbers, or even finite sequences of natural numbers, as natural numbers; and we can code functions from $\mathbb{N}$ to $\mathbb{N}$ and real numbers as sets of numbers (represented by rapidly converging Cauchy sequences of rationals). In the next section we would take some of such codings and their properties for granted and will define others from scratch. For now it is enough to know that in particular, RCA_0 is strong enough to code binary trees, an object useful to define the subsystem of the next section.

Definition 3.6 (binary tree). *The following definition is made in* RCA_0*. Let* $2^{<\mathbb{N}}$ *be the set of all (codes for) finite sequences of* $0'$*s and* $1'$*s. A* binary tree *is a set* $T \subseteq 2^{<\mathbb{N}}$ *which is closed under initial segment. A* path *through* T *is a function* $g : \mathbb{N} \to \mathbb{N}$ *such that* $g[n] \in T$ *for all* $n \in \mathbb{N}$*, where* $g[n] = \langle g(0), g(1), \ldots, g(n-1) \rangle$*.*

3.1.2 The system $\mathbf{WKL}_0$

We introduce a subsystem of second order arithmetic stronger than RCA_0, known as *weak König's lemma* and denoted by WKL_0.

Definition 3.7 (formal system WKL_0). *Let* $2^{<\mathbb{N}}$ *be the set of all (codes for) finite sequences of* $0'$*s and* $1'$*s. Weak König's lemma is the formalized statement that says that every infinite tree* $T \subseteq 2^{<\mathbb{N}}$ *has an infinite path (we say that a set* $X \subseteq N$ *is finite if it is bounded above, and we say that* X *is infinite otherwise). The axioms of* WKL_0 *are those of* RCA_0 *plus weak König's lemma.*

WKL_0 is a formal system capable of prove a great deal of theorems in analysis as we will see later. However, its power is restricted and another subsystem of $\mathbf{Z}_2$ will be needed.

3.1.3 The system $\mathbf{ACA}_0$

In what follows we define the system ACA_0, a formalized predicative system. We begin recalling arithmetical formulas.

Definition 3.8 (arithmetical formulas). *Let* φ *be a formula of* L_2*. We say that* φ *is* arithmetical *if it contains no set quantifiers. It may contain free set variables.*

Definition 3.9 (formal system ACA_0). *The axioms of* ACA_0 *are the basic axioms and the induction axiom of* Z_2 *plus the following:*

(ACA) *arithmetical comprehension axiom scheme:*

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ *is any arithmetical formula in which* X *does not occur freely.*

ACA_0 is a quite stronger axiom than WKL_0. However, in terms of mathematical practice, not as strong as the last subsystem we would be considering in this work.

3.1.4 The system Π^1_1-CA_0

Definition 3.10 (Π^1_1 formulas)**.** *Let φ be a formula of L_2. We say that φ is Π^1_1 if it is of the form $\forall X\,\theta$, where X is a set variable and θ is an arithmetical formula.*

Definition 3.11 (formal system Π^1_1-CA_0)**.** *The axioms of Π^1_1-CA_0 are the basic axioms and the induction axiom of Z_2 plus the following:*

(Π^1_1-CA) Π^1_1 *comprehension axiom scheme:*

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Π^1_1 formula in which X does not occur freely.

Π^1_1-CA_0 is a fully impredicative system that we will wish to avoid in the formalization of the MPT. In what follows, we will develop the bulk of Analysis needed to proof the MPT.

3.2 Development of Analysis in the Subsystems of $\mathbf{Z}_2$

As we already stated, when working in RCA_0 we can code a great deal of mathematical objects. We refer the reader to sections II.2, II.3 and the beggining of II.4 of [19] for details about the development within RCA_0 of properties of natural numbers, definitions and properties of codes for finite sequences of natural numbers, functions between sets of natural numbers and the definitions and properties of the systems of numbers $\mathbb{Z}$ and $\mathbb{Q}$. Particularly, in section II.3 it is proven within RCA_0 that the universe of total number theoretic functions is closed under composition, primitive recursion, and the least number operator i.e., minimization. Thus, RCA_0 can sustain a certain *arithmetization* of computation as expected by its name. Taking matters from there, we will show that it can sustain also the arithmetization of elementary analysis. To quote John Stillwell in his remarkable *Reverse Mathematics: Proofs from the Inside Out* [20], book that we emphatically recommend, “the remarkable convergence of analysis and computation to a common source in arithmetic is what makes the reverse mathematics of analysis possible."

3.2.1 Complete separable metric spaces in $\mathbf{Z}_2$

We start our development of the parts of Mathematical Analysis that will be useful for our purposes defining the system of real numbers $\mathbb{R}$.

Definition 3.12. *A* sequence of rational numbers *is defined in* RCA_0 *to be a function $f : \mathbb{N} \to \mathbb{Q}$. We denote such a sequence $(q_n)_{n\in\mathbb{N}}$ where $q_n = f(n)$.*

A quickly converging Cauchy sequence of rational numbers *is a sequence of rational numbers* $(q_n)_{n\in\mathbb{N}}$ *such that* $\forall n \forall m\,(n > m \to |q_n - q_m| \leq 2^{-m})$*. Here* $|q|$ *denotes the absolute value of a rational number* $q \in \mathbb{Q}$*, i.e.,* $|q| = q$ *if* $q \geq 0$*,* $-q$ *otherwise.*

Definition 3.13 (the system of real numbers)**.** *The following definition is made in* RCA_0*. A* real number *is defined to be a quickly converging Cauchy sequence of rational numbers* $(q_n)_{n\in\mathbb{N}}$*. Two real numbers* $(q_n)_{n\in\mathbb{N}}$ *and* $(q'_n)_{n\in\mathbb{N}}$ *are said to be equal if* $\forall n\,(|q_n - q'_n| \leq 2^{-n+1})$*. We then write* $x = y$*.*

When describing definitions or proofs within RCA_0 (or other subsystems of $\mathbf{Z}_2$), we shall sometimes use the symbol $\mathbb{R}$ informally to denote the set of all real numbers. Of course the set $\mathbb{R}$ does not formally exist, since RCA_0 is limited to the language L_2 of Second Order Arithmetic.

Similarly to how equality was defined, within RCA_0 we can define a sum, a difference, a product and an order to the effect that $\mathbb{R}$ can be proven to be an Archemedian ordered field (see Theorem II.4.5 of [19]); thus we shall use all of the usual properties of $\mathbb{R}$. We continue with more definitions.

Definition 3.14. *The following definition is made in* RCA_0*. A sequence of real numbers is a function* $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ *such that for each* $n \in \mathbb{N}$*, the function* $(f)_n : \mathbb{N} \to \mathbb{Q}$ *defined by* $(f)_n(k) = f((k,n))$ *is a real number. We shall employ notations such as* $(x_n)_{n\in\mathbb{N}}$ *for the sequence* f *with* $(f)_n = x_n$*.*

Now, we introduce some definitions that will allow us to develop some higher Analysis. We still follow [19] but use the notation of [11].

Definition 3.15 (complete separable metric space)**.** *The following definition is made in* RCA_0*. A (code for a)* complete separable metric space $\widehat{X}$ *is a non-empty set* $X \subseteq \mathbb{N}$ *together with a sequence of real numbers* $d : X \times X \to \mathbb{R}_{\geq 0}$ *such that* $d(a,a) = 0$*,* $d(a,b) = d(b,a)$ *and* $d(a,b) + d(b,c) \geq d(a,c)$ *for all* $a,b,c \in X$*. A point of* $\widehat{X}$ *is a sequence* $x = (x_n)_{n\in\mathbb{N}}$ *of elements of* X *such that for all* $n \leq m$*,* $d(x_n, x_m) \leq 2^{-n}$ *. We write* $x \in \widehat{X}$ *to mean that* x *is a point of* $\widehat{X}$*. We identify* $a \in X$ *with the constant sequence* $(a)_{n\in\mathbb{N}}$ *and consider* X *as a dense subset of* $\widehat{X}$*. We set* $d(x,y) = \lim_{n\to+\infty} d(x_n, y_n)$*, and write* $x =_{\widehat{X}} y$ *if* $d(x,y) = 0$ *(subscripts will be omitted if there is no confusion). Thus, for all* $x \in \widehat{X}$*, we have that* $d(x, x_n) \leq 2^{-n}$*, where* $x = (x_n)_{n\in\mathbb{N}}$*.*

Notice that we have defined a point x of a complete separable metric space $\widehat{X}$ as a *quickly convergent Cauchy sequences* of elements of X. The fact that for $x = (x_n)_{n\in\mathbb{N}} \in \widehat{X}$ we have that $d(x, x_n) \leq 2^{-n}$ for every $n \in \mathbb{N}$ justifies the designation of the space as *separable* with dense set X. To justify its denomination as *complete*, we must consider the following result. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\widehat{X}$ converges if there is a point $x \in \widehat{X}$ such that

$$\forall \varepsilon > 0\, \exists N\, \forall n\, (n \geq N \to d(x_n, x) < \varepsilon).$$

In that case, we write $x_n \to x$.

Theorem 3.16 (Exercise 10.9.2 of [8]). *The following is provable in* RCA_0. *Let $\widehat{X}$ be a complete separable metric space. Every quickly converging sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\widehat{X}$ converges.*

We have a case for completeness in terms of quickly convergent Cauchy sequences. Unfortunately, RCA_0 is not strong enough to prove that any Cauchy sequence (not necessarily quickly convergent) does converge. For this we need the power of ACA_0. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in \widehat{X}$, is said to be Cauchy if $\forall\varepsilon > 0\, \exists m\, \forall n\, (m < n \to d(x_m, x_n) \le \varepsilon)$.

Theorem 3.17 (Theorem III.2.7 of [19]). *The following is provable in* ACA_0. *In any complete separable metric space $\widehat{X}$, every Cauchy sequence is convergent.*

Sequences of real numbers are quite useful sets of reals. Next we present the definitions of some other important sets.

Definition 3.18 (open and closed sets). *Within* RCA_0, *let $\widehat{X}$ be a complete separable metric space. A (code for an)* open set *U in $\widehat{X}$ is a set $U \subseteq \mathbb{N} \times X \times \mathbb{Q}_{>0}$. A point $x \in \widehat{X}$ is said to belong to U (abbreviated $x \in U$) if $\exists n\, \exists a\, \exists r (d(x, a) < r \wedge (n, a, r) \in U)$.*

A closed set *in $\widehat{X}$ is defined to be the complement of an open set in $\widehat{X}$. In other words, we define a code for a closed set C to be the same thing as a code for an open set U, and we define $x \in C$ if and only if $x \notin U$.*

Definition 3.19 (separably closed set). *A (code for a)* separably closed set *in $\widehat{A}$ is a sequence $C = (x_n)_{n\in\mathbb{N}}$ of points of $\widehat{A}$. The separably closed set is then denoted by $\overline{C}$ and $x \in \overline{C}$ if and only if $\forall q \in \mathbb{Q}_{>0}\, \exists n\, d(x, x_n) < q$.*

The definitions of (codes for) closed sets and (codes for) separably closed sets intend to code the same mathematical object (closed sets) but their logical content is quite different as can be appreciated with the following theorems that will play an important role later on.

Theorem 3.20 (Theorem 7.1 of [13]). *The following are equivalent over* RCA_0:

1. Π^1_1-CA_0.
2. *For every complete separable metric space $\widehat{A}$ and every closed set C in $\widehat{A}$ there exists a continuous function $f_C : \widehat{A} \to \mathbb{R}$ such that for every $x \in \widehat{A}$ we have $f_C(x) = \inf\{d(x, y) : y \in C\}$.*

Theorem 3.21 (Theorem 7.3 of [13]). *The following are equivalent over* RCA_0:

1. ACA_0.

2. *For every complete separable metric space $\widehat{A}$ and every separably closed set $\overline{C}$ in $\widehat{A}$ there exists a continuous function $f_{\overline{C}} : \widehat{A} \to \mathbb{R}$ such that for every $x \in \widehat{A}$ we have $f_{\overline{C}}(x) = \inf\{d(x,y) : y \in \overline{C}\}$.*

We continue the first part of our review of Analysis with the definition of continuous functions within RCA_0.

Although the classical non-formalized "$\varepsilon - \delta$" definition of continuity is typically expressed in terms of strict inequalities, it can be stated equivalently with non-strict inequalities. Thus, Theorem 2.4 can be stated in terms of closed balls. In consequence, a function $F : X \to Y$ is continuous iff for all $x \in X$, for every closed ball B containing $F(x)$ there is a closed ball $\tilde{B}$ including x with $F(\tilde{B}) \subseteq B$.

Following this, we notice that to represent a continuous function f in second order arithmetic it is necessary to capture information about which closed balls map into which other closed balls without referring directly to points in the space. Since any closed ball is completely determined by a center x and a radius r, it follows from this characterization that we can represent a continuous function F with a sequence Φ of tuples of center-radius pairs $\langle (x,r),(y,s)\rangle$ such that enumerates certain facts of the form $F(\overline{B}_r(x)) \subseteq \overline{B}_s(y)$. The inclusion of a four-tuple $\langle (x,r),(y,s)\rangle$ in Φ needs to show that this set inclusion holds.

However, the code Φ does not need to include every such inclusion; it only needs to include enough information about F for us to recover a value for $F(x)$ for each x in the domain of F. In the Reverse Mathematics literature the following definition of such a code for Φ to represent continuous functions is fundamental.

Definition 3.22 (coded continuous function). *The following definition is made in* RCA_0. *Let $\widehat{X}$ and $\widehat{Y}$ be complete separable metric spaces. A continuous partial function $f : \widehat{X} \to \widehat{Y}$ is coded by a set $\Phi \subseteq \mathbb{N} \times X \times \mathbb{Q}_{>0} \times Y \times \mathbb{Q}_{>0}$ that satisfies the properties below. Let us write $B_r(a) \xrightarrow{\Phi} B_s(b)$ for the formula $\exists n((n,a,r,b,s) \in \Phi)$. Then, for all $a, a' \in X$, all $b, b' \in Y$, and all $r, r', s, s' \in \mathbb{Q}_{>0}$, Φ must satisfy:*

(CF1) if $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $B_r(a) \xrightarrow{\Phi} B_{s'}(b')$ then $d(b,b') \leq s + s'$;

(CF2) if $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $d(a',a) + r' < r$, then $B_{r'}(a') \xrightarrow{\Phi} B_s(b)$;

(CF3) if $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $d(b,b') + s < s'$, then $B_r(a) \xrightarrow{\Phi} B_{s'}(b')$.

A point $x \in \widehat{X}$ is in the domain of the function f coded by Φ if, for every $\varepsilon > 0$, there are $B_r(a) \xrightarrow{\Phi} B_s(b)$ such that $d(x,a) < r$ and $s < \varepsilon$. If $x \in dom(f)$, we define the value $f(x)$ to be the unique point $y \in Y$ such that $d(y,b) \leq s$ for all $B_r(a) \xrightarrow{\Phi} B_s(b)$ with $d(x,a) < r$. We say that f is totally defined on $\widehat{X}$ if $x \in dom(f)$ for all $x \in \widehat{X}$ and refer to such an object as a coded continuous function*.*

A wide range of usual functions can be proven to be coded continuous in RCA_0. We present an especially important result regarding the composition of coded continuous functions.

Theorem 3.23 (Lemma II.6.4 of [19]). *The following is provable in* RCA_0. *If* $f : \widehat{A} \to \widehat{B}$ *and* $g : \widehat{B} \to \widehat{C}$ *are continuous, then so is* $h = g \circ f : \widehat{A} \to \widehat{C}$ *given by* $h(x) = g(f(x))$.

There is a very useful characterization of continuity of functions between metric spaces regarding sequences of elements of the space and the sequences of the image. We can not prove the characterization within RCA_0 but it will be possible to prove one of the implications.

Theorem 3.24. *The following is provable in* RCA_0. *Let* $\widehat{X}$ *and* $\widehat{Y}$ *be two separable metric spaces. Let* $f : \widehat{X} \to \widehat{Y}$ *be a coded continuous function and let* $(x_n)_{n\in\mathbb{N}}$ *be a convergent sequence in* $\widehat{X}$, *i.e., there is* $x \in \widehat{X}$ *such that* $x_n \to x$. *Then the sequence* $(f(x_n))_{n\in\mathbb{N}}$ *of elements of* $\widehat{Y}$ *converges to* $f(x)$.

Proof. Denote by d and d' the metrics of $\widehat{X}$ and $\widehat{Y}$ respectively and let f be coded by the set $\Phi \subseteq \mathbb{N} \times X \times \mathbb{Q}_{>0} \times Y \times \mathbb{Q}_{>0}$. To show that $f(x_n) \to f(x)$, let $\varepsilon > 0$ (in $\mathbb{Q}$) and show that there is $N \in \mathbb{N}$ such that

$$n \geq N \to d'(f(x_n), f(x)) < \varepsilon.$$

On the one hand, since $x \in \widehat{X}$, then $x \in dom(f)$ and therefore there are $B_r(a) \xrightarrow{\Phi} B_s(b)$ such that $d(x,a) < r$ and $s < \frac{\varepsilon}{4}$. Whence, $d'(f(x),b) \leq s$. Adding s to both sides of the inequality we get that $d'(f(x),b) + s \leq 2s < \frac{\varepsilon}{2}$. Thus, we have

$$B_r(a) \xrightarrow{\Phi} B_s(b) \text{ and } d'(b, f(x)) < \frac{\varepsilon}{2};$$

hence, by (CF3), we conclude that

$$B_r(a) \xrightarrow{\Phi} B_{\varepsilon/2}(f(x)). \tag{3.1}$$

On the other hand, since $d(x,a) < r$, by density there is $\delta > 0$ such that $d(x,a) + \delta < r$. Thus, due to (CF2), the latter and (3.1) imply that $B_\delta(x) \xrightarrow{\Phi} B_{\varepsilon/2}(f(x))$. The fact that $x_n \to x$ provides an $N' \in \mathbb{N}$ such that $d(x_n, x) < \delta$ for every $n \geq N'$. Now, take $N = N'$ and let $n \geq N$. Since $d(x_n, x) < \delta$ and because $x_n \in dom(f)$ we have that $d(f(x_n), f(x)) \leq \frac{\varepsilon}{2} < \varepsilon$. This completes the proof. □

Other important definitions that are very useful in Analysis have their formalized counterpart. We review some of them.

Definition 3.25 (compactness). *The following definition is made in* RCA_0. *A compact metric space is a complete separable metric space* $\widehat{X}$ *such that there exists an infinite sequence of finite sequences*

$$((x_{ij})_{i \leq n_j})_{j \in \mathbb{N}}, \quad x_{ij} \in \widehat{X},$$

such that for all $y \in \widehat{X}$ *and* $j \in \mathbb{N}$ *there exists* $i \leq n_j$ *such that* $d(x_{ij}, y) < 2^{-j}$.

Remark 3.26. *Within* RCA_0 *one can prove that the sequence* $((i \cdot 2^{-j})_{i \leq 2^j})_{j \in \mathbb{N}}$ *shows that the closed unit interval* $[0,1] = \{x : 0 \leq x \leq 1\}$ *is compact. More generally, any closed bounded interval in* $\mathbb{R}$ *is compact. See Example III.2.4 of [19].*

Definition 3.27 (modulus of uniform continuity). *The following definition is made in* RCA_0. *Let* $\widehat{X}$ *and* $\widehat{Y}$ *be complete separable metric spaces, and let* F *be a coded continuous function from* $\widehat{X}$ *to* $\widehat{Y}$. *A* modulus of uniform continuity *for* F *is a function* $h : \mathbb{N} \to \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *and all* x *and* y *in* $\widehat{X}$, *if* $F(x)$ *and* $F(y)$ *are defined and* $d(x,y) < 2^{-h(n)}$, *then* $d(F(x), F(y)) < 2^{-n}$.

We will need to develop some insight on separable Banach spaces within RCA_0. Thus, we present the formalized definition.

Definition 3.28 (separable Banach space). *The following definition is made in* RCA_0. *A (code for a)* separable Banach space $\widehat{E}$ *consist in a countable vector space* E *over the rational field* $\mathbb{Q}$ *together with a sequence of real numbers* $\|\cdot\| : E \to \mathbb{R}$ *satisfying*

1. $\|q \cdot a\| = |q| \cdot \|a\|$ *for all* $q \in \mathbb{Q}$ *and* $a \in E$;

2. $\|a + b\| \leq \|a\| + \|b\|$ *for all* $a, b \in E$.

A point of $\widehat{E}$ *is a sequence* $x = (x_n)_{n \in \mathbb{N}}$ *of elements of* E *such that for all* $n \leq m$, $\|x_n - x_m\| \leq 2^{-n}$. *As usual, we define a pseudometric on* E *by* $d(a,b) = \|a - b\|$, *for all* $a, b \in E$. *Thus,* $\widehat{E}$ *is the complete separable metric space which is the completion of* E *under* d.

If $x = (x_n)_{n \in \mathbb{N}}$ *and* $y = (y_n)_{n \in \mathbb{N}}$ *are points of* $\widehat{E}$ *and* $\alpha = (q_n)_{n \in \mathbb{N}}$ *is a real number, we define* $\|x\| = \lim_{n \to +\infty} \|x_n\|$, $x + y = \lim_{n \to +\infty} (x_n + y_n)$ *and* $\alpha \cdot x = \lim_{n \to +\infty} (q_n \cdot x_n)$. *It is easy to show within* RCA_0 *that this limits exist and that* $\|\cdot\| : \widehat{E} \to \mathbb{R}$, $+ : \widehat{E} \times \widehat{E} \to \widehat{E}$ *and* $\mathbb{R} \times \widehat{E} \to \widehat{E}$ *can be coded as continuous functions. Thus,* $\widehat{E}$ *enjoys the usual properties of a normed vector space over* $\mathbb{R}$ *(see section II.10 of [19] for details about this definition and its properties).*

Now, to finish this section we consider three important results that will play a crucial role in what follows. Notice the increasing logical strength of each of them.

Theorem 3.29 (Theorem 10.4.1. of [8]). *The following is provable in* RCA_0. *If F is a coded continuous function from an interval $[a,b]$ to $\mathbb{R}$, and y is between $F(a)$ and $F(b)$, there is an $x \in [a,b]$ with $F(x) = y$.*

Theorem 3.30 (Theorem IV.2.2 of [19]). *The following is provable in* WKL_0. *Let $\widehat{X}$ be a compact metric space. Let C be a closed set in $\widehat{X}$, and let F be a continuous function from C into a complete separable metric space $\widehat{Y}$. Then F has a modulus of uniform continuity on C. If in addition $\widehat{X} = C$ and $\widehat{Y} = \mathbb{R}$, the F attains a maximum value.*

Theorem 3.31. *The following is provable in* ACA_0. *Every sequence of real numbers bounded from below has an infimum.*

Proof. Adapt proof of 4. from Theorem III.2.2 of [19]. □

3.2.2 The separable Banach spaces $C([0,1],\widehat{E})$

With the formalization of the proof of the MPT in sight, we would want to reason towards the existence of a version in RCA_0 of the separable Banach space of continuous functions $C([0,1],\widehat{E})$, where $\widehat{E}$ is a separable Banach space. However, RCA_0 will not be enough to do so. We eventually will have to reason within WKL_0. To begin with, consider E, the countable dense subset of $\widehat{E}$. Yet within RCA_0 define $|E| \subseteq \mathbb{N}$ to be the set of (codes for) nonempty finite sequences of elements of $\mathbb{Q} \times E$, $\langle r_0,\dots,r_n\rangle$, where $r_i = \langle q_i, x_i\rangle \in \mathbb{Q}\times E$ for each $i \in \{0,1,\dots,n\}$ and such that $q_0 = 0$, $q_n = 1$ and with $q_i < q_j$ if $i < j$. We define the "line" passing through the points r_k and r_{k+1} as follows:

$$\ell_{r_k}^{r_{k+1}}(x) = \left(\frac{x - q_k}{q_{k+1} - q_k}\right)\cdot(x_{k+1} - x_k) + x_k, \text{ for } x \in [q_k, q_{k+1}]\,. \tag{3.2}$$

Equation 3.2 is just a generalization of the classical equation of a straight line passing between to given points from Euclidean plane geometry.

We want to define an addition and a scalar product on $|E|$. Consider $\langle r_0,\dots,r_m\rangle$ and $\langle s_0,\dots,s_n\rangle$ on $|E|$, where $r_i = \langle q_i,x_i\rangle \in \mathbb{Q}\times E$ for each $i\in\{0,1,\dots,m\}$ and $s_i = \langle q_i', x_i'\rangle \in \mathbb{Q}\times E$ for each $i \in \{0,1,\dots,n\}$.

For the *addition* we set

$$\langle r_0,\dots,r_m\rangle + \langle s_0,\dots,s_n\rangle = \langle t_0,\dots,t_k\rangle$$

where $k = \max\{m,n\}$ and $t_i = \langle q_i'', x_i''\rangle$ is constructed as follows.

- $q_0'' = 0$ and $x_0'' = x_0 + x_0'$.
- $q_1'' = \min\{q_1, q_1'\}$ and

$$x_1'' = \begin{cases} x_1 + \ell_{s_0}^{s_1}(q_1) & \text{if } q_1'' = q_1, \\ x_1' + \ell_{r_0}^{r_1}(q_1') & \text{if } q_1'' = q_1'. \end{cases}$$

- $$q_2'' = \begin{cases} \min\{q_1', q_2\} & \text{if } q_1'' = q_1, \\ \min\{q_1, q_2'\} & \text{if } q_1'' = q_1', \end{cases}$$

  and

  $$x_2'' = \begin{cases} x_1' + \ell_{r_1}^{r_2}(q_1') & \text{if } q_2'' = q_1', \\ x_2 + \ell_{s_1}^{s_2}(q_2) & \text{if } q_2'' = q_2 \\ x_1 + \ell_{s_1}^{s_2}(q_1) & \text{if } q_2'' = q_1 \\ x_2' + \ell_{r_1}^{r_2}(q_2') & \text{if } q_2'' = q_2'. \end{cases}$$

We can continue in this way successively until we reach

- $q_k'' = 1$ and $x_k'' = x_m + x_n'$.

The previous definition of addition is projected to be used towards a computable definition of a step by step sum of piece-wise linear functions with breakpoints in the r_i and the s_i. The result will be another piece-wise linear function with breakpoints in the t_i.

For the *scalar product* we set:

$$q \cdot \langle r_0, \ldots, r_m \rangle = \langle 0 \rangle, \text{ if } q = 0$$

and

$$q \cdot \langle r_0, \ldots, r_m \rangle = \langle q \cdot r_0, \ldots, q \cdot r_m \rangle, \text{ if } q \in \mathbb{Q} \setminus \{0\}.$$

One can verify that $|E|$ is a vector space over $\mathbb{Q}$.

Now, continuing in RCA_0, for all $m \in \mathbb{N}$, given $\langle r_0, \ldots, r_m \rangle \in |E|$ we define a function $g : [0, 1] \to \widehat{E}$ by

- $g(q_k) = x_k$ for $k \in \{0, \ldots, m\}$, and
- $g(x) = \ell_{r_k}^{r_{k+1}}(x)$ for $x \in [q_k, q_{k+1}]$.

Notice that g is a piece-wise linear (in the "likewise a straight line" meaning of the term) and continuous function with rational breakpoints; we will call this functions polygonal when $\widehat{E} = \mathbb{R}$ and therefore $E = \mathbb{Q}$. It is easy to verify that g is a coded continuous function.

Up to here, we are asserting that finite sequences of points $\langle r_0, \ldots, r_m \rangle \in |E|$ serve as codes of piece-wise linear continuous functions like g: precisely the ones constructed using those points as breakpoints.

Now, we want to define a norm for $|E|$. To that aim, RCA_0 is not strong enough. We must use the power of WKL_0.

As defined above let $g : [0,1] \to \widehat{E}$ a piece-wise linear continuous function with rational breakpoints and consider:

$$\begin{aligned} G : [0,1] &\to \mathbb{R} \\ x &\mapsto \|g(x)\| \,. \end{aligned}$$

Since the norm $\|\cdot\|$ and g are coded continuous functions then G, as a composition, is also a coded continuous function. Using the Theorem 3.30 with $X = C = [0,1]$ and $Y = \mathbb{R}$, we have that $\max\limits_{x\in[0,1]} \|g(x)\| \in \mathbb{R}$. Thus, for $\langle r_0, \ldots, r_m \rangle \in |E|$, we set

$$\|\langle r_0, \ldots, r_m \rangle\|_1 = \max_{x\in[0,1]} \|g(x)\| \,.$$

We define $C([0,1], \widehat{E}) = \widehat{|E|}$, the completion of $|E|$ under the metric induced by this norm.

We want to assert that the points of our $C([0,1], \widehat{E})$ are in canonical one to one correspondence with the continuous $\widehat{E}$-valued functions on the closed unit interval $[0,1]$. For this, we prove the following.

Theorem 3.32. *The following is provable in* RCA_0. *Let F be a continuous $\widehat{E}$-valued function defined on* $[0,1]$.

1. *If $F(x)$ is uniformly continuous, then for each $\varepsilon > 0$ there exists a piece-wise linear continuous function f with rational breakpoints such that $\|F(x) - f(x)\| < \varepsilon$ for all $x \in [0,1]$.*

2. *If $F(x)$ possesses a modulus of uniform convergence, then there exists a sequence of piece-wise linear continuous functions with rational breakpoints $(f_n)_{n\in\mathbb{N}}$ such that $\|F(x) - f_n(x)\| < 2^{-n}$ for all $x \in [0,1]$.*

Proof. To prove 1, let $F : [0,1] \to \widehat{E}$ be a uniformly continuous function, i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$ then $\|F(x_1) - F(x_2)\| < \varepsilon$. Now, we fix $\varepsilon > 0$. Since $\frac{\varepsilon}{5} > 0$, we can find $\delta > 0$ as in the previous definition applied to $\frac{\varepsilon}{5} > 0$. Taking $m \in \mathbb{N}$ large enough so $\frac{1}{m} < \delta$, we have

$$\|F(x) - F(y)\| < \frac{\varepsilon}{5}, \text{ whenever } |x-y| < \frac{1}{m}, \text{ for all } x, y \in [0,1].$$

For such an m we construct the piece-wise linear continuous function $f : [0,1] \to \widehat{E}$ as follows. Let $k \in \{0, \ldots, m\}$. Since $F\left(\frac{k}{m}\right) \in \widehat{E}$, then $F\left(\frac{k}{m}\right) = (x_i^k)_{i\in\mathbb{N}}$ with $x_i^k \in E$, and we know that for every $i \in \mathbb{N}$, $\left\|F\left(\frac{k}{m}\right) - x_i^k\right\| \leq 2^{-i}$. We can take $i \in \mathbb{N}$ such that $2^{-i} < \frac{\varepsilon}{5}$ and define:

- $f\left(\frac{k}{m}\right) = x_i^k \in E$, for $k \in \{0, \ldots, m\}$.
- $f(x) = (mx - k) \cdot \left(f\left(\frac{k+1}{m}\right) - f\left(\frac{k}{m}\right)\right) + f\left(\frac{k}{m}\right)$, for $x \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$.

Thus, we have that $\left\| f\left(\frac{k}{m}\right) - F\left(\frac{k}{m}\right)\right\| < \frac{\varepsilon}{5}$, for each $k \in \{0, \ldots, m\}$.

Next, for $k \in \{0, \ldots, m\}$ and $x \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$, and because $0 \leq mx - k \leq 1$, notice that:

$$\begin{aligned}\left\| f\left(\tfrac{k}{m}\right) - f(x)\right\| &= \left\| f\left(\tfrac{k}{m}\right) - (mx-k)\cdot\left(f\left(\tfrac{k+1}{m}\right) - f\left(\tfrac{k}{m}\right)\right) - f\left(\tfrac{k}{m}\right)\right\| \\ &= |-(mx-k)| \left\| f\left(\tfrac{k+1}{m}\right) - f\left(\tfrac{k}{m}\right)\right\| \\ &\leq \left\| f\left(\tfrac{k+1}{m}\right) - f\left(\tfrac{k}{m}\right)\right\|.\end{aligned}$$

Finally, given $x \in [0,1]$, we know $x \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$ for some $k \in \{0, \ldots, m\}$. Therefore:

$$\begin{aligned}\|F(x) - f(x)\| &\leq \left\|F(x) - F\left(\tfrac{k}{m}\right)\right\| + \left\|F\left(\tfrac{k}{m}\right) - f\left(\tfrac{k}{m}\right)\right\| + \left\| f\left(\tfrac{k}{m}\right) - f(x)\right\| \\ &\leq \tfrac{\varepsilon}{5} + \tfrac{\varepsilon}{5} + \left\| f\left(\tfrac{k}{m}\right) - f\left(\tfrac{k+1}{m}\right)\right\| \\ &\leq \tfrac{2\varepsilon}{5} + \left\| f\left(\tfrac{k}{m}\right) - F\left(\tfrac{k}{m}\right)\right\| + \left\|F\left(\tfrac{k}{m}\right) - F\left(\tfrac{k+1}{m}\right)\right\| + \left\|F\left(\tfrac{k+1}{m}\right) - f\left(\tfrac{k+1}{m}\right)\right\| \\ &\leq \tfrac{2\varepsilon}{5} + \tfrac{\varepsilon}{5} + \tfrac{\varepsilon}{5} + \tfrac{\varepsilon}{5} = \varepsilon.\end{aligned}$$

For 2, since any continuous coded function that possesses a modulus of uniform continuity is uniformly continuous, we can use part 1 and induction with $\varphi(n)$, the Σ_1^0 formula that says that there exists a piece-wise linear continuous functions f_n with rational breakpoints such that $\|F(x) - f_n(x)\| < 2^{-n}$ for all $x \in [0,1]$. □

Due to Theorems 3.30 and 3.32 we assert that within WKL_0, the space $C([0,1], \widehat{E})$ is isomorphic to the space of uniformly continuous functions $F : [0,1] \to \widehat{E}$ that have moduli of uniform continuity and that the dense set is given by piece-wise linear continuous functions $f : [0,1] \to \widehat{E}$ with rational breakpoints, each represented by finitely many pairs $\langle x, f(x)\rangle \in \mathbb{Q} \times E$.

With a similar construction, it is straightforward to have access to the separable metric space $\Gamma = \widehat{A} = \{g \in C([0,1], \widehat{E}) : g(0) = 0,\ g(1) = v\}$ for any fixed $v \in \widehat{E}$, where the dense subset A consists in piece-wise linear continuous functions with rational breakpoints and such that the image of 0 is 0 and the image of 1 is v. Notice that this will not be a Banach space (because it will not be a vector space) but just a metric space. The metric being defined by

$$d(f,g) = \max_{t\in[0,1]} \|f(t) - g(t)\|.$$

Up to this point, we have set up the formalized framework where to develop the formalization of the MPT. In the next chapter, we formalize its first ingredient: the Deformation Lemma.

Chapter 4

Formalizing the Deformation Lemma

We have seen that one possible path to perform the proof of the MPT relies on the use of the Deformation Lemma. This is the path that we will take here although we again encourage the interested researcher to explore the path that uses the Ekeland $\varepsilon-$principle.

In order to formalize the proof of the Deformation Lemma (Lemma 4.24) we need to review some results from the theory of Ordinary Differential Equations (ODEs).

4.1 Ordinary Differential Equations

We begin with some preliminary results.

Theorem 4.1 (Generalized Theorem for Convergence for Series)**.** *The following is provable in* RCA_0*. Let* $\widehat{A}$ *be a separable metric space and let* $\widehat{E}$ *be a separable Banach space. Let* $\Sigma_{k=0}^{+\infty}\alpha_k$ *be a convergent series of nonnegative real numbers* $\alpha_k \geq 0$*. Let* $(\phi_k)_{k\in\mathbb{N}}$ *be a sequence of continuous functions* $\phi_k : \widehat{A} \to \widehat{E}$ *such that* $\|\phi_k(x)\| \leq \alpha_k$ *for all* $k \in \mathbb{N}$ *and* $x \in \widehat{A}$*. Then* $\phi = \Sigma_{k=0}^{+\infty}\phi_k : \widehat{A} \to \widehat{E}$ *is continuous, and* $\|\phi(x)\| \leq \Sigma_{k=0}^{+\infty}\alpha_k$ *for all* $x \in \widehat{A}$*.*

Proof. We rewrite the proof of Lemma II.6.5 of [19], where the theorem is proved for $\widehat{E} = \mathbb{R}$. We set $\Phi \subseteq A \times \mathbb{Q}_{>0} \times E \times \mathbb{Q}_{>0}$ such that $B_r(a) \xrightarrow{\Phi} B_s(b)$ if and only if there is some $m \in \mathbb{N}$ such that there exist $B_r(a) \xrightarrow{\Phi_k} B_{s_k}(b_k)$, $k < m$, such that $b = \sum_{k<m} b_k$ and

$$\sum_{k=0}^{\infty} s_k + \sum_{k=0}^{\infty} \alpha_k < s.$$

One can verify that this is a code for $\phi : \widehat{A} \to \widehat{E}$ as required. □

Remark 4.2. *The preceding theorem can be used to show that real-valued functions defined by power series, such as e^x, are also coded continuous functions.*

Next, we present a formalized version of the Riemann integral.

Definition 4.3 (Riemann integral). *The following definitions are made in* RCA_0. *A* partition *of the interval* $[a,b]$ *is a finite list of points* $\mathcal{P} = \{x_0, x_1, x_2, ..., x_k\}$ *with* $a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = b$. *In each subinterval* $I_j = [x_{j-1}, x_j]$, *select a computable point* ξ_j *(e.g. the average of the endpoints). Let* $\Delta x_j = x_j - x_{j-1}$. *Given a Banach space* $\widehat{E}$ *and a function* $f : [a,b] \to \widehat{E}$ *we define the* Riemann sum as

$$\mathcal{R}_{\mathcal{P}} = \sum_{i=0}^{k} f(\xi_j)\Delta x_j.$$

Define the mesh *of the partition* $\mathcal{P}$ *to be* $m(\mathcal{P}) = \max_j \Delta x_j$. *We say that the Riemann sums have a limit* $\ell \in \mathbb{R}$ *as the mesh of the partitions tends to zero if, given* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that, if* $m(\mathcal{P}) < \delta$, *then* $\|\mathcal{R}_{\mathcal{P}} - \ell\| < \delta$. *In that case, we call the limit* ℓ *the* Riemann integral *of the function* f *on the interval* $[a,b]$. *We write as usual*

$$\ell = \int_a^b f(x)\,dx$$

for the Riemann integral.

We are ready to prove a fundamental result.

Theorem 4.4 (Existence of the Riemann Integral). *The following is provable in* WKL_0. *Let* $\widehat{E}$ *be a separable Banach space and let* $F : [a,b] \to \widehat{E}$ *be a coded continuous function. Then the Riemann integral of* F *on* $[a,b]$ *exists.*

Proof. First, we are going to show a kind of Cauchy property for the Riemann integral following some ideas from [6] and [16]. Let $\varepsilon > 0$. Due to Theorem 3.30, we know that WKL_0 proves that the coded continuous function F is uniformly continuous on $[a,b]$. This allows us to choose $\delta > 0$ such that $|s - t| < \delta$ implies $\|F(s) - F(t)\| < \dfrac{\varepsilon}{2(b-a)}$. Now let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[a,b]$ with mesh less than δ. We show that

$$\|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{Q}}\| < \varepsilon.$$

Let $\mathcal{W}$ be the common refinement of $\mathcal{P}$ and $\mathcal{Q}$, defined to be $\mathcal{W} = \mathcal{P} \cup \mathcal{Q}$. Let the points of $\mathcal{P}$ be called $p_0, p_1, \ldots, p_k$ and let the points of $\mathcal{W}$ be called $w_0, w_1, \ldots, w_m$. The partition $\mathcal{P}$ gives rise to subintervals I_j, having lengths Δ_j, and the partition $\mathcal{W}$ gives rise to subintervals J_l, having lengths Δ'_l. For each j, let s_j be a computable point chosen from I_j and for each l, let t_l be a computable point chosen from J_l

(these points are chosen to enable us to write down the Riemann sums). Since the partition $\mathcal{W}$ contains every point of $\mathcal{P}$, plus some additional points as well, every J_l is contained in some I_j. Thus, consider that for each j:

$$\Delta_j = \sum_{J_l \subseteq I_j} \Delta'_l.$$

Now consider the following:

$$\begin{aligned}
\|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{W}}\| &= \left\| \sum_{j=1}^{k} F(s_j)\Delta_j - \sum_{l=1}^{m} F(t_l)\Delta'_l \right\| \\
&= \left\| \sum_{j=1}^{k} \left[F(s_j) \sum_{J_l \subseteq I_j} \Delta'_l \right] - \sum_{j=1}^{k} \left[\sum_{J_l \subseteq I_j} F(t_l)\Delta'_l \right] \right\| \\
&\leq \sum_{j=1}^{k} \sum_{J_l \subseteq I_j} \left\| F(s_j) - F(t_l) \right\| \Delta'_l.
\end{aligned}$$

Since each of the points s_j, t_l are in I_j, we have that $|s_j - t_l| < \delta$; therefore, $\left\|F(s_j) - F(t_l)\right\| < \dfrac{\varepsilon}{2(b-a)}$. Whence,

$$\begin{aligned}
\|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{W}}\| &< \sum_{j=1}^{k} \sum_{J_l \subseteq I_j} \frac{\varepsilon}{2(b-a)} \Delta'_l. \\
&= \frac{\varepsilon}{2(b-a)} \sum_{j=1}^{k} \sum_{J_l \subseteq I_j} \Delta'_l. \\
&= \frac{\varepsilon}{2(b-a)} \sum_{j=1}^{k} \Delta_j \\
&= \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.
\end{aligned}$$

Analogously, we can estimate that $\|\mathcal{R}_{\mathcal{Q}} - \mathcal{R}_{\mathcal{W}}\| < \dfrac{\varepsilon}{2}$ and therefore that $\|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{Q}}\| < \varepsilon$. Thus, we have shown that for all $\varepsilon > 0$, there is $\delta > 0$ such that for any two partitions $\mathcal{P}$ and $\mathcal{Q}$ with mesh less than δ, we have that $\|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{Q}}\| < \varepsilon$. We use this fact to prove that the Riemann integral exists. For each $n \in \mathbb{N}$, let $\delta_n > 0$ be such that for any two partitions $\mathcal{P}$ and $\mathcal{Q}$ with mesh less than δ_n, we have that

$$\|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{Q}}\| < \frac{1}{2^n}.$$

We can assume that $\delta_n \geq \delta_{n+1}$; otherwise, we replace δ_n by $\delta'_n = \min\{\delta_1, \ldots, \delta_n\}$. For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a partition with $m(\mathcal{P}) < \delta_n$. Clearly, if $m > n$ then both

$\mathcal{P}_m$ and $\mathcal{P}_n$ have mesh less that δ_n, thus

$$\|\mathcal{R}_{\mathcal{P}_m} - \mathcal{R}_{\mathcal{P}_n}\| < \frac{1}{2^n}. \tag{4.1}$$

This implies that the sequence $(\mathcal{R}_{\mathcal{P}_m})_{m\in\mathbb{N}}$ is a quickly converging Cauchy sequence in $\widehat{E}$. In consequence, by Theorem 3.16, this sequence converges in $\widehat{E}$, to a limit that we call $\ell = \lim_{m\to+\infty} \mathcal{R}_{\mathcal{P}_m}$. Taking the limit in (4.1) as $m \to +\infty$, we have that for all $n \in \mathbb{N}$:

$$\|\mathcal{R}_{\mathcal{P}_n} - \ell\| < \frac{1}{2^n}.$$

To see that ℓ is the Riemann integral we are looking for, fix $\varepsilon > 0$ and let $K \in \mathbb{N}$ be such that $2^{-K} < \varepsilon/2$. Let $\mathcal{P}$ be an arbitrary partition with $m(\mathcal{P}) < \delta_K$; this way we have that:

$$\begin{aligned}\|\mathcal{R}_{\mathcal{P}} - \ell\| &\leq \|\mathcal{R}_{\mathcal{P}} - \mathcal{R}_{\mathcal{P}_K}\| + \|\mathcal{R}_{\mathcal{P}_K} - \ell\| \\ &< \frac{1}{2^K} + \frac{1}{2^K} < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, we can conclude our proof. □

Some properties of the Riemann integral that will be used later are easy to prove in WKL_0. We state them in the following lemma:

Lemma 4.5. *The following is provable in* WKL_0*. Suppose that* $f, g : [a, b] \to \widehat{E}$ *are two coded continuous functions. Then,*

1. $\alpha \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b [\alpha f(x) + g(x)]\,dx$, *for* $\alpha \in \mathbb{R}$.
2. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, *for* $c \in (a, b)$.
3. $\left\|\int_a^b f(x)\,dx\right\| \leq \int_a^b \|f(x)\|\,dx$.
4. $\left\|\int_a^b f(x)\,dx\right\| \leq \max_{x\in[a,b]} \|f(x)\| \cdot (b - a)$.

Next, we want to establish a sort of Generalized Fundamental Theorem of Calculus. First, we must review some definitions on bounded and continuous linear operators and an important theorem relating both. All of this can be found in [19].

Definition 4.6 (bounded linear operators). *The following definition is made in* RCA_0*. Let* $\widehat{A}$ *and* $\widehat{B}$ *be separable Banach spaces. A (code for a)* bounded linear operator *from* $\widehat{A}$ *to* $\widehat{B}$ *is a sequence* $F : A \to \widehat{B}$ *of points of* $\widehat{B}$*, indexed by elements of* A*, such that (i)* $F(q_1a_1 + q_2a_2) = q_1F(a_1) + q_2F(a_2)$ *for all* $q_1, q_2 \in \mathbb{Q}$ *and* $a_1, a_2 \in A$*, (ii) there exists a real number* α *such that* $\|F(a)\| \leq \alpha \cdot \|a\|$ *for all* $a \in A$.

For F and α as above and $x = (x_i)_{i\in\mathbb{N}} \in \widehat{A}$, we define $F(x) = \lim_{k\to+\infty} F(a_k)$. Thus $\|F(x)\| \leq \alpha \cdot \|x\|$ for all $x \in \widehat{A}$. We write $F : \widehat{A} \to \widehat{B}$ to denote this state of affairs. If $\alpha \in \mathbb{R}$ is such that $\|F(x)\| \leq \alpha \cdot \|x\|$ for all $x \in \widehat{A}$, we write $\|F\| \leq \alpha$.

Definition 4.7 (continuous linear operators)**.** *The following definition is made in RCA_0. Let $\widehat{A}$ and $\widehat{B}$ be separable Banach spaces. A* continuous linear operator *from $\widehat{A}$ to $\widehat{B}$ is a totally defined continuous function $\varphi : \widehat{A} \to \widehat{B}$ such that $\varphi(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1\varphi(x_1) + \alpha_2\varphi(x_2)$ for all $\alpha_1, \alpha_2 \in \mathbb{R}$ and $x_1, x_2 \in \widehat{A}$.*

Theorem 4.8 (Theorem II.10.7. of [19])**.** *The following is provable in* RCA_0. *Given a continuous linear operator $\varphi : \widehat{A} \to \widehat{B}$, there exists a bounded linear operator $F : \widehat{A} \to \widehat{B}$ such that $F(x) = \varphi(x)$ for all $x \in \widehat{A}$. Conversely, given a bounded linear operator $F : \widehat{A} \to \widehat{B}$, there exists a continuous linear operator $\varphi : \widehat{A} \to \widehat{B}$ such that $\varphi(x) = F(x)$ for all $x \in \widehat{A}$.*

Thus, we know that RCA_0 proves that bounded linear operators are the same thing as continuous linear operators.

If $F : \widehat{A} \to \widehat{B}$ is a bounded linear operator we write $F \in \mathcal{B}(\widehat{A}, \widehat{B})$. A special case is the dual space of the separable Banach space $\widehat{A}$, when $\widehat{B} = \mathbb{R}$.

For $F \in \mathcal{B}(\widehat{A}, \widehat{B})$ we define its norm as:

$$\|F\|_{\mathcal{B}(\widehat{A},\widehat{B})} = \sup_{x\neq 0} \frac{\|F(x)\|_{\widehat{B}}}{\|x\|_{\widehat{A}}}.$$

Consider the following result.

Lemma 4.9 (Existence of norms for bounded linear functionals)**.** *The following is provable in* ACA_0. *Every bounded linear functional on a Banach space has a norm.*

Proof. This follows from (1) of Theorem 13.4 of [4]. □

We must mention that for arbitrary separable spaces $\widehat{A}$ and $\widehat{B}$, the Banach space $\mathcal{B}(\widehat{A}, \widehat{B})$ (and therefore the dual of a separable Banach space $\widehat{E}$, denoted $\widehat{E}'$) cannot be proven to exist in Second Order Arithmetic because there is no way to justify it would be separable in full generality. When using it as a set, we should assume it is also a separable Banach space in order to define continuously differentiable functions, which is what we do next.

Definition 4.10. *Let $\widehat{A}$ and $\widehat{B}$ be two Banach spaces, and $U \subseteq \widehat{A}$ be an open subset of $\widehat{A}$. A function $f : U \to \widehat{B}$ is called* Fréchet differentiable *(or just differentiable) at $x \in U$ if there exists a bounded linear operator $D : \widehat{A} \to \widehat{B}$ such that*

$$\lim_{\|h\|_{\widehat{B}}\to 0} \frac{\|f(x+h) - f(x) - D(h)\|_{\widehat{A}}}{\|h\|_{\widehat{B}}} = 0, \tag{4.2}$$

wich using the "little-O" notation of Landau can be written as follows:

$$f(x+h) - f(x) - D(h) = o(h), \text{ as } \|h\| \to 0.$$

If there exists such an operator, it is unique, so we write $f'(x) = D$ and call it the Fréchet derivative of f at x. A function f that is Fréchet differentiable for any point of $x \in U$ is said to be of class C^1 if the function

$$\begin{aligned} f' : U &\to \mathcal{B}(\widehat{A}, \widehat{B}) \\ x &\mapsto f'(x) \end{aligned}$$

is continuous.

Note that this is not the same as requiring that the map $f'(x) : \widehat{A} \to \widehat{B}$ be continuous for each value of x, which is already assumed since bounded and continuous are equivalent.

Now we are ready to prove a pretty fundamental theorem.

Theorem 4.11 (Generalized Fundamental Theorem of Calculus)**.** *The following is provable in* WKL_0*. Let $\widehat{E}$ be a separable Banach space and let $f : [a,b] \to \widehat{E}$ be a coded continuous function. Then the function*

$$\begin{aligned} F : [a,b] &\to \widehat{E} \\ x &\mapsto \int_a^x f(s)\, ds. \end{aligned}$$

is differentiable on (a,b) and $F'(x) = f(x)$.

Proof. By the previous theorem we know that WKL_0 is enough to define $F(x) = \int_a^x f(s)\, ds$ for each $x \in [a,b]$. Now, to show that its derivative is $f(x)$, let $x \in (a,b)$ and show that

$$F(x+h) - F(x) - f(x)h = o(h), \text{ as } |h| \to 0.$$

Let $\varepsilon > 0$. By the uniform continuity of f, we choose $\delta > 0$ such that $|s - t| < \delta$

implies $\|f(s) - f(t)\| < \varepsilon$. Now, assume that $|h| \leq \delta$ and consider the following,

$$\begin{aligned}
\frac{\|F(x+h) - F(x) - f(x)h\|}{|h|} &= \left|h^{-1}\right| \left\| \int_a^{x+h} f(s)\,ds - F\int_a^x f(s)\,ds - f(x)h \right\| \\
&\leq h^{-1} \left\| \int_x^{x+h} f(s)\,ds - f(x)h \right\| \\
&= h^{-1} \left\| \int_x^{x+h} f(s)\,ds - \int_x^{x+h} f(x)\,ds \right\| \\
&= h^{-1} \left\| \int_x^{x+h} [f(s) - f(x)]\,ds \right\| \\
&\leq h^{-1} \left[\max_{s\in[x,x+h]} \|f(s) - f(x)\| \right] (h) \\
&= \max_{s\in[x,x+h]} \|f(s) - f(x)\| \\
&= \left\| f(s') - f(x) \right\|,
\end{aligned}$$

where $s' \in [x, x+h]$ is where the maximum is attained. Since $x, s' \in [x, x+h]$, then $|s' - x| < h$, and therefore $\|f(s') - f(x)\| < \varepsilon$. □

The Fundamental Theorem of Calculus is an angular result in the development of Integration Theory. We recall a particular application of it that will be useful afterwards.

Theorem 4.12 (Grönwall's inequality). *The following is provable in* WKL_0. *Let* $\psi(t)$ *be a continuous function on* $[0, h]$ *which satisfies the integral inequality:*

$$0 \leq \psi(t) \leq \int_0^t (M\psi(s) + A)\,ds$$

for $t \in [0, h]$ *and for constants* $M, A \geq 0$. *Then*

$$\psi(t) \leq Ahe^{Mh}$$

for $0 \leq t \leq h$. *In particular, if*

$$\psi(t) \leq M \int_0^t \psi(s)\,ds$$

for $0 \leq t \leq h$, *then*

$$\psi(t) = 0.$$

Proof. The proof follows immediately using the Fundamental Theorem of Calculus as can be apreciated in the original proof by Grönwall on [14]. □

Now, we need a generalization of Picard's theorem for existence and uniqueness for initial value problems.

Theorem 4.13 (Generalized Picard's Theorem). *The following is provable in* RCA_0. *Let* $\widehat{E}$ *be a separable Banach space. Assume that* $f : \mathbb{R} \times \widehat{E} \to \widehat{E}$ *has a modulus of uniform continuity* $h : \mathbb{N} \to \mathbb{N}$ *and satisfies a Lipschitz condition with respect to the second variable:*

$$\|f(x, y_1) - f(x, y_2)\| \leq L \cdot \|y_1 - y_2\|.$$

Also, let $\|f(x,y)\| \leq M$ *whenever* $|x| \leq a$ *and* $\|y - u\| \leq b$ *for* $a, b \in \mathbb{R}$ *and* $u \in \widehat{E}$. *Then, the initial value problem:*

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(0) = u, \end{cases} \tag{4.3}$$

has a unique solution $y = \phi(x)$ *for the interval* $J = [-\alpha, \alpha]$, *where* $\alpha = \min\{a, b/M\}$. *Moreover* $\phi(x)$ *has a modulus of uniform continuity on this interval.*

Proof. In this proof we follow those from Theorems IV.8.1 and IV.8.3 of [19], where things are done for $\widehat{E} = \mathbb{R}$.

First, let $A = \{q_i : i \in \mathbb{N}\}$ be an enumeration of the rational numbers in the closed interval $[-\alpha, \alpha]$. Thus $\widehat{A} = [-\alpha, \alpha]$. We may safely assume that $q_0 = 0$. Let C be the closed convex set in $\widehat{E}^{\mathbb{N}}$ consisting of all sequences $(y_i)_{i \in N}$ such that $y_0 = u$ and $\|y_i - y_j\| \leq M|q_i - q_j|$ for all $i, j \in \mathbb{N}$. To each $(y_i)_{i \in \mathbb{N}} \in C$ is associated a continuous function

$$\phi : [-\alpha, \alpha] \to \widehat{E}$$

such that $\phi(q_i) = y_i$ for all $i \in \mathbb{N}$. Namely, the code Φ of ϕ is given by putting $B_r(q_i) \xrightarrow{\Phi} B_s(b)$ if and only if $Mr + \|b - y_i\| < s$. Thus we shall identify points of C with continuous functions $\phi : [-\alpha, \alpha] \to \widehat{E}$ satisfying $\phi(0) = u$ and

$$\|\phi(x) - \phi(x')\| \leq M|x - x'|$$

for $|x|, |x'| \leq \alpha$. We define a continuous function $F : C \to C$ as follows. For $(y_i)_{i \in \mathbb{N}}$, we put

$$F((y_i)_{i \in \mathbb{N}}) = \left(u + \int_0^{q_i} f(x, \phi(x))\, dx\right)_{i \in \mathbb{N}}$$

where $\phi : [-\alpha, \alpha] \to \widehat{E}$ is the continuous function associated to $(y_i)_{i \in \mathbb{N}}$ as above. Notice that for all $i, j \in \mathbb{N}$ we have that

$$\begin{aligned} &\|u + \textstyle\int_0^{q_i} f(x, \phi(x))\, dx - u - \int_0^{q_j} f(x, \phi(x))\, dx\| \\ &\quad = \left\| \textstyle\int_{q_i}^{q_j} f(x, \phi(x))\, dx \right\| \\ &\quad \leq M|q_i - q_j|, \end{aligned}$$

so $F((y_i)_{i\in\mathbb{N}}) \in C$. Using a modulus of uniform continuity for f, we can construct a code for F. Thus, we have the coded continuous function $F : C \to C$ defined by

$$F(\phi)(x) \mapsto u + \int_0^x f(s, \phi(s))\, ds.$$

We define a sequence of functions $\phi_n \in C$ by $\phi_0(x) = u$ for all $|x| \leq \alpha$ and $\phi_{n+1} = F(\phi_n)$ for all $n \in \mathbb{N}$. First, we prove that for all $n \in \mathbb{N}$, if $|x| \leq \alpha$, we have that

$$\|\phi_n(x) - u\| \leq b. \tag{4.4}$$

We proceed by induction. The base, when $n = 0$, is trivial. For the inductive step, let (4.4) hold; this implies that $\|f(s, \phi_n(s))\| \leq M$ for all $s \in [0, x]$. Now consider the following:

$$\begin{aligned}\|\phi_{n+1}(x) - u\| &= \left\| \int_0^x f(s, \phi_n(s))\, ds \right\| \\ &\leq \max_{s\in[0,x]} \|f(s, \phi_n(s)))\|\,(x - 0) \\ &\leq M|x| \\ &\leq M\alpha \leq b.\end{aligned}$$

Second, we want to prove that for all $n \in \mathbb{N}$, we have that

$$\|\phi_{n+1}(x) - \phi_n(x)\| \leq \frac{L^n M |x|^{n+1}}{(n+1)!}. \tag{4.5}$$

We shall use induction again. For the base case, we can see that:

$$\begin{aligned}\|\phi_1(x) - \phi_0(x)\| &= \left\| \int_0^x f(s, \phi_1(s))\, ds \right\| \\ &\leq \max_{s\in[0,x]} \|f(s, \phi_1(s))\|\,(x - 0) \\ &\leq M|x|.\end{aligned}$$

For the inductive step, assuming that (4.5) holds, we have that:

$$\begin{aligned}
\|\phi_{n+2}(x) - \phi_{n+1}(x)\| &= \left\| \int_0^x [f(s, \phi_{n+1}(s)) - f(s, \phi_n(s))]\, ds \right\| \\
&\leq \int_0^x \| f(s, \phi_{n+1}(s)) - f(s, \phi_n(s)) \| \, ds \\
&\leq L \int_0^x \|\phi_{n+1}(s) - \phi_n(s)\| \, ds \\
&\leq L \int_0^x \frac{L^n M |s|^{n+1}}{(n+1)!} \, ds. \\
&\leq \frac{L^{n+1} M}{(n+1)!} \int_0^x |s|^{n+1} \, ds \\
&= \frac{L^{n+1} M |x|^{n+2}}{(n+2)!}.
\end{aligned}$$

With (4.5) been proved to hold for all $n \in \mathbb{N}$, let $\psi_n = \phi_{n+1} - \phi_n$ and $\beta_n = \dfrac{L^n M |x|^{n+1}}{(n+1)!}$ for $n \in \mathbb{N}$. We have that $\sum_{n=0}^{+\infty} \beta_n$ is a convergent series and since $\|\psi_n(x)\| \leq \beta_n$ for all $n \in \mathbb{N}$ and all $x \in [-\alpha, \alpha]$, by Theorem 4.1, we have that $\psi = \sum_{n=0}^{+\infty} \psi_n : \widehat{A} \to \widehat{E}$ converges. Notice that $\sum_{k=0}^{n} \psi_n = \phi_n - \phi_0$; therefore ϕ_n converges to $\phi = \psi + \phi_0$, an element of C. We can verify that this limit is a fixed point of F. To see this, notice that on the one hand, since $\phi_n \to \phi$, we have by Theorem 3.24 that $F(\phi_n) \to F(\phi)$. On the other hand, since also $\phi_{n+1} \to \phi$, by the definition of the sequence we have that $F(\phi_n) \to \phi$. Unicity of the limit gives us that $F(\phi) = \phi$. Hence,

$$\phi(x) = u + \int_0^x f(s, \phi(s))\, ds.$$

Taking the derivative, we see that $y = \phi(x)$ is a solution of the initial value problem (4.3).

To prove uniqueness, suppose that $\tilde{y}$ is another solution. We define the function

$$\psi(t) = \|y(t) - \tilde{y}(t)\|.$$

Since,

$$\|y(t) - \tilde{y}(t)\| = \left\| \int_0^t f(s, y(s)) - f(s, \tilde{y}(s))\, ds \right\| \leq L \int_0^t \|y(s) - \tilde{y}(s)\| = L \int_0^t \psi(s)\, ds.$$

Therefore,

$$\psi(t) \leq L \int_0^t \psi(s)\, ds, \quad \forall\, t \geq 0,$$

whence, due to Gronwall's inequality implies that $\psi = 0$, guaranteeing the uniqueness. □

The solution $y = \phi(x)$ of the initial-value problem (4.3) depends not only on the current value of the independent variable x but also on the initial data u. Moreover, the uniqueness theorem ensures that, for a given value of x, solutions with distinct values of u must have distinct values of y, and conversely, distinct values of y require distinct values of u. The solution may be written $y = y(x, u)$ to emphasize that y is determined by its initial data, i.e., is a function of its initial data: for a fixed value of x, it maps initial data u to y. Let us call this function the solution map and write:

$$\begin{aligned} y : [-\alpha, +\alpha] \times \widehat{E} &\to \widehat{E} \\ (x, u) &\mapsto y(u, x). \end{aligned}$$

where y solves the problem

$$\begin{cases} y'(x,u) = f(x, y(x,u)) \\ y(0,u) = u. \end{cases} \tag{4.6}$$

To establish the Theorem of Existence of Solution of ODEs that will indeed be used in the Deformation Lemma we need a formalized version of the theorem that assures the continuity of solutions also with respect to initial values. We begin with the following result.

Lemma 4.14. *The following is provable in* RCA_0. *Under the hypotheses of Picard's theorem, let*

$$\begin{aligned} y : [-\alpha, +\alpha] \times \widehat{E} &\to \widehat{E} \\ (x, u) &\mapsto y(x, u). \end{aligned}$$

be the solution map. Then there exists $K > 0$ *such that*

$$\|y(x,u) - y(x,\overline{u})\| \leq K \|u - \overline{u}\|$$

for all $u, \overline{u}$ *and for all* $x \in [-\alpha, +\alpha]$.

Proof. Take $K = e^{L\alpha}$ and let $x \in [-\alpha, +\alpha]$ and $u, \overline{u} \in \widehat{E}$; consider that due to Picard's integral approximations we have

$$\begin{aligned} \|y(x,u) - y(x,\overline{u})\| &= \left\| u + \int_0^x f(s, y(s,u)) - \overline{u} - \int_0^x f(s, y(s,\overline{u}))\, ds \right\| \\ &\leq \|u - \overline{u}\| + \int_0^x \|f(s, y(s,u)) - f(s, y(s,\overline{u}))\|\, ds \\ &\leq \|u - \overline{u}\| + \int_0^x L\, \|y(s,u) - y(s,\overline{u})\|\, ds \\ &= \int_0^x \left(L\, \|y(s,u) - y(s,\overline{u})\| + \frac{\|u - \overline{u}\|}{x} \right) ds. \end{aligned}$$

Now, applying Theorem 4.12 with $M = L$ and $A = \dfrac{\|u - \overline{u}\|}{x}$, we get that

$$\|y(x,u) - y(x,\overline{u})\| \leq \|u - \overline{u}\| \, e^{Lx},$$

and since $e^{Lx} \leq e^{L\alpha} = K$, we get the result.

With this in mind, we prove the following theorem. □

Theorem 4.15. *The following is provable in* RCA_0*. Under the hypotheses of the generalized Picard's theorem, the solution map*

$$\begin{aligned} y : [-\alpha, +\alpha] \times \widehat{E} &\to \widehat{E} \\ (x,u) &\mapsto y(x,u). \end{aligned}$$

is a coded continuous function.

Proof. Recall following the same notation that in the proof of Picard's theorem that $\widehat{A} = [-\alpha, +\alpha]$. We denote by d the metric on $A \times E$, the dense subset of $[-\alpha, +\alpha] \times \widehat{E}$:

$$d((x,u),(x',u')) = |x - x'| + \|u - u'\|.$$

This is just the taxicab norm. We want to code y by $\Phi \subseteq (A \times E) \times \mathbb{Q}_{>0} \times E \times \mathbb{Q}_{>0}$. Let $\Psi \subseteq A \times \mathbb{Q}_{>0} \times E \times \mathbb{Q}_{>0}$ be the code of the function $\psi : A \to \widehat{E}$ that solves

$$\begin{cases} \psi'(x) = f(x, \psi(x)) \\ \psi(0) = u, \end{cases} \tag{4.7}$$

which exists by Picard's theorem. Now, let $B_r((x,u)) \xrightarrow{\Phi} B_s(b)$ if and only if $B_r(x) \xrightarrow{\Psi} B_{s-Kr}(b)$, where K is the constant given by the previous lemma. We first prove that this defines a code for a partial function:

(CF1) Let $B_r((x,u)) \xrightarrow{\Phi} B_s(b)$ and $B_r((x,u)) \xrightarrow{\Phi} B_{s'}(b')$. To show that $\|b - b'\| \leq s + s'$, notice that as we have that $B_r(x) \xrightarrow{\Psi} B_s(b)$ and $B_r(x) \xrightarrow{\Psi} B_{s'}(b')$, we can get $\|b - b'\| \leq s - Kr + s' - Kr$ and therefore $\|b - b'\| \leq s + s'$, as wished.

(CF2) Let $B_r((x,u)) \xrightarrow{\Phi} B_s(b)$ and $d((x,u),(x',u')) + r' < r$. We shall show that $B_{r'}((x',u')) \xrightarrow{\Phi} B_s(b)$. We have that $B_r(x) \xrightarrow{\Psi} B_{s-Kr}(b)$ and $|x - x'| + r' < r$. Therefore, by (CF2) applied to the code Ψ we obtain that $B_{r'}(x') \xrightarrow{\Psi} B_{s-Kr}(b)$. Since $r' < r$, we know that $s - Kr < s - Kr'$ so by (CF3) with respect to Ψ it is straightforward to get $B_{r'}(x') \xrightarrow{\Psi} B_{s-Kr'}(b)$ and thus $B_{r'}((x',u')) \xrightarrow{\Phi} B_s(b)$.

(CF3) Let $B_r((x,u)) \xrightarrow{\Phi} B_s(b)$ and $\|b-b'\|+s < s'$. This implies that $B_r(x) \xrightarrow{\Psi} B_{s-Kr}(b)$ and that $\|b-b'\|+s-Kr < s'-Kr$, so (CF3) with respect to Ψ gives us that $B_r(x) \xrightarrow{\Psi} B_{s'-Kr}(b')$. Whence, $B_r((x,u)) \xrightarrow{\Psi} B_{s'}(b')$, which is what we needed.

Now, we show that y is totally defined on $[-\alpha,+\alpha]\times\widehat{E}$. Let $(z,v)\in[-\alpha,+\alpha]\times\widehat{E}$ and show that $(z,v)\in dom(y)$. For this, we fix $\varepsilon>0$ and show there is $B_r((x,u)) \xrightarrow{\Phi} B_s(b)$ such that $d((z,v),(x,u)) < r$ and $s<\varepsilon$. On the one hand, let $\hat{r}>0$ such that $\hat{r}<\dfrac{\varepsilon}{K}$, so $\varepsilon-K\hat{r}>0$. Then, since $z\in\widehat{A}$, we know there are $B_{\bar{r}}(\bar{x}) \xrightarrow{\Psi} B_{\bar{s}}(\bar{b})$ such that $|z-\bar{x}|<\bar{r}$ and $\bar{s}<\varepsilon-K\hat{r}$. We can choose $\bar{r}$ as small as we want, particularly $\bar{r}<\hat{r}$. Now, since $v\in E$, we know that $v=(v_i)_{i\in I}\subseteq E$. Let $i\in\mathbb{N}$ be such that $|z-\bar{x}|+2^{-i}<\bar{r}$. Therefore $|z-\bar{x}|+\|v-v_i\|<\bar{r}$. Now, let s be such that $\bar{s}=s-K\bar{r}$. Thus,

$$s=\bar{s}+K\bar{r}<\varepsilon-K\hat{r}+K\bar{r}<\varepsilon-K\bar{r}+K\bar{r}=\varepsilon.$$

This way, taking $x=\bar{x}$, $u=v_i$, $r=\bar{r}$, $b=\bar{b}$ and the aforementioned s, we have that $B_r(x) \xrightarrow{\Psi} B_{s-Kr}(b)$, so $B_r(x) \xrightarrow{\Phi} B_b(s)$ and of course $d((z,v),(x,u)) < r$ and $\varepsilon<s$.

Finally, let $(z,v)\in\widehat{A}\times\widehat{E}$ and $B_r((x,u)) \xrightarrow{\Phi} B_s(b)$ such that $d((z,v),(x,u))<r$. We show that $\|y(z,v)-b\|\le s$. On the one hand we have that $B_r(x) \xrightarrow{\Psi} B_{s-Kr}(b)$ and $|z-x|<r$, so we know that $\|\phi(z)-b\|\le s-Kr$. Notice that there is $\bar{u}\in E$ such that $\|v-\bar{u}\|\le r$ and such that $y(z,\bar{u})=\phi(z)$. Thus, $\|y(z,\bar{u})-b\|\le s-Kr$. Therefore,

$$\begin{aligned}\|y(z,v)-b\| &\le \|y(z,v)-y(z,\bar{u})\|+\|y(z,\bar{u})-b\|\\ &\le K\,\|v-\overline{u}\|+s-Kr\\ &\le Kr+s-Kr=s.\end{aligned}$$

This concludes the proof. □

Thanks to Theorem 4.13, we have that Problem (4.3) has a unique solution $y(t)$ defined on an interval I centered in 0. We will denote by (ω_-,ω_+) the *maximal open interval* of existence of y; namely ω_- and ω_+ are such that there are no solutions of (4.3) defined on an open interval which contains strictly (ω_-,ω_+).

For what will follow when analyzing the Deformation Lemma, it will be important that the solutions of an ODE of the style of (4.3) is globally defined for positive t, i.e., $\omega_+=+\infty$. With this in mind, we present the following results formalizing the ideas of [1].

Lemma 4.16. *The following is provable in* WKL_0*. If* $\omega_+ < +\infty$ *(respectively* $\omega_- > -\infty$*) then* $y(t)$ *has no limit points as* $t \nearrow \omega_+$ *(respectively* $t \searrow \omega_-$*).*

Proof. By contradiction, assume that there is $v \in \widehat{E}$ such that $v = \lim_{t \nearrow \omega_+} y(t)$; then, one could define the problem

$$\begin{cases} \beta'(t) = f(s, \beta(s)), \\ \beta(\omega_+) = v. \end{cases} \tag{4.8}$$

which would have a solution β defined in an open maximal interval $(\omega_+ - \varepsilon, \omega_+ + \varepsilon)$, with $\varepsilon > 0$. But with this, the function

$$\tilde{y}(t) = \begin{cases} y(t), & \text{if } t \in (\omega_-, \omega_+), \\ \beta(t), & \text{if } t \in (\omega_+, \omega_+ + \varepsilon) \end{cases} \tag{4.9}$$

would be a continuous function that solves (4.3) in $(\omega_-, \omega_+ + \varepsilon)$, contradicting the maximality of the interval (ω_-, ω_+). The same argument holds for ω_-. $\square$

Theorem 4.17. *The following is probable in* ACA_0*. Under the hypotheses of Picard's theorem and its notations we have that* $\omega_+ = +\infty$*.*

Proof. Arguing by contradiction, suppose that $\omega_+ < +\infty$; i.e., $\omega_+ \in \mathbb{R}$, so $\omega_+ = (t_n)_{n \in \mathbb{N}}$ is a quickly converging Cauchy sequence of rational numbers with $t_n \in (\omega_-, \omega_+)$. Since (4.3) holds for every $t_j, t_i \in (t_n)_{n \in N}$, we have that given $j \leq i$:

$$y(t_j) - y(t_i) = \int_{t_i}^{t_j} \frac{d}{dt} y(s)\, ds = \int_{t_i}^{t_j} f(s, y(s))\, ds,$$

and therefore,

$$\begin{aligned} \|y(t_j) - y(t_i)\| &= \left\| \int_{t_i}^{t_j} f(s, y(s))\, ds \right\| \\ &\leq \int_{t_i}^{t_j} \|f(s, y(s))\|\, ds \\ &\leq M|t_j - t_i| < M \cdot 2^{-j}. \end{aligned}$$

The last inequality implies that $(y(t_n))_{n \in \mathbb{N}}$ is a Cauchy sequence on $\widehat{E}$. Being in a complete separable metric space, by Theorem (3.17), we know that $(y(t_n))_{n \in \mathbb{N}}$ converges, i.e., there is $v \in \widehat{E}$ such that $\lim_{n \to +\infty} y(t_n) = v$; since $t_n \to \omega_+$, we have that $v = \lim_{t \nearrow \omega_+} y(t)$ so by the previous lemma we reach a contradiction. This way, we can conclude that $\omega_+ = +\infty$, so the solution of (4.3) can be obtained for all $t \geq 0$. $\square$

Recall that a function $F : \widehat{E} \to \widehat{E}$ is called locally Lipschitz if for every $u \in \widehat{E}$ there are $\delta > 0$ and $L > 0$, depending on u, such that

$$(\|x-u\| < \delta \wedge \|y-u\| < \delta) \Rightarrow (\|F(x)-F(y)\| \leq L\,\|x-y\|),$$

and that it is called uniformly bounded if there is $C > 0$ such that $\|F(x)\| \leq C$ para todo $x \in \widehat{E}$.

With this in mind, we can summarize the results of this section with the following theorem.

Theorem 4.18. *The following is provable in* ACA_0. *Let $\widehat{E}$ be a separable Banach space and $F : \widehat{E} \to \widehat{E}$ be a locally Lipschitz and uniformly bounded coded continuous function that has a modulus of uniform continuity. Then, there is a unique coded continuous function $y : [0,+\infty) \times \widehat{E} \to \widehat{E}$ such that*

$$\begin{cases} y'(t,u) = F(y(t,u)), \\ y(0,u) = u. \end{cases} \tag{4.10}$$

This theorem will be used in the proof of the Deformation Lemma, result that we formalize next.

4.2 The Deformation Lemma in Hilbert Spaces

The Deformation Lemma is a result that can be sated and proved (in the metalanguage) for Banach spaces. However, the construction of a quiet technical concept, the pseudo-gradient vector field, must be used to do so. In the last chapter we will comment on it but not pursue a formalization.

One way to avoid pseudo-gradient vector fields is to work within Hilbert spaces, where we have a gradient for free instead. It will require an extra hypothesis but it is used in a great amount of Analysis and PDEs literature for theory and applications (see for example [5] and [9]); in this sense we are not making unnatural assumptions.

We first recall some basic concepts of the formalized theory of Separable Hilbert spaces.

Definition 4.19 (Hilbert space)**.** *The following definition is made in* RCA_0. *A (code for a real) separable* Hilbert space $\widehat{H}$ *consists of a countable vector space H over $\mathbb{Q}$ together with a function $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$ satisfying for all $x, y, z \in H$ and $a, b \in \mathbb{Q}$:*

1. $\langle x,x\rangle \geq 0$,

2. $\langle x,y\rangle = \langle y,x\rangle$,

3. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$.

Every Hilbert space can be viewed as a Banach space with norm $\|x\| = \langle x, x\rangle^{1/2}$*. The triangle inequality and the Cauchy-Schwartz inequality follow from the axiomatic characterization of the inner product, and then the inequality*

$$\|\langle x, y\rangle - \langle z, w\rangle\| = \|\langle x, y - w\rangle + \langle x - z, w\rangle\| \leq \|x\|\,\|y - w\| + \|x - z\|\,\|w\|$$

shows that the inner product is continuous. We view $\widehat{H}$ *as the completion of* H *as usual, and extend the inner product to the whole space by defining* $\langle x, y\rangle = \lim_{n\to+\infty}\langle x_n, y_n\rangle$ *for* $x = (x_n)_{n\in\mathbb{N}}$ *and* $y = (y_n)_{n\in\mathbb{N}}$*; the inequality above can be used to find an explicit code for the inner product as a continuous function on* $\widehat{H}\times\widehat{H}$ *(See [4] for details).*

We continue with a very important result.

Theorem 4.20 (Fréchet-Riesz representation). *The following is provable in* ACA_0*. Let* $\widehat{H}$ *be a Hilbert space and let* F *be a bounded linear functional (in short* $F \in \widehat{H}\,'$*). Then there is* $y \in \widehat{H}$ *such that*

$$F(x) = \langle y, x\rangle, \qquad \forall\, x \in \widehat{H}.$$

Proof. This follows from (6) of Theorem 13.4 of [4]. □

It is straightforward to prove within the same system that the representative y from above is unique. Thus, assuming that the dual space $\widehat{H}\,'$ exists (as a separable Banach space within our theory), we can define the Riesz operator R:

$$\begin{aligned} R : \widehat{H}\,' &\to \widehat{H} \\ F &\mapsto R(F) = y. \end{aligned}$$

This allows us to define the following

Definition 4.21 (gradient). *The following definition is made in* ACA_0*. Let* $\widehat{H}$ *be a separable Hilbert space with separable dual* $\widehat{H}\,'$*. Let* $U \subseteq \widehat{H}$ *be open and* $R : \widehat{H}\,' \to \widehat{H}$ *be the Riesz operator. Let* $I : U \to \mathbb{R}$ *be a differentiable functional on* x*. The element* $R(I'(x)) \in \widehat{H}$ *is known as the gradient of* I *on* x *and is denoted by* $\nabla I(x)$*. Therefore,*

$$I'(x)h = \langle \nabla I(x), h\rangle, \qquad \forall\, h \in \widehat{H}.$$

In other words, if I *is Fréchet differentiable on* $\widehat{H}$*, each element* $x \in \widehat{H}$ *has linked to it its gradient that is also an element of* $\widehat{H}$*:*

$$\begin{aligned} \nabla I : H &\to H \\ x &\mapsto \nabla I(x). \end{aligned}$$

where clearly $\nabla I = R \circ I'$*.*

We can summarize all of this in order to avoid assumptions about dual spaces with a definition that encapsulates everything of what is needed.

Definition 4.22 (derivative of a functional on a Hilbert space). *The following definition is made in* RCA_0. *Let* $\widehat{H}$ *be a separable Hilbert space. Let* $U \subseteq \widehat{H}$ *be an open subset and* $I : U \to \mathbb{R}$ *be a functional. We say that* I *is differentiable at* $x \in U$ *if there is* $y \in \widehat{H}$ *such that*

$$I(x+h) - I(x) - \langle x, y\rangle = o(h), \text{ as } \|h\| \to 0.$$

We write $I'(x) = y$, *as* y *can be proven to be unique and call it the* derivative *of* I *at* x.

The following definition is important.

Definition 4.23. *The following definition is made in* RCA_0. *Let* $\widehat{H}$ *be a separable Hilbert space and* $I : \widehat{H} \to \mathbb{R}$ *be a functional. We write that* $I \in C^{1,1}(\widehat{H})$ *if* I *has a derivative at every point* $x \in \widehat{H}$ *and* $I' : \widehat{H} \to \widehat{H}$ *is a Lipschitz continuous function (i.e., a coded continuous functions that satisfies the Lipschitz inequality).*

We are ready to prove the version for Hilbert spaces of the Deformation Lemma

Lemma 4.24 (Deformation Lemma). *The following is provable in* ACA_0. *Let* $\widehat{H}$ *be a separable Hilbert space and let* $I \in C^{1,1}(\widehat{H})$ *such that* I *satisfies* (PS). *Suppose* $c \in \mathbb{R}$ *and* $\bar{\varepsilon} > 0$. *If* c *is not a critical value of* I, *then there exists* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\eta \in C([0,1] \times \widehat{H}, \widehat{H})$ *such that:*

(a) $\eta(1, I^{c+\varepsilon}) \subseteq I^{c-\varepsilon}$.

(b) $\eta(1, u) = u$ *if* $I(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}.]$

Proof. In the first place we consider the following claim:

$$\exists\, \hat{\varepsilon} > 0 \text{ and } \exists\, b > 0 \text{ such that } \forall\, u \in I^{c+\hat{\varepsilon}} \backslash I^{c-\hat{\varepsilon}}, \ \|I'(u)\| \geq b. \tag{4.11}$$

To prove it, we proceed by contradiction and suppose that

$$\forall\, \hat{\varepsilon} \geq 0 \text{ and } \forall\, b \geq 0, \ \exists\, u \in I^{c+\hat{\varepsilon}} \backslash I^{c-\hat{\varepsilon}} \text{ such that } \|I'(u)\| < b.$$

Since for every $k \in \mathbb{N}$, $\hat{\varepsilon}_k = b_k = \dfrac{1}{2^k} > 0$, then $\exists\, u_k \in I^{c+\hat{\varepsilon}_k} \backslash I^{c-\hat{\varepsilon}_k}$ such that

$$\|I'(u_k)\| < b_k. \tag{4.12}$$

Thus, for every $k \in \mathbb{N}$, we have that $c - \hat{\varepsilon}_k \leq I(u_k) \leq c + \hat{\varepsilon}_k$ and $\|I'(u_k)\| < b_k$. By how we chose $\hat{\varepsilon}_k = b_k$ we can conclude that,

$$I(u_k) \to c, \text{ and },$$
$$I'(u_k) \to 0.$$

This means that the sequence $(u_k)_k$ of elements of $\widehat{H}$ is such that $(I(u_k))_k$ is a bounded sequence in $\mathbb{R}$ and such that $I'(u_k) \to 0$ (in $\widehat{H}$). Therefore, by (PS), $(u_k)_k$ has a convergent subsequence. The limit of this subsequence is a critical point at the level c, which contradicts the hypothesis. Therefore, we conclude (4.11).

Let $\tilde{\varepsilon} \le \hat{\varepsilon}$; then $c+\tilde{\varepsilon} \le c+\hat{\varepsilon}$ and $c-\tilde{\varepsilon} \ge c-\hat{\varepsilon}$. Let $u \in I^{c+\tilde{\varepsilon}} \backslash I^{c-\tilde{\varepsilon}}$, i.e. $c-\tilde{\varepsilon} < I(u) \le c+\tilde{\varepsilon}$; this implies that $c-\hat{\varepsilon} < I(u) \le c+\hat{\varepsilon}$, or in other words, $u \in I^{c+\hat{\varepsilon}} \backslash I^{c-\hat{\varepsilon}}$. Thus, due to (4.11), we have that $\|I(u)\| \ge b$. Since $\tilde{\varepsilon}$ and u were chosen arbitrarily, we conclude that no matter how small is $\tilde{\varepsilon}$, it follows that

$$\left\|I'(u)\right\| \ge b \quad \forall u \in I^{c+\tilde{\varepsilon}} \backslash I^{c-\tilde{\varepsilon}}. \tag{4.13}$$

We want to select $\tilde{\varepsilon}$ in such a way that the following sets are separably closed:

$$A = \{u \in \widehat{H} : I(u) \le c-\tilde{\varepsilon}\} \cup \{u \in \widehat{H} : I(u) \ge c+\tilde{\varepsilon}\}$$

and

$$B = \{u \in \widehat{H} : c-\varepsilon \le I(u) \le c+\varepsilon\}.$$

for $\varepsilon \in (0, \tilde{\varepsilon})$.

We will see that it would be enough to guarantee that there is $\delta > 0$ such that for all $0 < k \le \delta$ we have that $c-k$ and $c+k$ are not critical values of I. By contradiction assume that for all $\delta > 0$, there exists $0 < k \le \delta$ such that $c-k$ or $c+k$ are critical values of I. This allows us to find a sequence of positive numbers $(k_n)_{n\in\mathbb{N}}$ such that $k_n \to 0$ and such that that $c-k_n$ (or $c+k_n$) are critical values of I. This means that there is a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of $\widehat{E}$ such that

$$\begin{aligned} I(u_n) &\to c, \text{ and ,} \\ I'(u_n) &\to 0. \end{aligned}$$

Once again by (PS), $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence. The limit of this subsequence is a critical point at the level c, i.e., c is critical value of I which contradicts the hypothesis. Now, we know that the described δ does exists.

We take

$$0 < \tilde{\varepsilon} < \min\left\{\bar{\varepsilon}, \delta, \frac{b^2}{4}, \frac{b}{4}\right\} \tag{4.14}$$

and let $\varepsilon \in (0, \tilde{\varepsilon})$ to define the sets:

$$A = \{u \in \widehat{H} : I(u) \le c-\tilde{\varepsilon}\} \cup \{u \in \widehat{H} : I(u) \ge c+\tilde{\varepsilon}\}$$

and

$$B = \{u \in \widehat{H} : c-\varepsilon \le I(u) \le c+\varepsilon\}.$$

We claim that A and B are separably closed. For A, consider the countable set

$$C = \{u \in H : I(u) \leq c - \tilde{\varepsilon}\} \cup \{u \in H : I(u) \geq c + \tilde{\varepsilon}\}.$$

We want to show that $A = \overline{C}$. Let $u \in A$. If $I(u) < c - \tilde{\varepsilon}$, then by density and continuity we can find $\bar{u} \in \overline{C}$ such that for all $q \in \mathbb{Q}$ we have that $\|u - \bar{u}\| < q$; so we are done. In the case that $I(u) = c - \tilde{\varepsilon}$, the fact that $I'(u) \neq 0$ allows us to find $x \in \widehat{E}$ such that $I(u + x) < c - \tilde{\varepsilon}$ and therefore find $\bar{u} \in C$ as close as we want to u. For B, we can proceed analogously since $\varepsilon < \tilde{\varepsilon}$.

Now, notice that the relation between ε and $\tilde{\varepsilon}$ determine that $A \cap B = \varnothing$.

Consider thus the function:

$$g(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)},$$

where $\rho_D(x) = \inf\{\|x - y\| : y \in D\}$, for $D \subseteq \widehat{H}$. Here we use Theorem 3.21 and thanks to ACA_0 we have ρ_A and ρ_B as coded continuous functions.

It is possible to prove that g is locally Lipschitz. Notice that $g = 0$ on A, $g = 1$ on B and $0 \leq g(x) \leq 1$ for all $x \in \widehat{H}$.

Also, consider the function

$$s : [0, +\infty) \to [0, 1]$$
$$s \mapsto h(s) = \begin{cases} 1, & \text{if } \ 0 \leq s \leq 1, \\ 1/s, & \text{if } \ s > 1. \end{cases}$$

which is locally Lipschitz because every constant function is and $s \mapsto 1/s$ also is on $(1, +\infty)$.

Finally, we define:

$$W(x) = -g(x)h(\|I'(x)\|)I'(x) \quad (x \in \widehat{H}).$$

This is a locally Lipschitz continuous function, whence by Theorem 4.18 the problem

$$\begin{cases} \eta'(t, u) = W(\eta(t, u)), \\ \eta(0, u) = u, \end{cases} \tag{4.15}$$

possesses a unique solution $\eta \in C([0, +\infty) \times \widehat{H}, \widehat{H})$.

Restricting the domain we have a function $\eta \in C([0, 1] \times \widehat{H}, \widehat{H})$ such that $\eta(0, u) = u$.

We want to prove property (a). Using the Chain Rule we have that:

$$\begin{aligned} \frac{dI(\eta(t, u))}{dt} &= \langle I'(\eta(t, u)), \eta'(t, u)\rangle \\ &= \langle I'(\eta(t, u)), -g(\eta(t, u))h(\|I'(\eta(t, u))\|)I'(\eta(t, u))\rangle \\ &= -g(\eta(t, u))h(\|I(\eta(t, u))\|)\,\|I'(\eta(t, u))\|^2 \leq 0. \end{aligned}$$

This implies that $I(\eta(t,u))$ is decreasing on t. Thus, $\eta(t,u) \in I^{c-\varepsilon}$ for all $u \in I^{c-\varepsilon}$ and for all $t \in [0,1]$. To show $\eta(1, I^{c+\varepsilon}) \subseteq I^{c-\varepsilon}$, let $v \in \eta(1, I^{c+\varepsilon})$. Thus, there is $u \in I^{c+\varepsilon}$ such that $\eta(1,u) = v$. In the first place, if there is $t \in [0,1]$ such that $u \in I^{c-\varepsilon}$, we are done; therefore, we assume that $\eta(t,u) \in I^{c+\varepsilon} \backslash I^{c-\varepsilon}$ for all $t \in [0,1]$; this way $g(\eta(t,u)) = 1$ and

$$\frac{dI(\eta(t,u))}{dt} = -h(\|I'(\eta(t,u))\|)\,\|I'(\eta(t,u)))\|^2.$$

If on the one hand, $t \in (0,1)$, $\|I'(\eta(t,u))\| \leq 1$, then $h(\|I'(\eta(t,u))\|) = 1$, and therefore

$$\frac{d}{dt} I(\eta(t,u)) = -\|I'(\eta(t,u))\|^2 \leq -b^2 \leq -\frac{b^2}{2} \leq -2\varepsilon \tag{4.16}$$

If on the other hand, $t \in (0,1)$, $\|I'(\eta(t,u))\| > 1$, then $h(\|I'(\eta(t,u))\|) = \dfrac{1}{\|I'(\eta(t,u))\|}$, whence,

$$\frac{d}{dt} I(\eta(t,u)) = -\|I'(\eta(t,u))\| < -\frac{b}{2} \leq -2\varepsilon. \tag{4.17}$$

Hence, for any $t \in (0,1)$ we have that

$$\frac{d}{dt} I(\eta(t,u)) \leq -2\varepsilon.$$

Due to the Fundamental Theorem of Calculus, we have that

$$I(\eta(1,u)) \leq I(u) - 2\varepsilon \leq c + \varepsilon - 2\varepsilon \leq c - \varepsilon,$$

so $v = \eta(1,u) \in I^{c-\varepsilon}$ as desired.

To prove (b), let $u \in \widehat{H}$ be such that $I(u) \notin [c-\bar{\varepsilon}, c+\bar{\varepsilon}]$. Since $\bar{\varepsilon} > \tilde{\varepsilon}$ then $u \in A$ and since $g = 0$ on A, we have that $W(u) = 0$ for all $u \in \widehat{H}$ such that $I(u) \notin [c-\bar{\varepsilon}, c+\bar{\varepsilon}]$; thus, for those u, $\eta(t,u) = u$ is the unique solution to (4.15) obtaining what was needed. □

Once we have developed the formalized proof of the Deformation Lemma within ACA_0, we are ready to use it to formalize the MPT in the next chapter.

Chapter 5

Formalizing the Mountain Pass Theorem

In this chapter we use everything that we have reviewed so far to show that the the proof of the MPT can be formalized within ACA_0. We will first proceed by arguing that ACA_0 is strong enough to prove the existence of

$$c = \inf_{g\in\Gamma} \max_{u\in g([0,1])} I(u).$$

where

$$\Gamma = \{g \in C([\,0,1\,],E) : g(0) = 0,\ g(1) = v\}.$$

5.1 The existence of c in ACA_0

In the first place we need a formalized path to code as continuous the function

$$\begin{aligned} F : \Gamma &\to \mathbb{R}. \\ g &\mapsto \max_{t\in[0,1]} I(g(t)). \end{aligned}$$

We recall the non-formalized proof using the $\varepsilon - \delta$ definition in the metalanguage before going into the formalized argument.

Proof of the $\varepsilon - \delta$ continuity of F. Let $f \in \Gamma$ and let $\varepsilon > 0$ be fixed. We have to show that there is $\delta > 0$ such that for all $g \in \Gamma$,

$$d(g, f) < \delta \Rightarrow |F(g) - F(f)| < \varepsilon.$$

Since $[0,1]$ is compact and f is a continuous function on $[0,1]$, then the direct image $f([0,1])$ is also compact. Thus, I is uniformly continuous on $f([0,1]) \subseteq \widehat{E}$.

Therefore, there is $\delta_0 > 0$ such that for every $y \in f([0,1])$ and all $x \in \widehat{E}$ such that $\|x - y\| < \delta$, we have that
$$|I(x) - I(y)| < \varepsilon.$$
Now, let $\delta = \delta_0$ and assume that $d(g, f) < \delta$, i.e.,
$$\max_{t\in[0,1]} \|g(t) - f(t)\| < \delta.$$
Let $\bar{t} = \arg\max_{t\in[0,1]} I(g(t))$; then
$$F(g) - F(f) = I(g(\bar{t})) - \max_{t\in[0,1]} I(f(t)) \leq I(g(\bar{t})) - I(f(\bar{t})).$$
Now, we also have that
$$\left\|g(\bar{t}) - f(\bar{t})\right\| \leq d(g, f) < \delta,$$
and since $f(\bar{t}) \in f([0,1])$ we can conclude that
$$|I(g(\bar{t})) - I(f(\bar{t}))| < \varepsilon,$$
so
$$F(g) - F(f) < \varepsilon.$$
Reversing the roles of g and f we obtain $|F(g) - F(f)| < \varepsilon$. □

To proceed with the formalized proof that such an F is coded continuous, we would impose a Lipschitz condition. This would made the construction of the code more approachable.

Theorem 5.1. *The following is provable in* WKL_0. *Let $\widehat{E}$ be a Banach space and let $I : \widehat{E} \to \mathbb{R}$ be a coded continuous function. Let*
$$\widehat{A} = \{g \in C([0,1], \widehat{E}) : g(0) = 0,\, g(1) = v\}.$$
If I is Lipschitz with constant $L > 0$, then the following is a coded continuous function:
$$\begin{aligned} F : \widehat{A} &\to \mathbb{R}. \\ g &\mapsto F(g) = \max_{t\in[0,1]} I(g(t)). \end{aligned}$$

Proof. Consider the Banach space $\widehat{B} = C([0,1], \mathbb{R})$, where B is the dense set consisting in the polygonal functions with rational breakpoints. Let
$$\begin{aligned} m : \widehat{B} &\to \mathbb{R}. \\ f &\mapsto \max_{t\in[0,1]} f(t), \end{aligned}$$

and

$$\begin{aligned} h : \widehat{A} &\to \widehat{B}. \\ g &\mapsto I \circ g. \end{aligned}$$

Clearly $F = m \circ h$, so if we prove that m and h are coded continuous functions we are done because of Theorem 3.23.

Let us begin with m. We denote by d the metric on B:

$$d(a, a') = \max_{t \in [0,1]} |a(t) - a'(t)|.$$

We want to code m by $\Phi \subseteq B \times \mathbb{Q}_{>0} \times \mathbb{Q} \times \mathbb{Q}_{>0}$, so Let $B_r(a) \xrightarrow{\Phi} B_s(b)$ if and only if $(a, r, b, s) \in B \times \mathbb{Q}_{>0} \times \mathbb{Q} \times \mathbb{Q}_{>0}$ and $|\max a(t) - b| + r < s$. Here $\max a(t) = \max_{t \in [0,1]} a(t)$ is just the maximum of a finite list of rationals (the peaks of the polygonal function a) and thus a computable quantity.

We first proof that this defines a code for a continuous partial function:

(CF1) Let $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $B_r(a) \xrightarrow{\Phi} B_{s'}(b')$. We must show that $|b - b'| \leq s + s'$. Since $|\max a(t) - b| + r < s$ and $|\max a(t) - b'| + r < s'$, then:

$$\begin{aligned} |b - b'| &= |b - \max a(t) + \max a(t) - b'| \\ &\leq |b - \max a(t)| + r + |\max a(t) - b'| + r \\ &\leq s + s'. \end{aligned}$$

(CF2) Let $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $d(a', a) + r' < r$. We must show that $B_{r'}(a') \xrightarrow{\Phi} B_s(b)$, i.e., $|\max a'(t) - b| + r' < s$. Since $|\max a(t) - b| + r < s$, we have:

$$\begin{aligned} |\max a'(t) - b| + r' &= |\max a'(t) - \max a(t) + \max a(t) - b| + r' \\ &\leq |\max a(t) - \max a'(t)| + |\max a(t) - b| + r' \\ &\leq \max |a(t) - a'(t)| + |\max a(t) - b| + r' \\ &= d(a, a') + r' + |\max a(t) - b| \\ &\leq r + |\max a(t) - b| \\ &\leq s. \end{aligned}$$

(CF3) Let $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $|b - b'| + s < s'$. We must show that then $B_r(a) \xrightarrow{\Phi} B_{s'}(b')$, i.e., i.e., $|\max a(t) - b| + r < s'$. Since $|\max a(t) - b| + r < s$, we have:

$$\begin{aligned} |\max a(t) - b'| + r &= |\max a'(t) - b + b - b'| + r' \\ &\leq |\max a(t) - b| + r + |b - b'| \\ &\leq s + |b - b'| \\ &\leq s'. \end{aligned}$$

Now, we show that m is totally defined on $\widehat{B}$. Let $x \in \widehat{B}$ and show that $x \in dom(m)$. For this, we fix $\varepsilon > 0$ and show there is $B_r(a) \xrightarrow{\Phi} B_s(b)$ such that $d(x,a) < a$ and $s < \varepsilon$. Since $x \in \widehat{B}$, we know that $x = (x_i)_{i\in\mathbb{N}}$ with $x_i \in B$ for all $i \in \mathbb{N}$. Take $n \in \mathbb{N}$ such that $2^{-n+2} < \varepsilon$. Now, choose $a = x_n$, $b = \max a(t)$, $s = 2^{-n+2}$ and $r = 2^{-n+1}$. Thus, is obvious that $(a,r,b,s) \in B \times \mathbb{Q}_{>0} \times \mathbb{Q} \times \mathbb{Q}_{>0}$ and that $|\max a(t) - b| + r = r < s$ so $B_r(a) \xrightarrow{\Phi} B_s(b)$. Moreover, $d(x,a) \leq 2^{-n} < r$.

Finally, we have to show that the unique point $y \in \mathbb{R}$ such that $|y - b| \leq s$ for all $B_r(a) \xrightarrow{\Phi} B_s(b)$ with $d(x,a) < r$ is precisely $y = \max_{t\in[0,1]} x(t)$. Let $x \in \widehat{B}$ and $B_r(a) \xrightarrow{\Phi} B_s(b)$ such that $d(x,a) < r$. We want to show that $|\max_{t\in[0,1]} x(t) - b| \leq s$. Since $|\max a(t) - b| + r < s$, we have that:

$$\begin{aligned}\left|\max_{t\in[0,1]} x(t) - b\right| &\leq \left|\max_{t\in[0,1]} x(t) - \max a(t)\right| + |\max a(t) - b| \\ &\leq \max_{t\in[0,1]} |x(t) - a(t)| + |\max a(t) - b|.\end{aligned}$$

If we show that $d(x,a) = \lim_{i\to+\infty} \max_{t\in[0,1]} |x_i(t) - a(t)|$ is indeed equal to $\max_{t\in[0,1]} |x(t) - a(t)|$, we are done because we would have:

$$\begin{aligned}\left|\max_{t\in[0,1]} x(t) - b\right| &\leq d(x,a) + |\max a(t) - b| \\ &< r + |\max a(t) - b| < s.\end{aligned}$$

To show that

$$\lim_{i\to+\infty} \max_{t\in[0,1]} |x_i(t) - a(t)| = \max_{t\in[0,1]} |x(t) - a(t)|,$$

let $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that for all $i \geq N$,

$$\left|\max_{t\in[0,1]} |x_i(t) - a(t)| - \max_{t\in[0,1]} |x(t) - a(t)|\right| < \varepsilon.$$

Taking N such that $2^{-N} < \varepsilon$ suffices. Thus, we can conclude that $|\max_{t\in[0,1]} x(t) - b| \leq s$, as wished.

Next, we prove that h is a coded continuous function. As we already stated, A is the dense set consisting in the piece-wise linear continuous functions with rational breakpoints with fixed images for 0 and 1. Let Ψ be the code of the given coded continuous function I and let d_A denote the metric on A:

$$d_A(a,a') = \max_{t\in[0,1]} \left\|a(t) - a'(t)\right\|.$$

We want to code h by $\Phi \subseteq A \times \mathbb{Q}_{>0} \times B \times \mathbb{Q}_{>0}$, so let $B_r(a) \xrightarrow{\Phi} B_s(b)$ if and only if $(a,r,b,s) \in A \times \mathbb{Q}_{>0} \times B \times \mathbb{Q}_{>0}$ and $\forall t \in [0,1]$, $|b(t) - I(a(t))| + Lr < s$.

Let us see that this defines a code for a continuous partial function:

(CF1) Let $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $B_r(a) \xrightarrow{\Phi} B_{s'}(b')$. We prove that $d(b,b') \leq s+s'$. Since for all $t \in [0,1]$, $|b(t) - I(a(t))| + Lr < s$ and $|b'(t) - I(a(t))| + Lr < s'$, it follows straightforwardly due to the triangle inequality that for all $t \in [0,1]$, $|b(t) - b'(t)| \leq s+s'$ and therefore that $d(b,b') \leq s+s'$.

(CF2) Let $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $d_A(a',a) + r' < r$. We must show that $B_{r'}(a') \xrightarrow{\Phi} B_s(b)$. Let $t \in [0,1]$ and show $|b(t) - I(a'(t))| + Lr' < s$. On the one hand, since $d_A(a',a) + r' < r$, then

$$\|a(t) - a'(t)\| + r' < r.$$

On the other hand, using the latter and due to the Lipschitz condition, we get

$$\begin{aligned}|I(a(t)) - I(a'(t))| + Lr' &\leq L\,\|a(t) - a'(t)\| + Lr' \\ &= L(\|a(t) - a'(t)\| + r') \\ &< Lr.\end{aligned}$$

Therefore we have that $|I(a(t)) - I(a'(t))| + Lr' < Lr$. Then, as $|b(t) - I(a(t))| + Lr < s$, we get:

$$\begin{aligned}|b(t) - I(a'(t))| + Lr' &\leq |b(t) - I(a(t))| + |I(a(t)) - I(a'(t))| + Lr' \\ &\leq |b(t) - I(a(t))| + Lr < s.\end{aligned}$$

(CF3) Let $B_r(a) \xrightarrow{\Phi} B_s(b)$ and $d(b,b') + s < s'$. We show that then $B_r(a) \xrightarrow{\Phi} B_{s'}(b')$. We have that for all $t \in [0,1]$, $|b(t) - I(a(t))| + Lr < s$ and $|b(t) - b'(t)| + s < s'$. Thus, given $t \in [0,1]$:

$$\begin{aligned}|b'(t) - I(a(t))| + Lr &\leq |b'(t) - b(t)| + |b(t) - I(a(t))| + Lr. \\ &\leq |b(t) - b'(t)| + s \\ &\leq s'.\end{aligned}$$

Now, we show that h is totally defined on $\widehat{A}$. Let $x \in \widehat{A}$ and show that $x \in dom(h)$. For this, we fix $\varepsilon > 0$ and show there is $B_r(a) \xrightarrow{\Phi} B_s(b)$ such that $d(x,a) < r$ and $s < \varepsilon$. Since $x \in \widehat{A}$, we know that $x = (x_i)_{i \in \mathbb{N}}$ with $x_i \in A$ for all $i \in \mathbb{N}$. Take $n \in \mathbb{N}$ such that $2^{-n+1}(L+1) < \varepsilon$. Now, choose $a = x_n$. Since $I \circ a \in C([0,1],\mathbb{R})$, then by Theorem 3.32, there is a polygonal function b such that $|b(t) - (I \circ a)(t)| < 2^{-n+1}$ for all $t \in [0,1]$. Choosing $r = 2^{-n+1}$ and $s = 2^{-n+1}(L+1)$, is obvious that $(a,r,b,s) \in A \times \mathbb{Q}_{>0} \times B \times \mathbb{Q}_{>0}$ and that for all $t \in [0,1]$, $|b(t) - I(a(t))| + Lr < s$. Evidently, $d(x,a) \leq 2^{-n} < r$.

Finally, let $x \in \widehat{A}$ and $B_r(a) \xrightarrow{\Phi} B_s(b)$ such that $d_A(x,a) < r$. We show that $d(I \circ x, b) \leq s$. The fact that $d_A(x,a) < r$ implies that for all $t \in [0,1]$, we have that $\|x(t) - a(t)\| < r$.

Whence, since for all $t \in [0,1]$, $|b(t) - I(a(t))| + Lr < s$, we get that for all $t \in [0,1]$:

$$\begin{aligned} |(I \circ x)(t) - b(t)| &\leq |(I \circ x)(t) - I(a(t))| + |I(a(t)) - b(t)| \\ &= |I(x(t)) - I(a(t))| + |I(a(t)) - b(t)| \\ &\leq L\,\|x(t) - a(t)\| + |(I \circ a)(t) - b(t)| \\ &\leq Lr + |(I \circ a)(t) - b(t)| < s. \end{aligned}$$

Whence we get that $d(I \circ x, b) \leq s$, as wished. □

Now we are ready to prove the main result of this section, namely, the first part of the MPT.

Theorem 5.2. *(existence of c) The following is provable in* ACA_0*. Let* $\widehat{E}$ *be a Banach space and let* $I : \widehat{E} \to \mathbb{R}$ *be a Lipschitz coded continuous function. Suppose that* $I(0) = 0$ *and that there exists real numbers* $\rho, \alpha > 0$ *such that:*

1. *If* $\|u\| = \rho$ *then* $I(u) \geq \alpha$*,*
2. *There is* $v \in \widehat{E}$ *such that* $\|v\| > \rho$ *and* $I(v) \leq 0$*.*

Then, the following exist:

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0,\ g(1) = v\}.$$

Proof. Let $\Gamma = \widehat{A}$ as previously and let

$$\begin{aligned} F : \widehat{A} &\to \mathbb{R}. \\ g &\mapsto F(g) = \max_{t \in [0,1]} I(g(t)). \end{aligned}$$

This function is well defined because of Theorem 3.30 and is a coded continuous function thanks to Theorem 5.1. For $g \in \widehat{A}$, we have that $\|g(0)\| = \|0\| = 0$ and $\|g(1)\| = \|v\| > \rho$. The fact that g and the norm $\|\cdot\|$ are both coded continuous gives us due to Theorem 3.29 (the intermediate value theorem in RCA_0) the existence of $t_g \in [0,1]$ such that $\|g(t_g)\| = \rho$. Thanks to the condition 1., we have $(I(g(t_g)) \geq \alpha$ and therefore:

$$\max_{t \in [0,1]} I(g(t)) \geq \alpha.$$

Thus, we have that $F(g) \geq \alpha$ for all $g \in \widehat{A}$. Now, let $(a_i)_{i\in\mathbb{N}}$ be an enumeration of A and consider the sequence of real numbers $(r_n)_{n\in\mathbb{N}}$ where $r_n = F(a_n)$ for all $n \in \mathbb{N}$. Thus $(r_n)_{n\in\mathbb{N}}$ is bounded from bellow and by Theorem 3.31, we have that $\inf_{n\in\mathbb{N}} r_n$ exists. We claim that $\inf_{g\in\Gamma} F(g) = \inf_{n\in\mathbb{N}} r_n$. On the one hand, we show that $\inf_{n\in\mathbb{N}} r_n$ is a lower bound of $\{F(g) : g \in \Gamma\}$. Let $g \in \Gamma$. Then $g = (g_n)_{n\in\mathbb{N}}$ with $g_n \in A$ and $d(g, g_n) \leq 2^{-n}$. This implies that $g_n \to g$. Since F is coded continuous, Theorem 3.24 says that $F(g_n) \to F(g)$. This and the fact that $F(g_n) \geq \inf_{n\in\mathbb{N}} r_n$ for all $n \in \mathbb{N}$ gives us that $F(g) \geq \inf_{n\in\mathbb{N}} r_n$. On the other hand, we show that $\inf_{n\in\mathbb{N}} r_n$ is the greatest lower bound of $\{F(g) : g \in \Gamma\}$. In case there is another lower bound of $\{F(g) : g \in \Gamma\}$, say d, it is straightforward to see that $d \leq \inf_{n\in\mathbb{N}} r_n$. Thus, we have shown that

$$\inf_{g\in\Gamma} F(g) = \inf_{n\in\mathbb{N}} r_n.$$

Notice that c as defined in the statement of the theorem is just another form to write $\inf_{g\in\Gamma} F(g)$. Thus, the proof is complete. □

5.2 The formalized Mountain Pass Theorem in Hilbert Spaces

We are ready to present the formalized proof of the Mountain Pass Theorem.

Theorem 5.3 (Formalized Mountain Pass Theorem). *The following is provable in* ACA_0. *Let $\widehat{H}$ be a separable Hilbert space and let $I \in C^{1,1}(\widehat{H}, \mathbb{R})$ be a Lipschitz continuously differentiable functional that satisfies (PS). Suppose that $I(0) = 0$ and that there exist $\rho, \alpha > 0$ such that:*

1. *If $\|u\| = \rho$ then $I(u) \geq \alpha$,*
2. *There is $v \in E$ such that $\|v\| > \rho$ and $I(v) \leq 0$.*

Then I has a critical value $c \geq \alpha$. Moreover, c can be characterized as:

$$c = \inf_{g\in\Gamma} \max_{u\in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0,1], \widehat{H}) : g(0) = 0,\ g(1) = v\}.$$

Proof. In the first place, thanks to Theorem 5.2 we know that ACA_0 can prove that $c = \inf_{g\in\Gamma} \max_{u\in g([0,1])} I(u)$ indeed exists, i.e., $c \in \mathbb{R}$.

Now, towards proving that c is a critical value of I, we will use our formalized Deformation Lemma (Lemma 4.24).

By contradiction, suppose that c is not a critical value of I. By Lemma 4.24 with $\bar{\varepsilon} = \alpha/2$, we get that there exists $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times \widehat{H}, \widehat{H})$ as in the aforementioned result. By the characterization of the greatest lower bound, we can find $g \in \Gamma$ such that

$$F(g) < c + \varepsilon. \tag{5.1}$$

Consider the notation $\eta_t(x) = \eta(t, x)$ and define

$$\begin{aligned} \tilde{g} : [0,1] &\to E \\ t &\mapsto \eta_1(g(t)). \end{aligned}$$

We want to show that $\tilde{g} \in \Gamma$. Due to the continuity of η with respect to the second variable, since g is also continuous we can state that $\tilde{g}$ is continuous too. Notice that since $\bar{\varepsilon} = \alpha/2$, then $c - \bar{\varepsilon} > 0$. Therefore:

$$I(0) = 0 < c - \bar{\varepsilon} \text{ and } I(v) \le 0 < c - \bar{\varepsilon}.$$

Thus, by (b) of the deformation lemma, we have:

$$\tilde{g}(0) = \eta_1(g(0)) = \eta_1(0) = 0$$

and

$$\tilde{g}(1) = \eta_t(g(1)) = \eta_1(v) = v.$$

Thus, $\tilde{g} \in \Gamma$ and therefore,

$$c \le F(\tilde{g}). \tag{5.2}$$

By (5.1), $I(g(t)) \le c + \varepsilon$ for all $t \in [0,1]$. Thus, $g(t) \in I^{c+\varepsilon}$ for all $t \in [0,1]$ and therefore $\eta_1(g(t)) \in \eta_1(I^{c+\varepsilon})$ for all $t \in [0,1]$. By (a) of the deformation lemma, we have that $\eta_1(g(t)) \in I^{c-\varepsilon}$ for all $t \in [0,1]$, or in other words $\tilde{g}(t) \in I^{c-\varepsilon}$ for all $t \in [0,1]$. This of course means that $I(\tilde{g}(t)) \le c - \varepsilon$ for all $t \in [0,1]$; hence:

$$F(\tilde{g}) \le c - \varepsilon < c. \tag{5.3}$$

This contradicts (5.2), so we are done. □

We have achieved a complete formalization of a version the Mountain Pass Theorem in Hilbert spaces within the formal subsystem of second order arithmetc ACA_0.

Chapter 6

Conclusions and further research

6.1 Conclusions

We have presented an step by step formalized proof of the MPT in Hilbert spaces. Our main result is that ACA_0 proves the MPT in Hilbert spaces but we have achieved some partial results quite remarkable on their own to be summarized in what follows:

- WKL_0 gives us access to $C([0,1],\widehat{E})$, a space isomorphic to the space of uniformly continuous functions $F:[0,1]\to\widehat{E}$ that have moduli of uniform continuity and that the dense set is given by piece-wise linear continuous functions $f:[0,1]\to\widehat{E}$ with rational breakpoints, each represented by finitely many pairs $\langle x, f(x)\rangle \in \mathbb{Q}\times E$. This is the content of theorems 3.30 and 3.32.

- ACA_0 proves the existence of solutions of initial value problems defined in separable Banach spaces; the solutions being defined for every positive real and being continuous with respect to the independent variable and the initial values. This is the content of Theorem 4.18.

- ACA_0 proves the Deformation Lemma in Hilbert spaces. This is the content of Lemma 4.24.

- ACA_0 proves a version of the Mountain Pass Theorem in Hilbert spaces. This is the content of Theorem 5.3 where Theorem 5.2 and Lemma 4.24 are the basic ingredientes.

To close this chapter and monograph, we give some insight on the possible formalization of a more general result: the MPT in Banach spaces.

6.2 Further research: towards a formalization of the MPT in Banach spaces

The proof of the Deformation Lemma in Banach spaces relies on a tool called pseudo-gradient vector field which allows us to define a initial value problem in the fashion of problem (4.15) to obtain the deformation without using the Hilbert space structure.

To shed some light in a possible formalization, we describe in what follows some highlights in the path towards it that shall be considered.

6.2.1 The Pseudo-gradiente vector field

First, we introduce two important definitions.

Definition 6.1 (Pseudo-gradient vector). *Let $\widehat{E}$ be a separable Banach space. $U \subseteq \widehat{E}$ and let $I : U \to \mathbb{R}$ be C^1-functional; A vector $v \in \widehat{E}$ is called a pseudo-gradiente vector (in short p.g.) for I on $u \in U$ if,*

(i) $\|v\| \leq 2\,\|I'(u)\|_{\widehat{E}'}$ *and*

(ii) $I'(u)v \geq \|I'(u)\|^2_{\widehat{E}'}$.

Here we avoid the parenthesis in the application of $I'(u)$ on v and write $I'(u)v$ instead of $I'(u)(v)$.

Remark 6.2. *In general, a pseudo-gradient vector is not unique. In fact, any convex combination of any finite set of pseudo-gradient vectors is also a pseudo-gradient vector.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of $n \in \mathbb{N}$ pseudo-gradient vectors for I on $u \in U$ and let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of non-negative real numbers such that $\sum\limits_{i=1}^{n} \lambda_i = 1$. We want to show that $\sum_{i=1}^{n} \lambda_i v_i$ is a pseudo-gradient vector field for I on $u \in U$.

On the one hand, to show (i) consider:

$$\begin{aligned}
\left\|\sum_{i=1}^{n} \lambda_i v_i\right\| &\leq \sum_{i=1}^{n} \lambda_i \|v_i\| \\
&\leq \sum_{i=1}^{n} 2\lambda_i \left\|I'(u)\right\|_{\widehat{E}'} \\
&= 2\left\|I'(u)\right\|_{\widehat{E}'} \sum_{i=1}^{n} \lambda_i \\
&= 2\left\|I'(u)\right\|_{\widehat{E}'};
\end{aligned}$$

and on the other hand to verify (ii), we have:

$$\begin{aligned} I'(u)\left(\sum_{i=1}^{n}\lambda_i v_i\right) &= \sum_{i=1}^{n}\lambda_i I'(u)v_i \\ &\geq \sum_{i=1}^{n}\lambda_i \left\|I'(u)\right\|_{\widehat{E}'}^2. \\ &= \left\|I'(u)\right\|_{\widehat{E}'}^2 \sum_{i=1}^{n}\lambda_i \\ &= \left\|I'(u)\right\|_{\widehat{E}'}^2. \end{aligned}$$

□

Definition 6.3 (Pseudo-gradient vector field). *Let $I : \widehat{E} \to \mathbb{R}$ be C^1 and $\tilde{E} = \{u \in \widehat{E} : I'(u) \neq 0\}$. A map $V : \tilde{E} \to \widehat{E}$ is called a pseudo-gradient vector field on $\tilde{E}$ if it is locally Lipschitz and if $V(x)$ is a p.g. vector for I on x, for every $x \in \tilde{E}$.*

The concept of pseudo-gradient vector field is central in the proof of the Deformation Lemma for Banach spaces. But the existence of such a field needs of the following property: paracompactness of separable metric spaces. Fortunately it has been formalized.

Theorem 6.4 (Theorem II.7.2 of [19], paracompactness). *The following is provable in* RCA_0. *Let X a separable metric space. Given an open covering $\langle U_n : n \in \mathbb{N}\rangle$, we can effectively find an open covering $\langle V_n : n \in n \in N\rangle$ such that $V_n \subseteq U_n$ for all n, and $\langle V_n : n \in \mathbb{N}\rangle$ is locally finite, i.e., for all $x \in X$ there exists an open set W such that $x \in W$ and $W \cap V_n = \varnothing$ for all but finitely many n.*

Finally, we can state the non-formalized version of the main result of this section. In the metalanguage, to prove this theorem it is necessary to construct the pseudo-gradient vector field using distances between points and closed sets which are presumably not separably closed. Thus, a formalized version of the theorem as it stands seems to be provable in Π^1_1-CA_0 in principle. We present the non-formalized proof to witness the aforementioned difficulties.

Theorem 6.5 (Existence of pseudo-gradient vector fields). *For every functional $I \in C^1(\widehat{E})$, there is a pseudo-gradient vector field over $\tilde{E}$.*

Proof. Let $u \in \tilde{E}$, thus $I'(u) \neq 0$, whence $\|I'(u)\|_{\widehat{E}'} > 0$. Recall that the norm in $\widehat{E}'$ can be characterized in such a way that we can write

$$\left\|I'(u)\right\|_{\widehat{E}'} = \sup_{\substack{v\in\widehat{E} \\ \|v\|=1}} |I'(u)v|.$$

The characterization of the least upper bound of a set of real numbers A expresses that for all $\varepsilon > 0$ there is $\tilde{w} \in A$ such that $\sup A - \varepsilon < \tilde{w}$. Then, since $\frac{1}{6}\left\|I'(u)\right\|_{E'} > 0$ we have that there is $\tilde{w} \in \widehat{E}$ with $\|\tilde{w}\| = 1$ such that

$$\left\|I'(u)\right\|_{\widehat{E}'} - \frac{1}{6}\left\|I'(u)\right\|_{\widehat{E}'} < |I'(u)\tilde{w}|,$$

i.e.,

$$\frac{5}{6}\left\|I'(u)\right\|_{\widehat{E}'} < |I'(u)\tilde{w}|$$

In the case that $I'(u)\tilde{w} > 0$, then we can state that there is $w \in \widehat{E}$ with $\|w\| = 1$ such that

$$\frac{5}{6}\left\|I'(u)\right\|_{\widehat{E}'} < I'(u)w,$$

taking $w = \tilde{w}$.
In the case that $I'(u)\tilde{w} < 0$, then

$$\frac{5}{6}\left\|I'(u)\right\|_{\widehat{E}'} < -I'(u)\tilde{w} = I'(u)(-\tilde{w}).$$

Since $\|-\tilde{w}\| = 1$, we can affirm that there is $w \in \widehat{E}$ con$\|w\| = 1$ such that

$$\frac{5}{6}\left\|I'(u)\right\|_{\widehat{E}'} < I'(u)w,$$

taking $w = -\tilde{w}$.
The case $I'(u)\tilde{w} = 0$ leads to a contradiction with the positive behavior of $\|I'(u)\|_{\widehat{E}'}$. This way, we have shown that $\forall\, u \in \tilde{E},\ \exists\, w \in \widehat{E}$ such that $\|w\| = 1$ and $I'(u)w > \frac{5}{6}\|I'(u)\|_{\widehat{E}'}$. Hence, $v = \frac{6}{5}\|I'(u)\|_{\widehat{E}'} \cdot w$ is a pseudo-gradient vector for I on $\tilde{E}$. In fact, (i) holds:

$$\begin{aligned}\|v\| &= \left\|\frac{6}{5}\left\|I'(u)\right\|_{\widehat{E}'} \cdot w\right\| \\ &= \frac{6}{5}\left\|I'(u)\right\|_{\widehat{E}'}\|w\| \\ &\leq 2\left\|I'(u)\right\|_{\widehat{E}'};\end{aligned}$$

and also (ii):

$$\begin{aligned}I'(u)v &= I'(u)\left(\frac{6}{5}\left\|I'(u)\right\|_{\widehat{E}'} \cdot w\right) \\ &= \frac{6}{5}\left\|I'(u)\right\|_{\widehat{E}'} I'(u)w \\ &> \frac{6}{5}\left\|I'(u)\right\|_{\widehat{E}'} \frac{5}{6}\left\|I'(u)\right\|_{\widehat{E}'} \\ &= \left\|I'(u)\right\|^2_{\widehat{E}'}.\end{aligned}$$

Due to the continuity of I' there is a neighborhood N_u of u, such that

$$\forall z \in N_u, \ \ \|v\| \leq 2 \left\| I'(z) \right\|_{E'} \text{ and } I'(z)v \geq \left\| I'(z) \right\|_{E'}^2,$$

i.e., v is a pseudo-gradient vector for I on every $z \in N_u$. Let the family $\{N_i\}_{i \in J} = \{N_{u_i} \mid u_i \in \tilde{E}\}$. This is an open covering of $\tilde{E}$ and since $\tilde{E}$ is a metric space, it is is paracompact and we can guarantee the existence of a locally finite subcovering $\{M_i\}_{i \in J}$. We define for each $i \in J$:

$$\begin{aligned} \rho_i &: E \to \mathbb{R} \\ &x \mapsto \rho_i(x) = dist(x, \widehat{E} \backslash M_i), \end{aligned} \tag{6.1}$$

which can be proven to be Lipschitz continuous. By how the distance between a point and a set is defined, $\rho_i(x) = 0$ if $x \notin M_i$. Now, consider the following function

$$\begin{aligned} \beta_i &: \widehat{E} \to \mathbb{R} \\ &x \mapsto \beta_i(x) = \frac{\rho_i(x)}{\sum_j \rho_j(x)}, \end{aligned}$$

Since each $x \in \widehat{E}$ is only in a finite number of the subsets $\{M_i\}_{i \in I}$ we could only compute finitely many finite distances ρ_i and therefore the sum in the denominator is finite and positive. On the other hand, each of the subsets M_i is in some subset $\{N_i\}_{i \in J} = \{N_{u_i} \mid u_i \in \tilde{E}\}$ and then we can find $z_i = \frac{6}{5} \|I'(u_i)\|_{E'} \cdot w_i$ a pseudo-gradient vector for I on each M_i. Finally, we define $V(x) = \sum_i z_i \beta_i(x)$. Since for each i, $0 \leq \beta_i \leq 1$ and $\sum_i \beta_i(x) = 1$, then for all $x \in \tilde{E}$, $V(x)$ is a convex combination of pseudo-gradient vectors for I on x, and therefore a pseudo-gradient vector for I on x itself. To show that V is locally Lipschitz continuous, consider that ρ_i is Lipschitz continuous so the sum of all of them is too; thus the quotient β_i es locally Lipschitz. Then, by definition, for all $x_0 \in \widehat{E}$ there are $\delta_i > 0$ and $L_i > 0$ (depending on x_0) such that for all $x \in B_{\delta_i}(x_0)$ we have that

$$|\beta_i(x) - \beta_i(x_0)| \leq L_i \|x - x_0\|, \quad \forall i \in J.$$

Recall that that the following is finite: set of indexes i where the function β_i is different from zero for every argument $x \in \widehat{E}$. Therefore, we can take $\delta = \min_i\{\delta_i\}$

so $\delta > 0$; this way, for $x \in B_\delta(x_0)$,

$$\begin{aligned} \|V(x) - V(x_0)\| &= \left\| \sum_i z_i \beta_i(x) - \sum_i z_i \beta_i(x_0) \right\| \\ &= \left\| \sum_i (\beta_i(x) - \beta_i(x_0)) z_i \right\| \\ &\leq \sum_i |\beta_i(x) - \beta_i(x_0)| \, \|z_i\| \\ &\leq \|x - x_0\| \sum_i L_i \, \|z_i\| \, . \end{aligned}$$

Notice that the final sum is finite from the start because of the features of β_i so taking the Lipschitz constant $L = \sum_i L_i \, \|z_i\|$ we obtain the result. □

We will not pursue a formalization of the latter proof in this work. We just draw attention at how the functions ρ_i are defined in (6.1). Since in general the sets $\widehat{E} \setminus M_i$ are closed, then in a first instance it appears that $dist(x, \widehat{E} \setminus M_i)$ would need the power of Π^1_1-CA_0 to be coded continuous (Theorem 3.20); but of course, there is the possibility to tighten the result to ACA_0 by showing that the mentioned sets are in fact separably closed (Theorem 3.21). If achieved, we can use the pseudo-gradient vector field to construct a initial value problem as in our proof of the Deformation Lemma and prove the lemma itself for Banach spaces. At the end, we conjecture that a generalized MPT for Banach spaces could be formalized in ACA_0.

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